

Surface Area of Spheres - the Wok Problem

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1 Introduction

Since the first introduction to geometry in elementary school, students have investigated the volume, surface area, and many other geometric properties of 2-Dimensional, as well as 3-Dimensional shapes. Yet one area students do not intuitively understand is circles and spheres, especially the derivation of area formula, volume area, etc. As a part of the curriculum of second year Calculus course, today we look at one of these problems and approach from a Calculus point of view:

One such problem states that a semi-spherical wok exists with diameter of 16 cm and height 9 cm. We are to paint the outside and inside of 5000 woks with 0.5 mm thick of paint each. Calculate the volume of paint needed in liters.

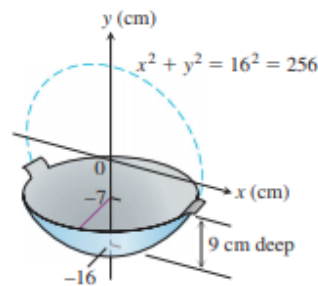


Figure 1: Wok Dimensions

2 Interpolation

The problem states that a wok needs to be painted on both its inside and outside. This meant the area to paint is the surface area of the wok's inside and outside. Additionally the paint is painted 0.5 mm thick, which meant the volume of paint needed on each side is the surface area multiplied by the height of the paint, which is 0.5 mm. This meant the real question this question is asking is what is the surface area of part of a sphere, since height is known.

Realizing this we can use the formula $A = \int 2\pi y \sqrt{1^2 + \left(\frac{dy}{dx}\right)^2} dx$ to calculate surface area. This indeed meant the calculation must be done in terms of dx , which will be addressed.

Thus we need to find $f'(x)$ with given equation

$$x^2 + y^2 = 256$$

Furthermore since the wok represents part of a sphere. And a sphere can be represented by rotating a function representing a semicircle around the x axis, if the function is in terms of x . To rewrite the equation of the entire circle as a function of x . Simplifying we get:

$$y = \sqrt{256 - x^2}$$

This equation does not match the diagram shown in the given graph, it is the rotation of the diagram. This satisfied the condition that the function is in terms of dx .

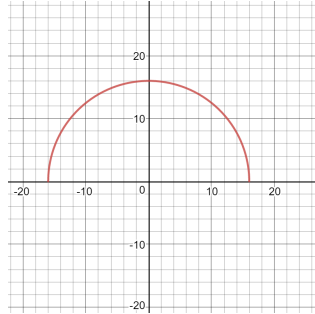


Figure 2: Equation of the wok

The wok is imagined to be lying between the first and fourth quadrant from $x = 7$ to $x = 16$. Thus only half of the wok is visible on the graph. This also meant o a full rotation contains the entire surface area of the sphere without overlapping. Using equations we got previously:

$$\frac{dy}{dx} = \frac{1}{2}(256 - x^2)^{-1/2}2x$$

We can now plug in the equation and the bounds from $x = 7$ to $x = 16$:

$$A = \int_7^{16} 2\pi\sqrt{256 - x^2}\sqrt{1^2 + \left(\frac{1}{2}(256 - x^2)^{-1/2}2x\right)^2} dx$$

Simplifying each term inside the integral:

$$A = 2\pi \int_7^{16} \sqrt{256 - x^2}\sqrt{1 + (256 - x^2)^{-1}x^2}dx$$

Changing negative exponent to a fraction:

$$A = 2\pi \int_7^{16} \sqrt{256 - x^2}\sqrt{1 + \frac{x^2}{256 - x^2}}dx$$

Rewriting 1 with common denominator $256 - x^2$:

$$A = 2\pi \int_7^{16} \sqrt{256 - x^2}\sqrt{\frac{256 - x^2}{256 - x^2} + \frac{x^2}{256 - x^2}}dx$$

Combining terms with common denominator: $256 - x^2$:

$$A = 2\pi \int_7^{16} \sqrt{256 - x^2}\sqrt{\frac{256 - x^2 + x^2}{256 - x^2}}dx$$

Canceling out similar terms under the radicant:

$$A = 2\pi \int_7^{16} \sqrt{256 - x^2}\sqrt{\frac{256}{256 - x^2}}dx$$

Pulling out the constant 256:

$$A = 2\pi \int_7^{16} \sqrt{256 - x^2}16\sqrt{\frac{1}{256 - x^2}}dx$$

Moving the constants out of the integral:

$$A = 32\pi \int_7^{16} \sqrt{256 - x^2}\sqrt{\frac{1}{256 - x^2}}dx$$

Rewriting radicant as exponents:

$$A = 32\pi \int_7^{16} (256 - x^2)^{1/2} (256 - x^2)^{-1/2} dx$$

Combining similar terms by adding up the coefficient:

$$A = 32\pi \int_7^{16} (256 - x^2)^0 dx$$

Simplifying the integrand, a function to the 0 power is 1:

$$A = 32\pi \int_7^{16} 1 dx$$

Taking the integral:

$$A = 32\pi [x]_7^{16}$$

Plugging in the bounds:

$$A = 32\pi(16 - 7) = 288\pi$$

Thus we have found the surface area of one side of the wok in cm^2 . To calculate the total volume of paint needed, we multiply the surface area. In order to convert final result to liters, we use dimensional analysis:

$$V = \frac{288\pi cm^2}{1side} \cdot \frac{0.5mm}{1side} \cdot \frac{1cm}{10mm} \cdot \frac{1side}{1wok} \cdot 5000wok \cdot \frac{1liter}{1000cm^3} = 72\pi/side$$

Thus to paint 5000 woks of this dimension we need 72π liters of paint for each side of the wok.

3 Conclusion

This problem investigates the properties of surface areas of spheres by connecting them to real life situations. During the calculation we investigated the properties of a sphere without using any equation specific to spheres and derived at the result that 5000 woks would require 72π liters of paint. The same process if used more generally can be used to derive the formula of the surface area of spheres: $4\pi r^2$.