TUT1_Linear_Programming_Problem-1

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TUT1_Linear _Program...

Learning Objective

Understand the structures of the linear programming problem in standard form and in canonical form. Be able to set up a linear programming model using suitable notation, and convert to different forms when needed.

Definitions

1. A **general linear programming problem** is of the form linear
   Find values of the **decision variables** $x_1, x_2, ..., x_n$ that will
   Max or min the **objective function** $z = c_1x_1 + ... + c_nx_n$
   subject to the **Constraints**
   $a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \le (\ge)(=) b_1$ no strict inequality
   $a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \le (\ge)(=) b_2$
   $\vdots$ multiple constraints are allowed.
   $a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \le (\ge)(=) b_m$
2. A **linear programming problem in standard form** is as follow
   Find values of the **decision variables** $x_1, x_2, ..., x_n$ that will
   Max the **objective function** $z = c_1x_1 + ... + c_nx_n$
   subject to the Constraints
   $a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \le b_1$ only this allowed.
   $a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \le b_2$
   $\vdots$
   $a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \le b_m$
   $\boxed{x_1, x_2, ..., x_n \ge 0}$ for every decision variable
3. A **linear programming problem in canonical form** is as follow
   Find values of the **decision variables** $x_1, x_2, ..., x_n$ that will
   Max the **objective function** $z = c_1x_1 + ... + c_nx_n$
   subject to the **Constraints** equality
   $a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$
   $a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$
   $\vdots$
   $a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m$
   $\boxed{x_1, x_2, ..., x_n \ge 0}$

Changing the forms of a LPP

Given an arbitrary linear programming problem, we can always reformulated as a standard linear programming problem or a canonical linear programming problem, using the following ideas. Make sure you understand why it is necessary to make the change and why it will work.

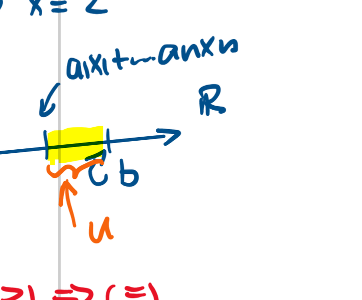
Type	General Idea and Example	
Minimization to Maximization	$\min \sum_{i=1}^{n} c_ix_i$	$-\max(-\sum_{i=1}^{n} c_ix_i)$
	min x, subject to $x \ge 2$	$-\max -x$ subject to $x \ge 2$
Reversing an Inequility ($\ge$ to $\le$)	$a_1x_1 + ... + a_nx_n \ge b$	$-a_1x_1 - ... - a_nx_n \le -b$
	$2x - 4y \ge 3$	$-2x + 4y \le -3$
Inequility to Equility ($\le$ to $=$)	$a_1x_1 + ... + a_nx_n \le b$	$a_1x_1 + ... + a_nx_n + u = b, \boxed{u \ge 0}$
Introduce Slack Variable	$-2x + 4y \le -3$	$-2x + 4y + u = -3, \boxed{u \ge 0}$
Equality to Inequility ($=$ to $\le$ and $\ge$)	$a_1x_1 + ... + a_nx_n = b$	$a_1x_1 + ... + a_nx_n \le b$ $a_1x_1 + ... + a_nx_n \ge b$
Ex: Canonical ⇒ Standard	$2x - 4y = 3$	$2x - 4y \le 3$ $2x - 4y \ge 3$
Unconstrained variables	$x \in \mathbb{R}$	Replace x by $x^+ - x^-$ $x^+, x^- \ge 0$
	max $2x$	max $2x^+ - 2x^-$ subject to $x^+, x^- \ge 0$

$x = 2$

$x \ge 2 \Rightarrow -x \le -2$

$\max(-x) \Rightarrow -x = -2$

$\Rightarrow x = 2$



$(\ge) \Rightarrow (=)$

$(\le)$

$\boxed{x = x^+ - x^-, \; x^+, x^- \ge 0}$

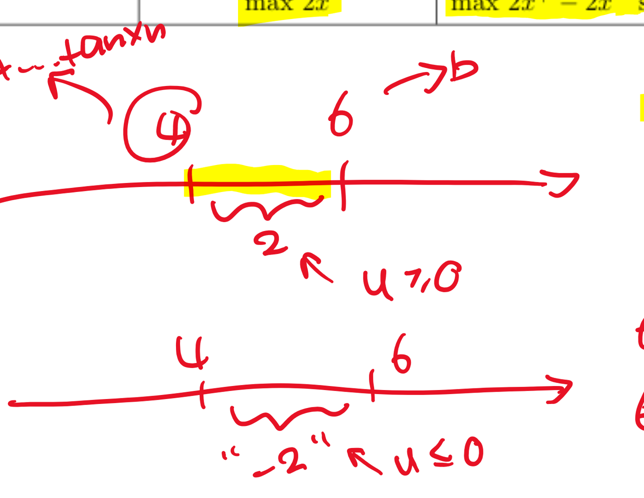
$x = -3, \; x^+ = 0, \; x^- = 3$

$x = 4, \; x^+ = 4, \; x^- = 0$

$x = -7, \; x^+ = 0, \; x^- = 7$

$x = 0, \; x^+ = 0, \; x^- = 0$

$x = 0, \; x^+ = 3, \; x^- = 3$



$4 \le 6$

$4 + 2 = 6$

$6 \ge 4 \Rightarrow -6 \le -4$

$6 - 2 = 4$

$-6 + 2 = -4$

$u = 2 \ge 0$

Matrix Notation

A standard linear programming problem:
Find values of the **decision variables** $x_1, x_2, ..., x_n$ that will
Max the **objective function** $z = c_1x_1 + ... + c_nx_n$
subject to the **Constraints**
$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \le b_1$
$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \le b_2$
$\vdots$
$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \le b_m$
$x_1, x_2, ..., x_n \ge 0$

can be written in matrix notation as:
Find a vector $\mathbf{x} \in \mathbb{R}^n$ that will
Max $\mathbf{z} = \mathbf{c}^T\mathbf{x}$
subject to the **Constraints**
$\mathbf{Ax} \le \mathbf{b}$
$\mathbf{x} \ge \mathbf{0}$

component wise "≤"

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

max x
s.t. $x \ge 2$

$x = x^+ - x^-$
max $x^+ - x^-$
s.t. $x^+ - x^- \ge 2$
$x^+, x^- \ge 0$

explain why

~~max~~ max x
s.t $x \ge 2$
$\boxed{x \ge 0}$

A canonical linear programming problem can be written in matrix notation similarly by replacing the $\le$ to $=$, and considering the slack variables, if any, when defining **A**, **x**, **b** and **c**.

Thinking question: What will happen to **A**, **x**, **b** and **c** when we change from a standard linear programming problem to a canonical linear programming problem?

minimize the cost

Question

model A 30 /min 8 \$15000 Required at least 320 /min

model B 50 /min 8 \$20000 No more than 12

Equipment purchasing problem (textbook 1.1.2), set up a linear programming model of the situation described, then convert it to the standard form and to the canonical form.

- **Understanding:** What are the decision variables? (What are the unknowns?)

  X = number of model A
  Y = number of model B

- **Understanding:** What is the objective function? (What are you trying to maximize or minimize?)

  min 15000X + 20000Y

- **Understanding:** What are the constraints? (What are the conditions?)

  $30X + 50Y \ge 320$ $\quad X \ge 0$
  $X + 2Y \le 12$ $\quad Y \ge 0$

- **Devising a plan:** What can be useful to solve the problem? i.e. your answer is correct because ...

  Definition of LPP, standard form and canonical form

- **Carrying out the plan:** Can you set up a linear programming model of the situation as described?

  Find X, Y
  min $15000X + 20000Y$
  s.t. $30X + 50Y \ge 320$
  $X + 2Y \le 12$
  $X \ge 0$
  $Y \ge 0$

- **Carrying out the plan:** Can you change the model to a standard linear programming problem? and write out it in matrix notation after.

  Find X, Y
  $-\max -15000X - 20000Y$
  s.t. $-30X - 50Y \le -320$
  $X + 2Y \le 12$
  $\boxed{X \ge 0 \\ Y \ge 0}$

  $\vec{x} = \begin{pmatrix} X \\ Y \end{pmatrix}$ $\quad c = \begin{pmatrix} -15000 \\ -20000 \end{pmatrix}$

  $A = \begin{pmatrix} -30 & -50 \\ 1 & 2 \end{pmatrix}$ $\quad b = \begin{pmatrix} -320 \\ 12 \end{pmatrix}$

  Find $\vec{x}$
  $-\max c^T\vec{x}$
  s.t. $A\vec{x} \le b$
  $\vec{x} \ge 0$

- **Carrying out the plan:** Can you change the model to a Canonical linear programming problem? and write out it in matrix notation after.

  Find X, Y, U, S
  $-\max -15000X - 20000Y + 0U + 0S$
  s.t. $-30X - 50Y + U = -320$
  $X + 2Y + S = 12$
  $X, Y, U, S \ge 0$

  $\vec{x} = \begin{pmatrix} X \\ Y \\ U \\ S \end{pmatrix}$ $\quad C = \begin{pmatrix} -15000 \\ -20000 \\ 0 \\ 0 \end{pmatrix}$

  $A = \begin{pmatrix} -30 & -50 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$ $\quad b = \begin{pmatrix} -320 \\ 12 \end{pmatrix}$

  Find $\vec{x}$
  $-\max c^T\vec{x}$
  s.t. $A\vec{x} = b$
  $\vec{x} \ge 0$

- **Looking Back:** Can you check your answer?