

Preface

Here are the solutions to the practice problems for my Calculus II notes. Some solutions will have more or less detail than other solutions. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you've reached the level of working the harder problems then you will probably already understand the basics fairly well and won't need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven't been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

Applications of Series

1. Determine a Taylor Series about $x = 0$ for the following integral.

$$\int \frac{e^x - 1}{x} dx$$

Step 1

This problem isn't quite as hard as it might first appear. We know how to integrate a series so all we really need to do here is find a Taylor series for the integrand and then integrate that.

Step 2

Okay, let's start out by noting that we are working about $x = 0$ and that means we can use the formula for the Taylor Series of the exponential function. For reference purposes this is,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Next, let's strip out the $n = 0$ term from this and then subtract one. Doing this gives,

$$e^x - 1 = \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \right] - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

Of course, in doing the above step all we really managed to do was eliminate the $n = 0$ term from the series. In fact, that was not a bad thing to have happened as well see shortly.

Finally, let's divide the whole thing by x . This gives,

$$\frac{e^x - 1}{x} = \frac{1}{x} \sum_{n=1}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$$

We moved the x that was outside the series into the series. This is required in order to do the integral of the series. We only want a single x in the problem and we now have that.

Also note that while the function on the left has a division by zero issue the series on the right does not have this problem. All of the x 's in the series have positive or zero exponents! This is a really good thing.

Of course the other good thing that we have at this point is that we've managed to find a series representation for the integrand!

Step 3

All we need to do now is compute the integral of the series to get a series representation of the integral.

$$\int \frac{e^x - 1}{x} dx = \int \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} dx = \boxed{c + \sum_{n=1}^{\infty} \frac{x^n}{(n)(n!)}}$$

2. Write down $T_2(x)$, $T_3(x)$ and $T_4(x)$ for the Taylor Series of $f(x) = e^{-6x}$ about $x = -4$. Graph all three of the Taylor polynomials and $f(x)$ on the same graph for the interval $[-8, -2]$.

Step 1

The first thing we need to do here is get the Taylor Series for $f(x) = e^{-6x}$ about $x = -4$. Luckily enough for us we did that in Problem 3 of the previous section. Here is the Taylor Series we derived in that problem.

$$e^{-6x} = \sum_{n=0}^{\infty} \frac{(-6)^n e^{24}}{n!} (x+4)^n$$

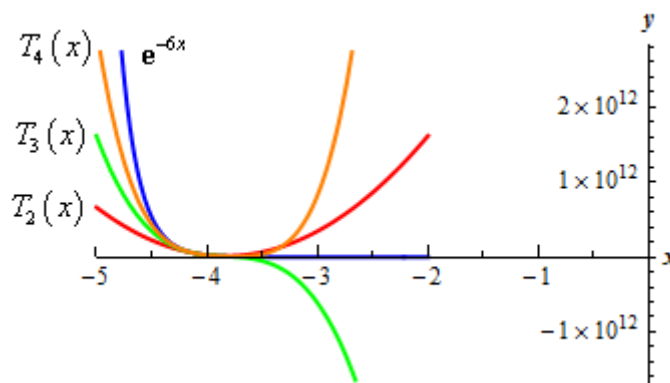
Step 2

Here are the three Taylor polynomials needed for this problem.

$$\begin{aligned} T_2(x) &= e^{24} - 6e^{24}(x+4) + 18e^{24}(x+4)^2 \\ T_3(x) &= e^{24} - 6e^{24}(x+4) + 18e^{24}(x+4)^2 - 36e^{24}(x+4)^3 \\ T_4(x) &= e^{24} - 6e^{24}(x+4) + 18e^{24}(x+4)^2 - 36e^{24}(x+4)^3 + 54e^{24}(x+4)^4 \end{aligned}$$

Step 3

Here is the graph for this problem.



We can see that as long as we stay “near” $x = -4$ the graphs of the polynomial are pretty close to the graph of the exponential function. However, if we get too far away the graphs really do start to diverge from the graph of the exponential function.

3. Write down $T_3(x)$, $T_4(x)$ and $T_5(x)$ for the Taylor Series of $f(x) = \ln(3+4x)$ about $x = 0$. Graph all three of the Taylor polynomials and $f(x)$ on the same graph for the interval $[-\frac{1}{2}, 2]$.

Step 1

The first thing we need to do here is get the Taylor Series for $f(x) = \ln(3+4x)$ about $x = 0$. Luckily enough for us we did that in Problem 4 of the previous section. Here is the Taylor Series we derived in that problem.

$$\ln(3+4x) = \ln(3) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \left(\frac{4}{3}\right)^n}{n} x^n$$

Step 2

Here are the three Taylor polynomials needed for this problem.

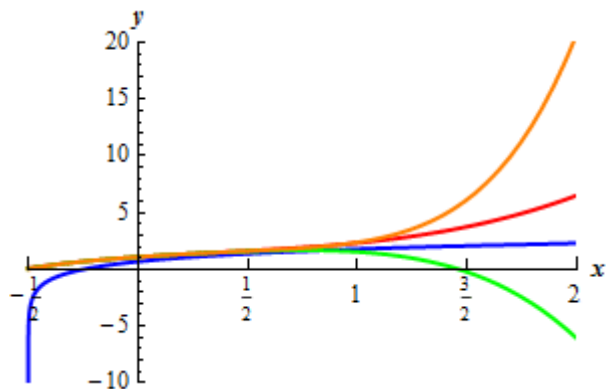
$$T_3(x) = \ln(3) + \frac{4}{3}x - \frac{8}{9}x^2 + \frac{64}{81}x^3$$

$$T_4(x) = \ln(3) + \frac{4}{3}x - \frac{8}{9}x^2 + \frac{64}{81}x^3 - \frac{64}{81}x^4$$

$$T_5(x) = \ln(3) + \frac{4}{3}x - \frac{8}{9}x^2 + \frac{64}{81}x^3 - \frac{64}{81}x^4 + \frac{1024}{1215}x^5$$

Step 3

Here is the graph for this problem.



We can see that as long as we stay “near” $x = 0$ the graphs of the polynomial are pretty close to the graph of the exponential function. However, if we get too far away the graphs really do start to diverge from the graph of the exponential function.
