# Interval and Radius of Convergence of a Power Series

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## Interval of Convergence

A power series is said to be centered about the real number $a$ if it has the form

(1.1) $$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

When $a = 0$, the power series is said to be centered about 0 and has the form

(1.2) $$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

The set of values of $x$ for which a power series converges forms an interval of real numbers which is referred to as the interval of convergence of the power series. The interval will have an associated center and radius, similar to a circle. Often, the radius of convergence will be a positive real number, while in some cases, it could be 0 or possibly $\infty$. The following table lists the nine types of intervals that exist.

|  | Interval $(a \le b)$ | Center | Radius | Inequality | Type |
|---|---|---|---|---|---|
| B1 | $(a,b)$ | $c = \dfrac{b+a}{2}$ | $r = \dfrac{b-a}{2}$ | $a < x < b$ | bounded / open |
| B2 | $[a,b)$ |  |  | $a \le x < b$ | bounded / half-closed / half-open |
| B3 | $(a,b]$ |  |  | $a < x \le b$ | bounded / half-closed / half-open |
| B4 | $[a,b]$ |  |  | $a \le x \le b$ | bounded / closed |
| U1 | $(a,\infty)$ | Not Applicable |  | $x > a$ | unbounded / open |
| U2 | $[a,\infty)$ |  |  | $x \ge a$ | unbounded / closed |
| U3 | $(-\infty,b)$ |  |  | $x < b$ | unbounded / open |
| U4 | $(-\infty,b]$ |  |  | $x \le b$ | unbounded / closed |
| U5 | $(-\infty,\infty)$ | $c = 0$ | $r = \infty$ |  | unbounded / open and closed |

Of these nine, only five may appear as the interval of convergence of a power series. If the interval of convergence is bounded, it will appear as one of the intervals from **B1** through **B4.** But if the interval of convergence is unbounded, then it will have the form of **U5**. In one very peculiar case, the interval will degenerate to a single point. It will be of type **B4** with $a = b$ so that we obtain $[a,b] = [a,a] = \{a\}$. The center and radius of the interval will always match the center and radius of the power series.

## The meaning behind $|x - c| < r$

The fundamental inequality, $|x - c| < r$, is an alternate way to describe open bounded intervals, that is, intervals of type **B1**. Let's begin by examining am example. Suppose that $|x - 5| < 2$. The inequality expresses the condition that $x$ be less than 2 units away from 5. In other words, $x$ must be less than $5 + 2 = 7$ but greater than $5 - 2 = 3$. This is stated more compactly as $3 < x < 7$. We conclude that $x$ lies in the bounded open interval $(3,7)$. To see that this is indeed true, let's formally solve the inequality. First, we unbind the absolute values bars by moving the opposite of 2 to the left of the bars so that $-2 < x - 5 < 2$. By adding 5 across the compound inequality, we find that $-2 + 5 < x - 5 + 5 < 2 + 5$ which leads directly to

$3 < x < 7$. Notice also that the center of the interval is given by $(7+3)/2 = 5$. We may define the radius of the interval as the distance from its center to either endpoint. In this case, $r = 7 - 5 = 2$, but also $r = 5 - 3 = 2$. Consequently, the radius is also half the length of the interval and may be calculated as $(7-3)/2 = 2$. It is not a coincidence that we are getting back the two numbers 5 and 2 that we started with. Here then is a summary of our findings.

**Center and Radius of an Open Bounded Interval**:

1. The inequality $|x - c| < r$ describes the interval $(c - r, c + r)$ which has center $c$ and radius $r$.

2. The interval $(a, b)$ has center $c = \dfrac{b+a}{2}$, radius $r = \dfrac{b-a}{2}$. It is described by the fundamental inequality $|x - c| < r$.

*Example 1:* $|x - 2| < 10$ describes an interval of radius 10 and center 2, namely the interval, $(2 - 10, 2 + 10) = (-8, 12)$.

*Example 2:* $|x| < 2.1$ Note that this is equivalent to $|x - 0| < 2.1$ which describes an interval of radius 2.1 with center 0, namely the interval, $(0 - 2.1, 0 + 2.1) = (-2.1, 2.1)$.

*Example 3:* The interval $(3, 11)$ has center $c = (11+3)/2 = 7$ and radius $r = (11-3)/2 = 4$ and is described by the inequality $|x - 7| < 4$.

# Finding the Radius and Interval of Convergence of a Power Series

A common way to find the radius and interval of convergence of a power series is by application of the famed ratio test for series. A similar approach may be taken with the root test. Consider the power series,

$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$, which is centered at $a$. The ratio test directs us to compute the limit $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$, whereas the root test would have us calculate $L = \lim_{n\to\infty} \sqrt[n]{\left| a_n \right|}$. Whichever approach is taken, the evaluation of the limit may lead to one of the following possible outcomes.

**I.** It happens that $L = 0$.

In this case, the power series converges for all values of $x$. The radius of convergence is $R = \infty$ and the interval of convergence is $I = (-\infty, \infty)$.

**II.** It happens that $L = \infty$.

In this case, the power series converges only at its center. The radius of convergence is $R = 0$ and the interval of convergence is $I = [a, a] = \{a\}$.

**III.** $L$ depends on $x$.

Setting $L < 1$ will lead to a fundamental inequality of the form $\left| x - c \right| < r$. The inequality describes an open bounded interval with center $c$ and radius $r$. The center of the interval will match the center of the power series so that $c = a$. The radius of convergence will be equal to the radius of the interval so that $R = r$. The interval forms the interior of the interval of convergence. To find the actual interval of convergence, the endpoints must be independently tested for inclusion within the interval of convergence.

## A Final Example

As our final example, we consider $\sum_{n=0}^{\infty} \frac{n}{2^n}(x-3)^n$ . Applying the ratio test to this series, we obtain

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-3)^{n+1}}{2^{n+1}}\cdot\frac{2^n}{n(x-3)^n}\right| = \lim_{n\to\infty}\frac{1}{2}\cdot\frac{n+1}{n}\left|x-3\right| = \frac{1}{2}\left|x-3\right|$$

Upon setting $L<1$, we are led to $\left|x-3\right|<2$ . This defines an open bounded interval whose center is 3 that matches the center of the power series. The radius of the interval is 2, hence the radius of convergence is $R=2$ . The interval defined by the inequality is $(3-2,3+2)=(1,5)$ . This forms the interior of the interval of convergence.

The endpoints are 1 and 5. They must be tested. For $x=1$ , we have

$\sum_{n=0}^{\infty}\frac{n}{2^n}(1-3)^n = \sum_{n=0}^{\infty}\frac{n}{2^n}\cdot(-1)^n 2^n = \sum_{n=0}^{\infty}(-1)^n n$ . This series diverges by the divergence test. For $x=5$ , we obtain $\sum_{n=0}^{\infty}\frac{n}{2^n}(5-3)^n = \sum_{n=0}^{\infty}\frac{n}{2^n}\cdot 2^n = \sum_{n=0}^{\infty} n$ , which also diverges by the divergence test. Thus, neither endpoint is to be included as part of the interval of convergence.

The interval of convergence is therefore $I=(1,5)$ .