# ES 155 Homework 5

1. (a) The observability matrix for the $2 \times 2$ case is $w_O = \begin{bmatrix} C \\ CA \end{bmatrix}$. From Homework 4, we had $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix}$, so

$$w_0 = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

which is full rank, so the system is observable.

(b) Using an observer for the system in the form

$$\dot{x} = A\hat{x} + Bu + L(y - C\hat{(x)})$$

where $\hat{x}$ is the observed state and $L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$. The error can bbe defined as $e(t) := x(t) - \hat{x}(t)$. The dynamics of $e(t)$ can be written

$$\begin{aligned} \dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t) \\ &= Ax + Bu - A\hat{x} - Bu - L(y - C\hat{x}) \\ &= Ax + Bu - A\hat{x} - Bu - L(Cx - C\hat{x}) \\ &= A(x - \hat{x}) - LC(x - \hat{x}) \\ \dot{e}(t) &= (A - LC)e(t) \end{aligned}$$

so $\dot{e}(t) = A_e e(t)$ where $A_e = (A - LC)$. Now to compute $L$ such that the eigenvalues of $A_e$ are the roots of the given equation $\lambda^2 + 2\zeta_e\omega_e\lambda + \omega_e^2 = 0$, find the characteristic equation of $A_e$.

$$A_e = A - LC = \begin{bmatrix} -3 & 2 - l_1 \\ 1 & -1 - l_2 \end{bmatrix}$$

$$\texttt{det}(A_e - \lambda I) = 0$$

$$\lambda^2 + (4 + l_2)\lambda + (1 + l_1 + 3l_2) = \lambda^2 + 2\zeta_e\omega_e\lambda + \omega_e^2$$

so

$$\begin{aligned} 4 + l_2 &= 2\zeta_e\omega_e \\ l2 &= 2\zeta_e\omega_e - 4 \end{aligned} \qquad \begin{aligned} 1 + l1 + 3l2 &= \omega_e^2 \\ l1 &= \omega_e^2 - 6\zeta_e\omega_e + 23 \end{aligned}$$

Therefore

$$L = \begin{bmatrix} l1 \\ l2 \end{bmatrix} = \begin{bmatrix} \omega_e^2 - 6\zeta_e\omega_e + 23 \\ 2\zeta_e\omega_e - 4 \end{bmatrix}$$

(c) Now using a controller with $u = -K\hat{x} + k_r r$ with $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 4\zeta_0\omega_0 - 8 & 2\omega_0^2 - 4\zeta_0\omega_0 + 6 \end{bmatrix}$ and $k_r = 2\omega_0^2$. Use the new state vector $\tilde{x} = \begin{bmatrix} x \\ e \end{bmatrix}$, input $\tilde{u} = r$ and output $\tilde{y} = x_1$. To find the state space model

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}\tilde{u} \\ \tilde{y} &= \tilde{C}\tilde{x} \end{aligned}$$

write the matrices $\tilde{A}, \tilde{B}, \tilde{C}$. Similar to Homework 4, $\tilde{B} = \begin{bmatrix} 0.5k_r \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\tilde{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$. Now $\tilde{A}$ can be written as

$$\begin{aligned} \tilde{A} &= \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \\ &= \begin{bmatrix} -3 - 0.5k_1 & 2 - 0.5k_2 & 0.5k_1 & 0.5k_2 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -3 & 2 - l_1 \\ 0 & 0 & 1 & -1 - l_2 \end{bmatrix} \end{aligned}$$

Using symbolic math to compute the eigenvalues in MATLAB gives

$$' \begin{pmatrix} \omega_0\sqrt{(\zeta_0 - 1)\ (\zeta_0 + 1)} - \omega_0\,\zeta_0 \\ -\omega_0\sqrt{(\zeta_0 - 1)\ (\zeta_0 + 1)} - \omega_0\,\zeta_0 \\ 3\,\omega_e\,\zeta_e - \frac{\sqrt{{\omega_e}^4 - 12\,{\omega_e}^3\,\zeta_e + 36\,{\omega_e}^2\,{\zeta_e}^2 + 50\,{\omega_e}^2 - 316\,\omega_e\,\zeta_e + 665}}{2} - \frac{{\omega_e}^2}{2} - \frac{27}{2} \\ 3\,\omega_e\,\zeta_e + \frac{\sqrt{{\omega_e}^4 - 12\,{\omega_e}^3\,\zeta_e + 36\,{\omega_e}^2\,{\zeta_e}^2 + 50\,{\omega_e}^2 - 316\,\omega_e\,\zeta_e + 665}}{2} - \frac{{\omega_e}^2}{2} - \frac{27}{2} \end{pmatrix}$$

Plugging in values of $\omega_0 = 0.1, \omega_e = 0.3, \zeta_0 = 0.2, \zeta_e = 0.4$, the real values of eigenvalues of $\tilde{A}$ are all negative, so the system is stable.

2. (a) For the 4x4 case, the observability matrix is $W_O = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}$. See attached MATLAB code for computation and value of this matrix. Because this matrix is full rank (also computed in MATLAB), the system is observable from this measurement output.

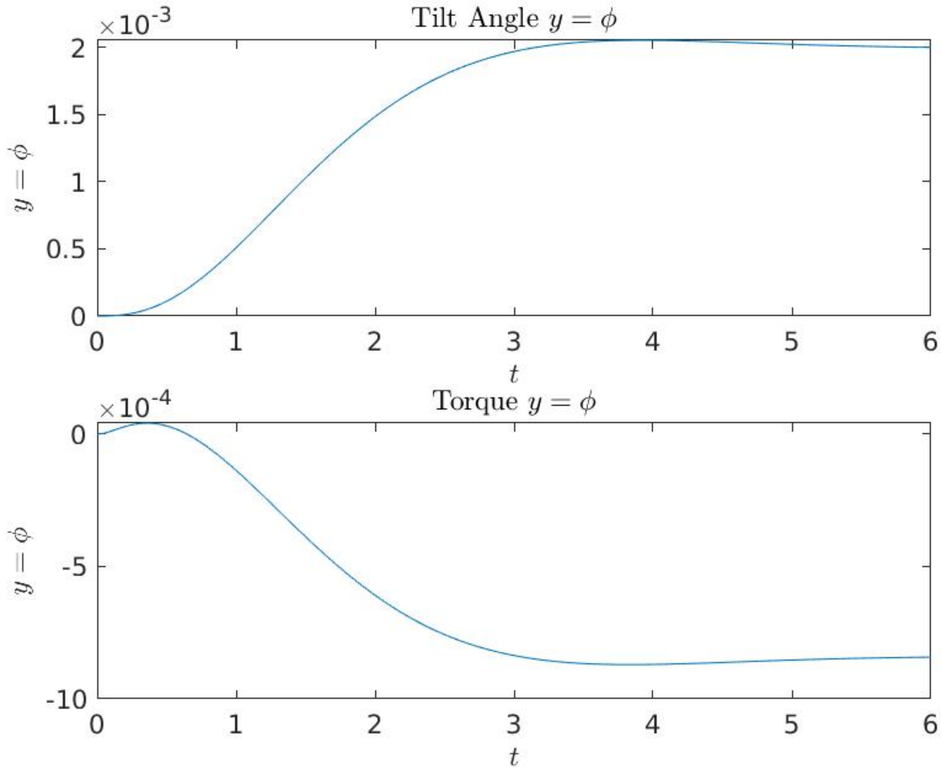


Figure 1: Steering Angle and Input Torque with step function input using observer feedback

(b) To design an observer with $\lambda = -4, -20, -2 \pm 2i$, use MATLAB's place function with $L^T = \texttt{place}(A^T, C^T, \{\lambda\})$. This gives

$$L = \begin{bmatrix} 14.4541 \\ -75.9787 \\ -19.1590 \\ 925.6916 \end{bmatrix}$$

(c) See MATLAB code for simulation. Figure 1 shows the steering angle and torque input with this observer. Compare this to Figure 2, which shows the same values without the observer. The output seems to be very similar, as expected. The observer feedback does a good job of replicating the system response if the true state is known.

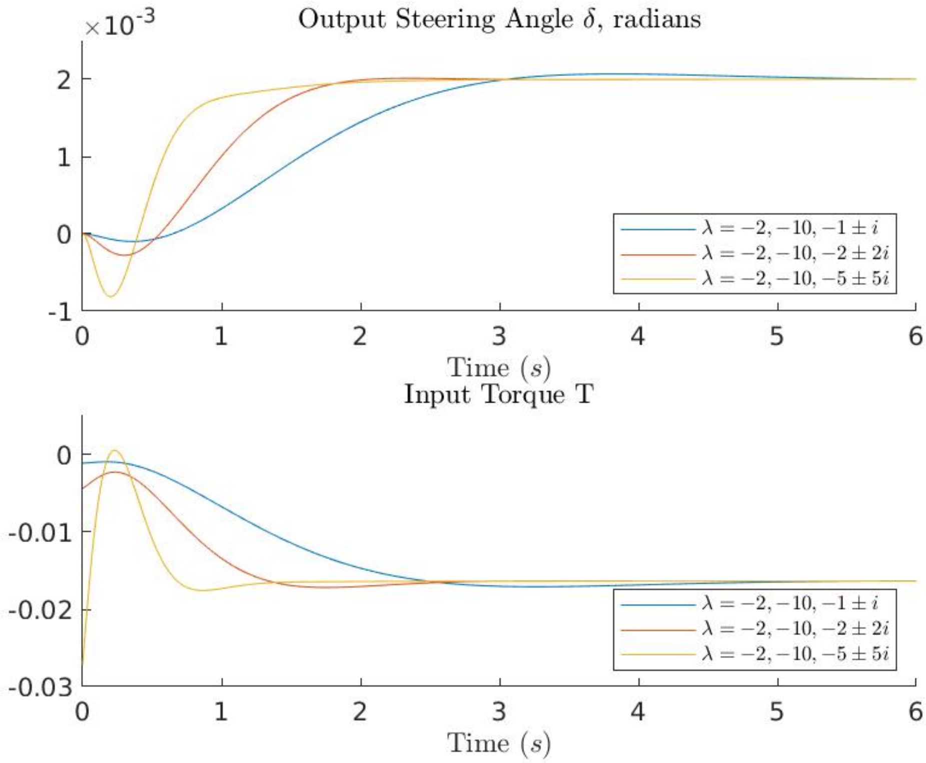


Figure 2: Steering Angle and Input Torque with step function input with no observer feedback. Only consider the blue line.