Chapter 5

Classical Theory for Propagation Through Random Media

Overview: In this chapter we introduce the *stochastic Helmholtz equation* as the governing partial differential equation for the scalar field of an optical wave propagating through a random medium. However, we provide only the foundational material here for the classical methods of solving the Helmholtz equation. It is interesting that all such methods are based on the same set of simplifying assumptions—backscattering and depolarization effects are negligible, the refractive index is delta correlated in the direction of propagation (Markov approximation), and the paraxial approximation can be invoked.

The Born and Rytov perturbation methods for solving the stochastic Helmholtz equation are introduced first. Whereas the *Born approximation* has limited utility in optical wave propagation, the *Rytov approximation* has successfully been used to predict all relevant statistical parameters associated with laser propagation throughout regimes featuring weak irradiance fluctuations. We also illustrate that the Rytov approximation can be generalized to include wave propagation through a train of optical elements that are all characterized by *ABCD* matrix representations. Methods applicable also under strong irradiance fluctuations are briefly discussed here but formulated in greater detail in Chap. 7. These methods are the *parabolic equation method*, which is based on the development of parabolic equations for each of the statistical moments of the field, and the *extended Huygens-Fresnel principle.*

Early probability density function (PDF) models developed for the irradiance of the optical wave include the *modified Rician distribution*, which follows from the Born approximation, and the *lognormal model*, which follows directly from the first Rytov approximation. Of these two PDFs, only the lognormal PDF model compares well with the lower-order irradiance moments calculated from experimental data under weak fluctuation conditions. Hence, in this regime it has been the most often-used model for calculating fade statistics associated with a fading communications channel. Nonetheless, more recent investigations of the lognormal PDF suggest that it may be optimistic in predicting fade probabilities, even in weak fluctuation regimes.

We end the chapter with a modification of the Rytov method called the *extended Rytov theory* that utilizes the two-scale behavior of the propagating wave encountered in regimes of strong irradiance fluctuations. The formalism of the method presented here permits the development of new models for beam wander and scintillation in subsequent chapters that are applicable under strong fluctuations.

5.1 Introduction

When an optical/IR wave propagates through a random medium like atmospheric turbulence, both the amplitude and phase of the electric field experience random fluctuations caused by small random changes in the index of refraction. Several different theoretical approaches have been developed for describing these random amplitude and phase fluctuations, based upon solving the wave equation (or some simplified form of it) for the electric field of the wave or for the various moments of the field. Unfortunately, these mathematically rigorous approaches in most cases have led to tractable analytic results supported by experimental data only in certain asymptotic regimes.

5.2 Stochastic Wave Equation

The classical problem of optical wave propagation in an unbounded continuous medium with smoothly varying stochastic refractive index has a governing differential equation with random coefficients [1–8]. In particular, by assuming a sinusoidal time variation (viz., a monochromatic wave) in the electric field, it has been shown that Maxwell's equations for the vector amplitude[1] $\mathbf{E}(\mathbf{R})$ of a propagating electromagnetic wave lead directly to [1,5,6]

$$\nabla^2\mathbf{E} + k^2n^2(\mathbf{R})\mathbf{E} + 2\nabla[\mathbf{E}\cdot\nabla\log n(\mathbf{R})] = \mathbf{0}, \tag{1}$$

where $\mathbf{R} = (x, y, z)$ denotes a point in space, $k = 2\pi/\lambda$ is the wave number of the electromagnetic wave, λ is the wavelength, $n(\mathbf{R})$ is the index of refraction whose time variations have been suppressed, and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplacian operator. We are assuming that time variations in the refractive index are sufficiently slow that a quasi steady-state approach can be used, which permits us to treat $n(\mathbf{R})$ as a function of position only.

Equation (1) can be reduced to a more rudimentary equation by imposing a simple set of fundamental assumptions on the propagating wave:

(*i*) backscattering of the wave can be neglected
(*ii*) depolarization effects can be neglected
(*iii*) the refractive index is delta correlated in the direction of propagation
(*iv*) the parabolic (paraxial) approximation can be invoked

Assumptions (*i*) and (*ii*) follow from the same idea. Namely, because the wavelength λ for optical/IR radiation is much smaller than the smallest scale of turbulence (i.e., the inner scale l_0), the maximum scattering angle is roughly $\lambda/l_0 \sim 10^{-4}$ rad. Hence, it follows that monochromatic radiation scattered by relatively weak, large-scale refractivity fluctuations is contained within a narrow cone about the forward scatter in the propagation direction. That is to say, backscattering of the wave requires a significant change in the "slowly varying"

[1]It is common practice in most works to also refer to the vector amplitude $\mathbf{E}(\mathbf{R})$ as the field of the electromagnetic wave.

refractive index over distances shorter than the wavelength, which doesn't occur in the atmosphere. As a further consequence, it has been shown that the last term on the left-hand side of Eq. (1) is negligible [5,9]. By dropping this term, which is related to the change in polarization of the wave as it propagates, Eq. (1) simplifies to

$$\nabla^2 \mathbf{E} + k^2 n^2(\mathbf{R})\mathbf{E} = \mathbf{0} \tag{2}$$

Equation (2) is now easily decomposed into three scalar equations, one for each component of the field **E**. If we let $U(\mathbf{R})$ denote one of the scalar components that is transverse to the direction of propagation along the positive z-axis, then (2) may be replaced by the scalar *stochastic Helmholtz equation*

$$\nabla^2 U + k^2 n^2(\mathbf{R})U = 0. \tag{3}$$

Under assumption (*iii*), it is assumed that the refractive index can be expressed as

$$n(\mathbf{R}) = n_0 + n_1(\mathrm{R}), \tag{4}$$

where $n_0 = \langle n(\mathrm{R})\rangle \cong 1, \langle n_1(\mathbf{R})\rangle = 0,$ and that the covariance function—delta correlated in the direction of propagation along the positive z-axis—can be expressed as

$$\langle n_1(\mathbf{R}_1)n_1(\mathbf{R}_2)\rangle = B_n(\mathbf{R}_1 - \mathbf{R}_2) \cong \delta(z_1 - z_2)A_n(\mathbf{r}_1 - \mathbf{r}_2). \tag{5}$$

Equation (5) is often referred to as the *Markov approximation.* In writing (5), we have assumed the covariance is statistically homogeneous so it is a function of only the difference $\mathbf{R}_1 - \mathbf{R}_2$, where $\mathbf{R}_j = (\mathbf{r}_j, z_j), j = 1, 2,$ and $A_n(\mathbf{r}_1 - \mathbf{r}_2)$ is a *two-dimensional covariance function.* We are further recognizing that if $n_1(\mathbf{R})$ is statistically homogeneous in three dimensions, it is also statistically homogeneous in two dimensions.

Even with the above simplifications, Eq. (3) has proven difficult to solve. Historically, the first approach to solving (3) was based on the *method of Green's function*, reducing (3) to an equivalent integral equation. However, exact solutions of (3) by this or any method have never been found! Some more fruitful early attempts to solve (3) were based on the *geometrical optics method* (GOM), which ignores diffraction effects, and on two perturbation theories widely known as the *Born approximation* and *Rytov approximation.* The Born approximation was first applied to the integral equation for scattering that can be derived directly from the Schröedinger equation. The Rytov approximation, also known as the *method of smooth perturbations* in the Russian literature, was originally developed for acoustic wave propagation. Both of these perturbation theories are restricted to regimes of *weak irradiance fluctuations* (Section 5.2.2), which normally limits the propagation path length to a few hundred meters or less on horizontal paths or to a few kilometers or less along certain slant paths. Other methods not bound by the limitations of weak irradiance fluctuations are briefly discussed in Section 5.8.

5.2.1 Covariance function of the refractive index

By definition, the refractive index perturbation term $n_1(\mathbf{R})$ has a zero mean value, and it is customary to assume that it is a *Gaussian* random field, although this is generally not required. In the former case, however, its characteristics are completely determined by the *correlation function* (or *covariance function*)

$$B_n(\mathbf{R}_1, \mathbf{R}_2) = \langle n_1(\mathbf{R}_1) n_1(\mathbf{R}_2) \rangle. \tag{6}$$

If the refractive-index fluctuations are *statistically homogeneous*, then the covariance function can be expressed as

$$B_n(\mathbf{R}_1, \mathbf{R}_2) \equiv B_n(\mathbf{R}_1 - \mathbf{R}_2) = \iiint_{-\infty}^{\infty} \Phi_n(\mathbf{K}) \exp[i\mathbf{K} \cdot (\mathbf{R}_1 - \mathbf{R}_2)]\, d^3\kappa, \tag{7}$$

where the function $\Phi_n(\mathbf{K})$ is the *three-dimensional spatial power spectrum* of the refractive-index fluctuations.

For the case when $n_1(\mathbf{R})$ is *delta correlated* in the direction of propagation [recall Eq. (5)], we have $\kappa_z = 0$ and the power spectrum reduces to $\Phi_n(\mathbf{K}) = \Phi_n(\kappa_x, \kappa_y, 0)$. Consequently, by setting $z = z_1 - z_2$ and $\mathbf{p} = \mathbf{r}_1 - \mathbf{r}_1$, Eq. (7) becomes

$$\begin{aligned} B_n(\mathbf{p}, z) \equiv \delta(z) A_n(\mathbf{p}) &= \iint_{-\infty}^{\infty} d^2\kappa \Phi_n(\mathbf{K}) \exp(i\mathbf{K} \cdot \mathbf{p}) \int_{-\infty}^{\infty} d\kappa_z \exp(iz\kappa_z) \\ &= 2\pi\delta(z) \iint_{-\infty}^{\infty} d^2\kappa \Phi_n(\mathbf{K}) \exp(i\mathbf{K} \cdot \mathbf{p}), \end{aligned} \tag{8}$$

where we have employed the Fourier transform identity for the delta function

$$\int_{-\infty}^{\infty} \exp(iz\kappa_z)\, d\kappa_z = \int_{-\infty}^{\infty} \cos(z\kappa_z)\, d\kappa_z = 2\pi\delta(z). \tag{9}$$

From Eq. (8) it now follows that

$$A_n(\mathbf{p}) = 2\pi \iint_{-\infty}^{\infty} \Phi_n(\mathbf{K}) \exp(i\mathbf{K} \cdot \mathbf{p})\, d^2\kappa. \tag{10}$$

If the refractive-index fluctuations are *isotropic* in addition to being statistically homogeneous, Eq. (10) can be further reduced to

$$A_n(\rho) = 4\pi^2 \int_0^{\infty} \kappa \Phi_n(\kappa) J_0(\kappa\rho)\, d\kappa, \tag{11}$$

where $\rho = |\mathbf{p}|$ and $J_0(x)$ is a Bessel function of the first kind and order zero.

In latter work it is useful to also introduce the *two-dimensional spectral density* $F_n(\mathbf{K}, z)$ defined by the transform relations

$$F_n(\kappa_x,\ \kappa_y,\ 0,\ |z|) = \int_{-\infty}^{\infty} \Phi_n(\kappa_x,\ \kappa_y,\ \kappa_z)\cos(z\kappa_z)\,d\kappa_z, \tag{12}$$

$$\Phi_n(\kappa_x,\ \kappa_y,\ \kappa_z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_n(\kappa_x,\ \kappa_y,\ 0,\ |z|)\cos(z\kappa_z)\,dz, \tag{13}$$

the latter of which, for $\kappa_z = 0$, reduces to

$$\int_{-\infty}^{\infty} F_n(\mathbf{K}, z)\,dz = 2\pi\Phi_n(\mathbf{K}). \tag{14}$$

Hence, we see that the delta-correlated covariance function $B_n(\mathbf{p}, z) = A_n(\mathbf{p})\delta(z)$ and the two-dimensional spectrum $F_n(\mathbf{K}, z)$ are Fourier transform pairs.

5.2.2 Weak and strong fluctuation conditions

Theoretical studies of optical wave propagation are traditionally classified as belonging to either weak or strong fluctuation theories. When using the Kolmogorov spectrum (Section 3.3.1) in the study of plane waves or spherical waves that have propagated over a path of length L, it is customary to distinguish between these cases by values of the *Rytov variance* (also commonly denoted by σ_1^2)

$$\sigma_R^2 = 1.23C_n^2 k^{7/6} L^{11/6}, \tag{15}$$

where C_n^2 is the refractive-index structure parameter. *Weak fluctuations* are associated with $\sigma_R^2 < 1$, and then the Rytov variance physically represents the irradiance fluctuations associated with an unbounded plane wave (Section 8.2). *Moderate fluctuation conditions* are characterized by $\sigma_R^2 \sim 1$, *strong fluctuations* are associated with $\sigma_R^2 > 1$, and the so-called *saturation regime* is defined by the condition $\sigma_R^2 \to \infty$.

The classification of weak and strong fluctuations based entirely on the Rytov variance is not adequate in the case of a Gaussian-beam wave, particularly for a focused beam. Fundamentally, weak fluctuation conditions correspond to regimes where the scintillation index throughout the beam profile is less than unity. Thus, for a Gaussian-beam wave, weak fluctuation regimes correspond to the set of conditions [10]

$$\sigma_R^2 < 1 \quad \text{and} \quad \sigma_R^2 \Lambda^{5/6} < 1, \tag{16}$$

where $\Lambda = 2L/kW^2$ and W is the (free space) beam radius at the receiver (see Chap. 4). If either of these conditions fails to exist, the fluctuations are classified as moderate to strong. For a Gaussian-beam wave and arbitrary refractive-index spectral model, weak fluctuations are also described by

$$q < 1 \quad \text{and} \quad q\Lambda < 1, \tag{17}$$

where the parameter $q = L/k\rho_{\mathrm{pl}}^2$ and ρ_{pl} is the plane wave spatial coherence radius (Section 6.4.1). As above, moderate-to-strong fluctuations are characterized by the reversal of either of these latter inequalities.

5.3 Born Approximation

The governing stochastic equation we wish to solve is Eq. (3), viz.,

$$\nabla^2 U + k^2 n^2(\mathbf{R}) U = 0. \tag{18}$$

The most well-known classical approaches to solving this equation are the Born and Rytov perturbation methods. The primary distinction between these two methods is that the Born approximation is based on the *addition* of perturbation terms to the unperturbed field, whereas the Rytov approximation involves the *multiplication* of perturbation terms.

To solve (18) by the *Born approximation*, we first write the square of the index of refraction term as

$$n^2(\mathbf{R}) = [n_0 + n_1(\mathbf{R})]^2 \cong 1 + 2n_1(\mathbf{R}), \qquad |n_1(\mathbf{R})| \ll 1. \tag{19}$$

In (19), we have assumed that the mean value satisfies $n_0 = \langle n(\mathbf{R})\rangle \cong 1$ and that $n_1(\mathbf{R})$ is a small random quantity with mean value zero; thus, we can neglect $n_1^2(\mathbf{R})$ as compared with $n_1(\mathbf{R})$. If the optical wave is propagating along the positive z-axis, we imagine the optical field at $z = L$ can be expressed as a sum of terms of the form

$$U(\mathbf{R}) = U_0(\mathbf{R}) + U_1(\mathbf{R}) + U_2(\mathbf{R}) + \cdots, \tag{20}$$

where $U_0(\mathbf{R})$ denotes the unperturbed (unscattered) portion of the field in the absence of turbulence and the remaining terms represent first-order scattering, second-order scattering, etc., caused by random inhomogeneities. It is generally assumed that $|U_2(\mathbf{r}, L)| \ll |U_1(\mathbf{r}, L)| \ll |U_0(\mathbf{r}, L)|$, although this may not occur in all propagation problems. Here we are assuming conditions of *weak fluctuations* for which Eqs. (16) or (17) are valid. The procedure we use is to substitute (19) and (20) into (18) and then equate terms of the same order. This action reduces (18) to the system of equations

$$\nabla^2 U_0 + k^2 U_0 = 0, \tag{21}$$

$$\nabla^2 U_1 + k^2 U_1 = -2k^2 n_1(\mathbf{R}) U_0(\mathbf{R}), \tag{22}$$

$$\nabla^2 U_2 + k^2 U_2 = -2k^2 n_1(\mathbf{R}) U_1(\mathbf{R}), \tag{23}$$

and so on for higher-order perturbations.

One of the major advantages of the above perturbation method is that we have transformed Eq. (18) with *random*, *space-dependent coefficients* to one homogeneous equation and a system of nonhomogeneous equations, all with *constant coefficients*. In particular, the random coefficient $n_1(\mathbf{R})$ has been transferred to the forcing term on the right-hand side of (22) and (23). The solution of the homogeneous Eq. (21) is, of course, simply the unperturbed field $U_0(\mathbf{R})$ (see Chap. 4), whereas each of the nonhomogeneous equations can be solved by the method of Green's function.

5.3.1 First-order perturbation

Given the unperturbed field $U_0(\mathbf{R})$, the solution of Eq. (22) can be expressed in the integral form

$$\begin{aligned} U_1(\mathbf{R}) &= \iiint_V G(\mathbf{S},\mathbf{R})\left[2k^2 n_1(\mathbf{S})U_0(\mathbf{S})\right] d^3S \\ &= 2k^2 \iiint_V G(\mathbf{S},\mathbf{R}) n_1(\mathbf{S})U_0(\mathbf{S})\, d^3S, \end{aligned} \tag{24}$$

where $G(\mathbf{S},\mathbf{R}) \equiv G(\mathbf{R},\mathbf{S})$ is the free-space *Green's function* defined by

$$G(\mathbf{S},\mathbf{R}) = \frac{1}{4\pi|\mathbf{R}-\mathbf{S}|}\exp(ik|\mathbf{R}-\mathbf{S}|). \tag{25}$$

Equation (24) represents the *first Born approximation* and has the physical interpretation that the field perturbation $U_1(\mathbf{R})$ is a sum of spherical waves generated at various points $\mathbf{S}$ throughout the scattering volume V, the strength of each such wave being proportional to the product of the unperturbed field term $U_0(\mathbf{S})$ and the refractive-index perturbation $n_1(\mathbf{S})$ at the point $\mathbf{S}$.

Because the scattering angle of the optical wave is very small, the maximum extent of atmospheric effects in the transverse distance from which scattered radiation is incident on a receiver is much less than the longitudinal distance from the scatterer to the receiver. Thus, we find it useful to introduce the cylindrical coordinate representations

$$\mathbf{R} = (\mathbf{r}, L), \quad \mathbf{S} = (\mathbf{s}, z), \tag{26}$$

and to use the *paraxial approximation* [see Section 4.2.1 and assumption (*iv*) in Section 5.2] to rewrite the Green's function (25) as

$$G(\mathbf{S},\mathbf{R}) \cong G(\mathbf{s},\mathbf{r};z,L) = \frac{1}{4\pi(L-z)}\exp\left[ik(L-z) + \frac{ik}{2(L-z)}|\mathbf{s}-\mathbf{r}|^2\right]. \tag{27}$$

By doing so, the first-order perturbation (24) takes the specific form

$$U_1(\mathbf{r},L) = \frac{k^2}{2\pi}\int_0^L dz \iint_{-\infty}^{\infty} d^2s \exp\left[ik(L-z) + \frac{ik|\mathbf{s}-\mathbf{r}|^2}{2(L-z)}\right] U_0(\mathbf{s},z)\frac{n_1(\mathbf{s},z)}{L-z}. \tag{28}$$

Because $\langle n_1(\mathbf{s},z)\rangle = 0$ by definition, it follows that the ensemble average of the first-order Born approximation also vanishes, i.e., $\langle U_1(\mathbf{r},L)\rangle = 0$. Next, let us consider higher-order perturbations.

5.3.2 Higher-order perturbations

In solving for the second-order perturbation in the Born approximation, we note that the term on the right-hand side of Eq. (23) is similar to that in Eq. (22). Hence, using the Green's function approach in much the same manner as we did for the first-order perturbation, we are led to

$$U_2(\mathbf{r},L)=\frac{k^2}{2\pi}\int_0^L dz\iint_{-\infty}^{\infty} d^2s\,\exp\left[ik(L-z)+\frac{ik|\mathbf{s}-\mathbf{r}|^2}{2(L-z)}\right]U_1(\mathbf{s},z)\frac{n_1(\mathbf{s},z)}{L-z},\quad (29)$$

where $U_1(\mathbf{s},z)$ denotes the first-order perturbation defined by Eq. (28). Unlike the first-order perturbation, we find that $\langle U_2(\mathbf{r},L)\rangle\neq 0$. In general, it readily follows that the mth-order perturbation term can be expressed in the form

$$U_m(\mathbf{r},L)=\frac{k^2}{2\pi}\int_0^L dz\iint_{-\infty}^{\infty} d^2s\,\exp\left[ik(L-z)+\frac{ik|\mathbf{s}-\mathbf{r}|^2}{2(L-z)}\right]U_{m-1}(\mathbf{s},z)\frac{n_1(\mathbf{s},z)}{L-z},$$
$$m=1,\,2,\,3,\ldots.\quad (30)$$

Although easy to write out expressions like (30) for $m>2$, their usefulness has thus far not been widely explored. One of the reasons why this is so is that it has been shown that the Born approximation has severe limitations on its applicability to optical wave propagation. Namely, it is valid only over extremely short propagation paths based on experimental data [11], thereby eliminating it as a useful technique for most applications of laser beam propagation.

5.4 Rytov Approximation

A different perturbational approach to solving Eq. (18), known as the *Rytov approximation* [12], was first applied to a problem of wave propagation in random media by Obukhov [13]. Later, the Rytov method was used in the well-known works of Tatarskii [2,14]. Restricted to weak fluctuation conditions, the Rytov method consists of writing the field of the electromagnetic wave as

$$U(\mathbf{R})\equiv U(\mathbf{r},L)=U_0(\mathbf{r},L)\exp[\psi(\mathbf{r},L)],\quad (31)$$

where ψ is a *complex phase perturbation* due to turbulence that takes the form

$$\psi(\mathbf{r},L)=\psi_1(\mathbf{r},L)+\psi_2(\mathbf{r},L)+\cdots.\quad (32)$$

We refer to ψ_1 and ψ_2 as the first-order and second-order complex phase perturbations, respectively. Note that "additivity" in the argument of the exponential function is equivalent to "multiplication" of exponential functions.

Historically, the Rytov method was applied directly to (18) and formal expressions were then developed for the first-order and second-order perturbations. Nonetheless, this is not necessary because we can relate these perturbation terms directly to the Born perturbations already calculated. To do so, it is convenient to introduce the *normalized Born perturbations* defined by

$$\Phi_m(\mathbf{r},L) = \frac{U_m(\mathbf{r},L)}{U_0(\mathbf{r},L)}, \quad m = 1,\ 2,\ 3,\ldots . \tag{33}$$

By equating the first-order Rytov and first-order Born perturbations according to

$$\begin{aligned} U_0(\mathbf{r},L)\exp[\psi_1(\mathbf{r},L)] &= U_0(\mathbf{r},L) + U_1(\mathbf{r},L) \\ &= U_0(\mathbf{r},L)[1+\Phi_1(\mathbf{r},L)], \end{aligned} \tag{34}$$

we find, upon dividing by $U_0(\mathbf{r},L)$ and taking the natural logarithm, that the first-order Rytov perturbation is equal to the normalized first-order Born perturbation, i.e.,

$$\begin{aligned} \psi_1(\mathbf{r},L) &= \ln[1+\Phi_1(\mathbf{r},L)] \\ &\cong \Phi_1(\mathbf{r},L), \quad |\Phi_1(\mathbf{r},L)| \ll 1, \end{aligned} \tag{35}$$

where

$$\begin{aligned} \Phi_1(\mathbf{r},L) &= \frac{U_1(\mathbf{r},L)}{U_0(\mathbf{r},L)} \\ &= \frac{k^2}{2\pi}\int_0^L dz \iint_{-\infty}^{\infty} d^2s \exp\left[ik(L-z) + \frac{ik|\mathbf{s}-\mathbf{r}|^2}{2(L-z)}\right]\frac{U_0(\mathbf{s},z)}{U_0(\mathbf{r},L)}\frac{n_1(\mathbf{s},z)}{(L-z)}. \end{aligned} \tag{36}$$

The quantity $U_0(\mathbf{r},L)$ in Eq. (36) denotes the optical field in the receiver plane (at $z = L$), whereas $U_0(\mathbf{s},z)$ represents the optical field at an arbitrary plane along the propagation path.

By equating the Born and Rytov perturbations through second-order terms, we see that

$$U_0(\mathbf{r},L)\exp[\psi_1(\mathbf{r},L)+\psi_2(\mathbf{r},L)] = U_0(\mathbf{r},L)[1+\Phi_1(\mathbf{r},L)+\Phi_2(\mathbf{r},L)]. \tag{37}$$

As before, if we divide both sides of (37) by $U_0(\mathbf{r},L)$ and take the natural logarithm of the resulting expression, we find that

$$\begin{aligned} \psi_1(\mathbf{r},L)+\psi_2(\mathbf{r},L) &= \ln[1+\Phi_1(\mathbf{r},L)+\Phi_2(\mathbf{r},L)] \\ &\cong \Phi_1(\mathbf{r},L)+\Phi_2(\mathbf{r},L) - \frac{1}{2}\Phi_1^2(\mathbf{r},L), \\ &\qquad |\Phi_1(\mathbf{r},L)| \ll 1,\ |\Phi_2(\mathbf{r},L)| \ll 1, \end{aligned} \tag{38}$$

where we have retained terms of the Maclaurin series on the right-hand side only up to second order. Thus, because $\psi_1(\mathbf{r},L) = \Phi_1(\mathbf{r},L)$, we deduce that the second-order Rytov perturbation can be approximated by the sum of second-order Born perturbations, i.e.,

$$\psi_2(\mathbf{r},L) = \Phi_2(\mathbf{r},L) - \frac{1}{2}\Phi_1^2(\mathbf{r},L), \tag{39}$$

where the integral representation for $\Phi_2(\mathbf{r}, L)$, obtained from Eq. (26) through division by $U_0(\mathbf{r}, L)$, is given by

$$\Phi_2(\mathbf{r}, L) = \frac{U_2(\mathbf{r}, L)}{U_0(\mathbf{r}, L)} = \frac{k^2}{2\pi}\int_0^L dz \iint_{-\infty}^{\infty} d^2s \exp\left[ik(L-z) + \frac{ik|\mathbf{s}-\mathbf{r}|^2}{2(L-z)}\right]$$
$$\times \frac{U_0(\mathbf{s}, z)}{U_0(\mathbf{r}, L)} \frac{\Phi_1(\mathbf{s}, z) n_1(\mathbf{s}, z)}{(L-z)}. \tag{40}$$

Most of the early work based on the Rytov theory made use of only the first-order perturbation ψ_1. Because it is directly proportional to the first Born approximation, it is also referred to as a *single scattering* approximation. The first-order perturbation is sufficient for calculating several of the statistical quantities of interest, such as the log-amplitude variance, phase variance, intensity and phase correlation functions, and the wave structure function. However, to obtain any of the statistical moments of the optical field from the Rytov theory, including the mean value $\langle U(\mathbf{r}, L)\rangle$, it is necessary to incorporate the second-order perturbation ψ_2 in addition to the first-order perturbation ψ_1 [15]. Hence, both Rytov perturbation terms will play a major role in our later calculations involving the second-order and fourth-order moments of the field.

5.4.1 First-order spectral representation

For the purpose of calculating statistical moments of the optical field, it is useful to first develop *spectral representations* for the above Born and Rytov perturbations. This is accomplished in part by writing the index-of-refraction fluctuation in the form of a *two-dimensional* Riemann-Stieltjes integral [2,3]

$$n_1(\mathbf{s}, z) = \iint_{-\infty}^{\infty} \exp(i\mathbf{K}\cdot\mathbf{s})\, d\nu(\mathbf{K}, z), \tag{41}$$

where $d\nu(\mathbf{K}, z)$ is the random amplitude of the refractive-index fluctuations and $\mathbf{K} = (\kappa_x, \kappa_y, 0)$ is the three-dimensional wave vector with $\kappa_z = 0$.

For the case of particular interest to us, the unperturbed field $U_0(\mathbf{r}, L)$ at propagation distance L is described by the Gaussian-beam wave (Section 4.2.3)

$$U_0(\mathbf{r}, L) = \frac{1}{p(L)} \exp\left[ikL - \frac{\alpha_0 k r^2}{2p(L)}\right], \tag{42}$$

where

$$p(L) = 1 + i\alpha_0 L, \quad \alpha_0 = \frac{2}{kW_0^2} + i\frac{1}{F_0}. \tag{43}$$

Here, W_0 and F_0 denote the beam radius and phase front radius of curvature, respectively, of the optical wave at the emitting aperture of the transmitter in the input plane ($z = 0$). Hence, we see that

$$\frac{U_0(\mathbf{s},z)}{U_0(\mathbf{r},L)} = \frac{p(L)}{p(z)}\exp\left[ik(z-L)\right]\exp\left[-\frac{\alpha_0 ks^2}{2p(z)}\right]\exp\left[\frac{\alpha_0 kr^2}{2p(L)}\right]. \tag{44}$$

By substituting (41) and (44) into (36) and changing the order of integration, we obtain

$$\Phi_1(\mathbf{r},L) = \frac{k^2}{2\pi}\int_0^L dz \iint_{-\infty}^{\infty} \frac{d\nu(\mathbf{K},z)}{\gamma(L-z)}\exp\left[\frac{i\gamma kr^2}{2(L-z)}\right]$$
$$\times \iint_{-\infty}^{\infty} d^2s \exp\left[i\left(\mathbf{K}-\frac{k\mathbf{r}}{L-z}\right)\cdot\mathbf{s}\right]\exp\left[\frac{iks^2}{2\gamma(L-z)}\right]. \tag{45}$$

In arriving at this last result we have used the identities

$$\exp\left[-\frac{\alpha_0 ks^2}{2(1+i\alpha_0 z)}\right]\exp\left[\frac{iks^2}{2(L-z)}\right] = \exp\left[\frac{iks^2}{2\gamma(L-z)}\right], \tag{46}$$

$$\exp\left[\frac{\alpha_0 kr^2}{2(1+i\alpha_0 L)}\right]\exp\left[\frac{ikr^2}{2(L-z)}\right] = \exp\left[\frac{i\gamma kr^2}{2(L-z)}\right], \tag{47}$$

where $\gamma = \gamma(z)$ is the complex *path amplitude weighting parameter* defined by

$$\gamma = \frac{p(z)}{p(L)} = \frac{1+i\alpha_0 z}{1+i\alpha_0 L}. \tag{48}$$

The evaluation of the inside integrals in (45) yields

$$\iint_{-\infty}^{\infty} d^2s \exp\left[i\left(\mathbf{K}-\frac{k\mathbf{r}}{L-z}\right)\cdot\mathbf{s}\right]\exp\left[\frac{iks^2}{2\gamma(L-z)}\right]$$
$$= \frac{2\pi i\gamma}{k}(L-z)\exp\left[i\gamma\mathbf{K}\cdot\mathbf{r} - \frac{i\kappa^2\gamma}{2k}(L-z)\right]\exp\left[-\frac{i\gamma kr^2}{2(L-z)}\right], \tag{49}$$

where $\kappa = |\mathbf{K}|$. Last, by combining results and recalling relation (35), we get the *first-order spectral representation*

$$\psi_1(\mathbf{r},L) = \Phi_1(\mathbf{r},L)$$
$$= ik\int_0^L dz \iint_{-\infty}^{\infty} d\nu(\mathbf{K},z)\exp\left[i\gamma\mathbf{K}\cdot\mathbf{r} - \frac{i\kappa^2\gamma}{2k}(L-z)\right]. \tag{50}$$

5.4.2 Second-order spectral representation

A spectral representation similar to (50) can likewise be obtained for the second-order perturbation defined by Eq. (40). We begin by inserting results (41), (44), and (50) into Eq. (40) to find

$$\Phi_2(\mathbf{r},\, L) = \frac{ik^3}{2\pi}\int_0^L dz \int_0^z dz' \iint_{-\infty}^{\infty} \iint_{-\infty}^{\infty} \frac{d\nu(\mathbf{K}, z)\, d\nu(\mathbf{K}',\, z')}{\gamma(L-z)}$$
$$\times \exp\left[\frac{i\gamma kr^2}{2(L-z)} - \frac{i\gamma'\kappa'^2}{2k}(z-z')\right]$$
$$\times \iint_{-\infty}^{\infty} d^2 s \exp\left[i\mathbf{s}\cdot\left(\mathbf{K} + \gamma'\mathbf{K}' - \frac{k\mathbf{r}}{L-z}\right)\right] \exp\left[\frac{iks^2}{2\gamma(L-z)}\right], \tag{51}$$

where $\gamma' = (1 + i\alpha_0 z')/(1 + i\alpha_0 z)$. The innermost integrals in (51) can be evaluated with the use of (49) once again. On doing so and recalling relation (39), we are led to the *second-order spectral representation* for the normalized second-order Born perturbation given by

$$\Phi_2(\mathbf{r}, L) = \psi_2(\mathbf{r}, L) + \frac{1}{2}\psi_1^2(\mathbf{r}, L) = -k^2 \int_0^L dz \int_0^z dz' \iint_{-\infty}^{\infty} \iint_{-\infty}^{\infty} d\nu(\mathbf{K}, z)\, d\nu(\mathbf{K}',\, z')$$
$$\times \exp\left[i\gamma(\mathbf{K} + \gamma'\mathbf{K}')\cdot\mathbf{r} - \frac{i\gamma|\mathbf{K} + \gamma'\mathbf{K}'|^2}{2k}(L-z) - \frac{i\gamma'\kappa'^2}{2k}(z-z')\right]. \tag{52}$$

From this expression we can also obtain the second-order spectral representation for $\psi_2(\mathbf{r}, L)$, although Eq. (52) is the result we actually need.

5.4.3 Statistical moments

From previous results, we have that $\langle\psi_1(\mathbf{r}, L)\rangle = \langle\Phi_1(\mathbf{r}, L)\rangle = 0$ as a consequence of $\langle n_1(\mathbf{s}, z)\rangle = 0$, or equivalently, $\langle d\nu(\mathbf{K}, z)\rangle = 0$. Ensemble averages of second-order approximations, however, do not vanish. Thus, if we let $\mathbf{r}_1$ and $\mathbf{r}_2$ denote two points in the transverse plane at $z = L$, it is notationally expedient in our following work to define the three second-order statistical moments:

$$\begin{aligned} E_1(\mathbf{r}, \mathbf{r}) &\equiv \langle\Phi_2(\mathbf{r}, L)\rangle = \langle\psi_2(\mathbf{r}, L)\rangle + \frac{1}{2}\langle\psi_1^2(\mathbf{r}, L)\rangle, \\ E_2(\mathbf{r}_1, \mathbf{r}_2) &\equiv \langle\Phi_1(\mathbf{r}_1, L)\Phi_1^*(\mathbf{r}_2, L)\rangle = \langle\psi_1(\mathbf{r}_1, L)\psi_1^*(\mathbf{r}_2, L)\rangle, \\ E_3(\mathbf{r}_1, \mathbf{r}_2) &\equiv \langle\Phi_1(\mathbf{r}_1, L)\Phi_1(\mathbf{r}_2, L)\rangle = \langle\psi_1(\mathbf{r}_1, L)\psi_1(\mathbf{r}_2, L)\rangle. \end{aligned} \tag{53}$$

The asterisk in (53) refers to the complex conjugate of the quantity.

For the second expression in (53), we find

$$E_2(\mathbf{r}_1, \mathbf{r}_2) = k^2 \int_0^L dz \int_0^L dz' \iint_{-\infty}^{\infty} \iint_{-\infty}^{\infty} \langle d\nu(\mathbf{K}, z)\, d\nu^*(\mathbf{K}', z')\rangle$$
$$\times \exp\left[i\gamma(z)\mathbf{K}\mathbf{r}_1 - i\gamma^*(z')\mathbf{K}'\cdot\mathbf{r}_2 - \frac{i\kappa^2\gamma(z)}{2k}(L-z) + \frac{i\kappa'^2\gamma^*(z')}{2k}(L-z')\right]. \tag{54}$$

Next, to ensure statistical homogeneity of the refractive index, we have

$$\langle d\nu(\mathbf{K},z)\, d\nu^*(\mathbf{K}',z')\rangle = F_n(\mathbf{K},|z-z'|)\delta(\mathbf{K}-\mathbf{K}')\, d^2\kappa\, d^2\kappa', \tag{55}$$

where δ is the delta function and $F_n(\mathbf{K},|z-z'|)$ is the *two-dimensional spectral density* of the index of refraction related to the three-dimensional spatial power spectrum according to Eq. (12). Therefore, Eq. (54) becomes

$$\begin{aligned} E_2(\mathbf{r}_1,\mathbf{r}_2) = k^2 \iint_{-\infty}^{\infty} d^2\kappa \int_0^L dz \int_0^L dz' F_n(\mathbf{K},|z-z'|) \\ \times \exp\left[i\gamma(z)\mathbf{K}\cdot\mathbf{r}_1 - i\gamma^*(z')\mathbf{K}\cdot\mathbf{r}_2 - \frac{i\kappa^2\gamma(z)}{2k}(L-z) + \frac{i\kappa^2\gamma^*(z')}{2k}(L-z') \right], \end{aligned} \tag{56}$$

where we have used (55) and also interchanged the order of integration.

To further simplify (56) we note that the function $F_n(\mathbf{K},|z-z'|)$ has some appreciable value only when the difference $|z-z'|$ is close to zero. And, because the spectral density $F_n(\mathbf{K},|z-z'|)$ depends only on the difference $z-z'$, it is useful to make the change of variables

$$\mu = z - z', \quad \eta = \frac{1}{2}(z+z'). \tag{57}$$

Doing so changes the region of integration from the square in Fig. 5.1 to the diamond-shaped region; hence, it follows that

$$\int_0^L dz \int_0^L dz' \Longrightarrow \int_0^L d\eta \int_{\xi_1(\eta)}^{\xi_2(\eta)} d\mu.$$

However, because the appreciable values of the function $F_n(\mathbf{K},|\mu|)$ exist only for μ within the correlation distance, shown by the shaded region in Fig. 5.1, it follows that the limits of integration on μ can be extended from $-\infty$ to ∞

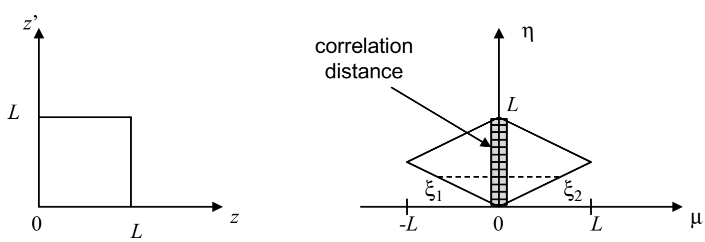

Figure 5.1 Range of integration for z and z' (square), and the corresponding range of integration for μ and η (diamond).

without significant error. In addition, we may write $z \cong z' \cong \eta$, which yields

$$\begin{aligned}
&\int_0^L dz \int_0^L dz' \, F_n(\mathbf{K},\, |z-z'|) \exp\left\{-\frac{i\kappa^2}{2k}[\gamma(z)(L-z)-\gamma^*(z')(L-z')]\right\} \\
&\quad = \int_0^L d\eta \int_{-\infty}^{\infty} d\mu F_n(\mathbf{K}, |\mu|) \exp\left[-\frac{i\kappa^2}{2k}(\gamma-\gamma^*)(L-\eta)\right] \\
&\quad = 2\pi \int_0^L d\eta \Phi_n(\mathbf{K}) \exp\left[-\frac{i\kappa^2}{2k}(\gamma-\gamma^*)(L-\eta)\right],
\end{aligned} \tag{58}$$

where we interpret $\gamma = \gamma(\eta)$ in the last step. From this result, we deduce that

$$\begin{aligned}
E_2(\mathbf{r}_1, \mathbf{r}_2) = 2\pi k^2 \int_0^L d\eta \iint_{-\infty}^{\infty} d^2\kappa \Phi_n(\mathbf{K}) \\
\times \exp\left[i\mathbf{K}\cdot(\gamma\mathbf{r}_1 - \gamma^*\mathbf{r}_2) - \frac{i\kappa^2}{2k}(\gamma-\gamma^*)(L-\eta)\right],
\end{aligned} \tag{59}$$

where we have once again interchanged the order of integration. Equation (59) represents our general expression for this statistical moment under the assumption of a statistically homogeneous random medium.

Along similar lines, the third statistical moment expression in (53) is

$$\begin{aligned}
E_3(\mathbf{r}_1, \mathbf{r}_2) = -k^2 \int_0^L dz \int_0^L dz' \iint_{-\infty}^{\infty} \iint_{-\infty}^{\infty} \langle d\nu(\mathbf{K}, z)\, d\nu(\mathbf{K}', z')\rangle \\
\times \exp\left[i\gamma(z)\mathbf{K}\cdot\mathbf{r}_1 + i\gamma(z')\mathbf{K}'\cdot\mathbf{r}_2 - \frac{i\kappa^2}{2k}\gamma(z)(L-z) - \frac{i\kappa'^2}{2k}\gamma(z')(L-z')\right],
\end{aligned} \tag{60}$$

and, based on the fact that $n_1(\mathbf{s}, z)$ is a real function, we can write

$$\begin{aligned}
\langle d\nu(\mathbf{K}, z)\, d\nu(\mathbf{K}',\, z'\rangle &= \langle d\nu(\mathbf{K}, z)\, d\nu^*(-\mathbf{K}',\, z')\rangle \\
&= F_n(\mathbf{K},\, |z-z'|)\delta(\mathbf{K}+\mathbf{K}')\, d^2\kappa\, d^2\kappa'.
\end{aligned} \tag{61}$$

Thus, by again making the change of variables (57), and recalling Eq. (12), it can be shown that Eq. (60) simplifies to

$$E_3(\mathbf{r}_1,\, \mathbf{r}_2) = -2\pi k^2 \int_0^L d\eta \iint_{-\infty}^{\infty} d^2\kappa \Phi_n(\mathbf{K}) \exp\left[i\gamma\mathbf{K}\cdot(\mathbf{r}_1-\mathbf{r}_2) - \frac{i\kappa^2\gamma}{k}(L-\eta)\right]. \tag{62}$$

For the first statistical moment in (53), we start with the result of Eq. (52) to get

$$\begin{aligned}
E_1(\mathbf{r}, \mathbf{r}) = -k^2 \int_0^L dz \int_0^z dz' \iint_{-\infty}^{\infty} \iint_{-\infty}^{\infty} \langle d\nu(\mathbf{K}, z)\, d\nu(\mathbf{K}', z')\rangle \\
\times \exp\left[i\gamma(\mathbf{K}+\gamma'\mathbf{K}')\cdot\mathbf{r} - \frac{i\gamma|\mathbf{K}+\gamma'\mathbf{K}'|^2}{2k}(L-z) - \frac{i\gamma'\kappa'^2}{2k}(z-z')\right].
\end{aligned} \tag{63}$$

By recalling Eq. (61) and then following an approach analogous to that used above in deriving $E_2(\mathbf{r}_1,\mathbf{r}_2)$ and $E_3(\mathbf{r}_1,\mathbf{r}_2)$, and finally by recognizing that

$$\gamma' = \frac{1+i\alpha_0 z'}{1+i\alpha_0 z} \cong 1, \quad (z' \cong z),$$

it readily follows that Eq. (63) reduces to

$$E_1(\mathbf{r},\mathbf{r}) \equiv E_1(0,0) = -\pi k^2 \int_0^L d\eta \iint_{-\infty}^{\infty} d^2\kappa \Phi_n(\mathbf{K}). \tag{64}$$

This last expression reveals that, up to second-order approximations, the average value of the normalized second-order Born approximation (52) is independent of the observation point $\mathbf{r}$ within the beam. Thus, we will use the symbol $E_1(0,0)$ in subsequent work to denote this fact.

In Eqs. (59), (62), and (64), the spectral density of the index of refraction $\Phi_n(\mathbf{K})$ is assumed to be independent of the position η along the propagation path. The structure constant C_n^2 in many cases, however, may vary along the propagation path as it does in vertical and slant paths. Consequently, the spectral density in such situations will also vary along the propagation path. To include this effect in our analysis, we can formally replace the spectral density $\Phi_n(\mathbf{K})$ with $\Phi_n(\mathbf{K},\eta)$. Also, if we assume the random medium is *statistically homogeneous* and *isotropic* in each transverse plane, then we can replace $\mathbf{K}$ with its scalar magnitude κ. Thus, we have

$$\mathbf{K}\cdot\mathbf{r} = \kappa r\cos\theta, \qquad d^2\kappa \implies \kappa\, d\theta\, d\kappa,$$

where θ is the angle between $\mathbf{K}$ and $\mathbf{r}$. Then, by the use of integral # 9 in Appendix II, Eqs. (64), (59), and (62) take the forms, respectively,

$$\begin{aligned} E_1(0,0) &= \langle\psi_2(\mathbf{r},L)\rangle + \frac{1}{2}\langle\psi_1^2(\mathbf{r},L)\rangle \\ &= -2\pi^2k^2\int_0^L d\eta\int_0^\infty d\kappa\kappa\Phi_n(\kappa,\eta), \end{aligned} \tag{65}$$

$$\begin{aligned} E_2(\mathbf{r}_1,\mathbf{r}_2) &= \langle\psi_1(\mathbf{r}_1,L)\psi_1^*(\mathbf{r}_2,L)\rangle \\ &= 4\pi^2k^2\int_0^L d\eta\int_0^\infty d\kappa\,\kappa\Phi_n(\kappa,\ \eta)J_0(\kappa|\gamma\mathbf{r}_1-\gamma^*\mathbf{r}_2|) \\ &\quad\times\exp\left[-\frac{i\kappa^2}{2k}(\gamma-\gamma^*)(L-\eta)\right], \end{aligned} \tag{66}$$

$$\begin{aligned} E_3(\mathbf{r}_1,\mathbf{r}_2) &= \langle\psi_1(\mathbf{r}_1,L)\psi_1(\mathbf{r}_2,L)\rangle \\ &= -4\pi^2k^2\int_0^L d\eta\int_0^\infty d\kappa\,\kappa\Phi_n(\kappa,\eta)J_0(\gamma\kappa|\mathbf{r}_1-\mathbf{r}_2|) \\ &\quad\times\exp\left[-\frac{i\kappa^2\gamma}{k}(L-\eta)\right], \end{aligned} \tag{67}$$

where $J_0(x)$ is a Bessel function of the first kind and order zero (see Appendix I). These three integrals form the basis for most of the development of statistical parameters introduced in the following chapters.

5.5 Linear Systems Analogy

The spectral representation (50) for the first-order Rytov approximation can be represented in a form similar to that used in *linear shift-invariant systems* involving the impulse response function and its Fourier transform known as the system transfer function. To see this, let us first return to (36) and express it in the form

$$\psi_1(\mathbf{r}, L) = \int_0^L dz \iint_{-\infty}^{\infty} d^2s\, h(\mathbf{s}, \mathbf{r}; z, L) n_1(\mathbf{s}, z), \tag{68}$$

where

$$\begin{aligned} h(\mathbf{s}, \mathbf{r}; z, L) &= 2k^2 G(\mathbf{s}, \mathbf{r}; z, L) \frac{U_0(\mathbf{s}, z)}{U_0(\mathbf{r}, L)} \\ &= \frac{k^2 \exp[ik(L - z)]}{2\pi(L - z)} \exp\left[\frac{ik}{2(L - z)} |\mathbf{s} - \mathbf{r}|^2\right] \frac{U_0(\mathbf{s}, z)}{U_0(\mathbf{r}, L)}, \end{aligned} \tag{69}$$

and $G(\mathbf{s}, \mathbf{r}; z, L)$ is the Green's function (27). By using Eqs. (44), (46), and (47) we can simplify this last result to

$$h(\mathbf{s}, \mathbf{r}; z, L) = \frac{k^2}{2\pi\gamma(L - z)} \exp\left[\frac{i\gamma k r^2}{2(L - z)}\right] \exp\left[\frac{iks^2}{2\gamma(L - z)}\right] \exp\left(-\frac{ik\mathbf{r}\cdot\mathbf{s}}{L - z}\right), \tag{70}$$

the details of which we leave to the reader (see Prob. 12). If we now replace the index of refraction in (68) with its spectral representation (41), and interchange the order of integration, we can write (68) in the form

$$\psi_1(\mathbf{r}, L) = \int_0^L dz \iint_{-\infty}^{\infty} d\nu(\mathbf{K}, z) H(\mathbf{K}, \mathbf{r}; z, L), \tag{71}$$

where

$$\begin{aligned} H(\mathbf{K}, \mathbf{r}; z, L) &= \iint_{-\infty}^{\infty} d^2s \exp\,(i\mathbf{K}\cdot\mathbf{s}) h(\mathbf{s}, \mathbf{r}; z, L) \\ &= ik \exp\left[i\gamma\mathbf{K}\cdot\mathbf{r} - \frac{i\kappa^2\gamma}{2k}(L - z)\right]. \end{aligned} \tag{72}$$

Thus, the functions $H(\mathbf{K}, \mathbf{r}; z, L)$ and $h(\mathbf{s}, \mathbf{r}; z, L)$ are similar to the *system transfer function* and *impulse response function* that occur in linear shift-invariant systems.

Another similar set of functions arise by writing

$$H(\mathbf{K}, \mathbf{r}; z, L) = \exp\,(i\gamma\mathbf{K}\cdot\mathbf{r}) H_0(\mathbf{K}; z, L), \tag{73}$$

where

$$H_0(\mathbf{K}; z, L) = ik \exp\left[-\frac{i\kappa^2\gamma}{2k}(L - z)\right]. \tag{74}$$

We can now express (71) in the alternate form

$$\psi_1(\mathbf{r}, L) = \int_0^L dz \int\int_{-\infty}^{\infty} d\nu(\mathbf{K}, z) \exp\,(i\gamma\mathbf{K}\cdot\mathbf{r}) H_0(\mathbf{K}; z, L), \tag{75}$$

where the related "impulse response function" is defined by (see Example 3 and Prob. 13)

$$\begin{aligned} h_0(\mathbf{r}; z, L) &= \frac{1}{(2\pi)^2}\int\int_{-\infty}^{\infty} d^2\kappa \exp\,(-i\mathbf{K}\cdot\mathbf{r}) H_0(\mathbf{K}; z, L) \\ &= 2k^2 G(\mathbf{r}, 0; z, L)\frac{U_0(\mathbf{r}, z)}{U_0(0, L)} \\ &= h(\mathbf{r}, 0; z, L). \end{aligned} \tag{76}$$

5.6 Rytov Approximation for *ABCD* Optical Systems

In Chap. 4 we introduced the *ABCD* ray-matrix representation method for free-space propagation through a train of optical elements along the path between the input and output planes. Here we wish to consider such systems in the presence of a weakly fluctuating random medium and, like in Sections 4.8 and 4.9, we will address only the case of optical systems featuring rotational symmetry. However, the general *ABCD* method is applicable to other geometries such as rectangular coordinates.

In many cases the random medium is assumed to exist everywhere along the propagation path between the input and output planes, but this is not necessary for the technique developed below. That is, the random medium may be confined to only a portion of the path between the input and output planes. If a random medium exists somewhere between the input plane at $z = 0$ and the output plane at $z = L$, the optical field at the output plane under the Rytov approximation is once again described by

$$U(\mathbf{r}, L) = U_0(\mathbf{r}, L) \exp\left[\psi_1(\mathbf{r}, L) + \psi_2(\mathbf{r}, L) + \cdots\right], \tag{77}$$

where $U_0(\mathbf{r}, L)$ in this case is the unperturbed field defined by (recall Section 4.8.2)

$$U_0(\mathbf{r}, L) = \frac{1}{A + i\alpha_0 B}\exp(ikL)\exp\left[-\left(\frac{\alpha_0 D - iC}{A + i\alpha_0 B}\right)\frac{kr^2}{2}\right]. \tag{78}$$

Here, $A = A(L), B = B(L), C = C(L)$, and $D = D(L)$ represent the matrix elements for the entire propagation path, and $\psi_1(\mathbf{r}, L)$ and $\psi_2(\mathbf{r}, L)$ represent first-order and second-order complex phase perturbations, respectively, caused by the random medium.

In the presence of optical elements, we have to reformulate the spectral representation (50) for the complex phase perturbation $\psi_1(\mathbf{r}, L)$ that accounts for optical elements along the path. This involves the development of a *generalized Green's function* in place of the line-of-sight expression (27). Equation (106) in Chap. 4 is such a generalization for free-space propagation through an *ABCD* system. In the case of the Rytov approximation, the Green's function that we need is a similar one that arises for a reciprocal propagating optical wave from the output plane at $z = L$ to an intermediate point at z along the propagation path. This leads to the interchange of $\mathbf{r}$ and $\mathbf{s}$, or equivalently, the matrix elements A and D, and hence

$$G(\mathbf{s}, \mathbf{r}; z, L) = \frac{1}{4\pi B(z;L)} \exp\left\{ ik(L-z) + \frac{ik}{2B(z;L)}\left[D(z;L)s^2 - 2\mathbf{r}\cdot\mathbf{s} + A(z;L)r^2\right]\right\}, \tag{79}$$

where $A(z;L), B(z;L)$, and $D(z;L)$ are the matrix elements that arise from reciprocal propagation from the output plane. Note that Eq. (79) reduces to Eq. (27) in the limiting case in which $A(z;L) = D(z;L) = 1$ and $B(z;L) = L - z$, which corresponds to line-of-sight propagation with no optical elements along the path.

Following the notation introduced in Section 5.5, we define

$$h(\mathbf{s}, \mathbf{r}; z, L) = 2k^2 G(\mathbf{s}, \mathbf{r}; z, L)\frac{U_0(\mathbf{s}, z)}{U_0(\mathbf{r}, L)}, \tag{80}$$

where, based on (78),

$$\begin{aligned}\frac{U_0(\mathbf{s}, z)}{U_0(\mathbf{r}, L)} &= \frac{p(L)}{p(z)}\exp[ik(z-L)] \\ &\times \exp\left\{-\left[\frac{\alpha_0 D(z) - iC(z)}{p(z)}\right]\frac{ks^2}{2}\right\}\exp\left\{\left[\frac{\alpha_0 D(L) - iC(L)}{p(L)}\right]\frac{kr^2}{2}\right\},\end{aligned} \tag{81}$$

and where $p(z) = A(z) + i\alpha_0 B(z)$. The quantities $A(z), B(z), C(z)$, and $D(z)$ represent matrix elements that characterize the path between $z = 0$ and an arbitrary point $z > 0$. The relation between the various matrix elements in the Green's function (79) and those in (81) is given by

$$\begin{aligned} A(z;L) &= A(z)D(L) - B(z)C(L), \quad & B(z;L) &= A(z)B(L) - A(L)B(z), \\ C(z;L) &= C(L)D(z) - C(z)D(L), \quad & D(z;L) &= A(L)D(z) - B(L)C(z), \end{aligned} \tag{82}$$

the details of which we leave to the reader (see Prob. 14). Upon simplification, Eq. (80) becomes

$$h(\mathbf{s}, \mathbf{r}; z, L) = \frac{k^2}{2\pi\gamma B(z;L)}\exp\left[\frac{iks^2}{2\gamma B(z;L)}\right]\exp\left[\frac{i\gamma kr^2}{2B(z;L)}\right]\exp\left[-\frac{ik\mathbf{r}\cdot\mathbf{s}}{B(z;L)}\right], \tag{83}$$

where $\gamma = p(z)/p(L)$, and by taking the two-dimensional Fourier transform, we have

$$H(\mathbf{K}, \mathbf{r}; z, L) = ik \exp\left[i\gamma \mathbf{K} \cdot \mathbf{r} - \frac{i\kappa^2}{2k}\gamma B(z; L)\right]. \tag{84}$$

Hence, by analogy with Eq. (71), we deduce that

$$\begin{aligned}\psi_1(\mathbf{r}, L) &= \int_0^L dz \int\int_{-\infty}^{\infty} d\nu(\mathbf{K}, z) H(\mathbf{K}, \mathbf{r}; z, L) \\ &= ik \int_0^L dz \int\int_{-\infty}^{\infty} d\nu(\mathbf{K}, z) \exp\left[i\gamma \mathbf{K} \cdot \mathbf{r} - \frac{i\kappa^2}{2k}\gamma B(z; L)\right].\end{aligned} \tag{85}$$

Last, by taking averages of the product of $\psi_1(\mathbf{r}, L)$ with its complex conjugate and with itself, we can obtain moments $E_2(\mathbf{r}_1, \mathbf{r}_2)$ and $E_3(\mathbf{r}_1, \mathbf{r}_2)$ similar to the corresponding expressions (59) and (62) (see Chap. 10). It can be shown, however, that $E_1(0, 0)$ for the present case reduces to (64) for line-of-sight propagation.

5.7 Classical Distribution Models

There has been considerable theoretical and experimental interest over the years in obtaining an expression for the *probability density function* (PDF) of the irradiance. In weak fluctuation regimes (Section 5.2.2), the results of both the Born and Rytov approximations lead to particular models for the PDF, but neither is presumed to be applicable in moderate-to-strong fluctuation regimes. Numerous theoretical studies have been conducted over the years in the hopes of developing mathematical models for the PDF that can predict the nature of the irradiance fluctuations, or scintillations, over a wide range of atmospheric conditions. These studies have led to myriad statistical models, most of which are inadequate in predicting the observed phenomena.

5.7.1 Modified Rician distribution

The first-order Born approximation leads to the field perturbation $U_1(\mathbf{r}, L)$ given by Eq. (28). If we assume the index of refraction $n_1(\mathbf{s}, z)$ is Gaussian distributed, then $U_1(\mathbf{r}, L)$ must also be Gaussian distributed. However, by virtue of the central limit theorem, we should expect the real and imaginary parts of this perturbation term to be approximately Gaussian distributed regardless of the statistics of the index of refraction. If we make the further assumption that the real and imaginary parts of U_1 are uncorrelated and have equal variances, then $U_1 = P_1 \exp(iS_1)$ is said to be *circular complex Gaussian.* In this case it is well known that the amplitude P_1 is a *Rayleigh* variate and the phase S_1 is *uniformly distributed* over 2π rad [16].

The irradiance of the field U along the optical axis can be expressed in the form

$$I = |U_0 + U_1|^2 = A^2 + P_1^2 + 2AP_1 \cos(S_1 - \varphi), \tag{86}$$

where A is the amplitude and φ is the phase of the unperturbed field component. Because the statistics of P_1 and S_1 are known, it is easily shown that the irradiance (86) has a PDF given by the *modified Rician distribution* (also called the *modified Rice-Nakagami distribution*) [16]

$$p_I(I) = \frac{1}{b}\exp\left[-\frac{(A^2+I)}{b}\right] I_0\left(\frac{2A}{b}\sqrt{I}\right), \quad I > 0, \tag{87}$$

where $b = \langle P_1^2 \rangle$ and $I_0(x)$ is a modified Bessel function of the first kind and order zero.

One method that was commonly used to compare theoretical models with experimental data was to compare them on the basis of the normalized statistical moments. The normalized statistical moments of the distribution (87) are given by

$$\frac{\langle I^n \rangle}{\langle I \rangle^n} = \frac{n!}{(1+x)^n} L_n(-x), \quad n = 1, 2, 3, \ldots, \tag{88}$$

where $L_n(x)$ is the nth Laguerre polynomial, $x = A^2/b$, and

$$\langle I^n \rangle = \int_0^\infty I^n p_I(I)\, dI, \quad n = 1, 2, 3, \ldots. \tag{89}$$

By plotting the normalized moments (88) for $n = 3, 4, 5$ as a function of the second normalized moment $\langle I^2 \rangle / \langle I \rangle^2$ and comparing these results with experimental data, Parry and Pusey [11] found that not only were the theoretical moments (88) too low in comparison with the data, but they also did not extend beyond the theoretical limit $\langle I^2 \rangle / \langle I \rangle^2 = 2$ as does measured data.[2] Figure 5.2 is a similar graph of experimental data under weak fluctuation conditions for which $\langle I^2 \rangle / \langle I \rangle^2 < 2$. These data, reported by Phillips and Andrews [19], are shown along with theoretical curves (dashed curves) predicted by Eq. (88). Here we see that the theoretical curves generally lie below the data. Comparisons made with other experimental data reveal similar mismatches, and, thus, the modified Rician distribution is not considered a suitable model for irradiance fluctuations except, possibly, under extremely weak fluctuations.

5.7.2 Lognormal distribution

Because the complex Gaussian model for the field as predicted by the Born approximation did not compare well with experimental data, greater attention was focused on the Rytov method for optical wave propagation. Based on the assumption that the first-order Born approximation U_1 is a circular complex Gaussian random variate, it follows that so is the first-order Rytov approximation $\psi_1 = \chi_1 + iS_1$, where χ_1 and S_1 denote the first-order *log amplitude* and *phase*,

[2] It has been shown that detector saturation [17] and finite sample size [18] may influence the measured values of higher-order scintillation moments for moderate-to-strong fluctuation regimes, causing misinterpretation of data in making comparisons with various mathematical models of the irradiance PDF. Consequently, this method of comparing normalized statistical moments between theory and measurements is seldom used anymore.

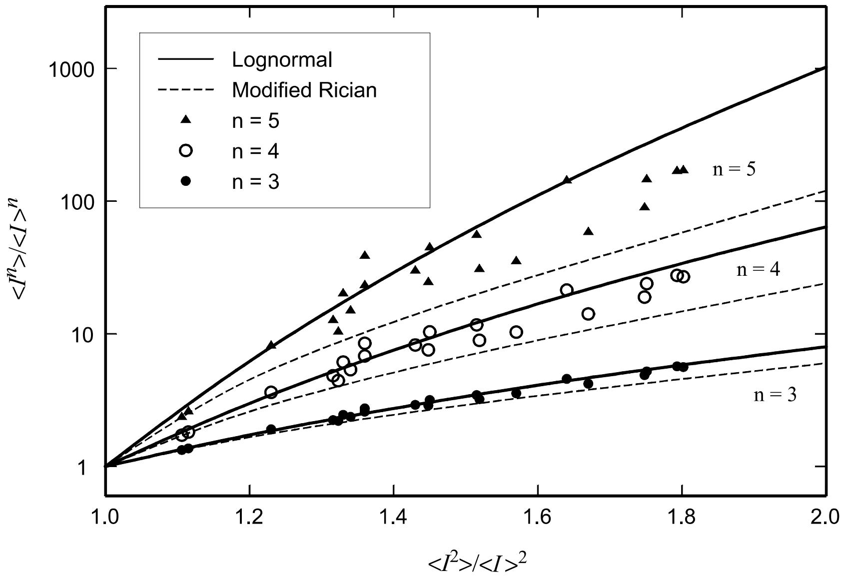

Figure 5.2 Measured values of the normalized third, fourth, and fifth moments as a function of the second normalized moment. The solid curves represent theoretical results predicted by the lognormal distribution and the dashed curves represent theoretical results predicted by the modified Rician distribution.

respectively, of the field. Retaining terms in the general complex phase perturbation up to second order, we write

$$\psi(\mathbf{r}, L) = \psi_1(\mathbf{r}, L) + \psi_2(\mathbf{r}, L) = \chi(\mathbf{r}, L) + iS(\mathbf{r}, L), \tag{90}$$

where $\chi = \chi_1 + \chi_2$ and $S = S_1 + S_2$. The irradiance of the field at a given propagation distance can then be expressed as

$$I = |U_0|^2 \exp(\psi + \psi^*) = (A^2 e^{2\chi_1}) e^{2\chi_2}, \tag{91}$$

where $A = |U_0|$ is the amplitude of the unperturbed field, and $e^{2\chi_2}$ acts like a random modulation of $A^2 e^{2\chi_1}$. Early studies [1,4], however, were based only on the first-order correction $I = A^2 e^{2\chi_1}$, leading to

$$\chi_1 = \frac{1}{2} \ln(I/A^2). \tag{92}$$

Equation (92) shows that, under the first-order Rytov approximation, the *logarithm* of the irradiance is Gaussian distributed, or the irradiance is said to be *lognormal.* That is, under the first-order Rytov approximation, the PDF for the irradiance fluctuations is the *lognormal distribution* (see Prob. 16)

$$p_I(I) = \frac{1}{2\sqrt{2\pi} I \sigma_\chi} \exp\left[-\frac{\left(\ln \frac{I}{A^2} - 2\langle\chi\rangle \right)^2}{8\sigma_\chi^2} \right], \quad I > 0, \tag{93}$$

where $\sigma_\chi^2 = \langle \chi_1^2 \rangle - \langle \chi_1 \rangle^2$ is the *variance of the log amplitude* χ_1. However, the second-order term χ_2 is *not* Gaussian distributed, and consequently the irradiance (91) is not truly lognormal [20].

Based on the lognormal distribution (93), the *normalized moments* of irradiance are readily found to be (see Prob. 17)

$$\frac{\langle I^n \rangle}{\langle I \rangle^n} = \mu^{n(n-1)/2}, \quad n = 1, 2, 3, \ldots, \tag{94}$$

where $\mu = \langle I^2 \rangle / \langle I \rangle^2$ is the second normalized moment. In particular, the normalized variance of the irradiance, or *scintillation index*, is related to the log-amplitude variance according to (see also Chap. 8)

$$\sigma_I^2 = \frac{\langle I^2 \rangle}{\langle I \rangle^2} - 1 = \exp\left(4\sigma_\chi^2\right) - 1. \tag{95}$$

The normalized moments (94) for $n = 3, 4, 5$ as a function of μ are shown in Fig. 5.2 (solid curves) along with experimental data from Ref. [18]. In this case, the theoretical curves generally provide a good fit with the experimental data, at least for $\mu < 1.5$. That this is so is probably a consequence of the fact that the normalized moments (94) only involve the first-order perturbation χ_1. Because of several similar comparisons with early experimental data, a great deal of optimism was prevalent for many years about the range of applicability of the lognormal model beyond the weak fluctuation regime. However, the pioneering work of Gracheva and Gurvich [21] in obtaining experimental data over long propagation paths proved that the lognormal model was not appropriate in strong fluctuation regimes.

5.8 Other Methods of Analysis

When theoretical results are compared with experimental data, the evidence more strongly supports the Rytov approximation over the Born approximation, but both approximation methods are limited to conditions of weak irradiance fluctuations. The weak fluctuation case applies to some applications involving propagation from the ground up to space or for horizontal paths in which the refractive-index structure constant C_n^2 is relatively small and/or the propagation distance L is relatively short. When the fluctuations of the optical wave become stronger, more sophisticated methods of analysis must be applied. Below we briefly introduce the parabolic equation method and the extended Huygens-Fresnel principle, the latter of which more closely resembles the Rytov approximation.

5.8.1 Parabolic equation method

One of the most general methods that is theoretically applicable under all atmospheric conditions is the *parabolic equation method* [6]. This method, based on establishing parabolic equations for each of the various moments of the optical field, has long been considered more fundamental than other formulations, although solutions of the parabolic equations have been obtained only in the case of first-order and second-order moments of the field. Unfortunately, even the second-order field moment has been solved by this method (or any method) only under certain restrictions.

The optical field associated with a Gaussian-beam wave propagating through a continuous medium with smoothly varying random refractive index is a solution of a stochastic differential equation. In particular, if we ignore depolarization effects and assume a monochromatic wave, the governing equation is once again the *stochastic Helmholtz equation* [recall Eqs. (3) and (19)]

$$\nabla^2 U + k^2[1 + 2n_1(\mathbf{R})]U = 0. \tag{96}$$

If we follow the development presented in Section 4.2, we start with the substitution $U(\mathbf{R}) = V(\mathbf{R})e^{ikz}$, from which we deduce

$$2ik\frac{\partial V(\mathbf{R})}{\partial z} + \nabla^2 V(\mathbf{R}) + 2k^2 n_1(\mathbf{R})V(\mathbf{R}) = 0. \tag{97}$$

The simplifying conditions given by Eqs. (8) in Chap. 4 permit us to further reduce (97) to the *parabolic equation*

$$2ik\frac{\partial V(\mathbf{R})}{\partial z} + \nabla_T^2 V(\mathbf{R}) + 2k^2 n_1(\mathbf{R})V(\mathbf{R}) = 0, \tag{98}$$

where $\nabla_T^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is a transverse Laplacian operator. In Eq. (98) we have essentially only eliminated the second derivative of the field in the direction of propagation, i.e., $\partial^2 V/\partial z^2 = 0$.

Based on this last expression and writing $\mathbf{R} = (\mathbf{r}, z)$, we see that the *mean field* $\langle V(\mathbf{r}, z)\rangle$ is a solution of

$$2ik\frac{\partial\langle V(\mathbf{r}, z)\rangle}{\partial z} + \nabla_T^2\langle V(\mathbf{r}, z)\rangle + 2k^2\langle n_1(\mathbf{r}, z)V(\mathbf{r}, z)\rangle = 0, \tag{99}$$

obtained by the term-by-term ensemble average of Eq. (98). The *second-order field moment*, also called the *mutual coherence function*, is defined by

$$\Gamma_2(\mathbf{r}_1, \mathbf{r}_2, z) = \langle V(\mathbf{r}_1, z)V^*(\mathbf{r}_2, z)\rangle, \tag{100}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ denote two points in the transverse plane at propagation distance z. The parabolic equation satisfied by the second field moment (100) is [6]

$$\begin{aligned}2ik\frac{\partial}{\partial z}\Gamma_2(\mathbf{r}_1, \mathbf{r}_2, z) &+ (\nabla_{T1}^2 - \nabla_{T2}^2)\Gamma_2(\mathbf{r}_1, \mathbf{r}_2, z)\\ &+ 2k^2\langle[n_1(\mathbf{r}_1, z) - n_1(\mathbf{r}_2, z)]V(\mathbf{r}_1, z)V^*(\mathbf{r}_2, z)\rangle = 0,\end{aligned} \tag{101}$$

where ∇_{T1}^2 and ∇_{T2}^2 are the transverse Laplacians with respect to $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively.

The basic problem with formulations (99) and (101) is that they are not closed, or mathematically complete. Namely, Eq. (99), for example, has introduced the unknown $\langle n_1(\mathbf{r},z)V(\mathbf{r},z)\rangle$ in addition to $\langle V(\mathbf{r},z)\rangle$. This difficulty has been overcome [14] and we will discuss it further in Section 7.2.

5.8.2 Extended Huygens-Fresnel principle

One approach to solving Eq. (3) by a different method was developed in the United States by Lutomirski and Yura [22] and in the former Soviet Union by Feizulin and Kravtsov [23]. This technique, called the *extended Huygens-Fresnel principle*, expresses the field in the form

$$U(\mathbf{r},L) = -\frac{ik}{2\pi L}\exp(ikL)\iint_{-\infty}^{\infty} d^2 s U_0(\mathbf{s},0)\exp\left[\frac{ik|\mathbf{s}-\mathbf{r}|^2}{2L} + \psi(\mathbf{r},\mathbf{s})\right], \qquad (102)$$

where $\psi(\mathbf{r},\mathbf{s})$ is the random part of the complex phase of a spherical wave propagating in the turbulent medium from the point $(\mathbf{s},0)$ to the point $(\mathbf{r},L)$. We recognize Eq. (102) as an extended version of the Huygens-Fresnel formula, hence its name.

It has been shown that the extended Huygens-Fresnel principle is applicable through first-order and second-order field moments under weak or strong fluctuation conditions of atmospheric turbulence. In fact, it has been established that, up to second-order field moments, the parabolic equation method and the extended Huygens-Fresnel principle yield the same results. In Chap. 7, we will use this latter method to develop results for these moments under general atmospheric conditions. For the fourth-order moment of the field, however, it has not been demonstrated that (102) is equivalent to the parabolic equation method or that (102) is applicable except under weak fluctuation conditions.

5.9 Extended Rytov Theory

Although exact solutions for the second-order moment of the field of a propagating optical wave have been found by several different approaches, this is not the case for the more elusive fourth-order moment. Nonetheless, until more recent times it was the conventional wisdom of the scientific community that an exact solution for the fourth-order moment of the optical field could eventually be developed that was applicable under general irradiance fluctuation conditions. The quest for such a solution has led to a number of failed attempts over many years. For that reason, it may be more useful to concentrate on simple approximate solutions rather than on an exact solution.

Only a few of the various approaches used by researchers to find expressions for the second-order and fourth-order field moments have been discussed here. Of these, the Rytov approximation is the most widely used method, but is limited to regimes of weak irradiance fluctuations. When the propagation channel involves moderate-to-strong irradiance fluctuations, the parabolic equation method and extended Huygens-Fresnel principle have generally been the most successful. Yet, the mathematical complications associated with both of these methods (as well as others) precludes the complete analysis for moderate levels of irradiance fluctuations that is often required in many applications. Of particular importance in this regard is the development of a scintillation model that can describe irradiance fluctuations throughout a long propagation path near the ground. In this section we introduce an approximation technique [24–28] that is based on a modification of the Rytov theory that permits us to extend many weak-fluctuation results into the moderate-to-strong fluctuation regimes. Although dependent largely on well accepted physical arguments and principles, the technique has remained primarily a heuristic approach to the problem of wave propagation. In this section we provide additional physical reasoning and mathematical modeling that support the theory.

5.9.1 Two-scale hybrid method

A random medium like the atmosphere contains random inhomogeneities (turbulent "eddies") of many different scale sizes, ranging from very large scales on the order of the outer scale to very small scales on the order of the inner scale. An optical wave propagating through such a medium will experience the effects of these random inhomogeneities in different ways, depending on the scale size. For example, large scales produce *refractive* (small-angle scattering and focusing) effects that tend to "steer" the beam in a slightly different direction (e.g., beam wander). Consequently, large-scale effects mostly distort the wave front (phase) of the propagating wave and this can generally be described by the method of geometrical optics. Small scales are mostly *diffractive* in nature and therefore distort the amplitude of the wave through beam spreading and amplitude (irradiance) fluctuations. Recognition of the existence of these two distinct components of inhomogeneity size and their influence on a propagating wave has led to what is now widely known as the *two-scale behavior* of the optical wave [5,29].

A method used to describe the scattering of waves at small inhomogeneities in the background of large inhomogeneities was first presented in the early 1970s and later summarized in a paper by Kratsov [30]. This technique, called a *hybrid approach*, relied on partitioning the permittivity (or index of refraction) into two components

$$n_1(\mathbf{R}) = n_{\mathrm{LS}}(\mathbf{R}) + n_{\mathrm{SS}}(\mathbf{R}), \tag{103}$$

where $n_{\mathrm{LS}}(\mathbf{R})$ represents the index of refraction associated with large scales (LS) and $n_{\mathrm{SS}}(\mathbf{R})$ is that associated with small scales (SS). In the hybrid approach it is assumed that $\langle n_{\mathrm{LS}}(\mathbf{R}_1)n_{\mathrm{SS}}(\mathbf{R}_2)\rangle = \langle n_{\mathrm{LS}}(\mathbf{R}_2)n_{\mathrm{SS}}(\mathbf{R}_1)\rangle = 0$. That is, the large-scale and small-scale components $n_{\mathrm{LS}}(\mathbf{R})$ and $n_{\mathrm{SS}}(\mathbf{R})$ are *uncorrelated.* Moreover, if $n_{\mathrm{LS}}(\mathbf{R})$ and $n_{\mathrm{SS}}(\mathrm{R})$ are Gaussian distributed, we can further conclude they are *statistically independent.* Based on uncorrelated components, the correlation function can be expressed as the sum

$$B_n(\mathbf{R}_1, \mathbf{R}_2) = \langle n_1(\mathbf{R}_1)n_1(\mathbf{R}_2)\rangle = B_{n,\mathrm{LS}}(\mathbf{R}_1, \mathbf{R}_2) + B_{n,\mathrm{SS}}(\mathbf{R}_1, \mathbf{R}_2). \tag{104}$$

If we now let $\Phi_{n,\mathrm{LS}}(\kappa)$ and $\Phi_{n,\mathrm{SS}}(\kappa)$ represent the spectral densities of refractive-index fluctuations associated with $n_{\mathrm{LS}}(\mathbf{R})$ and $n_{\mathrm{SS}}(\mathbf{R})$, respectively, it follows from (104) that the total spatial power spectrum of refractive-index fluctuations $n_1(\mathbf{R})$ is described by the sum

$$\Phi_n(\kappa) = \Phi_{n,\mathrm{LS}}(\kappa) + \Phi_{n,\mathrm{SS}}(\kappa). \tag{105}$$

In essence, the hybrid approach described above is an extension of the distorted-wave Born approximation used in quantum theory of scattering when applied to a random medium. One of the main outcomes of the hybrid method is that the mean irradiance deduced from this approach has a wider range of application than that from the conventional Born approximation, and that it can also describe the *enhanced backscatter phenomenon* important in double-passage problems [30] (see Chap. 13).

5.9.2 Spatial filters

Any attempt to analytically describe the two-scale effect on a propagating optical wave requires some knowledge of the transition scale sizes that separate large scales from small scales. This presents a particular difficulty in that knowledge of the effects of various scale sizes on the propagating wave must be known a priori. Consequently, the actual separation of turbulent eddies into large scales and small scales by identifying the transition scales becomes, to a great extend, a conditional process. To handle this difficulty, the theory developed here for moderate-to-strong irradiance fluctuations is based upon familiarity with the conventional Rytov theory for weak fluctuations so it can be used to help identify the transition scale sizes. Because it builds off the conventional Rytov theory, we call it the *extended Rytov theory.*

For developing the extended Rytov theory, we draw upon a key idea of the hybrid method and write the random index of refraction as the sum of large-scale and small-scale components analogous to (103). However, in the hybrid method it is assumed that *all* scale sizes act on a propagating wave, but this is the case only under weak irradiance fluctuations. That is, as a coherent optical wave propagates through a random medium, various eddies impress a spatial phase fluctuation on the wave front with an imprint of the scale size. The accumulation of such fluctuations on the phase leads to a reduction in the "smoothness" of the wave front. Hence, turbulent eddies further away experience a smoothness of the wave front only on the order of the transverse *spatial coherence radius*, which we denote by ρ_0 (see Chaps. 6 and 7). After a wave propagates a sufficient distance L, only those turbulent eddies on the order of ρ_0 or less are effective in producing further spreading and amplitude fluctuations on the wave. Under strong irradiance fluctuations the spatial coherence radius also identifies a related large-scale eddy size near the transmitter called the *scattering disk* $L/k\rho_0$. Basically, the scattering disk is defined by the refractive cell size l at which the focusing angle $\theta_F \sim l/L$ is equal to the average scattering angle $\theta_D \sim 1/k\rho_0$ (see Fig. 5.3). That is, the field within a coherence area of size ρ_0 at distance L from the transmitter is assumed

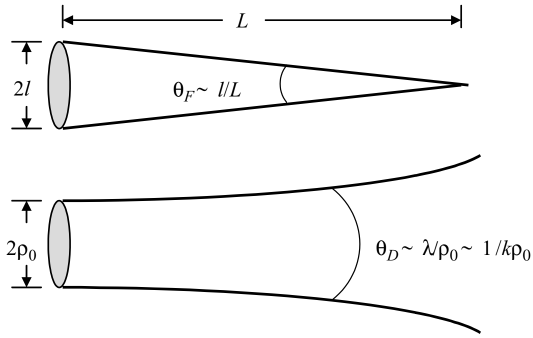

Figure 5.3 Schematic illustration of the focusing angle θ_F and diffraction angle θ_D.

to originate from a scattering disk $L/k\rho_0$ near the transmitter as illustrated in Fig. 5.4. We note that eddy sizes smaller than the scattering disk would correspond to coherence areas exceeding the size ρ_0—hence, they are excluded. Only eddy sizes equal to or larger than the scattering disk can contribute to the field within the coherence area.

As a consequence of the above physical description, we see that as a coherent wave propagates through a random medium from the regime of weak irradiance fluctuations into the regime of strong irradiance fluctuations there is the emergence of two dominate scales sizes—one denoted by l_X that defines the lower bound of the largest scales and l_Y which forms the upper bound of the smallest scales. Scale sizes that exist between l_Y and l_X are rendered ineffective for any meaningful refractive or diffractive influence on the propagating wave.

In addressing the loss of spatial coherence of the propagating wave in a more analytic manner, we introduce the notion of an "effective" refractive index defined by

$$n_{1,e}(\mathbf{R}) = n_X(\mathbf{R}) + n_Y(\mathbf{R}), \tag{106}$$

which differs from (103) in that it may not represent all scale sizes in the inertial range. Under weak irradiance fluctuations we assume (106) includes all scale sizes

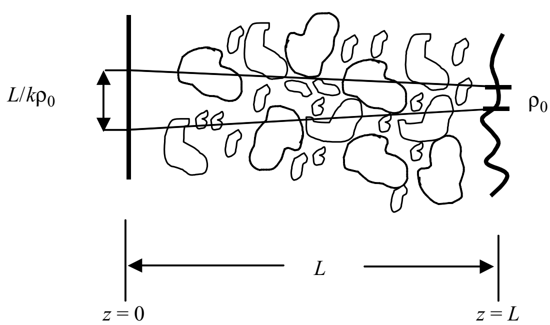

Figure 5.4 Schematic illustrating the relation of the scattering disk $L/k\rho_0$ to the spatial coherence radius ρ_0.

and is therefore the same as (103), but under moderate-to-strong fluctuations only the large-scale eddies greater than l_X and small-scale eddies less than l_Y are considered in (106). Corresponding to (106) is the notion of an effective three-dimensional spatial power spectrum. If the atmospheric turbulence is considered *statistically homogeneous* and *isotropic*, then we can use Eq. (52) in Chap. 2 to write

$$\begin{aligned}\Phi_{n,e}(\kappa) &= \frac{1}{2\pi^2\kappa}\int_0^\infty B_{n,e}(R)\sin(\kappa R)R\,dR \\ &= \frac{1}{2\pi^2\kappa}\int_0^\infty \left[B_{n,X}(R)+B_{n,Y}(R)\right]\sin(\kappa R)R\,dR,\end{aligned} \tag{107}$$

where $B_{n,X}(R) = \langle n_X(\mathbf{R}_1)n_X(\mathbf{R}_2)\rangle$ and $B_{n,Y}(R) = \langle n_Y(\mathbf{R}_1)n_Y(\mathbf{R}_2)\rangle$. From this result, we deduce that

$$\begin{aligned}\Phi_{n,e}(\kappa) &= \Phi_{n,X}(\kappa)+\Phi_{n,Y}(\kappa) \\ &= \Phi_n(\kappa)G_X(\kappa)+\Phi_n(\kappa)G_Y(\kappa).\end{aligned} \tag{108}$$

Like (106), we refer to (108) as the "effective" atmospheric spectrum that the optical wave experiences as it propagates. The quantities $G_X(\kappa)$ and $G_Y(\kappa)$ represent large-scale and small-scale *spatial filters*, respectively. The spatial filter $G_X(\kappa)$ permits scattering effects imposed on the optical wave only by scale sizes larger than the large-scale lower bound l_X and, similarly, $G_Y(\kappa)$ only permits scattering effects on the wave that might arise from scale sizes smaller than l_Y. Under weak irradiance fluctuations, all scale sizes contribute to various distortions of the propagating optical wave so that $G_X(\kappa)+G_Y(\kappa)\sim 1$. In this case the effective spectrum $\Phi_{n,e}(\kappa)$ is the same as the total atmospheric spectrum $\Phi_n(\kappa)$. However, because certain scale sizes in the atmosphere become ineffective at further corruption of the wave under moderate-to-strong irradiance fluctuations, it follows in these cases that $\Phi_{n,e}(\kappa)\neq\Phi_n(\kappa)$.

Based on the above representation for the index of refraction, the first-order and second-order Born approximations can be written as a sum of large-scale and small-scale components

$$\begin{aligned}U_1(\mathbf{R}) &= U_{1,X}(\mathbf{R})+U_{1,Y}(\mathbf{R}), \\ U_2(\mathbf{R}) &= U_{2,X}(\mathbf{R})+U_{2,Y}(\mathbf{R}).\end{aligned} \tag{109}$$

Similarly, the related Rytov approximation now becomes the extended Rytov theory defined by

$$\begin{aligned}U(\mathbf{r},L) &= U_0(\mathbf{r},L)\exp\left[\psi_X(\mathbf{r},L)+\psi_Y(\mathbf{r},L)\right] \\ &= U_0(\mathbf{r},L)\exp\left[\psi_X(\mathbf{r},L)\right]\exp\left[\psi_Y(\mathbf{r},L)\right],\end{aligned} \tag{110}$$

where we assume $\psi_X(\mathbf{r}, L)$ and $\psi_Y(\mathbf{r}, L)$ are statistically independent complex phase perturbations due only to large-scale and small-scale effects, respectively. Note that additivity in the argument of the exponential function in (110) is equivalent to a "modulation process" of the small-scale field fluctuations by large-scale field fluctuations.

At this point, we develop separate spectral representations for the first-order perturbations of $\psi_X(\mathbf{r}, L)$ and $\psi_Y(\mathbf{r}, L)$. In doing so, we introduce two-dimensional Riemann-Stieltjes integrals for each refractive-index component, i.e.,

$$\begin{aligned} n_X(\mathbf{s}, z) &= \iint_{-\infty}^{\infty} \exp(i\mathbf{K}\cdot\mathbf{s})\, d\nu_X(\mathbf{K}, z), \\ n_Y(\mathbf{s}, z) &= \iint_{-\infty}^{\infty} \exp(i\mathbf{K}\cdot\mathbf{s})\, d\nu_Y(\mathbf{K}, z), \end{aligned} \tag{111}$$

Then, by following the development in Section 5.4.1, we find that

$$\begin{aligned} \psi_{1,X}(\mathbf{r}, L) &= ik\int_0^L dz \iint_{-\infty}^{\infty} d\nu_X(\mathbf{K}, z) \exp\left[i\gamma\mathbf{K}\cdot\mathbf{r} - \frac{i\kappa^2\gamma}{2k}(L-z)\right], \\ \psi_{1,Y}(\mathbf{r}, L) &= ik\int_0^L dz \iint_{-\infty}^{\infty} d\nu_Y(\mathbf{K}, z) \exp\left[i\gamma\mathbf{K}\cdot\mathbf{r} - \frac{i\kappa^2\gamma}{2k}(L-z)\right]. \end{aligned} \tag{112}$$

Based on Equations (112), we can now develop expressions for the statistical moments $E_2(\mathbf{r}_1, \mathbf{r}_2)$ and $E_3(\mathbf{r}_1, \mathbf{r}_2)$. By writing $\psi_1(\mathbf{r}, L) = \psi_{1,X}(\mathbf{r}, L) + \psi_{1,Y}(\mathbf{r}, L)$ and recognizing that large-scale and small-scale perturbations are uncorrelated, we see that

$$\begin{aligned} E_2(\mathbf{r}_1, \mathbf{r}_2) &= \langle[\psi_{1,X}(\mathbf{r}_1, L) + \psi_{1,Y}(\mathbf{r}_1, L)][\psi^*_{1,X}(\mathbf{r}_2, L) + \psi^*_{1,Y}(\mathbf{r}_2, L)]\rangle \\ &= \langle\psi_{1,X}(\mathbf{r}_1, L)\psi^*_{1,X}(\mathbf{r}_2, L)\rangle + \langle\psi_{1,Y}(\mathbf{r}_1, L)\psi^*_{1,Y}(\mathbf{r}_2, L)\rangle. \end{aligned} \tag{113}$$

The further development of the statistical moment (113) requires that we examine the quantities

$$\begin{aligned} \langle d\nu_X(\mathbf{K}, z)\, d\nu^*_X(\mathbf{K}', z')\rangle &= F_{n,X}(\mathbf{K}, |z-z'|)\delta(\mathbf{K}-\mathbf{K}')\, d^2\kappa d^2\kappa', \\ \langle d\nu_Y(\mathbf{K}, z)\, d\nu^*_Y(\mathbf{K}', z')\rangle &= F_{n,Y}(\mathbf{K}, |z-z'|)\delta(\mathbf{K}-\mathbf{K}')\, d^2\kappa d^2\kappa', \end{aligned} \tag{114}$$

where $F_{n,X}(\mathbf{K}, z)$ and $F_{n,Y}(\mathbf{K}, z)$ are two-dimensional spectral densities describing large-scale and small-scale effects. The two-dimensional spectral densities are related to the three-dimensional spectral densities for the large-scale and small-scale inhomogeneities by

$$\begin{aligned} \int_{-\infty}^{\infty} F_{n,X}(\mathbf{K}, z)\, dz &= 2\pi\Phi_{n,X}(\mathbf{K}), \\ \int_{-\infty}^{\infty} F_{n,Y}(\mathbf{K}, z)\, dz &= 2\pi\Phi_{n,Y}(\mathbf{K}). \end{aligned} \tag{115}$$

From here the development of the three statistical moments defined for the conventional Rytov theory by (65)–(67) follows in essentially the same fashion. That is, the only difference in appearance is the formal replacement of the total atmospheric spectrum in (65)–(67) with the effective atmospheric spectrum given by (108).

5.9.3 Special scale sizes

The choice of actual filter cutoff spatial frequencies in the filter functions appearing in (108) requires some knowledge about the loss of spatial coherence that has not yet been addressed. We do recognize the existence of several scale sizes that will play an important role in the development of these filter functions. Including those already mentioned, these special scale sizes are given by:

- the *inner scale of turbulence* l_0
- the *spatial coherence radius* of the optical wave ρ_0
- the first *Fresnel zone* $\sqrt{L/k}$
- the *beam radius* W
- the *scattering disk* $L/k\rho_0$
- the *outer scale of turbulence* L_0

Depending on the statistical quantity of interest, we may use different definitions of what constitutes the large-scale spatial frequency cutoff l_X and what constitutes the small-scale spatial frequency cutoff l_Y in the development of the filter functions. For example, in the analysis of beam wander it is the size of the beam and the outer scale that are the important scale sizes, whereas in the analysis of scintillation it is the Fresnel zone, spatial coherence radius, and inner scale that are the significant scale sizes. In general, the identification of the appropriate cutoff spatial frequency is more important than the particular choice of mathematical function used to describe the filter.

5.10 Summary and Discussion

In most approaches, the starting point for describing the propagation of a monochromatic optical/IR wave through a turbulent medium with random index of refraction $n(\mathbf{R})$ is the *stochastic reduced wave equation*

$$\boxed{\nabla^2 U + k^2 n^2(\mathbf{R}) U = 0, \qquad (116)}$$

where $U = U(\mathbf{R})$ is a scalar component of the electric field. It is interesting that virtually all approaches to optical/IR propagation through a random medium rely on a simple set of fundamental assumptions:

- depolarization effects can be neglected
- backscattering of the wave can be neglected
- the wave equation may be approximated by the parabolic equation
- the refractive index is delta correlated in the direction of propagation

Historically, the first approach to solving Eq. (116) was based on the *method of Green's function*, reducing it to an integral equation [7]. More tractable solutions, however, can be obtained by the *geometrical optics method* (GOM) [1,8], and the *Born* and *Rytov approximations* [6,7]. The GOM is simple in that it ignores diffraction effects, but is generally limited to propagation paths in which

$L \ll l_0^2/\lambda$, where l_0 is the inner scale of turbulence. However, because phase fluctuations are most sensitive to large scale sizes, the GOM produces results for phase fluctuations similar to those of diffraction theories; consequently, the GOM is extensively used in various astronomical applications [31,32] and in adaptive optics developments [32–34]. Although we do not give separate treatment to the GOM in this text, we should point out that the results of this theory can usually be deduced from those of the diffraction theory by imposing the condition $\sqrt{\lambda L} \ll l_0$. Diffraction effects, important in the analysis of irradiance fluctuations sensitive to small scale sizes on the order of the Fresnel zone $\sqrt{\lambda L}$, are taken into account in both the Born and Rytov approximations, but the Born approximation was found to be restricted to extremely weak scattering conditions. The first method to give good agreement with scintillation data in the weak fluctuation regime was the Rytov approximation (and corresponding lognormal model for the irradiance), which is the standard method used today under these conditions.

In the Rytov method, the solution of (116) is assumed to take the form

$$\boxed{U(\mathbf{R}) \equiv U(\mathbf{r}, L) = U_0(\mathbf{r}, L)\exp[\psi_1(\mathbf{r}, L) + \psi_2(\mathbf{r}, L) + \cdots], \qquad (117)}$$

where $U_0(\mathbf{r}, L)$ is the unperturbed field and $\psi_1(\mathbf{r}, L)$ and $\psi_2(\mathbf{r}, L)$ represent first-order and second-order perturbations, respectively. These perturbations are directly related to the normalized Born approximations according to

$$\begin{aligned} \psi_1(\mathbf{r}, L) &= \frac{U_1(\mathbf{r}, L)}{U_0(\mathbf{r}, L)} = \Phi_1(\mathbf{r}, L), \\ \psi_2(\mathbf{r}, L) &= \frac{U_2(\mathbf{r}, L)}{U_0(\mathbf{r}, L)} - \frac{1}{2}\left[\frac{U_1(\mathbf{r}, L)}{U_0(\mathbf{r}, L)}\right]^2 = \Phi_2(\mathbf{r}, L) - \frac{1}{2}\Phi_1^2(\mathbf{r}, L). \end{aligned} \qquad (118)$$

Although direct use of the Born approximation to the optical wave propagation problem is not generally applicable, it is interesting that the Born approximation can play such a central role in the Rytov method. In particular, the three integrals

$$\begin{aligned} E_1(0,0) &= \langle \psi_2(\mathbf{r}, L)\rangle + \frac{1}{2}\langle \psi_1^2(\mathbf{r}, L)\rangle \\ &= -2\pi^2 k^2 \int_0^L dz \int_0^\infty d\kappa\, \kappa \Phi_n(\kappa, z), \end{aligned} \qquad (119)$$

$$\begin{aligned} E_2(\mathbf{r}_1, \mathbf{r}_2) &= \langle \psi_1(\mathbf{r}_1, L)\psi_1^*(\mathbf{r}_2, L)\rangle \\ &= 4\pi^2 k^2 \int_0^L dz \int_0^\infty d\kappa\, \kappa \Phi_n(\kappa,\ z) J_0(\kappa|\gamma \mathbf{r}_1 - \gamma^* \mathbf{r}_2|) \\ &\quad \times \exp\left[-\frac{i\kappa^2}{2k}(\gamma - \gamma^*)(L - z)\right], \end{aligned} \qquad (120)$$

$$\boxed{\begin{aligned} E_3(\mathbf{r}_1, \mathbf{r}_2) &= \langle \psi_1(\mathbf{r}_1, L)\psi_1(\mathbf{r}_2, L)\rangle \\ &= -4\pi^2 k^2 \int_0^L dz \int_0^\infty d\kappa\, \kappa\Phi_n(\kappa,\, z) J_0(\gamma\kappa|\mathbf{r}_1 - \mathbf{r}_2|) \\ &\quad \times \exp\left[-\frac{i\kappa^2\gamma}{k}(L-z)\right], \end{aligned} \qquad (121)}$$

that define second-order statistics for both the Born and Rytov approximations are used throughout most of the remaining chapters of this text to describe the fundamental statistical behavior of an optical wave propagating in a random medium. In fact, all statistical quantities of interest involving propagation of a Gaussian-beam wave in the weak fluctuation regime can be directly related to various linear combinations of the three second-order statistical moments (119) through (121).

Last, it has been established that the normalized moments of the irradiance (94) involve the first-order Rytov log-amplitude perturbation χ_1 but not the second-order perturbation χ_2. Consequently, the normalized moments can be closely approximated by using the lognormal distribution (93) under weak irradiance fluctuations. Nonetheless, because the second-order Rytov approximation is not Gaussian, the irradiance itself (91) is not truly a lognormal variate because it involves both first-order and second-order Rytov approximations. We believe this accounts for the fact that the lognormal model does not fit simulation data very well in the tails of the distribution (see Refs. [35–37]).

5.11 Worked Examples

Example 1: Under the assumption of a statistically homogeneous and isotropic medium, derive Eq. (11) from Eq. (10).

Solution: If we replace the rectangular coordinates κ_x, κ_y with polar coordinates defined by $\kappa_x = \kappa\cos\theta$, $\kappa_y = \kappa\sin\theta$, so that $d^2\kappa = \kappa\, d\theta\, d\kappa$, we find that Eq. (11) becomes

$$\begin{aligned} A_n(\mathbf{p}) &= 2\pi \iint_{-\infty}^{\infty} \Phi_n(\mathbf{K}) \exp(i\mathbf{K}\cdot\mathbf{p})\, d^2\kappa \\ &= 2\pi \int_0^\infty \int_0^{2\pi} \kappa\Phi_n(\kappa) \exp(i\kappa\rho\cos\theta)\, d\theta\, d\kappa. \end{aligned}$$

Performing the inside integration with the help of integral #9 in Appendix II, we are led to the intended result ($|\mathbf{p}| = \rho$)

$$A_n(\rho) = 4\pi^2 \int_0^\infty \kappa\Phi_n(\kappa) J_0(\kappa\rho)\, d\kappa.$$

❑

Example 2: Derive the set of equations (21) through (23) by inserting (19) and (20) into (18).

Solution: Conventional perturbation theory often makes use of a "small" perturbation parameter ε to help identify first-order terms, second-order terms, and so on. Sometimes the parameter appears naturally in the equation and at other times it is simply introduced. Initiating such a parameter here, we can rewrite Eqs. (20) and (19) as

$$U(\mathbf{R}) = U_0(\mathbf{R}) + \varepsilon U_1(\mathbf{R}) + \varepsilon^2 U_2(\mathbf{R}) + \cdots,$$

$$n^2(\mathbf{R}) \cong 1 + 2\varepsilon n_1(\mathbf{R}).$$

The substitution of these expressions into Eq. (18) yields

$$\nabla^2 U_0 + \varepsilon\nabla^2 U_1 + \varepsilon^2\nabla^2 U_2 + \cdots + k^2[1 + 2\,\varepsilon n_1(\mathbf{R}) + \cdots]$$
$$\times\,[U_0(\mathbf{R}) + \varepsilon U_1(\mathbf{R}) + \varepsilon^2 U_2(\mathbf{R}) + \cdots] = 0,$$

or, after rearranging terms,

$$\nabla^2 U_0 + \varepsilon\nabla^2 U_1 + \varepsilon^2\nabla^2 U_2 + \cdots + k^2(U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \cdots)$$
$$= -2k^2\varepsilon n_1(\mathbf{R})U_0(\mathbf{R}) - 2k^2\varepsilon^2 n_1(\mathbf{R})U_1(\mathbf{R}) + \cdots.$$

By equating like terms in ε^m, $m = 0, 1, 2, \ldots$, we deduce that

$$\varepsilon^0:\quad \nabla^2 U_0 + k^2 U_0 = 0,$$

$$\varepsilon:\quad \nabla^2 U_1 + k^2 U_1 = -2k^2 n_1(\mathbf{R})U_0(\mathbf{R}),$$

$$\varepsilon^2:\quad \nabla^2 U_2 + k^2 U_2 = -2k^2 n_1(\mathbf{R})U_1(\mathbf{R}).$$

❑

Example 3: By defining generalized position and wave-number vectors by

$$\tilde{\mathbf{r}} = \sqrt{\gamma}\mathbf{r}, \quad \tilde{\mathbf{s}} = \frac{\mathbf{s}}{\sqrt{\gamma}}, \quad \tilde{\mathbf{K}} = \sqrt{\gamma}\mathbf{K},$$

use the well-known Fourier transform shift property

$$\iint_{-\infty}^{\infty} d^2s \exp(i\mathbf{K}\cdot\mathbf{s})f(\mathbf{s}-\mathbf{r}) = \exp(i\mathbf{K}\cdot\mathbf{r})\iint_{-\infty}^{\infty} d^2s \exp(i\mathbf{K}\cdot\mathbf{s})f(\mathbf{s})$$

to establish the identity

$$\iint_{-\infty}^{\infty} d^2s \exp(i\mathbf{K}\cdot\mathbf{s})h(\mathbf{s},\mathbf{r}; z, L) = \exp(i\gamma\mathbf{K}\cdot\mathbf{r})\iint_{-\infty}^{\infty} d^2s \exp(i\mathbf{K}\cdot\mathbf{s})h_0(\mathbf{s}; z, L),$$

where $h(\mathbf{s},\mathbf{r};z,L)$ and $h_0(\mathbf{s};z,L)$ are the inverse Fourier transforms

$$h(\mathbf{s},\mathbf{r};z,L) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} d^2s\exp(-i\mathbf{K}\cdot\mathbf{s})H(\mathbf{K},\mathbf{r};z,L),$$
$$h_0(\mathbf{s};z,L) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} d^2s\exp(-i\mathbf{K}\cdot\mathbf{s})H_0(\mathbf{K};z,L).$$

Solution: To begin, we write the function defined by (70) in the form

$$\begin{aligned} h(\mathbf{s},\mathbf{r};z,L) &= \frac{k^2}{2\pi\gamma(L-z)}\exp\left[\frac{i\gamma kr^2}{2(L-z)}\right]\exp\left[\frac{iks^2}{2\gamma(L-z)}\right]\exp\left(-\frac{k\mathbf{r}\cdot\mathbf{s}}{L-z}\right) \\ &= \frac{k^2}{2\pi\gamma(L-z)}\exp\left[\frac{ik}{2(L-z)}\left|\sqrt{\gamma}\mathbf{r}-\frac{\mathbf{s}}{\sqrt{\gamma}}\right|^2\right], \end{aligned}$$

which, by defining the generalized vectors

$$\tilde{\mathbf{r}} = \sqrt{\gamma}\mathbf{r}, \quad \tilde{\mathbf{s}} = \frac{\mathbf{s}}{\sqrt{\gamma}},$$

we can interpret as

$$h(\mathbf{s},\mathbf{r};z,L) = \tilde{h}_0(\tilde{\mathbf{s}}-\tilde{\mathbf{r}};z,L) = \frac{k^2}{2\pi\gamma(L-z)}\exp\left[\frac{ik}{2(L-z)}|\tilde{\mathbf{s}}-\tilde{\mathbf{r}}|^2\right].$$

Therefore, by taking the two-dimensional Fourier transform of this last expression with transform variable $\tilde{\mathbf{K}} = \sqrt{\gamma}\mathbf{K}$, we are led to

$$\iint_{-\infty}^{\infty} d^2s\exp(i\tilde{\mathbf{K}}\cdot\tilde{\mathbf{s}})\tilde{h}_0(\tilde{\mathbf{s}}-\tilde{\mathbf{r}};z,L) = \exp(i\tilde{\mathbf{K}}\cdot\tilde{\mathbf{r}})\iint_{-\infty}^{\infty} d^2s\exp(i\tilde{\mathbf{K}}\cdot\tilde{\mathbf{s}})\tilde{h}_0(\tilde{\mathbf{s}};z,L),$$

which is the same as

$$\begin{aligned} \iint_{-\infty}^{\infty} d^2s\exp(i\mathbf{K}\cdot\mathbf{s})h(\mathbf{s},\mathbf{r};z,L) &= \exp(i\gamma\mathbf{K}\cdot\mathbf{r})\iint_{-\infty}^{\infty} d^2s\exp(i\mathbf{K}\cdot\mathbf{s})h_0(\mathbf{s};z,L) \\ &= \exp(i\gamma\mathbf{K}\cdot\mathbf{r})H_0(\mathbf{K};z,L). \end{aligned}$$

Because the left-hand side of this last expression defines the function $H(\mathbf{K},\mathbf{r};z,L)$, we have reproduced Eq. (73), which establishes the validity of our intended result. Doing so, we have also established the identity $h(\mathbf{s},\mathbf{r};z,L) = \exp(i\gamma\mathbf{K}\cdot\mathbf{r})h_0(\mathbf{s};z,L)$, from which we deduce the result implied by Eq. (76).

❑

Problems

Section 5.2

1. Under the assumption of isotropy, Eq. (10) takes the form (e.g., see Worked Example 1)

$$A_n(\rho) = 4\pi^2 \int_0^\infty \kappa\Phi_n(\kappa) J_0(\kappa\rho)\, d\kappa.$$

(*a*) Use the von Kármán spectrum model $\Phi_n(\kappa) = 0.033C_n^2(\kappa^2 + \kappa_0^2)^{-11/6}$ to deduce that the above result reduces to

$$A_n(\rho) = 0.78C_n^2\left(\frac{\rho}{\kappa_0}\right)^{5/6} K_{5/6}(\kappa_0\rho).$$

(*b*) Use asymptotic properties of the K Bessel function in (*a*) to show that (see Appendix I)

$$A_n(0) = \frac{0.78C_n^2}{\kappa_0^{5/3}}.$$

Section 5.3

2. By directly substituting the Rytov approximation written as $U = \exp(\psi)$ into the Helmholtz equation,

(*a*) show that it leads to a Ricatti equation given by

$$\nabla^2\psi + \nabla\psi \cdot \nabla\psi + k^2[1 + 2n_1(\mathbf{R})] = 0.$$

(*b*) Use the perturbation series $\psi = \psi_0 + \psi_1 + \psi_2 + \cdots$, where $\exp(\psi_0) = U_0$, to reduce the equation in (*a*) to a system of equations analogous to (21)–(23) for the Rytov approximations ψ_0, ψ_1, and ψ_2.

Section 5.4

3. Given that $\gamma = (1 + i\alpha_0 z)/(1 + i\alpha_0 L)$, verify the following identities:

(*a*) $$\exp\left[-\frac{\alpha_0 ks^2}{2(1 + i\alpha_0 z)}\right]\exp\left[\frac{iks^2}{2(L - z)}\right] = \exp\left[\frac{iks^2}{2\gamma(L - z)}\right].$$

(*b*) $$\exp\left[\frac{\alpha_0 kr^2}{2(1 + i\alpha_0 L)}\right]\exp\left[\frac{ikr^2}{2(L - z)}\right] = \exp\left[\frac{i\gamma kr^2}{2(L - z)}\right].$$

4. In terms of the Gaussian-beam parameters introduced in Chap. 4 show that

$$\gamma = \frac{1 + i\alpha_0 z}{1 + i\alpha_0 L} = 1 - (\overline{\Theta} + i\Lambda)(1 - z/L).$$

Hint: Use long division and the fact that $1/(1 + i\alpha_0 L) = \Theta - i\Lambda$.

5. By use of the integral formula [where $J_0(x)$ is a Bessel function]

$$\int_0^{2\pi} \exp(ix\cos\theta)\,d\theta = 2\pi J_0(x),$$

(*a*) deduce that

$$\iint_{-\infty}^{\infty} d^2s \exp\left[i\left(\mathbf{K} - \frac{k\mathbf{r}}{L-z}\right)\cdot \mathbf{s}\right] \exp\left[\frac{iks^2}{2\gamma(L-z)}\right]$$
$$= 2\pi \int_0^{\infty} s \exp\left[\frac{iks^2}{2\gamma(L-z)}\right] J_0\left(\left|\mathbf{K} - \frac{k\mathbf{r}}{L-z}\right| s\right) ds.$$

Hint: Change to polar coordinates $\mathbf{s} = (s_1, s_2) = (s\cos\theta, s\sin\theta)$.

(*b*) By completing the remaining integral in (*a*), show that

$$\iint_{-\infty}^{\infty} d^2s \exp\left[i\left(\mathbf{K} - \frac{k\mathbf{r}}{L-z}\right)\cdot \mathbf{s}\right] \exp\left[\frac{iks^2}{2\gamma(L-z)}\right]$$
$$= \frac{2\pi i\gamma}{k}(L-z)\exp\left[i\gamma\mathbf{K}\cdot\mathbf{r} - \frac{i\kappa^2\gamma}{2k}(L-z)\right] \exp\left[-\frac{i\gamma k r^2}{2(L-z)}\right].$$

6. Following the approach outlined in Prob. 5, provide the details that reduce Eq. (51) to Eq. (52).

7. By the use of Eq. (61),

(*a*) show that Eq. (60) becomes

$$E_3(\mathbf{r}_1,\mathbf{r}_2) = -k^2 \int_0^L dz \int_0^L dz' \iint_{-\infty}^{\infty} d^2\kappa F_n(\mathbf{K}, |z-z'|)$$
$$\times \exp\left[i\gamma(z)\mathbf{K}\cdot\mathbf{r}_1 - i\gamma(z')\mathbf{K}\cdot\mathbf{r}_2 - \frac{i\kappa^2}{2k}\gamma(z)(L-z) - \frac{i\kappa^2}{2k}\gamma(z')(L-z')\right].$$

(*b*) Following the change of variables $\eta = (1/2)(z+z')$ and $\mu = z - z'$, deduce that the result in (*a*) can be expressed as [where now $\gamma = \gamma(\eta)$]

$$E_3(\mathbf{r}_1,\mathbf{r}_2) = -2\pi k^2 \int_0^L d\eta \iint_{-\infty}^{\infty} d^2\kappa \Phi_n(\mathbf{K})$$
$$\times \exp\left[i\gamma\mathbf{K}\cdot(\mathbf{r}_1 - \mathbf{r}_2) - \frac{i\kappa^2\gamma}{k}(L-\eta)\right].$$

8. Under the assumption of statistical homogeneity and isotropy, show that

$$\iint_{-\infty}^{\infty} d^2\kappa \Phi_n(\mathbf{K}) \exp\left[i\mathbf{K}\cdot(\gamma\mathbf{r}_1 - \gamma^*\mathbf{r}_2) - \frac{i\kappa^2}{2k}(\gamma-\gamma^*)(L-\eta)\right]$$
$$= 2\pi \int_0^{\infty} \kappa\Phi_n(\kappa) J_0(\kappa|\gamma\mathbf{r}_1 - \gamma^*\mathbf{r}_2|) \exp\left[-\frac{i\kappa^2}{2k}(\gamma-\gamma^*)(L-\eta)\right] d\kappa.$$

Hint: Change to polar coordinates and use integral formula #9 in Appendix II.

9. From the general result given in Prob. 8,
   (a) deduce Eq. (65) from Eq. (64).
   (b) deduce Eq. (66) from Eq. (59).
   (c) deduce Eq. (67) from Eq. (62).

10. For an infinite plane wave,
   (a) show that the first-order Rytov approximation (36) reduces to

$$\psi_1(\mathbf{r},L)=\frac{k^2}{2\pi}\int_0^L dz\iint_{-\infty}^{\infty}\exp\left[\frac{ik|\mathbf{s}-\mathbf{r}|^2}{2(L-z)}\right]\frac{n_1(\mathbf{s},z)}{(L-z)}.$$

   (b) Use the relation $\langle n_1(\mathbf{s},z)n_1(\mathbf{s}',z')\rangle=\delta(z-z')A_n(\mathbf{s}-\mathbf{s}')$ to deduce that

$$\langle\psi_1(0,L)\psi_1^*(0,L)\rangle=\frac{k^4}{4\pi^2}\int_0^L dz\int_0^L dz'\,\delta(z-z')\iint_{-\infty}^{\infty}\frac{d^2s}{(L-z)}\iint_{-\infty}^{\infty}\frac{d^2s'}{(L-z')}A_n(\mathbf{s}-\mathbf{s}')\exp\left\{\frac{ik}{2}\left[\frac{s^2}{(L-z)}-\frac{s'^2}{(L-z')}\right]\right\}.$$

   (c) Use Eq. (10) to reduce the expression in (b) to

$$\langle\psi_1(0,L)\psi_1^*(0,L)\rangle=2\pi k^2L\iint_{-\infty}^{\infty}d^2\kappa\Phi_n(\mathbf{K}).$$

11. Use the technique illustrated in Prob. 10 to deduce that

$$\langle\psi_1(\mathbf{r}_1,L)\psi_1^*(\mathbf{r}_2,L)\rangle=2\pi k^2L\iint_{-\infty}^{\infty}d^2\kappa\Phi_n(\mathbf{K})\exp[i\mathbf{K}\cdot(\mathbf{r}_1-\mathbf{r}_2)].$$

Section 5.5

12. The normalized first-order Rytov approximation can be expressed in the form

$$\psi_1(\mathbf{r},L)=\int_0^L dz\iint_{-\infty}^{\infty}d^2s\,h(\mathbf{s},\mathbf{r};z,L)n_1(\mathbf{s},z),$$

   where

$$h(\mathbf{s},\mathbf{r};z,L)=2k^2G(\mathbf{s},\mathbf{r};z,L)\frac{U_0(\mathbf{s},z)}{U_0(\mathbf{r},L)}$$

   and $G(\mathbf{s},\,\mathbf{r};\,z,\,L)$ is the Green's function (27).
   (a) Show that

$$h(\mathbf{s},\mathbf{r};z,L)=\frac{k^2}{2\pi\gamma(L-z)}\exp\left[\frac{i\gamma kr^2}{2(L-z)}\right]\exp\left[\frac{iks^2}{2\gamma(L-z)}\right]\times\exp\left(-\frac{k\mathbf{r}\cdot\mathbf{s}}{L-z}\right).$$

(b) By replacing $n_1(\mathbf{s}, z)$ with its representation (41) and interchanging the order of integration, show that

$$\psi_1(\mathbf{r}, L) = \int_0^L dz \iint_{-\infty}^{\infty} d\nu(\mathbf{K}, z) H(\mathbf{K}, \mathbf{r}; z, L),$$

where

$$H(\mathbf{K}, \mathbf{r}; z, L) = \iint_{-\infty}^{\infty} d^2 s \exp(i\mathbf{K} \cdot \mathbf{s}) h(\mathbf{s}, \mathbf{r}; z, L).$$

(c) From the result of part (a), deduce that

$$H(\mathbf{K}, \mathbf{r}; z, L) = ik \exp\left[i\gamma\mathbf{K} \cdot \mathbf{r} - \frac{i\kappa^2\gamma}{2k}(L - z)\right].$$

13. Given the function defined by

$$h_0(\mathbf{r}; z, L) \equiv h(\mathbf{r}, 0; z, L) = 2k^2 G(\mathbf{r}, 0; z, L) \frac{U_0(\mathbf{r}, z)}{U_0(0, L)},$$

(a) use the Green's function (27) to show that

$$h_0(\mathbf{r}; z, L) = \frac{k^2}{2\pi\gamma(L - z)} \exp\left[\frac{ikr^2}{2\gamma(L - z)}\right].$$

(b) Show that the two-dimensional Fourier transform of $h_0(\mathbf{r}; z, L)$ leads to

$$\begin{aligned} H_0(\mathbf{K}; z, L) &= \iint_{-\infty}^{\infty} d^2 r \exp(i\mathbf{K} \cdot \mathbf{r}) h_0(\mathbf{r}; z, L) \\ &= ik \exp\left[-\frac{i\kappa^2\gamma}{2k}(L - z)\right], \end{aligned}$$

and deduce that the spectral representation (50) can be expressed in the form

$$\psi_1(\mathbf{r}, L) = \int_0^L dz \iint_{-\infty}^{\infty} d\nu(\mathbf{K}, z) \exp(i\gamma\mathbf{K} \cdot \mathbf{r}) H_0(\mathbf{K}; z, L).$$

Section 5.6

14. Let z be an arbitrary plane along the propagation path satisfying $0 < z < L$, where L represents the total propagation path length. If $A(z)$, $B(z)$, $C(z)$, and $D(z)$ denote the matrix elements of an optical system from input plane ($z = 0$) to position z along the propagation path and $A(z; L)$, $B(z; L)$, $C(z; L)$, and $D(z; L)$ denote the matrix elements of a reciprocal propagating ray from output plane L to z,

(a) then show that

$$\begin{pmatrix} A(L) & B(L) \\ C(L) & D(L) \end{pmatrix} = \begin{pmatrix} D(z; L) & B(z; L) \\ C(z; L) & A(z; L) \end{pmatrix} \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}.$$

(*b*) From the result of part (*a*), deduce that

$$A(z;L) = A(z)D(L) - B(z)C(L), \quad B(z;L) = A(z)B(L) - A(L)B(z),$$
$$C(z;L) = C(L)D(z) - C(z)D(L), \quad D(z;L) = A(L)D(z) - B(L)C(z).$$

(*c*) What do the matrix elements in part (*b*) become for line-of-sight propagation?

15. Given that

$$h(\mathbf{s},\mathbf{r};z,L) = 2k^2 G(\mathbf{s},\mathbf{r};z,L)\frac{U_0(\mathbf{s},z)}{U_0(\mathbf{r},L)}$$

and $G(\mathbf{s}, \mathbf{r}; z, L)$ is the generalized Green's function (79),

(*a*) show that

$$h(\mathbf{s},\mathbf{r};z,L) = \frac{k^2}{2\pi\gamma B(z;L)}\exp\left[\frac{i\gamma k r^2}{2B(z;L)}\right]\exp\left[\frac{iks^2}{2\gamma B(z;L)}\right]\exp\left[-\frac{k\mathbf{r}\cdot\mathbf{s}}{B(z;L)}\right].$$

(*b*) By replacing $n_1(\mathbf{s}, z)$ with its representation (41) and interchanging the order of integration, show that

$$\psi_1(\mathbf{r},L) = \int_0^L dz \iint_{-\infty}^{\infty} d\nu(\mathbf{K},z)H(\mathbf{K},\mathbf{r};z,L),$$

where

$$H(\mathbf{K},\mathbf{r};z,L) = \iint_{-\infty}^{\infty} d^2s\,\exp(i\mathbf{K}\cdot\mathbf{s})h(\mathbf{s},\mathbf{r};z,L).$$

(*c*) From the result of part (*a*), deduce that

$$H(\mathbf{K},\mathbf{r};z,L) = ik\exp\left[i\gamma\mathbf{K}\cdot\mathbf{r} - \frac{i\kappa^2\gamma}{2k}B(z;L)\right].$$

Section 5.7

16. Consider the transformation of random variables $r = \sqrt{x^2+y^2}$, where x and y are statistically independent Gaussian random variables with equal variances σ^2, and with mean values $\langle x\rangle = m$ and $\langle y\rangle = 0$. In this case the joint PDF for x and y becomes

$$p_{xy}(x,y) = \frac{1}{2\pi\sigma^2}\exp\left[-\frac{(x-m)^2+y^2}{2\sigma^2}\right].$$

(*a*) Make the change of variables in the above joint PDF to polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta,$$

and show that the resulting joint PDF becomes

$$p_{r\theta}(r,\theta) = \frac{r}{2\pi\sigma^2}\exp\left[-\frac{(r^2+m^2-2mr\cos\theta)}{2\sigma^2}\right].$$

(*b*) From the joint PDF in (*a*), show that the marginal PDF for r is given by

$$p_r(r) = \int_0^{2\pi} p_{r\theta}(r,\theta)\,d\theta = \frac{r}{\sigma^2}e^{-(r^2+m^2)/2\sigma^2}I_0\left(\frac{mr}{\sigma^2}\right), \qquad r > 0,$$

where $I_0(x)$ is a modified Bessel function of the first kind.

(*c*) By setting $I = r^2$, $b = \sigma^2$, and $A = m$, show that the answer in (*b*) becomes the modified Rician distribution (87).

17. Derive the normalized moments (88) for the modified Rician distribution (87).

18. Assume the density function of log amplitude χ is the Gaussian PDF

$$p(\chi) = \frac{1}{\sqrt{2\pi}\sigma_\chi}\exp\left[-\frac{(\chi - \langle\chi\rangle)^2}{2\sigma_\chi^2}\right],$$

where $\langle\chi\rangle$ is the mean value and σ_χ^2 is the variance. Given that intensity $I = A^2e^{2\chi}$, show that the PDF for intensity is the lognormal distribution (93).

19. By the use of the PDF given in Prob. 18,
(*a*) calculate the intensity moments

$$\langle I^n\rangle = A^{2n}\int_{-\infty}^{\infty} e^{2n\chi}p(\chi)\,d\chi, \qquad n = 1, 2, 3, \ldots.$$

(*b*) show that the normalized moments of intensity of the lognormal distribution can be expressed as

$$\frac{\langle I^n\rangle}{\langle I\rangle^n} = \mu^{n(n-1)/2}, \qquad n = 1,\ 2,\ 3,\ \ldots,$$

where $\mu = \langle I^2\rangle/\langle I\rangle^2$ is the second normalized moment.

20. From the result of Prob. 19, deduce that

$$\sigma_I^2 = \frac{\langle I^2\rangle}{\langle I\rangle^2} - 1 = \exp\left(4\sigma_\chi^2\right) - 1.$$

Section 5.8

21. Given that it can be shown that $[\mathbf{R} = (\mathbf{r}, z)]$

$$\langle n_1(\mathbf{R})V(\mathbf{R})\rangle = \frac{ik}{2}A_n(0)\langle V(\mathbf{r}, z)\rangle,$$

(*a*) use this result to conclude that (99) becomes

$$\left[2ik\frac{\partial}{\partial z} + \nabla_T^2 + ik^3A_n(0)\right]\langle V(\mathbf{r}, z)\rangle = 0.$$

(*b*) To solve the differential equation in part (*a*) together with the boundary condition $\langle V(\mathbf{r}, 0)\rangle = V_0(\mathbf{r}, 0) = U_0(\mathbf{r}, 0)$, we look for a solution of the form

$$\langle V(\mathbf{r},z)\rangle = w(\mathbf{r},z)\exp\left[-\frac{1}{2}k^2 z A_n(0)\right]$$
$$= w(\mathbf{r},z)\exp\left[-2\pi^2 k^2 z\int_0^\infty \kappa\Phi_n(\kappa)\,d\kappa\right],$$

where $w(\mathbf{r},z)$ is an unknown function. Show that the direct substitution of this expression into the differential equation leads to

$$\left[2ik\frac{\partial}{\partial z}+\nabla_T^2\right]w(\mathbf{r},z)=0.$$

(*c*) Note that the equation in (*b*) is the parabolic equation for the field in free space [recall Eq. (9) in Chap. 4]. Consequently, by writing $V(\mathbf{r},z)=U(\mathbf{r},z)e^{-ikz}$, deduce that the *mean field* can be written as

$$\langle U(\mathbf{r},z)\rangle = U_0(\mathbf{r},z)\exp\left[-2\pi^2 k^2 z\int_0^\infty \kappa\Phi_n(\kappa)\,d\kappa\right].$$

22. Given that

$$\langle\exp[\psi(\mathbf{r},\mathbf{s})]\rangle = \exp\left[-2\pi^2 k^2 z\int_0^\infty \kappa\Phi_n(\kappa)\,d\kappa\right],$$

show that the mean field deduced from (102) is

$$\langle U(\mathbf{r},z)\rangle = U_0(\mathbf{r},z)\exp\left[-2\pi^2 k^2 z\int_0^\infty \kappa\Phi_n(\kappa)\,d\kappa\right].$$

Section 5.9

23. For the case when the total atmospheric power spectrum $\Phi_n(\kappa)$ and the large-scale filter function $G_X(\kappa)$ are both Gaussian models, i.e.,

$$\Phi_n(\kappa)=\frac{\langle n_1^2\rangle l_n^3}{8\pi\sqrt{\pi}}\exp\left(-\frac{1}{4}l_n^2\kappa^2\right),\qquad G_X(\kappa)=\exp(-l_X^2\kappa^2),$$

show that the refractive index correlation function $B_{n,X}(R)$ under conditions of a statistically homogeneous and isotropic atmosphere is

$$B_{n,X}(R)=\frac{\langle n_1^2\rangle}{(1+4l_X^2/l_n^2)^{3/2}}\exp\left[-\frac{R^2}{l_n^2(1+4l_X^2/l_n^2)}\right].$$

24. If $l_X \gg l_n$, show that the correlation function in Prob. 23 reduces to

$$B_{n,X}(R)\cong\frac{\langle n_1^2\rangle l_n^3}{4l_X^3}\exp\left(\frac{-R^2}{4l_X^2}\right).$$

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