MAT337_TUT4

Tuesday, October 20, 2020 14:09

General strategy:

Draw a graph to get intuition, next convert your idea using mathematical languages.

MAT337_TUT4

Basic Definitions:

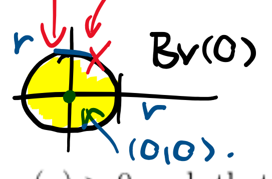
- 4.3.1 A point x is a **limit point** of a subset A of $\mathbb{R}^n$ if there is a sequence $(a_n)_{n=1}^{\infty}$ with $a_n \in A$ such that $x = \lim_{n\to\infty} a_n$. A set $A \subset \mathbb{R}^n$ is **closed** if it contains all of its limit points.
- If A is a subset of $\mathbb{R}^n$, the closure of A is the set $\bar{A}$ consisting of all limit points of A.
- The **ball about a in $\mathbb{R}^n$ of radius r** is the set

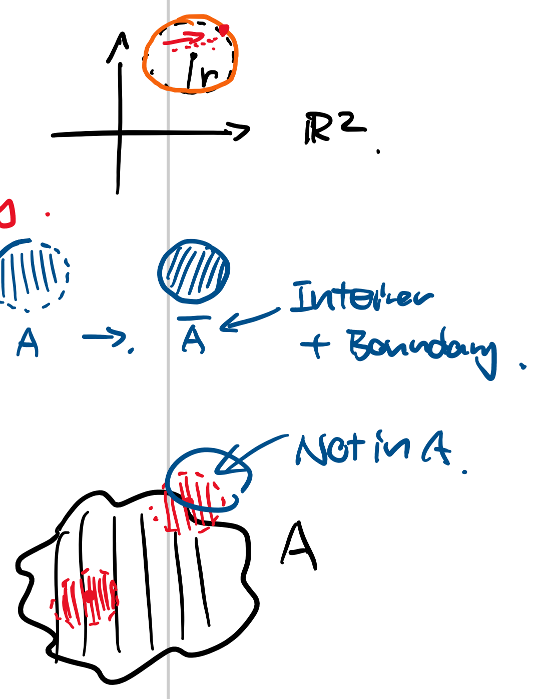
$$B_r(a) = \{x \in \mathbb{R}^n : ||x - a|| < r\}$$

distance between x and a.

A subset U of R^n is open if for every $a \in U$, there is some $r = r(a) > 0$ such that the ball $B_r(a)$ is contained in U.

on the boundary $||x-a|| = r$ Not including the boundary.





Questions:

1. Find the closure of the following sets:
   - $\mathbb{Q}$
   - $\{(x, y) \in \mathbb{R}^2 : xy < 1\}$
   - $\{(x, sin(\frac{1}{x})) : x > 0\}$
   - $\{(x, y) \in \mathbb{Q}^2 : x^2 + y^2 < 1\}$
2. Let $(a_n)_{n=1}^{\infty}$ be a sequence in ~~$\mathbb{R}$~~ R^k with $\lim_{n\to\infty} a_n = a$. Show that $\{a_n : n \geq 1\} \cup \{a\}$ is closed
3. Show that $U = \{(x, y) \in \mathbb{R}^2 : x^2 + 4y^2 < 4\}$ is open by explicitly finding a ball around each point that is contained in U.

#1(a) $\bar{\mathbb{Q}} = \mathbb{R}$

$Br(x)$ contains some rational and irrational, s.t. $x \in \partial\mathbb{Q}$

$x-r$ x $x+r$ $\mathbb{R}$

$\mathbb{R}$. rational approximation. x → limit point of $\mathbb{Q}$.

#1(b) $\{(x,y) \in \mathbb{R}^2 : xy < 1\} := A$

① If x or $y = 0$, then $xy = 0 < 1$

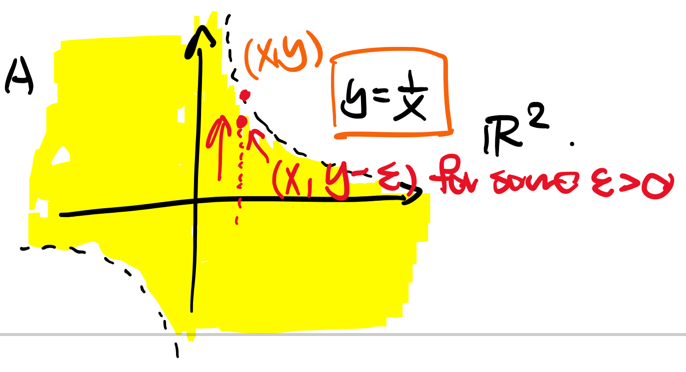
② If $x<0, y>0$, $xy<0<1$

If $x>0, y<0$, $xy<0<1$

③ If $x>0, y>0$. $y < \frac{1}{x}$ for $x>0$

If $x<0, y<0$, $y > \frac{1}{x}$ for $x<0$

$\Rightarrow A = \{(x,y) \in \mathbb{R}^2 : xy \leq 1\}$

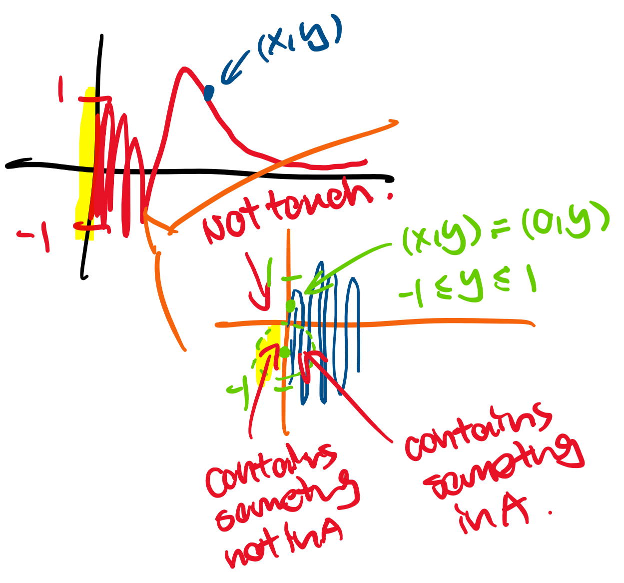


How to approach a given (x,y) in $\bar{A}$?

consider: $a_n = (x, y - \frac{1}{n}) \to (x,y)$ as $n \to \infty$.

s.t. $(x,y) \in \bar{A}$

#1(c) $\{(x, \sin(\frac{1}{x})) : x>0\} = A$ ← Graph of $y = \sin(\frac{1}{x})$ for $x>0$.

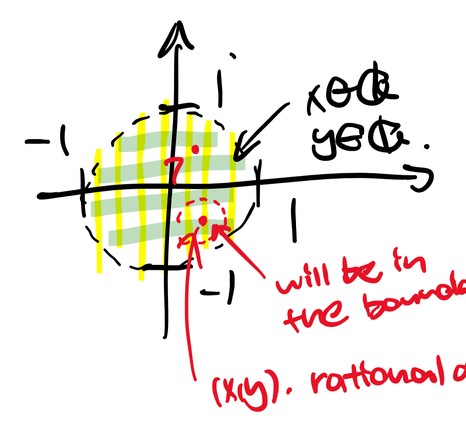


s.t. $\bar{A} = A \cup \{(0,y) : -1 \leq y \leq 1\}$

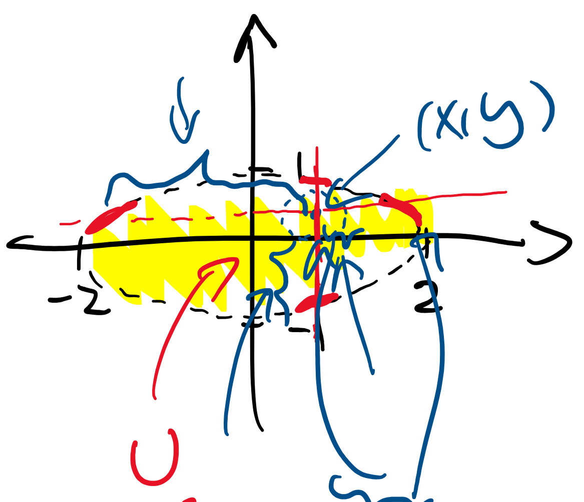
$(x_n, \sin(\frac{1}{x_n})) \to (0,y)$, where $-1 \leq y \leq 1$ by definition.

#1(d) $\{(x,y) \in \mathbb{Q}^2 : x^2+y^2<1\} := A$

$\bar{A} = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$.



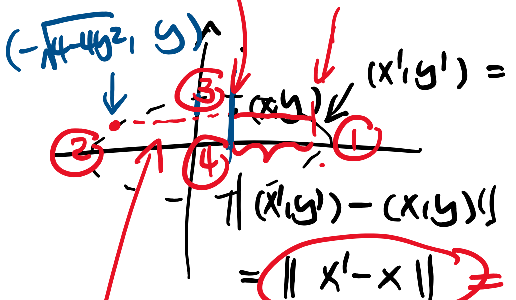
#3 $U = \{(x,y) \in \mathbb{R}^2 : x^2+4y^2<4\}$.



$4y^2 = 4-x^2$

$y^2 = 1 - \frac{x^2}{4}$ on the boundary,

$y = \pm\sqrt{1-\frac{x^2}{4}}$ $x^2+4y^2 = 4$



$(x',y') = (x', y)$ unknown. known.

$||(x',y') - (x,y)||$

$= ||x'-x|| = ||\sqrt{4-4y^2} - x||$

$\Rightarrow x^2 = 4-4y^2$.

$x = \pm\sqrt{4-4y^2} \to x' = \sqrt{4-4y^2}$

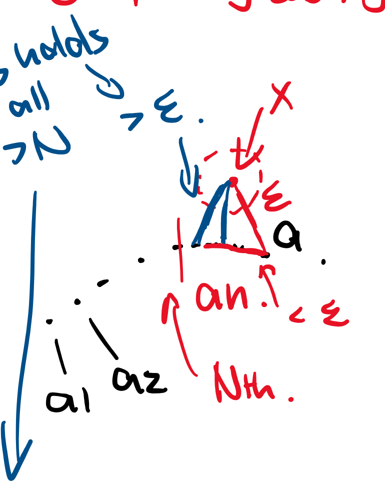
$||x - (-\sqrt{4-4y^2})||$

$$r = \frac{1}{2}\min\left(\underset{①}{|\sqrt{4-4y^2}-x|}, \underset{②}{|x-(-\sqrt{4-4y^2})|}, \underset{③}{|\sqrt{1-\frac{x^2}{4}}-y|}, \underset{④}{|y-(-\sqrt{1-\frac{x^2}{4}})|}\right)$$

s.t. $Br(x,y) \subseteq U$ for any such given $(x,y) \in U$ ∎

#2: Hint:

It makes sense,



To show U is closed equivalently show U^c is open

So we only have to deal with $a_1, a_2, \cdots, a_{n-1}$.

Consider $||x - a_i||$ for $i = 1,2,\cdots,n-1$

Take $r = \frac{1}{2}\min\{||x-a_1||, \cdots, ||x-a_{n-1}||, ||x-a||\}$ ∎