Computation Control 2025 Project: Linearized System for MPC

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We define the state vector to represent densities:

$$\boldsymbol{\rho}(k) = \begin{bmatrix} \rho_0(k) \\ \rho_1(k) \\ \rho_2(k) \\ \rho_3(k) \\ \rho_4(k) \end{bmatrix}, \tag{1}$$

where $\rho_0(k), \ldots, \rho_3(k)$ represent the density of peripheral regions with car flowing into the central region with density $\rho_4(k)$. **We assume there are only 4 connections to the center region, i.e. one for each peripheral region! We assume there is no connection among the peripheral region, i.e. the only flows are the flow from each peripheral region to the center region.** The state-space dynamics are most naturally expressed in terms of accumulation $n_i(k)$ (number of cars at time k). At each time step, the peripheral regions i sees an increase of cars of $T_s q_i$ and sees a decrease (outflow) of $T_s \phi_{i4}$. Similarly, the center region sees an increase of $T_s q_4$ due to demand, increase of $T_s \sum_{j=0}^{3} \phi_{ji}(k)$ due to the inflow from the peripheral regions. Additionally, a fraction (ϵ) of the cars currently in the center region completed their trips and are removed from the grid.

$$n_i(k+1) = n_i(k) + T_s(q_i(k) - \phi_{i4}(k)) \quad \text{for } i = 0, 1, 2, 3 \tag{2a}$$

$$n_4(k+1) = (\epsilon^{T_s}) n_4(k) + T_s \left(q_4(k) + \sum_{j=0}^{3} \phi_{ji}(k) \right) \tag{2b}$$

where:

- T_s is the sampling time,
- $\phi_{ji}(k)$ represents the flow from region j to region i at time k.
- $q_i(k)$ is the known demand at each time k, defined as the number of cars that "appear" in each region that are destined for the center region.[1]
- ϵ is an empirical parameter we choose ($\epsilon \leq 1$) to represent the fraction of cars in the center region that "disappear" (completed their trips) in one second. To get the decay for the entire sampling period T_s, we exponentiate ϵ to T_s.

[1] Note that in the traffic simulation, we have the full pairwise demand, i.e. cars originating from region i destined for region j. This also includes cars not destined for the center (region 4). However, since we assume no links between peripheral regions, we simply ignore all demands that are not destined for the center region. The rationale is that both the demand and the flow will be ignored, as these cars will likely not route through the center region.

To translate accumulation to density, we use

$$n_i(k) \text{ [\#vehicles]} = \rho_i(k)\text{n[\#vehicles/km]} \cdot \gamma_i \text{ [km]}, \tag{3}$$

where γ_i is a constant given by

$$\gamma_i \text{ [km]} \triangleq \text{(number of edges)} \cdot \text{(number of lanes/edge)} \cdot \text{(edge length)}. \tag{4}$$

The density state dynamics is thus:

$$\rho_i(k+1) = \rho_i(k) + \frac{T_s}{\gamma_i}(q_i(k) - \phi_{i4}(k)) \quad \text{for } i = 0,1,2,3 \tag{5a}$$

$$\rho_4(k+1) = (\epsilon^{T_s})\rho_4(k) + \frac{T_s}{\gamma_4}\left(q_4(k) + \sum_{j=0}^{3}\phi_{ji}(k)\right) \tag{5b}$$

We also know the relationship between inter-regional flow ϕ_{ij}, inter-regional speed v_{ij}, and inter-regional density ρ_{ij}:

$$\frac{\phi_{ji}}{v_{ji}} = \rho_{ji}. \tag{6}$$

Hence, we write:

$$\rho_i(k+1) = \rho_i(k) + \frac{T_s}{\gamma_i}\left(q_i(k) - \rho_{i4}(k)v_{i4}(k)\right) \quad \text{for } i = 0,1,2,3 \tag{7a}$$

$$\rho_4(k+1) = \epsilon^{T_s}\rho_4(k) + \frac{T_s}{\gamma_4}\left(q_4(k) + \sum_{j=0}^{3}\rho_{j4}(k)v_{j4}(k)\right). \tag{7b}$$

Note we do not have ρ_{ji}! Hence, we use the approximation $\rho_{ji} \approx \rho_j$ (rationale: the density of the link approximates the departing region),

$$\rho_i(k+1) = \rho_i(k) + \frac{T_s}{\gamma_i}\left(q_i(k) - \rho_i(k)v_{i4}(k)\right) \quad \text{for } i = 0,1,2,3 \tag{8a}$$

$$\rho_4(k+1) = \epsilon^{T_s}\rho_4(k) + \frac{T_s}{\gamma_4}\left(q_4(k) + \sum_{j=0}^{3}\rho_j(k)v_{j4}(k)\right). \tag{8b}$$

In matrix form:

$$\boldsymbol{\rho}(k+1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \epsilon^{T_s} \end{bmatrix} \boldsymbol{\rho}(k) + \begin{bmatrix} \frac{T_s}{\gamma_0} & 0 & 0 & 0 & 0 \\ 0 & \frac{T_s}{\gamma_1} & 0 & 0 & 0 \\ 0 & 0 & \frac{T_s}{\gamma_2} & 0 & 0 \\ 0 & 0 & 0 & \frac{T_s}{\gamma_3} & 0 \\ 0 & 0 & 0 & 0 & \frac{T_s}{\gamma_4} \end{bmatrix} \begin{bmatrix} q_0(k) \\ q_1(k) \\ q_2(k) \\ q_3(k) \\ q_4(k) \end{bmatrix}$$

$$+ \begin{bmatrix} -\frac{T_s}{\gamma_0} v_{04}(k)\rho_0(k) \\ -\frac{T_s}{\gamma_1} v_{14}(k)\rho_1(k) \\ -\frac{T_s}{\gamma_2} v_{24}(k)\rho_2(k) \\ -\frac{T_s}{\gamma_3} v_{34}(k)\rho_3(k) \\ \frac{T_s}{\gamma_4} \sum_{j=0}^{3} v_{j4}(k)\rho_j(k) \end{bmatrix}. \tag{9}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \epsilon^{T_s} \end{bmatrix} \boldsymbol{\rho}(k) + \begin{bmatrix} \frac{T_s}{\gamma_0} & 0 & 0 & 0 & 0 \\ 0 & \frac{T_s}{\gamma_1} & 0 & 0 & 0 \\ 0 & 0 & \frac{T_s}{\gamma_2} & 0 & 0 \\ 0 & 0 & 0 & \frac{T_s}{\gamma_3} & 0 \\ 0 & 0 & 0 & 0 & \frac{T_s}{\gamma_4} \end{bmatrix} \begin{bmatrix} q_0(k) \\ q_1(k) \\ q_2(k) \\ q_3(k) \\ q_4(k) \end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix} -\frac{50T_s}{\gamma_0} u_1(k)\rho_0(k) \\ -\frac{50T_s}{\gamma_1} (u_2(k)+u_3(k))\rho_1(k) \\ -\frac{50T_s}{\gamma_2} u_4(k)\rho_2(k) \\ -\frac{50T_s}{\gamma_3} u_0(k)\rho_3(k) \\ \frac{50T_s}{\gamma_4} \left[u_1(k)\rho_0(k) + (u_2(k)+u_3(k))\rho_1(k) + u_4(k)\rho_2(k) + u_0(k)\rho_3(k)\right] \end{bmatrix}}_{:=\boldsymbol{g}(\boldsymbol{\rho}(k),\boldsymbol{u}(k))} \tag{10}$$

where in (10) we apply the following mapping between inter-regional speed v and actuator input u.

$$u_0 = \frac{v_{34}}{50} \tag{11}$$

$$u_1 = \frac{v_{04}}{50} \tag{12}$$

$$u_2 = \frac{v_{14}}{50} \tag{13}$$

$$u_3 = \frac{v_{14}}{50} \tag{14}$$

$$u_4 = \frac{v_{24}}{50}. \tag{15}$$

Note the actuator inputs u are a factor to the nominal speed of 50 km/hr. In addition, region 1 has two actuators u_2 and u_3.

We linearize the nonlinear term (last term in (10), which we denote by $\boldsymbol{g}(\boldsymbol{\rho}(k), \boldsymbol{u}(k))$, around a nominal point $(\boldsymbol{\rho}^*, \boldsymbol{u}^*)$ using a first-order Taylor expansion. The linearization point is a design parameter (see Remark 2 in the following section).

$$\boldsymbol{g}(\boldsymbol{\rho}(k), \boldsymbol{u}(k)) \approx \boldsymbol{g}(\boldsymbol{\rho}^*, \boldsymbol{u}^*) + \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\rho}}\Big|_{(\boldsymbol{\rho}^*,\boldsymbol{u}^*)} (\boldsymbol{\rho}(k) - \boldsymbol{\rho}^*) + \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{u}}\Big|_{(\boldsymbol{\rho}^*,\boldsymbol{u}^*)} (\boldsymbol{u}(k) - \boldsymbol{u}^*). \tag{16}$$

The Jacobians are given by:

$$\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\rho}}\Big|_{(\boldsymbol{\rho}^*, \boldsymbol{u}^*)} = \frac{50T_s}{\gamma}\begin{bmatrix} -u_1^* & 0 & 0 & 0 & 0 \\ 0 & -(u_2^*+u_3^*) & 0 & 0 & 0 \\ 0 & 0 & -u_4^* & 0 & 0 \\ 0 & 0 & 0 & -u_0^* & 0 \\ u_1^* & (u_2^*+u_3^*) & u_4^* & u_0^* & 0 \end{bmatrix}, \tag{17}$$

$$\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{u}}\Big|_{(\boldsymbol{\rho}^*, \boldsymbol{u}^*)} = \frac{50T_s}{\gamma}\begin{bmatrix} 0 & -\rho_0^* & 0 & 0 & 0 \\ 0 & 0 & -\rho_1^* & -\rho_1^* & 0 \\ 0 & 0 & 0 & 0 & -\rho_2^* \\ -\rho_3^* & 0 & 0 & 0 & 0 \\ \rho_3^* & \rho_0^* & \rho_1^* & \rho_1^* & \rho_2^* \end{bmatrix}. \tag{18}$$

Thus, the linearized system is:

$$\begin{aligned} \boldsymbol{\rho}(k+1) \approx & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \epsilon^{T_s} \end{bmatrix}\boldsymbol{\rho}(k) + \begin{bmatrix} \frac{T_s}{\gamma_0} & 0 & 0 & 0 & 0 \\ 0 & \frac{T_s}{\gamma_1} & 0 & 0 & 0 \\ 0 & 0 & \frac{T_s}{\gamma_2} & 0 & 0 \\ 0 & 0 & 0 & \frac{T_s}{\gamma_3} & 0 \\ 0 & 0 & 0 & 0 & \frac{T_s}{\gamma_4} \end{bmatrix}\boldsymbol{q}(k) \\ & + \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\rho}}\Big|_{(\boldsymbol{\rho}^*, \boldsymbol{u}^*)}(\boldsymbol{\rho}(k) - \boldsymbol{\rho}^*) + \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{u}}\Big|_{(\boldsymbol{\rho}^*, \boldsymbol{u}^*)}(\boldsymbol{u}(k) - \boldsymbol{u}^*) + \boldsymbol{g}(\boldsymbol{\rho}^*, \boldsymbol{u}^*). \end{aligned} \tag{19}$$

Expanding everything out gives:

$$\boldsymbol{\rho}(k+1) = A\boldsymbol{\rho}(k) + B\boldsymbol{u}(k) + C\boldsymbol{q}(k) + d, \tag{20}$$

where

$$A = \begin{bmatrix} 1 - \frac{50T_s}{\gamma_0}u_1^* & 0 & 0 & 0 & 0 \\ 0 & 1 - \frac{50T_s}{\gamma_1}(u_2^*+u_3^*) & 0 & 0 & 0 \\ 0 & 0 & 1 - \frac{50T_s}{\gamma_2}u_4^* & 0 & 0 \\ 0 & 0 & 0 & 1 - \frac{50T_s}{\gamma_3}u_0^* & 0 \\ \frac{50T_s}{\gamma_4}u_1^* & \frac{50T_s}{\gamma_4}(u_2^*+u_3^*) & \frac{50T_s}{\gamma_4}u_4^* & \frac{50T_s}{\gamma_4}u_0^* & \epsilon^{T_s} \end{bmatrix}, \tag{21}$$

$$B = \begin{bmatrix} 0 & -\frac{50T_s}{\gamma_0}\rho_0^* & 0 & 0 & 0 \\ 0 & 0 & -\frac{50T_s}{\gamma_1}\rho_1^* & -\frac{50T_s}{\gamma_1}\rho_1^* & 0 \\ 0 & 0 & 0 & 0 & -\frac{50T_s}{\gamma_2}\rho_2^* \\ -\frac{50T_s}{\gamma_3}\rho_3^* & 0 & 0 & 0 & 0 \\ \frac{50T_s}{\gamma_4}\rho_3^* & \frac{50T_s}{\gamma_4}\rho_0^* & \frac{50T_s}{\gamma_4}\rho_1^* & \frac{50T_s}{\gamma_4}\rho_1^* & \frac{50T_s}{\gamma_4}\rho_2^* \end{bmatrix} \tag{22}$$

$$C = \begin{bmatrix} \frac{T_s}{\gamma_0} & 0 & 0 & 0 & 0 \\ 0 & \frac{T_s}{\gamma_1} & 0 & 0 & 0 \\ 0 & 0 & \frac{T_s}{\gamma_2} & 0 & 0 \\ 0 & 0 & 0 & \frac{T_s}{\gamma_3} & 0 \\ 0 & 0 & 0 & 0 & \frac{T_s}{\gamma_4} \end{bmatrix}, \tag{23}$$

$$d = \begin{bmatrix} \frac{50T_s}{\gamma_0}u_1^*\rho_0^* \\ \frac{50T_s}{\gamma_1}(u_2^*+u_3^*)\rho_1^* \\ \frac{50T_s}{\gamma_2}u_4^*\rho_2^* \\ \frac{50T_s}{\gamma_3}u_0^*\rho_3^* \\ -\frac{50T_s}{\gamma_4}\left(u_1^*\rho_0^* + (u_2^*+u_3^*)\rho_1^* + u_4^*\rho_2^* + u_0^*\rho_3^*\right) \end{bmatrix}. \tag{24}$$