Problem 1.

Consider a forward SDE

$$dx_t = f(x_t, t)\, dt + g(x_t, t)\, dW_t.$$

Show that the corresponding probability flow ODE can be written as

$$dx_t = \left[f(x_t, t) - \frac{1}{2}\frac{\partial}{\partial x} g^2(x_t, t) - \frac{g^2(x_t, t)}{2}\frac{\partial}{\partial x}\log p(x_t, t) \right] dt.$$

Solution.

Step 1. Fokker–Planck equation of the SDE.

For the Itô SDE

$$dx_t = f(x_t, t)\, dt + g(x_t, t)\, dW_t,$$

the marginal density $p(x, t)$ satisfies the Fokker–Planck equation

$$\partial_t p(x, t) = -\frac{\partial}{\partial x}\big(f(x, t)p(x, t)\big) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\big(g^2(x, t)p(x, t)\big).$$

Step 2. Continuity equation for a deterministic ODE.

If a deterministic ODE

$$\dot{x}_t = v(x_t, t)$$

has density $p(x, t)$, then p satisfies the Liouville (continuity) equation

$$\partial_t p(x, t) = -\frac{\partial}{\partial x}\big(v(x, t)p(x, t)\big).$$

Step 3. Equating (FP) and (CE).

To reproduce the same density evolution as the SDE, we set

$$-\frac{\partial}{\partial x}(vp) = -\frac{\partial}{\partial x}(fp) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(g^2 p).$$

Expanding the second derivative term:

$$\frac{\partial^2}{\partial x^2}(g^2 p) = \frac{\partial}{\partial x}\left(\frac{\partial g^2}{\partial x}p + g^2\frac{\partial p}{\partial x}\right).$$

Thus

$$-\frac{\partial}{\partial x}(vp) = -\frac{\partial}{\partial x}\left(fp - \tfrac{1}{2}\frac{\partial g^2}{\partial x}p - \tfrac{1}{2}g^2\frac{\partial p}{\partial x}\right).$$

Step 4. Solve for $v(x, t)$.

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Since the equality holds for all p, we have

$$v(x,t)p(x,t) = f\,p - \tfrac{1}{2}(\partial_x g^2)p - \tfrac{1}{2}g^2\,\partial_x p.$$

Dividing both sides by $p > 0$ and noting that $\partial_x p = p\,\partial_x \log p$,

$$v(x,t) = f(x,t) - \tfrac{1}{2}\,\partial_x g^2(x,t) - \tfrac{g^2(x,t)}{2}\,\partial_x \log p(x,t).$$

Step 5.

Hence the probability flow ODE is

$$\boxed{dx_t = \left[f(x_t,t) - \tfrac{1}{2}\,\partial_x g^2(x_t,t) - \tfrac{g^2(x_t,t)}{2}\,\partial_x \log p(x_t,t) \right] dt.}$$