pg 587, 19. obtain $\frac{x^3}{1-2x^2}$ from $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

"let $2x^2 = x$ and multiply by x^3"

then $\frac{x^3}{1-2x^2} = x^3 \sum_{n=0}^{\infty} (2x^2)^n$

$$\boxed{\frac{x^3}{1-2x^2} = \sum_{n=0}^{\infty} 2^n x^{2n+3}}$$

$$L = \lim_{n\to\infty} \frac{2^{n+1} x^{2(n+1)+3}}{2^n x^{2n+3}}$$

$$= \lim_{n\to\infty} 2 \frac{x^{2n+5}}{x^{2n+3}}$$

$$= 2 \lim_{n\to\infty} x^2$$

$$= 2x^2$$

$$|2x^2| < 1 \rightarrow \frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$$

By inspection if $\begin{cases} x = \frac{-1}{\sqrt{2}}, & \lim_{n\to\infty} 2^n \left(\frac{-1}{\sqrt{2}}\right)^{2n+3} \neq 0 \\ x = \frac{1}{\sqrt{2}}, & \lim_{n\to\infty} 2^n \left(\frac{1}{\sqrt{2}}\right)^{2n+3} \neq 0 \end{cases}$

$\therefore$ we need $\frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ for convergen

21. Determine the interval of convergence and the sum: $1-4x+16x^2-\ldots = \sum_{n=0}^{\infty} (-1)^n(4x)^n$

rewrite: $\sum_{n=0}^{\infty} (-4x)^n = \frac{1}{1-(-4x)} = \frac{1}{1+4x}$, so long as our series converges.

$$L = \lim_{n\to\infty} \frac{|-4x|^{n+1}}{|-4x|^n}$$

$$= \lim_{n\to\infty} |4x| = |4x|$$

$$|4x| < 1 \rightarrow \quad -1/4 < x < 1/4$$

By inspection, if $\begin{cases} x = -1/4, & \lim_{n\to\infty} (-4(\frac{-1}{4}))^n \neq 0 \\ x = 1/4, & \lim_{n\to\infty} (-4(\frac{1}{4}))^n \neq 0 \end{cases}$

$\therefore$ we need $\frac{-1}{4} < x < \frac{1}{4}$ for convergence.

<u>Extra Concept</u>

$$g(x) = \frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x}$$

$$= \frac{d}{dx}\sum_{n=0}^{\infty} x^n$$

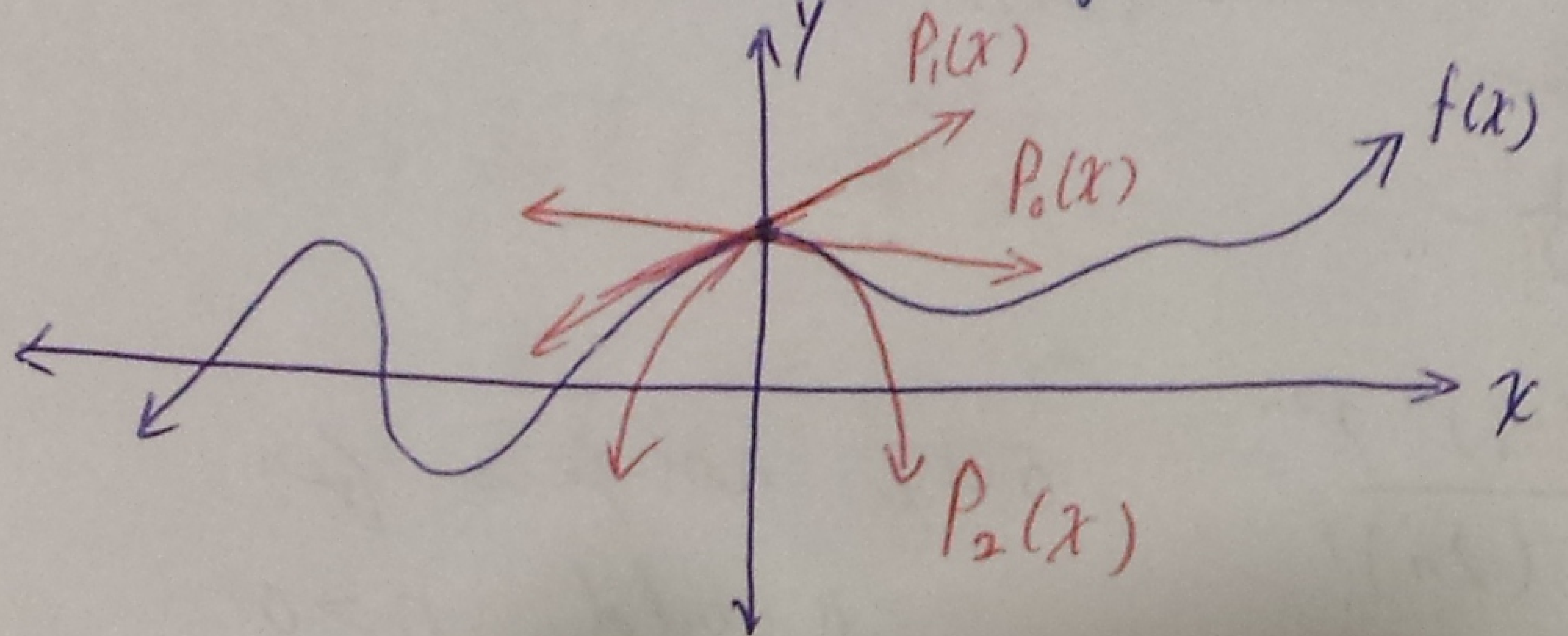
$$\boxed{g(x) = \sum_{n=0}^{\infty} n x^{n-1}}$$

You can show the radius of convergence is 1, but it will always be the same as the original function.

Taylor Series Intuition

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Let's consider $c=0$ for the sake of simplicity as we donot lose generality.



Suppose $f(x)$ is the measured position of a car at time x.

$P_0(x) = f(0)$ correctly models the position at $x=0$, useless elsewhere.

$P_1(x) = f(0) + x f'(0)$ correctly models position and velocity, useful for incremental analysis. THIS IS ACTUALLY EULER'S METHOD!

$P_2(x) = f(0) + x f'(0) + \frac{1}{2}x^2 f''(0)$ correctly models acceleration (curvature).

$P_3(x) = f(0) + x f'(0) + \frac{1}{2}x^2 f''(0) + \frac{1}{6}x^3 f'''(0)$ correctly models jerk.

What if our original $f(x)$ had constant acceleration like in PHYS131? Then $P_\infty(x) = f(0) + x f'(0) + \frac{1}{2}x^2 f''(0)$.

Does this look familiar? $S = S_0 + t v_0 + \frac{1}{2}t^2 a_0$.

Pg 545, 13. Find a Maclaurin Series For:

$\cosh x - \cos x$

$$= \frac{e^x + e^{-x}}{2} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$= \underbrace{\frac{1}{2}\sum_{n=0}^{\infty} \frac{x^n}{n!} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + 1 - \frac{x}{1!} + \frac{x^2}{2!} - \dots$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$ only non-zero for n odd, $n > 0$

$$= \sum_{n=0}^{\infty} \frac{x^{4n+2}}{(4n+2)!}$$, which converges for all x.

$$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} \dots$$
$$-\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots\right]$$

d the Taylor series for

$f(x) = \ln(2+x)$ in powers of $x-2$

$= \ln(4+(x-2))$

$= \ln\left[4\left(1+\frac{x-2}{4}\right)\right]$

$= \ln 4 + \ln\left(1+\frac{x-2}{4}\right)$

$= \ln 4 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-2)^n}{4^n n}$

recall $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n}$

$\cancel{\ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{n-1}}{n-1}}$

$\ln(1-x) = \sum_{n=1}^{\infty} \frac{-x^n}{n}$

$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

$L = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{4^{n+1}(n+1)} \cdot \frac{4^n n}{(x-2)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{x-2}{4} \frac{n}{n+1} \right|$

$= \left| \frac{x-2}{4} \right|$

$|x-2| < 4 \rightarrow -2 < x < 6$

sub $x=-2$

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-4)^n}{4^n n} = \sum_{n=1}^{\infty} (-1)^{2n-1} \frac{1}{n}$

$= \sum_{n=1}^{\infty} (-1)/n$ diverges

sub $x=6$

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(4)^n}{4^n n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

Converges by alt series

$\therefore (-2 < x \leq 6)$

25. $f(x) = x\ln(x)$ in powers of $x-1$

Sub $y = x-1$

$$x\ln x = (1+y)\ln(1+y)$$

$$= \ln(1+y) + y\ln(1+y)$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1}\frac{y^n}{n} + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{y^{n+1}}{n}$$

↓ index shift

$$x\ln x = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{y^n}{n} + \sum_{n=2}^{\infty}(-1)^{n-2}\frac{y^n}{n-1}$$

$$= y + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{y^n}{n} + (-1)^{n-2}\frac{y^n}{n-1}$$

$$= y + \sum_{n=2}^{\infty}(-1)^{n-1}\, y^n\left[\frac{1}{n} - \frac{1}{n-1}\right]$$

$$= (x-1) + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}(-1)}{n(n-1)}(x-1)^n$$

$$= (x-1) + \sum_{n=2}^{\infty}\frac{(-1)^n}{n(n-1)}(x-1)^n$$

$$L = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(n+1)(n)}(x-1)^{n+1}\cdot\frac{n(n-1)}{(x-1)^n(-1)^n}\right|$$

$$= \lim_{n\to\infty}\left|x-1\ \frac{n+1}{n-1}\right| = \lim_{n\to\infty}|x-1|$$

we need $|x-1| < 1$

$$0 < x < 2$$

sub $x = 0$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n(n-1)} (-1)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n(n-1)}$$ conv P-series

sub $x = 2$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n(n-1)} (1)^n$$

conv alt series test.

$\therefore \quad (0 \leq x \leq 2)$.