INFINITE SEQUENCES

Review

Let $\{a_n\}$ be a sequence. If $\lim\limits_{n\to+\infty} a_n = L$ then the sequence converges.

If $\lim\limits_{n\to+\infty} a_n$ results in either one of the following forms: $\frac{0}{0}, \frac{\infty}{\infty}, 0^0, \infty^0, 1^\infty, \infty-\infty$ use L'Hopital's Rule to evaluate the limit.

L'Hoptial's Rule

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Properties of Sequences

Suppose that $\{a_n\}$ and $\{b_n\}$ be two convergent sequences with $\lim\limits_{n\to+\infty} a_n = L$ and $\lim\limits_{n\to+\infty} b_n = M$ then the following results are true

Name	Property
Constant	$\lim\limits_{n\to+\infty} c = c$
Constant Multiple	$\lim\limits_{n\to+\infty} ca_n = c\lim\limits_{n\to+\infty} a_n = cL$
Additive	$\lim\limits_{n\to+\infty}(a_n+b_n) = \lim\limits_{n\to+\infty} a_n + \lim\limits_{n\to+\infty} b_n = L+M$
Difference	$\lim\limits_{n\to+\infty}(a_n-b_n) = \lim\limits_{n\to+\infty} a_n - \lim\limits_{n\to+\infty} b_n = L-M$
Product	$\lim\limits_{n\to+\infty}(a_nb_n) = \lim\limits_{n\to+\infty} a_n \bullet \lim\limits_{n\to+\infty} b_n = LM$
Quotient	$\lim\limits_{n\to+\infty}\frac{a_n}{bn} = \frac{\lim\limits_{n\to+\infty} a_n}{\lim\limits_{n\to+\infty} b_n} = \frac{L}{M}$ provided that $M \neq 0$
Squeeze Theorem For Sequences	Let $\{a_n\}, \{b_n\}$, and $\{c_n\}$ be sequences such that $\{a_n\} \le \{c_n\} \le \{b_n\}$. If $\lim\limits_{n\to+\infty} a_n = L$ and $\lim\limits_{n\to+\infty} b_n = L \Rightarrow \lim\limits_{n\to+\infty} c_n = L$

TESTS TO DETERMINE THE CONVERGENCE/DIVERGENCE OF A SERIES

Name	What Does It Say
Integral Test	$\sum a_k$ is a series of positive terms and $f(x) = a_k$. If the function is decreasing and continuous on $[a, +\infty)$, then the series converges if $\int_a^{+\infty} f(x)dx$ is defined. The series diverges if $\int_a^{+\infty} f(x)dx = \pm\infty$
Divergence Test	$\sum a_k$ diverges if $\lim\limits_{k\to+\infty} a_k \neq 0$
Ratio Test	$\sum a_k$ is a series of positive terms with $p = \lim\limits_{k\to+\infty}\frac{a_{k+1}}{a_k}$ If $p>1 \Rightarrow$ the series diverges. If $p<1 \Rightarrow$ the series converges. If $p=1 \Rightarrow$ no conclusion.
Root Test	$\sum a_k$ is a series of positive terms with $p = \lim\limits_{k\to+\infty}\sqrt[k]{a_k}$ If $p>1 \Rightarrow$ the series diverges. If $p<1 \Rightarrow$ the series converges. If $p=1 \Rightarrow$ no conclusion.
Comparison Test	Suppose that $\sum_1^\infty a_k$ and $\sum_1^\infty b_k$ are two series with positive terms such that $a_k \le b_k$ for all k. If $\sum_1^\infty b_k$ converges, then $\sum_1^\infty a_k$ converges. If $\sum_1^\infty a_k$ diverges, then also $\sum_1^\infty b_k$ diverges.
Limit Comparison Test	Suppose that $\sum_1^\infty a_k$ and $\sum_1^\infty b_k$ are two series with positive terms with $p = \lim\limits_{k\to+\infty}\frac{a_k}{b_k}$ If $p>0 \Rightarrow$ either both series converge, or both series diverge.
Name	What Does It Say
Alternating Series Test	Suppose that $a_k > 0$ for $k = 1,2,3,\ldots$ then the series has the form of either: $a_1 - a_2 + a_3 - a_4 + \ldots$ or $-a_1 + a_2 - a_3 + a_4 - \ldots$ The series converges if and only if 1. $a_{k+1} \le a_k$; the series is decreasing, and 2. $\lim\limits_{k\to+\infty} a_k = 0$
Ratio Test Absolute Convergence	$\sum a_k$ is a series of nonzero terms with $p = \lim\limits_{k\to+\infty}\left\lvert\frac{a_{k+1}}{a_k}\right\rvert$ If $p>1 \Rightarrow$ the series diverges. If $p<1 \Rightarrow$ the series converges. If $p=1 \Rightarrow$ no conclusion.

POWER OF SERIES AND RADIUS OF CONVERGENCE

Form	Sigma Notation	Radius of Convergence
$\frac{1}{1-x} = 1+x+x^2+x^3+x^4+\ldots$	$=\sum_1^\infty x^n$	$-1 \le x \le 1$
$\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\ldots$	$=\sum_1^\infty nx^{n-1}$	$-1 \le x \le 1$
$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$	$=\sum_1^\infty (-1)^n \frac{x^n}{n}$	$-1 \le x \le 1$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$	$=\sum_1^\infty (-1)^n \frac{x^{2n+1}}{2n+1}$	$-1 \le x \le 1$
$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$	$=\sum_1^\infty \frac{x^n}{n!}$	$(-\infty, \infty)$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \ldots$	$=\sum_1^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$(-\infty, \infty)$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$	$=\sum_1^\infty (-1)^n \frac{x^{2n}}{(2n)!}$	$(-\infty, \infty)$

VECTORS AND COORDINATE GEOMETRY

Vectors & Vector Dot Products

Let $\vec{u}$ be a vector having n components, then the length or norm of is $\vec{u}$ defined to be

$$\left|\vec{u}\right| = \sqrt{(n_1)^2 + (n_2)^2 + (n_3)^2 + \ldots + (n_n)^2}$$

Let $\vec{u}$ and $\vec{v}$ be two vectors, then their dot product is defined to be:

$$\vec{u}\cdot\vec{v}=|\vec{u}||\vec{v}|\cos\theta$$

Properties

Symmetry	$\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$
Multiplication with a scalar	$(k\vec{u})\cdot\vec{v}=k(\vec{u}\cdot\vec{v})$ k ∈ R
Linearity	$\vec{u}(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$

Uses of Dot Product

A. Show orthoganality: $\vec{u}\cdot\vec{v}=0 \Leftrightarrow \vec{u}\perp\vec{v}$
B. Finding the angle between two vectors:

$$\theta=\cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$$

Unit Vectors

Suppose that $\vec{u}$ is a vector having n components, then its unit vector is found be dividing each component of the vector by its norm. That is,

$$|\vec{u}|_{unit}=\frac{1}{|\vec{u}|}\vec{u}=\frac{1}{\sqrt{(n_1)^2+(n_2)^2+(n_3)^2+\ldots+(n_n)^2}}\vec{u}$$

VECTOR VALUED FUNCTIONS

Tangent Directions

If $\vec{r}(t)=x(t)\hat{i}+y(t)\hat{j}+z(t)\hat{k}$ is a position vector to the points on the curve $(x(t),y(t),z(t))$ then we can find the following quantities

Average Velocity	$v_{avg}=\frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t}$, $\Delta t=$ change in time
Velocity	$v(t)=r'(t)=\frac{dr}{dt}=\frac{dx}{dt}[x(t)]\hat{i}+\frac{dy}{dt}[y(t)]\hat{j}+\frac{dz}{dt}[z(t)]\hat{k}$
Speed	$\lvert v(t)\rvert$
Acceleration	$a(t)=r''(t)=\frac{d^2r}{dt^2}=\frac{dx}{dt}[x'(t)]\hat{i}+\frac{dy}{dy}[y'(t)]\hat{j}+\frac{dz}{dt}[z'(t)]\hat{k}$

Arc – length

$$s=s(t)=\int_{t_0}^{t}|\vec{r}'(t)|dt$$

Formulas for Curvature, Torsion, Radius of Curvature

Name	Formula
Unit Tangent Vector	$\overline{T}=\frac{\overline{v}}{\lvert\overline{v}\rvert}$
Curvature	$\kappa=\frac{\overline{v}\times\overline{a}}{\lvert v\rvert^3}$
Radius of Curvature	$\rho=\frac{1}{\kappa}$
Binormal Vector	$\overline{B}=\frac{\overline{v}\times\overline{a}}{\lvert\overline{v}\times\overline{a}\rvert}$
Unit Normal Vector	$\overline{N}=\overline{B}\times\overline{T}=\frac{\frac{d\overline{T}}{dt}}{\lvert\frac{d\overline{T}}{dt}\rvert}$
Tangential and Normal Components	$a=\frac{dv}{dt}\overline{T}+\lvert v\rvert^2\kappa\overline{N}$
Torsion	$\tau=\frac{(v\times a)\frac{da}{dt}}{\lvert v\times a\rvert^2}$

Frenet – Serret Formulas

$\frac{d\overline{T}}{ds}=\kappa N$	$\frac{d\overline{B}}{ds}=-\tau\overline{N}$	$\frac{d\overline{N}}{ds}=-\kappa\overline{T}+\tau\overline{B}$

PARTIAL DIFFERENTIATION

Definition of Limit on a plane

$$\lim_{(x,y)\to(a,b)} f(x,y)=L$$

Implies that

A. Every neighborhood of (a, b) contains points of the domain of $f(x,y)$ different from (a, b).
B. For every ε (positive) there exists a positive δ such that $|f(x,y)-L|<\varepsilon$ whenever (x, y) is in the domain of f and satisfies the inequality: $0<\sqrt{(x-a)^2+(y-b)^2}<\delta$

Definition of Continuity on a Plane

For continuity to happen, the following 2 conditions must hold:

1. $\lim_{(x,y)\to(a,b)} f(x,y)$ exists
2. $\lim_{(x,y)\to(a,b)} f(x,y)=f(a,b)$

Partial Derivatives

Given a function $f(x,y)$, the partial with respect to x, denoted as f_x, at the point (a,b) is defined to be

$$f_x(a,b)=\lim_{h\to 0}\frac{f(a+h,b)-f(a)}{h}$$

Similarly the partial with respect to y, at the point (a,b), denoted as f_y, is defined to be

$$f_y=\lim_{k\to 0}\frac{f(a,b+k)-f(b)}{k}$$

In practice we use the following

$f_x=f_1=\frac{\partial f}{\partial x}$	$\Rightarrow$	differentiate the function f with respect to x while holding all other variables as constants.
$f_y=f_2=\frac{\partial f}{\partial y}$	$\Rightarrow$	differentiate the function f with respect to y while holding all other variables as constants.

The idea can easily be extended to functions having more than 2 variables.

Higher Derivatives

Second derivatives for functions of 2 variables

$f_{xx}=f_{11}=\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$	Second derivative of f with respect to x
$f_{yy}=f_{22}=\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$	Second derivative of f with respect to y
$f_{xy}=f_{12}=\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$	Mixed partial
$f_{yx}=f_{21}=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$	Mixed partial

Chain Rule

$z=f(x,y)$ is a differentiable function of x and y with $x=x(t)$ and $y=y(t)$ then

$$z'(t)=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example: $z(x,y)=x\sin y$ where $x(t)=e^t$ and $y(t)=t^2$

Solution:

$$z'(t)=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial x}{\partial t}=e^t \qquad \frac{\partial y}{\partial t}=2t$$

$$\frac{\partial z}{\partial x}=\sin y \qquad \frac{\partial z}{\partial y}=x\cos y$$

$$z'(t)=e^t\sin y+2tx\cos y$$

The chain rule can be extended to include functions which have more than one variable.

Let z be a differentiable function of n variables, $x_1,x_2,x_3,\ldots,x_n$ and each x_j is a differentiable function of m variables, $t_1,t_2,t_3,\ldots,t_m$ then z is ultimately a function of $t_1,t_2,t_3,\ldots,t_m$. So

$$\frac{\partial z}{\partial t_i}=\frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_i}+\frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_i}+\ldots+\frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_m}$$

Tangent Planes & Gradients

Suppose that $z=f(x,y)$ has tangent planes everywhere, then the gradient is:

$$\nabla f(x,y)=\hat{i}\frac{\partial}{\partial x}(x,y)+\hat{j}\frac{\partial}{\partial y}(x,y)$$

$\nabla f(x,y)$ = Direction of the steepest slope upwards in the XY plane

$-\nabla f(x,y)$ = Direction of the steepest slope downwards in the XY plane.
$\|\nabla f(x,y)\|$ = Slope in steepest direction

$D_u f=\hat{u}\cdot\nabla f$ = Directional derivative of f with respect to $\hat{u}$

$\hat{u}+\hat{k}D_u f$ = Tangent vector with respect to $\hat{u}$ direction

$\hat{k}-\nabla f$ = Upward normal vector to the surface
= Perpendicular to all tangent vectors
= Normal vector to the tangent plane.

LOCAL EXTREMAS

Suppose that the surface $z=f(x,y)$ has tangent planes at every point. The points (x_0,y_0,z_0) on the surface where the tangent plane is horizontal $\Rightarrow \overline{\nabla}f\big|_{(x_0,y_0)}=0$, are called critical point (local extrema).

$D_u^2 f$ positive for all $\theta \Rightarrow$ nearby slopes are up
$\Rightarrow$ surface bends up
$\Rightarrow$ local minimum
$D_u^2 f$ negative for all $\theta \Rightarrow$ nearby slopes are down
$\Rightarrow$ surface bends down
$\Rightarrow$ local maximum
$D_u^2 f$ changes slope $\Rightarrow$ nearby slopes is up in some directions and down in other directions
$\Rightarrow$ surface bends up and down
$\Rightarrow$ saddle

The test can be expressed in terms of the Jacobian matrix

$$D=\begin{bmatrix}\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\partial y}\\ \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial y^2}\end{bmatrix}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

If det(D)<0 $\Rightarrow$ we have a saddle point

If det(D)<0 and a+c>0 $\Rightarrow$ we have a local minimum
If det(D)<0 and a+c<0 $\Rightarrow$ we have a local maximum

Lagrange Multipliers
To find the extreme values:

1. First get it into a Lagrangian function: $L(x,y)=f(x,y)+\lambda g(x,y)$
2. Solve for the critical point by finding out when
$$0=\frac{\partial L}{\partial x}=f_x(x,y)+\lambda g(x,y) \cdot 0=\frac{\partial L}{\partial y}=f_y(x,y)+\lambda g(x,y)\cdot \quad 0=\frac{\partial L}{\partial \lambda}g(x,y)$$
3. Once the critical points have been found, plug it back into the function and test to see which point satisfies the given criteria.

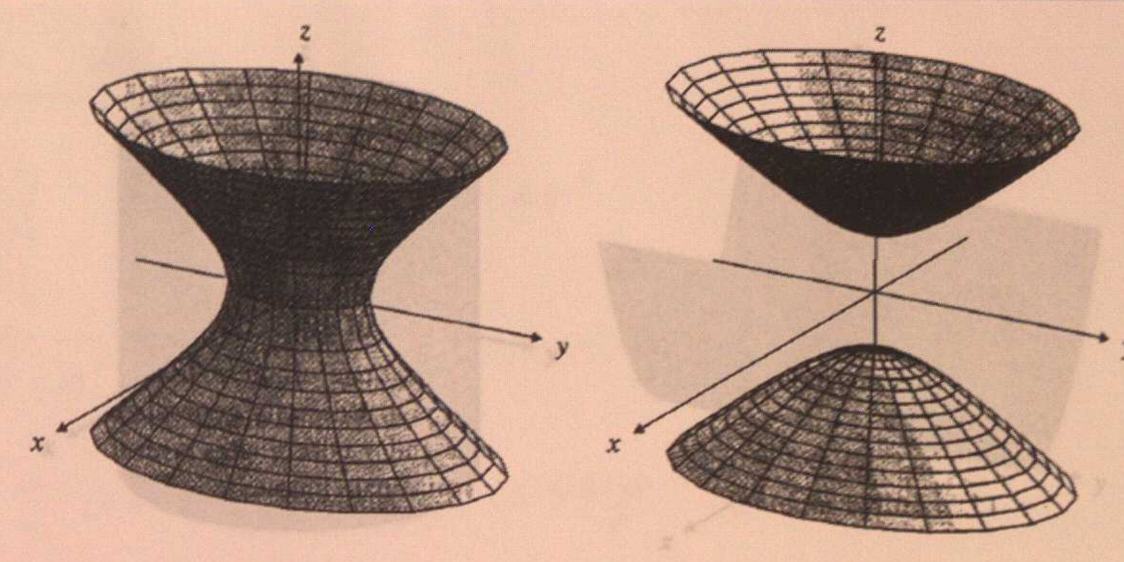
igure 10.37

(a) The hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

(b) The hyperboloid of two sheets

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$



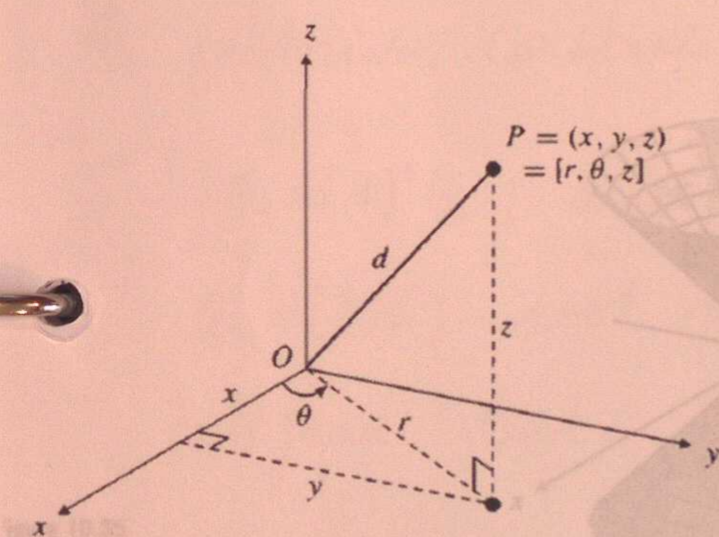


Figure 10.38 The cylindrical coordinates of a point

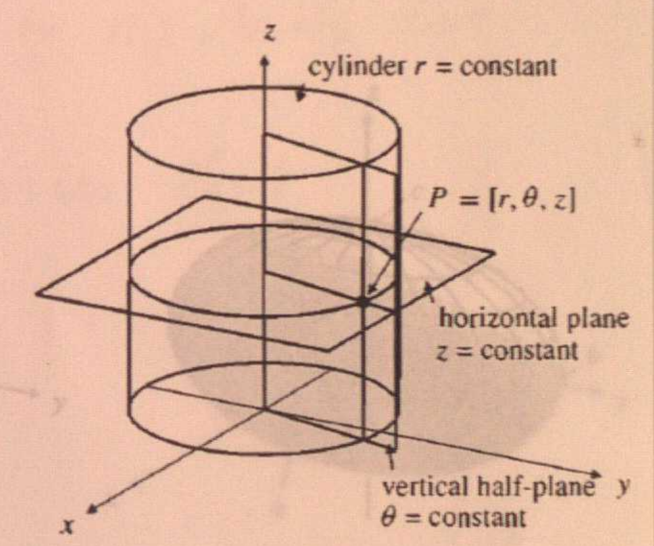


Figure 10.39 The coordinate surfaces for cylindrical coordinates

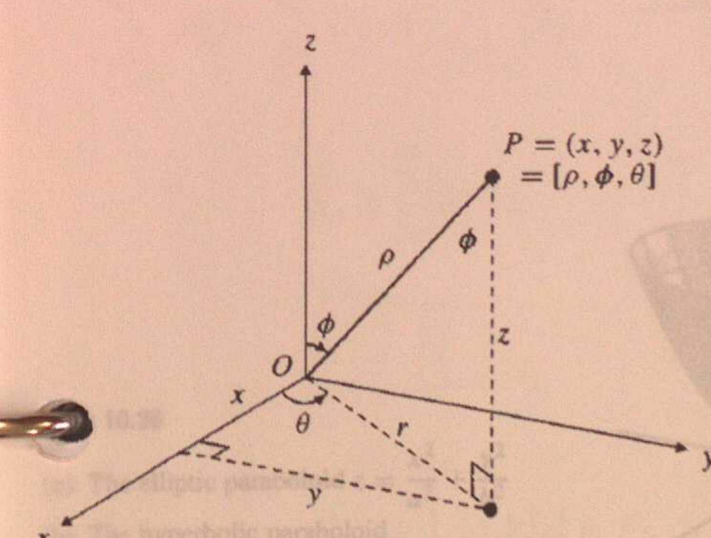


Figure 10.40 The spherical coordinates of a point

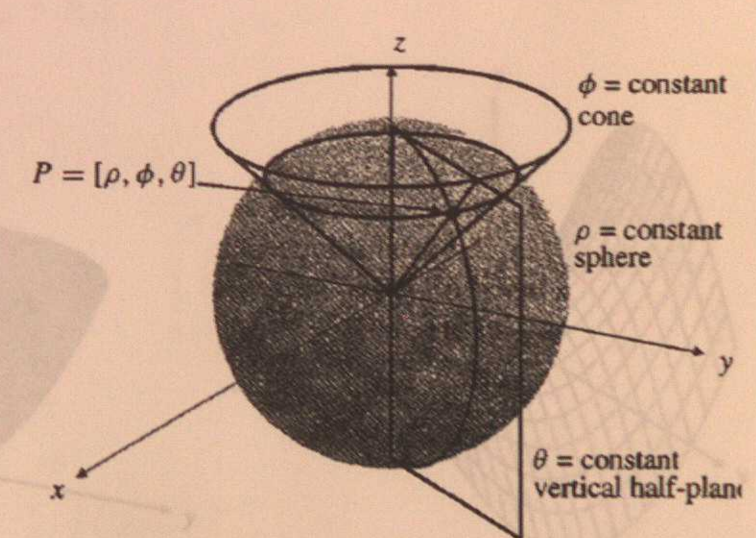


Figure 10.41 The coordinate surfaces for spherical coordinates

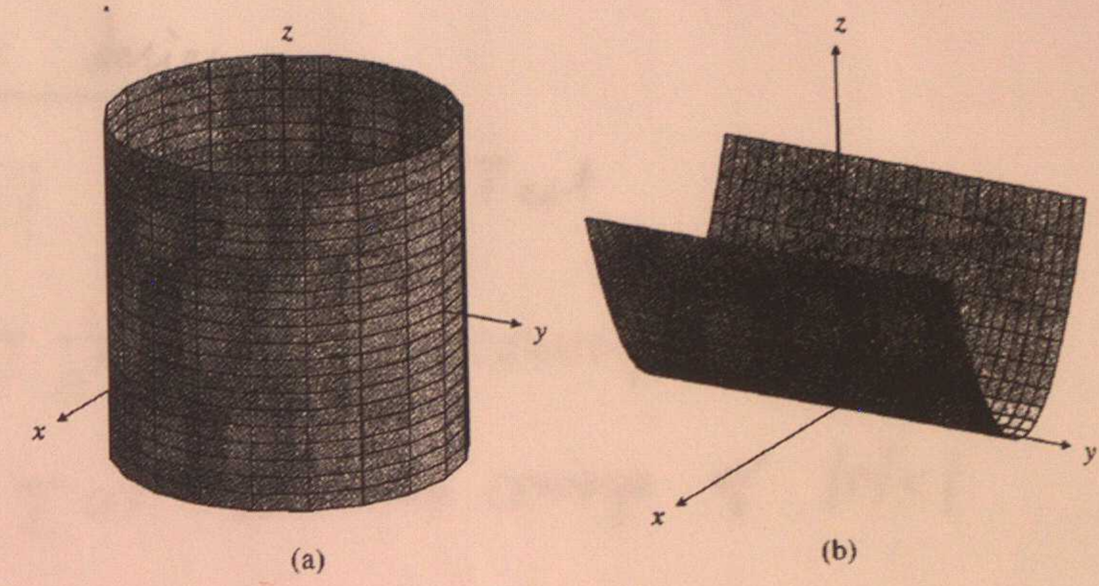


Figure 10.34

(a) The circular cylinder $x^2 + y^2 = a^2$

(b) The parabolic cylinder $z = x^2$

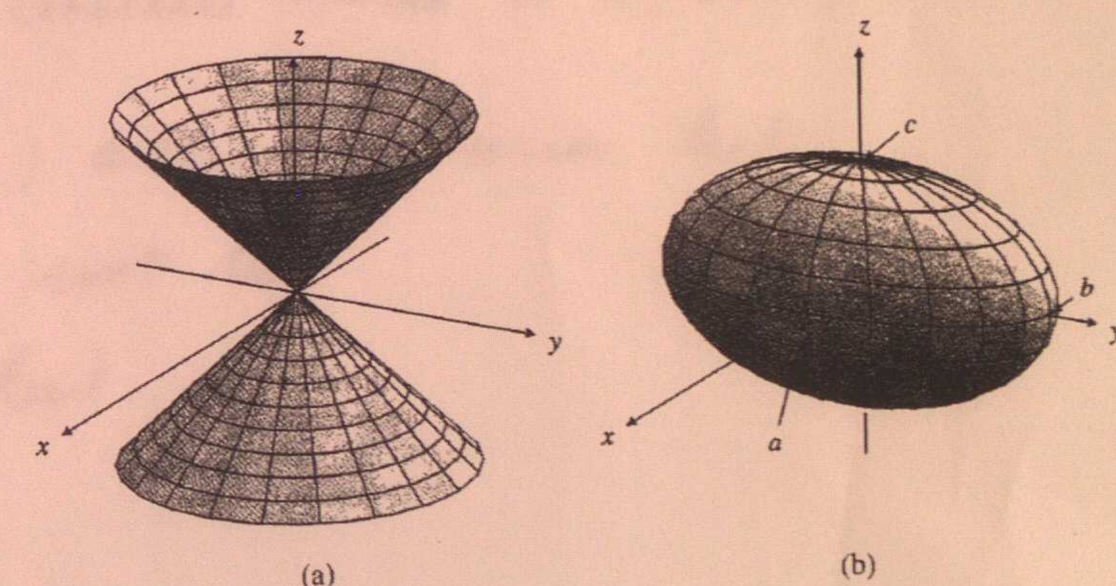


Figure 10.35

(a) The circular cone $a^2z^2 = x^2 + y^2$

(b) The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

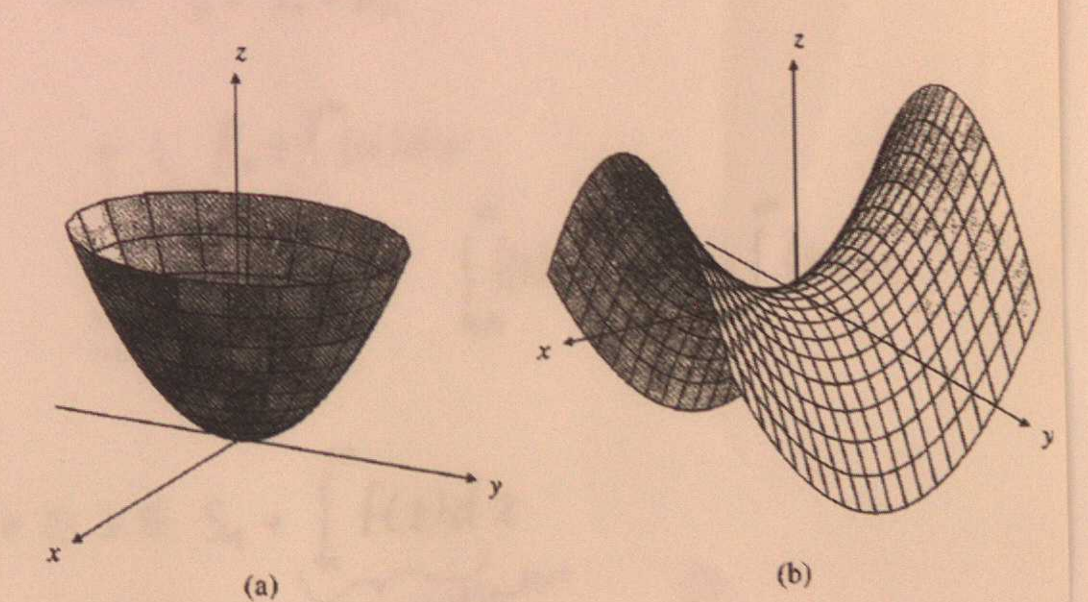


Figure 10.36

(a) The elliptic paraboloid $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

(b) The hyperbolic paraboloid $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

Strategy for Series

1. $\lim_{n\to\infty} a_n \neq 0$? Divergence Test

2. p-series: $\sum \frac{1}{n^p}$ will only converge if $p>1$

   geo-series: $\sum ar^n$ will only converge if $|r|<1$

3. compare to a p-series or geo-series?

4. factorials, constants raised to n; ratio test

5. $a_n=(-1)^n b_n$; alternating series test

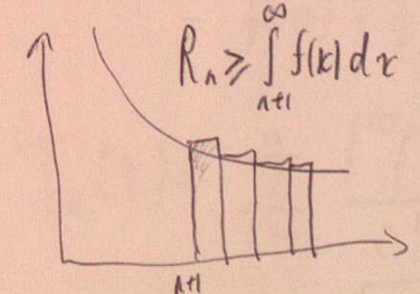
6. $a_n=(b_n)^n$; root test

7. Integral test

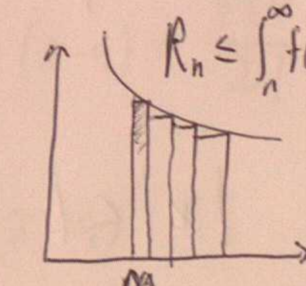
Estimation

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{n} a_i + \sum_{i=n+1}^{\infty} a_i \longrightarrow s = s_n + R_n$$

$R_n \geq \int_{n+1}^{\infty} f(x)\,dx$



$R_n \leq \int_{n}^{\infty} f(x)\,dx$



$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

$$s_n + \underbrace{\int_{n+1}^{\infty} f(x)\,dx}_{\text{under}} \leq s \leq s_n + \underbrace{\int_{n}^{\infty} f(x)\,dx}_{\text{overestimation}}$$

Estimation for Alt Series

$$|R_n| = |s - s_n| \le b_{n+1}$$

Power Series

$$\sum_{n=0}^{\infty} c_n (x-a)^n \qquad \underbrace{|x-a|<R}_{\text{converges}} \qquad \underbrace{|x-a|>R}_{\text{diverges}}$$

$a-R < x < a+R$ p-series converges

$x < a-R,\ x > a+R$ p-series diverges

$x = a$, p-series converges

Functions

Recall: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$, $|r|<1$

let $a=1$, $r=x$, then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x|<1$$

$$\boxed{\therefore\ f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1}$$

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

$$f'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} c_n(x-a)^n = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$$

$$\boxed{f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1}}$$

$$f''(x) = \sum_{n=0}^{\infty} n(n-1) c_n (x-a)^{n-2}$$

etc.

$$\int f(x)\,dx = \int \sum_{n=0}^{\infty} c_n (x-a)^n\,dx$$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

Taylor Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Calculus w/ Vector Functions

$$\lim_{t\to a} \vec{r}(t) = \lim_{t\to a} \langle f(t), g(t), h(t) \rangle$$

$$= \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle$$

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

$$\left.\begin{aligned} \frac{d}{dt}(\vec{u}\cdot\vec{v}) &= \vec{u}\,'\cdot\vec{v} + \vec{u}\cdot\vec{v}\,' \\ \frac{d}{dt}(\vec{u}\times\vec{v}) &= \vec{u}\,'\times\vec{v} + \vec{u}\times\vec{v}\,' \end{aligned}\right\} \text{ similar to product rule}$$

$$\frac{d}{dt}\left(\vec{u}(f(t))\right) = f'(t)\,\vec{u}\,'(f(t)) \quad \Big\} \text{ chain rule}$$

$$\int \vec{r}(t) = \langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \rangle + \vec{C}$$

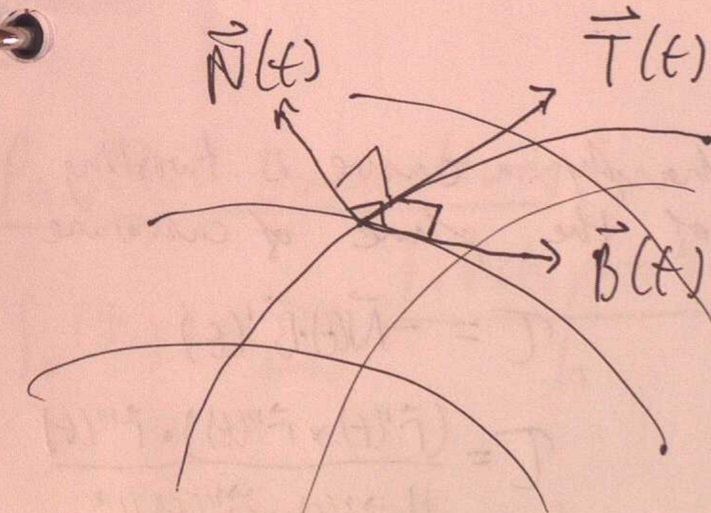
Tangent, Normal, Binormal "TNB"

$\vec{r}\,'(t)$ is the tangent vector, if $\vec{r}\,'(t) \neq 0$.

unit tangent vector $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$

unit normal vector $\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$

binormal vector $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$



$\vec{N}(t) = \vec{B}(t) \times \vec{T}(t)$

$\vec{T}(t) = \vec{N}(t) \times \vec{B}(t)$

Arc length

$$L = \int_a^b \|\vec{r}\,'(t)\|\, dt \qquad s(t) = \int_0^t \|\vec{r}\,'(u)\|\, du$$

Recall: $s(t) = \int ds = \int \sqrt{dx^2 + dy^2 + dz^2} = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

Reparametrize $\vec{r}(t)$ to $\vec{r}(s)$ after getting $s(t)$ function to tell us where we are on the curve after we've travelled a distance s along the curve.

Curvature, Torsion

how fast a curve is changing direction at a given point

how sharply a curve is twisting out of the plane of curvature

$$\tau = -\vec{N}(t)\cdot\vec{B}\,'(t)$$

$$K = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|}\right| = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} \qquad \tau = \frac{(\vec{r}\,'(t) \times \vec{r}\,''(t))\cdot \vec{r}\,'''(t)}{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|^2}$$

Velocity, Acceleration

$\vec{V}(t) = \vec{r}\,'(t)$, $\vec{a}(t) = \vec{r}\,''(t) = a_T\vec{T} + a_N\vec{N}$

$$a_T = V' = \frac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{\|\vec{r}\,'(t)\|} \qquad a_N = KV^2 = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|}$$

Multivariable Limits

Recall $\lim_{x\to a} f(x) = L$ iff $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$

A function $f(x,y)$ is <u>continuous</u> at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Find a path to evaluate the limit if the function is discontinuous at the limit.

Partial Derivatives

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \qquad f_y(x,y) = \lim_{h\to 0}\frac{f(x,y+h) - f(x,y)}{h}$$

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(f(x,y)) = z_x = \frac{\partial z}{\partial x} = D_x f$$

$f_x(x,y)$ represents the change of $f(x,y)$ as x changes and y is fixed.

$f_x(a,b)$ is the slope of the trace of $f(x,y)$ for the plane $y=b$ at the point (a,b).

Equation of Tangent Line

point $(a, b, f(a,b))$

$\vec{r}(x,y) = \langle x, y, z \rangle = \langle x, y, f(x,y) \rangle$

$\vec{r}_x(x,y) = \langle 1, 0, f_x(x,y) \rangle$

Higher Order Partial Derivatives

$f(x,y)$

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{xyx} = (f_{xy})_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial x \partial y \partial x}$$

Clairaut's Theorem

Suppose that f is defined on a disk D that contains the point (a,b). If f_{xy} and f_{yx} are continuous on this disk then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Can extend to any function and mixed partial derivatives.

$$f_{ssrtsrr} = f_{trsrssr}$$

Chain Rule

Recall: If $y = f(x)$, $x = g(t)$, then $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

Case 1: $z = f(x,y)$, $x = g(t)$, $y = h(t)$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

suppose $x = g(x)$

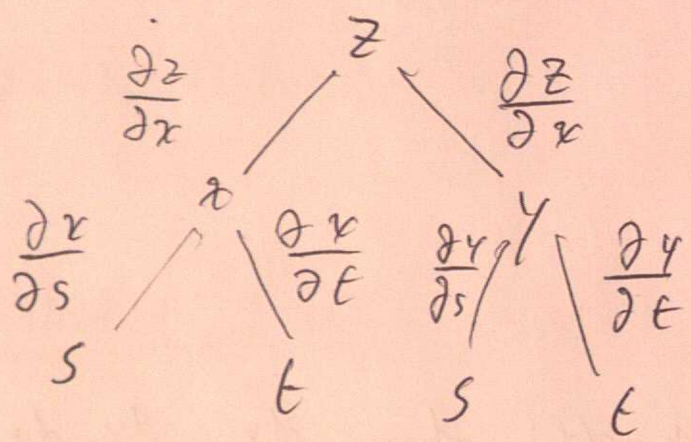
then $\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}\frac{dy}{dx}$.

Case 2: $z=f(x,y)$, $x=g(s,t)$, $y=h(s,t)$

$$\frac{\partial z}{\partial s}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$$

general: $z=f(x_1,x_2\ldots)$ $\quad x_i=g(t_1,t_2\ldots)$

$$\frac{\partial z}{\partial t_i}=\frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_i}+\frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_i}+\ldots$$



Implicit: $F(x,y)=0$,

$$F_x+F_y\frac{dy}{dx}=0 \qquad \frac{dy}{dx}=-\frac{F_x}{F_y}$$

$F(x,y,z)\overset{=0}{}$, $z=f(x,y)$.

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial F}{\partial y}\cancel{\frac{\partial y}{\partial x}}^{\,0}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0 \qquad \therefore \frac{\partial z}{\partial x}=-\frac{F_x}{F_z}$$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

Directional Derivatives

The rate of change of $f(x,y)$ in the direction of the unit vector $\vec{u}=\langle a,b\rangle$ is called the directional derivative $D_{\vec{u}}f(x,y)$.

$$D_{\vec{u}}f(x,y)=\lim_{h\to 0}\frac{f(x+ah,y+bh)-f(x,y)}{h}$$

more practically...

$x=x_0+az \quad y=y_0+bz$

$$D_{\vec{u}}f(x,y)=f_x(x,y)a+f_y(x,y)b$$

or

$$D_{\vec{u}}f(x,y,z)=\underbrace{\langle f_x,f_y,f_z\rangle}_{\text{gradient}}\cdot\langle a,b,c\rangle$$

$$\nabla f=\langle f_x,f_y,f_z\rangle=f_x\hat{i}+f_y\hat{j}+f_z\hat{k}$$

$$\therefore \quad D_{\vec{u}}f(\vec{x})=\nabla f\cdot\vec{u}$$

Max value of $D_{\vec{u}} f(\vec{x})$ (hence the max rate of change) is given by $\|\nabla f(\vec{x})\|$ and will occur in the direction $\nabla f(\vec{x})$.

ie) when $\vec{u} \parallel$ to $\nabla f(\vec{x})$.

Fact: The gradient vector $\nabla f(x_0, y_0, z_0)$ is $\perp$ to the level surface $f(x,y,z)=k$ at the point (x_0, y_0, z_0)

Recall: equation of a plane containing (x_0, y_0, z_0) w/ normal vector $\vec{n} = \langle a,b,c \rangle$ is given by

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

tangent plane

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0)$$

normal line

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t \nabla f(x_0, y_0, z_0)$$

Extrema

A function $f(x,y)$ has a relative minimum at point (a,b) if $f(x,y) \geq f(a,b)$ for all points (x,y) in some region around (a,b).

Critical Point "Stationary Point"

The point (a,b) is a critical point of $f(x,y)$ if $\nabla f(a,b) = \vec{0}$ $\begin{cases} f_x(a,b) = 0 \\ \text{and} \\ f_y(a,b) = 0 \end{cases}$

or

$f_x(a,b)$ DNE or $f_y(a,b)$ DNE

If (a,b) is a relative extrema of $f(x,y)$ then (a,b) is also a critical point of $f(x,y)$ and $\nabla f(a,b) = \vec{0}$.

Not all critical points are relative extrema!

Define $D(a,b)$

$$D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

If $D>0$ and $f_{xx}(a,b)>0$ then relative min.

If $D>0$ and $f_{xx}(a,b)<0$ then there is relative max

If $D<0$ then the point (a,b) is a saddle point.

If $D=0$, we gain no new information.

Saddle point curves up in one direction and down in a different direction.

Extreme Value Theorem

If $f(x,y)$ is continuous on some closed bounded set D in $\mathbb{R}^2$ then there are points in D, (x_1,y_1) and (x_2,y_2) so that $f(x_1,y_1)$ and $f(x_2,y_2)$ are absolute maximum and minimum respectively of the function in D.

Finding Absolute Extrema

1. Find all critical points and determine their values
2. Find all extrema of the function on the boundary.

Lagrange Multipliers

Let us optimize $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$

λ - Lagrange Multiplier

1. $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$
   $g(x,y,z)=k$

2. plug in all solutions (x,y,z) into f and identify max and min values.

If 2+ constraints then

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$$

$$g(x,y,z)=c$$

$$h(x,y,z)=k$$

2-D Hessian

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \qquad D_1 = f_{xx} \qquad D_2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

- If $D_1(a,b) > 0$ and $D_2(a,b) > 0$, then local min.
- If $D_1(a,b) < 0$ and $D_2(a,b) > 0$, then local max.
- If $D_2(a,b) < 0$, then saddle point.

3-D Hessian

$$H = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} \qquad D_1 = f_{xx} \qquad D_2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

$$D_3 = \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}$$

- If $D_1, D_2, D_3 > 0$, local min
- If $D_1 < 0$, $D_2 > 0$, $D_3 < 0$, local max
- If $D_3 \neq 0$ in any other case, saddle point

Double Integrals

Recall in 1-D:

$$A \approx f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x$$

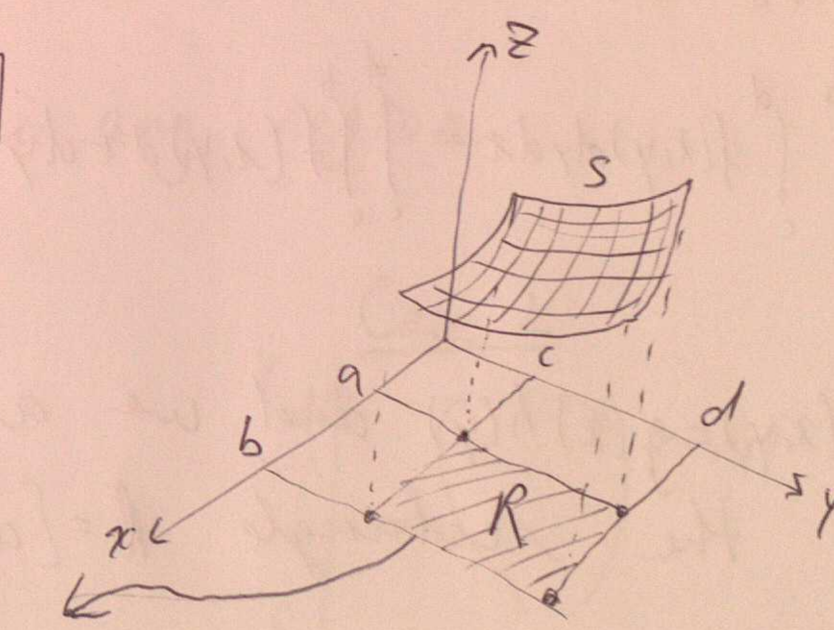
and as $n \to \infty$

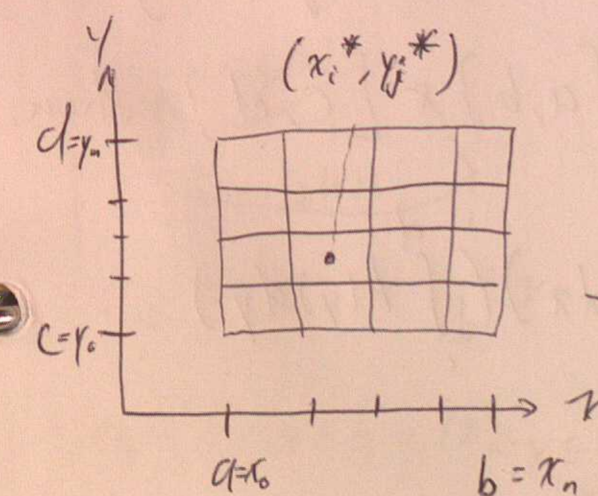
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

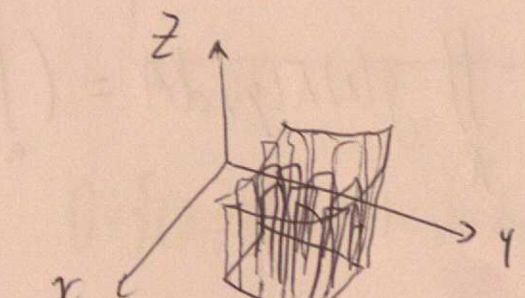
In one variable we integrated over an interval (1-D space), so in two variables, we integrate over $\mathbb{R}^2$.

$$R = [a,b] \times [c,d]$$

$a \le x \le b$
$c \le y \le d$







Each rectangle has base area ΔA, height of $f(x_i^*, y_j^*)$.

$$V \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_j^*)\Delta A$$

rectangle region

$$\boxed{\iint_R f(x,y)\,dA = \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_j^*)\Delta A}$$

Iterated Integrals

Fubini's Theorem: If $f(x,y)$ is continuous on $R=[a,b]\times[c,d]$ then,

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

If $f(x,y)=g(x)h(y)$ and we are integrating over the rectangle $R=[a,b]\times[c,d]$ then,

$$\iint_R f(x,y)\,dA = \iint_R g(x)h(y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)$$

Constants of Integration

Ex. $\int [x\sec^2(2y) + 4xy]\,dy$

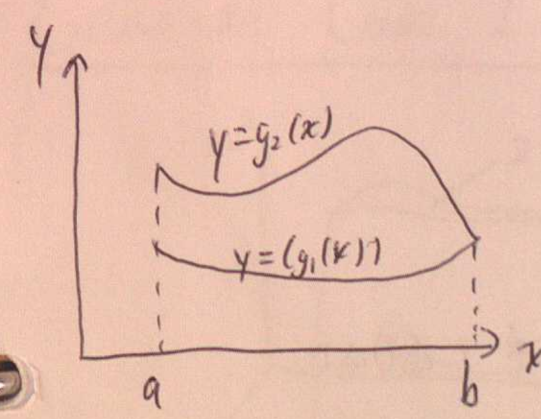
$= \frac{x}{2}\tan(2y) + 2xy^2 + g(x)$

Any function involving only x's will differentiate to zero, so when integrating wrt y, we need to acknowledge that there may have been a function of only x's in the function so the constant of integration is $g(x)$.

Double Integrals over General Regions

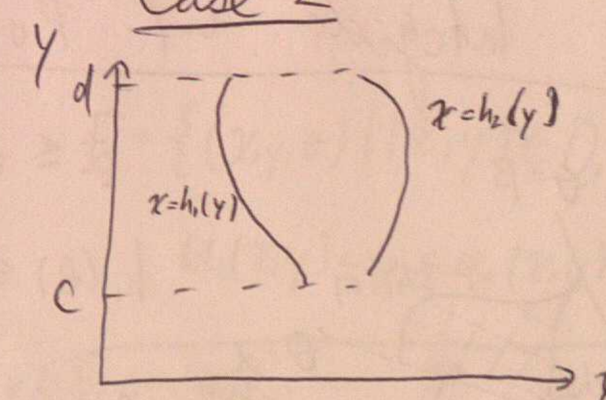
$\iint_D f(x,y)\,dA$, D is any region.

Case 1



$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

Case 2



$D = \{(x,y) \mid h_1(y) \le x \le h_2(y),\ c \le y \le d\}$

Case 1

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Case 2

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

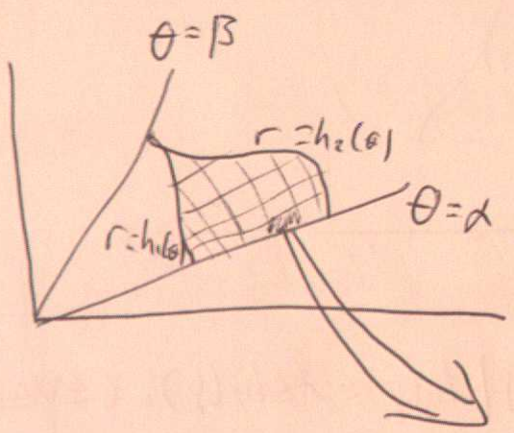
∴ The volume of the solid below the surface $z=f(x,y)$ and above the region D in the xy-plane is given by

$$V=\iint_D f(x,y)\,dA.$$

The area of region D is given by

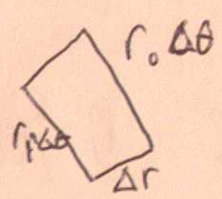
$$A \text{ of } D = \iint_D dA.$$

Double Integrals w/ Polar



$\alpha \le \theta \le \beta$

$h_1(\theta) \le r \le h_2(\theta)$



$\Delta A \doteq r\Delta\theta\Delta r$

$dA \approx \Delta A, \; d\theta \approx \Delta\theta, \; dr \approx \Delta r$

$\therefore \boxed{dA = r\,dr\,d\theta}$

Recall: $x=r\cos\theta, \quad y=r\sin\theta, \quad r^2=x^2+y^2$

$$\therefore \iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Triple Integrals

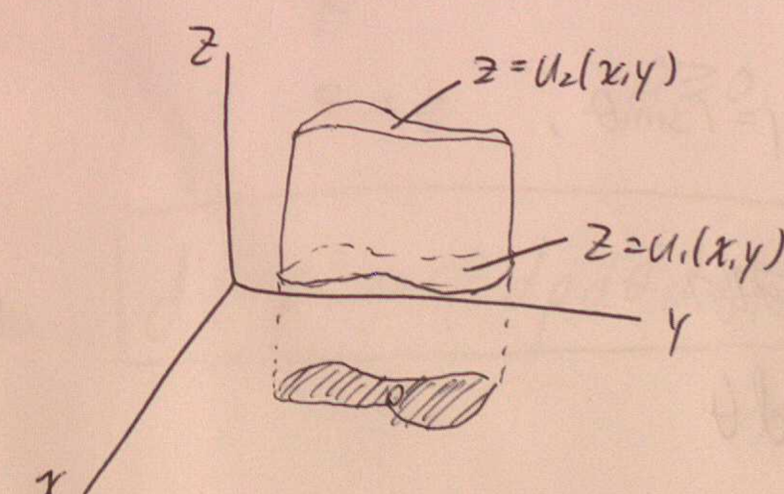
Lets start by integrating over the box

$B=[a,b]\times[c,d]\times[r,s]$, then

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

$$\boxed{\text{Volume}_E = \iiint_E dV}$$

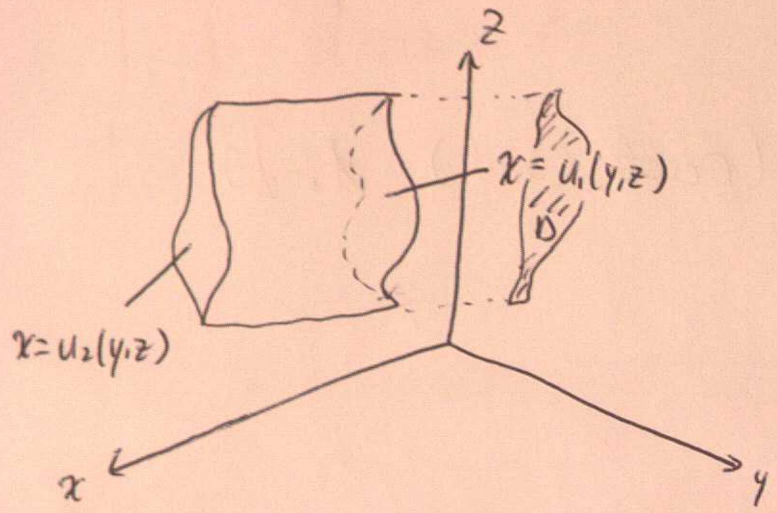
General Case 1



$E=\{(x,y,z) \mid (x,y)\in D, \; u_1(x,y)\le z \le u_2(x,y)\}$

$$\boxed{\iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \; dA}$$

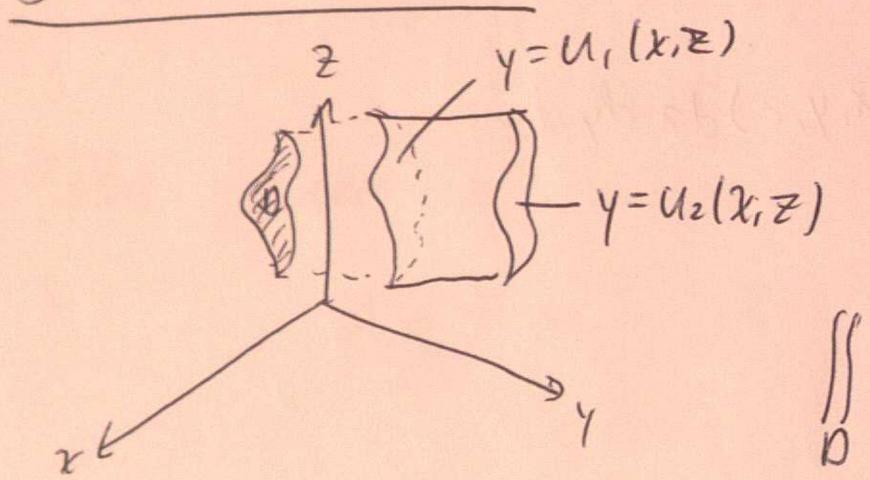
General Case 2



$$E = \{(x,y,z) \mid (y,z) \in D, \; u_1(y,z) \le x \le u_2(y,z)\}$$

$$\boxed{\iint_D \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx \; dA}$$

General Case 3



$$E = \{(x,y,z) \mid (x,z) \in D, \; u_1(x,z) \le y \le u_2(x,z)\}$$

$$\iint_D \int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy \; dA$$

Cylindrical Coordinates

Recall: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$\therefore$ $\cancel{dV = z\,d\theta\,dz\,dr\,d\theta}$

$dV = r\,dz\,dr\,d\theta$

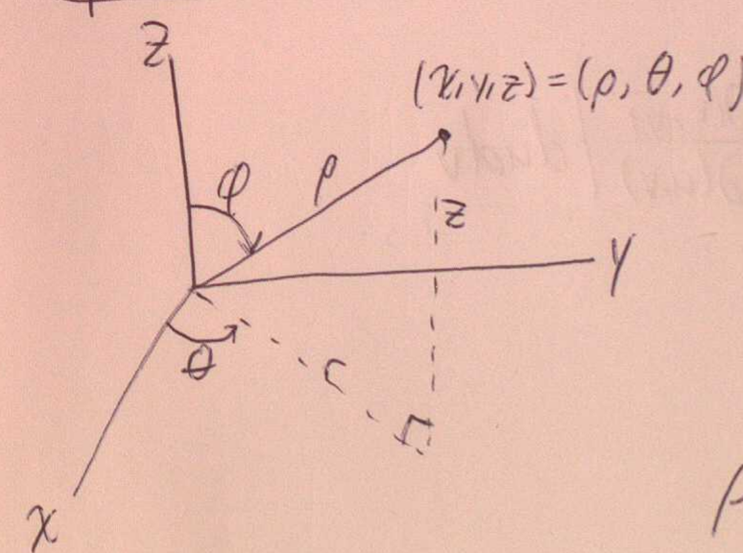
$$E = \{(x,y,z) \mid (x,y) \in D, \; u_1(x,y) \le z \le u_2(x,y)\}$$

$$= \{(r,\theta,z) \mid \alpha \le \theta \le \beta, \; h_1(\theta) \le r \le h_2(\theta), \; u_1(r\cos\theta, r\sin\theta) \le z \le u_2(r\cos\theta, r\sin\theta)\}$$

for D in x-y plane. This can be modified for yz or xz.

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} r\, f(r\cos\theta, r\sin\theta, z)\,dz\,dr\,d\theta}$$

Spherical Coordinates



$x = \rho\sin\varphi\cos\theta$

$y = \rho\sin\varphi\sin\theta$

$z = \rho\cos\varphi$

$x^2 + y^2 + z^2 = \rho^2$

$\rho \ge 0, \quad 0 \le \varphi \le \pi$

$$\boxed{dV = \rho^2 \sin\varphi\,d\rho\,d\theta\,d\varphi} \rightarrow \iiint_E f(x,y,z)\,dV$$

Recall

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_c^d f(u)\,du \quad , \quad u = g(x)$$

Change of Variables

The Jacobian of the transformation $x = g(u,v)$, $y = h(u,v)$

is $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$.

Under the transformation $x = g(u,v)$, $y = h(u,v)$:

$$\iint_D f(x,y)\,dA = \iint_S f(g(u,v), h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

For Triple:

$$dV = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} du\,dv\,dw$$

Surface Area

$$S = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\; dA \quad , \quad z = f(x,y)$$

$$\vec{r}(x,y) = \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix}$$

$$\underbrace{\left\| \vec{r}_x \times \vec{r}_y \right\|}_{\text{area}} = \sqrt{f_x^2 + f_y^2 + 1}$$