ES 155: Feedback Control System

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Homework 3, Due on Oct 5 2018, 5pm; Box outside of Lina's office, MD 345.

Note: In the upper left hand corner of the first page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. (*Predator-Prey Model*)

   Let's consider a predator-prey ecological system. In an isolated island, there are only two kinds of animals: tigers (predator) and hares (prey), and tigers have no choice but to hunt hares for food. Let $H(t)$ represent the number of hares and $G(t)$ represent the number of tigers. For simplification, $H(t)$ and $G(t)$ are treated as continuous variables instead of integer variables. The population dynamics of this predator-prey system are modeled as

$$\frac{dH(t)}{dt} = rH(t) - aG(t), \qquad H(t) \geq 0 \tag{1a}$$

$$\frac{dG(t)}{dt} = \frac{bH(t)G(t)}{c + H(t)} - eG(t), \qquad G(t) \geq 0 \tag{1b}$$

   In (1a), r is the growth coefficient of the hares and a represents the diminishing coefficient by the hunt of tigers. In (1b), b represents the growth coefficient of the tigers, c is a constant parameter that controls the prey consumption rate for low hare population, and e represents the mortality rate of the tigers.[1] In our following problem, r, a, c, b, e are all positive constant parameters with $b > e$.

   (a) Find the equilibrium points $\begin{bmatrix} H(t) - H^* \\ G(t) - G^* \end{bmatrix}$ for the above predator-prey dynamic system (1).[2] The number of equilibrium points may be more than one. Please write down the answer using the parameters.

   (b) Let $r = 0.1$, $e = 0.1$, $c = 100$, $b = 0.2$, $a = 0.5$. Calculate the equilibrium points $\begin{bmatrix} H^* \\ G^* \end{bmatrix}$ with these given constant values. Linearize the system model around the equilibrium points and write down the linearized model in the form of $\dot{x} = Ax$ where $x(t) := \begin{bmatrix} H(t) - H^* \\ G(t) - G^* \end{bmatrix}$. (You may obtain multiple different equilibrium points, please derive the linear system models for each equilibrium point)

   (c) Study the stability of the equilibrium points calculated in question (b) by checking the eigenvalues of the matrix A you obtain in question (b). (Please write down the steps how you get your conclusion so we can give you credits even if the final answer is not right.)

   (d) Suppose that you are a biologist entering this island and want to improve the ecological stability of the predator-prey system. Your efforts can be regarded as a control input $u(t)$ to the system. Recall that the state variable $x(t)$ is $x(t) := \begin{bmatrix} H(t) - H^* \\ G(t) - G^* \end{bmatrix}$ in the linear system $\dot{x} = Ax$ you obtained in question (b). Now consider the following system with a feedback controller,

$$\dot{x} = Ax + u(t) \qquad \text{with} \quad u(t) = \begin{bmatrix} w \cdot (H(t) - H^*) \\ 0 \end{bmatrix} \tag{2}$$

   where $\dot{x} = Ax$ is the linearized state-space model that you derive in question (b) **based on the nonzero equilibrium point**, w is the coefficient representing the additional growth rate of hares by your efforts.

[1] You might notice that the model is different from the model we constructed in class. Again, any model is only an approximation. In this homework assignment, we stick with abstract model in equation (1).

[2] The equilibrium points H^* and G^* should be nonnegative.

Write down the new system model (2) in an autonomous form of $\dot{x} = \bar{A}x$ by plugging $u(t)$ into $\dot{x} = Ax + u(t)$.[3]

Suppose $w = -0.61$. Study whether the new system (2) is stable or not.

2. (*Cart-Inverted Pendulum*).

Imagine the quintessential segway, in which a rod-like mass (the person) tries to balance on a moving base (the segway). We can begin to construct a simplified model of this system with a cart-inverted pendulum example. Consider an inverted pendulum on the top of a cart with the mass of $M = 10$ kg, where a force (input) F is applied to the cart. We assume that the rod has a length of $L = 1$ m, the mass at the end of the rod is $m = 80$ kg, and the moment of inertia of $I = 100$ kg m^2 about its center of gravity (see Fig. 1). The distance of the hing from the origin is given by p. Other parameters are given as follows: gravitational acceleration $g = 9.8\text{m/s}^2$, viscous friction of the cart $c = 0.1\text{N/m/sec}$, viscous friction at the hing $\gamma = 0.01\text{Nms}$. Suppose

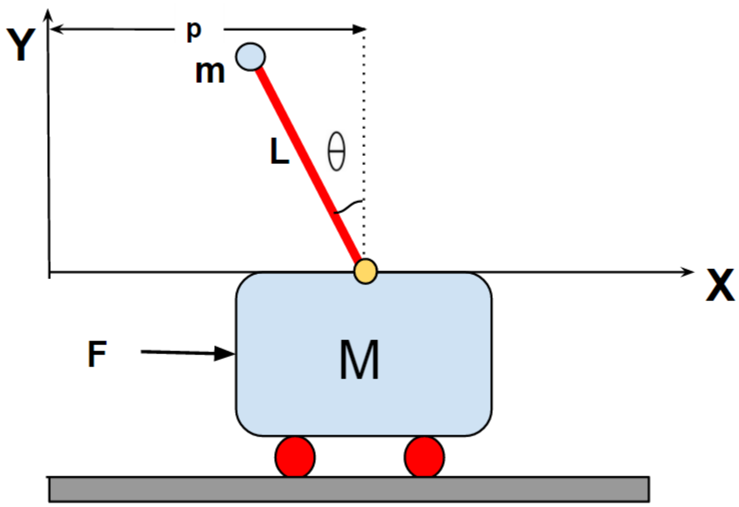

Figure 1: Free-body diagram

θ and p are the outputs of this system. The dynamical equations of motions for θ and p are given as follows:

$$(M+m)\ddot{p} - ml\cos(\theta)\ddot{\theta} + c\dot{p} + ml\sin(\theta)\dot{\theta}^2 = F \tag{3a}$$

$$-ml\cos(\theta)\ddot{p} + (I+ml^2)\ddot{\theta} + \gamma\dot{\theta} - mgl\sin(\theta) = 0 \tag{3b}$$

(a) Using the above equations (3) to derive the state-space model of the system in terms of $\dot{x} = f(x,u), y = h(x,u)$ where the control input $u = F$, output $y = \begin{bmatrix} p \\ \theta \end{bmatrix}$ and $x = \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix}$. Please write down the system model using the parameters (L, M, m, I, g, c) rather than using the specified parameter values.

(b) Assume that the external force is zero, i.e. $F(t) \equiv 0$. Calculate the equilibrium points and linearize the system around the equilibrium point(s). [Note: Please write down your results using the parameters (L, M, m, I, g, c) rather than using the specified parameter values. And please write down certain details about how you derive the results rather than merely providing the final results.]

(c) Study the stability of the equilibrium points calculated in (b) using the linearized state-space models obtained in (b). [Note: In this question, you can use the specified parameter values.]

(d) Now let us only focus on the equilibrium point with $\theta = 0$. A state feedback is defined as $F = -Kx$, i.e., the input force varies with the state of the system. Here the state of the system is $x = (p, \theta, \dot{p}, \dot{\theta})$. Show that the following state feedback stabilizes the *linearization* of the inverted pendulum on a cart: $K = [-15.3, \quad 1730, \quad -50, \quad 443]$. [Note: Please use the specified parameter values.]

[3] We call this as "close the loop".

(e) Now, simulate the *non-linear* system with the same state feedback as in part (d) (You can use ODE function in matlab to simulate). Try different initial conditions and show that the controller asymptotically stablizes the system for some initial conditions while it fails to do so for some others. That is, for the non-linear system, the state feedback controller *locally* stabilizes the system. Here *locally* means that the controller stabilizes the system if the initial condition is not very far away from the equilibrium point, but fails to stabilize the system if the initial condition is far away from the equilibrium point. Provide plots of $\theta(t)$ for both types of initial conditions.