Quiz12 Frobenius Form and Method

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Question: Find two independent solution of the equation $x^2y''+xy'+x^2y=0$ using power series solutions centered at $x_0=0$. Write the recurrence equations for the coefficients. Calculate the first few terms. Find the series convergence domain.

Idea: Frobenius form and method

Solution: $\boxed{y=x^r\sum_{n=0}^{\infty}a_nx^n}=\sum_{n=0}^{\infty}a_nx^{n+r}$ ← Always start with this guess, want to find r

$y'=\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}$ and $y''=\sum_{n=0}^{\infty}(n+r-1)(n+r)a_nx^{n+r-2}$

$x^2y''=\sum_{n=0}^{\infty}(n+r-1)(n+r)a_nx^{n+r}$

$xy'=\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}$ and $x^2y=\sum_{n=0}^{\infty}a_nx^{n+r+2}$

s.t. $0=x^2y''+xy'+x^2y=\left(\sum_{n=0}^{\infty}(n+r-1)(n+r)a_nx^{n+r}\right)+\left(\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}\right)+\left(\sum_{n=0}^{\infty}a_nx^{n+r+2}\right)$

$=\left(\sum_{n=0}^{\infty}(n+r-1)(n+r)a_nx^{n+r}\right)+\left(\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}\right)+\left(\sum_{n=2}^{\infty}(a_{n-2})x^{n+r}\right)$

$=(r(r-1)a_0+ra_0)x^r+(r(1+r)a_1+(1+r)a_1)x^{r+1}+\sum_{n=2}^{\infty}\left[(n+r)(n+r-1)a_n+(n+r)a_n+a_{n-2}\right]x^{n+r}$

Assume $a_0=1$ by convention — we can always multiply a constant to make the first non-zero a_n equals to 1

s.t.
$$\begin{cases} r(r-1)a_0+ra_0=0 & (1)\\ r(1+r)a_1+(1+r)a_1=0 & (2)\\ (n+r)(n+r-1)a_n+(n+r)a_n+a_{n-2}=0 \text{ for } n\geq 2 & (3)\end{cases}$$

(1) since $a_0=1$, s.t. $r(r-1)+r=0$ Indicial Equation

such that $r_1=r_2=0$ Index

(2) $(1+r)^2a_1=0$ and when $r=0$, $a_1(0)=0$

(3) $(n+r)^2a_n+a_{n-2}=0$ for $n\geq 2$

$a_n=-\frac{1}{(n+r)^2}a_{n-2}$ for $n\geq 2$

when $r=0$,

$a_3(0)=-\frac{1}{3^2}a_1(0)=0$

similarly, $a_n=0$ for $n=1,3,5,\dots$

and $a_2(0)=-\frac{1}{2^2}a_0(0)=-\frac{1}{4}$

similarly, $a_4(0)=-\frac{1}{4^2}a_2(0)=-\frac{1}{16}\cdot\left(-\frac{1}{4}\right)=\frac{1}{64}$

we have calculated the first few terms, when $r=0$

$a_0=1$, $a_1=0$, $a_2=-\frac{1}{4}$, $a_3=0$ and $a_4=\frac{1}{64}$

thus $y_1(x)=x^r\sum_{n=0}^{\infty}a_nx^n=1+0\cdot x-\frac{1}{4}x^2+0\cdot x^3+\frac{1}{64}x^4+\dots$

$=\sum_{n=0}^{\infty}a_{2n}(x^2)^n$

and $\lim_{n\to\infty}\frac{a_{2n}}{a_{2(n-1)}}=\lim_{n\to\infty}\frac{a_{2n}}{a_{2n-2}}=\lim_{n\to\infty}\left(-\frac{1}{n^2}\right)$ by (3) with $r=0$

$=0$

Therefore y_1 is well-defined for any x.

Now, we want to find y_2, and known

$y_2=y_1\cdot\log x+x^r\sum_{n=0}^{\infty}b_nx^n$, where $b_n=\partial_r a_n(r)\big|_{r=0}$

where,

$b_0=0$ b/c $a_0(r)=1$ which is independent of r

known $(1+r)^2a_1(r)=0$, s.t.

$2(1+r)a_1(r)+(1+r)^2\frac{\partial}{\partial r}(a_1(r))=0$ by implicit differentiation.

s.t. $2a_1(0)+\frac{\partial}{\partial r}(a_1(r))\big|_{r=0}=0 \Rightarrow \frac{\partial}{\partial r}(a_1(r))\big|_{r=0}=0$

therefore, $b_1=0$

and $b_2=\frac{\partial}{\partial r}(a_2(r))\big|_{r=0}=\frac{\partial}{\partial r}\left(-\frac{1}{(2+r)^2}a_0(r)\right)\Big|_{r=0}$ by (3)

$=\frac{\partial}{\partial r}\left(-\frac{1}{(2+r)^2}\right)\Big|_{r=0}$ b/c $a_0(r)=1$ for all r

$=\frac{2}{(2+r)^3}\Big|_{r=0}=\frac{2}{8}=\frac{1}{4}$

s.t. the first few terms are $b_0=0$, $b_1=0$, $b_2=\frac{1}{4}$