# ES 155: Systems and Control

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Homework 4, Due on October 19th 2018, Friday 5pm, Box outside of Lina's office, MD 345.

**Note: In the upper left hand corner of the first page of your homework set, please put the number of hours that you spent on this homework set (including reading).**

1. (*Nutrient*) First please read Section 4.6 (Drug Administration) in the textbook of *Feedback Systems: An Introduction for Scientists and Engineers* by Karl J. Åström and Richard M. Murray, *2nd edition* for a general background of compartment models. Now we will use it to model the nutrients such as vitamins and minerals in our body.

   We consider a simplified two-compartment model, e.g., the gastrointestinal (digestive) system and the blood circulatory system. We consider a nutrient with concentration $c_0$ that is injected in the digestive system at a volume flow rate of $u$ and that its concentration in the blood circulatory system is the output. Let $c_g$ and $c_b$ be the concentrations of the nutrient in the two compartments and let $V_1$ and $V_2$ be the volumes of the compartments. The dynamical model for the mass balance of the two compartments is

$$\begin{aligned} V_1 \frac{dc_g}{dt} &= q(c_b - c_g) - q_0 c_g + c_0 u \\ V_2 \frac{dc_b}{dt} &= q(c_g - c_b) \\ y &= c_b \end{aligned}$$

   where $q$ represents the flow rate between the compartments and $q_0$ represents the flow rate out of the digestive system that is not going to the blood circulatory system. Introducing the parameters $a_0 = q_0/V_1$, $a_1 = q/V_1$, $a_2 = q/V_2$, and $b_0 = c_0/V_1$ and let state variable $x = \begin{bmatrix} c_g \\ c_b \end{bmatrix}$. The model can be written as,

$$\frac{dx}{dt} = Ax + Bu = \begin{bmatrix} -a_0 - a_1 & a_1 \\ a_2 & -a_2 \end{bmatrix} x + \begin{bmatrix} b_0 \\ 0 \end{bmatrix} u, \qquad y = Cx = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

   Let us pick $a_1 = 2$, $a_2 = 1$, $a_0 = 1$, $b_0 = 0.5$.

   (a) Compute the reachability matrix for the system and show that the system is reachable.

   (b) Determine a controller $u = -Kx + k_r r$ (the first part $-Kx$ is a state-feedback control, where $K = [k_1, k_2]$ is a constant matrix to be determined; the second part $k_r r$ is a reference gain control, where $r$ is a fixed reference signal, and $k_r$ is a parameter to be determined). We require the controller to satisfy the following two properties: (i) the closed loop system has an unit static gain, i.e. after applying the controller, the steady-state output equals the reference signal ($y_{steady-state} = r$). (ii) the closed loop system's eigenvalues (i.e. the eigenvalues of $A - BK$) are the two roots of equation $\lambda^2 + 2\zeta_0\omega_0\lambda + \omega_0^2 = 0$, where $\zeta_0, \omega_0$ are considered given.

   (c) Set $\omega_0 = 1$ and $\zeta_0 = 0.1, 0.4, 0.7, 0.9$, then compute the eigenvalues for the resulting closed-loop system in part (b) and also the corresponding $K$ and $k_r$. Plug the controller $u = -Kx + k_r r$ into the system and get the closed-loop system $\dot{x} = Ax + B(-Kx + k_r r) = (A - BK)x + k_r Br, \quad y = Cx$. Plot the step response of the closed-loop systems on the same figure.[1] Discuss what you observe as $\zeta_0$ increases from 0.1 to 0.9.[2]

[1] Step response refers to the system output behavior under a step input signal $r(t) = r$ for all $t \geq 0$ where the initial value are zero.

[2] Hint: Since this is a linear system simulation, using matlab control toolbox command would be much easier than using ODE45. Firstly write down the closed loop system model $\dot{x} = Ax + B(-Kx + Kk_r r) = (A - BK)x + k_r Br, \quad y = Cx$. Then you can use matlab code, sys=ss($A - BK$, $k_r B$, $C$, $D$) where $D = 0$, and step(sys) to simulate the step response.

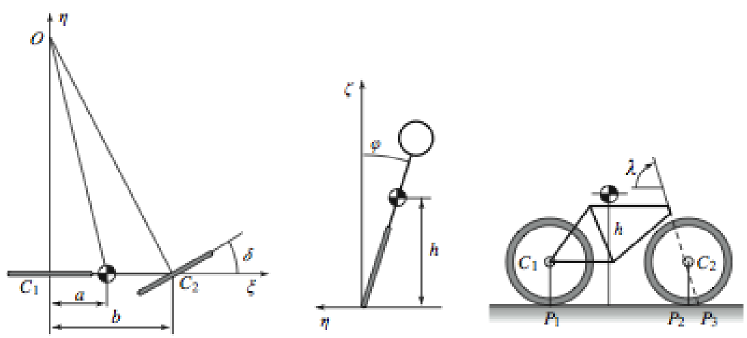
(d) (Open-ended and Bonus question)[3] As you observe from Parts (b) and (c), there are many controllers that would achieve our control objective, reaching a targeted value $r$ to ensure that there is enough nutrient in the blood circulation system. But different controllers come with different transient dynamics of the states (overshoot, oscillations before convergence, time to convergence) and different levels of the control input. A larger value of $u$ usually implies a larger economic cost of taking the nutrient. If you are a nutritionist or a policy maker in CDC (Center for Disease and Prevention), how would you advise people's decisions? When considering the economic cost, how should our society provide the same guarantee for everyone regardless of their income level? Also considering that different people would have different parameter values such as $q, q_0, V_1, V_2$, what are your suggestions on how we can design customized strategies for individuals?[4]

2. (*Whipple bicycle model*)

The following differential equation gives the whipple bicycle model

$$M\begin{bmatrix}\ddot{\phi}\\ \ddot{\delta}\end{bmatrix} + Cv_0\begin{bmatrix}\dot{\phi}\\ \dot{\delta}\end{bmatrix} + (K_0 + K_2 v_0^2)\begin{bmatrix}\phi\\ \delta\end{bmatrix} = \begin{bmatrix}0\\ T\end{bmatrix}$$

A full description of this model is given in Section 3.2 in Astrom and Murray. Roughly speaking, this model describes the dynamics of a bicycle, where $\phi$ is the tilt angle, $\delta$ is the steering angle, $v_0$ is the forward velocity, $T$ is the steering torque, and $M$, $C$, $K_0$ and $K_2$ are 2-by-2 matrices that depend on the geometry and the mass distribution of the bicycle.



(a) top view (b) rear view (c) side view

Download the MATLAB file 'Bicycle_whipple.m' from the course website. The model has already been written as a state space model in the file (See matrix $A$ and $B$ in lines 65-69, where the state vector is defined as $x = [\phi, \delta, \dot{\phi}, \dot{\delta}]$ and the input is $u = T$). Using the parameters in this file, the model is unstable at the velocity $v_0 = 5m/s$ and the open loop eigenvalues are $-1.84$, $-14.29$ and $1.30 \pm 4.60i$. Using MATLAB control toolbox (hint: you can use the 'place' function), find the feedback gain $K$ of a linear controller $u = -Kx + k_r r$ that stabilizes the bicycle and gives closed loop eigenvalues at each of the following three sets of eigenvalues

(i) $-2$, $-10$ and $-1 \pm i$.

(ii) $-2$, $-10$ and $-2 \pm 2i$.

(iii) $-2$, $-10$ and $-5 \pm 5i$

For each of the three cases, if we require that at steady state $\delta_{ss} = r$, calculate the reference gain $k_r$ and simulate the response of the system to a step change in the steering angle reference $r$ from 0 to 0.002 rad (assuming the initial condition is zero), and plot both the steering angle $\delta$ and the torque input $T$.

[3] This question is inspired by ABET criterion on promoting "Ethic" and "Economic" thinking. As long as you provide some thoughts, you will get some bonus points. Maximum bonus points are 3.

[4] This is a nice research and entrepreneurship idea considering the booming technologies in AI and data science plus the control you are learning!