ES 155 Homework 2

1. In this system model where $H(t)$ represents the population of hares and $G(t)$ represents the population of tigers at time t, the population dynamics of the system are given by

$$\frac{d}{dt}H(t) = rH(t) - aG(t) \tag{1}$$
$$\frac{d}{dt}G(t) = \frac{bH(t)G(t)}{c + H(t)} - eG(t) \tag{2}$$

where $H(t) \leq 0$ and $G(t) \leq 0$ (a population can't be negative). In this model, r is the growth coefficient of the hares, a is the diminishing coefficient on the hares by the tigers, b is the growth coefficient of the tigers, c is a constant controlling prey consumption when there is low hare population, and e represents the mortality rate of the tigers.

(a) The equilibrium points of this system satisfy the condition $\begin{bmatrix} \frac{d}{dt}H(t) \\ \frac{d}{dt}G(t) \end{bmatrix} = 0$:

$$\frac{d}{dt}H(t) = rH(t) - aG(t) = 0 \quad \text{solving for } G \tag{3}$$
$$G(t) = \frac{r}{a}H(t) \qquad \text{plug into } \frac{d}{dt}G(t) \tag{4}$$
$$\frac{d}{dt}G(t) = \frac{bH(t)G(t)}{c + H(t)} - eG(t) = 0 = \frac{bH(t)\frac{r}{a}H(t)}{c + H(t)} - e\frac{r}{a}H(t) \tag{5}$$
$$\frac{br}{a}H^2(t) = \frac{er}{a}H(t)\left(c + H(t)\right) \tag{6}$$
$$0 = (b - e)\frac{r}{a}H^2(t) - \frac{erc}{a}H(t) \tag{7}$$
$$H^* = \frac{ec}{b - e}, 0 \qquad \text{plug back in} \tag{8}$$
$$G^* = \frac{r}{a}\left(\frac{ec}{b - e}\right), 0 \tag{9}$$

There are two equilibirium points: $\begin{bmatrix} H^* \\ G^* \end{bmatrix} = \begin{bmatrix} \frac{ec}{b-e} \\ \frac{r}{a}\left(\frac{ec}{b-e}\right) \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(b) Let $r = 0.1, e = 0.1, c = 100, b = 0.2, a = 0.5$. To linearize the system, first find numeric values for the equilibirum points. Pluging these constants into the answer for part (a) gives

$$\begin{bmatrix} H^* \\ G^* \end{bmatrix} = \begin{bmatrix} \frac{ec}{b-e} \\ \frac{r}{a}\left(\frac{ec}{b-e}\right) \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 100 \\ 20 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{10}$$

Next, write down the A, B, C, D matrices. In this case, there is no control input yet, so B and D are 0. We also don't care about the output y now, which is not defined, so only A needs to be considered.

$$\frac{d}{dt}x(t) = \begin{bmatrix} \frac{\partial f_H}{\partial H} & \frac{\partial f_H}{\partial G} \\ \frac{\partial f_G}{\partial H} & \frac{\partial f_G}{\partial G} \end{bmatrix} \begin{bmatrix} H(t) - H^* \\ G(t) - G^* \end{bmatrix} \tag{11}$$

$$A = \begin{bmatrix} r & -a \\ \frac{bcG^*}{(c+H^*)^2} & \frac{bH^*}{c+H^*} - e \end{bmatrix} \tag{12}$$

Plugging in the constants and the two equilibrium points $(100, 20)$ and $(0, 0)$, repectively, we get

$$A = \begin{bmatrix} 0.1 & -0.5 \\ 0.01 & 0 \end{bmatrix}, \begin{bmatrix} 0.1 & -0.5 \\ 0.002 & -0.1 \end{bmatrix} \tag{13}$$

Now define $\tilde{x} = \begin{bmatrix} \tilde{H}(t) \\ \tilde{G}(t) \end{bmatrix} = \begin{bmatrix} H(t) - H^* \\ G(t) - G^* \end{bmatrix}$. For the two equilibrium points, we have

$$\begin{bmatrix} \tilde{H}(t) \\ \tilde{G}(t) \end{bmatrix} = \begin{bmatrix} H(t) - H^* \\ G(t) - G^* \end{bmatrix} = \begin{bmatrix} H(t) - 100 \\ G(t) - 20 \end{bmatrix}, \begin{bmatrix} H(t) \\ G(t) \end{bmatrix} \tag{14}$$

Therefore, the linearized model for the system is:

$$\begin{bmatrix} \dot{\tilde{H}}(t) \\ \dot{\tilde{G}}(t) \end{bmatrix} = \begin{bmatrix} 0.1 & -0.5 \\ 0.01 & 0 \end{bmatrix} \begin{bmatrix} \tilde{H}(t) \\ \tilde{G}(t) \end{bmatrix}, \begin{bmatrix} 0.1 & -0.5 \\ 0.002 & -0.1 \end{bmatrix} \begin{bmatrix} \tilde{H}(t) \\ \tilde{G}(t) \end{bmatrix} \tag{15}$$

for equilibirum points $(100, 20)$ and $(0, 0)$ respectively.

(c) The eigenvalues of the Jacobian matrix A indicate the stability of the system. For the nonzero equilibrium point $(100, 20)$, the eigenvalues of the $A = \begin{bmatrix} 0.1 & -0.5 \\ 0.01 & 0 \end{bmatrix}$ are $0.05 \pm 0.05i$. Because the real part of both eigenvalues is positive, this equilibrium is unstable.

The eigenvalues of the Jacaboian matrix $A = \begin{bmatrix} 0.1 & -0.5 \\ 0.002 & -0.1 \end{bmatrix}$ corresponding to the equilibrium point $(0, 0)$ are ± 0.949. Again, because one of these eigenvalues is positive, the equilibrium point is unstable.

Note: See attached MATALB code for eigenvalue calcualtion.

(d) By adding a control input $u(t)$ to the system, we have

$$\dot{(x)} = Ax + u(t) \quad \text{with} \quad u(t) = \begin{bmatrix} \omega \cdot \big(H(t) - H^*\big) \\ 0 \end{bmatrix} \tag{16}$$

Using only the nonzero equilibrium point $(100, 20)$, we can write the system in the form $\dot{x} = Ax + u(t)$:

$$\dot{x}(t) = \begin{bmatrix} 0.1 & -0.5 \\ 0.01 & 0 \end{bmatrix} \begin{bmatrix} H(t) - 100 \\ G(t) - 20 \end{bmatrix} + \begin{bmatrix} \omega \cdot \big(H(t) - H^*\big) \\ 0 \end{bmatrix} \tag{17}$$

This can also be rewritten in an autonomous form $\dot{x} = \bar{A}x$, which just involves adding ω to the first entry in A:

$$\dot{x}(t) = \begin{bmatrix} 0.1 + \omega & -0.5 \\ 0.01 & 0 \end{bmatrix} \begin{bmatrix} H(t) - 100 \\ G(t) - 20 \end{bmatrix} \tag{18}$$

For $\omega = -0.61$, the stability of the system can be determined by looking at the eigenvalues of $\bar{A}$:

$$\bar{A} = \begin{bmatrix} -0.51 & -0.5 \\ 0.01 & 0 \end{bmatrix} \tag{19}$$

$$\texttt{eig}\big(\bar{A}\big) = -0.5, -0.01 \tag{20}$$

Because the real parts of both eigenvalues are negative, the system is stable. Adding this control input stabalized the system.

2. The given dynamical equations for the cart inverted pendulum system are

$$(M + m)\ddot{p} - ml\cos(\theta)\ddot{\theta} + c\dot{p} + ml\sin(\theta)\dot{\theta}^2 = F \tag{21}$$

$$-ml\cos(\theta)\ddot{p} + \left(I + ml^2\right)\ddot{\theta} + \gamma\dot{\theta} - mgl\sin(\theta) = 0 \tag{22}$$

(a) To derive the state space model in terms of $\dot{x} = f(x, u), y = h(x, u)$ with control input $u = F$, output $y = \begin{bmatrix} p \\ \theta \end{bmatrix}$ and $x = \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix}$ and parameters L, M, m, I, g, c, first find $\dot{x}$:

$$\dot{x} = \frac{d}{dt}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{p} \\ \dot{\theta} \\ \ddot{p} \\ \ddot{\theta} \end{bmatrix} \tag{23}$$

The first two rows are easy. To find $\ddot{p}$, solve for $\ddot{\theta}$ in equation 22 then plug into equation 21.

$$\ddot{\theta} = \frac{1}{I + ml^2}\big(ml\cos(\theta)\ddot{p} - \gamma\dot{\theta} + mgl\sin(\theta)\big) \tag{24}$$

$$u = (M+m)\ddot{p} - ml\cos(\theta)\frac{ml\cos(\theta)\ddot{p} - \gamma\dot{\theta} + mgl\sin(\theta)}{I + ml^2} + c\dot{p} + ml\sin(\theta)\dot{\theta}^2 \tag{25}$$

$$\ddot{p} = \frac{u - ml\sin(\theta)\dot{\theta}^2 - c\dot{p} - ml\cos(\theta)\frac{\gamma}{I+ml^2}\dot{\theta} + \frac{m^2gl^2}{I+ml^2}\cos(\theta)\sin(\theta)}{M + m - \frac{m^2l^2\cos^2(\theta)}{I+ml^2}} \tag{26}$$

Similarly, solve for $\ddot{p}$ in equation 21 then plug into equation 22.

$$\ddot{p} = \frac{1}{M+m}\Big(u + ml\cos(\theta)\ddot{\theta} - c\dot{p} - ml\sin(\theta)\dot{\theta}^2\Big) \tag{27}$$

$$0 = -ml\cos(\theta)\frac{u + ml\cos(\theta)\ddot{\theta} - c\dot{p} - ml\sin(\theta)\dot{\theta}^2}{M+m} + \big(I + ml^2\big)\ddot{\theta} + \gamma\dot{\theta} - mgl\sin(\theta) \tag{28}$$

$$\ddot{\theta} = \frac{ml\cos(\theta)u - ml\cos(\theta)c\dot{p} - m^2l^2\cos(\theta)\sin(\theta)\dot{\theta}^2 - (M+m)(\gamma\dot{\theta} - mgl\sin(\theta))}{(M+m)(I+ml^2) - l^2\cos^2(\theta)} \tag{29}$$

Therefore, the state space model for this system can be written

$$\dot{x} = \begin{bmatrix} \dot{p} \\ \dot{\theta} \\ \ddot{p} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{p} \\ \dot{\theta} \\ \frac{u - ml\sin(\theta)\dot{\theta}^2 - c\dot{p} - ml\cos(\theta)\frac{\gamma}{I+ml^2}\dot{\theta} + \frac{m^2gl^2}{I+ml^2}\cos(\theta)\sin(\theta)}{M+m-\frac{m^2l^2\cos^2(\theta)}{I+ml^2}} \\ \frac{ml\cos(\theta)u - ml\cos(\theta)c\dot{p} - m^2l^2\cos(\theta)\sin(\theta)\dot{\theta}^2 - (M+m)(\gamma\dot{\theta} - mgl\sin(\theta))}{(M+m)(I+ml^2) - l^2\cos^2(\theta)} \end{bmatrix} \tag{30}$$

$$y = \begin{bmatrix} p \\ \theta \end{bmatrix} \tag{31}$$

(b) To find the equilibrium points for $F = 0$, set $\dot{x} = 0$. By equation 30, we can immediately see that $\dot{p} = 0$ and $\dot{\theta} = 0$. Plugging this into the last two entries of $\dot{x}$ gives

$$\ddot{p} = \frac{\frac{m^2gl^2}{I+ml^2}\cos(\theta)\sin(\theta)}{M+m-\frac{m^2l^2\cos^2(\theta)}{I+ml^2}} = 0 \tag{32}$$

$$\ddot{\theta} = \frac{(M+m)mgl\sin(\theta)}{(M+m)(I+ml^2)-l^2\cos^2(\theta)} = 0 \tag{33}$$

The latter equation implies that $\sin(\theta) = 0$ so $\theta = n\pi \forall n \in \mathbb{Z}$, which means that the cart-inverted pedulum has an equilibrium point at both the vertical top and bottom of the pendulum's swing. The prior equation agrees with this, as there is also a $\sin(\theta)$ multiplier. Note that the p position does not matter for either the dynamic equations or the equilibirium points, which makes sense: the system is "position invariant" in that offsets in position will not affect the dynamics. Thus, the system dynamics can be calculated for two equilibrium points:

$$x = \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} p^* \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} p^* \\ \pi \\ 0 \\ 0 \end{bmatrix} \tag{34}$$

where $\forall \quad p^* \in \mathbb{R}$.

Now to linearize the system near these two points, we must write the Jacobian matrix A for the system.

$$\dot{x} = Ax = \begin{bmatrix} \frac{\partial f_p}{\partial p} & \frac{\partial f_p}{\partial \theta} & \frac{\partial f_p}{\partial \dot{p}} & \frac{\partial f_p}{\partial \dot{\theta}} \\ \frac{\partial f_\theta}{\partial p} & \frac{\partial f_\theta}{\partial \theta} & \frac{\partial f_\theta}{\partial \dot{p}} & \frac{\partial f_\theta}{\partial \dot{\theta}} \\ \frac{\partial f_{\dot{p}}}{\partial p} & \frac{\partial f_{\dot{p}}}{\partial \theta} & \frac{\partial f_{\dot{p}}}{\partial \dot{p}} & \frac{\partial f_{\dot{p}}}{\partial \dot{\theta}} \\ \frac{\partial f_{\dot{\theta}}}{\partial p} & \frac{\partial f_{\dot{\theta}}}{\partial \theta} & \frac{\partial f_{\dot{\theta}}}{\partial \dot{p}} & \frac{\partial f_{\dot{\theta}}}{\partial \dot{\theta}} \end{bmatrix} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} \tag{35}$$

$$\tag{36}$$

At these two equilibrium points, where $\theta = 0, \pi$, we can make some similifying assumptions. For $\theta = 0$, we can approximate $\sin\theta \approx \theta = 0$ and $\cos\theta \approx= 1$. For $\theta = \pi$, we can approximate $\sin\theta \approx \pi - \theta$ and $\cos\theta \approx -1$. With these approximations, we can write

$$A_{\theta^*=0} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2l^2g}{(M-m)(I+ml^2)-m^2l^2} & -\frac{c(I+ml^2)}{(M-m)(I+ml^2)-m^2l^2} & -\frac{\gamma lm}{(M-m)(I+ml^2)-m^2l^2} \\ 0 & \frac{(M+m)mgl}{(M-m)(I+ml^2)-m^2l^2} & -\frac{clm}{(M-m)(I+ml^2)-m^2l^2} & -\frac{\gamma(M+m)}{(M-m)(I+ml^2)-m^2l^2} \end{bmatrix} \tag{37}$$

$$A_{\theta^*=\pi} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2l^2g}{(M-m)(I+ml^2)-m^2l^2} & -\frac{c(I+ml^2)}{(M-m)(I+ml^2)-m^2l^2} & \frac{\gamma lm}{(M-m)(I+ml^2)-m^2l^2} \\ 0 & -\frac{(M+m)mgl}{(M-m)(I+ml^2)-m^2l^2} & \frac{clm}{(M-m)(I+ml^2)-m^2l^2} & -\frac{\gamma(M+m)}{(M-m)(I+ml^2)-m^2l^2} \end{bmatrix} \tag{38}$$

The difference between these two Jacobians is only the sign of some of the entries.
To complete the system model, we need to write B, C, D. Because the control input $F = u = 0$, $B = 0$ as well. The C matrix to get y is simple because y is just the first two entries of x, so

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{39}$$

and D is also 0. With these matricies the system model can be written in the form

$$\dot{x} = Ax + Bu \tag{40}$$

$$y = Cx + Du \tag{41}$$

(c) We can plug in the parameters to determine the stability of the system at the two equilibrium points. One quantity that showed up often is $(M - m)(I + ml^2) - m^2l^2$, which for the given parameters $M = 10$ kg, $m = 80$ kg, $I = 100$ kg m^2, and $L = 1$ m is -19000 kg^2 m^2.

$$A_{\theta^*=0} = \begin{bmatrix} 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 \\ 0 & 6.4 & -0.00184 & -8.16 \times 10^{-5} \\ 0 & 7.2 & -8.16 \times 10^{-4} & -9.18 \times 10^{-5} \end{bmatrix} \tag{42}$$

$$\tag{43}$$

$$A_{\theta^*=\pi} = \begin{bmatrix} 0 & 0 & 1.0 & 0\ 0 & 0 & 0 & 1.0 \\ 0 & 6.4 & -0.00184 & 8.16 \times 10^{-5} & & & \\ 0 & -7.2 & 8.16 \times 10^{-4} & -9.18 \times 10^{-5} & & & \end{bmatrix} \tag{44}$$

$$\tag{45}$$

Taking the eigenvalues of these matrices, we can determine the stability of each of these equilibirium points. Plugging into MATLAB gives the eigenvalues of $A_{\theta=0}$ as

$$\begin{matrix} 0 \\ 2.6829 \\ -2.6837 \\ -0.0011 \end{matrix}$$

This shows that the upright pendulum is unstable, as we would expect. When plugging in the eigenvalues of $A_{\theta=\pi}$ I got

$$\begin{matrix} 0.0000 + 0.0000i \\ -0.0004 + 2.6833i \\ -0.0004 - 2.6833i \\ -0.0011 + 0.0000i \end{matrix}$$

This implies that this equilibrium point likely stable, though there is an eigenvalue with real part equal to zero which may complicate things.

(d) Examining the equilibrium point with $\theta = 0$ and defining state feedback $F = -Kx$ for $K = [-15.3, 1730, -50, 443]$. We can examine the stability of the system with the addition of the feedback by examining the eigenvalues of $\bar{A}$:

$$\dot{x} = Ax + Bu = Ax + B(-Kx) = (A - BK)x = \bar{A}x \tag{46}$$
$$\bar{A} = (A - BK) \tag{47}$$

Previously we had $B = \begin{bmatrix} 0 \\ 0 \\ \frac{I+ml^2}{(M-m)(I+ml^2)-m^2l^2} \\ \frac{lm}{(M-m)(I+ml^2)-m^2l^2} \end{bmatrix}$. With the given parameters, $B = \begin{bmatrix} 0 \\ 0 \\ 0.0184 \\ 0.0082 \end{bmatrix}$. Now we can compute $\bar{A}$.

$$\bar{A} = A - BK = \begin{bmatrix} 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 \\ 0.281 & -25.4 & 0.917 & -8.14 \\ 0.125 & -6.92 & 0.407 & -3.62 \end{bmatrix} \tag{48}$$

This matrix now describes the system with the feedback control: $\dot{x} = \bar{A}x$. Its eigenvalues are

$$\begin{aligned}&-0.9994 + 1.9990i\\&-0.9994 - 1.9990i\\&-0.3506 + 0.3495i\\&-0.3506 - 0.3495i\end{aligned}$$

Because the real part of all the eigenvalues are negative, the system is now stable. Adding the feedback controller stabalized the system.

(e) Simulating the nonlinear system in MATLAB can show the local effectiveness of the stabilization feedback control. Using Equation 30 and calculating $F = u = -Kx$ as in part (d), I simulated the system using `ode45`.

However, unlike the linearization of the system $\dot{x} = A_bar x$ which always remains stable despite the initial condition, the nonlinear model only stabalizes for certain initial conditions. For example, Figure 1 shows the system with initial condition $x_0 = \begin{bmatrix} 0 \\ 0.5 \\ 0 \\ 0 \end{bmatrix}$. In this case, the system stabilizes. However, Figure 2 shows the system with initial condition $x_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. In this case, the system is unstable, as the feedback control cannot keep up.

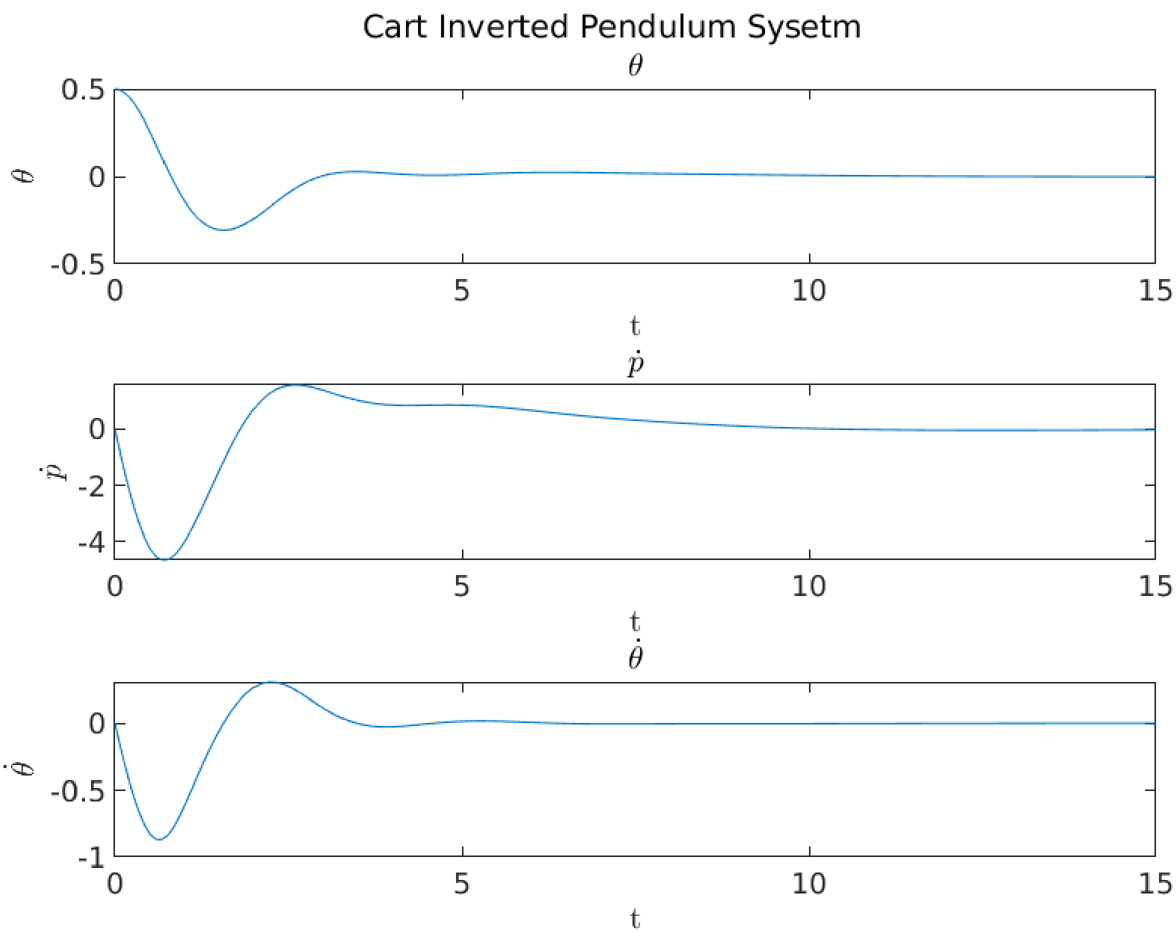

Figure 1: Stable Cart Inverted Pendulum with initial condition $x_0 = \begin{bmatrix} 0 \\ 0.5 \\ 0 \\ 0 \end{bmatrix}$

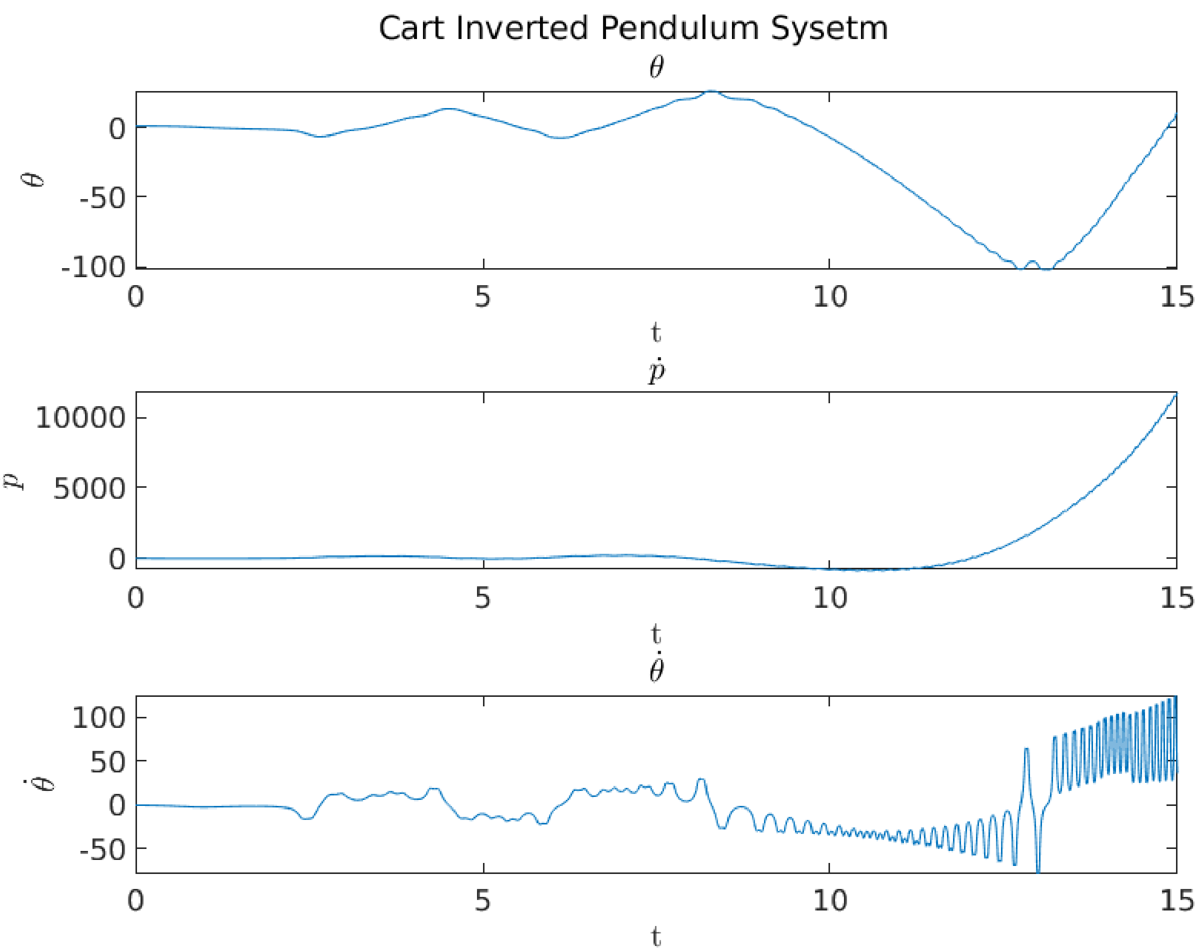

Figure 2: Unstable Cart Inverted Pendulum with initial condition $x_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$