Preface

Here are the solutions to the practice problems for my Calculus II notes. Some solutions will have more or less detail than other solutions. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you've reached the level of working the harder problems then you will probably already understand the basics fairly well and won't need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven't been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

Power Series

1. For the following power series determine the interval and radius of convergence.

$$\sum_{n=0}^{\infty} \frac{1}{(-3)^{2+n}\left(n^2+1\right)}(4x-12)^n$$

Step 1
Okay, let's start off with the Ratio Test to get our hands on *L*.

$$L = \lim_{n\to\infty}\left|\frac{(4x-12)^{n+1}}{(-3)^{3+n}\left((n+1)^2+1\right)}\frac{(-3)^{2+n}\left(n^2+1\right)}{(4x-12)^n}\right| = \lim_{n\to\infty}\left|\frac{(4x-12)}{(-3)\left((n+1)^2+1\right)}\frac{\left(n^2+1\right)}{1}\right|$$

$$= |4x-12|\lim_{n\to\infty}\frac{\left(n^2+1\right)}{3\left((n+1)^2+1\right)} = \frac{1}{3}|4x-12|$$

Step 2
So, we know that the series will converge if,

$$\frac{1}{3}|4x-12|<1 \quad\rightarrow\quad \frac{4}{3}|x-3|<1 \quad\rightarrow\quad |x-3|<\frac{3}{4}$$

Step 3

So, from the previous step we see that the radius of convergence is $\boxed{R = \frac{3}{4}}$.

Step 4
Now, let's start working on the interval of convergence. Let's break up the inequality we got in Step 2.

$$-\frac{3}{4} < x - 3 < \frac{3}{4} \qquad \rightarrow \qquad \frac{9}{4} < x < \frac{15}{4}$$

Step 5
To finalize the interval of convergence we need to check the end points of the inequality from Step 4.

$$x = \frac{9}{4} \; : \; \sum_{n=0}^{\infty} \frac{1}{(-3)^{2+n}\left(n^2+1\right)}(-3)^n = \sum_{n=0}^{\infty} \frac{1}{(-3)^2\left(n^2+1\right)} = \sum_{n=0}^{\infty} \frac{1}{9\left(n^2+1\right)}$$

$$x = \frac{15}{4} \; : \; \sum_{n=0}^{\infty} \frac{1}{(-1)^{2+n}(3)^{2+n}\left(n^2+1\right)}(3)^n = \sum_{n=0}^{\infty} \frac{1}{(-1)^{2+n}(3)^2\left(n^2+1\right)} = \sum_{n=0}^{\infty} \frac{(-1)^{2+n}}{9\left(n^2+1\right)}$$

Now, we can do a quick Comparison Test on the first series to see that it converges and we can do a quick Alternating Series Test on the second series to see that is also converges.

We'll leave it to you to verify both of these statements.

Step 6
The interval of convergence is below and for summary purposes the radius of convergence is also shown.

$$\boxed{\text{Interval} \quad : \quad \frac{9}{4} \le x \le \frac{15}{4}} \qquad \qquad \boxed{R = \frac{3}{4}}$$

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2. For the following power series determine the interval and radius of convergence.

$$\sum_{n=0}^{\infty} \frac{n^{2n+1}}{4^{3n}}(2x+17)^n$$

Step 1
Okay, let's start off with the Root Test to get our hands on *L*.

$$L = \lim_{n\to\infty} \left| \frac{n^{2n+1}}{4^{3n}}(2x+17)^n \right|^{\frac{1}{n}} = \lim_{n\to\infty} \left| \frac{n^{2+\frac{1}{n}}}{4^3}(2x+17) \right| = |2x+17| \lim_{n\to\infty} \frac{n^{2+\frac{1}{n}}}{4^3}$$

Okay, we can see that , in this case, L will be infinite provided $x \neq -\frac{17}{2}$ and so the series will diverge for $x \neq -\frac{17}{2}$. We also know that the power series will converge for $x = \frac{17}{2}$ (this is the value of a for this series!).

Step 2
Therefore we know that the interval of convergence is $\boxed{x = -\frac{17}{2}}$ and the radius of convergence is $\boxed{R = 0}$.

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3. For the following power series determine the interval and radius of convergence.

$$\sum_{n=0}^{\infty} \frac{n+1}{(2n+1)!}(x-2)^n$$

Step 1
Okay, let's start off with the Ratio Test to get our hands on L.

$$L = \lim_{n\to\infty}\left|\frac{(n+2)(x-2)^{n+1}}{(2n+3)!}\frac{(2n+1)!}{(n+1)(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)(x-2)}{(2n+3)(2n+2)(2n+1)!}\frac{(2n+1)!}{(n+1)}\right|$$
$$= |x-2|\lim_{n\to\infty}\frac{n+2}{(2n+3)(2n+2)(n+1)} = 0$$

Okay, we can see that , in this case, $L = 0$ for every x.

Step 2
Therefore we know that the interval of convergence is $\boxed{-\infty < x < \infty}$ and the radius of convergence is $\boxed{R = \infty}$.

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4. For the following power series determine the interval and radius of convergence.

$$\sum_{n=0}^{\infty} \frac{4^{1+2n}}{5^{n+1}}(x+3)^n$$

Step 1
Okay, let's start off with the Ratio Test to get our hands on L.

$$L = \lim_{n\to\infty}\left|\frac{4^{3+2n}(x+3)^{n+1}}{5^{n+2}}\frac{5^{n+1}}{4^{1+2n}(x+3)^n}\right| = \lim_{n\to\infty}\left|\frac{4^2(x+3)}{5}\right| = |x+3|\lim_{n\to\infty}\frac{16}{5} = \frac{16}{5}|x+3|$$

Step 2
So, we know that the series will converge if,

$$\frac{16}{5}\left|x+3\right|<1 \qquad \to \qquad \left|x+3\right|<\frac{5}{16}$$

Step 3
So, from the previous step we see that the radius of convergence is $\boxed{R=\frac{5}{16}}$.

Step 4
Now, let’s start working on the interval of convergence. Let’s break up the inequality we got in Step 2.

$$-\frac{5}{16}<x+3<\frac{5}{16} \qquad \to \qquad -\frac{53}{16}<x<-\frac{43}{16}$$

Step 5
To finalize the interval of convergence we need to check the end points of the inequality from Step 4.

$$x=-\frac{53}{16}\ :\ \sum_{n=0}^{\infty}\frac{4^1 4^{2n}}{5^n 5^1}\left(-\frac{5}{16}\right)^n=\sum_{n=0}^{\infty}\frac{4\left(16^n\right)}{5^n\left(5\right)}\frac{\left(-1\right)^n 5^n}{16^n}=\sum_{n=0}^{\infty}\frac{4\left(-1\right)^n}{5}$$

$$x=-\frac{43}{16}\ :\ \sum_{n=0}^{\infty}\frac{4^1 4^{2n}}{5^n 5^1}\left(\frac{5}{16}\right)^n=\sum_{n=0}^{\infty}\frac{4\left(16^n\right)}{5^n\left(5\right)}\frac{{}^n 5^n}{16^n}=\sum_{n=0}^{\infty}\frac{4}{5}$$

Now,

$$\lim_{n\to\infty}\frac{4\left(-1\right)^n}{5} \quad - \quad \text{Does not exist} \qquad\qquad \lim_{n\to\infty}\frac{4}{5}=\frac{4}{5}$$

Therefore each of these two series diverge by the Divergence Test.

Step 6
The interval of convergence is below and for summary purposes the radius of convergence is also shown.

$$\boxed{\text{Interval}\quad :\quad -\frac{53}{16}<x<-\frac{43}{16}} \qquad\qquad \boxed{R=\frac{5}{16}}$$

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5. For the following power series determine the interval and radius of convergence.

$$\sum_{n=0}^{\infty}\frac{6^n}{n}\left(4x-1\right)^{n-1}$$

Step 1

Okay, let's start off with the Ratio Test to get our hands on L.

$$L = \lim_{n\to\infty}\left|\frac{6^{n+1}\left(4x-1\right)^{n}}{n+1}\frac{n}{6^{n}\left(4x-1\right)^{n-1}}\right| = \lim_{n\to\infty}\left|\frac{6n\left(4x-1\right)}{n+1}\right| = \left|4x-1\right|\lim_{n\to\infty}\frac{6n}{n+1} = 6\left|4x-1\right|$$

Step 2
So, we know that the series will converge if,

$$6\left|4x-1\right| < 1 \qquad \to \qquad 24\left|x-\tfrac{1}{4}\right| < 1 \qquad \to \qquad \left|x-\tfrac{1}{4}\right| < \frac{1}{24}$$

Step 3
So, from the previous step we see that the radius of convergence is $\boxed{R = \frac{1}{24}}$.

Step 4
Now, let's start working on the interval of convergence. Let's break up the inequality we got in Step 2.

$$-\frac{1}{24} < x - \frac{1}{4} < \frac{1}{24} \qquad \to \qquad \frac{5}{24} < x < \frac{7}{24}$$

Step 5
To finalize the interval of convergence we need to check the end points of the inequality from Step 4.

$$x = \frac{5}{24} \;:\; \sum_{n=0}^{\infty}\frac{6^{n}}{n}\left(-\frac{1}{6}\right)^{n-1} = \sum_{n=0}^{\infty}\frac{6^{n}}{n}\frac{\left(-1\right)^{n-1}}{6^{n-1}} = \sum_{n=0}^{\infty}\frac{6\left(-1\right)^{n-1}}{n}$$

$$x = \frac{7}{24} \;:\; \sum_{n=0}^{\infty}\frac{6^{n}}{n}\left(\frac{1}{6}\right)^{n-1} = \sum_{n=0}^{\infty}\frac{6^{n}}{n}\frac{1}{6^{n-1}} = \sum_{n=0}^{\infty}\frac{6}{n}$$

Now, the first series is an alternating harmonic series which we know converges (or you could just do a quick Alternating Series Test to verify this) and the second series diverges by the *p*-series test.

Step 6
The interval of convergence is below and for summary purposes the radius of convergence is also shown.

$$\boxed{\text{Interval} \;:\; \frac{5}{24} \le x < \frac{7}{24}} \qquad\qquad \boxed{R = \frac{1}{24}}$$