Definition of Limit

- $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if

1. every neighbourhood of (a,b) contains points of the domain of f different from (a,b)

and

2. for every positive number ϵ there exists a positive number $\delta = \delta(\epsilon)$ st $|f(x,y) - L| < \epsilon$ holds whenever (x,y) is in the domain of f and satisfies $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$

p682, 7. Evaluate $\lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2}$

• If the limit exist it must be 0 as $y^3 \to 0$.
The function is defined everywhere except $(0,0)$.

$$|f(x,y)-0| = \left|\frac{y^3}{x^2+y^2}\right| \le \frac{y^2}{x^2+y^2}|y| \le |y| \le \sqrt{x^2+y^2} \to 0$$

as $(x,y) \to (0,0)$.

ie. if $\epsilon > 0$ and $\delta = \epsilon$ then $|f(x,y)-0| < \epsilon$

when $0 < \sqrt{x^2+y^2} < \delta$. $\therefore \lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} = 0$

9. Evaluate $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$

$f(0,y) = \frac{0}{y^2} = 0 \to 0$ as $x \to 0$

$f(x,x) = \frac{\sin x^2}{2x^2}$ ~~$\frac{...}{}$~~ $\lim_{x\to 0} \frac{\sin x^2}{2x^2}$ ~~$\frac{2x\sin x^2 \cos x^2}{4x}$~~

$= \lim_{x\to 0} \frac{2x\cos x^2}{4x} = \frac{1}{2} \to \frac{1}{2}$ as $x \to 0$

$\therefore \lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$ DNE

11. Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^4}$

If the limit exists it must be 0 as $x^2y^2 \to 0$.

The function is defined everywhere except $(0,0)$.

$$|f(x,y)-0| = \left|\frac{x^2y^2}{x^2+y^4}\right| = \frac{x^2y^2}{x^2+y^4} \le y^2 \le \sqrt{x^2+y^2} \to 0$$

as $(x,y)\to(0,0)$.

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^4} = 0.$$

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15. What is the domain of

$$f(x,y) = \frac{x-y}{x^2-y^2}$$

Evaluate $\lim_{(x,y)\to(1,1)} f(x,y)$.

Can the domain of f be extended st the resulting function ~~exists at~~ is continuous at $(1,1)$?

Can the domain be extended st the resulting function is continuous everywhere?

∴ cont. $f(x,y) = \dfrac{x-y}{x^2-y^2} = \dfrac{x-y}{(x-y)(x+y)}$

$f(x,y) = \dfrac{1}{x+y}$ for all $x \neq y$ and ~~f~~ is defined in the some points in the neighbourhood of $(1,1)$. It approaches $\frac{1}{2}$ as $(x,y) \to (1,1)$. ∴ the limit ~~exists~~ DNE

$\cancel{|f(x,y) - \frac{1}{2}| = \left|\frac{1}{x+y} - \frac{1}{2}\right|}$

If we define $f(1,1) = \frac{1}{2}$, then f becomes continuous $f(1,1) = \frac{1}{2}$.

If we define $f(x,y) = \frac{1}{2x}$ ~~f~~ for all $x = y$ except $(0,0)$ and become continuous at those points.

We cannot define $f(x,-x)$ st f becomes continuous on $y = -x$ ∵ $|f(x,y)| = \dfrac{1}{|x+y|} \to \infty$ as $y \to -x$.