MATH 262: Sample Final with Solutions

Problem 1. Determine the values of x for which the series

$$\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n}$$

converges absolutely, converges conditionally, or diverges.

Solution. Apply the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{(4x+1)^{n+1}}{n+1} \Big/ \frac{(4x+1)^n}{n} \right| = \lim_{n\to\infty} \left| \frac{(4x+1)^{n+1}\ n}{(4x+1)^n\ (n+1)} \right| = |4x+1| \cdot \lim_{n\to\infty} \frac{n}{n+1} = |4x+1|\,.$$

Hence, the series converges absolutely if $|4x+1| < 1$, and diverges if $|4x+1| > 1$.
Check the case when $|4x+1| = 1$. If $4x+1 = 1$ then

$$\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. If $4x+1 = -1$ then

$$\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges conditionally.

Problem 2. Given the function

$$F(x) = \int_0^x \frac{\sin t}{t}\, dt.$$

(a) find the Maclaurin series for $F(x)$,

(b) evaluate $F(1)$ with error less than 10^{-5}.

Solution. (a) We have

$$\sin t = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1}.$$

Hence,

$$\frac{\sin t}{t} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k}$$

and

$$F(x) = \int_0^x \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k} \right) dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{t^{2k+1}}{2k+1}.$$

(b)

$$F(1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\ (2k+1)!}$$

is an alternating series and we can use the alternating series error estimation. Hence,

$$|E(1) - s_n| \leq \left| \frac{(-1)^n}{(2n+1)\ (2n+1)!} \right| = \frac{1}{(2n+1)\ (2n+1)!},$$

where s_n is the sum of the first n terms of the series. Observe that the solution of the inequality

$$\frac{1}{(2n+1)\ (2n+1)!} < 10^{-5}$$

provides the required number of terms. The solution is $n \geq 4$, so we need at least 4 first terms of the series.

Problem 3.

(a) Without using l'Hopital's Rule, compute

$$\lim_{x \to 0} \frac{(e^{2x} - 1)^2}{\ln(1+x) - x}.$$

(b) Determine power series representation for the function

$$f(x) = \frac{1-x}{1+x}$$

in powers if $x - 2$.

Solution. (a) We have

$$e^{2x} = 1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \cdots,$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots,$$

so,

$$\lim_{x \to 0} \frac{(e^{2x} - 1)^2}{\ln(1+x) - x} = \lim_{x \to 0} \frac{\left(1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \cdots - 1\right)^2}{\left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) - x} = \lim_{x \to 0} \frac{4x^2 + O(x^3)}{-\frac{x^2}{2} + O(x^3)} = -8.$$

(b) We have

$$f(x) = \frac{1-x}{1+x} = \frac{2 - (1+x)}{1+x} = \frac{2}{1+x} - 1 = \frac{2}{3 + (x-2)} - 1 = \frac{2}{3} \cdot \frac{1}{1 - \left(\frac{-(x-2)}{3}\right)} - 1.$$

Since

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$$

then

$$\frac{1}{1-\left(\frac{-(x-2)}{3}\right)} = \sum_{n=0}^{\infty} \left(\frac{-(x-2)}{3}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n\ (x-2)^n}{3^n}.$$

Finally,

$$f(x) = \frac{2}{3} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n\ (x-2)^n}{3^n} - 1 = -\frac{1}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n\ 2\ (x-2)^n}{3^{n+1}}.$$

Problem 4. Given the curve

$$\mathbf{r}(t) = t^2\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$$

(a) find the arc length from $t = 0$ to $t = \frac{1}{3}$,

(b) find the unit tangent vector $\mathbf{T}$, principle normal $\mathbf{N}$, and binormal $\mathbf{B}$ of the curve at the point $t = \frac{1}{3}$,

(c) find the curvature κ, and torsion τ of the curve at the point $t = \frac{1}{3}$.

Solution. (a) We have

$$\mathbf{r}'(t) = 2t\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k},$$

so,

$$|\mathbf{r}(t)| = \sqrt{4t^2 + 4t^2 + 9t^4} = t\sqrt{8 + 9t^2}$$

and the required arc length is

$$\int_0^{\frac{1}{3}} t\sqrt{8+9t^2}\ dt = \left\{u = 8 + 9t^2,\ du = 18t\ dt,\ dt = \frac{du}{18t}\right\} = \int_8^9 \frac{t}{18t} u^{\frac{1}{2}}\ du$$

$$= \frac{1}{18} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \bigg|_8^9 = \frac{1}{27}\left(27 - 16\sqrt{2}\right).$$

(b) We have

$$\mathbf{r}'(t) = 2t\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k},$$
$$\mathbf{r}''(t) = 2\mathbf{i} + 2\mathbf{j} + 6t\mathbf{k},$$

so,

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = 6t^2\mathbf{i} - 6t^2\mathbf{j},$$
$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = 6\sqrt{2}\ t^2.$$

Hence,

$$\mathbf{T} = \frac{1}{t\sqrt{8+9t^2}} \cdot \left(2t,\ 2t,\ 3t^2\right) = \left(\frac{2}{\sqrt{8+9t^2}},\ \frac{2}{\sqrt{8+9t^2}},\ \frac{3t}{\sqrt{8+9t^2}}\right),$$

$$\mathbf{B} = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|} = \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0 \right),$$

and

$$\mathbf{N} = \mathbf{B} \times \mathbf{T} = \left(-\frac{3}{\sqrt{2}} \cdot \frac{t}{\sqrt{8+9t^2}},\ -\frac{3}{\sqrt{2}} \cdot \frac{t}{\sqrt{8+9t^2}},\ \frac{4}{\sqrt{2}} \cdot \frac{t}{\sqrt{8+9t^2}} \right).$$

Finally, at $t = \frac{1}{3}$ we have

$$\mathbf{T}\left(\frac{1}{3}\right) = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right),$$

$$\mathbf{B}\left(\frac{1}{3}\right) = \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0 \right),$$

$$\mathbf{N}\left(\frac{1}{3}\right) = \left(-\frac{1}{3\sqrt{2}},\ -\frac{1}{3\sqrt{2}},\ \frac{4}{9\sqrt{2}} \right).$$

(c) We have

$$\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{6\sqrt{2}\, t^2}{(t\sqrt{8+9t^2})^3}.$$

Hence

$$\kappa\left(\frac{1}{3}\right) = \frac{2\sqrt{2}}{3}.$$

Next,

$$\mathbf{r}'''(t) = 6\mathbf{k},$$

so,

$$\tau = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2} = \frac{(6t^2\mathbf{i} - 6t^2\mathbf{j}) \cdot 6\mathbf{k}}{t^2(8+9t^2)} = 0$$

everywhere.

Problem 5.

(a) Find the tangent plane to the surface

$$\frac{x^2}{4} - \frac{y^2}{2} - \frac{z^2}{4} = 1$$

at the point $(4, 2, 2)$.

(b) In what direction at the point $(2, 0)$ does the function $f(x, y) = xy$ have rate of change -1 ?

Solution. (a) The given surface can be considered as a level surface of the function

$$F(x, y, z) = \frac{x^2}{4} - \frac{y^2}{2} - \frac{z^2}{4}.$$

Hence, $\nabla F(4,2,2)$ is normal to the tangent plane in question. We have

$$\nabla F = \left(\frac{x}{2},\ -y,\ -\frac{z}{2}\right)$$

and

$$\nabla F(4,2,2) = (2,-2,-1).$$

So,

$$2(x-4) - 2(y-2) - (z-2) = 0$$

is the required tangent plane.
(b) The rate of change of $f(x,y)$ in the direction of a unit vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ is $D_{\mathbf{u}}f(2,0) = \mathbf{u}\cdot\nabla f(2,0)$. We have $\nabla f(x,y) = y\mathbf{i} + x\mathbf{j}$, so $\nabla f(2,0) = 2\mathbf{j}$ and

$$-1 = D_{\mathbf{u}}f(2,0) = (a\mathbf{i} + b\mathbf{j})\cdot 2\mathbf{j} = 2b,$$

which implies $b = -\frac{1}{2}$. Since $\mathbf{u}$ is a unit vector then $a^2 + b^2 = 1$ and we find $a = \pm\frac{\sqrt{3}}{2}$. Hence, $f(x,y)$ has rate of change -1 in the direction of the vectors $\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$ and $-\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$.

Problem 6. A surface $z = f(x,y)$ has the parametric representation

$$x = u + v^2,\ y = u^2 - v^3,\ z = 2uv.$$

Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ at the point (x,y,z) corresponding to $u = 2, v = 1$.

Solution. Observe that

$$x = u + v^2,\ y = u^2 - v^3$$

implicitly define $u = u(x,y),\ v = v(x,y)$, so that $z = 2u(x,y)v(x,y) = f(x,y)$. By the Chain Rule we have

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = 2v\frac{\partial u}{\partial x} + 2u\frac{\partial v}{\partial x},\\ \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = 2v\frac{\partial u}{\partial y} + 2u\frac{\partial v}{\partial y},\end{aligned}$$

so, everything reduces to finding $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$. Let

$$F(x,y,u,v) = u + v^2 - x,\ G(x,y,u,v) = u^2 - v^3 - y,$$

hence, by the Implicit Function Theorem we have

$$\frac{\partial u}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}},\ \frac{\partial u}{\partial y} = -\frac{\frac{\partial(F,G)}{\partial(y,v)}}{\frac{\partial(F,G)}{\partial(u,v)}},$$

$$\frac{\partial v}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(u,x)}}{\frac{\partial(F,G)}{\partial(u,v)}},\ \frac{\partial v}{\partial y} = -\frac{\frac{\partial(F,G)}{\partial(u,y)}}{\frac{\partial(F,G)}{\partial(u,v)}}.$$

Next,

$$\frac{\partial F}{\partial x} = -1, \ \frac{\partial F}{\partial y} = 0, \ \frac{\partial F}{\partial u} = 1, \ \frac{\partial F}{\partial v} = 2v,$$

$$\frac{\partial G}{\partial x} = 0, \ \frac{\partial G}{\partial y} = -1, \ \frac{\partial G}{\partial u} = 2u, \ \frac{\partial G}{\partial v} = -3v^2.$$

Hence,

$$\frac{\partial u}{\partial x} = \frac{-3v^2}{-3v^2 - 4uv}, \quad \frac{\partial u}{\partial y} = \frac{-2v}{-3v^2 - 4uv},$$

$$\frac{\partial v}{\partial x} = \frac{-2u}{-3v^2 - 4uv}, \quad \frac{\partial v}{\partial y} = \frac{1}{-3v^2 - 4uv},$$

and

$$\frac{\partial z}{\partial x} = \frac{-6v^3 - 4u^2}{-3v^2 - 4uv}, \quad \frac{\partial z}{\partial y} = \frac{-4v^2 + 2u}{-3v^2 - 4uv}.$$

Thus, when $u = 2, v = 1$ we have

$$\frac{\partial z}{\partial x} = 2, \quad \frac{\partial z}{\partial y} = 0.$$

Problem 7. Find and classify the critical points of the function $f(x, y) = 2x^3 - 6xy + 3y^2$.

Solution. We have

$$\frac{\partial f}{\partial x} = 6x^2 - 6y, \ \frac{\partial f}{\partial y} = -6x + 6y,$$

hence, to find all critical points of $f(x, y)$ we have to solve the system of equations

$$6x^2 - 6y = 0, \quad -6x + 6y = 0.$$

From the second equation we get $x = y$, so, after substitution into the first one we get $6x(x - 1) = 0$ from which it follows that there are two critical points $(0, 0)$ and $(1, 1)$. Next,

$$A = \frac{\partial^2 f}{\partial x^2} = 12x, \ B = \frac{\partial^2 f}{\partial x \partial y} = -6, \ C = \frac{\partial^2 f}{\partial y^2} = 6.$$

Hence, at $(0, 0)$ we have $A = 0, \ B = -6, \ C = 6$ so, $B^2 - AC = 36 > 0$ and $(0, 0)$ is a saddle point. At $(1, 1)$ we have $A = 12, \ B = -6, \ C = 6$, so $A > 0$, $B^2 - AC = 36 - 6 \cdot 12 = -36 < 0$ and $(1, 1)$ is a point of local minimum.

Problem 8. Using the Lagrange multiplier method, find the points on the curve $4x^2 - 2xy + 4y^2 = 1$ which are closest to and farthest from the origin.

Solution. In other words we have to minimize and maximaze $g(x, y) = \sqrt{x^2 + y^2}$, where (x, y) belongs to the curve $4x^2 - 2xy + 4y^2 = 1$. To reduce computations we are going to use $f(x, y) = x^2 + y^2$ instead of $g(x, y)$ since f and g attain their maxima and minima at the same points.

Maxima and minima of $f(x, y)$ occur as critical points of

$$L(x, y, \lambda) = x^2 + y^2 - \lambda(4x^2 - 2xy + 4y^2 - 1).$$

We have

$$\frac{\partial L}{\partial x} = 2x - \lambda(8x - 2y), \; \frac{\partial L}{\partial y} = 2y - \lambda(-2x + 8y), \; \frac{\partial L}{\partial \lambda} = 4x^2 - 2xy + 4y^2 - 1,$$

hence, to find critical points of $L(x, y, \lambda)$ we solve the system

$$2x - \lambda(8x - 2y) = 0, \; 2y - \lambda(-2x + 8y) = 0, \; 4x^2 - 2xy + 4y^2 - 1 = 0.$$

Observe that if either $8x - 2y = 0$ or $-2x + 8y = 0$ then $x = 0, \; y = 0$, and the last equation is not satisfied. Thus, $8x - 2y \neq 0, \; -2x + 8y \neq 0$ and expressing λ from the first two equations we get

$$\frac{2x}{8x - 2y} = \frac{2y}{-2x + 8y},$$

from which $x = \pm y$ follows. After substitution into the last equation we get four points

$$\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{10}}, -\frac{1}{\sqrt{10}}\right), \left(-\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right).$$

Now,

$$f\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = \frac{1}{3}, \; f\left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right) = \frac{1}{3}, \; f\left(\frac{1}{\sqrt{10}}, -\frac{1}{\sqrt{10}}\right) = \frac{1}{5}, \; f\left(-\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right) = \frac{1}{5},$$

so

$$\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$$

are farthest from the origin points on the curve, while

$$\left(\frac{1}{\sqrt{10}}, -\frac{1}{\sqrt{10}}\right), \left(-\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$$

are closest to the origin ones.