Chapter 2

Random Processes and Random Fields

Overview: Because the open channel through which we propagate electromagnetic radiation is often considered a turbulent medium, we present a brief review in this chapter of the main ideas associated with a *random field*, which in general is a function of a vector spatial variable **R** and time *t*. To begin, however, we start with the somewhat simpler concept of a *random process* and then present a parallel treatment for a random field.

Fundamental in the study of random processes is the introduction of *ensemble averages*, which are used to formulate *mean values, correlation functions*, and *covariance functions.* The development of these statistics is greatly simplified for a *stationary process*, which means that all statistics only depend on time differences and not the specific time origin. From a practical point of view, however, we usually consider just the weaker

condition of a *stationary process in the wide sense*, which demands only that the mean and covariance be invariant under translations in time.

Whereas theoretical treatments of a random process ordinarily involve the ensemble average, measurements of various statistics of a random process make use of the *long-time average.* Nonetheless, if a random process is *ergodic*, then we can equate long-time-average statistics to ensemble averages.

In addition to mean values, correlation functions, and covariance functions, we also introduce the notions of *structure function* and *power spectral density.* Structure functions, which involve averages of squared differences, are widely used in turbulence studies, particularly if the random process is not stationary but has stationary increments. The power spectral density is simply the Fourier transform of the covariance function and, consequently, contains the same information in a different form.

Last, our treatment of random fields is virtually identical to that of random process but there are some subtle differences between the two. For example the notion of "statistical homogeneity" is the spatial counterpart of the temporal "stationarity". It is also common to assume that a random field satisfies the additional property of *isotropy*, which means that the random field statistics depend only on the scalar distance between spatial points.

2.1 Introduction

We assume in this chapter that the reader is familiar with the idea of a random variable and the basics of probability theory. A natural generalization of the random variable concept is that of random process. A *random process*, also called a *stochastic process*, is a collection of time functions and an associated probability description [1–4]. The entire collection of such functions is called an *ensemble.* Ordinarily, we represent any particular member of the ensemble by simply $x(t)$, called a *sample function* or *realization* of the random process. For a fixed value of time, say t_1, the quantity $x_1 = x(t_1)$ can then be interpreted as a *random variable.*

A *continuous* random process is one in which the random variables x_1, $x_2, \ldots,$ can assume any value within a specified range of possible values. But, a *discrete* random process is one in which the random variables can assume only certain isolated values (possibly infinite in number). Here we are concerned only with continuous random processes.

One of the most common random processes occurring in engineering applications is *random noise*, e.g., a "randomly" fluctuating voltage or current at the input to a receiver that interferes with the reception of a radio or radar signal, or the current through a photoelectric detector, and so on. Although many treatments are limited to random processes of time t, we will find it necessary to extend these ideas to the notion of a *random field*, which in general is a function of both time t and space $\mathbf{R} = (x, y, z)$. Atmospheric wind velocity, temperature,

and index of refraction fluctuations are all examples of a random field important to optical wave propagation.

2.2 Probabilistic Description of Random Process

If we imagine "sampling" the random process $x(t)$ at a finite number of times $t_1, t_2, \ldots, t_n$, then we obtain the collection of random variables $x_k = x(t_k)$, $k = 1, 2, \ldots, n$. The probability measure associated with these random variables is described by the *joint probability density function* (PDF) of order n

$$p_x(x_1, t_1; x_2, t_2; \ldots; x_n, t_n).$$

In principle, we can develop the theory of a continuous random process by describing the joint probability density function of all orders. However, this is generally an impossible task so we usually settle for only first- and/or second-order distributions.

2.2.1 First- and second-order statistics

The quantity defined by the probability function

$$F_x(x, t) = \Pr[x(t) \leq x] \tag{1}$$

is called the first-order *distribution function* of the random process $x(t)$. The corresponding first-order PDF is

$$p_x(x, t) = \frac{\partial F_x(x, t)}{\partial x}. \tag{2}$$

Similarly, the second-order distribution function and corresponding PDF are defined, respectively, by

$$F_x(x_1, t_1; x_2, t_2) = \Pr[x(t_1) \leq x_1, x(t_2) \leq x_2], \tag{3}$$

$$p_x(x_1, t_1; x_2, t_2) = \frac{\partial^2 F_x(x_1, t_1; x_2, t_2)}{\partial x_1 \partial x_2}. \tag{4}$$

We note that $F_x(x_1, t_1; \infty, t_2) = F_x(x_1, t_1)$ and

$$p_{x_1}(x_1, t_1) = \int_{-\infty}^{\infty} p_x(x_1, t_1; x_2, t_2)\, dx_2. \tag{5}$$

Conditional PDFs and distributions associated with random processes can be defined in much the same manner as done for random variables. For example, the conditional PDF of $x_2 = x(t_2)$, given the process took on value x_1 at time t_1, is defined by

$$p_{x_2}(x_2, t_2 \,|\, x_1, t_1) = \frac{p_x(x_1, t_1; x_2, t_2)}{p_{x_1}(x_1, t_1)}. \tag{6}$$

2.2.2 Stationary random process

Suppose the first-order PDF does not depend on time, i.e., $p_x(x, t) = p_x(x)$, and further, that the second-order PDF has the form

$$p_x(x_1,t_1; x_2, t_2) = p_x(x_1, x_2; t_2 - t_1) \tag{7}$$

for all t_1 and t_2. That is, the second-order or joint PDF depends only on the time difference $\tau = t_2 - t_1$ but not on the specific times t_1 and t_2. If all marginal and joint PDFs depend only on the time difference $\tau = t_2 - t_1$, but not on the specific time origin, we have what is called a *stationary random process.* Such a process can also be described as one in which its moments are invariant under translations in time.

Truly stationary random processes do not exist in nature because there must be some finite time at which a process is stopped. Nonetheless, in some applications the process will not change significantly during the finite observation time, so we can treat it like a stationary process. Of course, if any of the PDFs associated with a random process do change with the choice of time origin, we say that process is *nonstationary.*

2.3 Ensemble Averages

In the following discussion we will use the bracket notation $\langle\ \rangle$ to denote an *ensemble average* of the quantity inside the brackets. We define the *mean*, also called the *expected value* or *ensemble average*, of the random process $x(t)$ by

$$\langle x(t)\rangle = m(t) = \int_{-\infty}^{\infty} x p_x(x, t)\, dx, \tag{8}$$

where we are emphasizing that the mean value in general may depend on time. Similarly, the variance defined by

$$\sigma_x^2(t) = \langle x^2(t)\rangle - m^2(t) = \int_{-\infty}^{\infty} [x(t) - m(t)]^2 p_x(x, t)\, dx \tag{9}$$

is also a function of time in the general case. However, if the random process is stationary, then its mean value and variance are both independent of time. In this latter case we write the mean as simply $\langle x(t)\rangle = m$ and the variance as σ_x^2.

2.3.1 Autocorrelation and autocovariance functions

Let x_1 and x_2 denote random variables taken from a *real stationary random process* $x(t)$ at times t_1 and t_2, respectively. We define the *autocorrelation function*

(also called simply the *correlation function*) by the expression

$$R_x(t_1, t_2) \equiv R_x(\tau) = \langle x(t_1)x(t_2)\rangle = \int\int_{-\infty}^{\infty} x_1 x_2\, p_x(x_1, x_2; \tau)\, dx_1\, dx_2, \tag{10}$$

where $\tau = t_2 - t_1$. If $x(t)$ is a *complex* stationary random process, then we define the correlation function by $R_x(\tau) = \langle x(t_1)x^*(t_2)\rangle$, where the asterisk $*$ denotes the complex conjugate of the quantity.

Similarly, the *autocovariance function* (or *covariance function*) is defined in general by the ensemble average

$$\begin{aligned} B_x(t_1, t_2) &= \langle [x(t_1) - \langle x(t_1)\rangle][x(t_2) - \langle x(t_2)\rangle]\rangle \\ &= \langle x(t_1)x(t_2)\rangle - m(t_1)m(t_2), \end{aligned} \tag{11}$$

from which, for a stationary process, we deduce

$$B_x(\tau) = R_x(\tau) - m^2. \tag{12}$$

Hence, when the mean of the random process is zero, the correlation and covariance functions are identical. Also, when $t_1 = t_2$ ($\tau = 0$), the covariance function (12) reduces to the variance (9) of the random variable x. It is customary in many cases to consider the *normalized covariance function* defined by the quotient

$$b_x(\tau) = \frac{B_x(\tau)}{B_x(0)}. \tag{13}$$

Because the maximum of the covariance function occurs at $\tau = 0$ [see Eq. (22) below], it follows that

$$-1 \le b_x(\tau) \le 1. \tag{14}$$

To be considered a *strict* stationary process, we require all marginal and joint density functions to be independent of the choice of time origin. However, this requirement is more stringent than necessary in most practical situations. If all we know is that the mean value $\langle x(t)\rangle$ is constant and the covariance function $B_x(\tau)$ depends only on the time interval $\tau = t_2 - t_1$, we say the random process $x(t)$ is stationary in the *wide sense*. Strict stationary processes are automatically wide-sense stationary, but the converse is not necessarily true. For most wide-sense stationary processes, it is usually the case that

$$B_x(\tau) \to 0, \quad |\tau| \to \infty. \tag{15}$$

For practical reasons, it is common in applications to assume the given random process is stationary, at least in the wide sense. That is the approach we generally take here.

2.3.2 Structure functions

Although random processes in practice are often approximated with sufficient accuracy by stationary random functions, there are some instances in which this is not the case. For example, atmospheric parameters such as wind velocity fluctuations and temperature fluctuations are not strictly stationary because their mean values are constant only over relatively short time periods. This difficulty can often be alleviated if the random process has *stationary increments.* In other words, rather than work directly with the random process $x(t)$ itself, we concentrate on the function $x(t + t_1) - x(t_1)$, which often behaves very much like a stationary process even though $x(t)$ may not be stationary. Such functions have what we consider a slowly varying mean and can be described most conveniently in terms of structure functions rather than covariance functions.

It is customary in the study of turbulence to write a random process $x(t)$ as a sum

$$x(t) = m(t) + x_1(t), \tag{16}$$

where $m(t)$ is the mean and $x_1(t)$ is the fluctuating part satisfying $\langle x_1(t)\rangle = 0$. The *structure function* associated with the random process $x(t)$ is defined by

$$\begin{aligned} D_x(t_1, t_2) &= \langle [x(t_1) - x(t_2)]^2 \rangle \\ &= [m(t_1) - m(t_2)]^2 + \langle [x_1(t_1) - x_1(t_2)]^2 \rangle. \end{aligned} \tag{17}$$

Here we see the utility of the structure function approach. If the mean value of $x(t)$ is "slowly varying," then the difference in means in Eq. (17) is nearly zero and the structure function reduces to

$$D_x(t_1, t_2) \cong \langle [x_1(t_1) - x_1(t_2)]^2 \rangle. \tag{18}$$

To put the above ideas into a more precise mathematical framework, let $x(t)$ be a random process such that the ensemble average of $x(t + \tau) - x(t)$ is independent of t. If it is also true that the ensemble average of $[x(t + \tau) - x(t)]^2$ is independent of t, we then call $x(t)$ a random process with *stationary increments.* We ordinarily characterize a random process with stationary increments by the structure function rather than by the covariance function.

A stationary process may be considered a special case of a process with stationary increments. For instance, if $x(t)$ is a stationary process, then its structure function and covariance function are directly related as are their respective spectra. In particular, it follows from definition that

$$\begin{aligned} D_x(\tau) &= \langle [x(t + \tau) - x(t)]^2 \rangle \\ &= \langle x^2(t + \tau) \rangle + \langle x^2(t) \rangle - 2\langle x(t + \tau)x(t) \rangle \\ &= 2[B_x(0) - B_x(\tau)]. \end{aligned} \tag{19}$$

2.3.3 Basic properties

Here we wish to present a few basic properties shared by all correlation and covariance functions of real stationary random processes. Because the proofs for

covariance functions are essentially the same as for correlation functions, we only present proofs of these properties for the latter. The fundamental properties are the following:

$R_x(\tau)$	$B_x(\tau)$
$R_x(0) = \langle x^2(t)\rangle \geq 0$	$B_x(0) = \sigma_x^2 \geq 0$ (20)
$R_x(-\tau) = R_x(\tau)$	$B_x(-\tau) = B_x(\tau)$ (21)
$\lvert R_x(\tau)\rvert \leq R_x(0)$	$\lvert B_x(\tau)\rvert \leq B_x(0)$ (22)

Equations (20) follow immediately from definition, e.g.,

$$R_x(0) = \langle x(t)x(t+\tau)\rangle\big|_{\tau=0} = \langle x^2(t)\rangle. \tag{23}$$

From (23), we deduce that the *total average power* of a stationary random process can always be found by setting $\tau = 0$ in the correlation function. Equations (21) follow from the observation that $R_x(t_1, t_2) = R_x(t_2, t_1)$. Basically, these relations imply that the correlation and covariance functions are *even functions* of τ. Finally, the validity of Eqs. (22) can be established by making the observation that

$$\begin{aligned}\langle [x(t) - x(t+\tau)]^2\rangle &= \langle x^2(t)\rangle + \langle x^2(t+\tau)\rangle - 2\langle x(t)x(t+\tau)\rangle \\ &= 2[R_x(0) - R_x(\tau)] \geq 0.\end{aligned} \tag{24}$$

Because the left-hand side of this last expression is nonnegative, it follows that the right-hand side is also nonnegative. Consequently, $R_x(\tau)$ has its maximum value at the origin.

2.4 Time Averages and Ergodicity

Up to this point we have considered what are called *ensemble averages*, such as the mean value and covariance of a given random process $x(t)$. Such averages depend upon knowledge of the various marginal and joint PDFs associated with the random process. In practice, however, we must usually deal with a single realization of a random process, such as a noisy radar signal, over some time period T that we will assume extends over $(-T/2, T/2)$. If $x(t)$ is a particular realization of a given random process, we can define its *finite-time average* by the integral

$$\overline{x_T(t)} = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt, \tag{25}$$

where the subscript T denotes that we observe the sample function only over time interval T. In the limit $T \to \infty$, we obtain the *long-time average*

$$\overline{x(t)} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt. \tag{26}$$

In the same manner, if $x(t)$ is a realization of a stationary random process, we define the *long-time-average* correlation function by

$$\mathfrak{R}_x(\tau) = \overline{x(t)x(t+\tau)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)\, dt. \tag{27}$$

Stationary random processes for which ensemble averages can be replaced by time averages are said to be *ergodic.* In particular, for an ergodic process, we have the equivalences

$$\overline{x(t)} = \langle x(t) \rangle, \tag{28}$$

$$\mathfrak{R}_x(\tau) = R_x(\tau). \tag{29}$$

To better understand the concept of ergodicity, let us imagine we have available a large number of identical noise generators that make up the ensemble. One method of determining the average voltage output of such generators would be to sample the output of a single generator (one realization) for a long period of time and then calculate its average. Or, we could sample the voltage output of all generators at a particular instant of time and then calculate the average value. If the process is ergodic, these two methods of computing averages always give the same value. If a random process is ergodic, it must also be stationary; however, if a random process is stationary, it may or may not be ergodic.

The theory of random processes is usually formulated in terms of ensemble averages, but actual measurements are ordinarily based on time averages. Therefore, the assumption that a stationary process is also ergodic turns out to be essential in most applications (whether or not it can be proved). In general, it is a difficult task to decide whether a given stationary process is also ergodic because there are no simple conditions that guarantee it. However, for a stationary *Gaussian random process* with zero mean and continuous correlation function, one simple condition that implies ergodicity is

$$\int_{-\infty}^{\infty} |R_x(\tau)|\, d\tau < \infty. \tag{30}$$

For a non-Gaussian process, no such condition exists.

2.5 Power Spectral Density Functions

A wide variety of engineering applications involving random processes are concerned with the determination of a particular correlation function or its Fourier transform, the latter of which is called a *power spectral density function.* Essentially, both of these functions provide the same information about the random process being studied, but historically they evolved from different groups of scientists. That is, correlation functions were primarily a product of mathematicians and statisticians, while spectral density functions were developed mostly as a tool of engineering.

Before introducing the power spectral density for a random function it may be useful to first review the basic notion of a Fourier transform and its inverse for

deterministic functions. Let us suppose that $f(t)$ represents a *deterministic* (nonrandom) function of time, such as a signal or voltage, and assume that $f(t)$ is absolutely integrable,[1] i.e.,

$$\int_{-\infty}^{\infty} \left| f(t) \right| dt < \infty. \tag{31}$$

The frequency content of the time function $f(t)$ can then be obtained from its *Fourier transform* [3]

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt, \tag{32}$$

where ω represents *angular frequency*. In this setting we say that $F(\omega)$ is the *spectrum* associated with the time signal $f(t)$. Based on its definition, it is clear that $F(\omega)$ is a complex function except in the special case when $f(t)$ is an even function. The original time function $f(t)$ can be fully recovered from $F(\omega)$ by the *inverse Fourier transform*

$$f(t) = \int_{-\infty}^{\infty} e^{i\omega t} F(\omega)\, d\omega. \tag{33}$$

Equation (33) is also considered an integral representation of the time function $f(t)$ in terms of its frequency function $F(\omega)$. We should point out that these definitions of Fourier transform are not universal. That is, the signs of the complex exponentials in Eqs. (32) and (33) are sometimes interchanged and the multiplicative constant $1/2\pi$ may appear in front of either integral or its square root in front of each expression. Nonetheless, all such variations are commonly called Fourier transform pairs. Also, these relations are valid for either real or complex functions.

2.5.1 Riemann-Stieltjes integral

Let us assume that $x(t)$ is a *complex stationary random process* with mean value zero. Clearly, random processes do not satisfy the basic condition (31) for the existence of a Fourier transform. However, a stationary random process can be represented in the form of a stochastic (random) *Riemann-Stieltjes integral* (also called a *Fourier-Stieltjes integral* or *spectral representation*) [1]

$$x(t) = \int_{-\infty}^{\infty} e^{i\omega t} dv(\omega), \tag{34}$$

where $dv(\omega)$ is a random complex amplitude. Of course, the random functions $x(t)$ and $dv(\omega)$ are not the same for each realization of the random process.[2] Because

[1] In some applications the basic assumption is that the energy is finite, i.e., $\int_{-\infty}^{\infty} |f(t)|^2 dt < \infty$.

[2] Note that Eq. (34) expresses the relation between a given realization of a random process and its spectrum in the same way that Eq. (33) does for a deterministic function provided we relate $dv(\omega)$ to $F(\omega)d\omega$.

we have assumed the mean value of $x(t)$ is zero, its covariance and correlation functions are the same. Thus, by using the Riemann-Stieltjes integral (34), the covariance (or correlation) function is given by

$$\begin{aligned} B_x(\tau) &= \langle x(t_1)x^*(t_2)\rangle \\ &= \int\int_{-\infty}^{\infty} \exp[i(\omega_1 t_1 - \omega_2 t_2)]\langle dv(\omega_1)\, dv^*(\omega_2)\rangle, \end{aligned} \tag{35}$$

where ω_1 and ω_2 are dummy variables of frequency and the asterisk $*$ denotes the complex conjugate quantity. In order that the covariance function (35) be a function of the time difference $\tau = t_2 - t_1$, characteristic of stationary functions, we must require that the random amplitude satisfies

$$\langle dv(\omega_1)\, dv^*(\omega_2)\rangle = \delta(\omega_2 - \omega_1) S_x(\omega_1)\, d\omega_2 d\omega_1, \tag{36}$$

where $\delta(\omega_2 - \omega_1)$ is the *Dirac delta function* [4] and $S_x(\omega_1) \geq 0$. By inserting Eq. (36) into Eq. (35), and utilizing the *sifting property* of the Dirac delta function

$$\int_{-\infty}^{\infty} \delta(\omega - a) g(\omega)\, d\omega = g(a), \tag{37}$$

we obtain the Fourier transform relation (now replacing ω_1 by ω)

$$B_x(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} S_x(\omega)\, d\omega. \tag{38}$$

We call $S_x(\omega)$ the *power spectrum* or *power spectral density* of the random process $x(t)$. It has also been shown that the power spectrum $S_x(\omega)$ is the inverse transform of the covariance function $B_x(\tau)$, i.e., that these functions are Fourier transform pairs. In this case, we can also write

$$S_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega\tau} B_x(\tau)\, d\tau. \tag{39}$$

The general transform relations (38) and (39) are widely known as the *Wiener-Khintchine theorem.*

If the random process $x(t)$ is *real*, then the covariance function is a real, even function and it can be shown that the power spectrum is also a real, even function. Thus, these two transform pairs in this special case can also be expressed as *cosine transforms*

$$B_x(\tau) = 2 \int_0^{\infty} S_x(\omega) \cos \omega\tau\, d\omega, \tag{40}$$

$$S_x(\omega) = \frac{1}{\pi} \int_0^{\infty} B_x(\tau) \cos \omega\tau\, d\tau. \tag{41}$$

In some discussions, the constants in front of the integrals in (40) and (41) are interchanged.

It is common practice to define the power spectral density as the Fourier transform of the *autocorrelation function* $R_x(\tau)$ rather than the *autocovariance function*

$B_x(\tau)$. However, these functions are the same for zero-mean processes, and, for nonzero-mean processes, the relationship between the two functions is given by Eq. (12). Hence, the Fourier transform of the autocorrelation function in this case will still be essentially the same as that of the autocovariance function, except the former will always contain an impulse function at dc ($\omega = 0$) originating from the mean. For this reason, it makes little difference which definition of power spectral density is used in practice.

The power spectrum $S_x(\omega)$ is a frequency domain representation, or spectral decomposition, of the temporal correlations of the complex random process $x(t)$. Because we are assuming that $x(t)$ is an ergodic process, the power spectrum can be estimated by time averages in practice.

2.6 Random Fields

A random function of a vector spatial variable $\mathbf{R} = (x, y, z)$ and, possibly, time t is called a *random field.* For the complete description of a random field it is necessary to know its joint probability distributions of all orders, the determination of which represents a very difficult problem. In the absence of such a family of probability distributions, it is customary to describe the random field in terms of its lower-order statistical moments. Our treatment of a random field given below will mostly parallel that presented above for a random process.

2.6.1 Spatial covariance function

It is the spatial variation in the complex random field that concerns us most in our applications involving optical wave propagation through a random medium. For that reason, we will suppress the time dependency of the random field in our subsequent treatment. To begin, we define the *mean* or *expected value* of the random field $u(\mathbf{R})$ by the symbol[3]

$$\langle u(\mathbf{R}) \rangle = m(\mathbf{R}), \tag{42}$$

where once again the brackets $\langle\ \rangle$ denote an *ensemble average.* The associated *spatial autocovariance function*, or simply the *covariance function*, is a two-point statistic defined by the ensemble average

$$B_u(\mathbf{R}_1, \mathbf{R}_2) = \langle [u(\mathbf{R}_1) - m(\mathbf{R}_1)][u^*(\mathbf{R}_2) - m^*(\mathbf{R}_2)] \rangle. \tag{43}$$

We say the random field $u(\mathbf{R})$ is *statistically homogeneous* if its moments are invariant under a spatial translation; that is, its mean value $\langle u(\mathbf{R}) \rangle = m$ is independent of the spatial position $\mathbf{R}$ and the covariance function depends only on the spatial vector separation $\mathbf{R} = \mathbf{R}_2 - \mathbf{R}_1$. In this case we

[3]To avoid confusion in this section with the spatial variable x, we will designate our general random field by the symbol $u(\mathbf{R})$ instead of $x(\mathbf{R})$.

write $B_u(\mathbf{R}_1, \mathbf{R}_2) = B_u(\mathbf{R}_2 - \mathbf{R}_1)$, or, equivalently,

$$B_u(\mathbf{R}) = \langle u(\mathbf{R}_1)\mathbf{u}^*(\mathbf{R}_1 + \mathbf{R})\rangle - |m|^2. \tag{44}$$

Thus, we see that the notion of "statistical homogeneity" is the spatial counterpart of "stationarity" in time. If the random field $u(\mathbf{R})$ also has invariance properties under rotations, it is *statistically isotropic.* In this case, the covariance function depends only on the scalar distance $R = |\mathbf{R}_2 - \mathbf{R}_2|$, i.e., $B_u(\mathbf{R}_1, \mathbf{R}_2) = B_u(R)$ and the ensemble properties in practice can be readily estimated by spatial averaging.

2.6.2 One-dimensional spatial power spectrum

Analogous to random process in Section 2.5.1, we can define a one-dimensional spatial power spectrum associated with a random field. That is, if $u(\mathbf{R})$ is a *statistically homogeneous* and *isotropic* complex random field with zero mean, its covariance function $B_u(R)$ can be expressed in the Fourier integral form

$$B_u(R) = \int_{-\infty}^{\infty} e^{i\kappa R} V_u(\kappa)\, d\kappa = 2\int_0^{\infty} \cos(\kappa R) V_u(\kappa)\, d\kappa, \tag{45}$$

where κ (in units of rad/m) denotes the wave number (spatial frequency) and $V_u(\kappa)$ is the *one-dimensional spectrum* of the random field $u(\mathbf{R})$. This relation can be derived in a manner completely analogous to the derivation of Eq. (31) for a random process. Hence, by Fourier transform relations, it follows that the spectrum $V_u(\kappa)$ is defined by the inverse Fourier transform

$$V_u(\kappa) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\kappa R} B_u(R)\, dR = \frac{1}{\pi}\int_0^{\infty} \cos(\kappa R) B_u(R)\, dR. \tag{46}$$

The spatial wave number κ, also called the *spatial frequency*, characterizes the scale size l associated with a particular random inhomogeneity, viz., $\kappa \sim 1/l$. Note that this is analogous to a similar reciprocal relation between frequency and time $\omega \sim 1/t$.

2.6.3 Three-dimensional spatial power spectrum

In making measurements, it is the one-dimensional spatial power spectrum that is of greatest interest; for theoretical developments, it is usually the three-dimensional spatial power spectrum that we need.

Let us assume that $u(\mathbf{R})$ is a *statistically homogeneous* complex random field with zero mean and Riemann-Stieltjes integral representation

$$u(\mathbf{R}) = \int\int\int_{-\infty}^{\infty} e^{i\mathbf{K}\cdot\mathbf{R}} dv(\mathbf{K}), \tag{47}$$

where $\mathbf{K} = (\kappa_x, \kappa_y, \kappa_z)$ is the vector wave number (in units of rad/m) and $dv(\mathbf{K})$ denotes the random amplitude of $u(\mathbf{R})$. By using the representation (47), the

covariance function (43) assumes the form

$$B_u(\mathbf{R}) = \langle u(\mathbf{R}_1)u^*(\mathbf{R}_2)\rangle$$
$$= \int\int\int\int\int\int_{-\infty}^{\infty} \exp[i(\mathbf{K}\cdot\mathbf{R}_1 - \mathbf{K}'\cdot\mathbf{R}_2)]\langle dv(\mathbf{K})\, dv^*(\mathbf{K}')\rangle. \tag{48}$$

To satisfy the conditions of statistical homogeneity, it follows that

$$\langle dv(\mathbf{K})\, dv^*(\mathbf{K}')\rangle = \delta(\mathbf{K}-\mathbf{K}')\Phi_u(\mathbf{K})\, d^3\kappa\, d^3\kappa', \tag{49}$$

in which case, Eq. (48) simplifies to

$$B_u(\mathbf{R}) = \int\int\int_{-\infty}^{\infty} e^{i\mathbf{K}\cdot\mathbf{R}}\Phi_u(\mathbf{K})\, d^3\kappa. \tag{50}$$

The function $\Phi_u(\mathbf{K})$ in (49) and (50) is the *three-dimensional spatial power spectrum* of the random field $u(\mathbf{R})$. This function can be obtained directly from the covariance function through the inverse Fourier transform relation

$$\Phi_u(\mathbf{K}) = \left(\frac{1}{2\pi}\right)^3 \int\int\int_{-\infty}^{\infty} e^{-i\mathbf{K}\cdot\mathbf{R}} B_u(\mathbf{R})\, d^3R. \tag{51}$$

Equations (50) and (51) are the three-dimensional analogues of the Weiner-Khintchin theorem for homogeneous random fields. For the special case in which the random field is *statistically homogeneous* and *isotropic*, these Fourier transform relations reduce to

$$\Phi_u(\kappa) = \frac{1}{2\pi^2\kappa}\int_0^{\infty} B_u(R)\sin(\kappa R)R\, dR,$$
$$B_u(R) = \frac{4\pi}{R}\int_0^{\infty} \Phi_u(\kappa)\sin(\kappa R)\kappa\, d\kappa, \tag{52}$$

where $\kappa = |\mathbf{K}|$ is the magnitude of the wave number vector.

Based on Eqs. (52), it can be shown that the relation between the three-dimensional spectrum and the one-dimensional spectrum (46) is given by [5]

$$\Phi_u(\kappa) = -\frac{1}{2\pi\kappa}\frac{dV_u(\kappa)}{d\kappa}. \tag{53}$$

This relation implies, for example, that a one-dimensional spectrum exhibiting a $\kappa^{-5/3}$ behavior corresponds to a three-dimensional spectrum with a $\kappa^{-11/3}$ behavior. However, convergence of the integrals in Eqs. (52) places certain restrictions on the allowed behavior of the three-dimensional spectrum and corresponding covariance function of the random field $u(\mathbf{R})$. In particular, if the spectrum has a singularity at $\kappa = 0$, convergence of the integral defining the covariance function, for instance, requires that $\Phi_u(\kappa) \sim \kappa^{-\alpha}$, where $\alpha < 3$, as $\kappa \to 0$.

In wave propagation through a homogeneous and isotropic random medium characterized by the three-dimensional spatial power spectrum $\Phi_u(\kappa_x, \kappa_y, \kappa_z)$, it often happens that we are also interested in the two-dimensional spectrum of fluctuations in a plane perpendicular to the direction of propagation.

If $F_u(\kappa_x, \kappa_y, 0; z)$ denotes the two-dimensional spectrum between transverse planes separated by distance z, it is related to the three-dimensional spectrum according to the Fourier transform relation [5,6]

$$F_u(\kappa_x, \kappa_y, 0; z) = \int_{-\infty}^{\infty} \Phi_u(\kappa_x, \kappa_y, \kappa_z) \cos(z\kappa_z)\, d\kappa_z. \tag{54}$$

Consequently, by properties of the Fourier transform, we also see that

$$\Phi_u(\kappa_1, \kappa_2, \kappa_3) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_u(\kappa_x, \kappa_y, 0; z) \cos(z\kappa_z)\, dz. \tag{55}$$

Because the spatial power spectrum $\Phi_u(\kappa_x, \kappa_y, \kappa_z)$ is an even function (by definition), we have chosen to express Eqs. (54) and (55) as cosine transforms.

2.6.4 Structure function

The theoretical treatment of spatial fluctuations of a random field in terms of the covariance function and power spectral density depends on the assumption of statistical homogeneity. Of particular importance is the assumption that the mean value of the field is constant over all space. In practice, however, the random field of interest may not have constant mean value over large distances. For instance, velocity fields in turbulence are not strictly homogeneous because the average velocity field cannot be constant over widely separated portions of the random medium. Nonetheless, it turns out that the velocity difference at two distinct points almost always behaves like a statistically homogeneous field.

To put this last statement in mathematical terms, let us consider a random field $u(\mathbf{R})$ that can be decomposed into the sum

$$u(\mathbf{R}) = m(\mathbf{R}) + u_1(\mathbf{R}), \tag{56}$$

where $m(\mathbf{R}) = \langle u(\mathbf{R}) \rangle$ is the (nonconstant) mean and $u_1(\mathbf{R})$ is a statistically homogeneous fluctuation with mean value $\langle u_1(\mathbf{R}) \rangle = 0$ for all $\mathbf{R}$. Random fields that permit a decomposition into a varying mean and a statistically homogeneous fluctuation are called *locally homogeneous*, which is the spatial equivalence of a random process with stationary increments. Hence, the theoretical framework developed in Section 2.3.2 for a random process with stationary increments can also be applied to a locally homogeneous random field.

Locally homogeneous fields are usually characterized not by the covariance function, but by the structure function. In general, the *structure function* for a locally homogeneous random field $u(\mathbf{R})$ can be expressed in the form (recall Section 2.3.2)

$$\begin{aligned} D_u(\mathbf{R}_1, \mathbf{R}_2) \equiv D_u(\mathbf{R}) &= \langle [u(\mathbf{R}_1) - u(\mathbf{R}_1 + \mathbf{R})]^2 \rangle \\ &\cong \langle [u_1(\mathbf{R}_1) - u_1(\mathbf{R}_1 + \mathbf{R})]^2 \rangle, \end{aligned} \tag{57}$$

and the spectrum is related to the structure function by [5,6]

$$D_u(\mathbf{R}) = 2\int\int\int_{-\infty}^{\infty} \Phi_u(\mathbf{K})[1 - \cos(\mathbf{K}\cdot\mathbf{R})]\, d^3\kappa. \tag{58}$$

Of course, in the special case in which the random field is *locally homogeneous* and *isotropic*, the structure function becomes a function of R alone and is related to the power spectrum according to

$$D_u(R) = 8\pi\int_0^{\infty} \kappa^2\Phi_u(\kappa)\left(1 - \frac{\sin\kappa R}{\kappa R}\right)d\kappa. \tag{59}$$

From this last relation, we observe that the term $1 - (\sin\kappa R)/\kappa R$ acts like a high-pass filter, where contributions from scale sizes larger than the separation distance are removed. This means the spectrum associated with the structure function can have a singularity at $\kappa = 0$ of the type $\kappa^{-\alpha}$, where[4] $\alpha < 5$. The inverse relation for a locally homogeneous and isotropic medium takes the form [7]

$$\Phi_u(\kappa) = \frac{1}{4\pi^2\kappa^2}\int_0^{\infty} \frac{\sin\kappa R}{\kappa R}\frac{d}{dR}\left[R^2\frac{d}{dR}D_u(R)\right]dR. \tag{60}$$

Last, in the case of a statistically homogeneous and isotropic field, it readily follows from the second result in (52) that the structure function and covariance function are directly related by the expression

$$D_u(R) = 2[B_u(0) - B_u(R)]. \tag{61}$$

2.7 Summary and Discussion

In this chapter we have reviewed the basic theory of a stationary random process and a statistically homogeneous and isotropic random field. We say a random process is *stationary* if all marginal and joint PDFs depend only on time differences τ but not on the specific times. A stationary (complex) random process $x(t)$ is generally described in the *time domain* by the *correlation function*

$$R_x(\tau) = \langle x(t)x^*(t+\tau)\rangle, \tag{62}$$

or the related *covariance function*

$$B_x(\tau) = R_x(\tau) - |m|^2, \tag{63}$$

[4]Recall that the spectrum associated with the covariance function requires $\alpha < 3$ (Section 2.6.3).

where $m = \langle x(t) \rangle$ is the mean value. An equivalent description in the *frequency domain* is given by the *power spectral density* $S_x(\omega)$, directly related to the covariance function through the *Weiner-Khintchine relations*

$$B_x(\tau) = 2\int_0^\infty S_x(\omega)\cos\omega\tau\, d\omega,$$
$$S_x(\omega) = \frac{1}{\pi}\int_0^\infty B_x(\tau)\cos\omega\tau\, d\tau. \tag{64}$$

If all time averages of a particular realization of a random process are equal to the corresponding ensemble averages, we say the process is *ergodic.*

In the case of a random field $u(\mathbf{R})$, the correlation and covariance functions represent a *spatial domain* description whereas the power spectrum is a *wave number representation.* If the random field is *statistically homogeneous* and *isotropic* (the equivalence of a stationary process), the *covariance function* $B_u(R)$ and *spatial power spectrum* $\Phi_u(\kappa)$ satisfy relations similar to (64) given by

$$\Phi_u(\kappa) = \frac{1}{2\pi^2\kappa}\int_0^\infty B_u(R)\sin(\kappa R)R\, dR,$$
$$B_u(R) = \frac{4\pi}{R}\int_0^\infty \Phi_u(\kappa)\sin(\kappa R)\kappa\, d\kappa, \tag{65}$$

Stationary processes ordinarily do not exist in the strict sense in practice, but we can almost always approximate a random process by a random function with stationary increments, or approximate a random field by a locally homogeneous field [5,6]. In either of these latter cases, the actual process is usually characterized by the *structure function* rather than by the covariance function. In general, however, the structure function contains less information than the covariance function. That is, if the covariance function is known we can always construct the structure function from it, but the covariance function cannot always be deduced from knowledge of the structure function alone.

Finally, we should point out that modern mathematics has devised a number of theories concerning probability theory and random processes. Although these rigorous theories permit great generality, they do so at the expense of added difficulty and abstractness. For most engineering applications, such an elaborate mathematical apparatus is not necessary. For instance, rather than introduce the general integral due to Lebesque, most results can readily be expressed in terms of the Riemann integral. When this is not possible, we can ordinarily use the Riemann-Stieltjes integral, which gives the same results as the Lebesque integral

when both types of integrals exist. Throughout the text, we generally rely on the simplest possible approach.

2.8 Worked Examples

EXAMPLE 1: Given the following covariance (correlation) functions, find the associated power spectrum for each using appropriate integrals listed in Appendix II:

$$(a) \quad B_x(\tau) = \exp(-|\tau/\tau_0|), \quad \tau_0 > 0$$

$$(b) \quad B_x(\tau) = \exp[-a^2(\tau/\tau_0)^2], \quad \tau_0 > 0, a > 0$$

Solution: The covariance functions in (*a*) and (*b*) (with $a = 1$) are shown below in Fig. 2.1. The time τ_0 required for the covariance function to reach $1/e$ of its value at $\tau = 0$ is called the *correlation time* of the underlying random process.

For the covariance function in part (*a*), we have (using integral #6 in Appendix II)

$$S_x(\omega) = \frac{1}{\pi}\int_0^\infty \cos(\omega\tau)\exp(-|\tau/\tau_0|)\,d\tau$$
$$= \frac{\tau_0}{\pi(1+\omega^2\tau_0^2)},$$

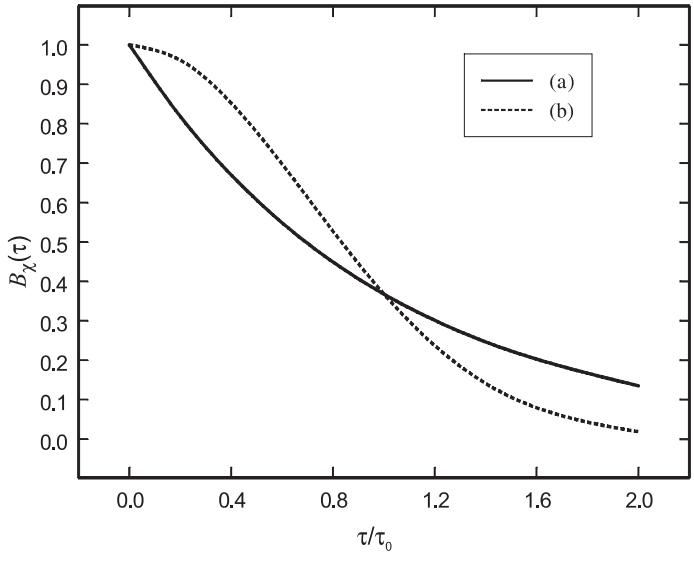

Figure 2.1 The covariance function (*a*) $B_x(\tau) = \exp(-|\tau/\tau_0|)$ and (*b*) $B_x(\tau) = \exp[-(\tau/\tau_0)^2]$.

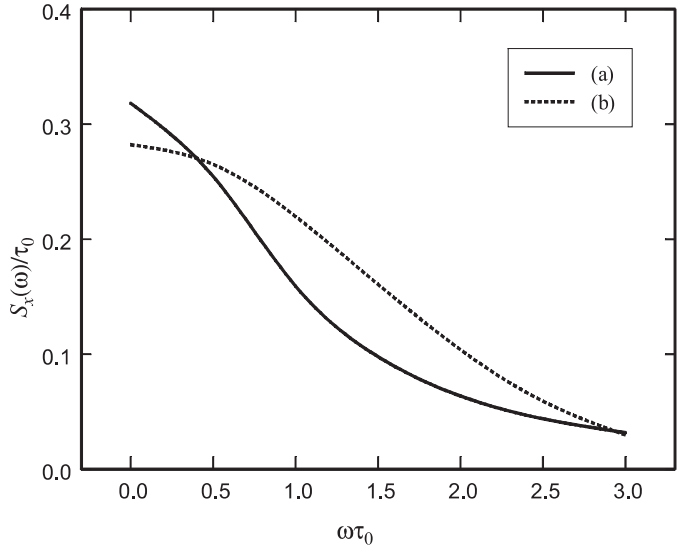

Figure 2.2 The power spectral density functions of the covariance functions in Fig. 2.1.

whereas the function in part (*b*) yields (using integral #8 in Appendix II)

$$S_x(\omega) = \frac{1}{\pi}\int_0^\infty \cos(\omega\tau)\exp[-a^2(\tau/\tau_0)^2]\,d\tau$$

$$= \frac{\tau_0}{2a\sqrt{\pi}}\exp\left(-\frac{\tau_0^2\omega^2}{4a^2}\right).$$

The corresponding spectrums for (*a*) and (*b*) are shown below in Fig 2.2 (once again with $a = 1$). As clearly seen from these examples, the correlation time τ_0 is directly related (inversely) to the width of the spectrum.

Problems

Section 2.3

1. Calculate the correlation and covariance functions of the nonstationary random process

$$x(t) = A\cos\omega_0 t,$$

where A is a random variable for which $\langle A\rangle = 1$ and $\langle A^2\rangle = 4$.

2. Consider the random process

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t,$$

where A and B are independent random variables with

$$\langle A\rangle = \langle B\rangle = 0,\ \langle A^2\rangle = \langle B^2\rangle = \sigma^2.$$

(a) Find the correlation function.
(b) Is this a stationary process (in the wide sense)?

3. Find the structure function associated with the following covariance functions of a stationary random process:

(a) $B_x(\tau) = b_0\exp(-|\tau/\tau_0|)$.

(b) $B_x(\tau) = \exp[-a^2(\tau/\tau_0)^2]$.

4. If $x(t)$ is a *complex* stationary random process with correlation function defined by

$$R_x(\tau) = \langle x(t)x^*(t+\tau)\rangle,$$

show that $R_x(-\tau) = R_x^*(\tau)$.

5. If $x(t)$ is a *complex* stationary random process with complex mean $\langle x(t)\rangle = m$, show that

$$B_x(\tau) = R_x(\tau) - |m|^2.$$

Section 2.4

6. Find the time-average correlation function for the random process in Problem 2. Is the process ergodic?

7. An ergodic random process $x(t)$ has the correlation function

$$R_x(\tau) = 25e^{-4|\tau|} + 9.$$

Deduce that

(a) $\langle x^2(t)\rangle = 34$.

(b) $\langle x(t)\rangle = \pm 3$.

Section 2.5

8. Show that the area under the power spectral density $S_x(\omega)$ equals the average power $\langle x^2(t)\rangle$ of the random process.

9. Show that the power spectral density associated with the correlation function has the form

$$S_x(\omega) + m^2\delta(\omega),$$

where m is the mean value of the stationary random process $x(t)$, $S_x(\omega)$ is the power spectral density, and $\delta(\omega)$ is the delta function.

10. Given that $B_x(\tau) = \cos\omega_0\tau$, find the corresponding power spectral density.

11. The correlation function of a stationary random process is

$$R_x(\tau) = \begin{cases} 1 - |\tau|/T, & |\tau| < T \\ 0, & |\tau| > T. \end{cases}$$

Show that the associated power spectral density is

$$S_x(\omega) = \frac{2\sin^2(\omega T/2)}{\pi T\omega^2}.$$

Section 2.6

12. Given that the covariance function of a statistically homogeneous and isotropic random field is described by $B_u(R) = \exp(-R^2/R_0^2)$, $R \geq 0$, where R_0 is a constant, show that the related power spectral density is

$$\Phi_u(\kappa) = \frac{R_0^3}{8\pi\sqrt{\pi}}\exp\left(-\frac{R_0^2\kappa^2}{4}\right).$$

Hint: Use the integral table in Appendix II.

13. Given that the covariance function of a statistically homogeneous and isotropic random field is described by $B_u(R) = \exp(-R/R_0)$, $R \geq 0$, where R_0 is a constant, show that the related power spectral density is

$$\Phi_u(\kappa) = \frac{R_0^3}{\pi^2(1 + R_0^2\kappa^2)^2}.$$

Hint: Use the integral table in Appendix II.

14. Given the structure function $D_u(R) = 2[1 - \exp(-R/R_0)]$, where R_0 is a constant, use Eq. (60) to show that

$$\Phi_u(\kappa) = \frac{R_0^3}{\pi^2(1 + R_0^2\kappa^2)^2}.$$

What is the equivalent one-dimensional spectrum $V_u(\kappa)$?

15. Given that $u(\mathbf{R})$ is a homogeneous and isotropic random field with power spectral density

$$\Phi_u(\kappa) = \frac{R_0^3}{8\pi\sqrt{\pi}} \exp\left(-\frac{R_0^2\kappa^2}{4}\right),$$

where R_0 is a constant, show that the structure function defined by Eq. (59) is

$$D_u(R) = 2[1 - \exp(-R^2/R_0^2)], \quad R \geq 0.$$

16. Assume as in Prob. 15 that

$$\Phi_u(\kappa) = \frac{R_0^3}{\pi^2(1 + R_0^2\kappa^2)^2},$$

and show that the structure function defined by Eq. (59) is

$$D_u(R) = 2[1 - \exp(-R/R_0)], \quad R \geq 0.$$

17. For a statistically homogeneous and isotropic field $u(\mathbf{R})$, verify that Eqs. (50) and (51) reduce to those given by Eqs. (52).
Hint: Introduce the spherical coordinate transformation equations

$$\begin{array}{ccc} x = R\cos\theta\sin\varphi, & y = R\sin\theta\sin\varphi, & z = R\cos\varphi, \\ 0 \leq \theta < 2\pi, & 0 \leq \varphi \leq \pi, & 0 \leq R < \infty. \end{array}$$

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