ECE549 / CS543 Computer Vision: Assignment 1

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1. **Vanishing Points and Vanishing Lines [5 pts].** Consider a plane defined by $\mathbf{N}^T\mathbf{X} = d$, that is undergoing perspective projection with focal length f. Show that the vanishing points of lines on this plane lie on the vanishing line of this plane (adapted from Jitendra Malik)

   Let $A = (A_X, A_Y, A_Z)$ denote a point in the space. Let $L_1(\lambda) = A + \lambda D = (A_X + \lambda D_X, A_Y + \lambda D_Y, A_Z + \lambda D_Z)$ denote a line, where D is the direction of the line. If we project this line onto 2D image, we have $l_1(\lambda) = \left(f\frac{A_X+\lambda D_X}{A_Z+\lambda D_Z}, f\frac{A_Y+\lambda D_Y}{A_Z+\lambda D_Z}\right)$. The vanishing point means $\lambda \to \infty$. So we have the vanishing point:

$$P(X,Y) = \lim_{\lambda\to\infty} l_1(\lambda) = \left(f\frac{D_X}{D_Z}, \frac{D_Y}{D_Z}\right) \tag{1}$$

   For our plane, we have $N_X x + N_Y y + N_Z z = d$. We can rewrite it as $\frac{N_X f X}{Z} + \frac{N_Y f Y}{Z} + fN_Z = \frac{fd}{Z}$. The vanishing line means $Z \to \infty$. So we have the vanishing line:

$$N_X x + N_Y y + fN_Z = 0 \tag{2}$$

   If the vanishing point is on the vanishing line, we would have $N_X \cdot D_X = N_Y \cdot D_Y = N_Z \cdot D_Z = 0$. Plug the vanishing point into vanishing line equation, we have:

$$\begin{aligned} f\frac{N_X D_X}{D_Z} + f\frac{N_Y D_Y}{D_Z} + fN_z &= \frac{1}{D_Z}\left(fN_X D_X + fN_Y D_Y + fN_Z D_Z\right) && (3)\\ &= \frac{1}{D_Z}(0+0+0) && (4)\\ &= 0 && (5)\end{aligned}$$

   Hence, the vanishing point is on the vanishing line.

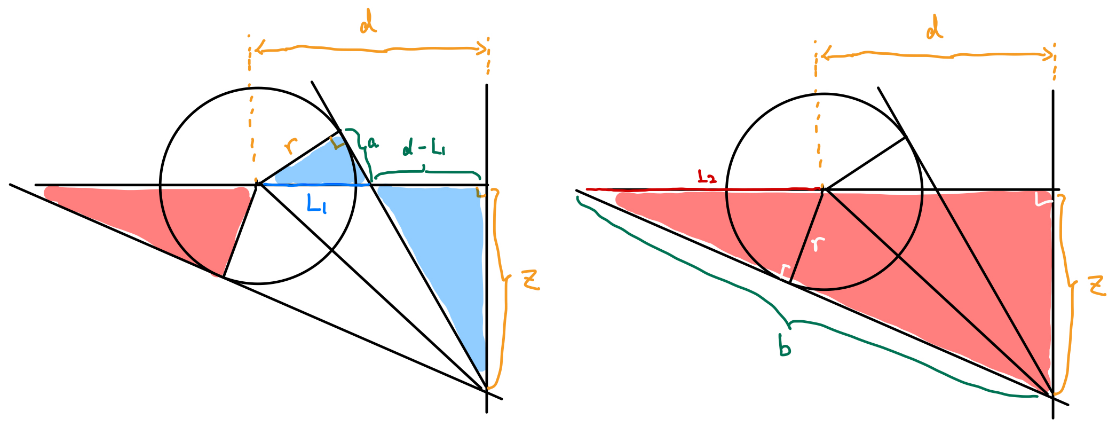
2. **Rectangle and Cylinder under Perspective Projection [15 pts].**

(a) **[5 pts]** Suppose a rectangle of width L is moving along the X axis. How does its width l on the image plan change w.r.t. its distance d to the origin? Recall that, under perspective projection a point (X, Y, Z) in 3D space maps to $(f\frac{X}{Z}, f\frac{Y}{Z})$ in the image, where f is the distance of the image plane from the pinhole.

The left side of the rectangle has a coordinate of $(d + L, 0, Z)$, and the right side of the rectangle has a coordinate of $(d, 0, Z)$. Using the perspective projection, the left side of the projected rectangle has a coordinate of $\left(f\frac{d+L}{Z}, 0\right)$, and the right side of the projected rectangle has a coordinate of $\left(f\frac{d}{Z}, 0\right)$. Hence, we have the width of the projected image:

$$l = f\frac{L}{Z} \tag{6}$$

(b) **[10 pts]** What if we replace the rectangle with a cylinder of radius r on the X axis, how does its width l on the image plane change w.r.t. its distance d to the origin? Show your work, and try to simplify the final result as much as possible. We won't take points off if your answer is correct and complete, but is only missing algebraic simplifications.



We would like to first figure out L_1 and L_2 so that we can have a equivalent rectangle projection. To calculate L_1, notice that two blue triangles on the left picture are similar triangles. Hence, we have:

$$\frac{d - L_1}{z} = \frac{a}{r} \tag{7}$$

Using trig identity, we also have:

$$r^2 + a^2 = L_1^2 \tag{8}$$

Put them together, we have:

$$r^2 + (d - L_1)^2\frac{r^2}{z^2} = L_1^2 \tag{9}$$

$$L_1 = \frac{zr\sqrt{d^2 - r^2 + z^2} - dr^2}{z^2 - r^2} \tag{10}$$

To calculate L_2, notice that two red triangles are similar triangles. Hence, we have:

$$\frac{b}{z} = \frac{L_2}{r} \tag{11}$$

Using trig identity, we also have:

$$b^2 = z^2 + (d + L_2)^2 \tag{12}$$

Put them together, we have:

$$L_2^2 \frac{z^2}{r^2} = z^2 + (d + L_2)^2 \tag{13}$$

$$L_2 = \frac{zr\sqrt{d^2 - r^2 + z^2} + dr^2}{z^2 - r^2} \tag{14}$$

Hence, the equivalent rectangle has width of L:

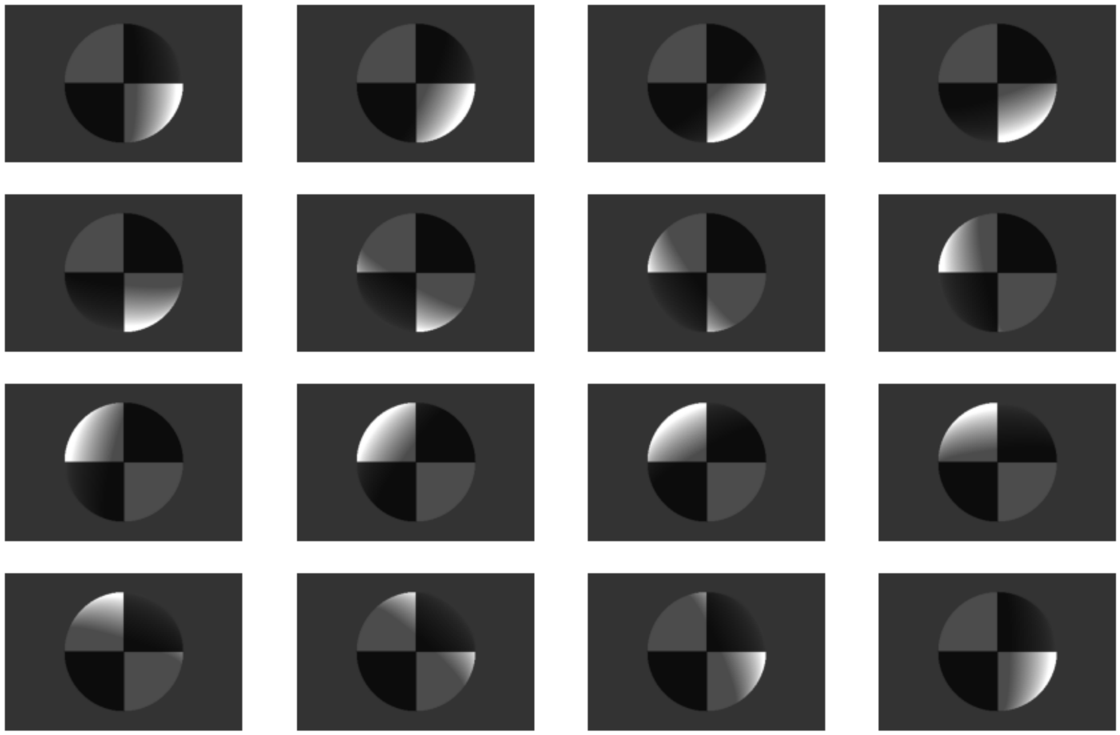
$$L = L_1 + L_2 = \frac{2zr\sqrt{d^2 - r^2 + z^2}}{z^2 - r^2} \tag{15}$$

Hence, the width of the projected image is:

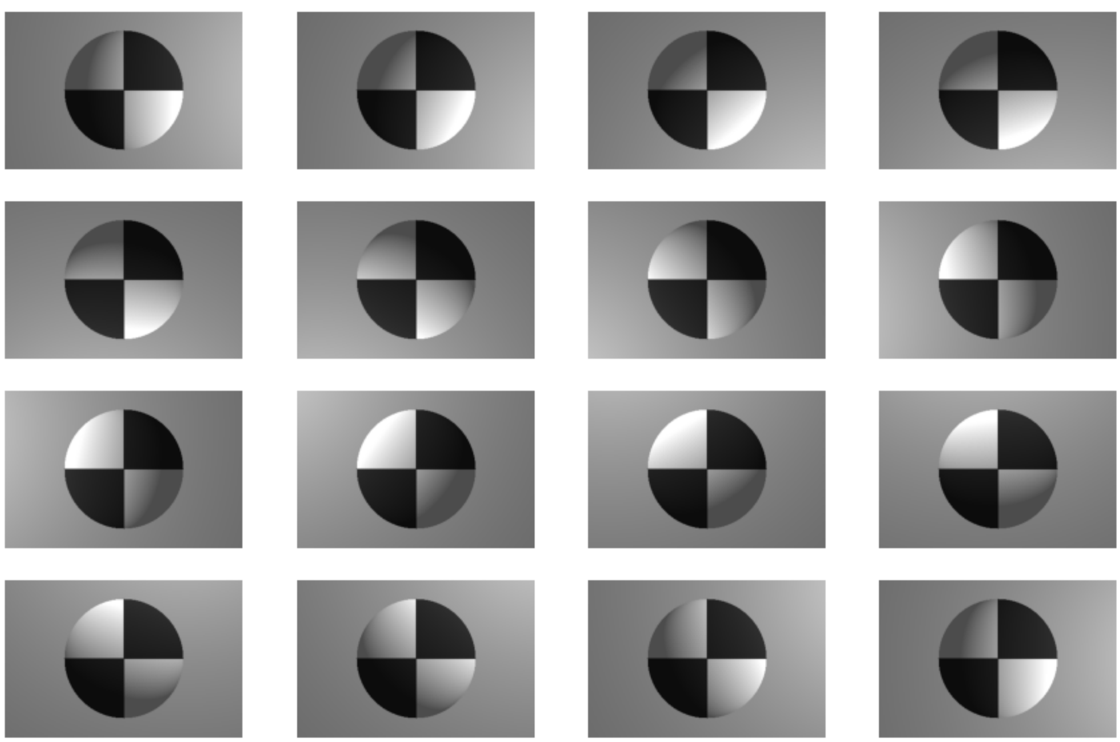
$$l = \frac{f}{z} \frac{2zr\sqrt{d^2 - r^2 + z^2}}{z^2 - r^2} \tag{16}$$

$$= \frac{2fr\sqrt{d^2 - r^2 + z^2}}{z^2 - r^2} \tag{17}$$

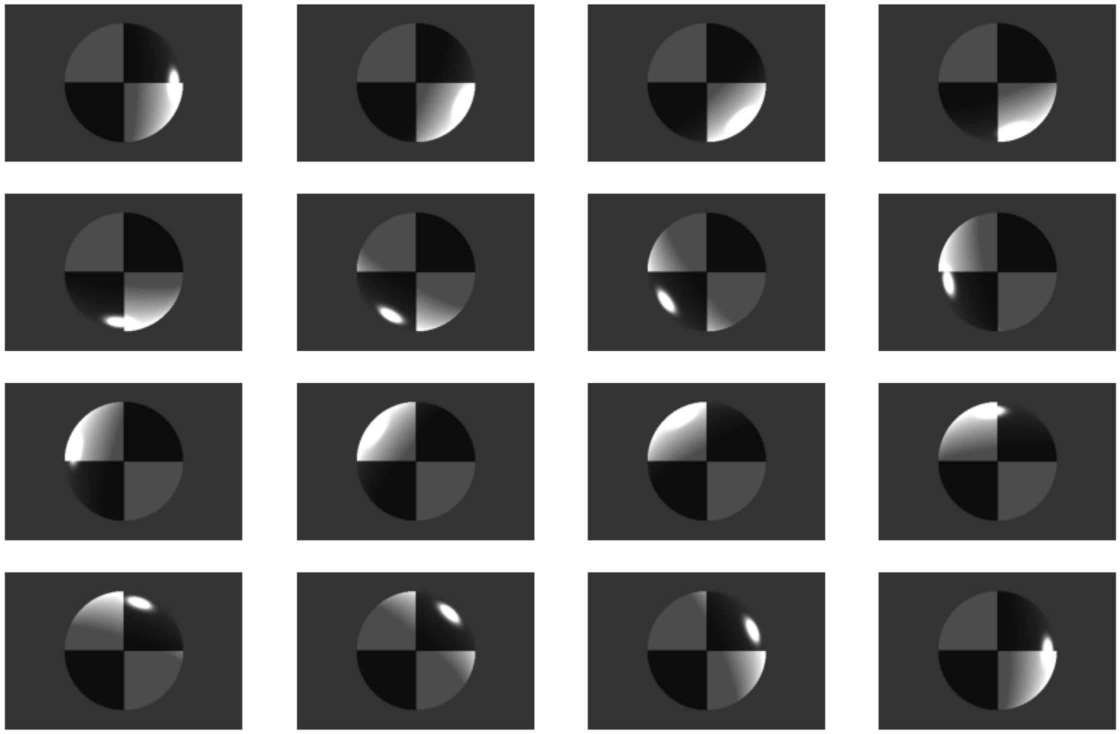
3. **Phong Shading Model [15 pts]**
Directional light source with no specular reflection:



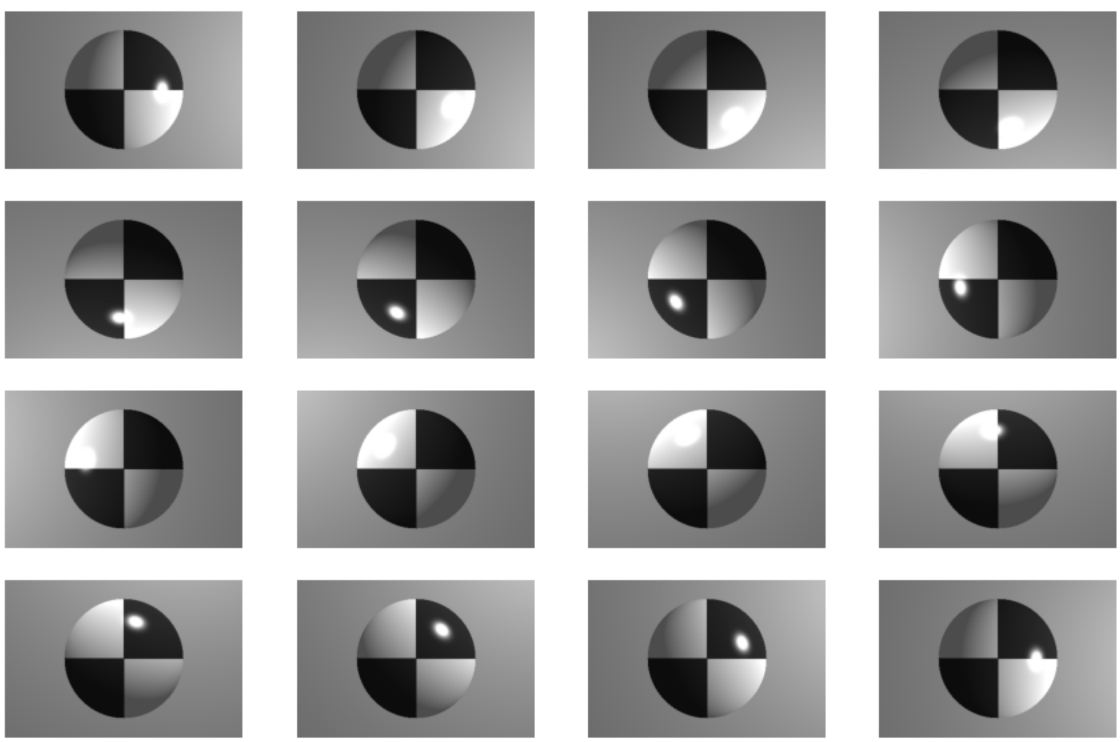
Point light source with no specular reflection:



Directional light source with specular reflection:



Point light source with specular reflection:



The surface normal, $\hat{n}$, is provided. We first create a vector $\hat{p}$ that contains the coordinate of each point in the scene. We can compute the coordinate of each point:

$$(X, Y, Z) = \left(\frac{Z(x - c_x)}{f}, \frac{Z(y - c_y)}{f}, Z \right) \tag{18}$$

Then we can get $\hat{v_i}$ for point light source and directional light source. For the point light source, we just need to use the point light source location minus the pixel location $\hat{p}$. For the directional light source, all pixels have the same $\hat{v_i}$ as the directional light inbound direction. For $\hat{v_r}$, it is essentially looking back from pixel location towards the camera, so $\hat{v_r} = -\hat{p}$. For $\hat{s_i}$, each pixel's value is computed with:

$$\hat{s_i} = (2\hat{n}\hat{n}^T - \hat{I}) \cdot \hat{v_i} \tag{19}$$

Where $\hat{I}$ is a 3×3 identity matrix. After computing all these matrices, we also need to normalize them to ensure correct result.

4. **Dynamic Perspective [15 pts]**

(a) **[5 pts]** For a point $P = (X, Y, Z)$ with translational velocity $t = (t_x, t_y, t_z)$ and angular velocity $\omega = (\omega_x, \omega_y, \omega_z)$, we have the linear velocity at point P:

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = - \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} - \begin{bmatrix} \omega_y Z - \omega_z Y \\ \omega_z X - \omega_x Z \\ \omega_x Y - \omega_y X \end{bmatrix} \tag{20}$$

We can project the linear velocity at point P to image with $x = \frac{fX}{Z}$ and $y = \frac{fY}{Z}$. To obtain the optic flow, we take the time derivative for x and y:

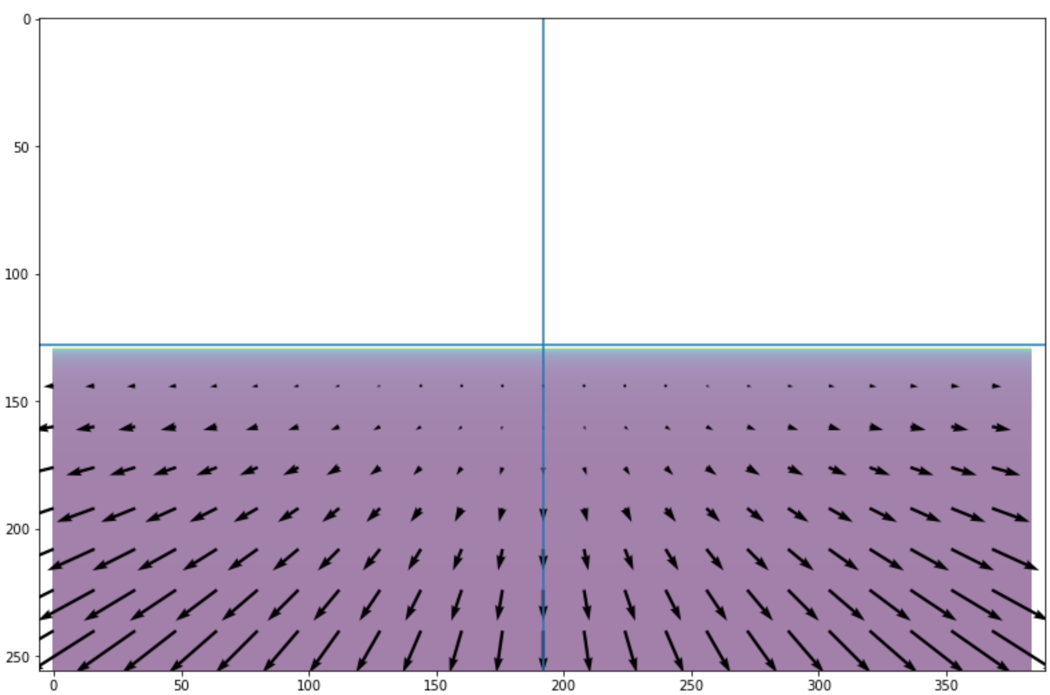
$$u = \dot{x} = f\frac{\dot{X}Z - \dot{Z}X}{Z^2}, v = \dot{y} = f\frac{\dot{Y}Z - \dot{Z}Y}{Z^2} \tag{21}$$

Expand everything and we will have:

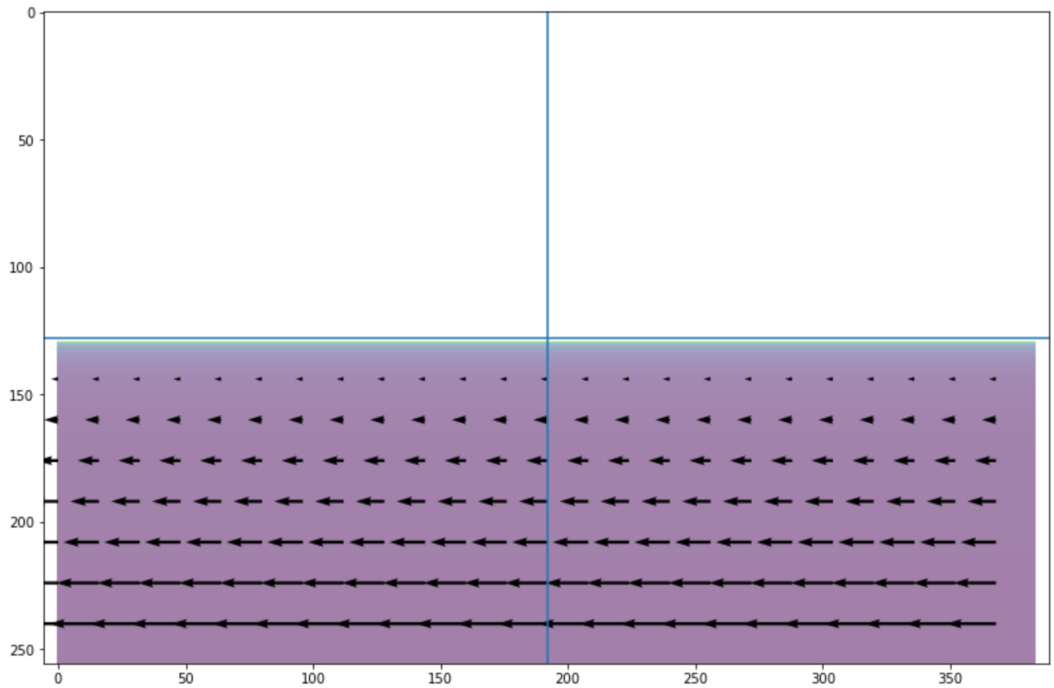
$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \frac{1}{Z} \begin{bmatrix} -f & 0 & x \\ 0 & -f & y \end{bmatrix} \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} + \begin{bmatrix} \frac{xy}{f} & -\left(f + \frac{x^2}{f}\right) & y \\ f + \frac{y^2}{f} & -\frac{xy}{f} & -x \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{22}$$

(b) **[10 pts]**

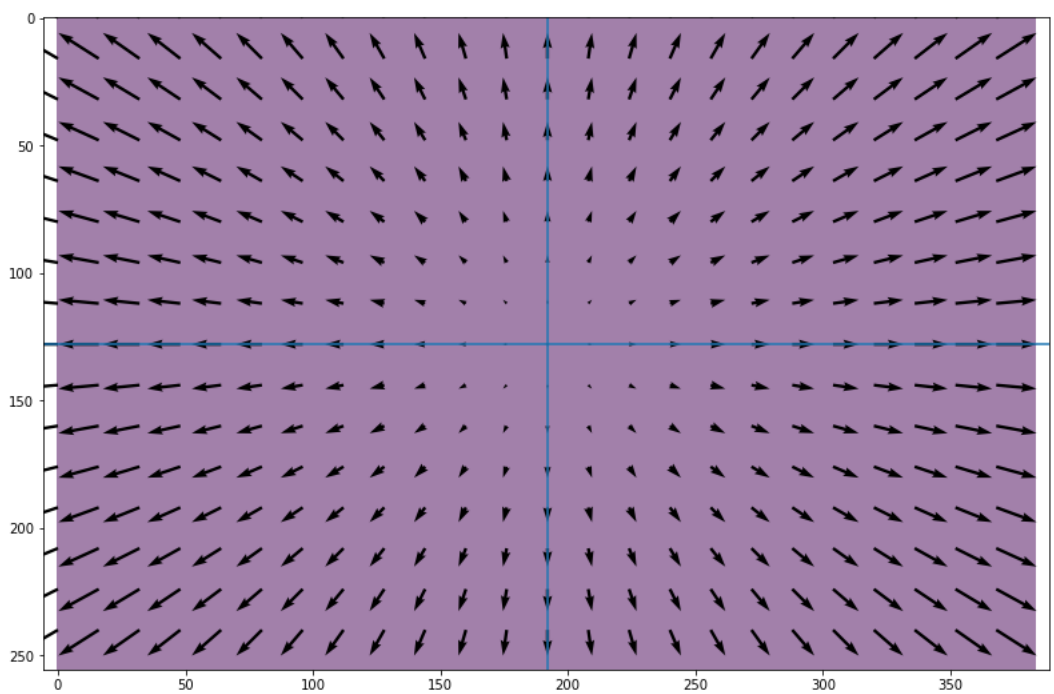
i. Looking forward on a horizontal plane while driving on a flat road



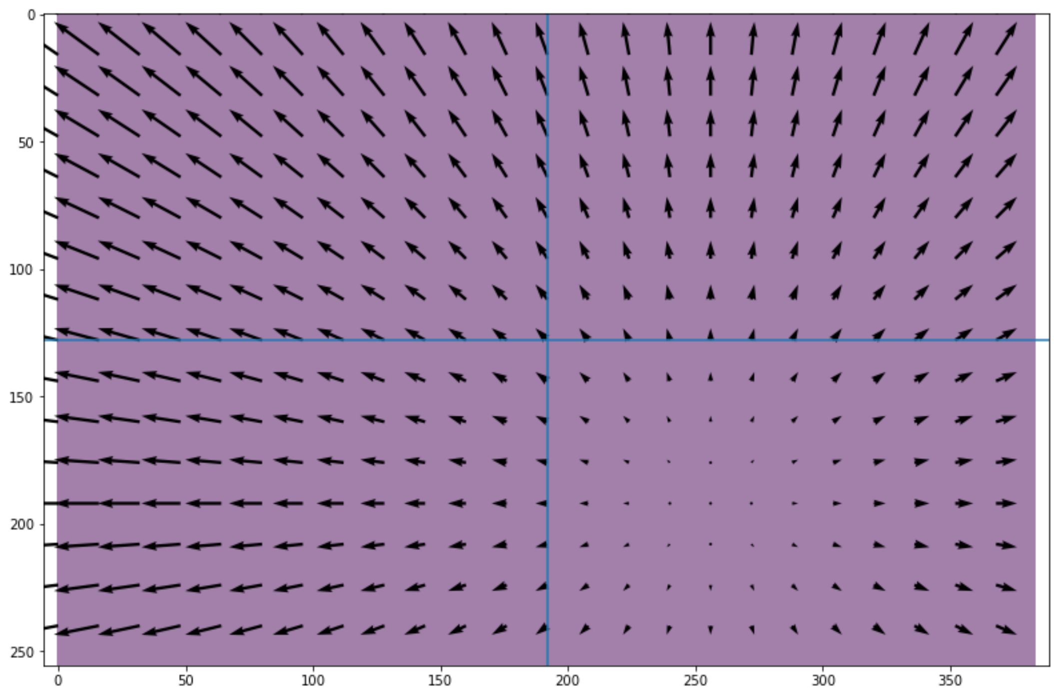
ii. Sitting in a train and looking out over a flat field from a side window (left side window)



iii. Flying into a wall head-on



iv. Flying into a wall but also translating horizontally, and vertically (towards right and down)



v. Counter-clockwise rotating in front of a wall about the Y-axis