# Appendix I
# Special Functions

Most of the functions encountered in introductory analysis belong to the class of *elementary functions.* This class is composed of polynomials, rational functions, transcendental functions (trigonometric, exponential, logarithmic, and so on), and functions constructed by combining two or more of these functions through addition, multiplication, division, or composition. Beyond these lies a class of *special functions* important in a variety of engineering and physics applications. Because special functions play such a central role in our analysis throughout the text, we provide in this appendix a brief summary of some of the special functions and their most important properties. A more complete treatment of the subject is provided in L. C. Andrews, *Special Functions of Mathematics for Engineers*, 2nd ed. (SPIE Optical Engineering Press, Bellingham, Wash.; Oxford University Press, Oxford, 1998).

## Gamma Function

Historically, the gamma function was discovered by Euler (1707–1783) in 1729. He was concerned with the problem of interpolating between the numbers

$$n! = \int_0^\infty e^{-t} t^n dt, \qquad n = 0, 1, 2, \ldots \tag{1}$$

with nonintegral values of $n$. His studies eventually led him to the *gamma function* defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \qquad \mathrm{Re}(z) > 0, \tag{2}$$

where $z$ is a complex variable and Re means the real part of the argument.

Some of the most important properties associated with the gamma function are listed below:

($G1$): $\Gamma(z+1) = z\,\Gamma(z)$

($G2$): $\Gamma(n+1) = n!, \quad n = 0, 1, 2, \ldots$

($G3$): $\Gamma(1/2) = \sqrt{\pi}$

$(G4)$: $\Gamma(n+1/2) = \dfrac{(2n)!}{2^{2n}n!}\sqrt{\pi}, \qquad n = 0, 1, 2, \ldots$

$(G5)$: $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}, \qquad (z \text{ nonintegral}).$

## Error Function

The *error function*, which derives its name from its importance in the theory of errors, also occurs in probability theory and in certain heat conduction problems, among other areas. It is defined by the integral

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt. \tag{3}$$

In some applications it is also useful to introduce the *complementary error function*

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt. \tag{4}$$

The primary properties associated with these functions are the following:

$(E1)$: $\operatorname{erf}(-z) = \operatorname{erf}(z)$

$(E2)$: $\operatorname{erf}(0) = 0$

$(E3)$: $\operatorname{erf}(\infty) = 1, \qquad |\arg(z)| < \pi/4$

$(E4)$: $\operatorname{erf}(z) = \dfrac{2}{\sqrt{\pi}} \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}, \qquad |z| < \infty$

$(E5)$: $\operatorname{erfc}(z) = 1 - \operatorname{erf}(z)$

$(E6)$: $\operatorname{erf}(z) \sim \dfrac{e^{-z^2}}{\sqrt{\pi z}}, \qquad \operatorname{Re}(z) \gg 1, \quad |\arg(z)| < 3\pi/4.$

## Pochhammer Symbol

Closely associated with the gamma function is the *Pochhammer symbol* defined by

$$(a)_0 = 1,$$

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad n = 1, 2, 3, \ldots \tag{5}$$

This symbol, which is fundamental in the study of generalized hypergeometric functions, satisfies the following relations:

$(P1)$: $(0)_0 = 1$

$(P2)$: $(1)_n = n!$

$$(P3):\quad (a)_{n+k} = (a)_k(a+k)_n$$

$$(P4):\quad (a)_{-n} = \frac{(-1)^n}{(1-a)_n}$$

$$(P5):\quad (-k)_n = \begin{cases} \dfrac{(-1)^n k!}{(k-n)!}, & 0 \le n \le k \\ 0, & n > k \end{cases}$$

$$(P6):\quad \binom{a}{n} = \frac{(-1)^n}{n!}(-a)_n$$

$$(P7):\quad (2n)! = 2^{2n}(1/2)_n n!$$

$$(P8):\quad (2n+1)! = 2^{2n}(3/2)_n n!$$

$$(P9):\quad (a)_{2n} = 2^{2n}(\tfrac{1}{2}a)_n(\tfrac{1}{2}+\tfrac{1}{2}a)_n.$$

## Hypergeometric Function

The major development of the theory of the *hypergeometric function* was carried out by Gauss (1777–1855). Specializations of this function include various elementary functions, such as the arcsine and natural log, and several orthogonal polynomial sets, such as the Chebyshev and Legendre polynomials. Its series definition is given by

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \qquad |z| < 1, \tag{6}$$

where $c \neq 0, -1, -2, \ldots$ and $z$ may be real or complex. The series (6) is restricted to values of the argument for which $|z| < 1$, although the hypergeometric function is defined for values outside the unit circle in the complex plane. For example, if $|z| > 1$, the analytic continuation formula may be used, which yields the result

$$\begin{aligned} {}_2F_1(a,b;c;z) = {} & \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\ {}_2F_1\left(a, 1-c+a; 1-b+a; \frac{1}{z}\right) \\ & + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\ {}_2F_1\left(b, 1-c+b; 1-a+b; \frac{1}{z}\right), \\ & |\arg(-z)| < \pi. \end{aligned} \tag{7}$$

Some additional relations involving the hypergeometric function are listed below:

(*H1*): ${}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z)$

(*H2*): $\dfrac{d^k}{dz^k}\,{}_2F_1(a,b;c;z) = \dfrac{(a)_k(b)_k}{(c)_k}\,{}_2F_1(a+k,b+k;c+k;z), \quad k = 1,2,3,\ldots$

(*H3*): ${}_2F_1(a,b;c;z) = \dfrac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\displaystyle\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt, \quad c > b > 0$

(*H4*): ${}_2F_1(a,b;c;1) = \dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$

(*H5*): ${}_2F_1(a,b;c;-z) = (1+z)^{-a}\,{}_2F_1\left(a, c-b; c; \dfrac{z}{1+z}\right)$

(*H6*): ${}_2F_1(1-a,1;2;-z) = \dfrac{(1+z)^a - 1}{az}.$

## Confluent Hypergeometric Functions

Kummer (1810–1893) is the name most closely associated with the confluent hypergeometric functions. For this reason, these functions are also known as Kummer's functions. The series representation for the *confluent hypergeometric function of the first kind* is given by

$$ {}_1F_1(a;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n}\frac{z^n}{n!}, \quad |z| < \infty, \tag{8}$$

where $c \neq 0, -1, -2, \ldots$ and $z$ may be real or complex. Specializations of this function include the exponential function, Hermite and Laguerre polynomial sets, and several Bessel functions, among others. The *confluent hypergeometric function of the second kind* is a linear combination of functions of the first kind that can be expressed as

$$\begin{aligned} U(a;c;z) = {} & \frac{\Gamma(1-c)}{\Gamma(1+a-c)}\ {}_1F_1(a;c;z) \\ & + \frac{\Gamma(c-1)}{\Gamma(a)} z^{1-c}\ {}_1F_1(1+a-c;2-c;z). \end{aligned} \tag{9}$$

Some useful properties associated with the confluent hypergeometric functions of the first and second kinds are listed below:

(*CH1*): $\dfrac{d^k}{dz^k}\ {}_1F_1(a;c;z) = \dfrac{(a)_k}{(c)_k}\ {}_1F_1(a+k;c+k;z), \quad k = 1,2,3,\ldots$

(*CH2*): ${}_1F_1(a;c;z) = \dfrac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\displaystyle\int_0^1 e^{zt}t^{a-1}(1-t)^{c-a-1}dt, \quad c > a > 0$

$$(CH3): \quad {}_1F_1(a;c;z) = e^z\ {}_1F_1(c-a;c;-z)$$

$$(CH4): \quad {}_1F_1(a;c;-z) \sim \begin{cases} 1 - \dfrac{az}{c}, & |z| \ll 1 \\[2ex] \dfrac{\Gamma(c)}{\Gamma(c-a)} z^{-a}, & \mathrm{Re}(z) \gg 1 \end{cases}$$

$$(CH5): \quad \frac{d}{dz}U(a;c;z) = -aU(a+1;c+1;z)$$

$$(CH6): \quad U(a;c;z) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-zt}t^{a-1}(1+t)^{c-a-1}dt, \qquad a > 0, \quad \mathrm{Re}(z) > 0$$

$$(CH7): \quad U(a;c;z) = z^{1-c}U(1+a-c;2-c;z)$$

$$(CH8): \quad U(a;c;z) \sim \begin{cases} \dfrac{\Gamma(1-c)}{\Gamma(1+a-c)} + \dfrac{\Gamma(c-1)}{\Gamma(a)} z^{1-c}, & |z| \ll 1 \\[2ex] z^{-a}, \quad \mathrm{Re}(z) \gg 1. \end{cases}$$

## Generalized Hypergeometric Functions

During the last 70 years there has been considerable interest in working with generalized hypergeometric functions, of which the hypergeometric and confluent hypergeometric functions are special cases. In general, we say a power series $\sum A_n z^n$ is a *series of hypergeometric type* if $A_{n+1}/A_n$ is a rational function of $n$. A general series of this type is

$$ {}_pF_q(a_1,\ldots,a_p;\ c_1,\ldots,c_q;z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(c_1)_n \cdots (c_q)_n} \frac{z^n}{n!}, \tag{10}$$

where $p$ and $q$ are nonnegative integers and no $c_k$ ($k = 1,\ldots,q$) is zero or a negative integer. This function, which we denote by simply ${}_pF_q$, is called a *generalized hypergeometric function.* Provided the series (10) does not terminate, it can be established by the ratio test of calculus that

1. If $p < q+1$, the series *converges* for all $|z| < \infty$.
2. If $p = q+1$, the series *converges* for $|z| < 1$ and diverges for $|z| > 1$.
3. If $p > q+1$, the series *diverges* for all $z$ except $z = 0$.

Many elementary functions, as well as special functions, are specializations of some generalized hypergeometric function. Some of these relations are listed below[1]:

$$(GH1): \quad {}_0F_0(-;-;z) = e^z$$

$$(GH2): \quad {}_1F_0(a;-;z) = (1-z)^{-a}$$

[1] The absence of a numerator or denominator parameter in any ${}_pF_q$ function is emphasized by a dash.

$$(GH3): \quad {}_0F_1\left(-;\ \frac{1}{2};\ -\frac{z^2}{4}\right) = \cos z$$

$$(GH4): \quad {}_0F_1\left(-;\ \frac{3}{2};\ -\frac{z^2}{4}\right) = \frac{\sin z}{z}$$

$$(GH5): \quad {}_0F_1\left(-;1;\ -\frac{z^2}{4}\right) = J_0(z) \quad \text{(Bessel function of first kind)}$$

$$(GH6): \quad {}_0F_1\left(-;a+\frac{1}{2};\ -\frac{z^2}{4}\right) = \left(\frac{z}{2}\right)^{1/2-a}\Gamma\left(a+\frac{1}{2}\right)J_{a-1/2}(z)$$
$$\text{(Bessel function of first kind)}$$

$$(GH7): \quad {}_1F_1(1;2;\ -z) = \frac{1}{z}(1-e^{-z})$$

$$(GH8): \quad {}_1F_1(-n;a+1;z) = \frac{n!}{(a+1)_n}L_n^{(a)}(z) \quad \text{(associated Laguerre function)}$$

$$(GH9): \quad {}_1F_1\left(\frac{1}{2};\frac{3}{2};-z^2\right) = \frac{\sqrt{\pi}}{2z}\operatorname{erf}(z)$$

$$(GH10): \quad {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\ z^2\right) = \frac{\sin^{-1} z}{z}$$

$$(GH11): \quad {}_1F_2\left(\frac{1}{2};1,1;-x^2\right) = [J_0(x)]^2.$$

## Bessel Functions of the First Kind

*Bessel functions* are named in honor of the German astronomer F. W. Bessel (1784–1846), who in 1824 carried out the first systematic study of their properties. There are several families of Bessel functions, which are also known as *cylinder functions.*

The *Bessel function of the first kind* has the series representation

$$J_p(z) = \sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{2k+p}}{k!\Gamma(k+p+1)}, \qquad |z| < \infty, \tag{11}$$

where $p$ denotes the *order* of the function. Some of the basic properties associated with this function are listed below:

$$(BJ1): \quad J_0(0) = 1; \quad J_p(0) = 0,\ p > 0$$

$$(BJ2): \quad J_{-n}(z) = (-1)^n J_n(z), \quad n = 1,2,3,\ldots$$

$$(BJ3): \quad e^{iz\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(z)e^{in\theta}$$

$$(BJ4):\quad J_p{}'(z) + \frac{p}{z}J_p(z) = J_{p-1}(z)$$

$$(BJ5):\quad J_p{}'(z) - \frac{p}{z}J_p(z) = -J_{p+1}(z)$$

$$(BJ6):\quad J_{p-1}(z) - J_{p+1}(z) = 2J_p{}'(z)$$

$$(BJ7):\quad J_{p-1}(z) + J_{p+1}(z) = \frac{2p}{z}J_p(z)$$

$$(BJ8):\quad J_n(x+y) = \sum_{k=-\infty}^{\infty} J_k(x)J_{n-k}(y)$$

$$(BJ9):\quad J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{\pm iz\cos\theta}d\theta$$

$$(BJ10):\quad J_p(z) \sim \frac{(z/2)^p}{\Gamma(1+p)}, \qquad p \neq -1,-2,-3,\ldots,\ \ z \to 0+$$

$$(BJ11):\quad J_p(z) \sim \sqrt{\frac{2}{\pi z}}\cos\Big[z - (p+1/2)\frac{\pi}{2}\Big], \qquad |z| \gg 1,\ \ |\arg(z)| < \pi.$$

## Modified Bessel Functions

The *modified Bessel functions of the first* and *second kind*, respectively, are defined by

$$I_p(z) = i^{-p}J_p(iz) = \sum_{k=0}^{\infty}\frac{(z/2)^{2k+p}}{k!\Gamma(k+p+1)}, \qquad |z| < \infty \tag{12}$$

and

$$K_p(z) = \frac{\pi}{2}\frac{I_{-p}(z) - I_p(z)}{\sin p\pi}, \tag{13}$$

where once again $p$ denotes the order. Because of its definition, the modified Bessel function $I_p(z)$ has many properties in common with the standard Bessel function $J_p(z)$. Some basic properties associated with both kinds of modified Bessel function are listed below:

$$(BI1):\quad I_0(0) = 1; \quad I_p(0) = 0,\ p > 0$$

$$(BI2):\quad I_{-n}(z) = I_n(z), \qquad n = 1,2,3,\ldots$$

$$(BI3):\quad e^{z\cos\theta} = \sum_{n=-\infty}^{\infty} I_n(z)\cos n\theta$$

$$(BI4):\quad I_p{}'(z) + \frac{p}{z}I_p(z) = I_{p-1}(z)$$

$$(BI5):\quad I_p{}'(z) - \frac{p}{z} I_p(z) = I_{p+1}(z)$$

$$(BI6):\quad I_{p-1}(z) + I_{p+1}(z) = 2I_p{}'(z)$$

$$(BI7):\quad I_{p-1}(z) - I_{p+1}(z) = \frac{2p}{z} I_p(z)$$

$$(BI8):\quad I_n(x+y) = \sum_{n=-\infty}^{\infty} I_k(x) I_{n-k}(y)$$

$$(BI9):\quad I_0(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{\pm z\cos\theta} d\theta$$

$$(BI10):\quad I_p(z) \sim \frac{(z/2)^p}{\Gamma(1+p)}, \qquad p \neq -1, -2, -3, \ldots, \quad z \to 0+$$

$$(BI11):\quad I_p(z) \sim \frac{e^z}{\sqrt{2\pi z}}, \qquad |z| \gg 1, \quad |\arg(z)| < \pi/2$$

$$(BK1):\quad K_{-p}(z) = K_p(z)$$

$$(BK2):\quad K_p{}'(z) + \frac{p}{z} K_p(z) = -K_{p-1}(z)$$

$$(BK3):\quad K_p{}'(z) - \frac{p}{z} K_p(z) = -K_{p+1}(z)$$

$$(BK4):\quad K_{p-1}(z) + K_{p+1}(z) = -2K_p{}'(z)$$

$$(BK5):\quad K_{p-1}(z) - K_{p+1}(z) = -\frac{2p}{z} K_p(z)$$

$$(BK6):\quad K_0(z) \sim -\ln z, \qquad z \to 0+$$

$$(BK7):\quad K_p(z) \sim \frac{\Gamma(p)}{2} \left(\frac{2}{z}\right)^p, \quad p > 0, \; z \to 0+$$

$$(BK8):\quad K_p(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}, \qquad |z| \gg 1, \quad |\arg(z)| < \pi/2.$$