Algorithm for Series

1. $\lim_{n\to\infty} a_n \neq 0$?
2. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ or can be compared?
3. $\sum_{n=0}^{\infty} ar^n$ or can be compared?
4. factorials / constants raised to n? → ratio test
5. $a_n = (-1)^n b_n$ → alternating series test
6. $a_n = (b_n)^n$ → root test
7. Integral test

Pretty much everything relates back to p-series or geo series. Even the proofs for the ratio and root test show this.

21. $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{3^n \ln n}$

$$\rho = \lim_{n\to\infty} \frac{\sqrt{n+1}}{3^{n+1}|\ln(n+1)|} \cdot \frac{3^n |\ln n|}{\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{1}{3} \sqrt{\frac{n+1}{n}} \, \frac{\ln(n)}{\ln(n+1)}$$

$$= \frac{1}{3} \cdot \lim_{n\to\infty} \cancel{\sqrt{\frac{1+1/n}{1}}} \to 1 \cdot \lim_{n\to\infty} \cancel{\frac{\ln(n)}{\ln(n+1)}} \to 1 \text{ by l'Hôpital's}$$

$= \frac{1}{3} < 1$ ∴ the ~~series~~ converges by ratio test.

25. $\sum_{n=4}^{\infty} \frac{2^n}{3^n - n^3}$

$$\rho = \lim_{n\to\infty} \frac{2^{n+1}}{3^{n+1}-(n+1)^3} \cdot \frac{3^n - n^3}{2^n}$$

$$= \lim_{n\to\infty} \frac{2}{3} \, \frac{3^n - n^3}{3^n - \frac{(n+1)^3}{3}}$$

$$= \frac{2}{3} \lim_{n\to\infty} \frac{1 - \frac{n^3}{3^n}}{1 - \frac{(n+1)^3}{3^{n+1}}} = 2/3 < 1$$

∴ the series converges by ratio test.

$$\sum_{k=1}^{\infty} \frac{3^k (k+1)^{10}}{4^k}$$

$$\rho = \lim_{k\to\infty} \frac{3^{k+1}(k+2)^{10}}{4^{k+1}} \cdot \frac{4^k}{3^k (k+1)^{10}}$$

$$= \lim_{k\to\infty} \frac{3}{4} \frac{(k+2)^{10}}{(k+1)^{10}}$$

$$= \lim_{k\to\infty} \frac{3}{4} \frac{(1+\frac{2}{k})^{10}}{(1+\frac{1}{k})^{10}} = 3/4 < 1$$ $\therefore$ the series converges by ratio test.

$$\sum_{k=1}^{\infty} \frac{(2k^2+6)^k}{(3k^2+100k+7)^k}$$

$$L = \lim_{k\to\infty} |a_k|^{1/k} = \lim_{k\to\infty} \frac{2k^2+6}{3k^2+100k+7} = \frac{2}{3} < 1$$

$\therefore$ the series converges by root test.

...tio Test

Assume $L<1$, and r exists st $L<r<1$.

define $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

then $\left|\frac{a_{n+1}}{a_n}\right| < r \rightarrow |a_{n+1}| < r|a_n|$ for sufficiently large n.

$|a_{N+1}| < r|a_N|$, N is sufficiently large integer

$|a_{N+2}| < r^1 |a_{N+1}| < r^2 |a_N|$

$|a_{N+k}| < r|a_{N+k-1}| < r^k |a_N|$, $k>0, k \in \mathbb{Z}$

$\therefore \boxed{|a_{N+k}| < r^k |a_N|}$

Consider $\sum_{k=0}^{\infty} |a_N| r^k$, a convergent geo series.

$\because |a_{N+k}| < |a_N| r^k$, $\therefore \sum_{k=0}^{\infty} |a_{N+k}|$ converges

so does our original $\sum_{k=0}^{\infty} a_n$.

Assume $L>1$, $|a_{n+1}| > |a_n|$ for large n.

$\therefore \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \neq 0$.

Root Test

Assume $L<1$, and r exists st $L<r<1$.

define $L=\lim_{n\to\infty}|a_n|^{1/n}$

then $|a_n|^{1/n}<r \to \boxed{|a_n|<r^n}$ for sufficiently large n.

Consider $\sum_{n=0}^{\infty} r^n$, a convergent geo series.

$\because |a_n|<r^n$, $\therefore \sum_{k=0}^{\infty}|a_{n+k}|$ converges.

so does $\sum_{k=0}^{\infty} a_n$.

Assume $L>1$, $|a_n|^{1/n}>1$ for larger n.

$\therefore |a_n|>1^n=1$

so $\lim_{n\to\infty} a_n \neq 0$.