Definition

You should always check the lecture note and the textbook for the definition, and make sure the definitions are consistents.

1. Let $v_1, \cdots, v_n$ be vectors in V. A **linear combination** of $v_1, \cdots, v_n$ is a vector of the form $a_1v_1 + \cdots + a_nv_n$ for some $a_1, \cdots, a_n \in \mathbb{F}$.
2. A set of vectors $\{v_1, \cdots, v_n\}$ in V is called a **spanning set** of V if $\forall v \in V, \exists a_1, \cdots, a_n \in \mathbb{F}$ such that $v = a_1v_1 + \cdots + a_nv_n$
3. Vectors $v_1, \cdots, v_n$ are said to be **linearly independent** if it is true that if $a_1v_1+\cdots+a_nv_n = 0$ for some $a_1, \cdots, a_n \in \mathbb{F}$ then $a_1 = \cdots = a_n = 0$. Otherwise the vectors are said to be **linearly dependent**.
4. A set of vectors $\{v_1, \cdots, v_n\}$ in V is called a **basis** of V if $\forall v \in V, \exists$ unique $a_1, \cdots, a_n \in \mathbb{F}$ such that $v = a_1v_1 + \cdots + a_nv_n$. We call $a_1, \cdots, a_n$ the coordinates of v (with respect to the basis).

Theorem

Let V be a vector space. A set of vectors $\{v_1, \cdots, v_n\}$ in V is a basic if and only if it is a linearly independent set and a spanning set of V.

Discussion

0.1 Understanding the definition

1. What is the difference between any of the following statements and the definition of linearly independent?

   (a) $v_1, \cdots, v_n$ are vectors in V, such that $a_1v_1 + \cdots + a_nv_n = 0$ and $a_1 = \cdots = a_n = 0$.

   (b) $v_1, \cdots, v_n$ are vectors in V, such that if $a_1 = \cdots = a_n = 0$ then $a_1v_1 + \cdots + a_nv_n = 0$.

2. Give a precise definition of linearly dependent from the definition of linealy independent. In other words, explain what it means by “otherwise” the vectors are linearly dependent.

3. There is an equivalent definition of linearly depedent, which said:
   A set of vectors $\{v_1, \cdots, v_n\}$ in V is linearly dependent, if at least one of those vectors can be written as a linear combination of the other vectors.
   Prove that this definition is equivalent to the one that you provided in the previous question.

0.2 Examples

1. Let $\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ 4 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \}$ be a set of vectors in $\mathbb{R}^4$. Find a largest possible subset of the given set, such that the subset is linearly independent. Check your answer.

   Remark. *If you are using a computational approach, think about why it would work.*

2. Let $1, x, x^2 \in \mathbb{P}_n(\mathbb{R})$, show that they are linearly independent.

3. Let $\left\{ \begin{pmatrix}1\\2\\3\\0\end{pmatrix}, \begin{pmatrix}9\\4\\3\\0\end{pmatrix}, \begin{pmatrix}4\\1\\0\\0\end{pmatrix}, \begin{pmatrix}8\\2\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\1\end{pmatrix} \right\}$ be a set of vectors in $\mathbb{R}^4$. In the previous question, you find a linearly independent subset. Check whether the subset is a spanning set of $\mathbb{R}^4$ by the definition of spanning set. If it is not, extend it to a linearly independent spanning set of $\mathbb{R}^4$.

4. Show that $\{1, 1+x, 1+x+x^2, \cdots, 1+x+x^2+\cdots+x^n\}$ is a basis of $\mathbb{P}_n(\mathbb{R})$. You may use the fact that $\{1, x, \cdots, x^n\}$ is a basis of $\mathbb{P}_n(\mathbb{R})$.

0.3 More questions

1. Let S_1, S_2 be linearly indepdent subsets of a vector space V, show that $S_1 \cap S_2$ is also linearly indepdent.

2. Find an example of linearly independent sets S_1 and S_2, such that $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2$ is linearly dependent

Remark. *By doing the questions, you should see the difference between a set union and a set intersection, and how definitions can be apply to those set operations. As further exercise, check the following statements.*

Let S_1, S_2 be spanning sets of a vector space V, check whether

(a) $S_1 \cap S_2$ is a spanning set

(b) $S_1 \cup S_2$ is a spanning set

0.4 Extend and review

At the end of the tutorial, you should learn how the definitions of spanning set, linearly independent and basis can be applied to a more general vector space, rather than $\mathbb{R}^n$ which is what we are familiar to. The key here is to follow whatever the definitions say. Besides, with the definitions, you should now understand why the "computational algorithms" work. Lastly, with the idea of coordinate and its uniqueness in the definition of basis, you might have some sense of how a general vector space can be related to $\mathbb{R}^n$, and everything that we have learned before apply. (We will talk more about this later)