McGill University April 2014
Faculty of Science Final Examination

Calculus 3

Math 222

Tuesday April 29, 2014
Solutions

Problem 1 (6 points)
Let $\mathbf{r}(t) = (t, \cos^2 t, \sin^2 t)$.

i. *Find the velocity* $\mathbf{r}'(t)$ *and the acceleration* $\mathbf{r}''(t)$.

ii. *Find the tangential and normal components of the acceleration*

Hint: *you can use the formulas* $\sin(2\alpha) = 2\sin\alpha \cdot \cos\alpha$ *and* $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$.

Solution: We find that $\mathbf{r}'(t) = (1, -2\cos t \sin t, 2\sin t, \cos t) = (1, -\sin(2t), \sin(2t))$. Next $\mathbf{r}''(t) = (0, -2\cos(2t), 2\cos(2t))$. We note that $||\mathbf{r}'(t)|| = \sqrt{1 + 2\sin^2(2t)}$.
The tangential component of the acceleration is given by the formula

$$a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{||\mathbf{r}'(t)||} = \frac{4\sin(2t)\cos(2t)}{\sqrt{1 + 2\sin^2(2t)}} = \frac{2\sin(4t)}{\sqrt{1 + 2\sin^2(2t)}}.$$

The normal component of the acceleration is

$$a_N = \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||} = \frac{||(-2)(0, \cos(2t), \cos(2t))||}{\sqrt{1 + 2\sin^2(2t)}} = \frac{2\sqrt{2}|\cos(2t)|}{\sqrt{1 + 2\sin^2(2t)}}$$

Problem 2 (6 points)

i. *Write the Taylor series about* $x = 0$ *of* $\ln(1 + x)$. *You do not need to justify your answer.*

ii. *Use part i. to write the Taylor series about* $x = 0$ *of* $\ln(1 + x^3)$.

iii. *Write the Taylor series about* $x = 0$ *of* $f(x) = \int_0^x \ln(1 + t^3)dt$.

Solution: We have $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}x^n/n = x - x^2/2 + x^3/3 \pm \ldots$. It follows that $\ln(1+x^3) = \sum_{n=1}^{\infty}(-1)^{n-1}x^{3n}/n = x^3 - x^6/2 + x^9/3 \pm \ldots$. Integrating term by term, we get

$$\int_0^x \ln(1+t^3)dt = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{3n+1}}{n(3n+1)} = x^4/4 - x^7/14 + x^{10}/30 \pm \ldots$$

Problem 3 (6 points)
Let $f(x,y,z) = x^2\cos(y)z^3$.

i. *Find the gradient $\nabla f(p)$ at the point $p = (1,\pi,-1)$. In which direction does the function f increase the most?*

ii. *Find the directional derivative $D_u f(p)$ where u is the vector $(1,2,1)$.*

iii. *Let S be the level surface of f passing through $(1,\pi,-1)$. Find an equation for the tangent plane to S at $(1,\pi,-1)$.*

Solution: We have $\nabla f = (f_x, f_y, f_z) = (2x\cos(y)z^3, -x^2\sin(y)z^3, 3x^2\cos(y)z^2)$. At the point $p = (1,\pi,-1)$, $\nabla f = (2,0,-3)$. The function f increases most in the direction $\nabla f(p) = (2,0,-3)$. The directional derivative is equal to

$$DUf(p) = \nabla f(p)\cdot u = (2,0,-3)\cdot(1,2,1) = 2-3 = -1.$$

The equation of the tangent plane at the point $p = (x_0,y_0,z_0)$ is $(x-x_0, y-y_0, z-z_0)\cdot\nabla f(p) = 0$. In our case it becomes

$$0 = (x-1, y-\pi, z+1)\cdot(2,0,-3) = 2x-2-3z-3 = 2x-3z-5,$$

or $2x - 3z = 5$.

Problem 4 (6 points)
Let x, y, z satisfy an equation: $x^2 - y^2 + z^2 - 2z = 4$. Find $\partial z/\partial x$ and $\partial z/\partial y$.

Solution: Differentiating with respect to x and using implicit differentiation, we find that $2x + 2z\cdot z_x - 2z_x = 0$. It follows that $z_x = x/(1-z)$.
Differentiating with respect to y and using implicit differentiation, we find that $-2y + 2z\cdot z_y - 2z_y = 0$. It follows that $z_y = y/(z-1)$.

Problem 5 (6 points)
Find all local maxima, local minima, and saddle points of the function

$$f(x,y) = x\cos y - x^3/3$$

in the region $\{(x,y) : |y| < \pi\}$.

Solution: At a critical point,

$$\nabla f = (\cos y - x^2, -x \sin y) = (0, 0).$$

Since $x \sin y = 0$, either $x = 0$ (case 1); or $\sin y = 0$, which for $|y| < \pi$ only happens when $y = 0$ (case 2).
In case 1, we $\cos y - 0^2 = 0$, so $\cos y = 0, y = \pm\pi/2$, so we get points $(0, \pi/2)$ and $(0, -\pi/2)$ (we have used that $|y| < \pi$).
In case 2, we have $\cos y = \cos 0 = 1$, so $x^2 = 1, x = \pm 1$; we get points $(1, 0)$ and $(-1, 0)$.
The Hessian of f is equal to

$$\begin{pmatrix} -2x & -\sin y \\ -\sin y & -x\cos y \end{pmatrix}$$

At the points $(0, \pm\pi/2)$ the determinant is equal to $-\sin^2(\pm\pi/2) = -1$, so we get saddle points by the Hessian criterion. At the point $(1, 0)$, the Hessian is

$$\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix},$$

so that point is a local maximum.
At the point $(-1, 0)$, the Hessian is

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},$$

so that point is a local minimum.

Problem 6 (6 points)
Use the method of **Lagrange multipliers** *to find the minimum value of the function*

$$f(x, y) = (x^3 + y^3)/3$$

subject to the constraint $g(x, y) = xy = 4$.

Solution: The hyperbola $g(x, y) = xy = 4$ has two branches: on the first branch $x > 0, y > 0$, and so $f(x, y) > 0$, and also it is easy to see that $f(x, y) \to \infty$ as either $x \to \infty$ or $y \to \infty$; we shall restrict ourselves to that branch.
On the 2nd branch, $x < 0, y < 0$, and it is easy to see that $f(x, y) \to -\infty$ as either $x \to -\infty$ or $y \to -\infty$; so the minimum of f restricted to that branch is $-\infty$.
On the first branch, we define $F(x, y) := f - \lambda g = (x^3 + y^3)/3 - \lambda(xy - 4)$. We have

$$\begin{aligned} F_x &= x^2 - \lambda y = 0, \\ F_y &= y^2 - \lambda x = 0, \\ F_\lambda &= xy - 4 = 0. \end{aligned}$$

If $\lambda = 0$, then from the first equation we have $x = 0$, and so the constraint equation $xy = 4$ is not satisfied, therefore $\lambda \neq 0$. Dividing the first two equations, we get $x^2/y^2 = y/x$, or $x^3 = y^3$. This implies $x = y$. Substituting into the last equation, we find $x = y = 2$. The minimum is $f(2, 2) = 16/3$.

Problem 7 (6 points)
Use **polar coordinates** *to find the volume of the region bounded by the paraboloids* $z = 3x^2 + 3y^2$ *and* $z = 4 - x^2 - y^2$.

Solution: We use polar coordinates (r, θ) in the (x, y)-plane, or, equivalently, cylindrical coordinates in $\mathbf{R}^3$. The equations of the paraboloids become $z = 3r^2$ and $z = 4 - r^2$. They intersect above the circle $4 - r^2 = 3r^2$ or $r = 1$. The paraboloid $z = 3r^2$ lies *below* the paraboloid $z = 4 - r^2$, so the limits of integration for z will be $3r^r \leq z \leq 4 - r^2$.
The volume is given by

$$\int_{r=0}^{1} \int_{\theta=0}^{2\pi} \int_{z=3r^2}^{4-r^2} dz r dr d\theta = 2\pi \int_0^1 (4 - 4r^2) r dr = 8\pi [r^2/2 - r^4/4]_0^1 = 2\pi.$$

Problem 8 (6 points)
Let D be the bounded region of the plane which is enclosed by the curves $y = 0, y = x^2$ and $x = 1$. Evaluate the following double integral:

$$\iint_D x \sin y \; dA.$$

Solution: The limits of integration are $x \in [0, 1]$, $y \in [0, x^2]$. The integral is

$$\int_{x=0}^{1} \int_{y=0}^{x^2} x \sin y dy dx = \int_0^1 x[-\cos(y)]_0^{x^2} dx = \int_0^1 x(1 - \cos(x^2)) dx = \frac{1}{2} - \int_0^1 x \cos(x^2) dx.$$

To compute the last integral, we note that $(d/dx)(-\sin(x^2)/2) = -x\cos(x^2)$, therefore the previous expression becomes

$$\frac{1}{2} - \int_0^1 (d/dx)(\sin(x^2)/2) dx = 1/2 - [\sin(x^2)/2]_0^1 = \frac{1 - \sin(1)}{2}.$$

Problem 9 (6 points)
Use **spherical coordinates** *to evaluate the triple integral*

$$\iiint_D xyz \; dV,$$

where D is the region lying between the spheres of radius $\rho = 2$ and $\rho = 4$, and above the cone $\phi = \pi/3$.

Solution: The limits of integration are $\phi \in [0, \pi/3], \rho \in [2, 4], \theta \in [0, 2\pi]$. The function $xyz = \rho^3 \sin^2\phi \cos\phi \sin\theta \cos\theta = \rho^3 \sin^2\phi \cdot \sin(2\theta)/2$. The integral is

$$\int_2^4 \int_0^{\frac{\pi}{3}} \int_0^{2\pi} \rho^3 \sin^2\phi \cos\phi \frac{\sin(2\theta)}{2} \rho^2 \sin\phi d\theta d\phi d\rho = \int_2^4 \rho^5 d\rho \int_0^{\frac{\pi}{3}} \sin^3\phi \cos\phi d\phi \int_0^{2\pi} \frac{\sin(2\theta)}{2} d\theta.$$

The last integral is equal to $[-\cos(2\theta)/4]_0^{2\pi} = 0$, hence the product of the three integrals is also equal to 0. Answer: 0.

Problem 10 (6 points)
Use the transformation $x = u/v, y = v$ to evaluate the double integral

$$\iint_D xy \; dA,$$

where D is the region in the first quadrant bounded by the lines $y = x$ and $y = 3x$ and the hyperbolas $xy = 1$ and $xy = 3$.

Solution: The integrand $xy = u$ in the new coordinates. The limits of integration are $xy = u \in [1, 3]$; straight lines $y = kx$ become $v = ku/v$ or $u = v^2/k$, so $v^2/u \in [1, 3]$ or equivalently $v \in [\sqrt{u}, \sqrt{3u}]$. The Jacobian is

$$\det \begin{pmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{pmatrix} = 1/v.$$

The integral is equal to

$$\int_{u=1}^{3} \int_{v=\sqrt{u}}^{\sqrt{3u}} \frac{u}{v} dv du = \int_1^3 u[\ln v]_{\sqrt{u}}^{\sqrt{3u}} du = \int_1^3 u \frac{(\ln 3 + \ln u - \ln u)}{2} du = \frac{\ln 3}{2} \int_1^3 u du$$

That expression is equal to

$$\frac{\ln 3(9-1)}{2 \cdot 2} = 2 \ln 3.$$