Introduction

Recall, we require the condition that $b \geq 0$ (in other words, $b_i \geq 0$ for all i) when running the simplex algorithm, because we have a way to choose the pivot (positive a_ij with smallest ratio) following from the intepretation when $b \geq 0$. However, if one of the $b_i < 0$, then we are not sure how to choose the pivot without violating $Ax = b$. Besides, if $b_i < 0$, then we do not have a starting BFS simply by making all slack variables to be the basic variables.
In this tutorial, we would introduce a method called the two-phased method which allows us to solve a LPP with some $b_i < 0$ by introducing an artificial variable x_0. The goal of the two-phased method is to find a starting BFS for the orignal LPP when using the simplex algorithm.
Notation: The two-phased method that we would be using is different from the one introduced in the textbook or other resources, however the ideas are equivalently the same. Besides, the notation for the simplex method follows from the one given in lectures. That is, we have c^T as the objective row, and the bottom right corner represents the negative of current cost.

Two-phased Method

Given an arbitrary LPP in its standard form, where $b \in \mathbb{R}^m$.
Max the **objective function** $z = c_1x_1 + ... + c_nx_n$
subject to the **Constraints**

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \leq b_1\\
&a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \leq b_2\\
&\qquad\qquad\vdots\\
&a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \leq b_m\\
&x_1, x_2, ..., x_n \geq 0
\end{aligned}$$

Consider the following auxiliary problem by introducing an artificial variable x_0.
min the **objective function** $w = x_0$
subject to the **Constraints**

$$\begin{aligned}
&-x_0 + a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \leq b_1\\
&-x_0 + a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \leq b_2\\
&\qquad\qquad\vdots\\
&-x_0 + a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \leq b_m\\
&x_0, x_1, x_2, ..., x_n \geq 0
\end{aligned}$$

The connection between the auxiliary problem and the original problem is as follow. You will be given the following thinking questions in the worksheet.

- One could observe that the auxiliary problem must have an optimal solution given the constraints. Why? (Hint: $x_0 \geq 0$ and this is a minimization problem.)
- If the auxiliary problem has an optimal solution $x_0 \neq 0$, then the original LPP has an empty feasible region. Why? (Hint: Think about the original LPP in canonical form, and the auxiliary problem be the one followed from the canonical LPP, if there is an x satisfying $Ax = b, x \geq 0$ in the original LPP, then the optimal solution for the auxiliary problem would have $x_0 = 0$ where you can find the optimal solution given x. How?)
- If the auxiliary problem has an optimal solution $x_0 = 0$, then there is a way to find a starting BFS which we will discuss next.

By considering the original LPP in its canonical form instead, and introduce the artificial variable to have the auxiliary problem, we want to solve the auxiliary problem using the simplex algorithm. Consider the following example:
max $z = x_1 + 2x_2 - x_3$
subject to

$2x_1 - 6x_2 + 3x_3 \leq 12$
$-2x_2 + x_3 \leq -2$
$x_1, x_2, x_3 \geq 0$

1. Convert the original LPP into canonical form and introduce the artificial variable to have the auxiliary problem.

2. Starting tableaux. We want to find a starting BFS for the auxiliary problem.

$$\left(\begin{array}{cc|c} -1 & A & b \\ \hline 0 & c^T & 0 \\ -1 & 0 & 0 \end{array}\right)$$

- What are the A, c, b of the given example?
- What is the current solution, and why this is not a BFS?
- What is the starting tableaux of the given example?

3. Consider the auxiliary problem by looking at the second objective row. Introduce x_0 as a basic variable and choose the a_ij (this can be any real, not necessary positive in this case) with the smallest negative b_i. If there is no negative b_i, you can solve the original LPP directly using the simplex algorithm without introducing the auxiliary problem. Consider the one that was choosen as pivot, and apply row operations (where multiplying -1 is allowed), and you will

get a starting BFS of the auxiliary problem. At this stage, $b \geq 0$ (which of the above step guarantee this?).

Thinking question:

- Recall the simplex algorithm, you will get a negative basic variable if you multiply -1 to the positive a_ij with smallest ratio when introducing a new basic variable, resulting a non-feasible solution. However, we are allowed (only once at this stage) to multiply -1 and still end up getting the feasible solution (in fact a BFS), why? (Hint: Think about the process of entering and leaving variables)

To this particular example, what is the starting BFS to the auxiliary problem after applying the above steps?

4. Now you have a starting BFS for the auxiliary problem with $b \geq 0$, solve it using the simplex algorithm

5. If the optimal solution occurs when $x_0 \neq 0$, then the original LPP has an empty feasible region. If the optimal solution occurs when $x_0 = 0$ and x_0 is not a basic variable, then you reach a BFS to the original LPP. We will talk about the case when $x_0 = 0$ and x_0 is a basic variable in the worksheet.

6. Now you have a starting BFS to the original LPP, you can solve the LPP using the simplex algorithm now.