Chapter 4
Fraunhofer Diffraction and Lenses

To obtain accurate results, evaluating the Fresnel diffraction integral numerically requires some care. Therefore, this chapter first deals with two simpler topics: diffraction with the Fraunhofer approximation and diffraction with lenses. This allows some optical examples of FTs to be demonstrated without the significant algorithm development and sampling analysis required for simulating Fresnel diffraction. Vacuum propagation algorithms and sampling analysis for Fresnel propagation are the subjects Chs. 6–8. Computing diffracted fields in the Fraunhofer approximation or when a lens is present does not require quite so much analysis up front. Additionally, these simple computations involve only a single DFT for each pattern. Chapter 2 provides the requisite background. Consequently, readers may notice that the MATLAB code listings in this chapter are quite simple.

4.1 Fraunhofer Diffraction

When light propagates very far from its source aperture, the optical field in the observation plane is very closely approximated by the Fraunhofer diffraction integral, given in Ch. 1 and repeated here for convenience:

$$U\left(x_2, y_2\right) = \frac{e^{ik\Delta z} e^{i\frac{k}{2\Delta z}\left(x_2^2+y_2^2\right)}}{i\lambda\Delta z} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U\left(x_1, y_1\right) e^{-i\frac{k}{\Delta z}\left(x_1 x_2 + y_1 y_2\right)}\, dx_1 dy_1. \tag{4.1}$$

According to Goodman,[5] "very far" is defined by the inequality

$$\Delta z > \frac{2D^2}{\lambda}, \tag{4.2}$$

where Δz is the propagation distance, D is the diameter of the source aperture, and λ is the optical wavelength. This is a good approximation because the quadratic phase is nearly flat over the source.

The Fraunhofer integral can be cast in the form of an FT that makes use of the

Table 4.1 Definition of symbols for optical propagation.

symbol	meaning
$\mathbf{r}_1 = (x_1, y_1)$	source-plane coordinates
$\mathbf{r}_2 = (x_2, y_2)$	observation-plane coordinates
δ_1	grid spacing in source plane
δ_2	grid spacing in observation plane
Δz	distance between source plane and observation plane

Listing 4.1 Code for performing a Fraunhofer propagation in MATLAB.

```
function [Uout x2 y2] = ...
    fraunhofer_prop(Uin, wvl, d1, Dz)
% function [Uout x2 y2] = ...
%     fraunhofer_prop(Uin, wvl, d1, Dz)

    N = size(Uin, 1);   % assume square grid
    k = 2*pi / wvl;  % optical wavevector
    fX = (-N/2 : N/2-1) / (N*d1);
    % observation-plane coordinates
    [x2 y2] = meshgrid(wvl * Dz * fX);
    clear('fX');
    Uout = exp(i*k/(2*Dz)*(x2.^2+y2.^2)) ...
        / (i*wvl*Dz) .* ft2(Uin, d1);
```

lessons from Ch. 2:

$$U(x_2, y_2) = \frac{e^{ik\Delta z} e^{i\frac{k}{2\Delta z}\left(x_2^2+y_2^2\right)}}{i\lambda\Delta z} \mathcal{F}\left\{U(x_1, y_1)\right\}\Big|_{f_{x1}=\frac{x_2}{\lambda\Delta z}, f_{y1}=\frac{y_2}{\lambda\Delta z}}. \tag{4.3}$$

To evaluate this on a grid, we must define the grid properties. We call the grid spacings δ_1 and δ_2 in the source and observation planes, respectively. The spatial-frequency variable for the source plane is $\mathbf{f}_1 = (f_{x1}, f_{y1})$, and its grid spacing is δ_{f1}. Now, the reader should notice that these spatial frequencies are directly mapped to the observation plane's spatial coordinates x_2 and y_2. These symbols are summarized in Table 4.1 and depicted in Fig. 1.2.

Now, numerically evaluating a Fraunhofer diffraction integral is a simple matter of performing an FT with the appropriate multipliers and spatial scaling. Listing 4.1 gives the MATLAB function `fraunhofer_prop` that can be used to numerically perform a wave-optics propagation when the Fraunhofer diffraction integral is valid, i.e., when Eq. (4.2) is true. In the Listing, the factor $\exp(ik\Delta z)$ has been ignored because it is just the on-axis phase. Readers should notice that the code takes advantage of the `ft2` function developed in Ch. 2.

Listing 4.2 demonstrates use of the `fraunhofer_prop` function. The example simulates propagation of a monochromatic plane wave from a circular aperture

Listing 4.2 MATLAB example of simulating a Fraunhofer diffraction pattern with comparison to the analytic result.

```
% example_fraunhofer_circ.m

N = 512;     % number of grid points per side
L = 7.5e-3;    % total size of the grid [m]
d1 = L / N;  % source-plane grid spacing [m]
D = 1e-3;    % diameter of the aperture [m]
wvl = 1e-6;  % optical wavelength [m]
k = 2*pi / wvl;
Dz = 20;     % propagation distance [m]

[x1 y1] = meshgrid((-N/2 : N/2-1) * d1);
Uin = circ(x1, y1, D);
[Uout x2 y2] = fraunhofer_prop(Uin, wvl, d1, Dz);

% analytic result
Uout_th = exp(i*k/(2*Dz)*(x2.^2+y2.^2)) ...
    / (i*wvl*Dz) * D^2*pi/4 ...
    .* jinc(D*sqrt(x2.^2+y2.^2)/(wvl*Dz));
```

to a distant observation plane. The $y_2 = 0$ slice of the resulting field's amplitude is shown in Fig. 4.1. The numerical results shown in Fig. 4.1 closely match the analytic results. However, if a large region was shown, the edges would begin to show some discrepancy. This is due to aliasing, as discussed in Sec. 2.3. If the example code was modeling a real system with a target board sensor that was only 0.4 m in diameter, then aliasing would not significantly affect the comparison between the numerical prediction and the experimentally measured diffraction pattern. The chosen grid spacing and number of grid points would be sufficient for that purpose.

To state this more concretely, the geometry of the propagation imposes a limit on the observable spatial frequency content of the source. The observation-plane coordinates are related to the spatial frequency of the source via

$$x_2 = \lambda \Delta z f_{x1} \tag{4.4a}$$

$$y_2 = \lambda \Delta z f_{y1}. \tag{4.4b}$$

Then, if a sensor in the $x_2 - y_2$ plane is 0.4 m wide, the maximum values of the observation-plane coordinates are $x_{max} = 0.2$ m and $y_{max} = 0.2$ m. This leads to maximum observable values of the source's spatial frequency $f_{x1,max}$ and $f_{y1,max}$ given by

$$f_{x1,max} = \frac{x_{2,max}}{\lambda \Delta z} \tag{4.5a}$$

$$f_{y1,max} = \frac{y_{2,max}}{\lambda \Delta z}. \tag{4.5b}$$

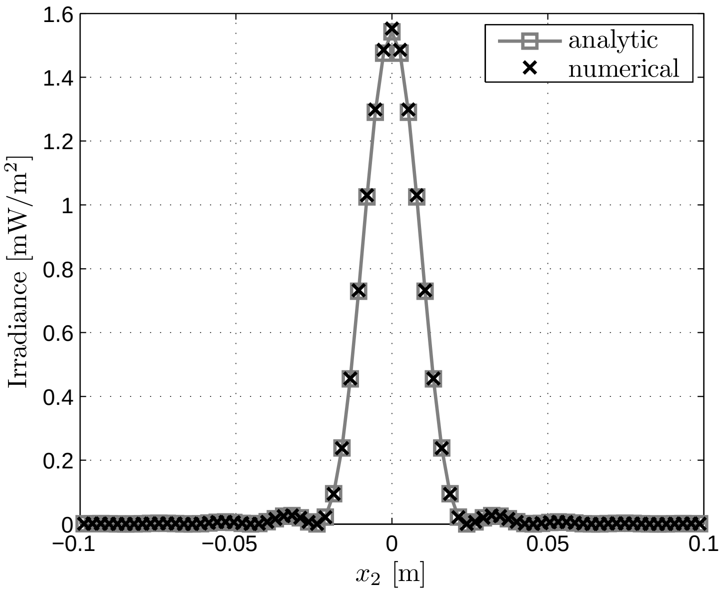

Figure 4.1 The $y_2 = 0$ slice of the amplitude of the Fraunhofer diffraction pattern for a circular aperture. Both the numerical and analytic results are shown for comparison.

As a result, in simulation, propagating a bandlimited (or filtered) version of the real source with spatial frequencies $\leq f_{x1,max}$ and $f_{y1,max}$ would produce the same observation-plane diffraction pattern as one would observe in a laboratory. This principle is used extensively in Ch. 7.

4.2 Fourier-Transforming Properties of Lenses

In this section, the discussion moves to near-field diffraction, governed by the Fresnel diffraction integral in the paraxial approximation for monochromatic waves. This is given in Eq. (1.57) and repeated here for reference:

$$U(x_2, y_2) = \frac{e^{ik\Delta z}}{i\lambda\Delta z} e^{i\frac{k}{2\Delta z}\left(x_2^2+y_2^2\right)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x_1, y_1) \times e^{i\frac{k}{2\Delta z}\left(x_1^2+y_1^2\right)} e^{-i\frac{2\pi}{\lambda\Delta z}(x_2x_1+y_2y_1)}\, dx_1\, dy_1. \tag{4.6}$$

Applying the Fraunhofer approximation in Eq. (4.2) removes the quadratic phase exponential in Eq. (4.6), resulting in the Fraunhofer diffraction integral. However, this approximation is not valid for the scenarios discussed in this section.

In the paraxial approximation, the phase delay imparted by a perfect, spherical (in the paraxial sense), thin lens is given by[5]

$$\phi(x, y) = -\frac{k}{2f_l}\left(x^2 + y^2\right), \tag{4.7}$$

where x and y are coordinates in the exit-pupil plane of the lens, and f_l is the focal length. In this section, a planar transparent object is placed in one of three locations: against (before), the lens, in front of the lens, and behind the lens. The object is illuminated by a normally incident, infinite-extent, uniform-amplitude plane wave. Equation (4.6) is used to propagate the light that passes through the object to the back focal plane of the lens. As a result, the phase term in Eq. (4.7) becomes a part of $U(x_1, y_1)$ inside the Fresnel diffraction integral, resulting in some simplifications as discussed in the next few subsections.

4.2.1 Object against the lens

When the object is placed against the lens as shown in Fig. 4.2, the optical field in the plane just after the lens is

$$U(x_1, y_1) = t_A(x_1, y_1)\, P(x_1, y_1)\, e^{-i\frac{k}{2f_l}\left(x_1^2+y_1^2\right)}, \tag{4.8}$$

where $t_A(x_1, y_1)$ is the aperture transmittance of the object and $P(x_1, y_1)$ is a real function that accounts for apodization by the lens. When Eq. (4.8) is substituted into Eq. (4.6), assuming propagation to the back focal plane, the result is

$$\begin{aligned} U(x_2, y_2) = \frac{1}{i\lambda f_l} e^{i\frac{k}{2f_l}\left(x_2^2+y_2^2\right)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t_A(x_1, y_1) \\ \times P(x_1, y_1)\, e^{-i\frac{2\pi}{\lambda f_l}\left(x_2 x_1 + y_2 y_1\right)}\, dx_1\, dy_1. \end{aligned} \tag{4.9}$$

Like in Sec. 4.1, this can be cast in terms of an FT so that

$$U(x_2, y_2) = \frac{1}{i\lambda f_l} e^{i\frac{k}{2f_l}\left(x_2^2+y_2^2\right)} \mathcal{F}\left\{t_A(x_1, y_1)\, P(x_1, y_1)\right\}\Big|_{f_x=\frac{x_2}{\lambda f_l}, f_y=\frac{y_2}{\lambda f_l}}. \tag{4.10}$$

This is not an exact FT relationship because of the quadratic phase factor outside the integral. Nonetheless, we can use a DFT to compute diffracted field.

Listing 4.3 gives the MATLAB function `lens_against_ft` from the object plane to the focal plane for an object placed against a converging lens. Notice that the implementation is very similar to `fraunhofer_prop`, which takes advantage of the function `ft2`.

4.2.2 Object before the lens

A more general situation is obtained when the object is placed a distance d before the lens as shown in Fig. 4.3. When the light propagates to the focal plane, the result is

$$U(x_2, y_2) = \frac{1}{i\lambda f_l} e^{i\frac{k}{2f_l}\left(1-\frac{d}{f_l}\right)\left(x_2^2+y_2^2\right)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t_A(x_1, y_1)$$

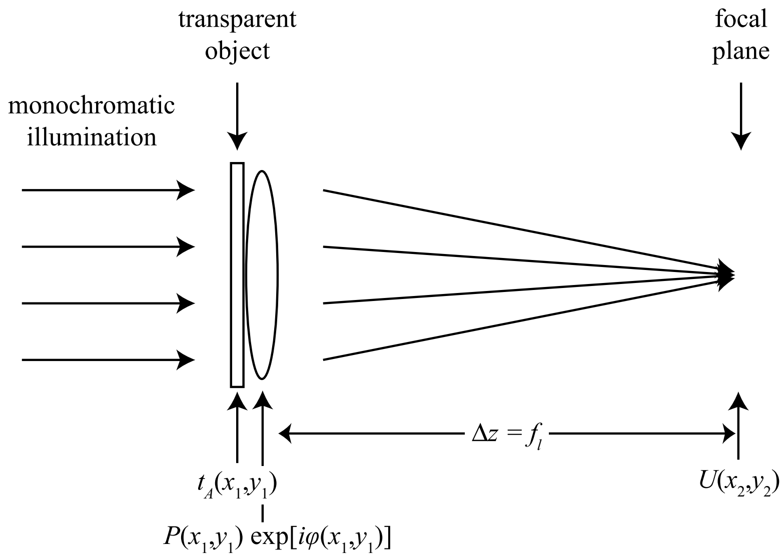

Figure 4.2 Diagram of lens geometry for an object placed against the lens.

Listing 4.3 Code for performing a propagation from the pupil plane to the focal plane for an object placed against (and just before a lens) in MATLAB.

```
function [Uout x2 y2] = ...
    lens_against_ft(Uin, wvl, d1, f)
% function [Uout x2 y2] = ...
%     lens_against_ft(Uin, wvl, d1, f)

    N = size(Uin, 1);   % assume square grid
    k = 2*pi/wvl;    % optical wavevector
    fX = (-N/2 : 1 : N/2 - 1) / (N * d1);
    % observation plane coordinates
    [x2 y2] = meshgrid(wvl * f * fX);
    clear('fX');

    % evaluate the Fresnel-Kirchhoff integral but with
    % the quadratic phase factor inside cancelled by the
    % phase of the lens
    Uout = exp(i*k/(2*f)*(x2.^2 + y2.^2)) ...
        / (i*wvl*f) .* ft2(Uin, d1);
```

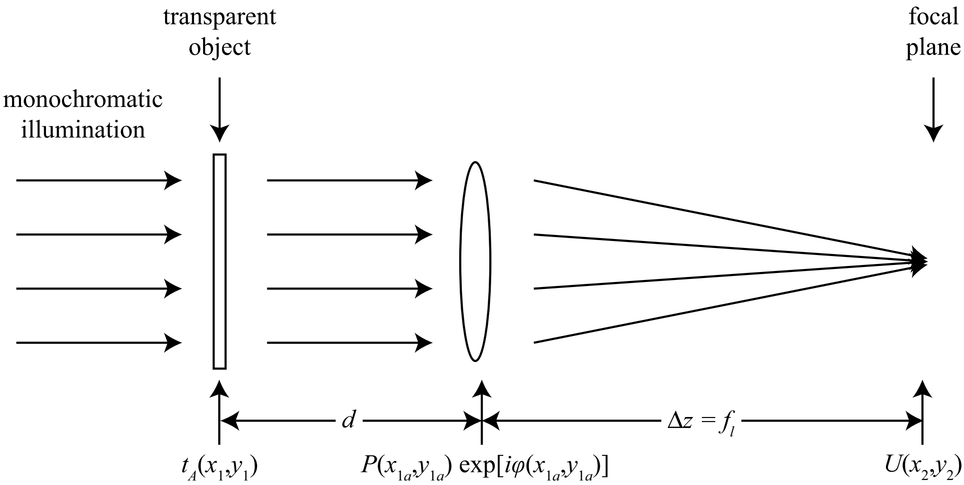

Figure 4.3 Diagram of lens geometry for an object placed before the lens.

$$\times P\left(x_1 + \frac{d}{f_l}x_2, y_1 + \frac{d}{f_l}y_2\right) e^{-i\frac{2\pi}{\lambda f_l}(x_2x_1+y_2y_1)}\, dx_1\, dy_1, \quad (4.11)$$

where the shifted argument of the pupil function accounts for vignetting of the object by the lens aperture. Each point in the focal plane experiences different vignetting with the least occurring for the point on the optical axis. The reader is referred to Goodman (Ref. 5) for more detail. Like in Sec. 4.1, this can be cast in terms of an FT so that

$$U(x_2, y_2) = \frac{1}{i\lambda f_l} e^{i\frac{k}{2f_l}\left(1-\frac{d}{f_l}\right)\left(x_2^2+y_2^2\right)} \times \mathcal{F}\left\{t_A(x_1, y_1)\, P\left(x_1 + \frac{d}{f_l}x_2, y_1 + \frac{d}{f_l}y_2\right)\right\}\bigg|_{f_x=\frac{x_2}{\lambda f_l}, f_y=\frac{y_2}{\lambda f_l}}. \quad (4.12)$$

There two are interesting cases. First, when the object is placed against the lens, $d = 0$, and so Eq. (4.12) reduces to the solution found in Eq. (4.10). Second, when the object is placed in the front focal plane of the lens, $d = f_l$, so the exponential phase factor outside of the integral becomes 1, leaving an exact FT relationship. Listing 4.4 gives the MATLAB function `lens_in_front` for an object placed a distance d in front of a converging lens.

4.2.3 Object behind the lens

When the object is placed behind the lens a distance d away from the focal plane as shown in Fig. 4.4, the optical field $U_s(x_1, y_1)$ just before the object is (in the geometric-optics approximation) a converging spherical wave given by

$$U_s(x_1, y_1) = \frac{f_l}{d} P\left(\frac{f_l}{d}x_1, \frac{f_l}{d}y_1\right) e^{-i\frac{k}{2d}\left(x_1^2+y_1^2\right)}. \quad (4.13)$$

Listing 4.4 Code for performing a propagation from the pupil plane to the focal plane for an object placed in front of a lens in MATLAB.

```
function [x2 y2 Uout] ...
    = lens_in_front_ft(Uin, wvl, d1, f, d)
% function [x2 y2 U_out] ...
%     = lens_in_front_ft(Uin, wvl, d1, f, d)

    N = size(Uin, 1);   % assume square grid
    k = 2*pi/wvl;    % optical wavevector
    fX = (-N/2 : 1 : N/2 - 1) / (N * d1);
    % observation plane coordinates
    [x2 y2] = meshgrid(wvl * f * fX);
    clear('fX');

    % evaluate the Fresnel-Kirchhoff integral but with
    % the quadratic phase factor inside cancelled by the
    % phase of the lens
    Uout = 1 / (i*wvl*f)...
        .* exp(i*k/(2*f) * (1-d/f) * (x2.^2 + y2.^2)) ...
        .* ft2(Uin, d1);
```

This is valid when the distance $d \ll f_l$. Then, the field just after the object is

$$U\left(x_1, y_1\right) = \frac{f_l}{d} P\left(\frac{f_l}{d} x_1, \frac{f_l}{d} y_1\right) e^{-i\frac{k}{2d}\left(x_1^2+y_1^2\right)} t_A\left(x_1, y_1\right). \tag{4.14}$$

Finally, propagating from the object to the focal plane using Eq. (4.6) yields

$$\begin{aligned} U\left(x_2, y_2\right) = {} & \frac{f_l}{d}\frac{1}{i\lambda d} e^{i\frac{k}{2d}\left(x_2^2+y_2^2\right)} \\ & \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t_A\left(x_1, y_1\right) P\left(\frac{f_l}{d} x_1, \frac{f_l}{d} y_1\right) e^{-i\frac{2\pi}{\lambda d}\left(x_2 x_1 + y_2 y_1\right)}\, dx_1\, dy_1. \end{aligned} \tag{4.15}$$

As before, this can be cast in terms of an FT so that

$$U\left(x_2, y_2\right) = \frac{f_l}{d}\frac{1}{i\lambda d} e^{i\frac{k}{2d}\left(x_2^2+y_2^2\right)} \mathcal{F}\left\{ t_A\left(x_1, y_1\right) P\left(\frac{f_l}{d} x_1, \frac{f_l}{d} y_1\right)\right\}\Bigg|_{f_x=\frac{x_2}{\lambda d}, f_y=\frac{y_2}{\lambda d}}. \tag{4.16}$$

Listing 4.5 gives the MATLAB function `lens_behind_ft` from the object plane to the focal plane.

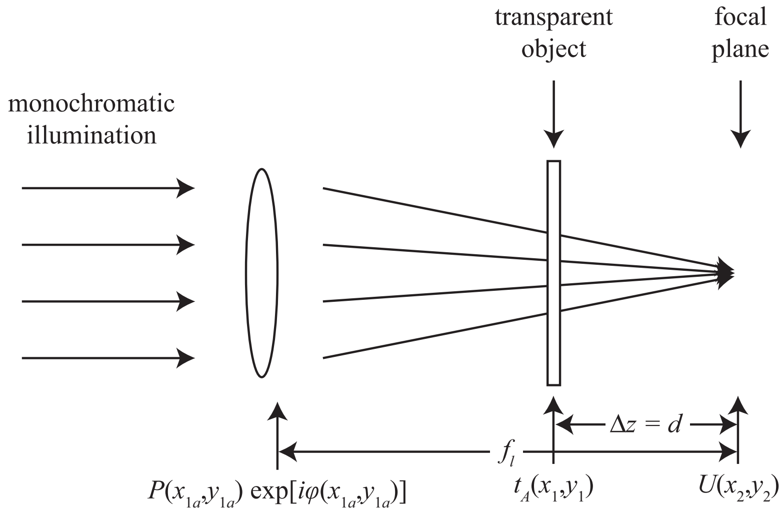

Figure 4.4 Diagram of lens geometry for an object placed behind the lens.

Listing 4.5 Code for performing a propagation from the pupil plane to the focal plane for an object placed behind a converging lens in MATLAB.

```
function [x2 y2 Uout] ...
    = lens_behind_ft(Uin, wvl, d1, f)
% function [x2 y2 Uout] ...
%     = lens_behind_ft(Uin, wvl, d1, d, f)

    N = size(Uin, 1);   % assume square grid
    k = 2*pi/wvl;    % optical wavevector
    fX = (-N/2 : 1 : N/2 - 1) / (N * d1);
    % observation plane coordinates
    [x2 y2] = meshgrid(wvl * d * fX);
    clear('fX');

    % evaluate the Fresnel-Kirchhoff integral but with
    % the quadratic phase factor inside cancelled by the
    % phase of the lens
    Uout = f/d * 1  / (i*wvl*d)...
        .* exp(i*k/(2*d)*(x2.^2 + y2.^2)) .* ft2(Uin, d1);
```

4.3 Problems

1. Repeat the example in Sec. 4.1 for a 1 mm $\times$ 1 mm square aperture in the source plane. Show the numerical and analytic results together on the same plot.

2. Repeat the example in Sec. 4.1 for a two-slit aperture consisting of two 1 mm $\times$ 1 mm square apertures spaced 0.5 mm apart in the source plane. Show the numerical and analytic results together on the same plot.

3. Repeat the example in Sec. 4.1 for a 1 mm $\times$ 1 mm square amplitude grating in the source plane. Let the amplitude transmittance be

$$t_A(x_1, y_1) = \frac{1}{2}\left[1 + \cos\left(2\pi f_0 x_1\right)\right] \operatorname{rect}\left(\frac{x_1}{D}\right) \operatorname{rect}\left(\frac{y_1}{D}\right), \tag{4.18}$$

   where $f_0 = 10/D$. Show the numerical and analytic results together on the same plot.

4. Repeat the example in Sec. 4.1 for a 1 mm $\times$ 1 mm square phase grating in the source plane. Let the amplitude transmittance be

$$t_A(x_1, y_1) = e^{i2\pi \cos(2\pi f_0 x_1)} \operatorname{rect}\left(\frac{x_1}{D}\right) \operatorname{rect}\left(\frac{y_1}{D}\right), \tag{4.19}$$

   where $f_0 = 10/D$. Show the numerical and analytic results together on the same plot.

5. A 1-μm wavelength Gaussian laser beam is normally incident on a lens. The beam waist is at the lens with width $w = 2$ cm, and the lens's focal length is 1 m. Assuming that the lens has an infinite diameter, numerically and analytically compute the diffraction pattern in the focal plane. Show the numerical and analytic results together on the same plot.