The Second Derivative Test in n variables.

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1 Introduction.

Stewart's *Calculus* textbook does a fine job of addressing of addressing the differential calculus of functions of n variables $f : \mathbb{R}^n \to \mathbb{R}$ in the cases of $n = 2$ or 3. (As such, it's usually easy to guess how these formulas generalise for arbitrary n.) However, it is curious that Stewart only covers the second derivative test in the case of $n = 2$ variables. After reviewing the cases of $n = 1$ and 2, I'll address what happens in general.

2 Second Derivative Test in 1 or 2 variables

Before stating the second derivative test as mentioned in Stewart, recall that for a function $y = f(x)$, the second derivative test uses concavity of the function at a critical point to determine whether we have a local maximum or minimum value at the said point.

Second Derivative Test, Single variable case:
Suppose that f'' is continuous near c, where $f'(c) = 0$ (that is, c is a critical point of f).
(a) If $f''(c) > 0$, then f has a local minimum at c.
(b) If $f''(c) < 0$, then f has a local maximum at c. ■

Note that the theorem is silent in the case that $f''(c) = 0$. In such a case, one has to try a different method to determine the nature of the critical point.

Here is the analogous statement for *two* variables.

Second Derivative Test, Two variable case:
Suppose that the second partial derivatives of $f : \mathbb{R}^2 \to \mathbb{R}$ are continuous on a disk with centre (a, b), where $f_x(a, b) = 0$ and $f_y(a, b) = 0$ (that is, (a, b) is a critical point of f).

Let H denote the "Hessian" matrix of second partial derivatives

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix},$$

and let $D_1 = f_{xx}$ and $D_2 = \det H = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$.
(a) If $D_1(a, b) > 0$ and $D_2(a, b) > 0$, then f has a local minimum at (a, b).
(b) If $D_1(a, b) < 0$ and $D_2(a, b) > 0$, then f has a local maximum at (a, b).
(c) If $D_2(a, b) < 0$, then f has a saddle point at (a, b). ■

A couple of remarks are in order. First, I write D_2 in terms of a determinant because not only is it easier to remember (which is always a good thing), but also it helps to point the way to further generalisations. Second, the test is inconclusive when $D_2(a, b) = 0$.

Example: Find and classify the critical points of the function
$f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 1$.

Solution: First, we set the first partial derivatives equal to zero:

$$f_x = 6xy - 6x = 0 \quad \text{and} \quad f_y = 3x^2 + 3y^2 - 6y = 0.$$

This yields four critical points $(0, 0), (0, 2), (1, 1), (-1, 1)$. Next, we compute the matrix of second partial derivatives:

$$H = \begin{bmatrix} 6y - 6 & 6x \\ 6x & 6y - 6 \end{bmatrix}.$$

Now, we apply the second derivative test to each critical point.

1. At $(0, 0)$, $D_1(0, 0) = -6 < 0$, $D_2(0, 0) = \begin{vmatrix} -6 & 0 \\ 0 & -6 \end{vmatrix} = 36 > 0$. Hence, f has a local maximum at $(0, 0)$.
2. At $(0, 2)$, $D_1(0, 2) = 6 > 0$, $D_2(0, 0) = \begin{vmatrix} 6 & 0 \\ 0 & 6 \end{vmatrix} = 36 > 0$. Hence, f has a local minimum at $(0, 0)$.
3. At $(\pm 1, 1)$, $D_2(\pm 1, 1) = \begin{vmatrix} 0 & \pm 6 \\ \pm 6 & 0 \end{vmatrix} = -36 < 0$. Hence, f has saddle points at $(\pm 1, 1)$. ■

3 Second Derivative Test in 3 or more variables

By using the Hessian matrix, stating the second derivative test in more than 2 variables is not too difficult to do. Before stating the general theorem, we will first state it in 3 variables (so the pattern is clear) and work an example.

Second Derivative Test, Three variable case:
Suppose that the second partial derivatives of $f : \mathbb{R}^3 \to \mathbb{R}$ are continuous on a ball with centre (a, b, c), where $f_x(a, b, c) = 0$ and $f_y(a, b, c) = 0$, and $f_z(a, b, c) = 0$ (that is, (a, b, c) is a critical point of f).

Let H denote the Hessian matrix of second partial derivatives

$$H = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix},$$

and let $D_1 = f_{xx}$, $D_2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$, and $D_3 = \det H = \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}$.

(a) If $D_1(a, b, c) > 0$, $D_2(a, b, c) > 0$, and $D_3(a, b, c) > 0$, then f has a local minimum at (a, b, c).
(b) If $D_1(a, b, c) < 0$, $D_2(a, b, c) > 0$, and $D_3(a, b, c) < 0$, then f has a local maximum at (a, b, c).
(c) In any other case where $D_3(a, b, c) \neq 0$, f has a saddle point at (a, b, c). ■

Example: Find and classify the critical points of the function
$f(x, y, z) = x^2 + y^2 + 7z^2 - xy - 3yz$.

Solution: First, we set the first partial derivatives equal to zero:

$$f_x = 2x - y = 0, f_y = 2y - x - 3z = 0, \quad \text{and} \quad f_z = 14z - 3y = 0.$$

This yields exactly one critical point $(0, 0, 0)$. Next, we compute the matrix of second partial derivatives:

$$H = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -3 & 14 \end{bmatrix}.$$

Now, we apply the second derivative test to the critical point $(0, 0, 0)$.
$D_1(0, 0, 0) = 2 > 0$, $D_2(0, 0, 0) = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0$, and

$D_3(0, 0, 0) = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -3 & 14 \end{vmatrix} = 24 > 0.$

Hence, f has a local minimum at $(0, 0, 0)$. ■

Now, we can state the general version of the second derivative test. Observe how this theorem indeed does summarise the forms of this test given in the specific cases. For brevity, let $\vec{x} = (x_1, x_2, ..., x_n)$, and use the gradient of f to concisely list the partial derivatives.

Second Derivative Test, the general n variable version:
Suppose that the second partial derivatives of $f : \mathbb{R}^n \to \mathbb{R}$ are continuous on a ball with centre $\vec{c}$, where $\nabla f(\vec{c}) = \vec{0}$ (that is, $\vec{c}$ is a critical point of f).

Let H denote the Hessian matrix of second partial derivatives, and for each $k = 1, 2, ..., n$, let D_k denote the determinant of the Hessian in the variables x_1, x_2, ..., x_k. Assume that $|H(\vec{c})| \neq 0$.

(a) If $D_k(\vec{c}) > 0$ for all $k = 1, 2, ..., n$, then f has a local minimum at $\vec{c}$.
(b) $(-1)^k \cdot D_k(\vec{c}) > 0$ for all $k = 1, 2, ..., n$, then f has a local maximum at $\vec{c}$.
(c) Otherwise, f has a saddle point at $\vec{c}$. ■

This theorem is usually proved using the quadratic approximation of the (multivariable) Taylor Series for f centred at $\vec{x} = \vec{c}$ and understandably involves a good amount of Linear Algebra. The curious student may consult an Advanced Calculus textbook for a proof of this theorem.