Appendix III

Tables of Beam Statistics

Listed below are several tables of approximations for various statistical quantities, based on different power spectrum models [Eqs. (18), (20), and (22) in Chap. 3]. The results are valid for line-of-sight propagation along a horizontal path where the refractive-index structure parameter C_n^2 can be treated as a constant.

Table I Wave Structure Function: Plane Wave

$$D(\rho, L) = 8\pi^2 k^2 L \int_0^\infty \kappa \Phi_n(\kappa)[1 - J_0(\kappa\rho)]d\kappa$$

Spectrum	Wave Structure Function for a Plane Wave
Kolmogorov	$D(\rho, L) = 2.914 C_n^2 k^2 L \rho^{5/3}$
von Kármán	$D(\rho, L) = 3.280 C_n^2 k^2 L l_0^{-1/3} \rho^2 \left[\frac{1}{(1 + 2.033\rho^2/l_0^2)^{1/6}} - 0.715(\kappa_0 l_0)^{1/3}\right]$
Modified Atmospheric	$D(\rho, L) = 2.700 C_n^2 k^2 L l_0^{-1/3} \rho^2 \left[\frac{1}{(1 + 0.632\rho^2/l_0^2)^{1/6}} + \frac{0.438}{(1 + 0.442\rho^2/l_0^2)^{2/3}} - \frac{0.056}{(1 + 0.376\rho^2/l_0^2)^{3/4}} - 0.868(\kappa_0 l_0)^{1/3}\right]$

Table II Wave Structure Function: Spherical Wave

$$D(\rho, L) = 8\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa)[1 - J_0(\kappa\rho\xi)]d\kappa\, d\xi$$

Spectrum	Wave Structure Function for a Spherical Wave
Kolmogorov	$D(\rho, L) = 1.093 C_n^2 k^2 L \rho^{5/3}$
von Kármán	$D(\rho, L) = 1.093 C_n^2 k^2 L l_0^{-1/3} \rho^2 \left[\frac{1}{(1 + \rho^2/l_0^2)^{1/6}} - 0.715(\kappa_0 l_0)^{1/3}\right]$
Modified Atmospheric	$D(\rho, L) = 0.900 C_n^2 k^2 L l_0^{-1/3} \rho^2 \left[\frac{1}{(1 + 0.311\rho^2/l_0^2)^{1/6}} + \frac{0.438}{(1 + 0.183\rho^2/l_0^2)^{2/3}} - \frac{0.056}{(1 + 0.149\rho^2/l_0^2)^{3/4}} - 0.868(\kappa_0 l_0)^{1/3}\right]$

Table III Wave Structure Function: Gaussian-Beam Wave

$$D(\rho, L) = 8\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) e^{-\Lambda L \kappa^2 \xi^2 / k} \{ I_0(\Lambda \kappa \xi \rho) - J_0[(1 - \overline{\Theta}\xi)\kappa\rho] \} d\kappa \, d\xi, \quad (\mathbf{r}_1 = -\mathbf{r}_2)$$

Spectrum	Wave Structure Function for a Gaussian-Beam Wave: $\Theta_0 \neq 0$
Kolmogorov	$D(\rho, L) = 1.093 C_n^2 k^{7/6} L^{11/6} \left[a \left(\frac{k\rho^2}{L} \right)^{5/6} + 0.618 \Lambda^{11/6} \left(\frac{k\rho^2}{L} \right) \right]$
von Kármán	$\begin{aligned} D(\rho, L) = 1.093 C_n^2 k^2 L l_0^{-1/3} \rho^2 \Bigg\{ & \frac{\Lambda^2}{(1 + 0.52 \Lambda Q_m)^{1/6}} \\ & -0.715(1 + \Theta + \Theta^2 + \Lambda^2)(\kappa_0 l_0)^{1/3} + \frac{1}{(1 - \Theta)} \\ & \times \left[\frac{1}{(1 + 0.11 \Lambda Q_m + \rho^2 / l_0^2)^{1/6}} - \frac{\Theta^3}{(1 + 0.11 \Lambda Q_m + \Theta^2 \rho^2 / l_0^2)^{1/6}} \right] \Bigg\} \end{aligned}$
Modified Atmospheric	$\begin{aligned} D(\rho, L) = 0.900 C_n^2 k^2 L l_0^{-1/3} \rho^2 \Bigg\{ & \frac{\Lambda^2}{(1 + 0.52 \Lambda Q_l)^{1/6}} + \frac{0.438 (\Lambda Q_l)^{1/6}}{(1 + 0.70 \Lambda Q_l)^{2/3}} - \frac{0.056 (\Lambda Q_l)^{1/6}}{(1 + 0.70 \Lambda Q_l)^{3/4}} \\ & - 0.868(1 + \Theta + \Theta^2 + \Lambda^2)(\kappa_0 l_0)^{1/3} \\ & + \frac{1}{1 - \Theta} \left[\frac{1}{(1 + 0.11 \Lambda Q_l + 0.311 \rho^2 / l_0^2)^{1/6}} - \frac{\Theta^3}{(1 + 0.11 \Lambda Q_l + 0.311 \Theta^2 \rho^2 / l_0^2)^{1/6}} \right] \\ & + \frac{0.438}{1 - \Theta} \left[\frac{1}{(1 + 0.21 \Lambda Q_l + 0.183 \rho^2 / l_0^2)^{2/3}} - \frac{\Theta^3}{(1 + 0.21 \Lambda Q_l + 0.183 \Theta^2 \rho^2 / l_0^2)^{2/3}} \right] \\ & - \frac{0.056}{1 - \Theta} \left[\frac{1}{(1 + 0.38 \Lambda_l Q_l + 0.149 \rho^2 / l_0^2)^{3/4}} - \frac{\Theta^3}{(1 + 0.38 \Lambda Q_l + 0.149 \Theta^2 \rho^2 / l_0^2)^{3/4}} \right] \Bigg\} \end{aligned}$

$$\Lambda_0 = \frac{2L}{kW_0^2}, \quad \Theta_0 = 1 - \frac{L}{F_0}; \quad \Lambda = \frac{2L}{kW^2} = \frac{\Lambda_0}{\Theta_0^2 + \Lambda_0^2}, \quad \Theta = 1 + \frac{L}{F} = \frac{\Theta_0}{\Theta_0^2 + \Lambda_0^2}, \ \overline{\Theta} = 1 - \Theta$$

$$Q_m = \frac{L\kappa_m^2}{k} = \frac{35.04L}{kl_0^2}, \quad Q_l = \frac{L\kappa_l^2}{k} = \frac{10.89L}{kl_0^2}, \quad Q_0 = \frac{L\kappa_0^2}{k}$$

$$a = \begin{cases} \dfrac{1 - \Theta^{8/3}}{1 - \Theta}, & \Theta \geq 0 \\[2ex] \dfrac{1 + |\Theta|^{8/3}}{1 - \Theta}, & \Theta < 0 \end{cases}$$

Table IV Spatial Coherence Radius: Plane Wave

Spectrum	Spatial Coherence Radius for a Plane Wave: $\kappa_0 = 0$
Kolmogorov	$\rho_{pl} = \left(1.46C_n^2k^2L\right)^{-3/5}, \quad l_0 \ll \rho_{pl} \ll L_0$
von Kármán	$\rho_{pl} = \begin{cases} \left(1.64C_n^2k^2Ll_0^{-1/3}\right)^{-1/2}, & \rho_{pl} \ll l_0 \\ \left(1.46C_n^2k^2L\right)^{-3/5}, & l_0 \ll \rho_{pl} \ll L_0 \end{cases}$
Modified Atmospheric	$\rho_{pl} = \begin{cases} \left(1.87C_n^2k^2Ll_0^{-1/3}\right)^{-1/2}, & \rho_{pl} \ll l_0 \\ \left(1.46C_n^2k^2L\right)^{-3/5}, & l_0 \ll \rho_{pl} \ll L_0 \end{cases}$

Note: Fried's parameter is related by $r_0 = 2.1\rho_{pl}$.

Table V Spatial Coherence Radius: Spherical Wave

Spectrum	Spatial Coherence Radius for a Spherical Wave: $\kappa_0 = 0$
Kolmogorov	$\rho_{sp} = (0.55C_n^2k^2L)^{-3/5}, \qquad l_0 \ll \rho_{sp} \ll L_0$
von Kármán	$\rho_{sp} = \begin{cases} \left(0.55C_n^2k^2Ll_0^{-1/3}\right)^{-1/2}, & \rho_{sp} \ll l_0 \\ \left(0.55C_n^2k^2L\right)^{-3/5}, & l_0 \ll \rho_{sp} \ll L_0 \end{cases}$
Modified Atmospheric	$\rho_{sp} = \begin{cases} (0.62C_n^2k^2Ll_0^{-1/3})^{-1/2}, & \rho_{sp} \ll l_0, \\ (0.55C_n^2k^2L)^{-3/5}, & l_0 \ll \rho_{sp} \ll L_0, \end{cases}$

Table VI Spatial Coherence Radius: Gaussian-Beam Wave

Spectrum	Spatial Coherence Radius for a Gaussian-Beam Wave: $\kappa_0 = 0$, $\Theta_0 \neq 0$
Kolmogorov	$\rho_0 = \left[\dfrac{8}{3(a+0.618\Lambda^{11/6})}\right]^{3/5} \left(1.46C_n^2 k^2 L\right)^{-3/5}, \qquad l_0 \ll \rho_0 \ll L_0$
von Kármán	$\rho_0 = \begin{cases} \left[\dfrac{3}{1+\Theta+\overline{\Theta}^2+\Lambda^2}\right]^{1/2} \left(1.64C_n^2 k^2 L l_0^{-1/3}\right)^{-1/2}, & \rho_0 \ll l_0 \\ \left[\dfrac{8}{3(a+0.618\Lambda^{11/6})}\right]^{3/5} \left(1.46C_n^2 k^2 L\right)^{-3/5}, & l_0 \ll \rho_0 \ll L_0 \end{cases}$
Modified Atmospheric	$\rho_0 = \begin{cases} \left[\dfrac{3}{1+\Theta+\overline{\Theta}^2+\Lambda^2}\right]^{1/2} \left(1.87C_n^2 k^2 L l_0^{-1/3}\right)^{-1/2}, & \rho_0 \ll l_0 \\ \left[\dfrac{8}{3(a+0.618\Lambda^{11/6})}\right]^{3/5} \left(1.46C_n^2 k^2 L\right)^{-3/5}, & l_0 \ll \rho_0 \ll L_0 \end{cases}$

$$\Lambda_0 = \frac{2L}{kW_0^2}, \quad \Theta_0 = 1 - \frac{L}{F_0}; \quad \Lambda = \frac{2L}{kW^2}, \quad \Theta = 1 + \frac{L}{F}, \quad \overline{\Theta} = 1 - \Theta$$

$$Q_m = \frac{L\kappa_m^2}{k} = \frac{35.04L}{kl_0^2}, \quad Q_l = \frac{L\kappa_l^2}{k} = \frac{10.89L}{kl_0^2},$$

$$a = \begin{cases} \dfrac{1-\Theta^{8/3}}{1-\Theta}, & \Theta \geq 0 \\ \dfrac{1+|\Theta|^{8/3}}{1-\Theta}, & \Theta < 0 \end{cases}$$

Note: For strong turbulence conditions, the diffractive beam parameters Θ and Λ can be replaced by the effective beam parameters Θ_e and Λ_e defined by

$$\Theta_e = 1 + \frac{L}{F_{\mathrm{LT}}} = \frac{\Theta - 2q\Lambda/3}{1+4q\Lambda/3},$$

$$\Lambda_e = \frac{2L}{kW_{\mathrm{LT}}^2} = \frac{\Lambda}{1+4q\Lambda/3}, \quad q = \frac{L}{k\rho_{\mathrm{pl}}^2}$$

$$a_e = \begin{cases} \dfrac{1-\Theta_e^{8/3}}{1-\Theta_e}, & \Theta_e \geq 0 \\ \dfrac{1+|\Theta_e|^{8/3}}{1-\Theta_e}, & \Theta_e < 0 \end{cases}$$

Table VII(a) Scintillation Index (Weak Fluctuations): Plane Wave

$$\sigma_I^2(L) = 8\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa)\left[1 - \cos\left(\frac{L\kappa^2}{k}\xi\right)\right] d\kappa \, d\xi$$

Spectrum	Normalized Irradiance Variance for a Plane Wave
Kolmogorov	$\sigma_I^2(L) = \sigma_R^2$
von Kármán	$\sigma_I^2(L) = 3.86\sigma_R^2\left[(1+1/Q_m^2)^{11/12}\sin\left(\frac{11}{6}\tan^{-1}Q_m\right) - \frac{11}{6}Q_m^{-5/6}\right]$
Modified Atmospheric	$\sigma_I^2(L) \equiv \sigma_{PL}^2 = 3.86\sigma_R^2\left\{(1+1/Q_l^2)^{11/12}\left[\sin\left(\frac{11}{6}\tan^{-1}Q_l\right) + \frac{1.507}{(1+Q_l^2)^{1/4}}\sin\left(\frac{4}{3}\tan^{-1}Q_l\right) - \frac{0.273}{(1+Q_l^2)^{7/24}}\sin\left(\frac{5}{4}\tan^{-1}Q_l\right)\right] - 3.50Q_l^{-5/6}\right\}$

$Q_m = \frac{L\kappa_m^2}{k} = \frac{35.04L}{kl_0^2}, \quad Q_l = \frac{L\kappa_l^2}{k} = \frac{10.89L}{kl_0^2}; \quad \sigma_R^2 = 1.23C_n^2 k^{7/6} L^{11/6}$

Table VII(b) Scintillation Index (Strong Fluctuations): Plane Wave

Spectrum	Normalized Irradiance Variance for a Plane Wave
Kolmogorov	$\sigma_{I,\mathrm{pl}}^2(L) = \exp\left[\frac{0.49\sigma_R^2}{\left(1+1.11\sigma_R^{12/5}\right)^{7/6}} + \frac{0.51\sigma_R^2}{\left(1+0.69\sigma_R^{12/5}\right)^{5/6}}\right] - 1$
Modified Atmospheric	$\sigma_{I,\mathrm{PL}}^2(L) = \exp\left[\sigma_{\ln X}^2(l_0) - \sigma_{\ln X}^2(L_0) + \frac{0.51\sigma_{PL}^2}{\left(1+0.69\sigma_{pl}^{12/5}\right)^{5/6}}\right] - 1$

$$\sigma_{\ln X}^2(l_0) = 0.16\sigma_R^2\left(\frac{2.61Q_l}{2.61+Q_l+0.45\sigma_R^2 Q_l^{7/6}}\right)^{7/6}\left[1 + 1.75\left(\frac{2.61}{2.61+Q_l+0.45\sigma_R^2 Q_l^{7/6}}\right)^{1/2} - 0.25\left(\frac{2.61}{2.61+Q_l+0.45\sigma_R^2 Q_l^{7/6}}\right)^{7/12}\right]$$

$$\sigma_{\ln X}^2(L_0) = 0.16\sigma_R^2\left[\frac{2.61Q_0Q_l}{2.61(Q_0+Q_l)+Q_0Q_l\left(1+0.45\sigma_R^2 Q_l^{7/6}\right)}\right]^{7/6} \times \left\{1 + 1.75\left[\frac{2.61Q_0}{2.61(Q_0+Q_l)+Q_0Q_l\left(1+0.45\sigma_R^2 Q_l^{7/6}\right)}\right]^{1/2} - 0.25\left[\frac{2.61Q_0}{2.61(Q_0+Q_l)+Q_0Q_l\left(1+0.45\sigma_R^2 Q_l^{7/6}\right)}\right]^{7/12}\right\}$$

$$Q_0 = \frac{L\kappa_0^2}{k}$$

Table VIIIa Scintillation Index (Weak Fluctuations): Spherical Wave

$$\sigma_I^2(L) = 8\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \left\{ 1 - \cos\left[\frac{L\kappa^2}{k} \xi(1-\xi) \right] \right\} d\kappa \, d\xi$$

Spectrum	Normalized Irradiance Variance for a Spherical Wave
Kolmogorov	$\sigma_I^2(L) \equiv \beta_0^2 = 0.4\sigma_R^2$
von Kármán	$\sigma_I^2(L) = 9.65\beta_0^2 \left[0.40(1 + 9/Q_m^2)^{11/12} \sin\left(\frac{11}{6} \tan^{-1} \frac{Q_m}{3} \right) - \frac{11}{6} Q_m^{-5/6} \right]$
Modified Atmospheric	$\sigma_I^2(L) \equiv \sigma_{SP}^2 = 9.65\beta_0^2 \left\{ 0.40(1 + 9/Q_l^2)^{11/12} \left[\sin\left(\frac{11}{6} \tan^{-1} \frac{Q_l}{3} \right) + \frac{2.610}{(9 + Q_l^2)^{1/4}} \times \sin\left(\frac{4}{3} \tan^{-1} \frac{Q_l}{3} \right) - \frac{0.518}{(9 + Q_l^2)^{7/24}} \sin\left(\frac{5}{4} \tan^{-1} \frac{Q_l}{3} \right) \right] - 3.50 Q_l^{-5/6} \right\}$

$Q_m = \frac{L\kappa_m^2}{k} = \frac{35.04L}{kl_0^2}, \quad Q_1 = \frac{L\kappa_l^2}{k} = \frac{10.89L}{kl_0^2}; \quad \beta_0^2 = 0.5 C_n^2 k^{7/6} L^{11/6}$

Table VIIIb Scintillation Index (Strong Fluctuations): Spherical Wave

Spectrum	Normalized Irradiance Variance for a Spherical Wave
Kolmogorov	$\sigma_{I,\mathrm{sp}}^2(L) = \exp\left[\frac{0.49\beta_0^2}{\left(1 + 0.56\beta_0^{12/5}\right)^{7/6}} + \frac{0.51\beta_0^2}{\left(1 + 0.69\beta_0^{12/5}\right)^{5/6}} \right] - 1$
Modified Atmospheric	$\sigma_{I,\mathrm{sp}}^2(L) = \exp\left[\sigma_{\ln X}^2(l_0) - \sigma_{\ln X}^2(L_0) + \frac{0.51\sigma_{SP}^2}{\left(1 + 0.69\sigma_{SP}^{12/5}\right)^{5/6}} \right] - 1$

$$\sigma_{\ln X}^2(l_0) = 0.04\beta_0^2 \left(\frac{8.56 Q_l}{8.56 + Q_l + 0.20\beta_0^2 Q_l^{7/6}} \right)^{7/6} \left[1 + 1.75 \left(\frac{8.56}{8.56 + Q_l + 0.20\beta_0^2 Q_l^{7/6}} \right)^{1/2} - 0.25 \left(\frac{8.56}{8.56 + Q_l + 0.20\beta_0^2 Q_l^{7/6}} \right)^{7/12} \right]$$

$$\sigma_{\ln X}^2(L_0) = 0.04\beta_0^2 \left[\frac{8.56 Q_0 Q_l}{8.56(Q_0 + Q_l) + Q_0 Q_l (1 + 0.20\beta_0^2 Q_l^{7/6})} \right]^{7/6} \times \left\{ 1 + 1.75 \left[\frac{8.56 Q_0}{8.56(Q_0 + Q_l) + Q_0 Q_l (1 + 0.20\beta_0^2 Q_l^{7/6})} \right]^{1/2} - 0.25 \left[\frac{8.56 Q_0}{8.56(Q_0 + Q_l) + Q_0 Q_l (1 + 0.20\beta_0^2 Q_l^{7/6})} \right]^{7/12} \right\}$$

$$Q_0 = \frac{L\kappa_0^2}{k}$$

Table IX(a) Scintillation Index (Weak Fluctuations): Gaussian-Beam Wave

$$\sigma_I^2(r,L) = 8\pi^2k^2L\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)e^{-\Lambda L\kappa^2\xi^2/k}\left\{I_0(2\Lambda r\kappa\xi) - \cos\left[\frac{L\kappa^2}{k}\xi(1-\bar{\Theta}\xi)\right]\right\}d\kappa\,d\xi$$

Spectrum	Normalized Irradiance Variance for a Gaussian Beam Wave $\Theta_0 \neq 0$[1]
Kolmogorov	$\sigma_I^2(r,L) = 4.42\sigma_R^2\Lambda^{5/6}\frac{r^2}{W^2} + 3.86\sigma_R^2\left\{0.40\left[(1+2\Theta)^2+4\Lambda^2\right]^{5/12} \times\cos\left[\frac{5}{6}\tan^{-1}\left(\frac{1+2\Theta}{2\Lambda}\right)\right] - \frac{11}{16}\Lambda^{5/6}\right\}$
von Kármán	$\sigma_I^2(r,L) = 3.93\sigma_R^2\Lambda^{5/6}\left[\left(\frac{\Lambda Q_m}{1+0.52\Lambda Q_m}\right)^{1/6} - 1.29(\Lambda Q_0)^{1/6}\right]\frac{r^2}{W^2} + 3.86\sigma_R^2\left\{0.40\frac{[(1+2\Theta)^2+(2\Lambda+3/Q_m)^2]^{11/12}}{[(1+2\Theta)^2+4\Lambda^2]^{1/2}}\sin\left(\frac{11}{6}\varphi_1+\varphi_2\right) - \frac{6\Lambda}{Q_m^{11/6}[(1+2\Theta)^2+4\Lambda^2]} - \frac{11}{6}\left(\frac{1+0.31\Lambda Q_m}{Q_m}\right)^{5/6}\right\}$
Modified Atmospheric	$\sigma_I^2(r,L) = 3.93\sigma_R^2\Lambda^{5/6}\left\{\left[\left(\frac{\Lambda Q_l}{1+0.52\Lambda Q_l}\right)^{1/6} + \frac{0.438(\Lambda Q_l)^{1/6}}{(1+0.70\Lambda Q_l)^{2/3}} - \frac{0.056(\Lambda Q_l)^{1/6}}{(1+0.70\Lambda Q_l)^{3/4}}\right] - 1.29(\Lambda Q_0)^{1/6}\right\}\frac{r^2}{W^2} + 3.86\sigma_R^2\left\{0.40\frac{[(1+2\Theta)^2+(2\Lambda+3/Q_l)^2]^{11/12}}{[(1+2\Theta)^2+4\Lambda^2]^{1/2}}\left[\sin\left(\frac{11}{6}\varphi_3+\varphi_2\right) + \frac{2.610}{[(1+2\Theta)^2Q_l^2+(3+2\Lambda Q_l)^2]^{1/4}}\sin\left(\frac{4}{3}\varphi_3+\varphi_2\right) - \frac{0.518}{[(1+2\Theta)^2Q_l^2+(3+2\Lambda Q_l)^2]^{7/24}}\sin\left(\frac{5}{4}\varphi_3+\varphi_2\right)\right] - \frac{13.401\Lambda}{Q_l^{11/6}[(1+2\Theta)^2+4\Lambda^2]} - \frac{11}{6}\left[\left(\frac{1+0.31\Lambda Q_l}{Q_l}\right)^{5/6} + \frac{1.096(1+0.27\Lambda Q_l)^{1/3}}{Q_l^{5/6}} - \frac{0.186(1+0.24\Lambda Q_l)^{1/4}}{Q_l^{5/6}}\right]\right\}$

$$\Lambda_0 = \frac{2L}{kW_0^2},\quad \Theta_0 = 1-\frac{L}{F_0},\quad \Lambda = \frac{2L}{kW^2},\quad \Theta = 1+\frac{L}{F},\quad \bar{\Theta} = 1-\Theta$$

$$Q_m = \frac{L\kappa_m^2}{k} = \frac{35.04L}{kl_0^2},\quad Q_l = \frac{L\kappa_l^2}{k} = \frac{10.89L}{kl_0^2},\quad Q_0 = \frac{L\kappa_0^2}{k}$$

$$\varphi_1 = \tan^{-1}\left[\frac{(1+2\Theta)Q_m}{3+2\Lambda Q_m}\right],\quad \varphi_2 = \tan^{-1}\left[\frac{2\Lambda}{1+2\Theta}\right],$$

$$\varphi_3 = \tan^{-1}\left[\frac{(1+2\Theta)Q_l}{3+2\Lambda Q_l}\right];\quad \sigma_R^2 = 1.23C_n^2k^{7/6}L^{11/6}$$

[1]Based on an untracked beam but ignoring beam wander effects.

Table IX(b) Scintillation Index (Strong Fluctuations): Gaussian-Beam Wave

Spectrum	Normalized Irradiance Variance for a Gaussian-Beam Wave: $\Theta_0 \neq 0$
Kolmogorov	$\sigma_I^2(\mathbf{r},L)_{\text{untracked}} = \exp\left\{\dfrac{0.49\sigma_B^2}{\left[1+0.56(1+\Theta)\sigma_B^{12/5}\right]^{7/6}} + \dfrac{0.51\sigma_B^2}{\left(1+0.69\sigma_B^{12/5}\right)^{5/6}}\right\} - 1 + 4.42\sigma_R^2\Lambda_e^{5/6}\left(\dfrac{\sigma_{pe}}{W_{\text{LT}}}\right)^2 + 4.42\sigma_R^2\Lambda_e^{5/6}\left(\dfrac{r-\sigma_{pe}}{W_{\text{LT}}}\right)^2, \quad \sigma_{pe} \le r < W$
Kolmogorov	$\sigma_I^2(\mathbf{r},L)_{\text{tracked}} = \exp\left\{\dfrac{0.49\sigma_B^2}{\left[1+0.56(1+\Theta)\sigma_B^{12/5}\right]^{7/6}} + \dfrac{0.51\sigma_B^2}{\left(1+0.69\sigma_B^{12/5}\right)^{5/6}}\right\} - 1 + 4.42\sigma_R^2\Lambda_e^{5/6}\left(\dfrac{r-\sqrt{\langle r_c^2\rangle}}{W_{\text{LT}}}\right)^2, \quad \sqrt{\langle r_c^2\rangle} \le r < W$
Modified Atmospheric	$\sigma_I^2(\mathbf{r},L)_{\text{untracked}} = 4.42\sigma_R^2\Lambda_e^{5/6}\left[1-1.15\left(\dfrac{\Lambda_e L}{kL_0^2}\right)^{1/6}\right]\left(\dfrac{r-\sigma_{pe}}{W_{\text{LT}}}\right)^2 + 4.42\sigma_R^2\Lambda_e^{5/6}\left(\dfrac{\sigma_{pe}}{W_{\text{LT}}}\right)^2 + \exp\left[\sigma_{\ln X}^2(l_0) - \sigma_{\ln X}^2(L_0) + \dfrac{0.51\sigma_G^2}{\left(1+0.69\sigma_G^{12/5}\right)^{5/6}}\right] - 1, \ \sigma_{pe} \le r < W$
Modified Atmospheric	$\sigma_I^2(\mathbf{r},L)_{\text{tracked}} = 4.42\sigma_R^2\Lambda_e^{5/6}\left[1-1.15\left(\dfrac{\Lambda_e L}{kL_0^2}\right)^{1/6}\right]\left(\dfrac{r-\sqrt{\langle r_c^2\rangle}}{W_{\text{LT}}}\right)^2 + \exp\left[\sigma_{\ln X}^2(l_0) - \sigma_{\ln X}^2(L_0) + \dfrac{0.51\sigma_G^2}{\left(1+0.69\sigma_G^{12/5}\right)^{5/6}}\right] - 1, \ \sqrt{\langle r_c^2\rangle} \le r < W$

Note: The longitudinal component of the scintillation index arises in all cases above by setting r equal to its smallest value in the specified interval.

$$\Lambda_0 = \frac{2L}{kW_0^2}, \quad \Theta_0 = 1 - \frac{L}{F_0}, \quad \Lambda = \frac{2L}{kW^2}, \quad \Theta = 1 + \frac{L}{F}, \quad \overline{\Theta} = 1 - \Theta;$$

$$\Lambda_e = \frac{\Lambda}{1+4q\Lambda/3}, \quad q = \frac{L}{k\rho_{\text{pl}}^2}$$

$$\sigma_B^2 = 3.86\sigma_R^2 \operatorname{Re}\left[i^{5/6}\, {}_2F_1\left(-\frac{5}{6}, \frac{11}{6}; \frac{17}{6}; \overline{\Theta} + i\Lambda\right) - \frac{11}{16}\Lambda^{5/6}\right]$$

$$\sigma_{pe}^2 = 7.25C_n^2 L^3 W_0^{-1/3}\int_0^1 \xi^2 \left\{\frac{1}{\left|\Theta_0 + \overline{\Theta}_0\xi\right|^{1/3}} - \left[\frac{\kappa_r^2 W_0^2}{1+\kappa_r^2 W_0^2\left(\Theta_0+\overline{\Theta}_0\xi\right)^2}\right]^{1/6}\right\}d\xi, \quad \kappa_r = 2\pi/r_0$$

$$\langle r_c^2\rangle = 7.25C_n^2 L^3 W_0^{-1/3}\int_0^1 \xi^2 \left\{\frac{1}{\left[\left(\Theta_0+\overline{\Theta}_0\xi\right)^2 + 1.63\sigma_R^{12/5}\Lambda_0(1-\xi)^{16/5}\right]^{1/6}} - \frac{(\kappa_0 W_0)^{1/3}}{\left\{1+\kappa_o W_0^2\left[(\Theta+\overline{\Theta}\xi)^2 + 1.63\sigma_R^{12/5}\Lambda_0(1-\xi)^{16/5}\right]\right\}^{1/6}}\right\}d\xi$$

$$\sigma_G^2 = 3.86\sigma_R^2\left\{0.40\frac{\left[(1+2\Theta)^2+(2\Lambda+3/Q_l)^2\right]^{11/12}}{\left[(1+2\Theta)^2+4\Lambda^2\right]^{1/2}}\left[\sin\left(\frac{11}{6}\varphi_2+\varphi_1\right)\right.\right.$$

$$+\frac{2.61}{\left[(1+2\Theta)^2Q_l^2+(3+2\Lambda Q_l)^2\right]^{1/4}}\sin\left(\frac{4}{3}\varphi_2+\varphi_1\right)$$

$$\left.-\frac{0.52}{\left[(1+2\Theta)^2Q_l^2+(3+2\Lambda Q_l)^2\right]^{7/24}}\sin\left(\frac{5}{4}\varphi_2+\varphi_1\right)\right]$$

$$-\frac{13.40\Lambda}{Q_l^{11/6}\left[(1+2\Theta)^2+4\Lambda^2\right]}-\frac{11}{6}\left[\left(\frac{1+0.31\Lambda Q_l}{Q_l}\right)^{5/6}\right.$$

$$\left.\left.+\frac{1.10(1+0.27\Lambda Q_l)^{1/3}}{Q_l^{5/6}}-\frac{0.19(1+0.24\Lambda Q_l)^{1/4}}{Q_l^{5/6}}\right]\right\}$$

$$\sigma_{\ln X}^2(l_0) = 0.49\sigma_R^2\left(\frac{1}{3}-\frac{1}{2}\overline{\Theta}+\frac{1}{5}\overline{\Theta}^2\right)\left(\frac{\eta_X Q_l}{\eta_X+Q_l}\right)^{7/6}\left[1+1.75\left(\frac{\eta_X}{\eta_X+Q_l}\right)^{1/2}-0.25\left(\frac{\eta_X}{\eta_X+Q_l}\right)^{7/12}\right]$$

$$\sigma_{\ln X}^2(L_0) = 0.49\sigma_R^2\left(\frac{1}{3}-\frac{1}{2}\overline{\Theta}+\frac{1}{5}\overline{\Theta}^2\right)\left(\frac{\eta_{X0} Q_l}{\eta_{X0}+Q_l}\right)^{7/6}\left[1+1.75\left(\frac{\eta_{X0}}{\eta_{X0}+Q_l}\right)^{1/2}-0.25\left(\frac{\eta_{X0}}{\eta_{X0}+Q_l}\right)^{7/12}\right]$$

$$\eta_X = \left[\frac{0.38}{1-3.21\overline{\Theta}+5.29\overline{\Theta}^2}+0.47\sigma_R^2 Q_l^{1/6}\left(\frac{\frac{1}{3}-\frac{1}{2}\overline{\Theta}+\frac{1}{5}\overline{\Theta}^2}{1+2.20\overline{\Theta}}\right)^{6/7}\right]^{-1}, \quad \eta_{X0} = \frac{\eta_X Q_0}{\eta_X+Q_0}$$

$$Q_l = \frac{L\kappa_l^2}{k} = \frac{10.89L}{kl_0^2}, \quad Q_0 = \frac{64\pi^2 L}{kL_0^2}$$

$$\varphi_1 = \tan^{-1}\left(\frac{2\Lambda}{1+2\Theta}\right), \quad \varphi_2 = \tan^{-1}\left[\frac{(1+2\Theta)Q_l}{3+2\Lambda Q_l}\right].$$