Chapter 18

Pulse Propagation

Overview: In this chapter we briefly investigate atmospheric effects on ultrashort pulses, i.e., on the order of femto-seconds. Longer pulses (e.g., picosecond pulses) can usually be modeled as if they were continuous wave (CW) signals during the time when the pulse is transmitted. However, this is not the case for ultrashort pulses.

Our treatment here is limited to *temporal pulse spreading* in the near field and far field, and to far-field *temporal scintillations.* Because of the increased bandwidth associated with short pulses, much of the analysis has to be performed in the frequency domain. This requires development of the *two-frequency mutual coherence function* and the *four-frequency cross-coherence function.*

18.1 Introduction

One of the major advantages of free-space optical (FSO) communication systems over conventional radio frequency (RF) systems is their high antenna gain which permits higher data transmission rates. Typical operating frequencies for RF communications and radar extend from 540 kHz to approximately 300 Ghz. Laser systems operate in the much higher terahertz regime ($\sim 3 \times 10^{14}$ Hz), which provides potentially larger bandwidths that can accommodate the higher transmission rates. However, owing to the shorter wavelengths associated with FSO systems, the transmitted optical pulses experience greater degradation than RF pulses, resulting in higher bit error rates (BER), which diminish the performance level of the FSO system. An understanding of these degrading effects on the optical pulses is necessary in designing a reliable optical communication system. Two of the aspects considered here are (*i*) *temporal broadening* and (*ii*) *temporal scintillation.* Temporal broadening can be deduced from knowledge of the two-frequency mutual coherence function (MCF) whereas temporal scintillation requires the four-frequency cross-coherence function.

18.2 Background

Because data rates are inversely proportional to pulse widths, ultrashort pulses allow for greater data rates than longer pulses. Unfortunately, atmospheric effects are more deleterious to the ultrashort pulses and their analysis is also more complicated. That is, picosecond pulses can generally be analyzed using a continuous wave (CW) model such as that used in previous chapters. For ultrashort pulses on the order of femtoseconds, however, it is necessary to do the analysis in the frequency domain using the two-dimensional and four-dimensional coherence functions.

Many researchers have studied pulse propagation over several years [1–17]. A number of these researchers studied only the evolution of the temporal characteristics of a Gaussian pulse in free space [12–17]. Most of the other studies have been concerned with pulse broadening which can be inferred from knowledge of the two-frequency MCF, but analytic solutions have been developed only in special cases. For example, an analytic model for the two-frequency MCF of a *plane wave* was developed by Sreenivasiah et al. [2], and Young et al. [7] developed an analytic model for the two-frequency MCF of a *Gaussian-beam wave* under weak irradiance fluctuations. In the late 1970s, Liu and Yeh [5,6] presented the temporal moments method to study the mean pulse width and arrival time fluctuations of an optical pulse. Their result for pulse broadening is valid for all types of pulses and conditions of atmospheric turbulence, but their arrival time statistics are valid for only a plane wave and the extreme case of very strong scattering. More recently, Young et al. [8] used the two-frequency MCF approach to obtain an analytical solution for the near-field temporal broadening of an ultrashort Gaussian pulse propagating through weak optical turbulence and a similar analysis for the far-field beam broadening was done by Kelly and Andrews [10]. This latter analysis also included scintillation statistics.

18.2.1 Atmospheric propagation model

Let us consider an input pulse in the plane of the transmitter ($z = 0$) that is propagated through a random medium to a receiver located at distance L from the source.

We take the case where the input pulse is a modulated signal with carrier (angular) frequency ω_0 that can be represented by

$$p_i(t) = v_i(t)e^{-i\omega_0 t}, \tag{1}$$

where the amplitude $v_i(t)$ represents the pulse shape. If the complex envelope of the output pulse is $v_0(t)$, the output pulse at the receiver can be similarly described by

$$p_0(t) = v_0(t)e^{-i\omega_0 t}. \tag{2}$$

The Fourier transform of the input pulse (1) is given by the expression

$$\begin{aligned} P_i(\omega) &= \int_{-\infty}^{\infty} p_i(t)e^{i\omega t}\,dt = \int_{-\infty}^{\infty} v_i(t)e^{i(\omega-\omega_0)t}\,dt \\ &= V_i(\omega - \omega_0), \end{aligned} \tag{3}$$

where $V_i(\omega)$ is the Fourier transform of the amplitude $v_i(t)$. In the same fashion, the Fourier transform of the output (2) leads to $P_0(\omega) = V_0(\omega - \omega_0)$, where $V_0(\omega)$ is the Fourier transform of the complex envelope of the output pulse. By using a linear systems approach, it follows that the input (1) and output (2) are related in the frequency domain according to

$$P_0(\omega) = H(\omega)P_i(\omega), \tag{4}$$

where $H(\omega)$ is the *system function* (or frequency transfer function) of the atmosphere. In the time domain, this last relation can be expressed as

$$v_0(t)e^{-i\omega_0 t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} V_i(\omega - \omega_0)H(\omega)e^{-i\omega t}\,d\omega, \tag{5}$$

from which we deduce

$$v_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V_i(\omega)H(\omega + \omega_0)e^{-i\omega t}\,d\omega. \tag{6}$$

Equation (6) represents the general relation between the complex envelope $v_0(t)$ of the output pulse and the amplitude $v_i(t)$ of the input pulse.

To understand the physical meaning of the time-varying random medium system function $H(\omega)$, we note that, if the input waveform is a simple time-harmonic function $\exp(-i\omega t)$, then the output is $H(\omega)\exp(-i\omega t)$. Hence, the system function is a random function that represents the response of the random medium to a time-harmonic input function. For the case of a monochromatic optical wave propagating distance L along the positive z-axis, the electromagnetic field of the wave at the plane of the receiver can be expressed in the form

$$u(\mathbf{r}, L; \omega, t) = U(\mathbf{r}, L; \omega)e^{-i\omega t}, \tag{7}$$

where $U(\mathbf{r}, L; \omega)$ is the complex amplitude of the wave in atmospheric turbulence, ω is angular frequency related to wave number k by $\omega = kc$, and the constant c is the speed of light (3×10^8 m/s). Hence, the system function of the random medium can be identified as $H(\omega) = U(\mathbf{r}, L; \omega)$. With this interpretation of the system function, we can now express the complex envelope of the output pulse in the form

$$v_0(\mathbf{r}, L; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V_i(\omega) U(\mathbf{r}, L; \omega + \omega_0) e^{-i\omega t}\, d\omega. \tag{8}$$

18.3 Two-Frequency Mutual Coherence Function

Much of the effort over the years concerning pulse propagation has been directed at calculating the two-frequency MCF, which provides a measure of the coherence bandwidth and coherence time.

The *two-point two-time correlation function* of the complex envelope of the output pulse (8) is defined by the ensemble average

$$\begin{aligned} R_v(\mathbf{r}_1, \mathbf{r}_2, L; t_1, t_2) &= \langle v_0(\mathbf{r}_1, L; t_1) v_0^*(\mathbf{r}_2, L; t_2) \rangle \\ &= \frac{1}{(2\pi)^2} \int\!\!\int_{-\infty}^{\infty} V_i(\omega_1) V_i^*(\omega_2) \Gamma_2(\mathbf{r}_1, \mathbf{r}_2, L; \omega_1 + \omega_0, \omega_2 + \omega_0) \\ &\quad \times \exp(-i\omega_1 t_1 + i\omega_2 t_2)\, d\omega_1\, d\omega_2, \end{aligned} \tag{9}$$

where Γ_2 is the two-frequency MCF defined by

$$\Gamma_2(\mathbf{r}_1, \mathbf{r}_2, L; \omega_1 + \omega_0, \omega_2 + \omega_0) = \langle U(\mathbf{r}_1, L; \omega_1 + \omega_0) U^*(\mathbf{r}_2, L; \omega_2 + \omega_0) \rangle. \tag{10}$$

The two-frequency MCF (10) plays an important role in determining the basic characteristics associated with pulse propagation in random media. To simplify notation in the following analysis, let

$$k_1 = \frac{\omega_1 + \omega_0}{c}, \quad k_2 = \frac{\omega_2 + \omega_0}{c}, \tag{11}$$

and then express (10) as

$$\begin{aligned} \Gamma_2(\mathbf{r}_1, \mathbf{r}_2, L; k_1, k_2) &= U_0(\mathbf{r}_1, L; k_1) U_0^*(\mathbf{r}_2, L; k_2) \langle \exp[\psi(\mathbf{r}_1, L; k_1) + \psi^*(\mathbf{r}_2, L; k_2)] \rangle \\ &= \Gamma_2^0(\mathbf{r}_1, \mathbf{r}_2, L; k_1, k_2) M_2(\mathbf{r}_1, \mathbf{r}_2, L; k_1, k_2), \end{aligned} \tag{12}$$

where the free-space two-frequency MCF is described by

$$\begin{aligned} \Gamma_2^0(\mathbf{r}_1, \mathbf{r}_2, L; k_1, k_2) &= U_0(\mathbf{r}_1, L; k_1) U_0^*(\mathbf{r}_2, L; k_2) \\ &= (\Theta_1 - i\Lambda_1)(\Theta_2 + i\Lambda_2) \exp[i(k_1 - k_2)L] \\ &\quad \times \exp\left[\frac{ik_1}{2L}(\bar{\Theta}_1 + i\Lambda_1) r_1^2\right] \exp\left[-\frac{ik_2}{2L}(\bar{\Theta}_2 - i\Lambda_2) r_2^2\right]. \end{aligned} \tag{13}$$

In Eq. (13), we have introduced the Gaussian beam parameters

$$
\begin{aligned}
\Theta_m &= \frac{\Theta_0}{\Theta_0^2 + \Lambda_{0,m}^2}, \quad \bar{\Theta}_m = 1 - \Theta_m; \quad m = 1, 2, \\
\Lambda_m &= \frac{\Lambda_{0,m}}{\Theta_0^2 + \Lambda_{0,m}^2}; \quad m = 1, 2, \\
\Theta_0 &= 1 - \frac{L}{F_0}, \quad \Lambda_{0,m} = \frac{2L}{k_m W_0^2}; \quad m = 1, 2.
\end{aligned} \tag{14}
$$

Although the notation is similar, the Gaussian beam parameters (14) should not be confused with those used to describe propagation through a train of optical elements (see Chap. 10). As before, F_0 denotes the phase front radius of curvature of the beam wave at the transmitter and W_0 is the beam radius. The remaining factor in Eq. (12), caused by atmospheric turbulence, is

$$
\begin{aligned}
M_2(\mathbf{r}_1, \mathbf{r}_2, L; k_1, k_2) = \exp\Bigg\{ &-2\pi^2 \left(k_1^2 + k_2^2\right) L \int_0^\infty \kappa \Phi_n(\kappa)\, d\kappa \\
&+ 4\pi^2 k_1 k_2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) J_0\left(\kappa |\gamma_1 \mathbf{r}_1 - \gamma_2^* \mathbf{r}_2|\right) \\
&\times \exp\left[-\frac{iL\kappa^2}{2} \xi \left(\frac{\gamma_1}{k_1} - \frac{\gamma_2^*}{k_2}\right)\right] d\kappa\, d\xi \Bigg\},
\end{aligned} \tag{15}
$$

where

$$
\begin{aligned}
\gamma_1 &= 1 - (\bar{\Theta}_1 + i\Lambda_1)\xi, \\
\gamma_2^* &= 1 - (\bar{\Theta}_2 - i\Lambda_2)\xi.
\end{aligned} \tag{16}
$$

An approximate form for the two-frequency MCF can readily be obtained from the above results in the special cases of near-field and far-field approximations. In the case of a collimated beam, for example, the *near-field approximation* corresponds to

$$
\Lambda_{0,m} \ll 1, \quad W_m \cong W_0 \quad (m = 1, 2), \tag{17}
$$

where W_m is the spot radius of the Gaussian beam at the receiver. Thus, it follows that $\Theta_m \cong 1$, $\gamma_m \cong 1$, and $k_m \Lambda_m / 2L \cong 1/W_0^2$, from which we deduce

$$
\begin{aligned}
M_2(\mathbf{r}_1, \mathbf{r}_2, L; k_1, k_2) \cong \exp\Bigg\{ &-2\pi^2 (k_1^2 + k_2^2) L \int_0^\infty \kappa \Phi_n(\kappa) d\kappa \\
&+ 4\pi^2 k_1 k_2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) J_0(\kappa\rho) \\
&\times \exp\left[-\frac{iL\kappa^2}{2} \left(\frac{1}{k_1} - \frac{1}{k_2}\right) \xi\right] d\kappa\, d\xi \Bigg\},
\end{aligned} \tag{18}
$$

$$\Gamma_2^0(\mathbf{r}_1, \mathbf{r}_2, L; k_1, k_2) \cong \exp\left[i(k_1 - k_2)L - \frac{\left(r_1^2 + r_2^2\right)}{W_0^2}\right], \tag{19}$$

where $\rho = |\mathbf{r}_1 - \mathbf{r}_2|$. Under the *far-field approximation*

$$\Lambda_{0,m} \gg 1, \quad W_m \cong W_0 \Lambda_{0,m} \quad (m = 1, 2), \tag{20}$$

it follows that

$$\Gamma_2^0(\mathbf{r}_1, \mathbf{r}_2, L; k_1, k_2) \cong \left(\frac{W_0^2}{2L}\right)^2 k_1 k_2 \exp[i(k_1 - k_2)L]$$

$$\times \exp\left[-\left(\frac{W_0}{2L}\right)^2 \left(k_1^2 r_1^2 + k_2^2 r_2^2\right) + \frac{i\left(k_1 r_1^2 - k_2 r_2^2\right)}{2L}\right], \tag{21}$$

$$M_2(\mathbf{r}_1, \mathbf{r}_2, L; k_1, k_2) \cong \exp\left\{-2\pi^2 (k_1^2 + k_2^2) L \int_0^\infty \kappa \Phi_n(\kappa)\, d\kappa\right.$$

$$+ 4\pi^2 k_1 k_2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) J_0(\kappa\rho\xi)$$

$$\left.\times \exp\left[-\frac{iL\kappa^2}{2}\left(\frac{1}{k_1} - \frac{1}{k_2}\right)\xi(1 - \xi)\right] d\kappa\, d\xi\right\}. \tag{22}$$

18.3.1 Mean irradiance and temporal pulse broadening

In Section 6.3.2 we discussed the spatial spreading of a Gaussian-beam wave caused by the random medium in which it propagates. Here we wish to extend that analysis to the temporal spreading of a *Gaussian pulse* defined by

$$v_i(t) = \exp\left(-t^2/T_0^2\right), \tag{23}$$

where we identify the quantity T_0 with the input pulse *half-width.* The Fourier transform of (23) is

$$V_i(\omega) = \int_{-\infty}^{\infty} e^{-t^2/T_0^2} e^{i\omega t}\, dt = \sqrt{\pi} T_0 \exp\left(-\frac{1}{4}\omega^2 T_0^2\right), \tag{24}$$

which has spectral half-width $\Delta\omega = 2/T_0$. Although very short pulses (e.g., on the order of femtoseconds[1]) are ordinarily classified as *wideband*, the transmitted waveform may still be considered *narrowband* if $\Delta\omega \ll \omega_0$, where ω_0 is the carrier frequency. For example, at optical frequencies on the order of $\omega_0 = 2\pi c/\lambda \sim 12\pi \times 10^{14}$, the transmitted waveform is narrowband relative to the carrier frequency under the condition $T_0 \geq 20$ fs. Temporal spreading of a pulse in a random medium is caused primarily by two mechanisms—the scattering process of the medium (i.e., dispersive spreading produced by multiple paths)

[1]One femtosecond (fs) is equal to 10^{-15} s.

and the wandering of the pulse [5]. The second mechanism, which is the dominant factor in weak fluctuations, is due to the difference in arrival time from one member of the ensemble to another. The combined effects of the two mechanisms can be deduced from the mean irradiance of the pulse obtained from Eq. (12) by setting $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}$ and $t_1 = t_2 = t$. This action leads to

$$\langle I(\mathbf{r}, L; t)\rangle = \frac{T_0^2}{4\pi}\int\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\omega_c^2 T_0^2\right)\exp\left(-\frac{1}{8}\omega_d^2 T_0^2\right) \times \Gamma_2\left(\mathbf{r}, \mathbf{r}, L; \omega_0 + \omega_c + \frac{1}{2}\omega_d, \omega_0 + \omega_c - \frac{1}{2}\omega_d\right)\exp(-i\omega_d t)\, d\omega_c\, d\omega_d, \tag{25}$$

where we have introduced the sum and difference frequencies

$$\omega_c = \frac{1}{2}(\omega_1 + \omega_2), \quad \omega_d = \omega_1 - \omega_2. \tag{26}$$

Let us consider the free-space irradiance of a collimated beam under the near-field and far-field approximations (17) and (20). Using Eqs. (19), (21), and (23), the near-field and far-field approximations yield, respectively,

$$I^0(\mathbf{r}, L; t) \cong \exp\left(-\frac{2r^2}{W_0^2}\right)\exp\left[-\frac{2(t - L/c)^2}{T_0^2}\right], \tag{27}$$

$$I^0(\mathbf{r}, L; t) \cong T_0^2\left(\frac{W_0^2}{2Lc}\right)^2 \frac{\omega_0^2 T_0^4 + 4(t - L/c - r^2/2Lc)^2}{[T_0^2 + (W_0 r/Lc)^2]^3} \times \exp\left[-\frac{\omega_0^2 T_0^2 W_0^2 r^2/2L^2c^2}{T_0^2 + (W_0 r/Lc)^2}\right]\exp\left[-\frac{2(t - L/c - r^2/2Lc)^2}{T_0^2 + (W_0 r/Lc)^2}\right]. \tag{28}$$

Expressions (27) and (28), which are not based on the narrowband assumption, were previously derived by Ziolkowski and Judkins [13]. As they point out, in the absence of turbulence the initial pulsed beam retains its form in the near field of the transmitter aperture, but as the field evolves from the near-field to the far-field region it acquires a time-derivative form that decays as $1/L$ in the far field. Thus, the distortion from the Gaussian form exhibited by Eq. (28) is expected from the radiation process itself for wideband pulses. On the other hand, if we impose the narrowband assumption $\omega_d^2 \ll \omega_c^2$, then the free-space irradiance in the near-field approximation is again that given by (27), but in the far-field approximation the mean irradiance is described by the undistorted Gaussian function

$$I^0(\mathbf{r}, L; t) \cong T_0\left(\frac{W_0^2}{2Lc}\right)^2 \frac{\omega_0^2 T_0^4 + T_0^2 + (W_0 r/Lc)^2}{[T_0^2 + (W_0 r/Lc)^2]^{5/2}} \times \exp\left[-\frac{\omega_0^2 T_0^2 W_0^2 r^2/2L^2c^2}{T_0^2 + (W_0 r/Lc)^2}\right]\exp\left[-\frac{2(t - L/c - r^2/2Lc)^2}{T_0^2}\right]. \tag{29}$$

For a collimated beam ($\Theta_0 = 1$) propagating in a random medium characterized by the von Kármán spectrum with inner scale zero, the two-frequency MCF in the near field takes the approximate form

$$\Gamma_2\left(\mathbf{r},\mathbf{r},L;\omega_0+\omega_c+\frac{1}{2}\omega_d,\omega_0+\omega_c-\frac{1}{2}\omega_d\right)\cong\exp\left(-\frac{2r^2}{W_0^2}\right)\exp\left(\frac{i\omega_d L}{c}-a_1\omega_d^2\right), \tag{30}$$

where we have also invoked the narrowband assumption $\omega_d^2 \ll \omega_c^2$ and introduced

$$a_1 = \frac{2\pi^2 L}{c^2}\int_0^\infty \kappa\Phi_n(\kappa)\,d\kappa \cong \frac{0.39C_n^2 L\kappa_0^{-5/3}}{c^2}. \tag{31}$$

The quantity $1/\sqrt{a_1}$ can be identified as a measure of the coherence bandwidth [18,19]. Young et al. [8] showed under these conditions that the resulting mean irradiance is

$$\langle I(\mathbf{r},L;t)\rangle \cong \frac{T_0}{T_1}\exp\left(-\frac{2r^2}{W_0^2}\right)\exp\left[-\frac{2(t-L/c)^2}{T_1^2}\right], \tag{32}$$

where

$$T_1 = \sqrt{T_0^2 + 8a_1}. \tag{33}$$

The quantity T_1 provides an estimate at the receiver of the turbulence-induced pulse half-width caused by a combination of beam wander and first-order scattering.

Invoking the narrowband assumption ($\omega_d^2 \ll \omega_c^2$), the two-frequency MCF for a collimated beam in the far-field approximation can be written as

$$\begin{aligned}\Gamma_2\left(\mathbf{r},\mathbf{r},L;\omega_0+\omega_c+\frac{1}{2}\omega_d,\omega_0+\omega_c-\frac{1}{2}\omega_d\right) &\cong \left(\frac{W_0^2}{2Lc}\right)^2(\omega_0+\omega_c)^2\\ \times\exp\left[-\frac{1}{2}\left(\frac{W_0 r}{Lc}\right)^2(\omega_0+\omega_c)^2\right]&\exp\left[i\left(\frac{L}{c}+\frac{r^2}{2Lc}\right)\omega_d - a_1\omega_d^2\right],\end{aligned} \tag{34}$$

where a_1 is defined by (31). Using Eq. (34), Kelly and Andrews [10] developed the expression

$$\begin{aligned}\langle I(\mathbf{r},L;t)\rangle \cong \frac{T_0^2}{T_1}\left(\frac{W_0^2}{2Lc}\right)^2 &\frac{\omega_0^2T_0^4 + T_0^2 + (W_0 r/Lc)^2}{[T_0^2+(W_0r/Lc)^2]^{5/2}}\\ \times\exp&\left[-\frac{\omega_0^2T_0^2W_0^2r^2/2L^2c^2}{T_0^2+(W_0r/Lc)^2}\right]\exp\left[-\frac{2(t-L/c-r^2/2Lc)^2}{T_1^2}\right].\end{aligned} \tag{35}$$

Thus, turbulence-induced temporal pulse spreading in both the near-field and far-field approximations under the narrowband assumption is described by the quantity T_1.

Note that the turbulence-induced quantity a_1, defined by Eq. (31) and used to determine T_1, is independent of the initial pulse width, wavelength, general beam characteristics, and near-field or far-field assumptions. It is simply a phase effect, not a diffraction effect, as is more clearly revealed by the analysis in Section 18.3.2 below.

18.3.2 Pulse arrival time

In most applications it is important to have certain information concerning the arrival time of a pulse as observed by a fixed observer. That is, the pulse arrival time t_a is a random variable that fluctuates about some mean value. Following Liu and Yeh [6], pulse statistics concerning the random arrival time can be described in terms of the *average temporal moments*

$$\langle M^{(n)}(\mathbf{r},L)\rangle = \int_{-\infty}^{\infty} t^n \langle I(\mathbf{r},L;t)\rangle\, dt, \quad n = 0,1,2,\ \ldots. \tag{36}$$

The zeroth moment relates to the total energy of the pulse, whereas the first moment relates to the mean arrival time or "time centroid" of the pulse. Similarly, pulse spreading caused by the random medium can be inferred from the second temporal moment [20]. Hence, the temporal moment method provides an alternative to calculating the temporal pulse spreading as compared with the analysis provided above using the mean irradiance.

By using properties of the Dirac delta function

$$\begin{gathered}\int_{-\infty}^{\infty} t^n e^{i(\omega_2-\omega_1)t}\, dt = 2\pi(-i)^n \delta^{(n)}(\omega_2-\omega_1),\\ \int_{-\infty}^{\infty} \delta^{(n)}(\omega_2-\omega_1) f(\omega_2)\, d\omega_2 = (-1)^n \frac{\partial^n}{\partial\omega_2^n} f(\omega_2)\bigg|_{\omega_2=\omega_1},\end{gathered} \tag{37}$$

it follows that the moments (36) in general can be calculated from

$$\langle M^{(n)}(\mathbf{r},L)\rangle = \frac{i^n}{2\pi}\int_{-\infty}^{\infty} V_i(\omega_1)\frac{\partial^n}{\partial\omega_2^n}\left[V_i^*(\omega_2)\Gamma_2(\mathbf{r},\mathbf{r},L;\omega_1+\omega_0,\omega_2+\omega_0)\right]\bigg|_{\omega_2=\omega_1} d\omega_1, \tag{38}$$

where we have used Eq. (9) and interchanged the order of integration. Equation (38) is theoretically valid under all turbulence conditions with the appropriate MCF.

By using weak fluctuation theory, near-field and far-field approximations, and narrowband assumption, the mean arrival time at transverse position $r = 0$ leads to [6]

$$\langle t_a\rangle = \frac{M^{(1)}(0,L)}{M^{(0)}(0,L)} = \frac{L}{c}, \tag{39}$$

the same as in free space and in Eqs. (32) and (35). Under the same conditions, the nominal duration of the pulse can be deduced from

$$(\Delta t)^2 = \frac{M^{(2)}(0,L)}{M^{(0)}(0,L)} - \langle t_a \rangle^2 = \frac{T_1^2}{4}, \tag{40}$$

which is proportional to the mean square pulse width defined by Eq. (33).

18.4 Four-Frequency Cross-Coherence Function

Whereas calculation of temporal broadening involves the two-frequency MCF, the calculation of temporal scintillation involves the four-frequency cross-coherence function. To begin, we calculate the *four-point four-time correlation function*

$$\begin{aligned}
&C_v(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4, L; t_1, t_2, t_3, t_4) \\
&\quad = \langle v_0(\mathbf{r}_1, L; t_1) v_0^*(\mathbf{r}_2, L; t_2) v_0(\mathbf{r}_3, L; t_3) v_0^*(\mathbf{r}_4, L; t_4) \rangle \\
&\quad = \frac{1}{(2\pi)^4} \int\int\int\int_{-\infty}^{\infty} V_i(\omega_1) V_i^*(\omega_2 V_i(\omega_3) V_i^*(\omega_4) \\
&\qquad \times \Gamma_4(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4, L; \omega_1 + \omega_0, \omega_2 + \omega_0, \omega_3 + \omega_0, \omega_4 + \omega_0) \\
&\qquad \times \exp(-i\omega_1 t_1 + i\omega_2 t_2 - i\omega_3 t_3 + i\omega_4 t_4)\, d\omega_1\, d\omega_2\, d\omega_3\, d\omega_4,
\end{aligned} \tag{41}$$

where Γ_4 is the *four-frequency cross-coherence function* which we can write in the form

$$\begin{aligned}
&\Gamma_4(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4, L; \omega_1 + \omega_0, \omega_2 + \omega_0, \omega_3 + \omega_0, \omega_4 + \omega_0) \\
&\quad = \Gamma_2(\mathbf{r}_1, \mathbf{r}_2, L; k_1, k_2) \Gamma_2(\mathbf{r}_3, \mathbf{r}_4, L; k_3, k_4) M_4(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4, L; k_1, k_2, k_3, k_4).
\end{aligned} \tag{42}$$

On the right-hand side of (42) we have introduced the wave numbers

$$k_m = \frac{\omega_m + \omega_0}{c}, \quad m = 1, 2, 3, 4. \tag{43}$$

The factor M_4 in (42) is a consequence of the effect of atmospheric turbulence and is defined by

$$\begin{aligned}
M_4(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4, L; k_1, k_2, k_3, k_4) = \exp\Big[&2E_1(0, 0, L; k_1, k_2) + 2E_1(0, 0, L; k_3, k_4) \\
&+ E_2(\mathbf{r}_1, \mathbf{r}_2, L; k_1, k_2) + E_2(\mathbf{r}_1, \mathbf{r}_4, L; k_1, k_4) \\
&+ E_2(\mathbf{r}_3, \mathbf{r}_2, L; k_3, k_2) + E_2(\mathbf{r}_3, \mathbf{r}_4, L; k_3, k_4) \\
&+ E_3(\mathbf{r}_1, \mathbf{r}_3, L; k_1, k_3) + E_3^*(\mathbf{r}_2, \mathbf{r}_4, L; k_2, k_4) \Big],
\end{aligned} \tag{44}$$

where

$$E_1(0,0,L;k_1,k_2) = -\pi^2(k_1^2+k_2^2)L\int_0^\infty \kappa\Phi_n(\kappa)d\kappa, \tag{45}$$

$$E_2(\mathbf{r}_1,\mathbf{r}_2,L;k_1,k_2) = 4\pi^2 k_1 k_2 L\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)J_0(\kappa|\gamma_1\mathbf{r}_1-\gamma_2^*\mathbf{r}_2|) \times \exp\left[-\frac{iL\kappa^2}{2}\xi\left(\frac{\gamma_1}{k_1}-\frac{\gamma_2^*}{k_2}\right)\right]d\kappa\, d\xi, \tag{46}$$

$$E_3(\mathbf{r}_1,\mathbf{r}_2,L;k_1,k_2) = -4\pi^2 k_1 k_2 L\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)J_0(|\kappa\gamma_1\mathbf{r}_1-\gamma_2\mathbf{r}_2|) \times \exp\left[-\frac{iL\kappa^2}{2}\xi\left(\frac{\gamma_1}{k_1}+\frac{\gamma_2}{k_2}\right)\right]d\kappa\, d\xi. \tag{47}$$

18.4.1 Temporal scintillation index

The *temporal scintillation index* is defined by the normalized variance

$$\sigma_I^2(\mathbf{r},L;t) = \frac{\langle I^2(\mathbf{r},L;t)\rangle}{\langle I(\mathbf{r},L;t)\rangle^2} - 1, \tag{48}$$

where the second moment $\langle I^2(\mathbf{r},L;t)\rangle$ can be deduced from the correlation function (41) by setting $\mathbf{r}_1=\mathbf{r}_2=\mathbf{r}_3=\mathbf{r}_4=\mathbf{r}$ and $t_1=t_2=t_3=t_4=t$. In particular, we find

$$\begin{aligned}\langle I^2(\mathbf{r},L;t)\rangle = \frac{T_0^4}{16\pi^2}\int\int\int\int_{-\infty}^{\infty} &\exp\left[-\frac{1}{2}T_0^2\left(\omega_{12}^2+\omega_{34}^2\right)-\frac{1}{8}T_0^2\left(\tilde{\omega}_{12}^2+\tilde{\omega}_{34}^2\right)\right]\\ &\times\Gamma_4(\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{r},L;\omega_0+\omega_{12},\omega_0+\omega_{34},\tilde{\omega}_{12},\tilde{\omega}_{34})\\ &\times\exp[-i(\tilde{\omega}_{12}+\tilde{\omega}_{34})t]\,d\omega_{12}\,d\omega_{34}\,d\tilde{\omega}_{12}\,d\tilde{\omega}_{34},\end{aligned} \tag{49}$$

where

$$\omega_{mn} = \frac{1}{2}(\omega_m+\omega_n), \quad \tilde{\omega}_{mn} = \omega_m-\omega_n \quad (m,n=1,2,3,4;\ m\neq n). \tag{50}$$

Under the narrowband assumption and far-field approximation the fourth moment (44) simplifies to

$$\begin{aligned}M_4\left(\mathbf{r},\mathbf{r},\mathbf{r},\mathbf{r},L;k_{12},k_{24},\tilde{k}_{12},\tilde{k}_{34}\right) &= \exp\left[-2\pi^2(\tilde{k}_{12}+\tilde{k}_{34})^2L\int_0^\infty\kappa\Phi_n(\kappa)d\kappa\right]\\ &\times\exp\left\{4\pi^2k_{13}^2L\int_0^1\int_0^\infty\kappa\Phi_n(\kappa)\left[1-\exp\left(-\frac{iL\kappa^2}{k_{13}}\xi(1-\xi)\right)\right]d\kappa\,d\xi\right\}\\ &\times\exp\left\{4\pi^2k_{24}^2L\int_0^1\int_0^\infty\kappa\Phi_n(\kappa)\left[1-\exp\left(\frac{iL\kappa^2}{k_{24}}\xi(1-\xi)\right)\right]d\kappa\,d\xi\right\},\end{aligned} \tag{51}$$

where

$$k_{mn} = \frac{1}{2}(k_m + k_n), \quad \tilde{k}_{mn} = k_m - k_n \quad (m, n = 1, 2, 3, 4;\ m \neq n). \tag{52}$$

Evaluation of the integrals in (51) based on the von Kármán spectrum yields

$$\begin{aligned} &M_4(\mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, L; \omega_0 + \omega_{12}, \omega_0 + \omega_{34}, \tilde{\omega}_{12}, \tilde{\omega}_{34}) \\ &\quad \cong \exp\big\{-a_1(\tilde{\omega}_{12} + \tilde{\omega}_{34})^2 - i(\Delta - 1.87\varepsilon)(\tilde{\omega}_{12} + \tilde{\omega}_{34}) \\ &\qquad + \varepsilon[(\omega_0 + \omega_{12}) + (\omega_0 + \omega_{34})]\big\}, \end{aligned} \tag{53}$$

where

$$\Delta = \frac{0.39 C_n^2 L \kappa_0^{1/3}}{c}, \quad \varepsilon = \frac{0.25 C_n^2 L^{11/6} \omega_0^{1/6}}{c^{7/6}}. \tag{54}$$

From Eqs. (34), (42), and (53), it follows that

$$\begin{aligned} &\Gamma_4(\mathbf{r}, \mathbf{r}, \mathbf{r}, \mathbf{r}, L;\ \omega_0 + \omega_{12}, \omega_0 + \omega_{34}, \tilde{\omega}_{12}, \tilde{\omega}_{34}) \\ &\quad \cong \left(\frac{W_0^2}{2Lc}\right)^4 (\omega_0 + \omega_{12})^2 (\omega_0 + \omega_{34})^2 \\ &\quad \times \exp\left\{-2\left(\frac{W_0 r}{2Lc}\right)^2 \left[(\omega_0 + \omega_{12})^2 + (\omega_0 + \omega_{34})^2\right] + \varepsilon\left[(\omega_0 + \omega_{12}) + (\omega_0 + \omega_{34})\right]\right\} \\ &\quad \times \exp\left[-a_1(\tilde{\omega}_{12} + \tilde{\omega}_{34})^2 - i\left(\Delta - 1.87\varepsilon - \frac{L}{c} - \frac{r^2}{2Lc}\right)(\tilde{\omega}_{12} + \tilde{\omega}_{34})\right]. \end{aligned} \tag{55}$$

Thus, the subsequent substitution of (55) into (49) yields the approximation

$$\begin{aligned} \langle I^2(\mathbf{r}, L; t)\rangle = &\left(\frac{W_0^2}{2Lc}\right)^4 \frac{1}{\sqrt{T_0^2 + 16a_1}} \frac{T_0^3\left[T_0^2 + (W_0 r/Lc)^2 + (\omega_0 T_0^2 + \varepsilon)^2\right]^2}{\left[T_0^2 + (W_0 r/Lc)^2\right]^5} \\ &\times \exp\left[-\omega_0^2 T_0^2 + \frac{\left(\omega_0^2 T_0^2 + \varepsilon\right)^2}{T_0^2 + (W_0 r/Lc)^2}\right] \\ &\times \exp\left[-\frac{4(t - L/c - r^2/2Lc + \Delta - 1.87\varepsilon)^2}{T_0^2 + 16a_1}\right]. \end{aligned} \tag{56}$$

Finally, from (48) we obtain the temporal scintillation index

$$\begin{aligned} \sigma_I^2(\mathbf{r}, L; t) = &\frac{T_0^2 + 8a_1}{T_0\sqrt{T_0^2 + 16a_1}} \frac{\left[T_0^2 + (W_0 r/Lc)^2 + (\omega_0 T_0^2 + \varepsilon)^2\right]^2}{\left[\omega_0^2 T_0^4 + T_0^2 + (W_0 r/Lc)^2\right]^2} \\ &\times \exp\left\{\frac{\varepsilon(2\omega_0 T_0^2 + \varepsilon)}{T_0^2 + (W_0 r/Lc)^2} - \frac{[a_1(T_0^2 + 8a_1) + 8](\Delta - 1.87\varepsilon)^2}{2(T_0^2 + 16a_1)}\right\} \\ &\times \exp\left\{\frac{a_1\left[8(t - L/c - r^2/2Lc) - (\Delta - 1.87\varepsilon)(T_0^2 + 8a_1)\right]^2}{2(T_0^2 + 8a_1)(T_0^2 + 16a_1)}\right\} - 1. \end{aligned} \tag{57}$$

Investigation of the scintillation index (57) reveals that the smallest scintillation values occur slightly before the mean arrival time ($\Delta - 1.87\varepsilon < 0$) and that scintillation increases at the leading and trailing edges of the pulse. Furthermore, the scintillation index contains both a linear and quadratic term in time t. In the asymptotic limit $T_0 \to \infty$, the far-field temporal scintillation index approaches

$$\sigma_I^2(\mathbf{r}, L; t) \longrightarrow \exp(2\varepsilon\omega_0) - 1 \cong 2\varepsilon\omega_0, \tag{58}$$

or, by using (54) for ε,

$$\sigma_I^2(\mathbf{r}, L; t) \longrightarrow 0.50 C_n^2 k^{7/6} L^{11/6}, \quad T_0 \longrightarrow \infty. \tag{59}$$

We recognize this last expression as the scintillation index of a CW spherical wave.

18.5 Summary and Discussion

In this chapter we have studied the temporal broadening and temporal scintillation of ultrashort space-time Gaussian pulses propagating through weak optical turbulence. This analysis involves first calculating the two-frequency MCF for beam broadening and mean arrival time, and the four-frequency cross-coherence function for temporal scintillations. We examined both near-field and far-field cases of beam broadening and deduced that the turbulence-induced beam broadening is independent of wavelength and decreases with decreasing outer scale. The far-field temporal scintillation index is minimized at approximately the mean arrival time of the pulse (39) and increases in the leading and trailing edges of the pulse. Both temporal broadening and temporal scintillation increase dramatically for ultrashort pulses on the order of 10–30 fs. For longer pulses the far-field scintillation index approaches that of a CW spherical wave.

As a final comment, we point out the analysis provided here involves only optical turbulence effects; it does not include dispersion effects. It is possible that dispersion effects are just as significant as (or, possibly, more significant than) atmospheric effects. Further analysis is required in this regard.

Problems

Section 18.3

1. Show that the turbulence factor (15) of the two-frequency MCF in the special case $\mathbf{r}_2 = -\mathbf{r}_1$ can be expressed in the form

$$M_2(\rho, L; k_1, k_2) = \exp(I_1 + I_2 + I_3),$$

where

$$\begin{aligned}
I_1 &= -2\pi^2 k_1^2 L\left(\frac{k_1 - k_2}{k_1}\right)^2 \int_0^\infty \kappa\Phi_n(\kappa)d\kappa,\\
I_2 &= -4\pi^2 k_1^2 L\left(\frac{k_2}{k_1}\right)\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\left[1-\exp\left(-\frac{L\Lambda_{\mathrm{av}}\kappa^2\xi^2}{k_1}\right)\right]d\kappa d\xi,\\
I_3 &= -4\pi^2 k_1^2 L\left(\frac{k_2}{k_1}\right)\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\exp\left(-\frac{L\Lambda_{\mathrm{av}}\kappa^2\xi^2}{k_1}\right)\\
&\quad\times\left\{1 - J_0[(1-\bar{\Theta}_{\mathrm{av}}\xi)\kappa\rho]\right\}d\kappa d\xi,\\
\Lambda_{\mathrm{av}} &= \frac{1}{2}\left(\Lambda_1 + \frac{k_1}{k_2}\Lambda_2\right),\ \bar{\Theta}_{\mathrm{av}} = 1 - \frac{1}{2}(\Theta_1 + \Theta_2).
\end{aligned}$$

2. Show that the turbulence factor (15) of the two-frequency MCF in the special case $\mathbf{r}_2 = \mathbf{r}_1 = \mathbf{r}$ can be expressed in the form

$$M_2(\mathbf{r}, L; k_1, k_2) = \exp(J_1 + J_2 + J_3),$$

where

$$\begin{aligned}
J_1 &= -2\pi^2 k_1^2 L\left(\frac{k_1 - k_2}{k_1}\right)^2 \int_0^\infty \kappa\Phi_n(\kappa)d\kappa,\\
J_2 &= -4\pi^2 k_1^2 L\left(\frac{k_2}{k_1}\right)\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\left[1-\exp\left(-\frac{L\Lambda_{\mathrm{av}}\kappa^2\xi^2}{k_1}\right)\right]d\kappa d\xi,\\
J_3 &= 4\pi^2 k_1^2 L\left(\frac{k_2}{k_1}\right)\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\exp\left(-\frac{L\Lambda_{\mathrm{av}}\kappa^2\xi^2}{k_1}\right)\{I_0[(\Lambda_1+\Lambda_2)\kappa r\xi]-1\}d\kappa d\xi,\\
\Lambda_{\mathrm{av}} &= \frac{1}{2}\left(\Lambda_1 + \frac{k_1}{k_2}\Lambda_2\right).
\end{aligned}$$

3. For the special case of a plane wave for which $U_0(\mathbf{r}, L; \omega) = e^{ikL} = e^{i\omega L/c}$, show that the complex envelope of a Gaussian pulse [Eq. (23)] propagating distance L in free space is

$$v_0(\mathbf{r}, L; t) = \exp\left[-\frac{(t - L/c)^2}{T_0^2}\right]\exp\left(-\frac{i\omega_0 L}{c}\right).$$

4. For the special case of a spherical wave for which

$$U_0(\mathbf{r}, L; \omega) = \frac{1}{4\pi L}\exp\left(\frac{i\omega L}{c} + \frac{i\omega r^2}{2Lc}\right),$$

show that the complex envelope of a Gaussian pulse [Eq. (23)] propagating distance L in free space is

$$v_0(\mathbf{r}, L; t) = \frac{1}{4\pi L}\exp\left[-\frac{(t - L/c - r^2/2Lc)^2}{T_0^2}\right]\exp\left(-\frac{i\omega_0 L}{c}\right).$$

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