A *failed* attempt to construct exotic smooth structures in S^4 or $\#^n \mathbb{CP}^2$

Main idea:

Find a pair of knots (K, K') such that:

$\begin{cases} ① \ \exists\ \varphi: S^3_0(K) \xrightarrow{\cong} S^3_0(K') \\ ② \ K \text{ is } \underline{H\text{-slice}}. \\ ③ \ K' \text{ is not } H\text{-slice}. \end{cases}$

slice: bounds a disc Δ smoothly embedded in D^4.

H-slice: bounds a disc Δ smoothly embedded in $\underbrace{W \setminus \mathring{B}^4}_{W^\circ}$, and $[\Delta] = 0 \in H_2(W^\circ, \partial W^\circ)$

Then, ① + ② $\Rightarrow \exists$ $\overset{\text{homeomorphic}}{\Uparrow}$ homotopy S^4 or $\#^n \mathbb{CP}^2$ $\underline{X}$ s.t. K' is slice in W.

Thrm: W smooth, closed, oriented, simply-connected 4-manifold. If there exists $\varphi: S^3_0(K) \xrightarrow{\cong} S^3_0(K')$ and K is H-slice in W, then we can construct a 4-manifold X homotopy equivalent to W, and K' is H-slice in X.

proof: $V := W^\circ \setminus \nu(\underset{\text{the disk } K \text{ bounds in } W^\circ}{\Delta})^\circ$

note: $\partial V \cong S^3_0(K)$
$\pi_1(V)$ is generated by $i_*(\pi_1(\partial V))$.

$X := \underbrace{X(-K')}_{\text{the knot trace}} \cup_\varphi V$

note: $-\partial(X(-K')) \cong S^3_0(K')$

· Think of (X, V) as a relative handle decomposition.

$$X = V \cup \underbrace{2\text{-handle}}_{\text{along } \varphi(\mu_{K'})} (0\text{-framed}) \cup 4\text{-handle}.$$

($\mu_{K'}$: meridian)

$\Rightarrow \pi_1(X) = \pi_1(V) / \langle [\varphi(\mu_{K'})] \rangle = 1.$ $\Leftarrow$ $\begin{cases} \pi_1(S^3_0(K')) / \langle [\mu_{K'}] \rangle = 1 \\ \Downarrow \\ \pi_1(\partial V) / \langle [\varphi(\mu_{K'})] \rangle = 1. \\ \text{(generates } \pi_1(V)\text{)} \end{cases}$

· $H_*(X) \cong H_*(W)$

$\Rightarrow \boxed{X \simeq_{htpy} W.}$

- K' bounds a disc D in $X^\circ(-K) \rightsquigarrow \begin{cases} \text{survives to } X \\ [D]=0 \in H_2(X,\partial X). \end{cases} \Rightarrow H\text{-slice}.$

$\boxtimes$

<u>Now</u>: $X \simeq_{Top} W$, K' H-slice in W (①+②+ thm)
K' not H-slice in X ((③)) $\Rightarrow X \not\cong_{diff} W$.

Thus find such a pair $(K,K') \Rightarrow$ find an exotic S^4 or $\#^n \mathbb{CP}^2$.

Unfortunately, no such pair has been found.

$\Rightarrow$ Take a step back satisfying 2 out of 3 conditions and the last is undetermined.

<u>Two constructions by Manolescu & Piccirillo</u>.

<u>I. Using RBG links</u>.

$S^3_0(K) \xrightarrow{\cong} S^3_0(K')$

<u>Idea</u>: Find a pair (K,K') s.t. $\begin{cases} \text{① associated to a RBG link} \\ \text{② Sliceness of } K \text{ is } \underline{\text{not determined}} \\ \text{③ } \underline{s(K')} < 0 \Rightarrow \text{obstructs } K' \text{ from being } H\text{-slice.} \end{cases}$

Rasmussen's s-invariant.

In particular, they used a family of <u>small-RBG links</u> $L(a,b,c,d,e,f)$ (called Manolescu-Piccirillo family of knot) w/ $R=U$ to construct ①.

unknot.

<u>However</u>, Nakamura proved (2022) that in this case, K is never H-slice.

<u>Thm</u>: For a small RGB link L w/ $R=U$, w/ associated homeomorphism $\varphi: S^3_0(K) \longrightarrow S^3_0(K')$

- If K is H-slice in $\#^n \mathbb{CP}^2 \Rightarrow s(K') \geq 0$.
- If K is H-slice in $\#^n \overline{\mathbb{CP}^2} \Rightarrow s(K') \leq 0$.

- If K is slice $\Rightarrow s(K') = 0$.

This eliminates the possibility of Manolescu/Piccirillo's attempt of construction.

<u>What is still left open?</u>

- the Manolescu-Piccirillo family $L(a,b,c,d,e,f)$ might still provide a pair (K, K') yielding an exotic manifold, <u>but</u> we have to use obstructions <u>other than Rassmussen's s-invariant</u> to satisfy property ③.

- Rassmussen's s-invariant can still be used to obstruct sliceness of K' for general pairs of (K, K') (or even those yielded by general RBG links)

<u>Details</u>:

<u>key</u>: $\{\text{RBG links}\} \longleftrightarrow \left\{\begin{matrix}\text{0-surgery}\\ \text{homeomorphism}\end{matrix}\right\}$.

<u>defn</u>: (RBG links).

An RBG link $\mathcal{L} = R \cup B \cup G \subseteq S^3$ is a 3-component <u>rationally framed link</u> w/ framings r, b, g s.t.
$$\begin{cases} \psi_B : S^3_{r,g}(R \cup G) \xrightarrow{\cong} S^3 \\ \psi_G : S^3_{r,b}(R \cup B) \xrightarrow{\cong} S^3 \\ H_1(S^3_{r,b,g}(R \cup B \cup G); \mathbb{Z}) \cong \mathbb{Z}. \end{cases}$$

<u>Thm</u>: $\{\text{RBG links } \mathcal{L}\} \longleftrightarrow \left\{\begin{matrix}(K, K') \text{ s.t.} \\ \varphi : S^3_0(K) \xrightarrow{\cong} S^3_0(K')\end{matrix}\right\}$.

<u>proof</u>:

$(\rightarrow)$: define $\begin{cases} K_B \text{ to be the knot s.t. } \psi_B : S^3_{r,\infty,g}(\mathcal{L}) \xrightarrow{\cong} S^3_*(K_B) \\ f_b \text{ to be the knot s.t. } \psi_B : S^3_{r,b,g}(\mathcal{L}) \xrightarrow{\cong} S^3_{f_b}(K_B) \end{cases}$

Similarly define K_G and f_g.

$\psi_B \circ \psi_G^{-1}$ is the $\varphi_{\mathcal{L}}$ we are looking for.

($f_b = f_g = 0$ by homology condition)

$(\leftarrow)$ define $B = K'$, $b = 0$. (R, r) be the image of $(\mu_K, 0)$ under φ.

$\Rightarrow h \circ \varphi^{-1} : S^3_{0,r}(B, R) \xrightarrow{\cong} S^3$ is a homeo (slam-dunk homeo).

define $(G, g) = 0$-framed meridian of R.

$\square$

Problem: Trace embedding Lemma.

1) If φ extends to trace diffeomorphism, then K' is slice iff K is.

- In particular, we are interested in φ s.t. it extends to trace homeomorphism but not trace diffeomorphism.

reason: It is well-known $\begin{cases} \text{when } \varphi \text{ does not extend to trace homeo} \\ \text{when trace homeo extends to diffeo.} \end{cases}$

2) For RBG links $\mathcal{L}$ in full generality $\begin{cases} \text{problem 1) is hard to fix.} \\ \text{the associated knots } K, K' \text{ are complicated.} \end{cases}$

solution: We study {small RGB links} $\subseteq$ {special RGB links} $\subseteq$ {RBG links}

In details,

- When trace homeomorphism?

defn (parity)

$\varphi: S^3_0(K) \xrightarrow{\cong} S^3_0(K')$ is even if $X(K') \cup_\varphi -X(K)$ has even intersection form, and is odd otherwise.

An RBG link $\mathcal{L}$ is even (odd) if the associated 0-surgery homeomorphism is even (odd).

Thm. $\varphi: S^3_0(K) \longrightarrow S^3_0(K')$ extends to a trace homeomorphism $\Phi: X(K) \longrightarrow X(K')$ iff φ is even.

proof of ($\Rightarrow$):

φ extends to $\Phi: X(K) \xrightarrow{\cong} X(K')$

$\Rightarrow W := X(K) \cup$ 0-framed 2-handle along μ_K has even intersection form.

$\downarrow \cong$

$Z := X(K') \cup_\varphi -X(K)$ $\Rightarrow$ Z has even intersection form $\blacksquare$

Lemma. $\varphi: S^3_0(K) \xrightarrow{\cong} S^3_0(K)$ odd $\Rightarrow Arf(K) = Arf(K') = 0$.

proof.

Combine the two results, we have that

- $\mathrm{Arf}(K)=1 \Rightarrow$ every $\varphi: S^3_0(K) \xrightarrow{\cong} S^3_0(K)$ extends to a trace homeomorphism

- <u>When trace diffeomorphism?</u>

<u>defn</u> (Property $\mathcal{U}$)

Given $\varphi: S^3_0(K) \xrightarrow{\cong} S^3_0(K')$, denote $\underline{\gamma} \subseteq S^3_0(K')$ be the framed knot given by $\varphi((\mu_K, 0))$.

Then φ <u>has property $\mathcal{U}$</u> if $\exists$ a diagramatic choice of γ in the standard diagram of $S^3_0(K')$ s.t. γ is 0-framed and is unknotted.

<u>Theorem</u>: Property $\mathcal{U} \Rightarrow \exists$ diffeomorphism $\Phi: X(K) \xrightarrow{\cong} X(K')$ s.t. $\Phi|_\partial = \varphi$.

<u>proof</u>: define
$$\begin{cases} X \subset X(K) \text{ as the tubular neighbourhood of core} \cong D^2 \times D^2. \\ X' \subseteq X \quad \cong D^2 \times \mathring{D}^2 \\ Y := \nu(D) \subseteq X(K') \text{ w/ } D \text{ is the standard slice disk} \cong D^2 \times D^2 \\ Y' \subseteq Y \quad \cong D^2 \times \mathring{D}^2 \end{cases}$$

F' is the natural bundle diffeomorphism $X \xrightarrow{\cong} Y$

<u>note</u>: $\begin{cases} X(K) \setminus X' \cong B^4 & \text{obviously} \\ X(K') \setminus Y' \cong B^4 & \text{proved later.} \end{cases}$

<u>Then</u>, $\varphi|_{\partial X(K) \setminus \nu(\mu_K)} \cup F'|_{D^2 \times \partial D^2} =: f$ gives a piecewise homeomorphism $\underset{\substack{\cong \\ S^3}}{\partial X_1} \xrightarrow{\cong} \underset{\substack{\cong \\ S^3}}{\partial Y_1}$.

This extends to a diffeomorphism $F: \underset{\substack{\cong \\ B^4}}{X(K) \setminus X'} \xrightarrow{\cong} \underset{\substack{\cong \\ B^4}}{X(K') \setminus Y'}$.

(since $\exists !$ isotopy class of homeo $S^3 \to S^3$).

Why is $X(K')\setminus Y' \cong B^4$?

$X(K')\setminus Y'$ has boundary S^3 and standard handle decomposition $\underline{K}'$ and carving out $\varphi(\mu_K)$.

$\rightsquigarrow$ the boundary is naturally described as $S^3_{0,0}(K' \cup \varphi(\mu_K))$.

- $S^3_0(\varphi(\mu_K)) \cong S^1 \times S^2$.

  $\rightsquigarrow$ Can think of $\partial(X(K')\setminus Y')$ as S^3 obtained by some knot l in $S^1\times S^2$.

  $\Rightarrow l$ is isotopic to $S^1 \times *$. [Gabai's proof]

This isotopy gives rise to a cancelling 1-2 pair. ✓.

$\square$

- **Special RBG links.**

  defn: An RBG link L is *special* if $b=g=0$, $r\in\mathbb{Z}$, and

  $$R\cup B \cong R\cup \mu_R \cong R\cup G.$$

  remark: For a special RBG link, we can build K_G/K_B as follows:

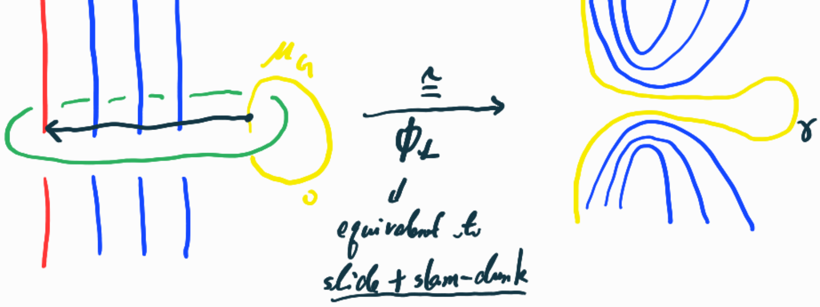
  - Think of every knot as a 2-handle
    - Slide G over R (multiple times) until G no longer intersects the disk Δ_B bounded by B. $\Rightarrow K_G$.
    - Cancel $R\frown B$
    - Similarly, reversing the role G and B, we have K_B.
  - The crossing numbers of the associated knots K_B and K_G is highly related to $|\Delta_B \cap G|$ and $|\Delta_G \cap B|$.

Lemma: A special RBG link is even iff r is.

proof: $X(K_B)\cup_{\varphi_L} X(-K_G) = \underbrace{K_B \cup \varphi_L((\mu_{K_G},0))}_{\text{2-handles}} \cup$ 4-handle.

$\Rightarrow$ intersection form is even iff the framing of γ is even.

claim that this framing is r. (γ is a "framed ghost" of R)



$\square$

Lemma: If $R = \mathcal{U}$, $r = 0$ in a special RBG link, then φ_L has property $\mathcal{U}$.

proof: Exactly as before, γ has the knot type & framing of R. $\square$

- **Small RBG links**.

defn: A special RBG link is small if.

- B bounds a properly embedded disc Δ_B. $\begin{cases} |\Delta_B \cap R| = 1 \\ |\Delta_B \cap G| \leq 2. \end{cases}$
- G bounds a properly embedded disc Δ_G. $\begin{cases} |\Delta_G \cap R| = 1 \\ |\Delta_G \cap B| \leq 2. \end{cases}$

(All intersections are transverse).

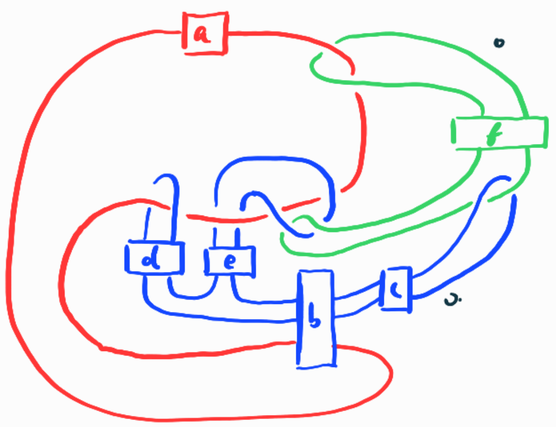
remark: This definition is established to minimize the crossing number of the associated K_B / K_G — given by the following Lemma.

Lemma: if L small RBG link w/ $|\Delta_B \cap G| < 2$, then $K_B = K_G$.

proof omitted?

$\square$

Manolescu-Piccirillo 6-parameter family.



- $r = a + b$
- This is small-RBG link.
- $a, c, e \in [-2, 2]$, $b, d, f \in [-1, 1]$ $\Rightarrow$ crossing # of K_B, $K_G \leq 55$.
- $b = -1 \Rightarrow G$ and B are symmetric $\Rightarrow K_G \cong K_B$ ✗.
- $a + b = 0 \Rightarrow$ property $\mathcal{U} \Rightarrow$ diffeo ✗.

Mathematica / Sage / Snappy $\Rightarrow$ 24 pairs of knots w/ $\begin{cases} s(K) = 0 \\ s(K') = -2 \end{cases}$

(23 of which have undetermined sliceness).

- 5 of which has vanishing Alexander Polynomial $\Rightarrow$ possibly slice.
- the rest 18 are possibly $\mathcal{H}$-slice in $\#^n \mathbb{CP}^2$.

$\boxed{\text{Nakamura's Work (2022)}}$: all of these are not $\mathcal{H}$-slice in $\#^n \overline{\mathbb{CP}}^2$.

<u>defn</u> (sliceness for framed knots).

A <u>framed knot</u> $(K, k) \subseteq S^3$ is slice in W (or W°) if K bounds a smoothly, properly embedded disk D which <u>induces the framing k</u> <u>on K</u>.

push off K in $\nu(D)$ defines a parallel knot $\Downarrow$ defines k.

<u>Lemma</u>: (framed trace embedding lemma).

A framed knot (K, k) in S^3 is smoothly slice in W iff $-X_k(K)$ smoothly embeds in W.

<u>Lemma</u>: (<u>stable</u> trace diffeomorphism)

W a smooth, closed, oriented 4-manifold.

$L = (R, r) \cup (B, 0) \cup (G, 0)$ a special RBG link. w/ $\varphi_L : S^3_0(K_B) \xrightarrow{\cong} S^3_0(K_G)$

Then (R, r) slice in $W \Rightarrow \varphi_L$ extends to $\Phi_L : X_0(K_B) \# W \cong_{\text{diff}} X_0(K_G) \# W$

<u>proof</u>: $Z := \underbrace{W^\circ \setminus \nu(D)}_{\text{slice disk}} \cup \underbrace{B \cup G}_{\text{2-handles}}$. <u>note</u> $\partial Z \cong S^3_{r,0,0}(R, B, G)$

<u>note</u>: slam dunk homeo $\psi_B : S^3_{r,0}(R, G) \xrightarrow{\cong} S^3$ extends to a diffeomorphism

$\Psi_B : (W^\circ \setminus \nu(D)) \cup_{(G,0)} 2\text{-}h \longrightarrow W^\circ$.

$\Rightarrow$ induces a diffeomorphism $Z \xrightarrow{\cong} W^\circ \cup_{(K_B,0)} 2\text{-}h$

$\cong \downarrow$ τ as the image $\psi_B(B, 0)$.

Similarly, $Z \xrightarrow{\cong} W \# X_0(K_G)$ extending ψ_G.

$W \# X_0(K_?)$

extend ψ_B $W \# X_0(K_B)$

(since $W^\circ = W \# B^4$)

$\Rightarrow X_0(K_B) \# W \cong X_0(K_G) \# W$ extending $\varphi_L = \psi_B \circ \psi_G^{-1}$.

$\boxtimes$

Corollary: X smooth, closed, oriented. K_B H-slice in $X \leftrightsquigarrow K_G$ H-slice in $(X \setminus \nu(D)) \cup_{\varphi_L} -X_0(K_G) =: X'$. If (R, r) is slice in W, then $X' \# -W \cong X \# -W$, and thus K_G is H-slice in $X \# -W$.

Theorem: The 23 knots $K_1, \dots, K_{23}$ are NOT H-slice in $\#^n \mathbb{CP}^2$ $\forall n$.

proof: $(R, r) = (\mathcal{U}, r) \Rightarrow$ slice in $W := \#^r \overline{\mathbb{CP}}^2$. note: $r \geq 0$ in all 23 knots.

Then K_i H-slice in $\#_n \mathbb{CP}^2 \Rightarrow K_i'$ H-slice in $\#^{(n+r)} \mathbb{CP}^2$.

$\Rightarrow s(K_i') \geq 0$ $\times$ yielding a contradiction for $s(K_i') = -2$

$\boxtimes$

Problem: The proof only works when $r \geq 0$.

- If we want to generalize the theorem to general small RBG links, we need to discuss the case $r < 0$.

Lemma: Suppose L is a small RBG-link w/ $R = \mathcal{U}$ w/ $r < 0$. Then both K and K' are (-1)-slice in $\#^{|r|} \mathbb{CP}^2$ w/ slice disks $D, D' \subset (\#^{|r|} \mathbb{CP}^2)^\circ$ that intersects
$\begin{cases} \text{one of the } |r| \text{ exceptional spheres } \mathbb{CP}^1 \text{ geometrically in 3 pts} \\ \text{remaining } (|r|-1) \text{ exceptional spheres } \mathbb{CP}^1 \text{ nullhomologously.} \end{cases}$

proof: WLOG, assume $|\Delta_B \cap G| = 2 = |\Delta_G \cap B|$ (else it just yield not interesting cases).

By induction on r.

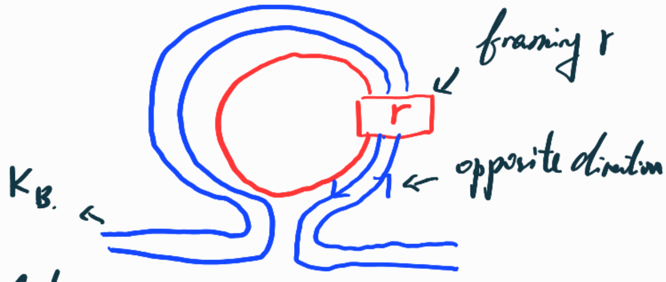
Inductive step: $(r < -1)$.

observe: For special RGB link L, $H_1(S^3_{r,b,g}(L)) \cong \mathbb{Z}$

$\Rightarrow \det(\text{linking matrix}) = \begin{vmatrix} r & 1 & 1 \\ 1 & 0 & l \\ 1 & l & 0 \end{vmatrix} = l \cdot (2 - rl) = 0.$

$\Rightarrow rl = 2$ or $l = 0$

* In this case, $|\mathcal{U}| = 0$ or 2 (2 intersections w/ signs), and $r < -1$
  $\Rightarrow r\ell \neq 2 \Rightarrow \ell = 0$. ($B, G$ are unlinked).
* ℓ records the number of slides needed to get K_B (counted w/ signs).
  $\Rightarrow$ locally, it looks like.



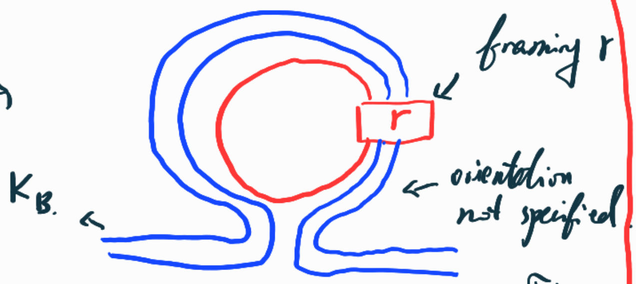
- Let $\mathcal{L}^*$ be the similar RBG link w/ $r \rightsquigarrow r+1$
  $\Rightarrow$ The resulting knot K_B^* yields from ↙ but w/ <u>r+1</u> framing.

$\Rightarrow$. K_B is obtained by K_B^* w/ an extra null-homologous, negative twist.

• If $(K_B^*, -1)$ bounds disk D^* in $\#^{|r+1|}\mathbb{CP}^2$ intersecting a $\mathbb{CP}^1$ in 3 pts
  $\Rightarrow$ Construct D for $(K_B, -1)$ as follows:
  * $D \subseteq |r|\,\mathbb{CP}^2$ is D^* attaching a $(+1)$-framed 2-handle to $\#^{|r+1|}\mathbb{CP}^2$ along <u>an unknot surrounding the twist box</u>
  * <u>note</u> $\begin{cases} (\#^{|r+1|}\mathbb{CP}^2)^\circ \cup 2\text{-}h \cong (\#^{|r+1|}\mathbb{CP}^2)^\circ \\ \partial D = K_B^* \text{ w/ an extra twist} \end{cases}$ ✓.

<u>Base case</u>: $(r = -1)$

The picture in this case is ⟶



Think of E as concordance.
$$\begin{cases} S^3 \times I \underset{\substack{(R,r) \\ = (\mathcal{U},-1)}}{\cup} 2\text{-}h \cong \overline{\mathbb{CP}^2} \setminus \mathrm{int}(B^4 \amalg B^4) \\ \text{define } E \subseteq \overline{\mathbb{CP}^2} \text{ to be the image of } K_B \times I. \end{cases}$$

Think of "sliding" as concordance $\rightsquigarrow$. E as concordance btw $(\mathcal{U}, 0)$ and K_B
$$\begin{cases} \text{slide } E \text{ over the 2-handle} \rightsquigarrow (K_B, -1) \\ \text{Undo/reverse the two } \underline{\text{slam-dunk slides}} \Rightarrow \\ \underline{\text{Slide}}\ (B, -1) \text{ off } (R, r) \Rightarrow \underline{(\mathcal{U}, 0)} \end{cases}$$

in $\mathbb{CP}^1$.

Turning $\bar{E}$ upside down and connect it w/ the unknot slice disc.

$\Rightarrow$ get the slice disk $\underline{D}$. (3 intersections because 3 slides) $\boxed{\square}$

<u>Corollary</u>: L small RBG link w/ $R = \mathcal{U}$, $r < 0$. Then $s(K) \cdot s(K') \geq 0$.

<u>proof</u>:

<u>Lemma</u> [MMSW 19] If $\Sigma \subset \#^n \mathbb{C}P^2 \setminus (\mathrm{int}(B^4 \amalg B^4))$ is a null-homologous oriented cobordism from L_1 to L_2, with each component of Σ having a boundary component in L_1, then $s(L_2) - s(L_1) \geq \chi(\Sigma)$.

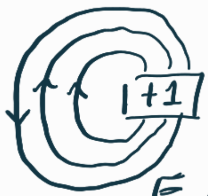
$\Rightarrow$ K H-slice in $\#^n \mathbb{C}P^2 \Rightarrow s(K) \geq 0$.

<u>proof</u>: Omitted.

Let D be the disc in the above lemma. D intersects a $\mathbb{C}P^1$ in 3 pts.

Delete a suitably small neighbourhood $\nu(\mathbb{C}P^1)$ s.t.

$$D \cap \partial \nu(\mathbb{C}P^1) = F_{2,1}(1). \quad \text{[MMSW 19]}$$


$F_{2,1}(1)$.

$\Rightarrow$ D is a homologous cobordism C in $\#(|r|-1)\mathbb{C}P^2$ from $\underline{-F_{2,1}(1) \to K}$

$$\Rightarrow s(K) - \underset{\substack{\| \\ -2}}{s(-F_{2,1}(1))} \geq \chi(C) = -2 \Rightarrow s(K) \geq 0.$$ $\boxed{\square}$

<u>Thus</u>,

<u>Thrm</u>: For a small RGB link L w/ $R = \mathcal{U}$, w/ associated homeomorphism $\varphi: S^3_0(K) \to S^3_0(K')$

- If K is H-slice in $\#^n \mathbb{C}P^2 \Rightarrow s(K') \geq 0$.
- If K is H-slice in $\#^n \overline{\mathbb{C}P^2} \Rightarrow s(K') \leq 0$.
- If K is slice $\Rightarrow s(K') = 0$.

This eliminates the possibility of ...

This eliminates the possibility of Manolescu/Piccirillo's attempt of construction.

Problem, can we generalize this non-existing result to general special RBG links?

Answer. depends on the conjecture.

Conjecture: K (-1)-slice in $\#^n \mathbb{CP}^2 \overset{?}{\Rightarrow} s(K) \geq 0$?

The associated part for the corollary above is:

Lemma (?) If $r < \underline{\mathbb{PF}_-(R)}$, then K, K' are both (-1)-slice in $\#^n \mathbb{CP}^2$ $\overset{?}{\Rightarrow} s(K), s(K') \geq 0$.

↪ the largest negative framing r s.t. (R, r) is slice in some $\#^n \mathbb{CP}^2$.

Similarly define $\mathbb{PF}_+$

* This exists by a stronger version of

Thus if the conjecture is true,

Thm. $\varphi: S^3_0(K) \xrightarrow{\cong} S^3_0(K')$, arising from special RBG-link, then ?

- $r < \mathbb{PF}_-(R)$ or $r \geq \mathbb{PF}_+(R)$, K H-slice in $\# n \mathbb{CP}^2 \Rightarrow s(K) \geq 0$
- $r \leq \mathbb{PF}_-(R)$ or $r \geq \mathbb{PF}_+(R)$ ———— $\overline{\mathbb{CP}^2} \Rightarrow s(K) \leq 0$.
- $r < \mathbb{PF}_-(R)$ or $r > \mathbb{PF}_+(R)$, K slice $\Rightarrow s(K') = 0$.

Annulus Twist. $\Rightarrow$ An infinite family of knots $\{K_n\}$ s.t. $S^3_0(K_i) \cong S^3_0(K_j)$.

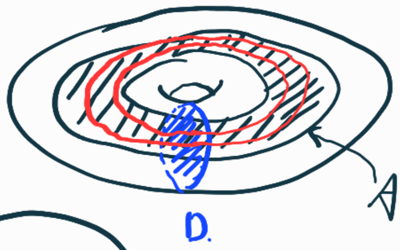
- Start with a ribbon knot. $K = K_0$
- Exotic S^4 problem $\Rightarrow$ $\{K_n\}$ is slice?

Construction: [Osoinach 06'].

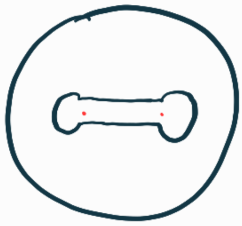
Thm: Let A be an annulus in the std solid torus V. Let $V' := V_{1/n, -1/n}(\partial A)$, $n \in \mathbb{Z}$. Then V' is a solid torus, and any meridian for V is a meridian for V'.

proof: (sketch).

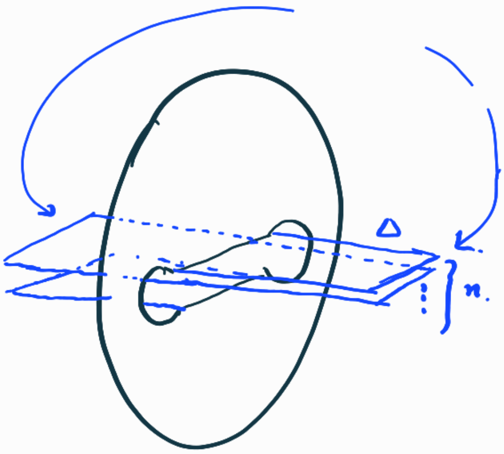
We map a meridian disk $D \subseteq V$ to $D' \subseteq V'$ constructed as follows.



- Remove $\nu(\partial A) \rightsquigarrow$ have annulus $A_c \subseteq A$
- Remove $\nu(A_c) \rightsquigarrow D$ becomes



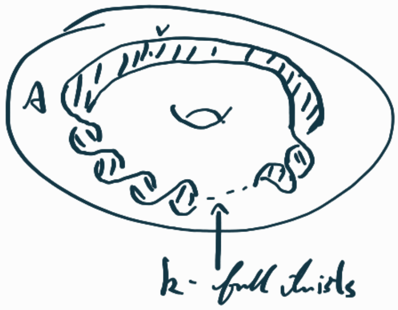
- Add in a disk Δ as follows.



- D becomes a twice punctured disk (pair of pants).
- Cap off the 2 punctures by $\pm 1/n$. Dehn Surgery. $\rightsquigarrow D'$.

$\blacksquare$

Corollary. $V_{k \pm 1/n}(\partial A)$ is solid torus, and any meridian for V is a meridian for V'.



proof: Exactly as before except for slopes.

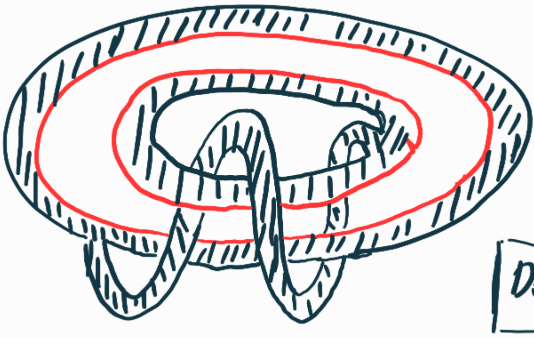
$\blacksquare$

Put into the case of 3-manifolds. (i.e. $V = \nu(K)$).

start with A.

Add in A_c. (red)
($\partial A_c = C_1 \cup C_2$)

Band sum the curves.
⇝ get knot K.



is a pair of pants  w/ boundary $C_1 \cup C_2 \cup K$.
$= A'$.

$\boxed{\text{Doing zero-surgery on } K} = \boxed{\text{Capping off } A' \text{ along } K.}$

Twisting along A_c in $S^3_0(K)$ $\Rightarrow$ $S^3_0(K) \xrightarrow{\cong} S^3_{0,\pm 1/n}(K, C_1, C_2)$

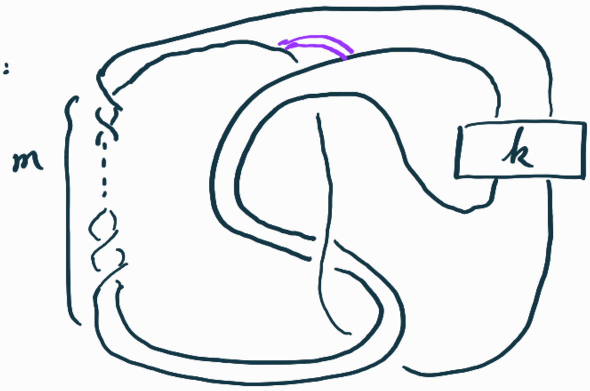
Theorem $\Rightarrow$ $\exists$ homeos $f_n : S^3_{\pm 1/n}(C_1, C_2) \xrightarrow{\cong} S^3$. (twisting along A_c n times)

<u>defn</u>: $K_n := f_n(\iota_n(K))$, w/ $\iota_n : S^1 \to S^3_{\pm 1/n}(C_1, C_2)$ is the embedding given by the above diagram.

Surgering along the knot yields $S^3_{0,\pm 1/n}(K, C_1, C_2) \xrightarrow{\cong} S^3_0(K_n)$
↑ again by homology condition.

Combine everything ⇝ $\boxed{S^3_0(K_n) \cong S^3_0(K)}$

<u>e.g.</u>:



$K = J_m[k]$

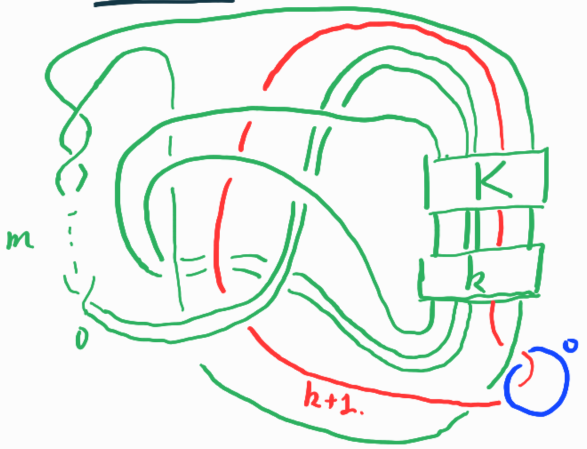
- the embedding of $\partial A \to S^3$ are unknots.
- k twists for A_c.
- m twists for K.
- <u>ribbon</u> if $m = 3 - 2k$

$J_m[k]_1$

$J_m[k]_{-1}$.

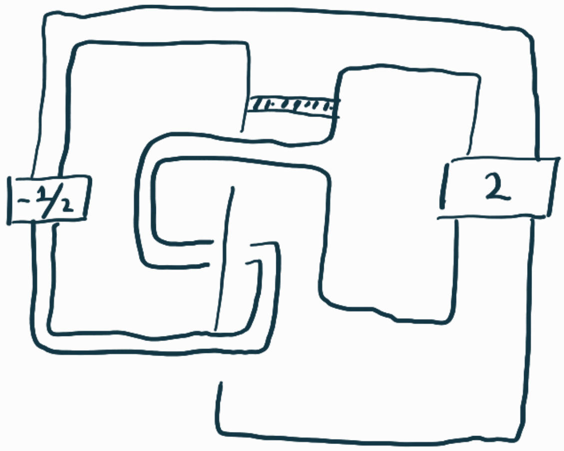
Again, we search for pairs of knots $J_m[k]$ and $J_m[k]_n$ s.t. only one of them is slice. $\rightsquigarrow$ Yields $X := X(-K') \cup_\varphi E(D)$ as htpy 4-sphere.
↖ slice disk exterior

<u>Remark</u>: Each annulus twisting knot K $\rightsquigarrow$ associated RBG link.

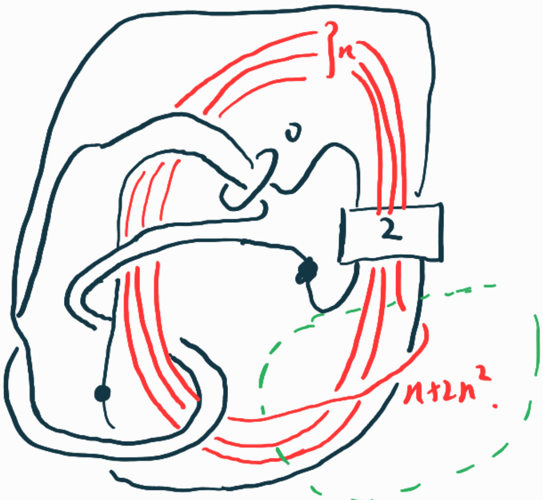


• is <u>NOT</u> special if <u>$K \neq U$</u> or $k \neq \pm 1$.

<u>One specific example</u>: $K = J_{-1}[2] = 8_8$ is ribbon $\rightsquigarrow$ slice.



$\rightsquigarrow$



change every crossing if $n<0$.

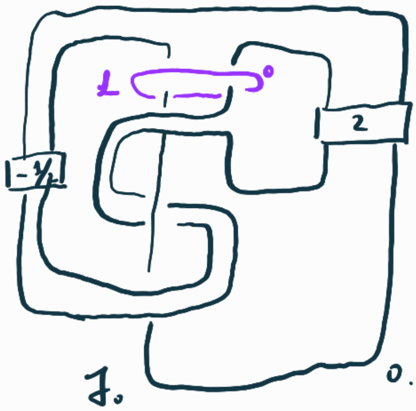
Homotopy sphere associated to $(8_8)_n =: Z_n$.

<u>Nakamura</u>: All Z_n's are standard $\Rightarrow$ all $(8_8)_n$ are slice.

<u>Trick</u>: Draw the Kirby diagram of Z_n upside down!

• By defn. $Z_n = E(D_0) \cup -X_0(J_n) \rightsquigarrow -Z_n = X_0(J_n) \cup -E(D_0)$.

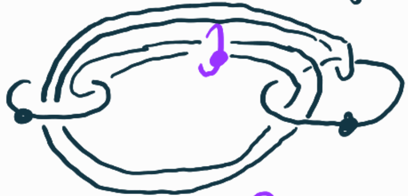
• Z_0 is the standard S^4 w/ kirby diagram.



J_0

$\cup$ 3-4-handles.

(reason: For ribbon knots, e.g. 6_1

$Z(D) =$ 

$\cong$ 

$\cong$ 

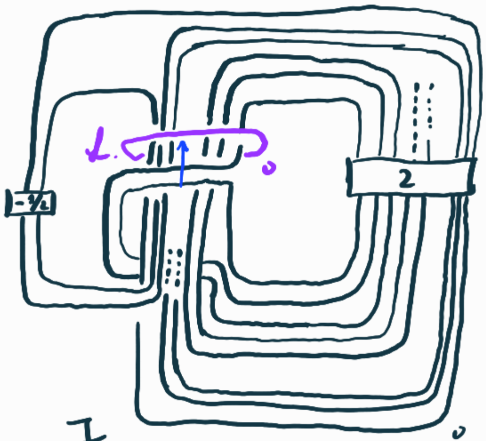
$\rightsquigarrow$ Become 3 and 4-handles after turning upside down.

$\rightsquigarrow X_0(K) \cup -E(D)$ look like 

$\cup$ 3-4.)

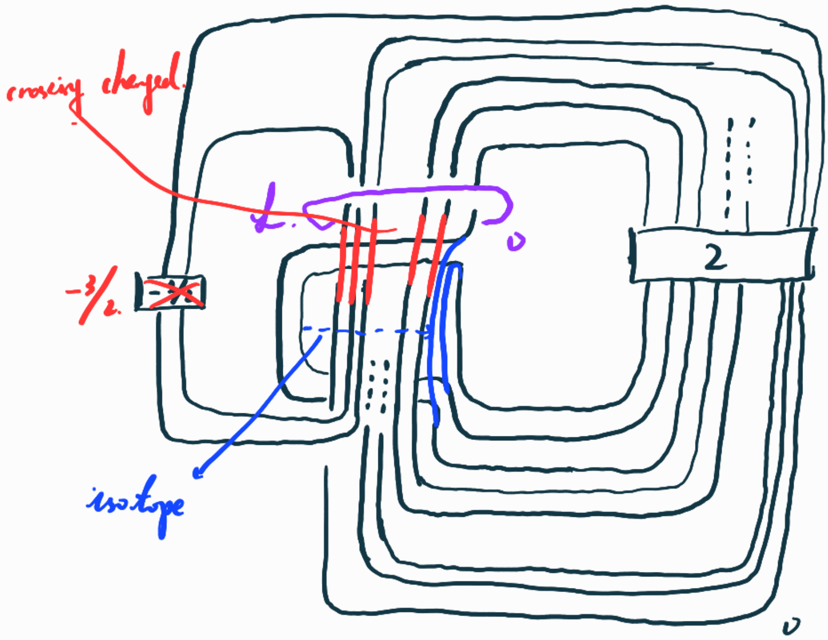
• Z_n has Kirby diagram

(reason: annulus twist homeomorphism is supported in $\nu(A)$ disjoint from $\mathcal{L}$).

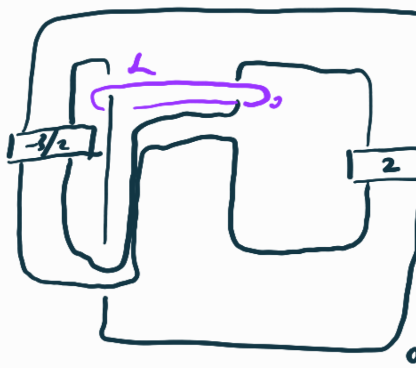


J_n

• Do the blue slide. yield.



• Isotope as blue arrow and unwind. yield



• Thus all Z_n has same kirby diagrams, thus is $Z_0 \cong S^4$.

Summary

(-1)-slice in $\overline{\mathbb{CP}}^2 \Rightarrow s \geq 0$

RBG links
- special RBG link: Depend on a conjecture. Cannot yield exotic structure unless $r \in [\mathbb{PF}_-(R), \mathbb{PF}_+(R)]$.
- general.
  - coming from annulus twisting: Cannot yield exotic structure as long as no known slice knots not ribbon.
  - more general: Too complicated/no viable algorithm.