Combined notes for exotic Smooth Structures on Closed 4-manifolds

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Chapter 1

Knot Surgery

In this chapter, we will discuss the first method of producing exotic closed 4-manifolds: knot surgery. The main idea is to cut out a homologically interesting thickened torus, and replace it by a homologically non-distinguishable thickened S^3-knot complement.

To study knot surgeries, the two main tools that we will use are the Seiberg-Witten invariants, and the Alexander polynomial. The basics of Seiberg-Witten theory are discussed in the appendix, while some specific applications (like the Laurent series form and its gluing results) will be discussed in the first section. The rest of the section 1 will be focusing on other definitions and properties we might need during the study of knot surgeries.

In section 2, we will rigorously define the knot surgery, and state the main theorem first proved by Fintushel and Stern in [5], the proof of which will be delayed till section 5. In section 3, we will see some examples and applications of the knot surgery. And at last in section 4, we will study the action of knot surgeries on Kirby diagrams. The basics of Kirby diagrams are also discussed in the appendices.

1.1 Preparations

1.1.1 The Near Cusp Embedded Torus

1.1.2 The Laurent Series Form of Seiberg-Witten Invariants

1.1.3 Alexander Polynomials

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Appendix A

A Note on Seiberg-Witten Invariants

A.1 Overview

The Seiberg-Witten invariant on a smooth 4-manifold X is a smooth structure invariant that takes the form of a function: $\mathcal{SW}_X : \{\text{Spin}^{\mathbb{C}} \text{ structures on } X\} \to \mathbb{Z}$. The invariant can be defined for any smooth 4-manifold X but only provides non-trivial information for spaces X such that $b^+(X) + b_1(X) + 1$ is odd. The Seiberg-Witten invariants play an important role for distinguishing exotic smooth structures of a closed 4-manifold.

Seiberg-Witten Theory is an easier gauge theory approach to the topology of 4-manifolds campared to Donaldson Theory. Seiberg-Witten was originally developed based on theoretical Physics ideas: the magnetic monopoles equations.

A.2 Backgrounds

To understand Seiberg-Witten theory, it is essential to understand:

- Almost complex structures.
- $\text{Spin}^{\mathbb{C}}$ structures.

A.2.1 Almost Complex Structures

A 4-manifold M is said to have an almost complex structure if its tangent bundle τ_M is a complex plane bundle. Equivalently, M has an almost complex structure if there exists a bundle automorphism $J : \tau_M \to \tau_M$ that is fibre-preserving and that $J^2 = -id$.

Definition A.1 (Integrable)**.** An almost complex structure is said to be integrable if it comes from an actual complex manifold.

Chern classes

The idea of an almost complex structure is to generalize the notion of a complex manifold. One important invariant about complex manifolds is the Chern classes. However, it is not necessary to have a complex structure to define a Chern class for M. Instead, an almost complex structure is enough.

Indeed, the second Chern class $c_2(\tau_M)$ is equal to the Euler class of the tangent bundle $e(\tau_M)$. Moreover, we define the first Chern class of the almost complex structure as the first Chern class of the tangent bundle $c_1(J) := c_1(\tau_M)$. Note that this first Chern class is exactly the the first Chern class corresponding to the determinant line bundle K_J^* of the tangent bundle.

J-invariant Curves

The idea of J-invariant curves in almost complex 4-manifolds is a generalization of the notion of complex curves in complex surfaces. A curve S is said to be J-invariant if its tangent subbundle is invariant under J, i.e. $J[\tau_S] = \tau_S$. (Of course we can similarly define a J-invariant normal bundle). For a J-invariant curve S, we alwasy have the following identity:

$$\chi(S) + S \cdot S = c_1(J) \cdot S$$

. This is called the adjunction formula for S.

Existence of Almost Complex Structures

We know that for an almost complex structure J, the identity $c_1(J) \cdot c_1(J) = 3sign(M) + 2\chi(M)$ holds.

The converse is in fact also true. An almost complex structure exists if and only if there exists an integral lift $\underline{w}$ of the second Stiefel-Whitney class $w_2(M)$ satisfying $\underline{w} \cdot \underline{w} = 3sign(M) + 2\chi(M)$. In this case, the integral lift $\underline{w}$ can be seen as the first Chern class of the corresponding almost complex structure.

Note that if the manifold M is simply connected or indefinite, such an integral lift exists if and only if $b^+ + b_1$ is odd.

Also note that an integral lift (satisfying the above identity or not) always define a partial almost complex structure on the three skeleton for the smooth manifold. This partial almost complex structure is in fact a Spin$^{\mathbb{C}}$-structure on M, which is discussed in the next section.

Two out of Three

For the three different structures on the smooth manifold M: almost complex structures, Riemannian metrics, and an exterior 2-form satisfy the two-out-of-three rule: whenever given two of the three structures compatible with each other, they together determines the third.

1) Almost complex + Riemannian $\Rightarrow$ 2-form:
2) 2-form + Riemannian $\Rightarrow$ Almost complex :
3) Almost-complex + 2-form $\Rightarrow$ Riemannian:

A.2.2 Spin$^{\mathbb{C}}$ structures.

A.3 Seiberg-Witten Invariants

We fix a smooth manifold M, a Riemannian structure g, and a $Spin^{\mathbb{C}}$ structure $\mathfrak{s}$ on M. Then we observe the following set of Seiberg-Witten Equations (following the notation from last section):

$$\begin{cases} \mathcal{D}^A \varphi = 0 \\ F_A^+ = \sigma(\varphi) \end{cases}$$

, where A is a $U(1)$ connection, $\varphi \in \Gamma(\mathcal{W}^+)$ is a self-dual spinor field, $\mathcal{D}^A$ is the Dirac operator mentioned earlier, and σ is the squaring map mentioned before.

The solutions of the set of PDEs are called **Seiberg-Witten monopoles**, with its name coming from the monopole equations in Physics. The space of monopoles is denoted as $\mathfrak{S}$, which can be embedded in the configuration space $\Gamma(\mathcal{W}^+) \times \mathcal{A}$, where $\mathcal{A}$ is the space of all connections (Following the notation of [2]). The gauge equivalence classes $\mathcal{M} := \mathfrak{S}/\mathcal{G}$ is called the Seiberg-Witten moduli space.

We wish to define a numerical invariant (depending only on the chosen $Spin^{\mathbb{C}}$ structure) on the 4-manifold using Seiberg-Witten moduli space. The main idea is to use the moduli space to represent a homology class in the configuration space, but we need a bit more restrictions, which goes as follows:

- We wish to avoid reducible solutionsin the moduli space $\mathcal{M}$ via transversality results. If so, for any two choices of Riemannian metrics in the first step, we have a cobordism in the configuration space. thus we have a numerical invariant independent of the choice of the Riemannian metric.
- The moduli space $\mathcal{M}$ has to be compact.
- The moduli space $\mathcal{M}$ should be orientable since we want to define a numerical invariant other than $\mathbb{Z}/2$.

A.3.1 Transversality

A.3.2 Compactness

A.3.3 Orientability

A.3.4 Definition of Seiberg-Witten Invariants

With the above results, we can finally finish with the definition of the Seberg-Witten invariant. We start with the simply-connected case, and a fact about the ambient space $\Gamma(\mathcal{W}^+) \times \mathcal{A}/\mathcal{G}$.

Fact. When the manifold M is simply connected, the ambient space of gauge equivalence classes of connection-spinor pairs $\Gamma(\mathcal{W}^+) \times \mathcal{A}/\mathcal{G}$ has the homotopy type of $\mathbb{CP}^2$.

Thus its homology ring is $\mathbb{Z}[u]$ with $|u| = 2$. Thus we can define the following:

Definition A.2. Seiberg-Witten Invariant For a simply commected 4-manifold X and a fixed $Spin^{\mathbb{C}}$ structure $\mathfrak{s}$, its Seiberg-Witten moduli space is denoted $\mathcal{M}$. Then the Seiberg-Witten invariant is defined by:

$$\mathcal{SW}_X(\mathfrak{s}) := \begin{cases} \int_{\mathcal{M}} u \wedge ... \wedge u & b_2^+ \text{ is odd } \Leftrightarrow dim(\mathcal{M}) \text{ is even} \\ 0 & b_2^+ \text{ is even } \Leftrightarrow dim(\mathcal{M}) \text{ is odd} \end{cases}$$

Note that for the non-simply connected case, the condition is similar. In this case, the parity of b_2^+ is modified to the parity of $b_1 + b_2^+$.

At last we define an important property called "simple type".

Definition A.3. Simple Type A 4-manifold X is said to be of simple type if all non-empty moduli spaces either:

- are 0-dimensional.
- the associated $Spin^{\mathbb{C}}$ structure $\mathfrak{s}$ comes from almost complex structures.
- the associated $Spin^{\mathbb{C}}$ structure satisfies $c_1(\mathfrak{s}) \cdot c_1(\mathfrak{s}) = 2 \cdot \chi(X) + 3 \cdot sgn(X)$.

We finish this section by a conjecture that all simply-connected 4-manifolds of $b_2^+ \geq 2$ is of simple type.

A.4 Properties and Applications

A.4.1 General properties

We know from the previous section that for 4-manifolds with $b_2^+ \geq 2$, the Seiberg-Witten invariant can be seen as a function $\mathcal{SW}_M : \{Spin^{\mathbb{C}} \text{ structures }\} \to \mathbb{Z}$ (by counting the solutions of Seiberg-Witten equations).

If $H^2(M)$ has no 2-torsion, the set of $Spin^{\mathbb{C}}$ is uniquely characterized by the integral lifts of the second Stiefel-Whitney class w_2. In this sense, we can treat the Seiberg-Witten invariant as a function

$$\mathcal{SW}_M : \{\underline{w} \in H^2(M) | \underline{w} = w_2(M) \text{ mod } 2\} \to \mathbb{Z}$$

.

Also note that if $H^2(M)$ has 2-torsion, then there might be more than one associated $Spin^{\mathbb{C}}$ structures. We can still get a function in the above form by summing up all $Spin^{\mathbb{C}}$ structures associated to $\underline{w}$.

Note. We can also study a parameterized version of the above function. We pick a distinguished $Spin^{\mathbb{C}}$ structure $\mathfrak{s}_o$ and write the function as $\mathcal{SW}_M(\mathfrak{s}_o + \cdot) : H^2(M) \to \mathbb{Z}$

Note. There is another form of the Seiberg-Witten invariants taking the form of Laurent series, which is studied in the section involving knot surgeries.

Definition A.4. Basic classes An integral cohomology class $\kappa \in H^2(M)$ is called a basic class if $SW_M(\kappa) \neq 0$.

An immediate consequence is that any 4-manifold has at most a finite number of basic classes.

Next we list a number of general properties of Seiberg-Witten invariants with only a sketch of the proofs.

Theorem A.5 (Involution Lemma)**.** *If $b_2^+ \geq 2$, and κ is a basic class, then we have $\mathcal{SW}_M(-\kappa) = \pm\mathcal{SW}_M(\kappa)$*

Proof. □

Theorem A.6 (Vanishing Theorem for Scalar Curvature)**.** *If $b_2^+ \geq 2$, and if M admits a Riemannian metric with everywhere-positive scalar curvature, then $\mathcal{SW}_M \equiv 0$.*

Proof. The theorem follows directly from the curvature bound mentioned in the previous sections. □

Theorem A.7 (Vanishing Theorem for Connected Sums)**.** *If a smooth 4-manifold M is diffeomorphic to $N' \sharp N''$, where $b_2^+(N') \geq 1$ and $b_2^+(N'') \geq 1$, then $\mathcal{SW}_M \equiv 0$.*

Proof. The theorem follows from the following three facts.

- All Seiberg-Witten solutions vanish on the connecting cylinder $S^3 \times [0, 1]$.
- $\mathcal{M}_{N' \sharp N''} = \mathcal{M}_{N'} \times \mathcal{M}_{N''}$ for any fixed choice of $Spin^{\mathbb{C}}$ structures on N' and N''.
- $dim\mathcal{M}_{N' \sharp N''} = dim\mathcal{M}_{N'} + dim\mathcal{M}_{N''} + 1$ by the virtual dimension formula.

□

Theorem A.8 (Blow-up Formula)**.** *For a simply-connected 4-manifold M with $b_2^+(M) \geq 2$ and is of simple type. Let $\{\kappa_i\}$ be the set of basic classes of M, then the topological blow-up $M \sharp \overline{CP}^2$ has basic classes $\{\kappa_i \pm E\}$, where E is the class of $\overline{\mathbb{CP}^1} \subset \overline{\mathbb{CP}^2}$. Moreover, we have*

$$\mathcal{SW}_{M \sharp \overline{CP}^2}(\kappa_i \pm E) = \pm\mathcal{SW}_M(\kappa_i)$$

.

Proof. The formula follows from the same procedure as the last theorem. □

Theorem A.9 (Non-vanishing Theorem for Symplectic Manifolds)**.** *If M is a simply connected 4-manifold such that $b_2^+(M) \geq 2$, and admits a symplectic structure ω, then $K_M^* = c_1(\omega)$ is a basic class and $\mathcal{SW}_M(\pm K_M^*) = \pm 1$.*

Proof. □

Then the following corollary follows directly from the non-vanishing theorem for symplectic manifolds and the vanishing theorem for connected sums.

Corollary A.10. *A symplectic 4-manifold cannot split into $N' \sharp N''$ where $b_2^+(N') \geq 1$ and $b_2^+(N'') \geq 1$.*

Theorem A.11 (Adjunction Formula)**.** *If M is a 4-manifold such that $b_2^+(M) \geq 2$, and if S is a connected surface embedded in M such that either of the following holds*

- *$S \cdot S \geq 0$ and S is homologically non-trivial.*
- *M is of simple type and S is not the 2-sphere.*

Then for any basic class κ, we have the following inequality:

$$\chi(S) + S \cdot S \leq -|\kappa \cdot S|$$

.

Proof. □

Note. The adjunction formula gives a lower bound for the genus of S fixing a homology class represented by S.

Note. If we use the adjunction formula backwards, we can fix S and give restrictions to the basic class κ.

Corollary A.12. *If a 4-manifold M contains a homologically non-trivial embedded sphere S such that $S \cdot S \geq 0$, then $\mathcal{SW}_M \equiv 0$.*

A.4.2 Properties for Symplectic Manifolds

A.4.3 Properties for Complex Surfaces

Appendix B

A Note on Kirby Calculus

B.1 Basic ideas

B.2 Handle moves

B.2.1 Handle Slide

B.2.2 Handle Cancellation

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