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## INFINITE SEQUENCES

Review

Let  $\{a_n\}$  be a sequence. If  $\lim_{n \rightarrow \infty} a_n = L$  then the sequence converges.

If  $\lim_{n \rightarrow \infty} a_n$  results in either one of the following

forms:  $\frac{0}{0}, \frac{\infty}{\infty}, 0^0, \infty^0, 1^\infty, \infty - \infty$  use

L'Hopital's Rule to evaluate the limit.

### L'Hopital's Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

### Properties of Sequences

Suppose that  $\{a_n\}$  and  $\{b_n\}$  be two convergent sequences with  $\lim_{n \rightarrow \infty} a_n = L$  and

$\lim_{n \rightarrow \infty} b_n = M$  then the following results are true

Name	Property
Constant	$\lim_{n \rightarrow \infty} c = c$
Constant Multiple	$\lim_{n \rightarrow \infty} ca_n = c \lim_{n \rightarrow \infty} a_n = cL$
Additive	$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n = L + M$
Difference	$\lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n = L - M$
Product	$\lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n = LM$
Quotient	$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} = \frac{L}{M}$ provided that $M \neq 0$

Squeeze Theorem For Sequences Let  $\{a_n\}$ ,  $\{b_n\}$ , and  $\{c_n\}$  be sequences such that  $\{a_n\} \leq \{c_n\} \leq \{b_n\}$ . If  $\lim_{n \rightarrow \infty} a_n = L$  and  $\lim_{n \rightarrow \infty} b_n = L \Rightarrow \lim_{n \rightarrow \infty} c_n = L$

## TESTS TO DETERMINE THE CONVERGENCE/DIVERGENCE OF A SERIES

Name	What Does It Say
Integral Test	$\sum a_k$ is a series of positive terms

and  
 $f(x) = a_k$  - If the function is decreasing and continuous on  $[a, \infty)$ , then the series converges if

$\int_a^{+\infty} f(x) dx$  is defined. The series diverges if  $\int_a^{+\infty} f(x) dx = \pm \infty$

Divergence Test  $\sum a_k$  diverges if  $\lim_{k \rightarrow \infty} a_k \neq 0$

Ratio Test  $\sum a_k$  is a series of positive terms with  $p = \lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k}$

If  $p > 1 \Rightarrow$  the series diverges.  
 If  $p < 1 \Rightarrow$  the series converges.  
 If  $p = 1 \Rightarrow$  no conclusion.

Root Test  $\sum a_k$  is a series of positive terms with  $p = \lim_{k \rightarrow \infty} \sqrt[k]{a_k}$

If  $p > 1 \Rightarrow$  the series diverges.  
 If  $p < 1 \Rightarrow$  the series converges.  
 If  $p = 1 \Rightarrow$  no conclusion.

Comparison Test Suppose that  $\sum a_k$  and  $\sum b_k$  are two series with positive terms such that  $a_k \leq b_k$  for all  $k$ .

If  $\sum b_k$  converges, then  $\sum a_k$  converges.  
 If  $\sum a_k$  diverges, then also  $\sum b_k$  diverges.

Limit Comparison Test Suppose that  $\sum a_k$  and  $\sum b_k$  are two series with positive terms with

$p = \lim_{k \rightarrow \infty} \frac{a_k}{b_k}$   
 If  $p > 0 \Rightarrow$  either both series converge, or both series diverge.

Name What Does It Say

Alternating Series Test Suppose that  $a_k > 0$  for  $k = 1, 2, 3, \dots$  then the series has the form of either:  
 $a_1 - a_2 + a_3 - a_4 + \dots$   
 or

$-a_1 + a_2 - a_3 + a_4 - \dots$   
 The series converges if and only if 1.  $a_{k+1} \leq a_k$ : the series is decreasing, and 2.  $\lim_{k \rightarrow \infty} a_k = 0$

Ratio Test Absolute Convergence  $\sum a_k$  is a series of nonzero terms with  $p = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right|$   
 If  $p > 1 \Rightarrow$  the series diverges.  
 If  $p < 1 \Rightarrow$  the series converges.  
 If  $p = 1 \Rightarrow$  no conclusion.

## POWER OF SERIES AND RADIUS OF CONVERGENCE

Form	Sigma Notation	Radius of Convergence
$\frac{1}{1-x} = 1+x+x^2+x^3+x^4+\dots$	$= \sum_1^{\infty} x^n$	$-1 \leq x \leq 1$
$\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\dots$	$= \sum_1^{\infty} nx^{n-1}$	$-1 \leq x \leq 1$
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$	$= \sum_1^{\infty} (-1)^{n-1} \frac{x^n}{n}$	$-1 \leq x \leq 1$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$	$= \sum_1^{\infty} (-1)^{n-1} \frac{x^{2n+1}}{2n+1}$	$-1 \leq x \leq 1$
$e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$	$= \sum_1^{\infty} \frac{x^n}{n!}$	$(-\infty, \infty)$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$	$= \sum_1^{\infty} (-1)^{n-1} \frac{x^{2n+1}}{(2n+1)!}$	$(-\infty, \infty)$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$	$= \sum_1^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$(-\infty, \infty)$

## VECTORS AND COORDINATE GEOMETRY

Vectors & Vector Dot Products

Let  $\vec{u}$  be a vector having  $n$  components, then the length or norm of is  $\|\vec{u}\|$  defined to be

$$\|\vec{u}\| = \sqrt{(n_1)^2 + (n_2)^2 + (n_3)^2 + \dots + (n_n)^2}$$

Let  $\vec{u}$  and  $\vec{v}$  be two vectors, then their dot product is defined to be:

$$\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta$$

#### Properties

Symmetry	$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
Multiplication with a scalar	$(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v})$ $k \in \mathbb{R}$
Linearity	$\vec{u}(\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

#### Uses of Dot Product

- A. Show orthogonality:  $\vec{u} \cdot \vec{v} = 0 \Leftrightarrow \vec{u} \perp \vec{v}$
- B. Finding the angle between two vectors:

$$\theta = \cos^{-1} \left( \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} \right)$$

#### Unit Vectors

Suppose that  $\vec{u}$  is a vector having  $n$  components, then its unit vector is found by dividing each component of the vector by its norm. That is,

$$|\vec{u}|_{\text{unit}} = \frac{1}{|\vec{u}|} \vec{u} = \frac{1}{\sqrt{(n_1)^2 + (n_2)^2 + (n_3)^2 + \dots + (n_n)^2}} \vec{u}$$

## VECTOR VALUED FUNCTIONS

#### Tangent Directions

If  $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$  is a position vector to the points on the curve  $(x(t), y(t), z(t))$  then we can find the following quantities

Average Velocity	$v_{\text{avg}} = \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}$ , $\Delta t = \text{change in time}$
Velocity	$v(t) = \vec{r}'(t) = \frac{dx}{dt}[\vec{x}(t)] + \frac{dy}{dt}[\vec{y}(t)] + \frac{dz}{dt}[\vec{z}(t)]$
Speed	$ v(t) $
Acceleration	$a(t) = \vec{r}''(t) = \frac{d^2\vec{r}}{dt^2} = \frac{dx}{dt^2}[\vec{x}(t)] + \frac{dy}{dt^2}[\vec{y}(t)] + \frac{dz}{dt^2}[\vec{z}(t)]$

#### Arc length

$$s = s(t) = \int_{t_0}^t |\vec{r}'(t)| dt$$

#### Formulas for Curvature, Torsion, Radius of Curvature

Name	Formula
Unit Tangent Vector	$\vec{T} = \frac{\vec{v}}{ \vec{v} }$
Curvature	$\kappa = \frac{ \vec{v} \times \vec{a} }{ \vec{v} ^3}$
Radius of Curvature	$\rho = \frac{1}{\kappa}$
Binormal Vector	$\vec{B} = \frac{\vec{v} \times \vec{a}}{ \vec{v} \times \vec{a} }$
Unit Normal Vector	$\vec{N} = \vec{B} \times \vec{T} = \frac{d\vec{T}}{dt}$
Tangential and Normal Components	$a = \frac{d\vec{v}}{dt} \cdot \vec{T} +  \vec{v} ^2 \kappa \vec{N}$
Torsion	$\tau = \frac{(\vec{v} \times \vec{a}) \cdot \frac{d\vec{a}}{dt}}{ \vec{v} \times \vec{a} ^2}$

#### Frenet – Serret Formulas

$$\begin{aligned} \frac{d\vec{T}}{ds} &= \kappa \vec{N} \\ \frac{d\vec{B}}{ds} &= -\tau \vec{N} \\ \frac{d\vec{N}}{ds} &= -\kappa \vec{T} + \tau \vec{B} \end{aligned}$$

## PARTIAL DIFFERENTIATION

#### Definition of Limit on a plane

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$$

Implies that

- A. Every neighborhood of  $(a, b)$  contains points of the domain of  $f(x, y)$  different from  $(a, b)$ .
- B. For every  $\epsilon$  (positive) there exists a positive  $\delta$  such that  $|f(x, y) - L| < \epsilon$  whenever  $(x, y)$  is in the domain of  $f$  and satisfies the

inequality:  $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$

#### Definition of Continuity on a Plane

For continuity to happen, the following 2 conditions must hold:

$$1. \lim_{(x,y) \rightarrow (a,b)} f(x,y) \text{ exists}$$

$$2. \lim_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b)$$

#### Partial Derivatives

Given a function  $f(x, y)$ , the partial with respect to  $x$ , denoted as  $f_x$ , at the point  $(a, b)$  is defined to be

$$f_x(a, b) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h}$$

Similarly the partial with respect to  $y$ , at the point  $(a, b)$ , denoted as  $f_y$ , is defined to be

$$f_y = \lim_{k \rightarrow 0} \frac{f(a, b+k) - f(a, b)}{k}$$

In practice we use the following

$f_x = f_1 = \frac{\partial f}{\partial x}$	$\Rightarrow$	differentiate the function $f$ with respect to $x$ while holding all other variables as constants.
$f_y = f_2 = \frac{\partial f}{\partial y}$	$\Rightarrow$	differentiate the function $f$ with respect to $y$ while holding all other variables as constants.

The idea can easily be extended to functions having more than 2 variables.

#### Higher Derivatives

#### Second derivatives for functions of 2 variables

$f_{xx} = f_{11} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right)$	Second derivative of $f$ with respect to $x$
$f_{yy} = f_{22} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right)$	Second derivative of $f$ with respect to $y$
$f_{xy} = f_{12} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)$	Mixed partial
$f_{yx} = f_{21} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)$	Mixed partial

#### Chain Rule

$z = f(x, y)$  is a differentiable function of  $x$  and  $y$  with  $x = x(t)$  and  $y = y(t)$  then

$$z'(t) = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

Example:  $z(x, y) = x \sin y$  where  $x(t) = e^t$  and  $y(t) = t^2$

Solution:

$$z'(t) = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

$$\frac{\partial x}{\partial t} = e^t \quad \frac{\partial y}{\partial t} = 2t$$

$$\frac{\partial z}{\partial x} = \sin y \quad \frac{\partial z}{\partial y} = x \cos y$$

$$z'(t) = e^t \sin y + 2tx \cos y$$

The chain rule can be extended to include functions which have more than one variable.

Let  $z$  be a differentiable function of  $n$  variables,  $x_1, x_2, x_3, \dots, x_n$  and each  $x_j$  is a

differentiable function of  $m$  variables,  $t_1, t_2, t_3, \dots, t_m$  then  $z$  is ultimately a function of  $t_1, t_2, t_3, \dots, t_m$ . So

$$\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial z}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial z}{\partial x_n} \frac{\partial x_n}{\partial t_i}$$

#### Tangent Planes & Gradients

Suppose that  $z = f(x, y)$  has tangent planes everywhere, then the gradient is:

$$\nabla f(x, y) = \hat{i} \frac{\partial}{\partial x}(x, y) + \hat{j} \frac{\partial}{\partial y}(x, y)$$

$\nabla f(x, y) = \text{Direction of the steepest slope upwards in the XY plane}$

## LOCAL EXTREMAS

Suppose that the surface  $z = f(x, y)$  has tangent planes at every point. The points  $(x_0, y_0, z_0)$  on the surface where the tangent plane is horizontal

$$\Rightarrow \nabla f \Big|_{(x_0, y_0)} = 0$$

, are called critical point (local extrema).

$D_u f$  positive for all  $\theta \Rightarrow$  nearby slopes are up

$\Rightarrow$  surface bends up

$\Rightarrow$  local minimum

$D_u f$  negative for all  $\theta \Rightarrow$  nearby slopes are down

$\Rightarrow$  surface bends down

$\Rightarrow$  local maximum

$D_u f$  changes slope  $\Rightarrow$  nearby slopes are up in some directions and down in other directions

$\Rightarrow$  surface bends up and down

$\Rightarrow$  saddle

The test can be expressed in terms of the Jacobian matrix

$$D = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If  $\det(D) < 0 \Rightarrow$  we have a saddle point

If  $\det(D) < 0$  and  $a+c > 0 \Rightarrow$  we have a local minimum

If  $\det(D) < 0$  and  $a+c < 0 \Rightarrow$  we have a local maximum

#### Lagrange Multipliers

To find the extreme values:

1. First get it into a Lagrangian function:  $L(x, y) = f(x, y) + \lambda g(x, y)$

2. Solve for the critical point by finding out when

$$0 = \frac{\partial L}{\partial x} = f_x(x, y) + \lambda g(x, y) \quad 0 = \frac{\partial L}{\partial y} = f_y(x, y) + \lambda g(x, y)$$

3. Once the critical points have been found, plug it back into the function and test to see which point satisfies the given criteria.

$-\nabla f(x, y) = \text{Direction of the steepest slope downwards in the XY plane.}$

$\|\nabla f(x, y)\| = \text{Slope in steepest direction}$

$D_u f = \hat{u} \cdot \nabla f = \text{Directional derivative of } f \text{ with respect to } \hat{u}$

$\hat{u} + \hat{k} D_u f = \text{Tangent vector with respect to } \hat{u}$

$\hat{k} - \nabla f = \text{Upward normal vector to the surface}$   
 $= \text{Perpendicular to all tangent vectors}$   
 $= \text{Normal vector to the tangent plane.}$

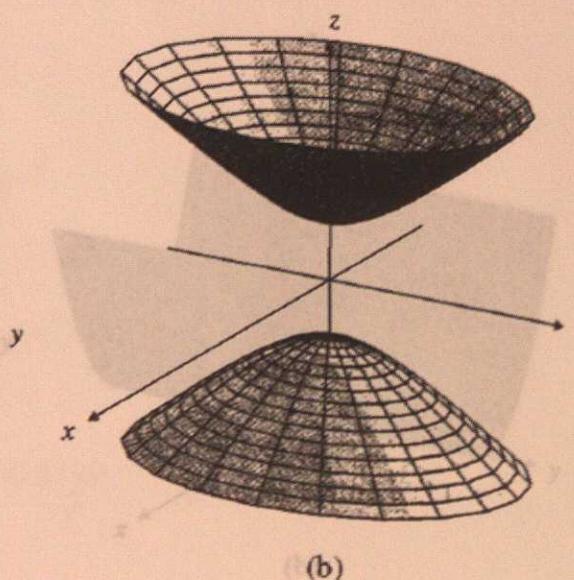
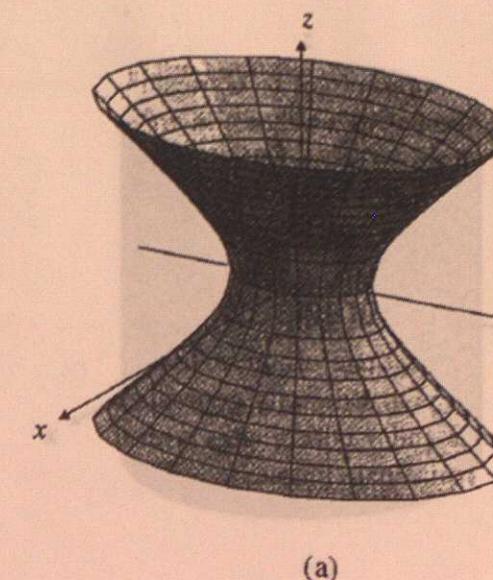
figure 10.37

(a) The hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

(b) The hyperboloid of two sheets

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$



(a)

(b)

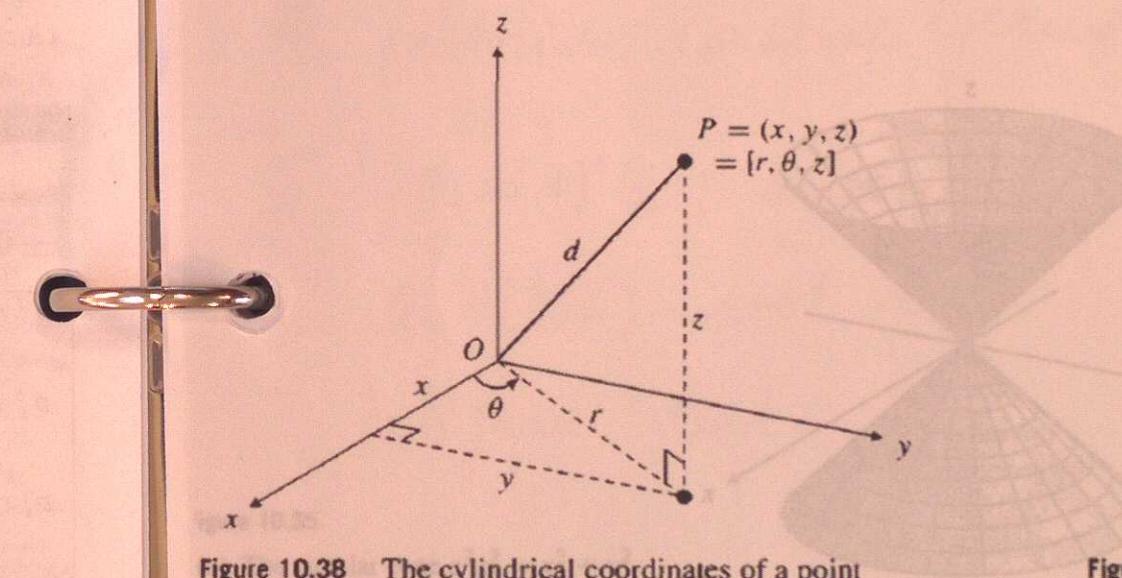


Figure 10.38 The cylindrical coordinates of a point

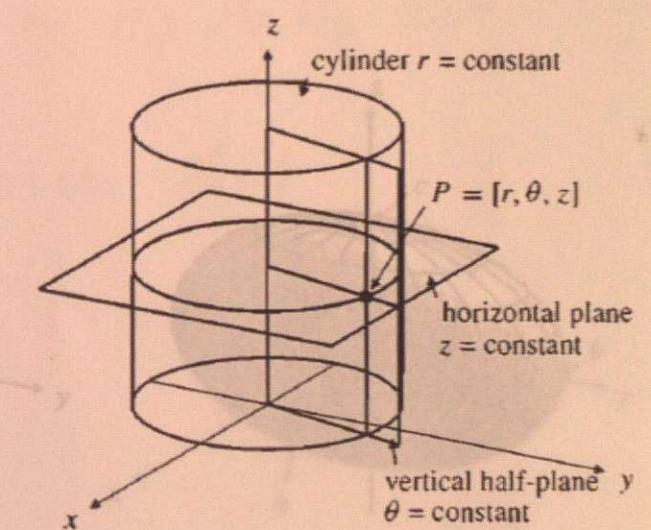


Figure 10.39 The coordinate surfaces for cylindrical coordinates

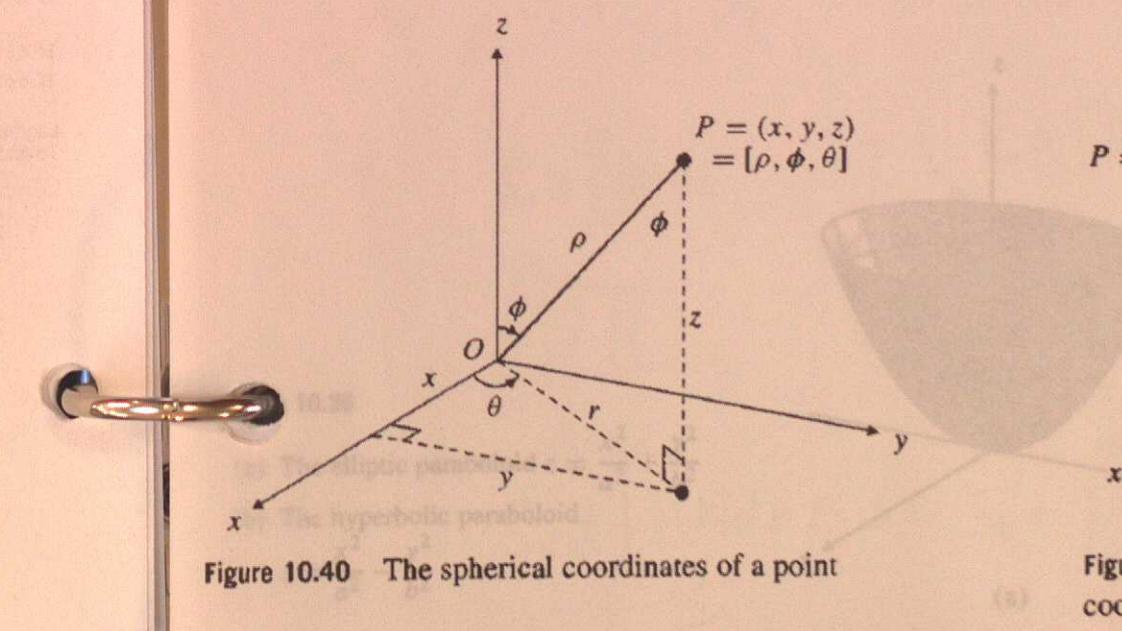


Figure 10.40 The spherical coordinates of a point

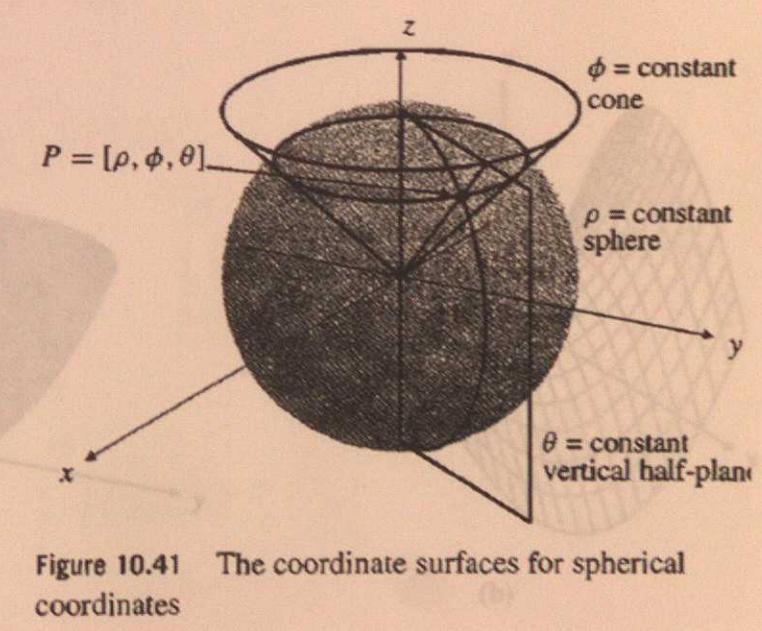
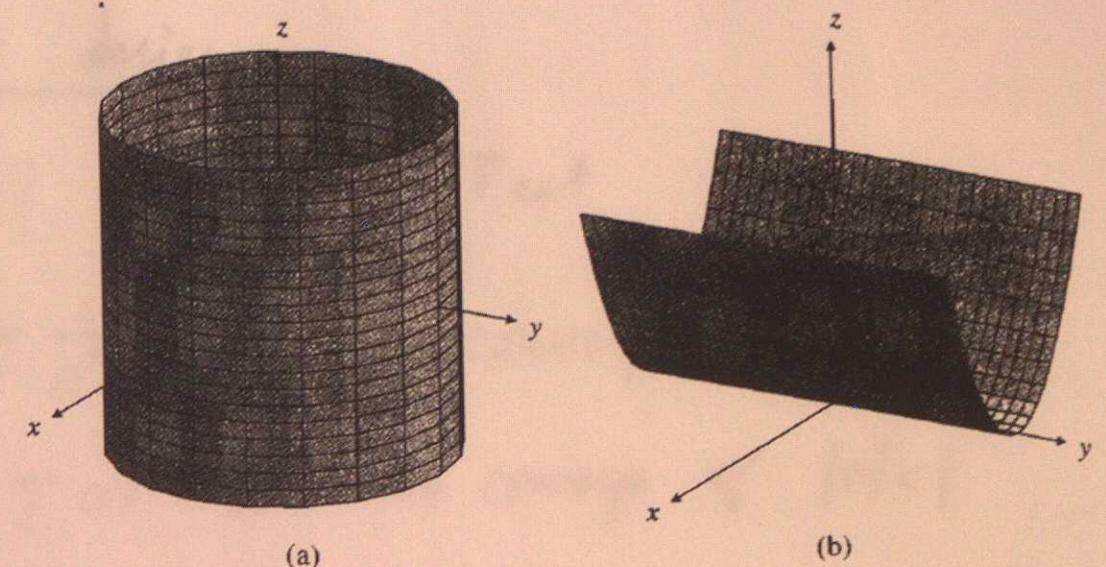
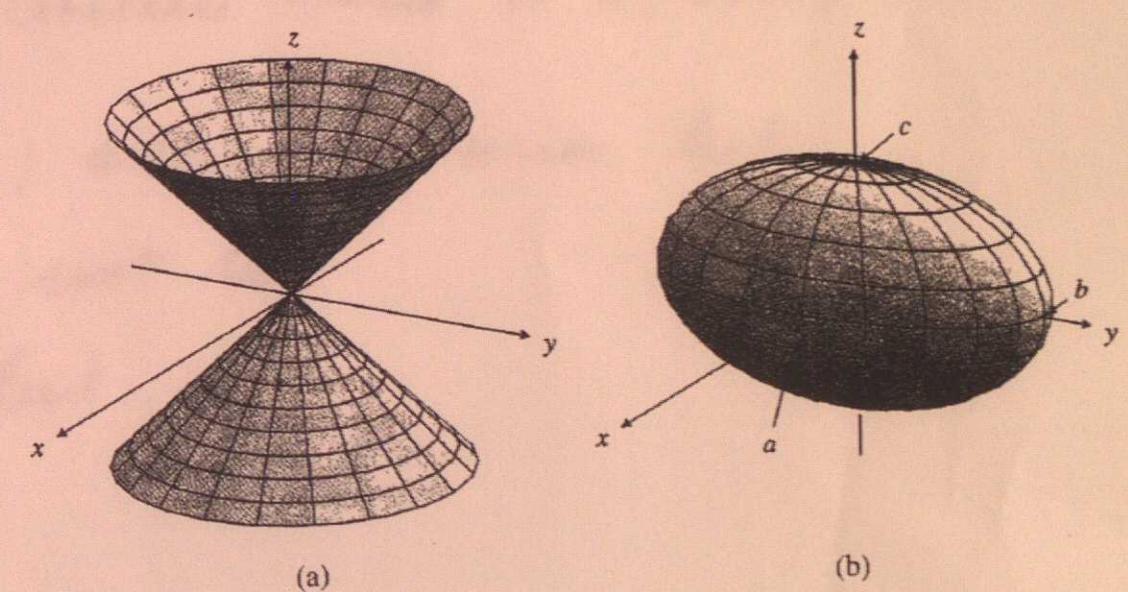


Figure 10.41 The coordinate surfaces for spherical coordinates



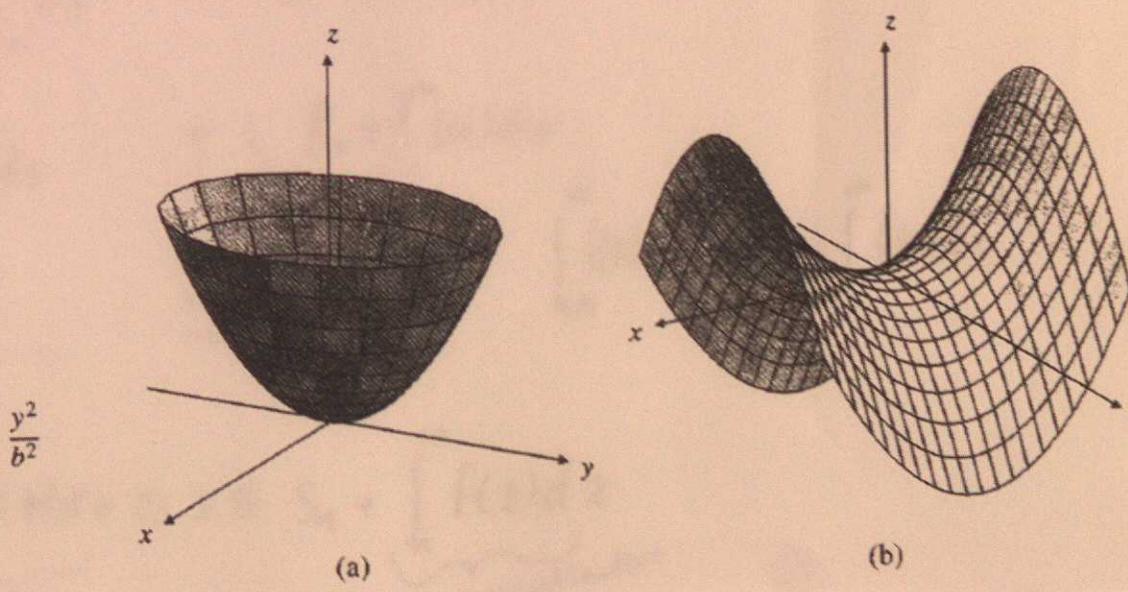
**Figure 10.34**

- (a) The circular cylinder  
 $x^2 + y^2 = a^2$
- (b) The parabolic cylinder  $z = x^2$



**Figure 10.35**

- (a) The circular cone  $a^2 z^2 = x^2 + y^2$
- (b) The ellipsoid  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



**10.36**

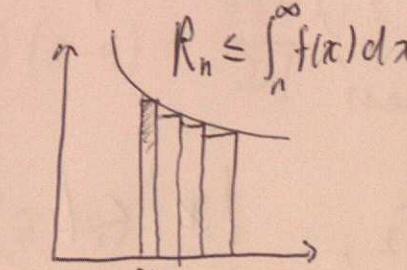
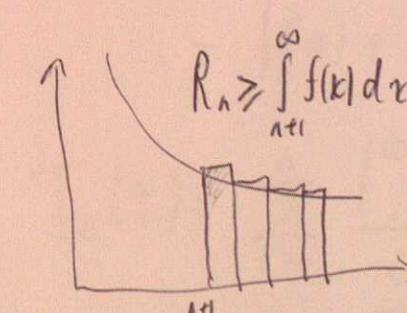
- (a) The elliptic paraboloid  $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
- (b) The hyperbolic paraboloid  
$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

## Strategy for Series

1.  $\lim_{n \rightarrow \infty} a_n \neq 0$ ? Divergence Test
2. p-series:  $\sum \frac{1}{n^p}$  will only converge if  $p > 1$   
geo-series:  $\sum ar^n$  will only converge if  $|r| < 1$
3. compare to a p-series or geo-series?
4. factorials, constants raised to n; ratio test
5.  $a_n = (-1)^n b_n$ ; alternating series test
6.  $a_n = (b_n)^n$ ; root test
7. Integral test

## Estimation

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^n a_i + \sum_{i=n+1}^{\infty} a_i \rightarrow s = s_n + R_n$$



$$\int_n^{\infty} f(x) dx \leq R_n \leq \int_{n+1}^{\infty} f(x) dx$$

$$s_n + \int_{n+1}^{\infty} f(x) dx \leq s \leq s_n + \int_n^{\infty} f(x) dx$$

*under* *overestimate*

## Estimation for Alt Series

$$|R_n| = |S - S_n| \leq b_{n+1}$$

## Power Series

$$\sum_{n=0}^{\infty} C_n (x-a)^n$$

$ x-a  < R$	$ x-a  > R$
converges	diverges

$a-R < x < a+R$  P-series converges  
 $x < a-R, x > a+R$  P-series diverges

$x=a$ , P-series converges

## Functions

$$\text{Recall: } \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad |r| < 1$$

let  $a=1, r=x$ , then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1$$

$$\therefore f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + \dots$$

$$f'(x) = \frac{d}{dx} \sum_{n=0}^{\infty} C_n (x-a)^n = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \dots$$

$$f''(x) = \sum_{n=0}^{\infty} n(n-1) C_n (x-a)^{n-2}$$

etc.

$$\begin{aligned} \int f(x) dx &= \int \sum_{n=0}^{\infty} C_n (x-a)^n dx \\ &= C + \sum_{n=0}^{\infty} C_n \frac{(x-a)^{n+1}}{n+1} \end{aligned}$$

## Taylor Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$T_n(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x-a)^i$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

## Calculus w/ Vector Functions

$$\begin{aligned}\lim_{t \rightarrow a} \vec{r}(t) &= \lim_{t \rightarrow a} \langle f(t), g(t), h(t) \rangle \\ &= \left\langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \right\rangle\end{aligned}$$

$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

$$\begin{aligned}\frac{d}{dt}(\vec{u} \cdot \vec{v}) &= \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}' \\ \frac{d}{dt}(\vec{u} \times \vec{v}) &= \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'\end{aligned}\quad \left. \begin{array}{l} \text{similar to} \\ \text{product rule} \end{array} \right\}$$

$$\frac{d}{dt}(\vec{u}(f(t))) = f'(t) \vec{u}'(f(t)) \quad \left. \begin{array}{l} \\ \text{chain rule} \end{array} \right\}$$

$$\int \vec{r}(t) = \langle \int f(t) dt, \int g(t) dt, \int h(t) dt \rangle + \vec{c}$$

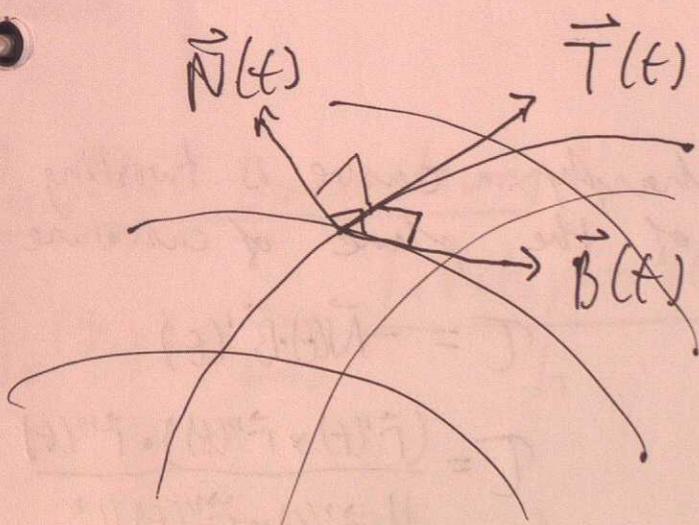
Tangent, Normal, Binormal  $\overset{\alpha}{TNB}$

$\vec{r}'(t)$  is the tangent vector, if  $\vec{r}'(t) \neq 0$ .

$$\text{Unit tangent vector } \vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

$$\text{Unit normal vector } \vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

$$\text{Binormal vector } \vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$



$$\vec{N}(t) = \vec{B}(t) \times \vec{T}(t)$$

$$\vec{T}(t) = \vec{N}(t) \times \vec{B}(t)$$

## Arc length

$$L = \int_a^b \|\vec{r}'(t)\| dt \quad s(t) = \int_0^t \|\vec{r}'(u)\| du$$

$$\text{Recall: } s(t) = \int ds = \int \sqrt{dx^2 + dy^2 + dz^2} = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

Reparametrize  $\vec{r}(t)$  to  $\vec{r}(s)$  after getting  $s(t)$  function to tell us where we are on the curve after we've travelled  $s$  distance  $s$  along the curve.

## Curvature, Torsion

how fast a curve is changing direction at a given point

$$K = \left| \frac{d\vec{T}}{ds} \right| = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

how sharply a curve is twisting out of the plane of curvature

$$\tau = -\vec{N}(t) \cdot \vec{B}'(t)$$

$$\tau = \frac{(\vec{r}'(t) \times \vec{r}''(t)) \cdot \vec{r}'''(t)}{\|\vec{r}'(t) \times \vec{r}''(t)\|^2}$$

## Velocity, Acceleration

$$\vec{v}(t) = \vec{r}'(t), \quad \vec{a}(t) = \vec{r}''(t) = a_T \vec{T} + a_N \vec{N}$$

$$a_T = v' = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{\|\vec{r}'(t)\|} \quad a_N = Kv^2 = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|}$$

## Multivariable Limits

Recall  $\lim_{x \rightarrow a} f(x) = L$  iff  $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L$

A function  $f(x, y)$  is continuous at  $(a, b)$  if  $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$

Find a path to evaluate the limit if the function is discontinuous at the limit.

## Partial Derivatives

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} \quad f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

$$f_z(x, y) = f_z = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z} (f(x, y)) = z_x = \frac{\partial z}{\partial x} = D_x f$$

$f_x(x, y)$  represents the change of  $f(x, y)$  as  $x$  changes and  $y$  is fixed.

$f_x(a,b)$  is the slope of the trace of  $f(x,y)$  for the plane  $y=b$  at the point  $(a,b)$ .

### Equation of Tangent Line

point  $(a, b, f(a,b))$

$$\vec{r}(x,y) = \langle x, y, z \rangle = \langle x, y, f(x,y) \rangle$$

$$\vec{r}_x(x,y) = \langle 1, 0, f_x(x,y) \rangle$$

### Higher Order Partial Derivatives

$$\begin{aligned} (f_x)_x &= f_{xx} = \frac{\partial}{\partial x} \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} \\ (f_x)_y &= f_{xy} = \frac{\partial}{\partial y} \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} \\ (f_y)_x &= f_{yx} = \frac{\partial}{\partial x} \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} \end{aligned}$$

$$f_{xyx} = (f_{xy})_x = \frac{\partial}{\partial x} \left( \frac{\partial^2 f}{\partial y \partial x} \right) = \frac{\partial^3 f}{\partial x \partial y \partial x}$$

### Clairaut's Theorem

Suppose that  $f$  is defined on a disk  $D$  that contains the point  $(a,b)$ . If  $f_{xy}$  and  $f_{yx}$  are continuous on this disk then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

(can extend to any function and mixed partial derivatives.)

$$f_{ssrtssr} = f_{trsrsr}$$

### Chain Rule

Recall: If  $y=f(x)$ ,  $x=g(t)$ , then  $\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$

Case 1:  $z=f(x,y)$ ,  $x=g(t)$ ,  $y=h(t)$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

Suppose  $x=g(r)$

$$\text{then } \frac{dz}{dr} = \frac{\partial f}{\partial x} \frac{dx}{dr} + \frac{\partial f}{\partial y} \frac{dy}{dr} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} \frac{dx}{dr}$$

Case 2:  $z = f(x, y)$ ,  $x = g(s, t)$ ,  $y = h(s, t)$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \quad \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}$$

general:  $z = f(x_1, x_2, \dots)$   $x_i = g(t_1, t_2, \dots)$

$$\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial z}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \dots$$

$$\begin{array}{ccccc} & z & & & \\ \frac{\partial z}{\partial x} & & & & \frac{\partial z}{\partial x} \\ & x & & y & \\ \frac{\partial x}{\partial s} & & \frac{\partial x}{\partial t} & & \frac{\partial y}{\partial s} \\ s & & t & & s \\ & & & & t \end{array}$$

Implicit:  $F(x, y) = 0$ ,

$$F_x + F_y \frac{dy}{dx} = 0 \quad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$F(x, y, z) = 0, \quad z = f(x, y), \quad \frac{\partial F}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} = 0 \quad \therefore \quad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

### Directional Derivatives

The rate of change of  $f(x, y)$  in the direction of the unit vector  $\vec{u} = \langle a, b \rangle$  is called the directional derivative  $D_{\vec{u}} f(x, y)$ .

$$D_{\vec{u}} f(x, y) = \lim_{h \rightarrow 0} \frac{f(x+ah, y+bh) - f(x, y)}{h}$$

more practically...

$$x = x_0 + a_2 \quad y = y_0 + b_2$$

$$D_{\vec{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

or

$$D_{\vec{u}} f(x, y, z) = \underbrace{\langle f_x, f_y, f_z \rangle}_{\text{gradient}} \cdot \langle a, b, c \rangle$$

$$\nabla f = \langle f_x, f_y, f_z \rangle = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$$

$$\therefore D_{\vec{u}} f(\vec{x}) = \nabla f \cdot \vec{u}$$

Max value of  $D_{\vec{u}} f(\vec{x})$  (hence the max rate of change) is given by  $\|\nabla f(\vec{x})\|$  and will occur in the direction  $\nabla f(\vec{x})$ .

i.e. when  $\vec{u} \parallel \nabla f(\vec{x})$ .

Fact: The gradient vector  $\nabla f(x_0, y_0, z_0)$  is  $\perp$  to the level surface  $f(x, y, z) = k$  at the point  $(x_0, y_0, z_0)$

Recall: equation of a plane containing  $(x_0, y_0, z_0)$  w/ normal vector  $\vec{n} = \langle a, b, c \rangle$  is given by

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

tangent plane

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

normal line

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t \nabla f(x_0, y_0, z_0)$$

### Extrema

A function  $f(x, y)$  has a relative minimum at point  $(a, b)$  if  $f(x, y) \geq f(a, b)$  for all points  $(x, y)$  in some region around  $(a, b)$ .

### Critical Point "Stationary Point"

The point  $(a, b)$  is a critical point of  $f(x, y)$  if  $\nabla f(a, b) = \vec{0}$   $\left\{ \begin{array}{l} f_x(a, b) = 0 \\ f_y(a, b) = 0 \end{array} \right.$

or

$f_x(a, b)$  DNE or  $f_y(a, b)$  DNE

If  $(a, b)$  is a relative extremum of  $f(x, y)$  then  $(a, b)$  is also a critical point of  $f(x, y)$  and  $\nabla f(a, b) = \vec{0}$ .

Not all critical points are relative extrema!

Define  $D(a,b)$

$$D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

If  $D > 0$  and  $f_{xx}(a,b) > 0$  then relative min.

If  $D > 0$  and  $f_{xx}(a,b) < 0$  then there is relative max

If  $D < 0$  then the point  $(a,b)$  is a saddle point.

If  $D=0$ , we gain no new information.

Saddle point curves up in one direction and down in a different direction.

### Extreme Value Theorem

If  $f(x,y)$  is continuous on some closed bounded set  $D$  in  $\mathbb{R}^2$  then there are points in  $D$ ,  $(x_1, y_1)$  and  $(x_2, y_2)$  so that  $f(x_1, y_1)$  and  $f(x_2, y_2)$  are absolute maximum and minimum respectively of the function in  $D$ .

### Finding Absolute Extrema

1. Find all critical points and determine their values
2. Find all extrema of the function on the boundary.

### Lagrange Multipliers

Let us optimize  $f(x,y,z)$  subject to the constraint  $g(x,y,z)=k$

$$\begin{aligned} 1. \quad \nabla f(x,y,z) &= \lambda \nabla g(x,y,z) \\ g(x,y,z) &= k \end{aligned}$$

2. plug in all solutions  $(x,y,z)$  into  $f$  and identify max and min values.

If  $\exists t$  constraints then

$$\begin{aligned} \nabla f(x,y,z) &= \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \\ g(x,y,z) &= c \\ h(x,y,z) &= k \end{aligned}$$

## 2-D Hessian

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \quad D_1 = f_{xx} \quad D_2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

- If  $D_1(a,b) > 0$  and  $D_2(a,b) > 0$ , then local min.
- If  $D_1(a,b) < 0$  and  $D_2(a,b) > 0$ , then local max.
- If  $D_2(a,b) < 0$ , then saddle point.

## 3-D Hessian

$$H = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} \quad D_1 = f_{xx} \quad D_2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \quad D_3 = \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}$$

- If  $D_1, D_2, D_3 > 0$ , local min
- If  $D_1 < 0, D_2 > 0, D_3 < 0$ , local max
- If  $D_3 \neq 0$  in any other case, saddle point

## Double Integrals

Recall in 1-D:

$$A \approx f(x_1^*) \Delta x + f(x_2^*) \Delta x + \dots + f(x_n^*) \Delta x$$

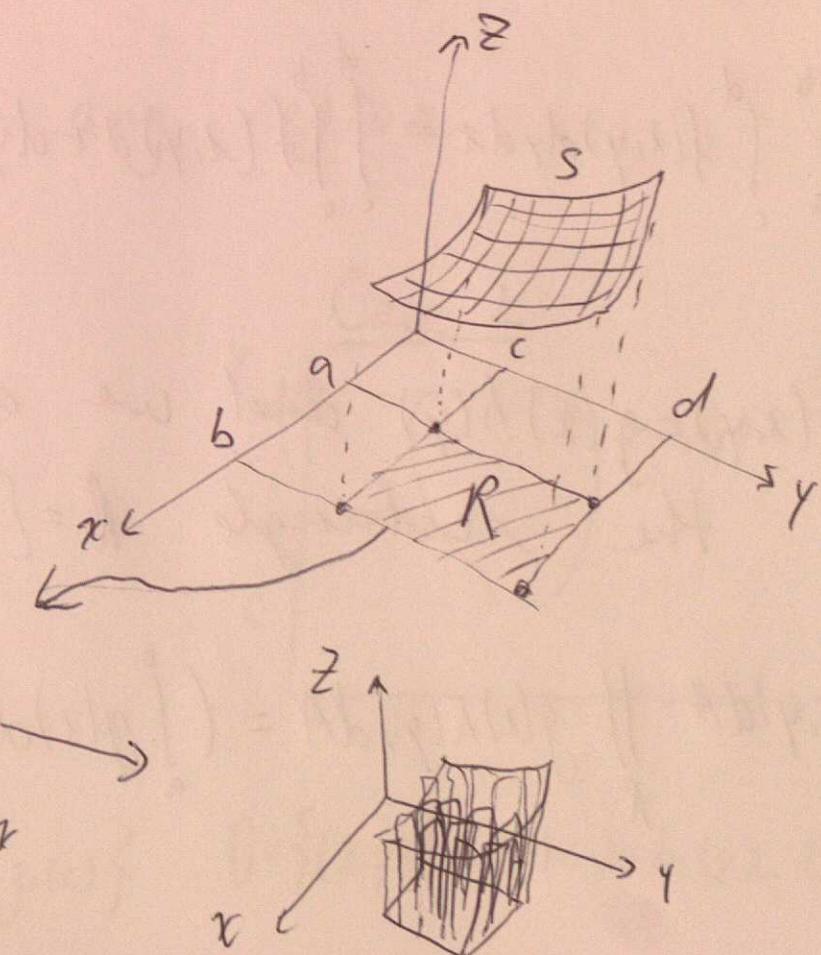
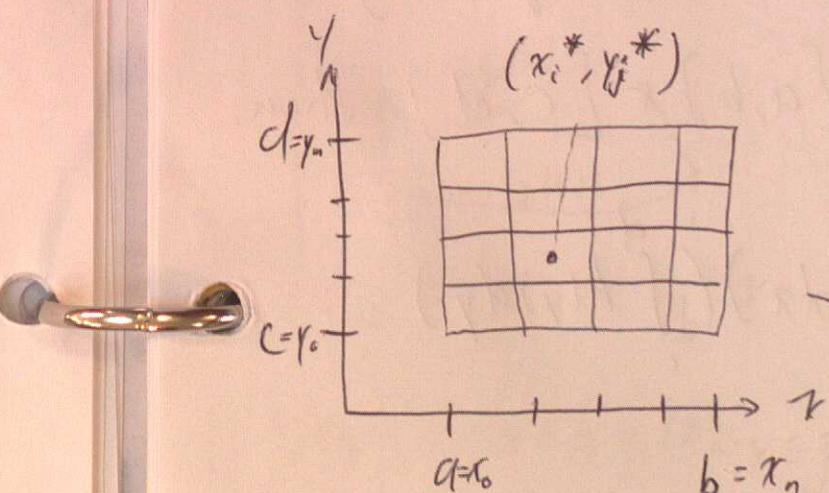
and as  $n \rightarrow \infty$

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

In one variable we integrated over an interval (1-D space), so in two variables, we integrate over  $\mathbb{R}^2$ .

$$R = [a, b] \times [c, d]$$

$$\begin{aligned} a &\leq x \leq b \\ c &\leq y \leq d \end{aligned}$$



Each rectangle has base area  $\Delta A$ ,  
height of  $f(x_i^*, y_j^*)$ .

$$V \approx \sum_{i=1}^n \sum_{j=1}^m f(x_i^*, y_j^*) \Delta A$$

rectangle region

$$\iint_R f(x, y) dA = \lim_{n,m \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m f(x_i^*, y_j^*) \Delta A$$

### Iterated Integrals

Fubini's Theorem: If  $f(x, y)$  is continuous on  $R = [a, b] \times [c, d]$  then,

$$\int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

If  $f(x, y) = g(x) h(y)$  and we are integrating over the rectangle  $R = [a, b] \times [c, d]$  then,

$$\iint_R f(x, y) dA = \iint_R g(x) h(y) dA = \left( \int_a^b g(x) dx \right) \left( \int_c^d h(y) dy \right)$$

### Constants of Integration

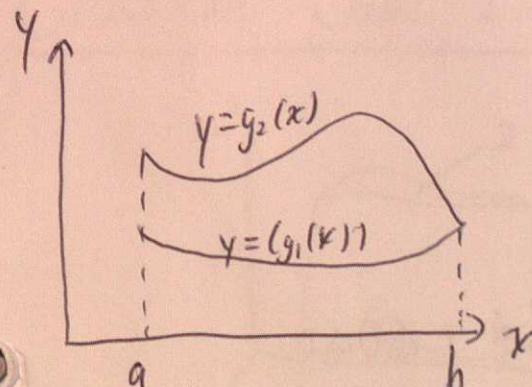
$$\begin{aligned} \text{Ex. } & \int [x \sec^2(2y) + 4xy] dy \\ &= \frac{x}{2} \tan(2y) + 2xy^2 + g(x) \end{aligned}$$

Any function involving only  $x$ 's will differentiate to zero, so when integrating wrt  $y$ , we need to acknowledge that there may have been a function of only  $x$ 's in the function, so the constant of integration is  $g(x)$ .

### Double Integrals over General Regions

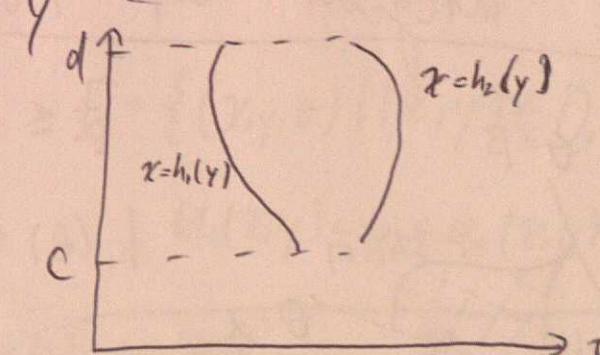
$$\iint_D f(x, y) dA, \quad D \text{ is any region.}$$

#### Case 1



$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

#### Case 2



$$D = \{(x, y) \mid h_1(y) \leq x \leq h_2(y), c \leq y \leq d\}$$

Case 1

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) dy dx$$

Case 2

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y) dx dy$$

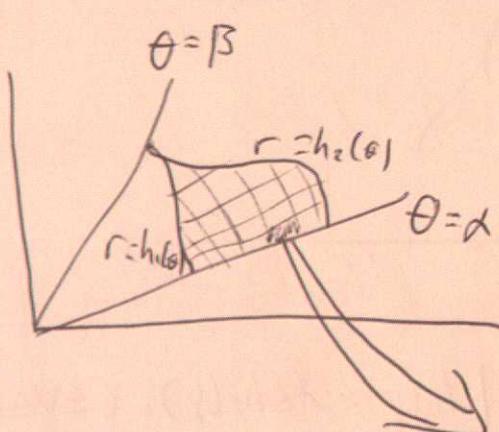
∴ The volume of the solid below the surface  $z = f(x,y)$  and above the region  $D$  in the  $xy$ -plane is given by

$$V = \iint_D f(x,y) dA.$$

The area of region  $D$  is given by

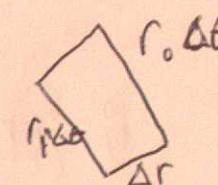
$$A \text{ of } D = \iint_D dA.$$

Double Integrals w/ Polar



$$\alpha \leq \theta \leq \beta$$

$$h_1(\theta) \leq r \leq h_2(\theta)$$



$$\Delta A \approx r \Delta \theta \Delta r$$

$$dA \approx \Delta A, \quad d\theta \approx \Delta \theta, \quad dr \approx \Delta r$$

$$\therefore [dA = r dr d\theta]$$

$$\text{Recall: } x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2$$

$$\therefore \iint_D f(x,y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta$$

### Triple Integrals

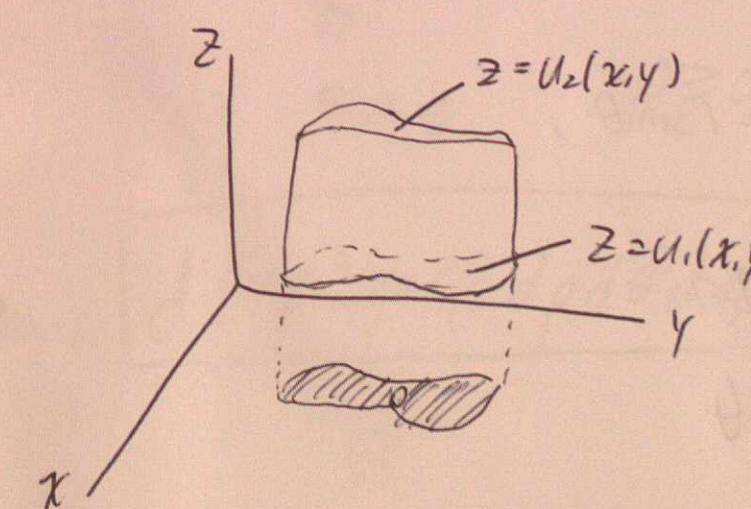
Let's start by integrating over the box

$$B = [a,b] \times [c,d] \times [r,s], \quad \text{then}$$

$$\iiint_B f(x,y,z) dV = \int_r^s \int_c^d \int_a^b f(x,y,z) dx dy dz$$

$$\boxed{\text{Volume}_E = \iiint_E dV}$$

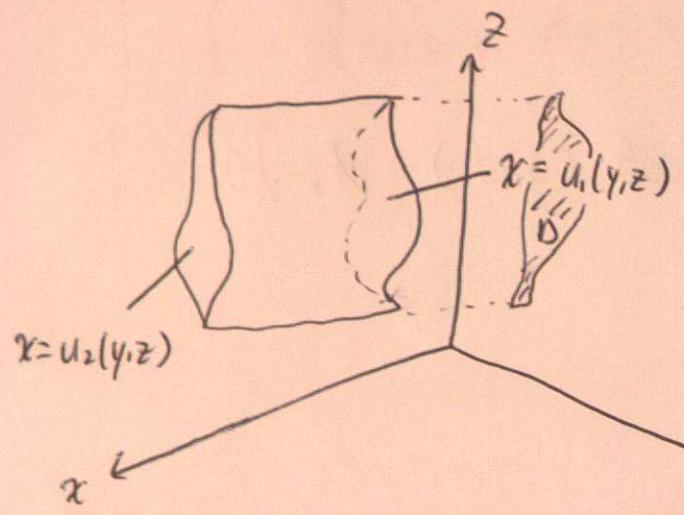
General Case 1



$$E = \{(x,y,z) | (x,y) \in D, u_1(x,y) \leq z \leq u_2(x,y)\}$$

$$\boxed{\int_0^s \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z) dz dA}$$

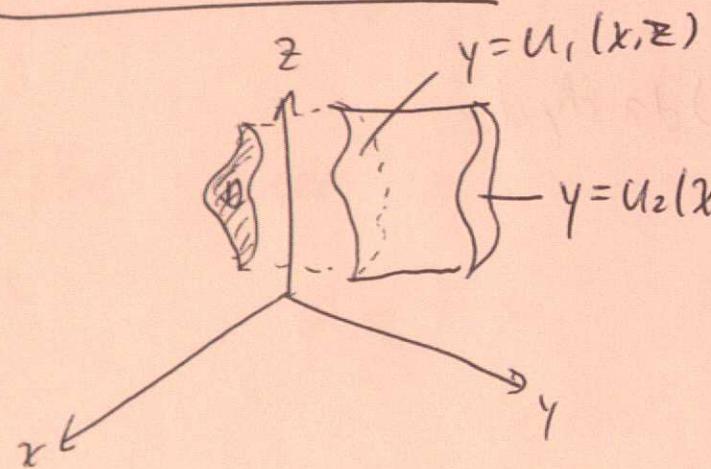
### General Case 2



$$E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$$

$$\iint_D \int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z) dx dy dz$$

### General Case 3



$$E = \{(x, y, z) \mid (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$$

$$\iint_D \int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z) dy dx dz$$

### Cylindrical Coordinates

$$\text{Recall: } x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$$

~~$$dV = z dA dz$$~~

$$dV = r dz dr d\theta$$

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

$$= \{(r, \theta, z) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta), u_1(r \cos \theta, r \sin \theta) \leq z \leq u_2(r \cos \theta, r \sin \theta)\}$$

for  $D$  in  $x$ - $y$  plane. This can be modified  
for  $yz$  or  $xz$ .

$$\iiint_E f(x, y, z) dV = \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} r f(r \cos \theta, r \sin \theta, z) dz dr d\theta$$

### Spherical Coordinates

$$(x, y, z) = (\rho, \theta, \phi)$$

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

$$\rho^2 + y^2 + z^2 = \rho^2$$

$$\rho \geq 0, \quad 0 \leq \phi \leq \pi$$

$$dV = \rho^2 \sin \phi d\rho d\theta d\phi \rightarrow \iiint_E f(x, y, z) dV$$

Recall

$$\int_a^b f(g(x)) g'(x) dx = \int_c^d f(u) du, \quad u = g(x)$$

### Change of Variables

The Jacobian of the transformation  $x=g(u,v), y=h(u,v)$

is  $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$

Under the transformation  $x=g(u,v), y=h(u,v)$ :

$$\iint_D f(x,y) dA = \iint_S f(g(u,v), h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$$

For Triple:

$$dV = \begin{vmatrix} \frac{\partial v}{\partial u} & \frac{\partial v}{\partial v} & \frac{\partial v}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} du dv dw$$

### Surface Area

$$S = \iint_D \sqrt{f_x^2 + f_y^2 + 1} dA, \quad z = f(x,y)$$

$$\vec{r}(x,y) = \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix}$$

$$\|\vec{r}_x \times \vec{r}_y\| = \sqrt{f_x^2 + f_y^2 + 1}$$

area