Minimization of Time Required for Object in Free Fall to Reach Ground in Three Dimensional Space

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Part I
Interpretation of the problem

1 Introduction

Imagine a squad of paratroopers about jump off a plane. Their goal is to reach an objective in the shortest amount of time possible. As the plane quickly approaches the objective, the paratroopers face a interesting question: what is the best time to jump?

This is a similar scenario to popular video games of the battle royal genre, in which 100 players are dropped off a plane at the beginning of each round. The same question is asked by millions of players everyday, since being able to land earlier grants players a significant advantage, and an even more significant disadvantage to those landed late. At the time I first asked this question, I decided it is a trivial problem that would be naturally answered by experience. But after rounds and rounds, it came clear that while experience would certainly suffice the goal of not being the last to land, to be the first, require a rigorous mathematical model. With this opportunity, I will attempt to solve this problem.

A more methodical definition of this problem is given as follows:

An object moving in a three dimensional space, that is having three components to its displacement: x,y,z, is following a straight line. It is deciding at what time to deviate from its current path in order to move towards another point any where in the three dimensional space. To maximize its velocity or minimize its path in order to achieve the shortest time.

2 Mathematical reproduction of the problem

F urthermore, we must define the problem in strict mathematical terms:

As shown in the image below, a particle(plane), P, with a constant velocity vector $\vec{u} =< x_v, y_v, z_v >$, a point(objective) O, shown as a red point below, exists within $\mathbb{R}^3$ at coordinate $O(x_o, y_o, z_o)$. The position of point P is given as a parametric equation of t where $\vec{s} =< x(t), y(t), z(t) >$, shown as a black vector in the image below. Another point Q has position $\vec{r}$ and $\vec{r} = \vec{s} =< x(t), y(t), z(t) >, \{t < k | k \subseteq \mathbb{N}\}$. At $t = k$ point Q changes its direction and obtains a new velocity $\vec{v} =< x(t), y(t), z(t) >$. The new position of point Q is $\vec{r_2} =< x(t), y(t), z(t) >$. Find the minimal value of t given the components of $\vec{u}$ and O.

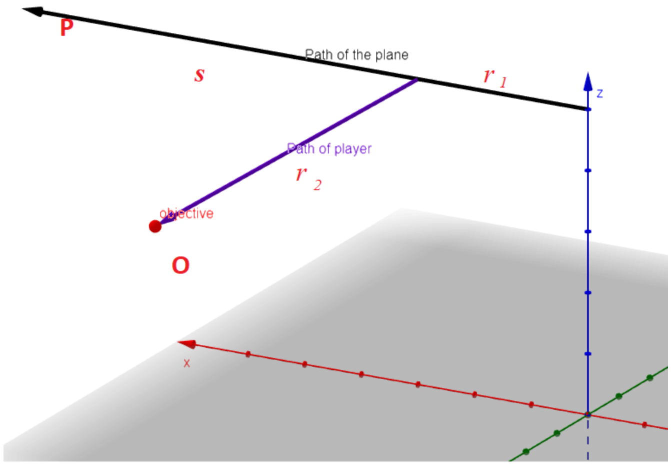

Figure 1: Demonstration of the problem

In order to find the minimal time it would take for the player to reach its objective, we must fist define the variables that would contribute to time.

First and foremost the path take by the player would have the greatest impact on time. We can easily see that velocity, another factor that determines time, is independent of the path chosen. Or in other words the path the player choose will not affect how fast he can go. Additionally it is also easy to see that the shorter the path, the shorter the time. Thus we conclude that:

The path that would lead to the shortest time is are straight line segments.

The second factor that would contribute to time is the velocity. We can easily see that the particle Q experiences 2 different velocities, $\vec{r}$ and $\vec{r_2}$. And since $\vec{r}$ a constant equal to $\vec{u}$, the only variable in this problem is $\vec{r_2}$. Thus naturally it may seem that the greater the magnitude of $\vec{r_2}$, the faster the particle will reach point O. But another crucial observation must be made to understand that because the particle is falling, its change in velocity, $\vec{a}$ equals to gravity. This would mean that the earlier the particle Q deviates from the path of P, the greater the magnitude of $\vec{r_2}$. Then we make the conjecture that:

To minimize time we need to maximize the magnitude of $\vec{r_2}$.

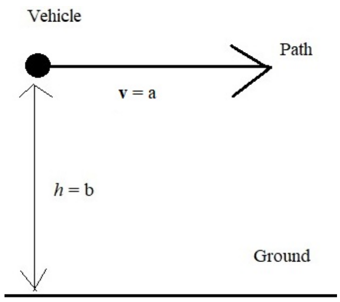

Figure 2: Stage1: the object remains on vehicle and travels at velocity v=a at height h = b

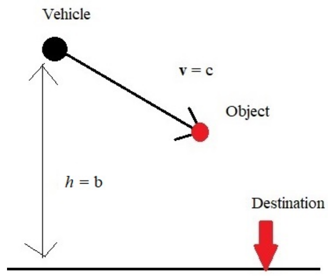

Figure 3: Stage2: the object leaves the vehicle at initial height h = b and travels to velocity v =c

During stage 1 since the path and velocity are constant, thus the duration of stage is only dependent on the time at which stage 2 begins, or the arbitrary time the object leaves the vehicle. Thus if stage 2 starts at $t = a$, then the duration of stage 1 is a. Furthermore we can conclude that since velocity is constant, the time required to complete stage1, t_{stage1}, is linear.

Stage 2 consists of 2 sub-stages, first sub-stage is when the object is first thrown off the vehicle and is in a controlled free fall, where the angle of falling can be manipulated. During this stage the velocity will be taken as a vector quantity, and the three components, or the composite movement in x,y,and z direction will determine overall velocity. The second sub-stage is when the object reach a certain altitude and opens its parachute, which will slow this object to a constant speed, and its velocity will be evaluated similarly to the first sub-stage as a vector. The duration of both stages will mostly be determined by the angle of approach, which changes the horizontal distance traveled throughout the entire stage.

Based on these three observations, the following equation of T_{total} at a point P where t_{stage1} and t_{stage2} are known can be deduced:

$$T_{total} = t_{stage1} + t_{stage2}$$

Thus T_{total} at another point can also be deduced in relation to point P:

$$T_{total2} = t_{stage1} + dt_{stage1} + t_{stage2} + dt_{stage2}$$

That T_{total2} is the sum of T_{total} at point P and the difference in duration of t_{stage1} and the difference in duration of t_{stage2}. And the desire is that $T_{total2} < T_{total}$, thus:

$$T_{total2} - T_{total} \leq 0$$

$$t_{stage1} + dt_{stage1} + t_{stage2} + dt_{stage2} - t_{stage1} - t_{stage2} \leq 0$$

$$dt_{stage1} + dt_{stage2} \leq 0$$

This inequality will be further examined in later discussions.

3 Assumptions and definitions

The combination of the two stages produce a simplified mathematical representation of the problem. But with further thought we would realize that this representation avoided consideration of factors such as air resistance and terminal velocity. Such factors, as mentioned in the introduction, need to be considered differently in the two different scenarios presented.

In the scenario of a computer simulated environment, such values are completely arbitrary to the designer's will. Since this work will not be specific to a single instance of such computer games, some general observations will be made. Thus in this calculation we will redefine the following for the purpose of simplification, and to best reconcile the video game physics and real world physics:

- Air resistance: Air resistance will be ignored.
- Initiation: At the instant the object leaves the vehicle, the object assumes 0 velocity and is stationary relative to the ground.
- Terminal velocity: The falling object will reach an arbitrarily defined terminal velocity, and keep falling at this velocity in stage 1 until stage 2 is reached.
- Deceleration: The instance stage 2 is reached, the object will release a parachute, and will slow down and descend at terminal velocity, while horizontal velocity will not affect on terminal velocity.
- Velocity:The object will travel in any direction at a constant velocity.

The definition of Terminal Velocity needs to be further examined. Terminal Velocity is the velocity at which an object will reach in an atmosphere, where the air resistance is equal to the gravitational acceleration. Terminal Velocity will be reached quickly after an object begins its free fall, or in a different word, the time it takes for the object to reach Terminal Velocity is finite, and as the height of fall increases, this time do not increase, and remain a constant c. Thus this time compared to the total time of the fall, as the height of fall changes and approaches infinity, is

$$\lim_{x \to \infty} \frac{c}{x} = 0$$

Thus we will assume the time it would take for the object to reach Terminal Velocity is 0.

Similarly to Terminal Velocity, the final stages of the fall, after the parachute is released, the player experiences a limited period of deceleration. This period of time, again, is independent of the height of the fall, and is completely arbitrary. In reality one skydiver can release its parachute 3000 ft off the ground, another can release it 2000 ft off the ground. In the context of this problem, we ignore this period of time and assumes the object can instantly decelerate.

Another concept that has to be examined is the velocity of the vehicle in relation to the velocity of the object. It is commonsense to see that it is unrealistic that the object would travel at a greater velocity than the vehicle, or in reality if a skydiver can fly faster than a plane. Thus in this problem the, strictly speaking, horizontal velocity is always significantly less than that of the vehicle.

By setting up the two scenarios of the same problem in this fashion we have greatly simplify the problem into its essence. Nuances like ignoring air resistance will definitely introduce a greater error to the overall calculation, but we hope it is negligible compared to other more important factors.

Part II
Speculations

4 Speculations

With the problem and variables defined, we will now look at the problem as a whole and provide some speculations. As explained in Part I Mathematical Reproduction of the Problem, the overall time is the duration of stage 1 and 2 combined, and the aim is to minimize this sum. Of these two stages it is easy to see that stage 2 is symmetrical to the normal line to the path of the vehicle that passes through the destination point as shown in figure 4.

Thus the duration of stage2 at any point after the vehicle crosses the normal line has an identical, symmetrical point before the vehicle passes through the normal line. Furthermore the duration of stage1 at any point after the normal line is greater than that before crossing the normal line, as we've concluded in Part I that the duration of stage1 is linear, and the longer the object remains on the vehicle, the longer stage1 is. Thus it is safe to conclude that the smallest T_{total} can only be achieved before crossing the normal line, and t_{stage1} is bounded:

$$0 \geq t_{stage1} \geq t_{crossing}$$

A further speculation can be made that the furthest distance the object can reach is a radius around the point where the object leaves the vehicle as shown in figure 6, and this radius is limited. Thus if the object leaves the vehicle way too early, or from a mathematical perspective, as the beginning time of stage 2 shifts further towards negative infinity, the object will never reach the desired location. Thus we can conclude that one mustn't leave the vehicle too early. Thus it can be further deduced that the duration of stage1 is

$$t_{maxRadius} \geq t_{stage1} \geq t_{crossing}$$

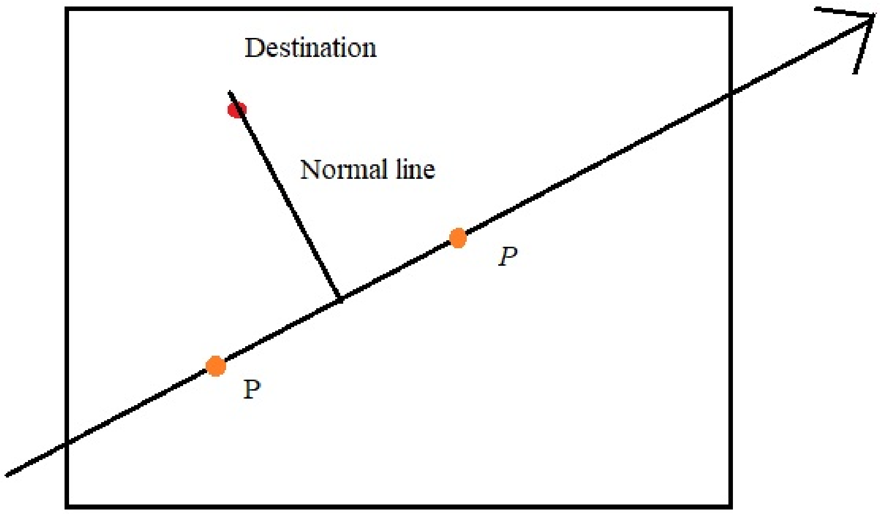

Figure 4: As shown above, point P and point P are symmetrical to the normal line

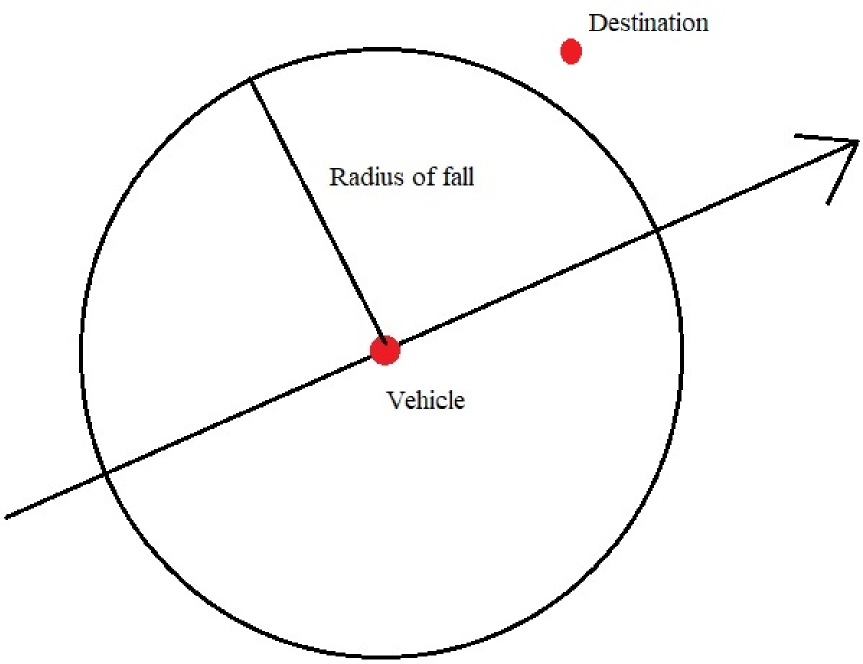

Figure 5: As shown above, the object cannot reach the destination that is outside the maximum radius

Part III
Calculations

To solve this problem, we need to look back at the inequality we derived in Part I:

$$dt_{stage1} + dt_{stage2} \leq 0$$

Since the duration of stage1 is linear as discussed above, thus the change in duration of stage1 is a constant:

$$\frac{d}{dT_{total}} t_{stage1} = 1$$

Thus the inequality can be rewritten as:

$$1 + dt_{stage2} \leq 0$$

$$1 \leq -dt_{stage2}$$

$$dt_{stage2} \leq -1$$

To elaborate on this new inequality, as the vehicle moves along the path, the duration of stage 2 is decreasing. But there will be a point where they amount of duration that would be decreases if the object remained on the vehicle is less than the increase of duration of stage1 by remaining on the vehicle. Thus by finding this point where $dt_{stage2} > -1$, we find the optimal position to leave the vehicle.

To calculate the total distance as shown in figure 6, d would be given since it can be measured as the distance between the destination and the path. We also need to know what the value of l is, and as we've established, the point P is where stage1 ends, and stage1 is bounded in $t_{maxRadius} \geq t_{stage1} \geq t_{crossing}$. Thus we can define t as time before reaching the normal line, and $l = t * v$, where v is the velocity of the vehicle, and:

$$r = \sqrt{(tv)^2 + d^2}$$

Referring to figure 7, we can see that k is dependent upon h, which is a constant that would be known is a specific example, and r, which was just calculated. Thus

$$k = \sqrt{b^2 + r^2}$$

Plugging in r:

$$k = \sqrt{b^2 + \sqrt{(tv)^2 + d^2}^2}$$

Simplifying:

$$k = \sqrt{b^2 + (tv)^2 + d^2}$$

Because b, v, d are all constants, we can take them out and replace them with another constant, where $\alpha = v^2$, $\beta = b^2 + d^2$:

$$k = \sqrt{\alpha t^2 + \beta}$$

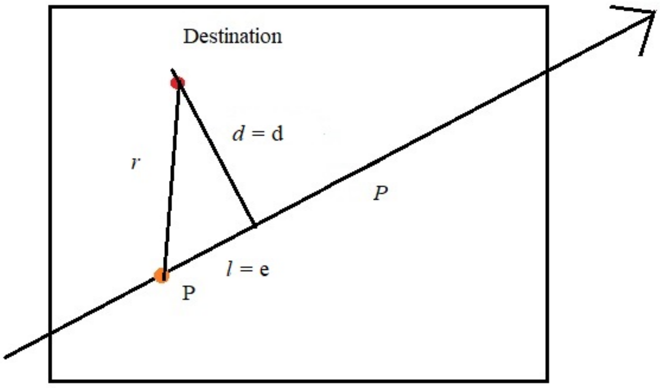

Figure 6: The horizontal distance, r, the object has to travel

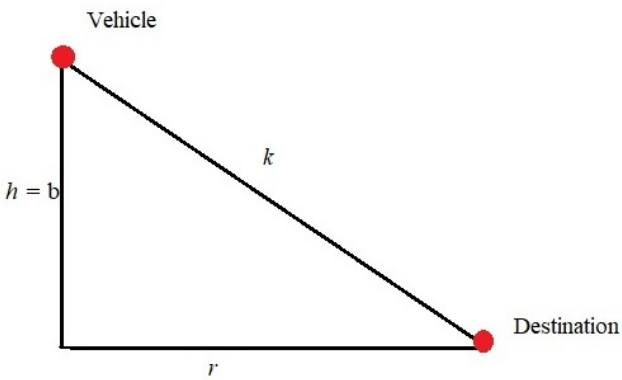

Figure 7: The total distance, k, the object has to travel

Referring to assumptions made in Part I, we assume the object falls at a constant velocity despite the angle, thus $t_{stage2} = \frac{k}{v_{terminal}}$, and $v_{terminal}$ is arbitrarily defined. Plugging in k:

$$\frac{\sqrt{\alpha t^2 + \beta}}{v_{terminal}}$$

Simplifying:

$$k = \gamma\sqrt{\alpha t^2 + \beta}, where \gamma = \frac{1}{v_{terminal}}$$

Taking derivative of k:

$$k' = 2\alpha t(\alpha t^2 + \beta)^{-1/2}$$

Thus plugging back into the inequality:

$$2\alpha t(\alpha t^2 + \beta)^{-1/2} = -1$$

Thus solving for t will produce the optimal time.

Part IV
Conclusion

5 Conclusion

Upon inspection after completion of the calculations, flaws with this problem can be seen. One assumption in particular about the velocity at which the object falls is especially flawed. It assumes the the object falls at constant velocity at any given angle, which is accurate when the object travels at a very sharp angle to the ground, but if the angle of travel is beyond a certain point, it will behave drastically differently from what is assumed. Additional assumptions would have introduced even greater error to this already flawed solution. With hindsight a better approach to this problem is with computer simulations combined with a better, more robust mathematical model that account for other variables that were not accounted, which would most certainly be significantly more difficult execute. But apart from the inherent complexity of this problem itself, the result concluded from this investigation should very well represent what is described, and achieve what is explicitly desired by the assumptions in this report.