Complex Vector Space

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$\mathbb{C} = \{a+ib : a,b \in \mathbb{R}\}$

Addition: $(a_1+ib_1) + (a_2+ib_2) = (a_1+a_2) + i(b_1+b_2)$

Multiplication: usual multiplication, with $i^2 = -1$

① $\mathbb{C}^n$ as a complex vector space: ← $\mathbb{F}$

Addition: usual one

Scalar multiplication: zv where $z \in \mathbb{F} = \mathbb{C}$, $v \in \mathbb{C}^n$, $zv = z(v_1 \dots v_n) := (zv_1 \dots zv_n)$ ← depends on $\mathbb{F}$

Basis: $\{e_1, \dots, e_n\}$, where $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ ← ith term. $\Rightarrow \dim_{\mathbb{C}} \mathbb{C}^n = n$

so that $\mathbb{C}^n = \{ \sum_{i=1}^{n} a_i e_i \mid a_i \in \mathbb{C} \}$ = span of basis above

② $\mathbb{C}^n$ as a real vector space: ← $\mathbb{F}$

Addition: usual one

Scalar multiplication: zv, where $z \in \mathbb{F} = \mathbb{R}$, $v \in \mathbb{C}^n$, $zv = z(v_1 \dots v_n) := (zv_1 \dots zv_n)$

Basis: $\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{e_1} \dots \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}}_{e_n}, \underbrace{\begin{pmatrix} i \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{v_1}, \dots \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ i \end{pmatrix}}_{v_n} \}$ (call this) $\Rightarrow \dim_{\mathbb{R}} \mathbb{C}^n = 2n$

so that $\mathbb{C}^n = \{ \sum_{i=1}^{n} a_i e_i + b_i v_i \mid a_i, b_i \in \mathbb{R} \}$ = span of basis above

TRUE/FALSE:

(c) $\mathbb{R}^2$ is a subset of $\mathbb{C}^2$

Pf: let $(x,y) \in \mathbb{R}^2$, where $x,y \in \mathbb{R}$
and known $\mathbb{R} \subseteq \mathbb{C}$, s.t. $x,y \in \mathbb{C}$
then $(x,y) \in \mathbb{C}^2$ as required.

(d) $\mathbb{R}^2$ is a subspace of $\mathbb{C}^2$ as a complex vector space ← $\mathbb{F}$

subspace ← subset + itself being a v.s.

{False} $(1,0) \in \mathbb{R}^2$, $i \in \mathbb{F} = \mathbb{C}$
then $i(1,0) = (i,0) \notin \mathbb{R}^2$
This contradicts $\mathbb{R}^2$ being closed under scalar multiplication!!

(e) Check: $\mathbb{R}^2$ is a subspace of $\mathbb{C}^2$ as a real vector space

① $(0,0) \in \mathbb{R}^2$, s.t. $\mathbb{R}^2 \neq \emptyset$

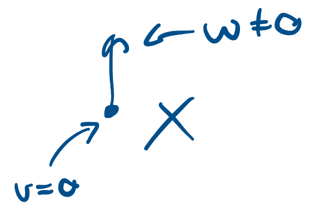
② $(a,b) + (c,d) = (a+c, b+d) \in \mathbb{R}^2$ for any $(a,b), (c,d) \in \mathbb{R}^2$ } closed under addition

③ $k(a,b) = (ka, kb) \in \mathbb{R}^2$ for any $(a,b) \in \mathbb{R}^2$, $k \in \mathbb{F} = \mathbb{R}$ } closed under scalar multiplication

Euclidean Inner Product on $\mathbb{C}^n$

For any $x = (x_1 \dots x_n)$, $y = (y_1 \dots y_n) \in \mathbb{C}^n$,

$\langle x,y \rangle := \sum_{i=1}^{n} x_i \bar{y_i}$ ← This is important!!



Avoid $\|v\| = 0$

Orthogonality: depends on the choice of inner product

Perpendicular: Two vectors v, w are perpendicular, if they are orthogonal and the angle between makes sense.

$\langle v, w \rangle = 0$

$\arccos\left(\frac{\langle v,w \rangle}{\|v\| \|w\|} \right)$

$\|v\| = \sqrt{\langle v,v \rangle}$

Example: whether or not $z_1 = \begin{pmatrix} 1+i \\ 1-i \end{pmatrix}$ and $z_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ perpendicular, parallel or neither?