ES 155: Systems and Control

Lecturer: Na Li (nali@seas.harvard.edu)

Homework 2, Due on September 26 2018, In class.

Note: In the upper left hand corner of the first page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. Consider the RLC circuit in Figure 1. In Figure 1, V_S is the source voltage, V_L is the voltage on the inductor,

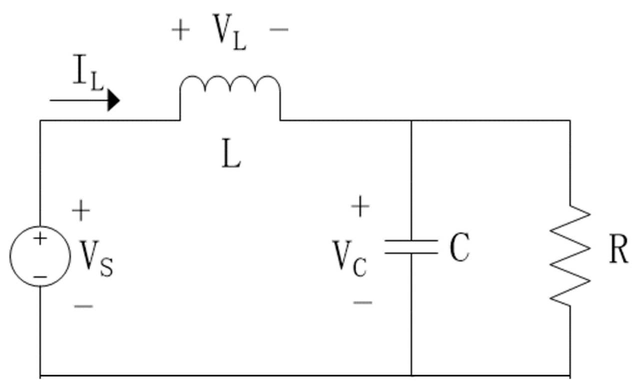

Figure 1: RLC circuit

V_C is the voltage on the capacitor and I_L is the current through the inductor. You may also introduce other circuit variables. Resistance of the resistor is R, inductance of the inductor is L, capacitance of the capacitor is C. Now we would like to know how $V_C(t)$ changes when $V_S(t)$ changes. Note that the voltage resource $V_S(t)$ can be either DC or AC or any other time-varying signal.

   (a) Let the control input $u(t) = V_s(t)$, output be $y(t) = V_c(t)$. Define the state $x(t)$ and derive the state space model in the form of $\dot{x}(t) = f(x(t), u(t)), y(t) = h(x(t), u(t))$. (Hint: You might need to use Kirchhoff's Circuit Laws, Ohm's Law, the capacitor equation and the inductor equation. You will have a 2nd-order system, meaning that $x(t) \in \mathbb{R}^2$. If you find it difficult, please come to office hours or schedule individual meetings with TFs or Lina. We will help you step by step.)

   (b) Let $R = 1\Omega$, $L = 0.1H$, $C = 0.2F$. Please draw the phase portrait of the system (We will provide sample code on the course website about drawing phase portrait.).

   (c) Let $R = 1\Omega$, $L = 0.1H$, $C = 0.2F$. Now assume at beginning $V_S(t = 0) = 0V$, $I_L(t = 0) = 0A$, $V_C(t = 0) = 0V$. At time $t = 0$, V_S is suddenly changed from $0V$ to $1V$, i.e. $V_S(t) = 1V, \forall t \in (0, \infty)$. Please calculate the equilibrium point(s), and use ODE function in Matlab to simulate and plot the system state trajectory. (Note: Mathematically, this question asks you to study the system dynamics given initial condition $x(0) = [0, 0]$ and constant input $V_S(t) = 1V$ for all $t > 0$. We call this response of the system as "step response". The input is called as "step input".)

2. (Autonomous Driving System) In this problem, we will go through a very simplified version of autonomous driving control. Now assume that you are the Uber chief scientist to develop a controller for driving a car to follow a given trajectory, e.g., a road. Let's consider a car shown as Figure 1. The position of this car is

specified by its coordinate variables (x, y). v and ω are the forward tangential and angular velocities of the car, which can be directly controlled by the car.[1] θ is the angle between x-axis and the forward tangential velocity. By the definition of angular velocity, we have the connection between θ and ω through $\dot{\theta}(t) = \omega(t)$. The control task is to control v and ω to let the car position follow a desired trajectory.

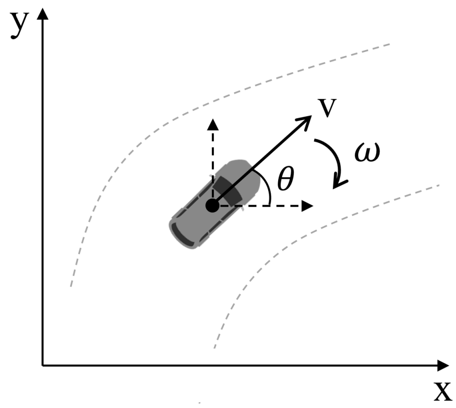

Figure 2: Illustration of car motion.

a. Let the control input be $u(t) = \begin{bmatrix} v(t) \\ \omega(t) \end{bmatrix}$, state be $z(t) = \begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix}$. Please write down the dynamics of the state in terms of the state space form $\dot{z} = f(z, u)$. (hint: the velocities in the directions of x-axis and y-axis are $v_x = v \cdot \cos\theta$ and $v_y = v \cdot \sin\theta$ respectively.)

b. If we set the angular velocity $\omega(t) \equiv 0$ and the forward tangential velocity $v(t) \equiv c$ for all time $t \geq 0$ where c is a constant. The initial position $(x(0), y(0))$ is the origin $(0, 0)$ and the initial $\theta(0)$ is 0. Derive the moving trajectory for the car, i.e., derive the solution $(x(t), y(t), \theta(t))$ of the dynamics that you derive in Part a of this question.

c. If we set the angular velocity $\omega(t) \equiv b$ and the forward tangential velocity $v(t) \equiv c$ for all time $t \geq 0$ where b and c are two constants. The initial position $(x(0), y(0))$ is $(0, -\frac{c}{b})$ and the initial $\theta(0)$ is 0. Derive the moving trajectory for the car, i.e., derive the solution $(x(t), y(t), \theta(t))$ of the dynamics that you derive in Part a of this question. (Hint: You might need to use the fact that $\frac{d\sin(wt)}{dt} = w\cos(wt)$ and $\frac{d\cos(wt)}{dt} = -w\sin(wt)$.) If you compute the radius of the position, i.e., $\sqrt{x(t)^2 + y(t)^2}$, what do you notice?

3. (Linear Algebra review) Feel free to use computer softwares (such as Mathematica, Matlab, or Python) to answer the following questions. But hope you can roughly understand the math behind the answers. You can check the section reading materials or just google the terms..

(a) What is the **determinant** of matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$?

(b) What is the **rank** of matrix $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$?

(c) What is the **rank** of matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$?

(d) What is **row rank** of matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$?

(e) What is **column rank** of matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$?

[1]For the purpose of simplification, here we assume v and w can be freely determined(controlled) instead of through their accelerations like what we did in class.

(f) What is **row rank** of matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$?

(g) What is **column rank** of matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$?

(h) What do you notice through the answers of Part (d) (e) (f) and (g)?