Introduction to Machine Learning (IML)

LECTURE #7: OPTIMIZATION

236756 – 2022-2023 WINTER – TECHNION

LECTURER: YONATAN BELINKOV

Today

- **Part II**: *the different aspects of learning*
  1. Statistics: generalization and PAC theory
  2. Modeling:
     - Error decomposition
     - Regularization
     - Model selection
  3. Optimization: convexity, gradient descent (today)
  4. Practical aspects and potential pitfalls

Optimization

Optimizing the learning objective

- **Regularized Risk Minimization** (RRM):

$$\operatorname*{argmin}_{w\in\mathbb{R}^d} \boxed{\frac{1}{m}\sum_{i=1}^{m} \underbrace{\ell(y_i, w^\top x_i)}_{\text{loss}}} + \underbrace{\lambda R(w)}_{\text{regularization}}$$

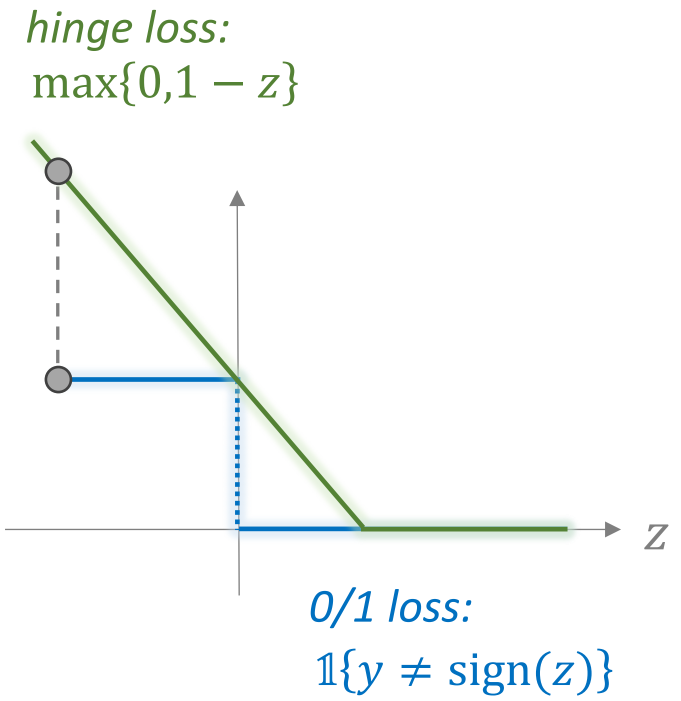
- **Recall**: ERM of 0/1 is hard (discrete)
- **Solution**: continuous proxy

Optimizing the learning objective

- **Regularized Risk Minimization** (RRM):

$$\underset{w\in\mathbb{R}^d}{\operatorname{argmin}} \boxed{\frac{1}{m}\sum_{i=1}^{m}\underbrace{\ell(y_i, w^\top x_i)}_{\textbf{loss}}} + \underbrace{\lambda R(w)}_{\textbf{regularization}}$$

- **Recall**: ERM of 0/1 is hard (discrete)
- **Solution**: continuous proxy
- Soft SVM's *hinge loss* is one option
- **Problem**: unnecessarily large penalties for far from margin

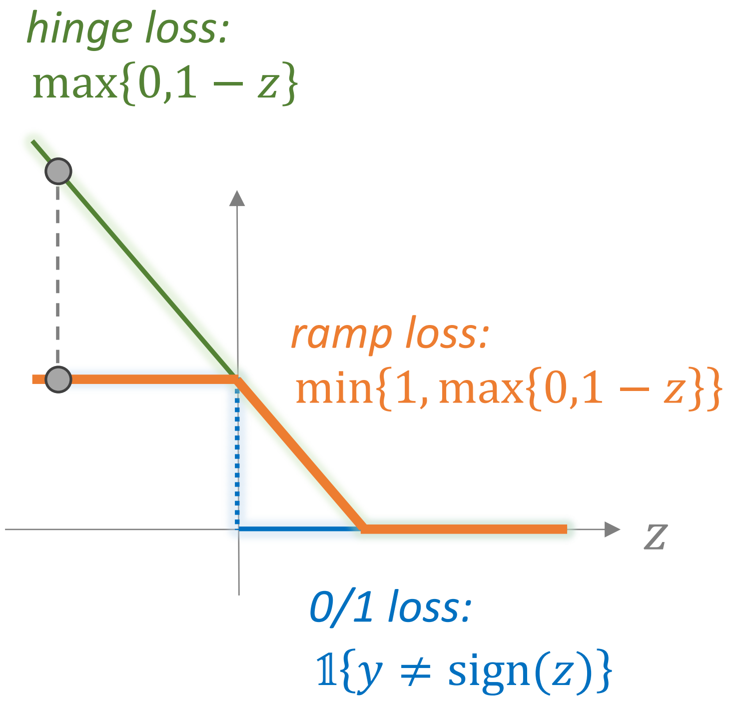


Optimizing the learning objective

- **Regularized Risk Minimization** (RRM):

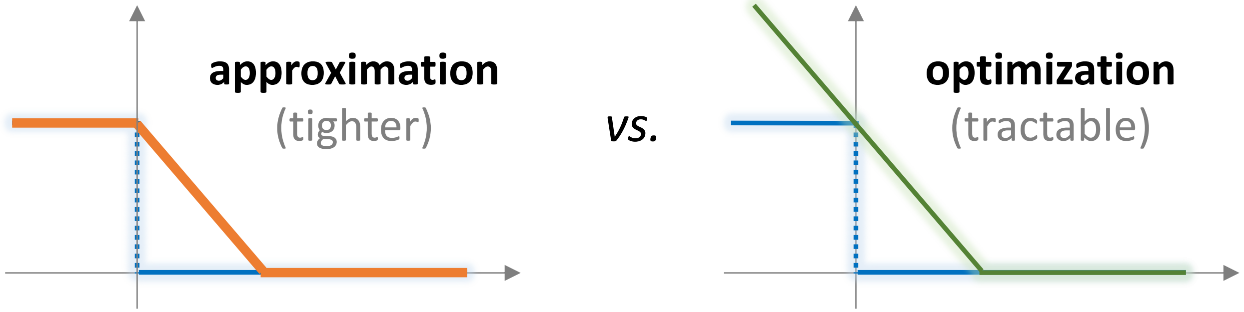
$$\underset{w\in\mathbb{R}^d}{\operatorname{argmin}} \boxed{\frac{1}{m}\sum_{i=1}^{m} \underset{\textbf{loss}}{\ell(y_i, w^\top x_i)}} + \underset{\textbf{regularization}}{\lambda R(w)}$$

- **Recall**: ERM of 0/1 is hard (discrete)
- **Solution**: continuous proxy
- Soft SVM's *hinge loss* is one option
- **Problem**: unnecessarily large penalties for far from margin
- **Alternative**: *ramp loss*
- **Problem**: optimization… (today we'll see why)



Optimizing the learning objective

- **Tradeoff**:



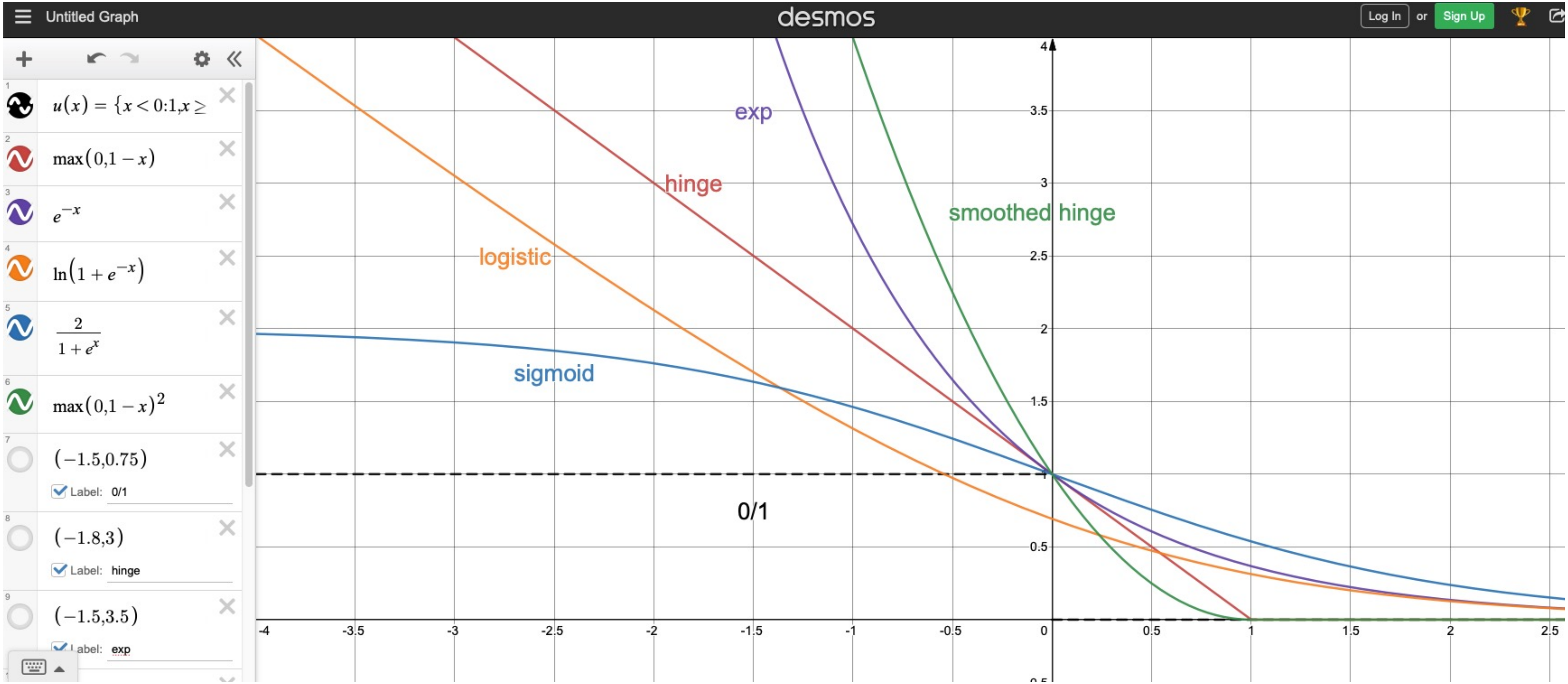
approximation
(tighter)

vs.

optimization
(tractable)

- Many proxies out there!
- [DESMOS]

Proxy losses



Untitled Graph

1 $u(x) = \{x < 0{:}1, x \geq$

2 $\max(0,1-x)$

3 e^{-x}

4 $\ln(1+e^{-x})$

5 $\frac{2}{1+e^{x}}$

6 $\max(0,1-x)^2$

7 $(-1.5,0.75)$

☑ Label: 0/1

8 $(-1.8,3)$

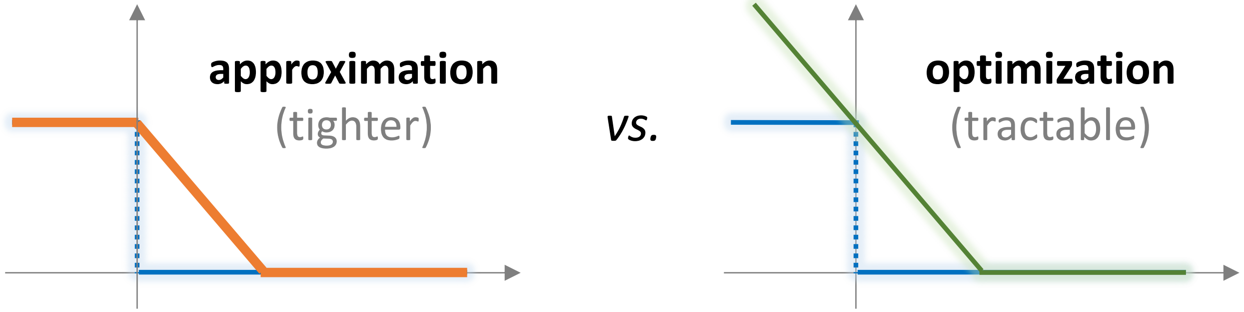
☑ Label: hinge

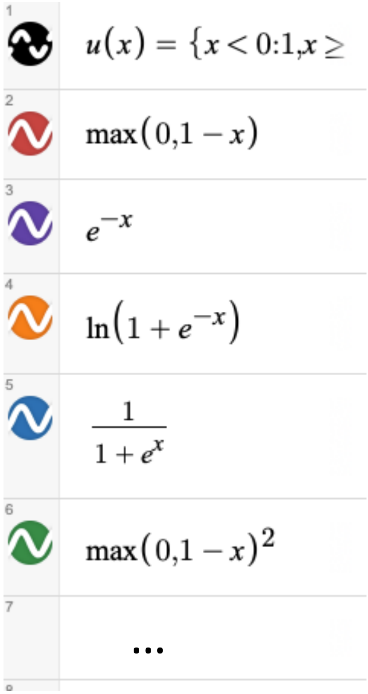
9 $(-1.5,3.5)$

☑ Label: exp

Optimizing the learning objective

- **Tradeoff**:





1 $u(x) = \{x < 0{:}1, x \geq$

2 $\max(0, 1 - x)$

3 e^{-x}

4 $\ln\left(1 + e^{-x}\right)$

5 $\frac{1}{1 + e^{x}}$

6 $\max(0, 1 - x)^2$

7 …

- Many proxies out there – which is better?
- No definite answer
- But we can say some things sometimes
- **Today**: focus on optimization (but also generalization)
- Functional properties that give optimization guarantees:
  - (sub-)differentiable
  - convex
  - Lipschitz
  - smooth
  - bounded

Gradient descent

https://en.wikipedia.org/wiki/Gradient_descent

One algorithm to rule them all

- **Goal**: solve $\min_{w \in \mathbb{R}^d} f(w)$
- (in learning: f is the learning objective, w are model parameters)

- **Approach**: Gradient Descent (GD) [~Cauchy 1847]

$$w_{t+1} = w_t - \eta \nabla f(w_t)$$

- "First order iterative method"
- Forms the basis for many other optimization algorithms
  (we'll see some today)

Gradient descent

- **Definition**:
  The *gradient* of f w.r.t. w is a vector function of partial derivatives,

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial w_1} \\ \vdots \\ \frac{\partial f}{\partial w_d} \end{pmatrix}$$

- *Gradient evaluation* at $\overline{w} \in \mathbb{R}^d$:

$$\nabla f(\overline{w}) \in \mathbb{R}^d$$

Gradient Descent (GD)

- Initialize w_0 (e.g., $w_0 = \vec{0}$)
- Repeat:
  - $w_{t+1} = w_t - \eta \nabla f(w_t)$
- Until convergence (e.g., $||w_{t+1} - w_t|| \leq \epsilon$)

Gradient descent

- **View I**: fog in mountains
- **View II**: local greedy descent
- **View III**: optimize simple approximations

$$@t: \;\; f(w) \approx f(w_t) + \nabla f(w_t)^\top (w - w_t)$$

- **View IV**: Taylor expansion + tradeoff
  - $f(w_t + \delta) \approx f(w_t) + \nabla f(w_t)^\top \delta, \;\; \delta \in \mathbb{R}^d$
  - $w_{t+1} = w_t + \delta$ close to w_t, i.e., $\|\delta\|_2$ is small
  - $w_{t+1} = \underset{w}{\operatorname{argmin}} \frac{1}{2} \|w - w_t\|_2^2 + \eta\big(f(w_t) + \nabla f(w_t)^\top (w - w_t)\big)$
  - $\partial = 0 \Rightarrow w_{t+1} = w_t - \eta \nabla f(w_t)$

Gradient Descent (GD)

- Initialize w_0 (e.g., $w_0 = \vec{0}$)
- Repeat:
  - $w_{t+1} = w_t - \eta \nabla f(w_t)$
- Until convergence (e.g., $\|w_{t+1} - w_t\| \leq \epsilon$)

Gradient descent

- **View I**: fog in mountains
- **View II**: local greedy descent
- **View III**: optimize simple approximations

$$@t:\ \ f(w) \approx \nabla f(w_t)^\top w$$

- **View IV**: Taylor expansion + tradeoff
  - $f(w_t + \delta) \approx f(w_t) + \nabla f(w_t)^\top \delta,\ \ \delta \in \mathbb{R}^d$
  - $w_{t+1} = w_t + \delta$ close to w_t, i.e., $\|\delta\|_2$ is small
  - $w_{t+1} = \underset{w}{\operatorname{argmin}} \frac{1}{2}\|w_t - w\|_2^2 + \eta\big(f(w_t) + \nabla f(w_t)^\top (w - w_t)\big)$
  - $\partial = 0 \Rightarrow w_{t+1} = w_t - \eta \nabla f(w_t)$

Gradient Descent (GD)

- Initialize w_0 (e.g., $w_0 = \vec{0}$)
- Repeat:
  - $w_{t+1} = w_t - \eta \nabla f(w_t)$
- Until convergence (e.g., $\|w_{t+1} - w_t\| \leq \epsilon$)

Need to determine:

1. Step size or learning rate η
2. Stopping criterion

Gradient descent

- **Definition**: Let $B_\epsilon(w) = \{v : \ \|w - v\| \leq \epsilon\}$ be a ball of radius ϵ around w. Then w is a ***local minimum*** of f if

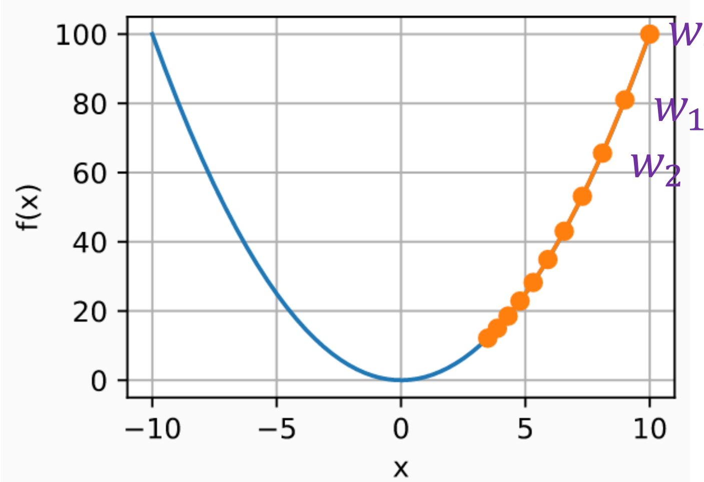
$$\exists\epsilon > 0\ \forall v \in B_\epsilon(w),\ \ f(w) \leq f(v)$$

- **Claim**: if η is small enough, then $f(w_{t+1}) < f(w_t)$ for any non-minimum w_t (this is the “descent” part of “gradient descent”)
- **Proof**: consider non-minimum w_t and negate

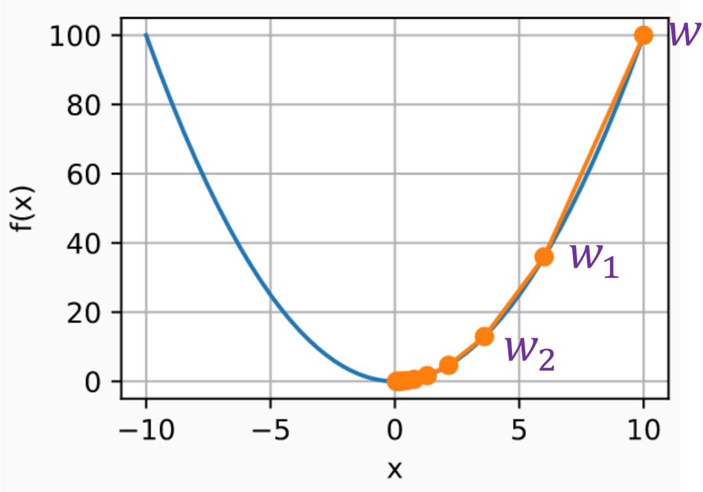
- **Corollary**: GD converges* to a local minimum (or saddle point!)
  * for an appropriate choice of η
- **Trick**: use time-varying learning rate η_t, for example, $\eta_t = 1/t$
- **Note**: if w_t is a local minimum then $\nabla f(w_t) = 0$ (but not iff!)

Learning rate

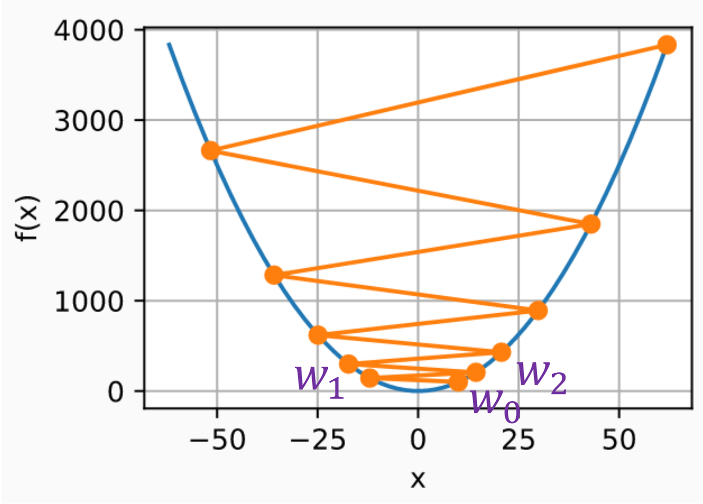
- **Q1**: So GD converges for learning rate $\eta_t = 1/t$. Are we done?
- **A1**: No!



$\eta_t = 1/t$
slow convergence
(too slow!)



$\eta_t = 1/\sqrt{t}$
“just right”
(how can we know?)



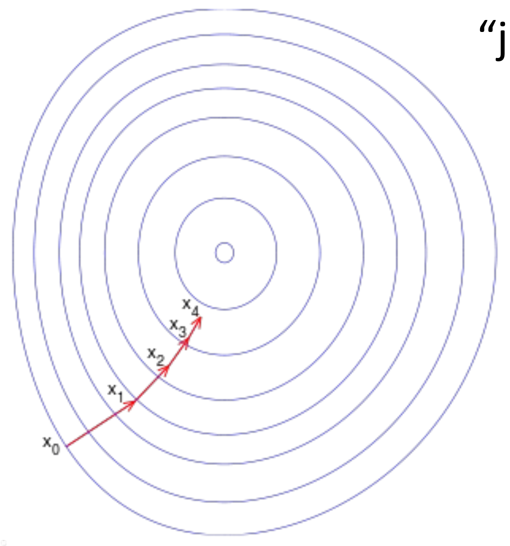
$\eta_t = 10$
divergence
(too fast!)

(illustrative)

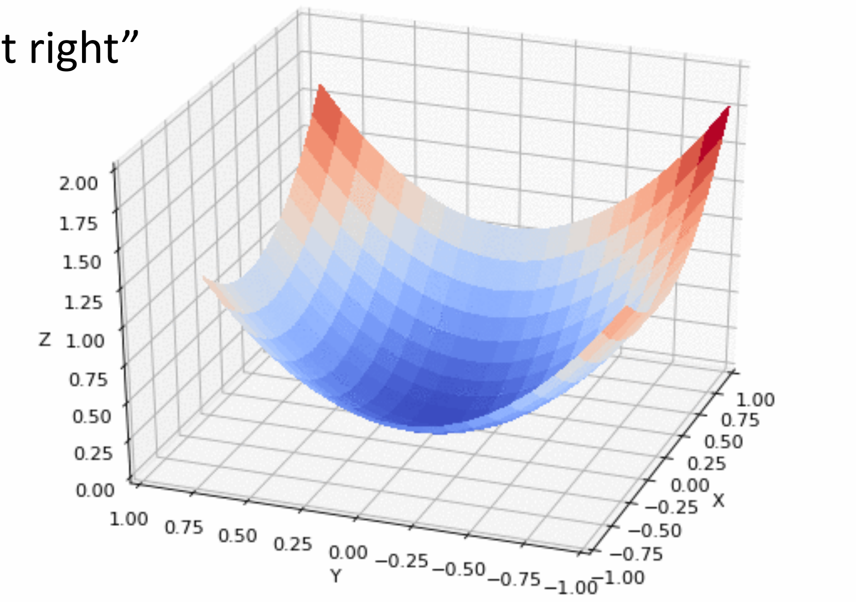
https://d2l.ai/chapter_optimization/

Learning rate

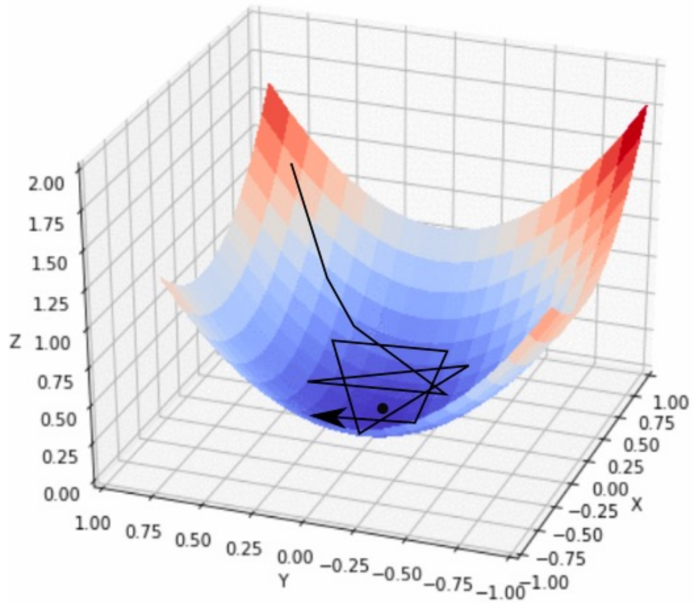
- **Q1**: So GD converges for learning rate $\eta_t = 1/t$. Are we done?
- **A1**: No!



“just right”



divergence?
slow convergence?



- Choosing the learning rate can be tricky – we will return to this.

https://blog.paperspace.com/intro-to-optimization-in-deep-learning-gradient-descent/

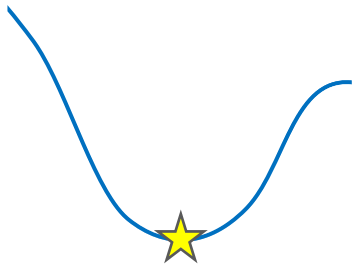
When to stop

- **Q2**: So GD converges in theory. But in practice, how do we know when (and if) it did?
- **A2**: Can't "know", but can estimate.

- Set criteria for stopping, for example:
  - stop when $\|w_{t+1} - w_t\| \leq \epsilon$
  - or when $\|f(w_{t+1}) - f(w_t)\| \leq \epsilon$
  - or when $\|\nabla f(w_t)\| \leq \epsilon$
  - or …
- Actually, knowing when to stop is (also) tricky – and has statistical implications! We will return to this as well.

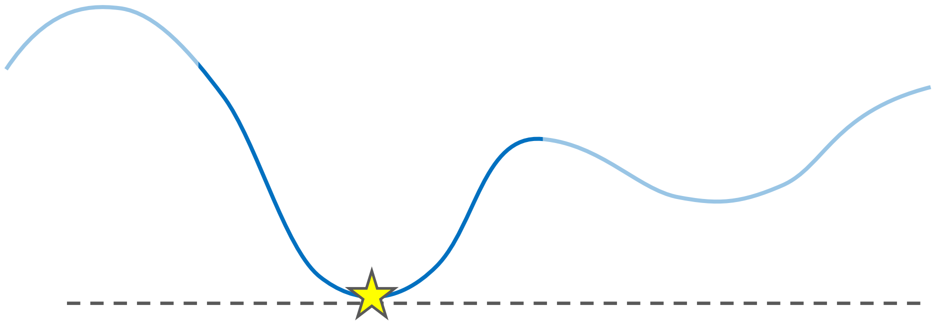
Local minima

- **Q3**: So GD can converge to a local minimum. Should we be happy?
- **A3**: It depends…



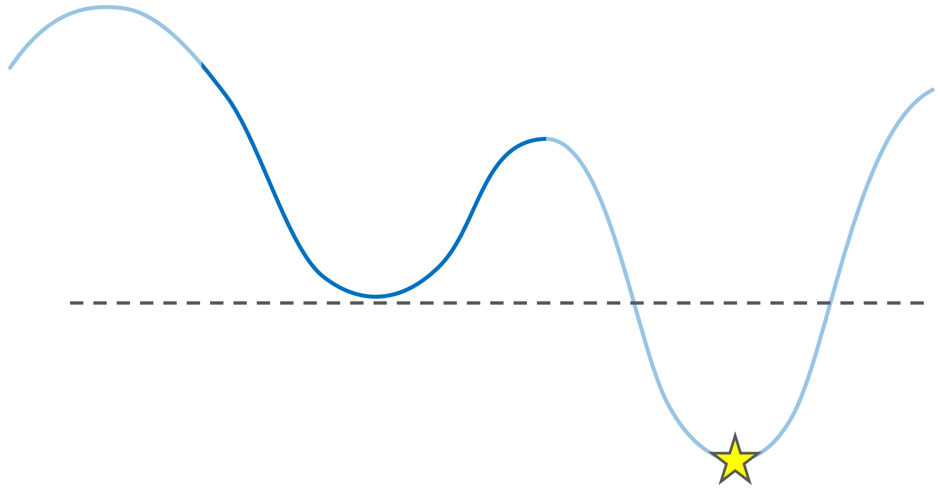
Local minima

- **Q3**: So GD can converge to a local minimum. Should we be happy?
- **A3**: It depends…



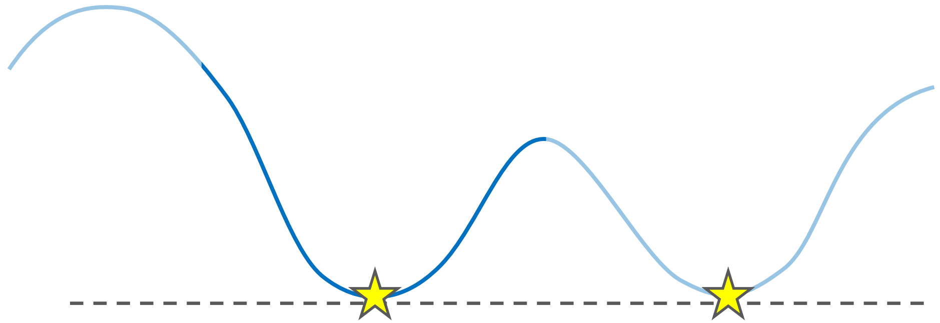
Local minima

- **Q3**: So GD can converge to a local minimum. Should we be happy?
- **A3**: It depends…



Local minima

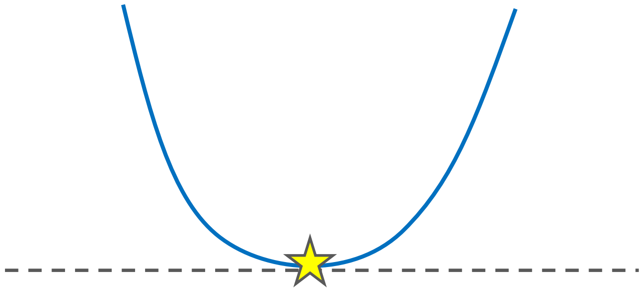
- **Q3**: So GD can converge to a local minimum. Should we be happy?
- **A3**: It depends…



- **Goal:** find conditions on f for which GD is guaranteed to work well

Local minima

- **Q3**: So GD can converge to a local minimum. Should we be happy?
- **A3**: It depends…



- **Goal:** find conditions on f for which GD is guaranteed to work well

Convexity

Descending with guarantees

- Recall foggy mountain story
- When will the "going down" approach help you reach your cabin?

- When mountain range is:
    1. continuous
    2. smooth
    3. **has a single valley**

- Enter **convexity.**
- [on board]

https://en.wikipedia.org/wiki/Gradient_descent

Convex sets

- **Definition:** *convex combination*:

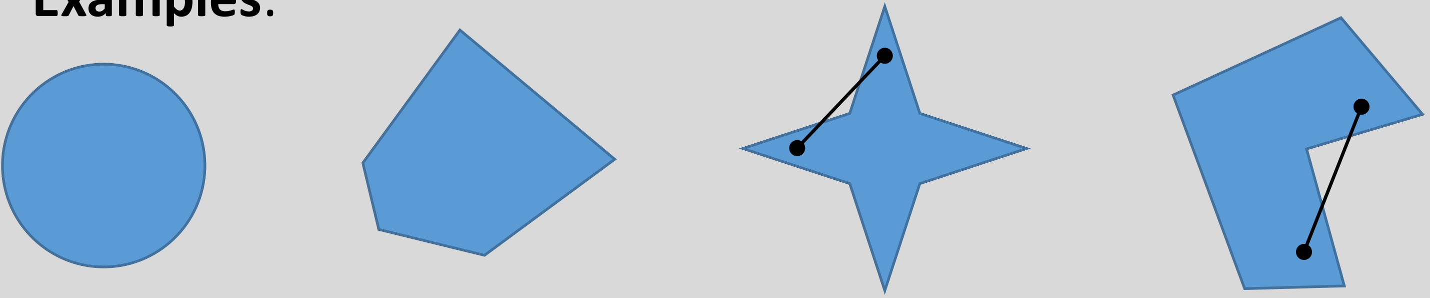
$$\alpha u + (1 - \alpha) v \quad \text{for} \quad u, v \in \mathbb{R}^d, \alpha \in [0,1]$$

- As interpolation – point on line between u, v
- **Definition**: C is a *convex set* if

$$\forall u, v \in C, \text{ any convex combination of } u, v \text{ is also in } C$$

$$\mathbb{R}^d$$

- **Examples**:



$$H_{w,b} = \{x : w^\top x \geq b\}$$

- **Definition:** For C convex set, $f: C \to \mathbb{R}$ is *convex function* if $\forall u, v \in C$, $\alpha \in [0,1]$:

$$f(\alpha u + (1 - \alpha) v) \leq \alpha f(u) + (1 - \alpha) f(v)$$

GD convergence rates

- β-Smooth: $f(x_t) - f(x^*) \le \frac{2\beta\|x_0 - x^*\|}{t+4} = O\left(\frac{\beta}{t}\right)$
  - Lower bound: $\Omega\left(\frac{\beta}{t^2}\right)$
- L-Lipschitz: $f(x_t) - f(x^*) = O\left(\frac{L}{\sqrt{t}}\right)$
- σ-Strongly convex: $f(x_t) - f(x^*) = O(e^{-2\sigma t})$

- **Depends on:**
  - Type
  - Parameter
  - Initial guess!

f is β-Smooth if

$$\|\nabla f(u) - \nabla f(w)\| \le \beta\|u - w\|$$

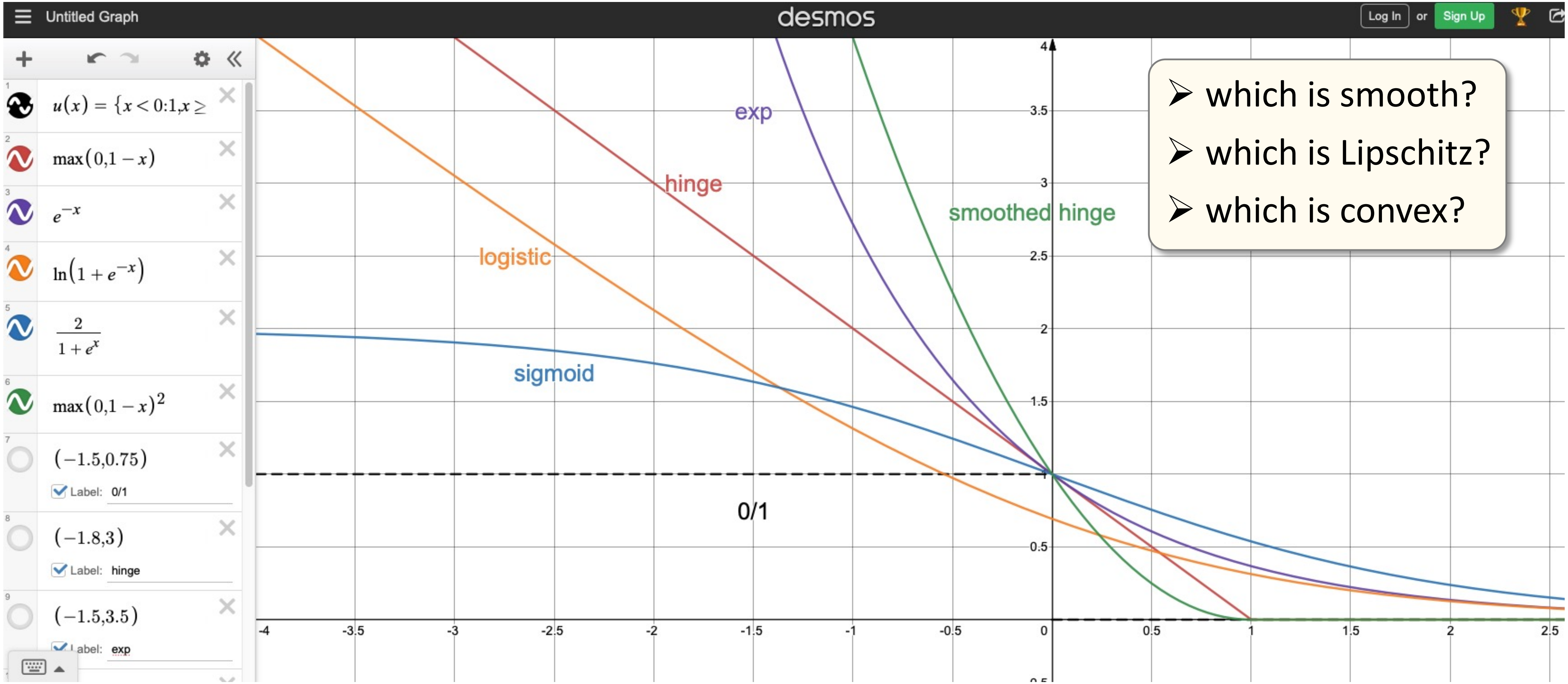
f is L-Lipschitz if

$$\|f(u) - f(w)\| \le L\|u - w\|$$

f is σ-Strongly convex if

$$f(\alpha w + (1-\alpha)u)$$
$$\le \alpha f(w) + (1-\alpha)f(u) - \frac{\sigma}{2}\alpha(1-\alpha)\|u - w\|^2$$

GD convergence rates



1. $u(x) = \{x < 0 : 1, x \geq$
2. $\max(0, 1 - x)$
3. e^{-x}
4. $\ln\left(1 + e^{-x}\right)$
5. $\frac{2}{1 + e^{x}}$
6. $\max(0, 1 - x)^2$
7. $(-1.5, 0.75)$ ☑ Label: 0/1
8. $(-1.8, 3)$ ☑ Label: hinge
9. $(-1.5, 3.5)$ ☑ Label: exp

- ➢ which is smooth?
- ➢ which is Lipschitz?
- ➢ which is convex?

https://www.desmos.com/calculator/t3tbluo2rf

Optimization for learning

Convex learning problems

- **Recall ERM/RLM:**

$$\underset{\theta}{\operatorname{argmin}} \frac{1}{m} \sum_{i=1}^{m} \ell\big(y_i, f_\theta(x_i)\big) + \lambda R(\theta)$$

- How can we tell if the learning objective is convex?

Composition rules

- **Convexity-preserving operations:**
  1. **scaling**: f convex, $\alpha \geq 0 \Rightarrow \alpha f$ convex [easy]
  2. **sum**: f, g convex $\Rightarrow f + g$ convex [tirgul]
  3. **noneg. weighted sum**: f_i convex, $\alpha_i \geq 0 \Rightarrow \sum_i \alpha_i f_i$ convex
  4. **composition**: f convex, g linear $\Rightarrow f \circ g$ convex [on board]
  5. …
- (these are sufficient conditions, but not necessary)

Composition of convex and linear is convex

- **Claim**: if g is convex and h is linear, then $f = g \circ h$ is convex
- **Proof**: $f(\alpha u + (1-\alpha)v) = g\big(h(\alpha u + (1-\alpha)v)\big) =_{h\ linear}$ $g\big((\alpha u + (1-\alpha)v)^T x + b\big) = g(\alpha u^T x + (1-\alpha)v^T x + b) =$ $g(\alpha u^T x + (1-\alpha)v^T x + (\alpha + 1 - \alpha)b) =$ $g(\alpha(u^T x + b) + (1-\alpha)(v^T x + b)) = g\big(\alpha h(u) + (1-\alpha)h(v)\big)$ $\leq_{g\ convex} \alpha g\big(h(u)\big) + (1-\alpha)g\big(h(v)\big) = \alpha f(u) + (1-\alpha)f(v)$

Convex learning problems

- **Claim**: The following conditions are sufficient for the learning objective to be convex (in w):
  1. $\ell(\cdot,\cdot)$ is convex (in it's second argument)
  2. $R(\cdot)$ is convex
  3. f_w is linear (in w, i.e., $f_w(x) = w^\top x$)

Convex learning problems

- **Claim**: The following conditions are sufficient for the learning objective to be convex (in w):
  1. $\ell(\cdot,\cdot)$ is convex (in it's second argument)
  2. $R(\cdot)$ is convex
  3. f_w is linear (in w, i.e., $f_w(x) = w^\top x$)
- **Let's try** →

$$\underbrace{\underset{\substack{\uparrow\\ \text{conv}}}{\underset{w\in\mathbb{R}^d}{\operatorname{argmin}}} \underbrace{\overset{\substack{\geq 0\\ \downarrow}}{\frac{1}{m}} \sum_{i=1}^{m} \underbrace{\overset{\substack{\text{conv}\\ \downarrow}}{\ell}\left(y_i, \overset{\substack{\text{lin}\\ \downarrow}}{f_w}(x_i)\right)}_{\text{conv}}}_{\text{conv}} + \underbrace{\overset{\substack{\geq 0\\ \downarrow}}{\lambda}\overset{\substack{\text{conv}\\ \downarrow}}{R}(w)}_{\text{conv}}}_{\text{conv}}$$

Convex learning problems

- **Claim**: The following conditions are sufficient for the learning objective to be convex (in w):
  1. $\ell(\cdot,\cdot)$ is convex (in it's second argument)
  2. $R(\cdot)$ is convex
  3. f_w is linear (in w, i.e., $f_w(x) = w^\top x$)

$$\underbrace{\underset{\substack{\uparrow\\ \text{conv}}}{\operatorname*{argmin}_{w\in\mathbb{R}^d}} \underbrace{\overset{\substack{\geq 0\\ \downarrow}}{\frac{1}{m}} \sum_{i=1}^{m} \underbrace{\overset{\substack{\text{conv}\\ \downarrow}}{\ell}\left(y_i, \overset{\substack{\text{lin}\\ \downarrow}}{f_w}(x_i)\right)}_{\text{conv}}}_{\text{conv}} + \underbrace{\overset{\substack{\geq 0\\ \downarrow}}{\lambda}\overset{\substack{\text{conv}\\ \downarrow}}{R}(w)}_{\text{conv}}}_{\text{conv}}$$

- **Let's try** →
- **Claim:** The Soft SVM objective is convex
- **Need to prove:**
  1. $\max\{0,1-z\}$ is convex [ex]
  2. ℓ_2-norm squared is convex [tirgul]
- **Corollary**: can solve with GD!

$$\operatorname*{argmin}_{w\in\mathbb{R}^d} \frac{1}{m}\sum_{i=1}^{m} \max\{0,1-y_i\cdot w^\top x_i\} + \lambda\|w\|_2^2$$

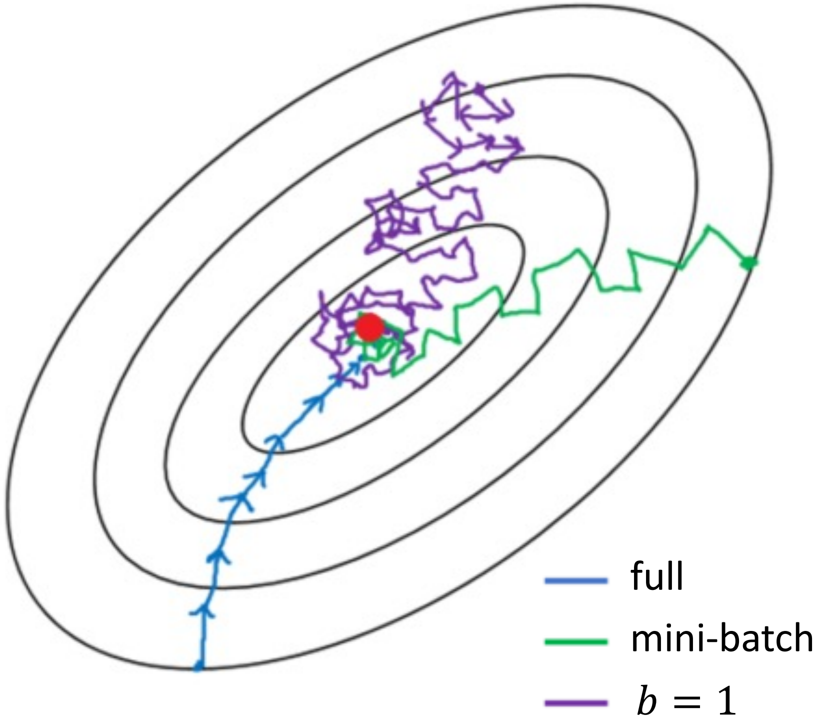
Variations, extensions, and beyond

The computational cost of GD

- For running GD we need to compute gradients of the learning objective.
- **Note**: $\nabla\left(\frac{1}{m}\sum_{i=1}^{m}\ell(y_i, f_w(x_i))\right) = \frac{1}{m}\sum_{i=1}^{m}\underbrace{\nabla\ell(y_i, f_w(x_i))}_{\nabla_i}$
- **For Soft SVM**:
  - Each $\nabla \max\{0,1-y_i\cdot w^\top x_i\}$ costs: $O(d)$
  - Overall: $O(dm)$
- GD can be costly!
- **Solution**: use *approximate* gradients

Taking approximate gradient steps

- **Observation**: sampling average is unbiased (but noisy) estimator
- **Idea**: replace full gradient with average of small random set of examples
- **Stochastic Gradient Descent (SGD):***
    1. Sample small random “mini-batch” $B \subset S$ of size b
    2. Compute average gradient $\bar{\nabla} = \frac{1}{b}\sum_{i \in B} \nabla_i$
    3. Apply *approximate* gradient step $x_{t+1} = x_t - \eta\bar{\nabla}$
- **Pros**: reduces compute time (significantly!)
- **Cons**: adds noise (but often worth it)
- Hyper parameter b trades off compute time with noise



---

* The name “SGD” typically refers to the $b = 1$ case

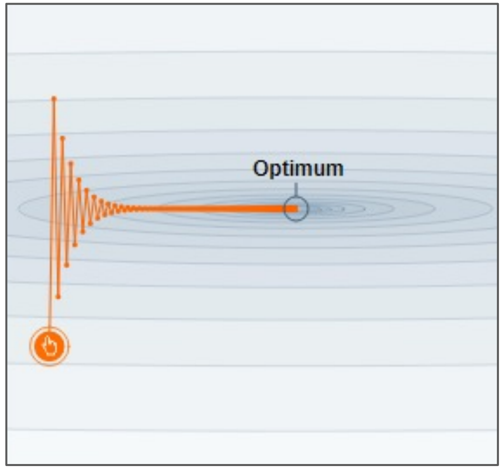
Looking back

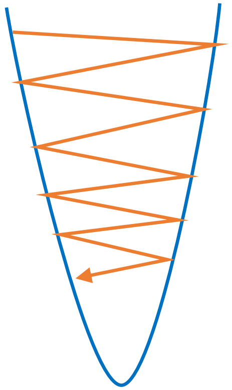
- Consider 2D quadratic with highly varying curvatures
  - On low-curvature dimension, GD crawls hesitantly
  - On high-curvature dimension, GD oscillates frantically
- Adding “momentum” sorts this out:

$$
\begin{aligned}
v_{t+1} &= \beta v_t - \eta \nabla f(w_t) && \rightarrow \textit{velocity/“memory”} \\
w_{t+1} &= w_t + v_{t+1} && \rightarrow \textit{position} \\
&= w_t - \eta \nabla f(w_t) + \beta v_t
\end{aligned}
$$

(imagine a rolling ball)

- Effects of momentum:
  - *increases* in dimensions where gradient preserves direction
  - *decreases* in dimensions where gradient direction varies
- Typical β values: 0.9, 0.95, 0.99

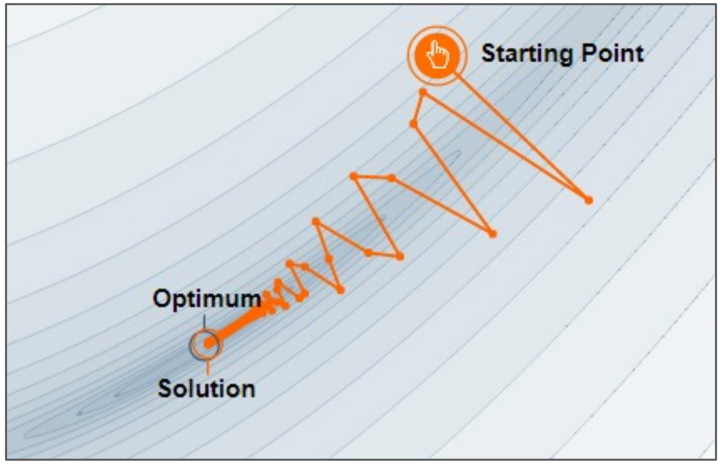


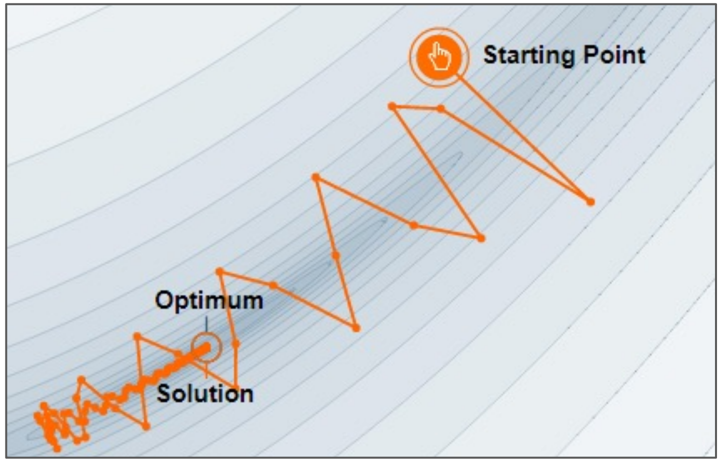


https://distill.pub/2017/momentum/

Looking forward

- Momentum is great, but can overshoot
- **Solution**: look into *future*
- **Idea:** *imagine* momentum has been applied, then compute gradient





https://distill.pub/2017/momentum/

Looking forward

- Momentum is great, but can overshoot
- **Solution**: look into *future*
- **Idea:** *imagine* momentum has been applied, then compute gradient

- **Nesterov's accelerated gradient:**

$$\tilde{w}_{t+1} = w_t + \beta v_t$$

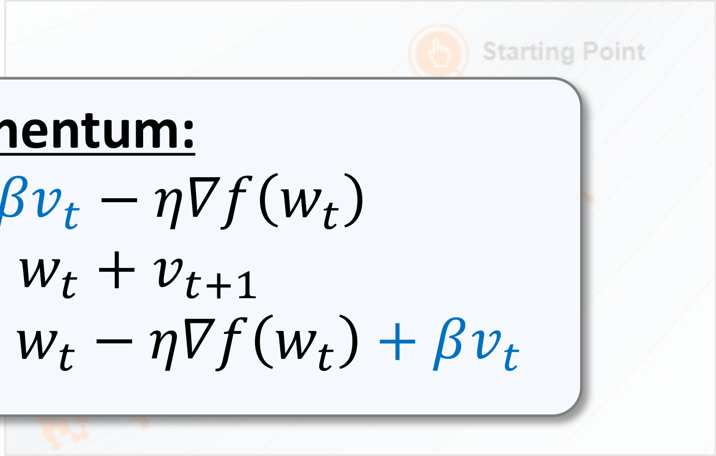
$$v_{t+1} = \beta v_t - \eta \nabla f(\tilde{w}_{t+1})$$

$$w_{t+1} = w_t + v_{t+1}$$

- **Equivalent**:

$$v_{t+1} = \beta v_t - \eta \nabla f(w_t)$$

$$w_{t+1} = w_t + (1+\beta) v_{t+1} - \beta v_t$$



Vs. momentum:

$$v_{t+1} = \beta v_t - \eta \nabla f(w_t)$$

$$w_{t+1} = w_t + v_{t+1}$$

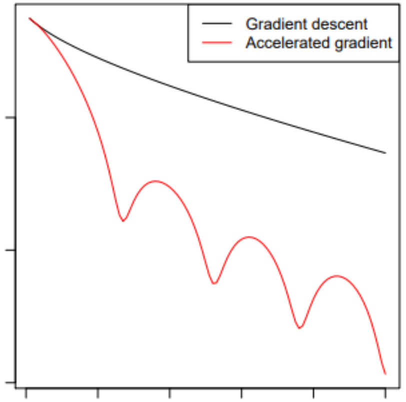
$$= w_t - \eta \nabla f(w_t) + \beta v_t$$

Looking forward

- **Nesterov's accelerated gradient:**

$$v_{t+1} = \beta v_t - \eta \nabla f(w_t)$$
$$w_{t+1} = w_t + (1+\beta) v_{t+1} - \beta v_t$$

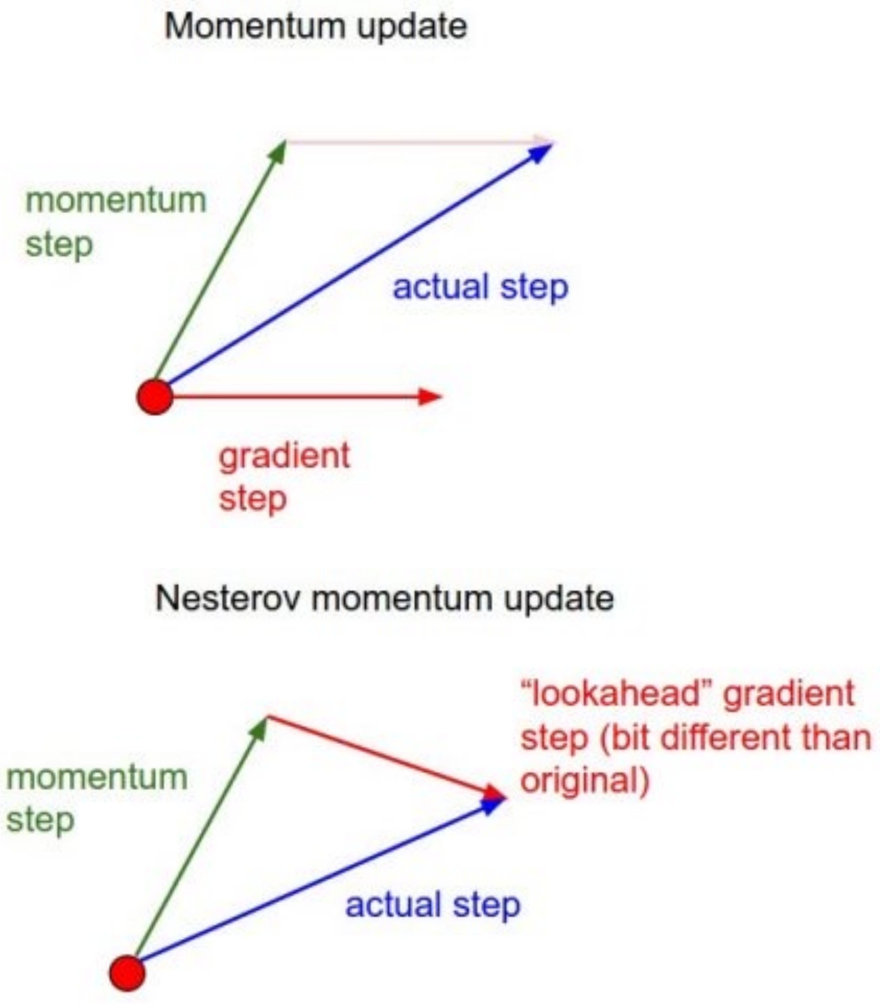
- This is not a descent method!



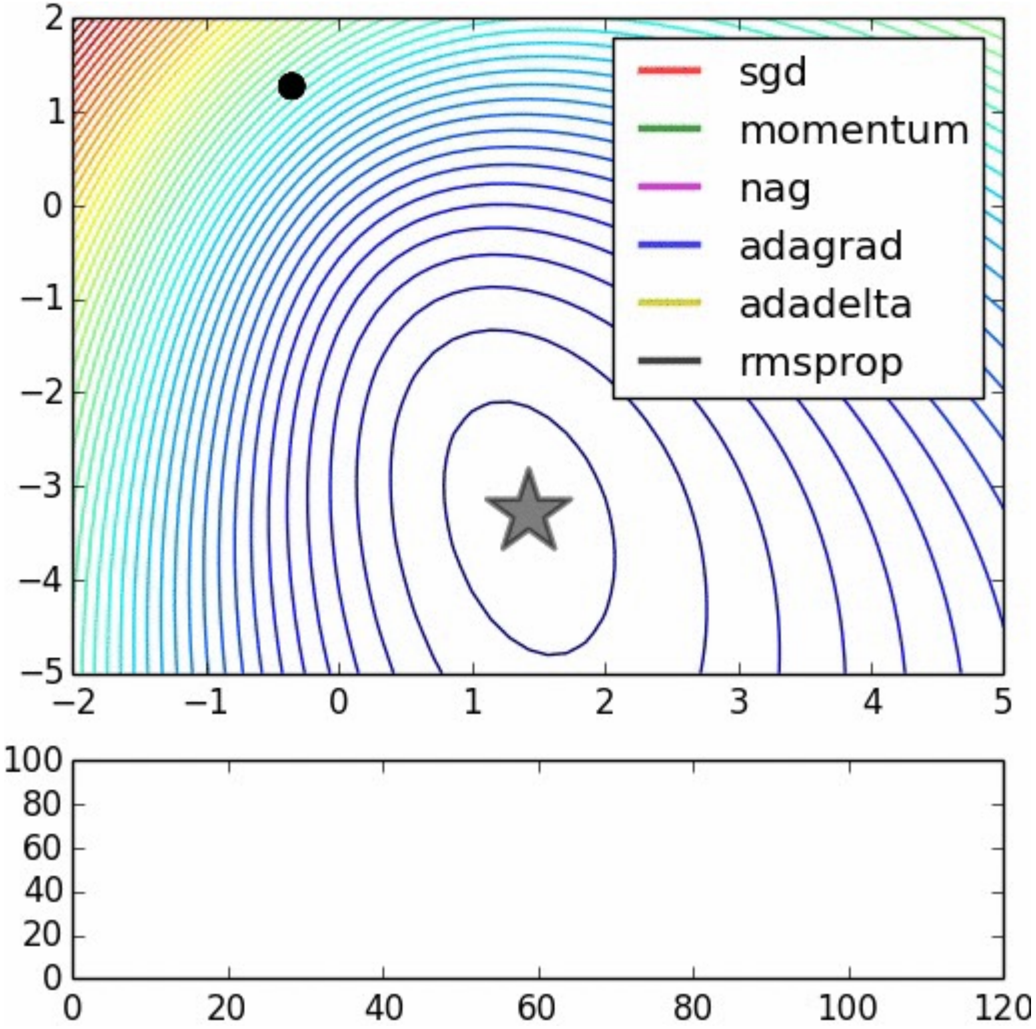
- Nonetheless, converges faster:

$$f(x_t) - f(x^*) \leq \frac{2\beta \|x_0 - x^*\|}{t^2} = O\left(\frac{\beta}{t^2}\right)$$

← *matches lower bound!*



https://cs231n.github.io/neural-networks-3/


2
1
0
−1
−2
−3
−4
−5
−2
−1
0
1
2
3
4
5
sgd
momentum
nag
adagrad
adadelta
rmsprop
100
80
60
40
20
0
0
20
40
60
80
100
120

Beyond convexity

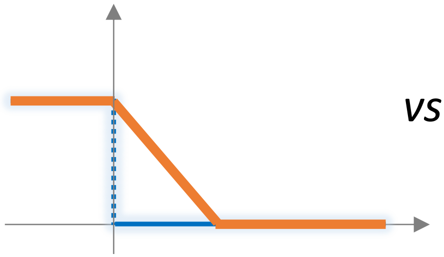
Back to modelling

- **Recall**:
  - Really want to optimize 0/1 loss
  - Instead optimize a continuous proxy
  - Proxies trade off in approximation vs. optimization
- GD provides strong guarantees for *convex* objectives
- But can still be applied to non-convex objectives!
  - ➢ non-convex losses
  - ➢ non-convex regularizers
  - ➢ non-convex predictive models

(just need differentiability)

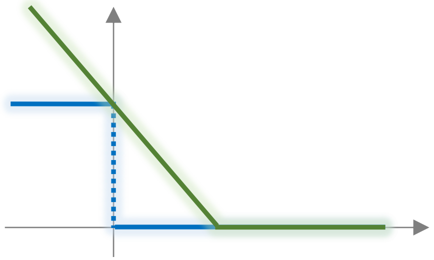
- Finds only <u>local</u> minimum, but many “tricks” for ending up at good local minimum

$$\underset{w \in \mathbb{R}^d}{\operatorname{argmin}} \frac{1}{m} \sum_{i=1}^{m} \overset{\textit{conv}}{\overset{\downarrow}{\ell}}\Big(y_i, \overset{\textit{lin}}{\overset{\downarrow}{f_w}}(x_i)\Big) + \lambda \overset{\textit{conv}}{\overset{\downarrow}{R}}(w)$$



approximation
(tighter)

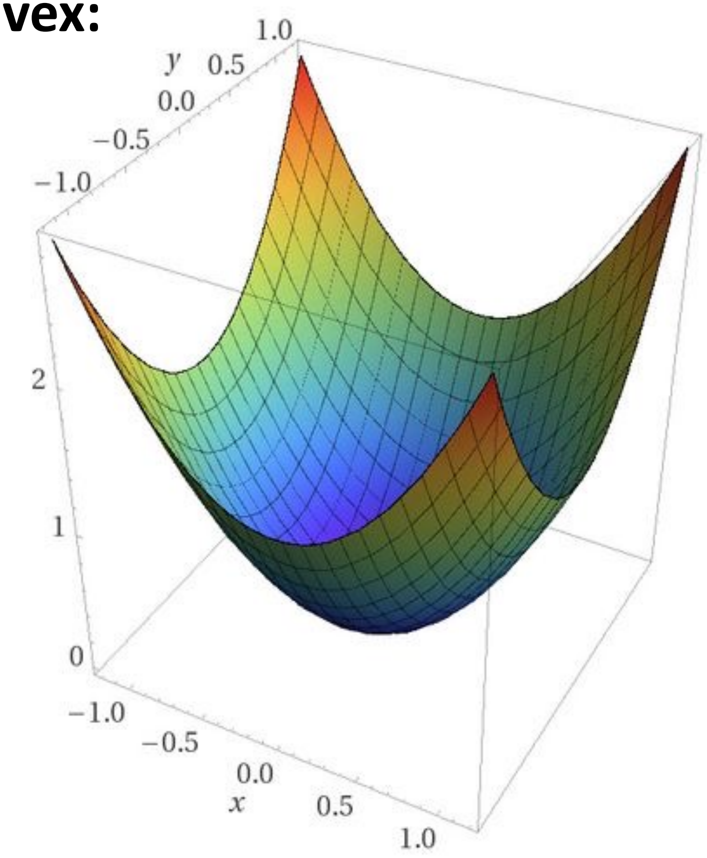
vs.



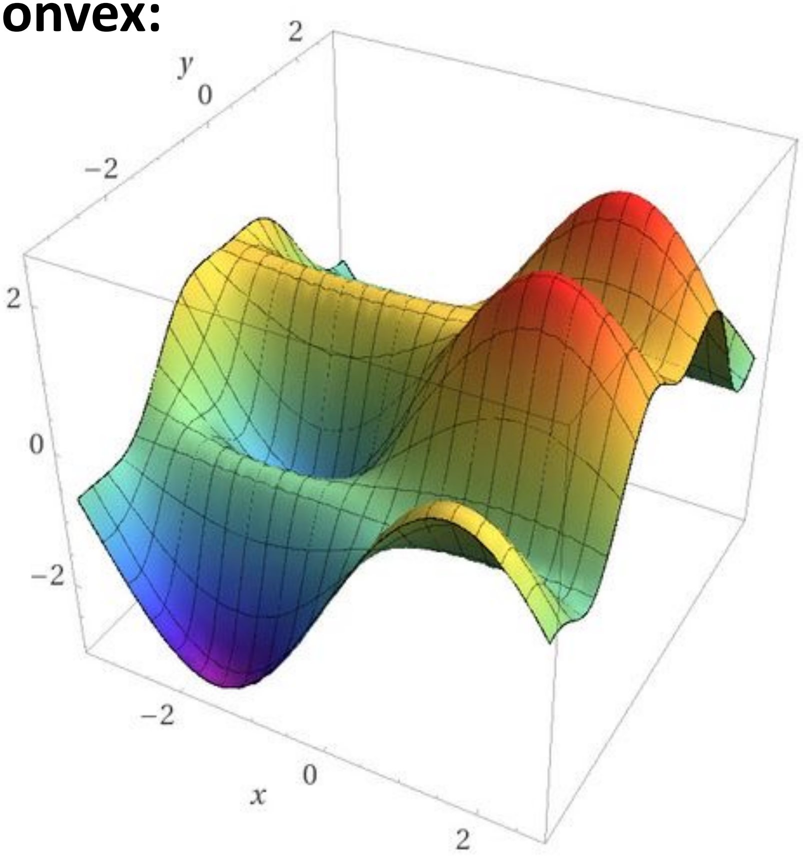
optimization
(tractable)

Optimization landscape

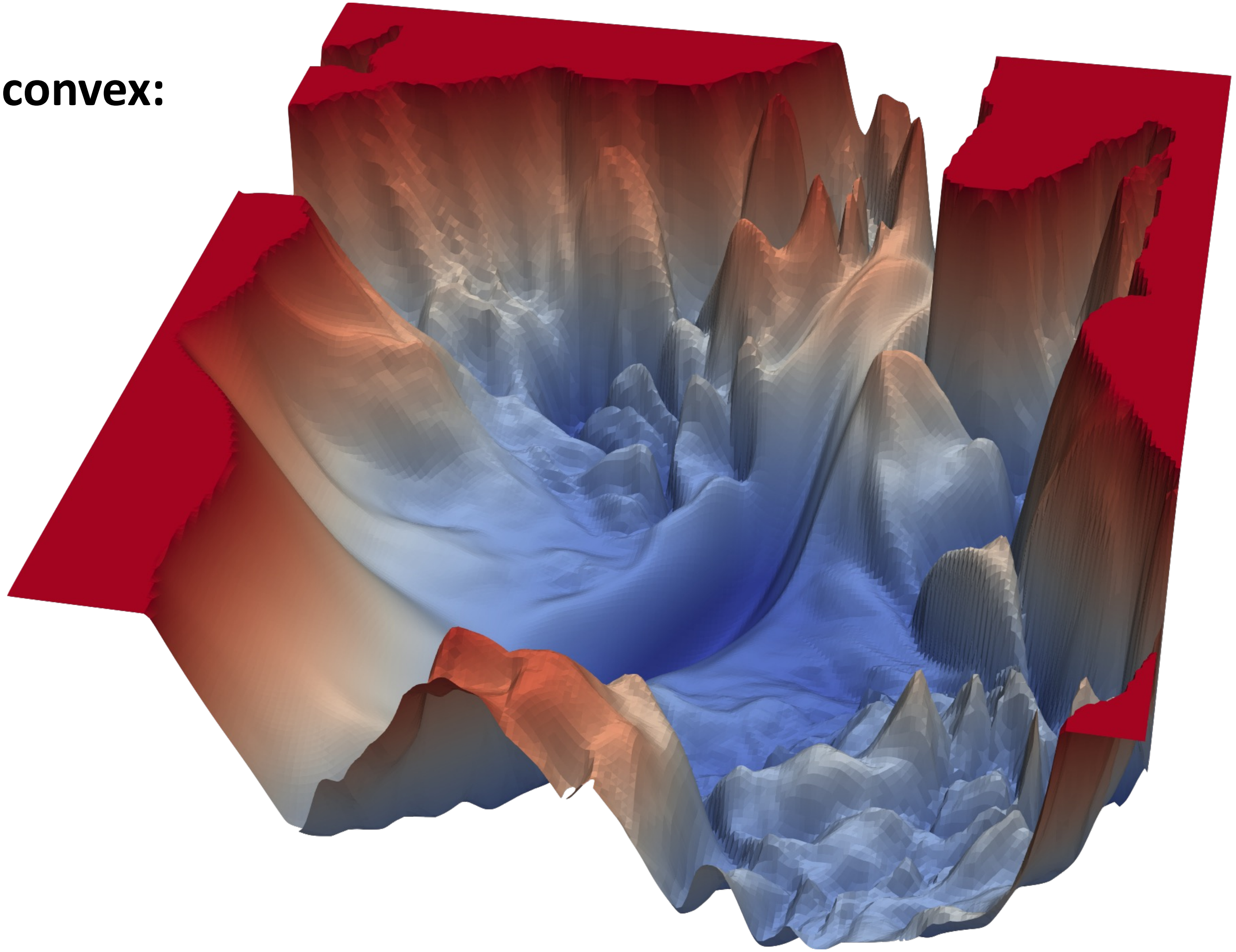
convex:

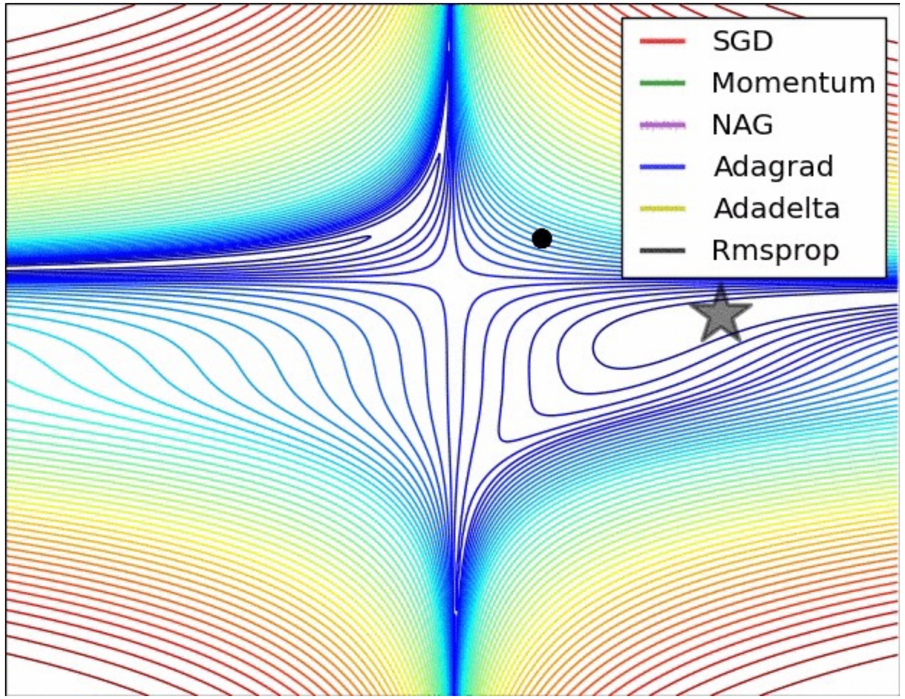


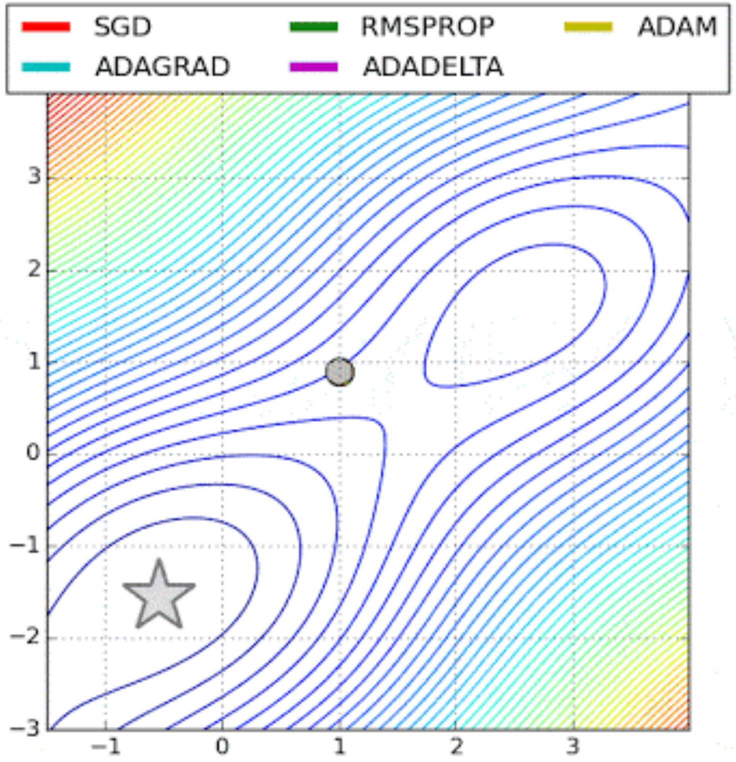
non-convex:



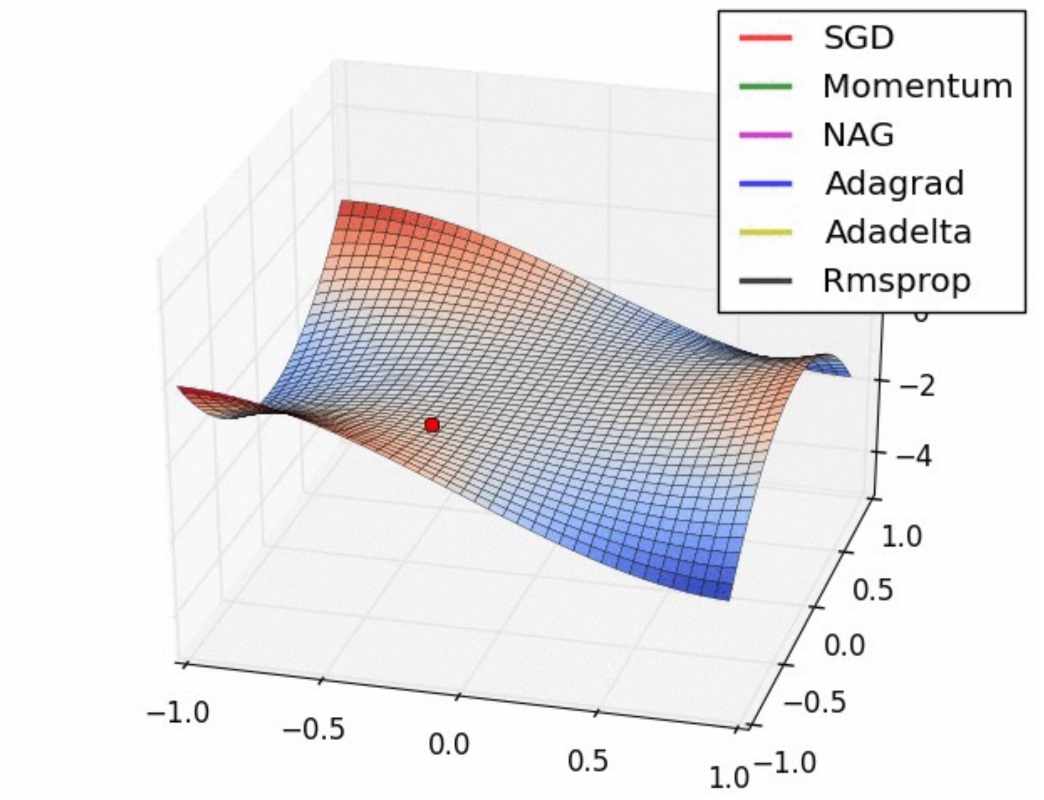
highly non-convex:

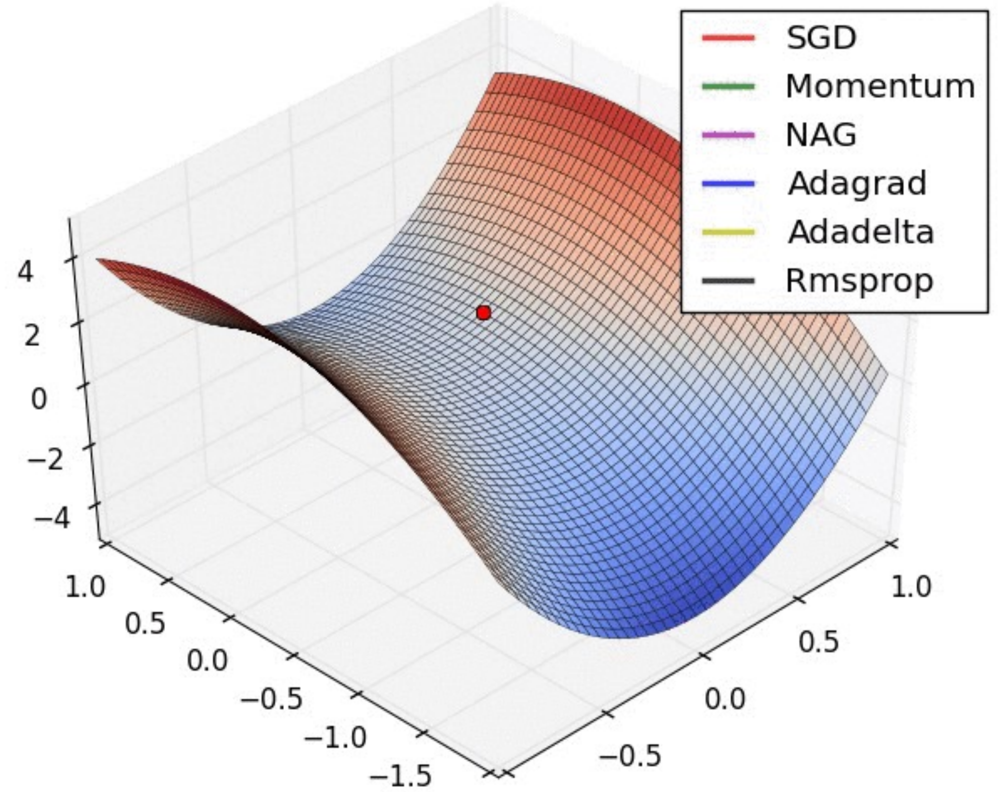






Can still apply gradient methods!

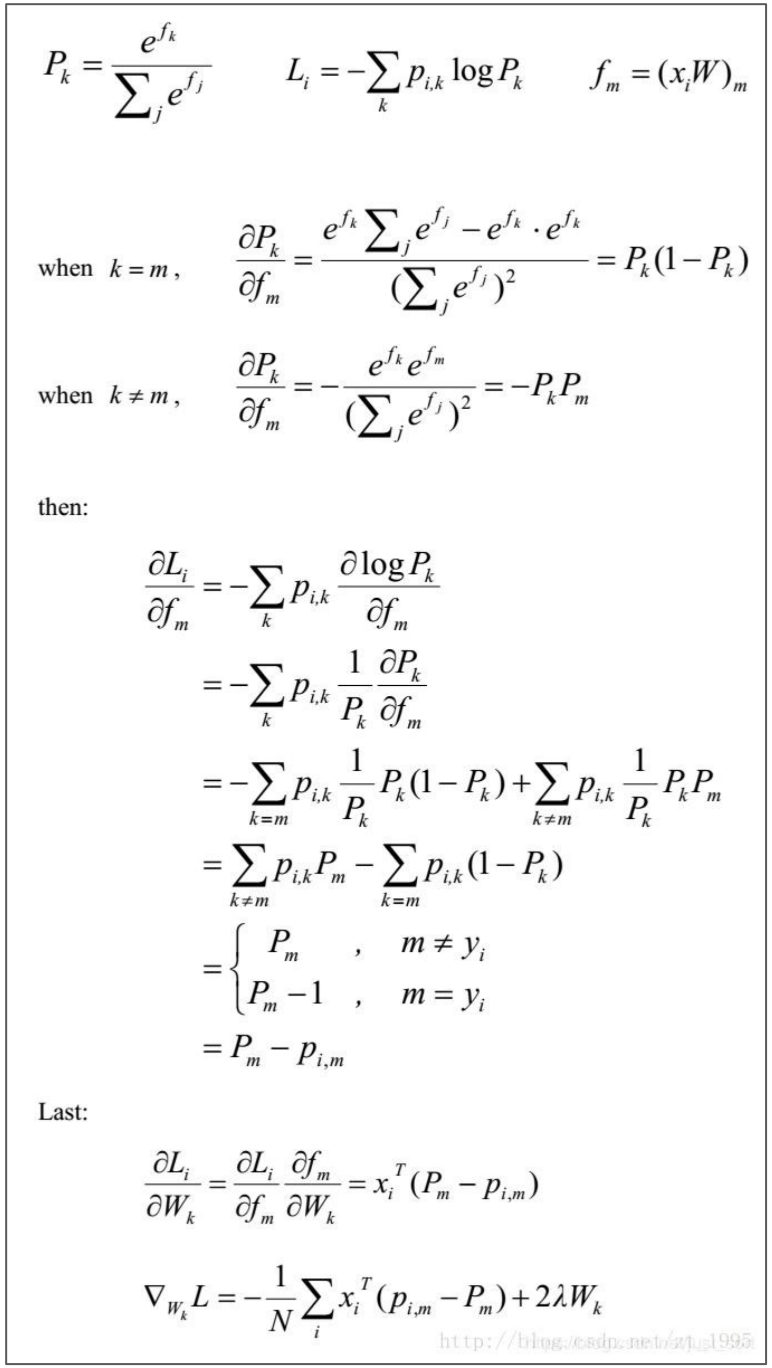




https://imgur.com/a/Hqolp

Automatic differentiation

- GD requires access to gradients
- **Stone age**: had to compute gradients by hand



$$P_k = \frac{e^{f_k}}{\sum_j e^{f_j}} \qquad L_i = -\sum_k p_{i,k} \log P_k \qquad f_m = (x_i W)_m$$

when $k = m$, $$\frac{\partial P_k}{\partial f_m} = \frac{e^{f_k} \sum_j e^{f_j} - e^{f_k} \cdot e^{f_k}}{(\sum_j e^{f_j})^2} = P_k(1 - P_k)$$

when $k \neq m$, $$\frac{\partial P_k}{\partial f_m} = -\frac{e^{f_k} e^{f_m}}{(\sum_j e^{f_j})^2} = -P_k P_m$$

then:

$$\begin{aligned}
\frac{\partial L_i}{\partial f_m} &= -\sum_k p_{i,k} \frac{\partial \log P_k}{\partial f_m} \\
&= -\sum_k p_{i,k} \frac{1}{P_k} \frac{\partial P_k}{\partial f_m} \\
&= -\sum_{k=m} p_{i,k} \frac{1}{P_k} P_k(1 - P_k) + \sum_{k \neq m} p_{i,k} \frac{1}{P_k} P_k P_m \\
&= \sum_{k \neq m} p_{i,k} P_m - \sum_{k=m} p_{i,k}(1 - P_k) \\
&= \begin{cases} P_m & , \quad m \neq y_i \\ P_m - 1 & , \quad m = y_i \end{cases} \\
&= P_m - p_{i,m}
\end{aligned}$$

Last:

$$\frac{\partial L_i}{\partial W_k} = \frac{\partial L_i}{\partial f_m} \frac{\partial f_m}{\partial W_k} = x_i^{\,T}(P_m - p_{i,m})$$

$$\nabla_{W_k} L = -\frac{1}{N} \sum_i x_i^{\,T}(p_{i,m} - P_m) + 2\lambda W_k$$

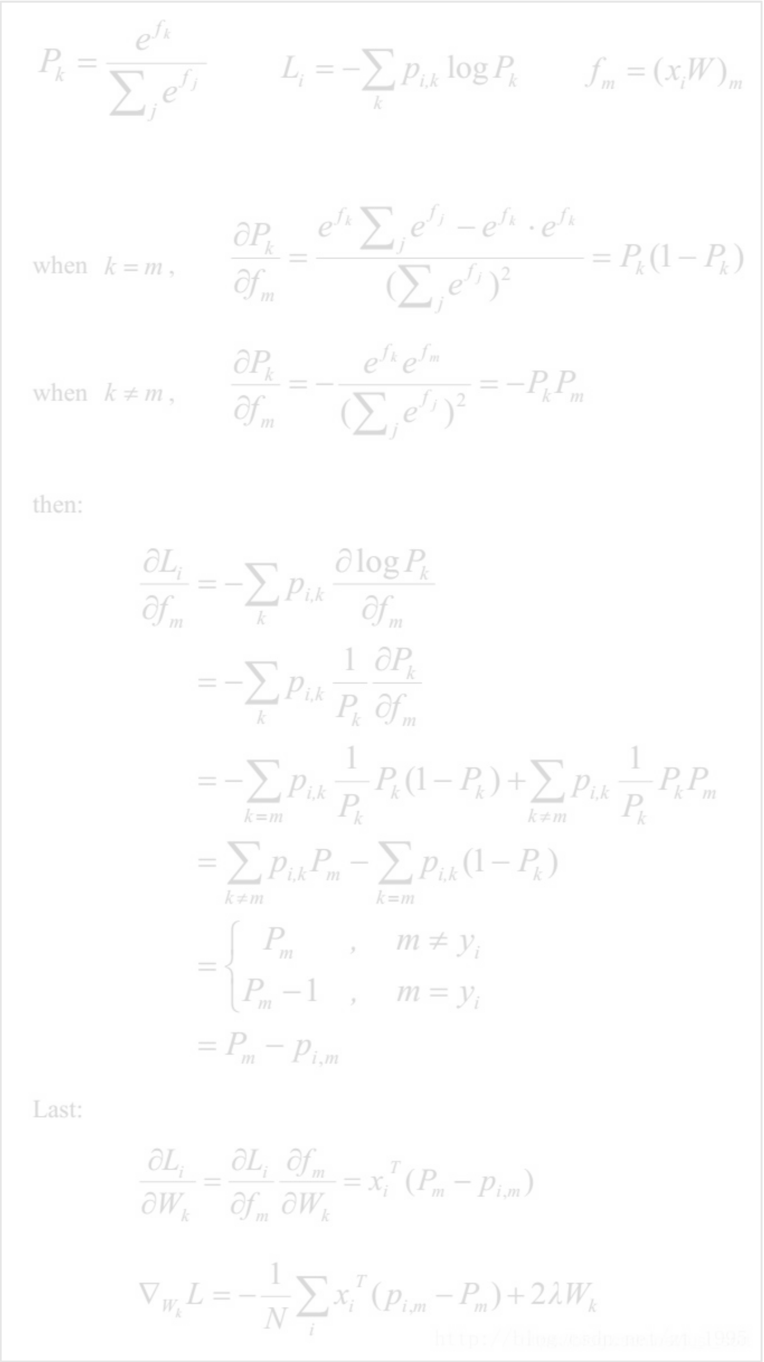
http://blog.csdn.net/zju_1995

Automatic differentiation

- GD requires access to gradients
- **Stone age**: had to compute gradients by hand
- **Modern age**: automatic differentiation (AutoDiff)

forward
`(y, dy/dx) = foo(x)`
backward

- Gradient computation completely abstracted away
- Building blocks + composition = **differentiable programs**
- We'll return to this when we discuss deep learning



$$P_k = \frac{e^{f_k}}{\sum_j e^{f_j}} \qquad L_i = -\sum_k p_{i,k} \log P_k \qquad f_m = (x_i W)_m$$

when $k = m$, $$\frac{\partial P_k}{\partial f_m} = \frac{e^{f_k} \sum_j e^{f_j} - e^{f_k} \cdot e^{f_k}}{(\sum_j e^{f_j})^2} = P_k(1 - P_k)$$

when $k \neq m$, $$\frac{\partial P_k}{\partial f_m} = -\frac{e^{f_k} e^{f_m}}{(\sum_j e^{f_j})^2} = -P_k P_m$$

then:

$$\begin{aligned}
\frac{\partial L_i}{\partial f_m} &= -\sum_k p_{i,k} \frac{\partial \log P_k}{\partial f_m} \\
&= -\sum_k p_{i,k} \frac{1}{P_k} \frac{\partial P_k}{\partial f_m} \\
&= -\sum_{k=m} p_{i,k} \frac{1}{P_k} P_k(1 - P_k) + \sum_{k \neq m} p_{i,k} \frac{1}{P_k} P_k P_m \\
&= \sum_{k \neq m} p_{i,k} P_m - \sum_{k=m} p_{i,k}(1 - P_k) \\
&= \begin{cases} P_m & , \quad m \neq y_i \\ P_m - 1 & , \quad m = y_i \end{cases} \\
&= P_m - p_{i,m}
\end{aligned}$$

Last:

$$\frac{\partial L_i}{\partial W_k} = \frac{\partial L_i}{\partial f_m} \frac{\partial f_m}{\partial W_k} = x_i^{\ T}(P_m - p_{i,m})$$

$$\nabla_{W_k} L = -\frac{1}{N} \sum_i x_i^{\ T}(p_{i,m} - P_m) + 2\lambda W_k$$

Up next

- **Part II**: *the different aspects of learning*
  1. Statistics: generalization and PAC theory
  2. Modeling:
     - Error decomposition
     - Regularization
     - Model selection
  3. Optimization: convexity, gradient descent
  4. Practical aspects and potential pitfalls

Perceptron

```
input: sample set $S = \{(x_i, y_i)\}_{i=1}^{m}$

algorithm:
• initialize $w_0 = \vec{0}$
• for $t = 1,2, \ldots$
    • if $\exists i \in [m]$  s.t.  $y_i w_t^{\top} x_i \leq 0$   #wrong classification
        • $w_{t+1} = w_t + y_i x_i$
    • else
        • return $w_t$
```