Objectives

Happy Halloween! The purpose of this tutorial is to understand the definitions of limit and continuity in higher dimension, with intuition, and be able to visualize limit approaching in higher dimension. Given a limit, how can you determine if it exists or not intuitively. How can you practically show that a limit does not exist, or if exist, find its limit?

Definition of Limit in Higher Dimension

Even though proving is not the main scope of mat235, you should still be familiar with the intuitive idea of the concepts. As I mentioned before, you can not setup model properly without understanding the intuition behind. If you are not comfortable with the precise definition, a picture may help. I will draw a picture for intuition in class.

1. **Definition of limit exist:** Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). Then we say that the limit of f(x,y) as (x,y) approaches (a,b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

   if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \epsilon$

2. **How to show the limit does not exist:** Follow from the definition in 1, limit does not exist if $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, i.e. limits are different along different paths in the space of the domain

3. **What are the paths you can choose:** There are infinitely many paths you can use to approach along. For example, When approaching $(0, 0)$, we can use $x = 0$, $y = 0$, $y = x^2$, $y = x^3$, etc. In fact, any paths containing $(0, 0)$.

Questions

1. Show that $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ does not exist

2. Show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist

3. Show that $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$ does not exist

4. Show that $\lim_{(x,y)\to(0,0)} \frac{5y^4cos^2x}{x^4+y^4}$ does not exist

Thinking questions: How can you determine if a limit exist or not intuitively?

Existence of limit Intuitively

Given a function $f : \mathbb{R}^2 \to \mathbb{R}$, how can we determine if a limit exist or not intuitively, before practically start working on it. Consider the examples from previous section:

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}, \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}, \lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}, \lim_{(x,y)\to(0,0)} \frac{5y^4cos^2x}{x^4+y^4}$$

Those limits do not exist. Look close inside the functions. Let's think about the functions as rational functions, i.e. $f(x,y) = \frac{g(x,y)}{h(x,y)}$. What is something special that occur in the functions of the examples? The 'degree' of h is greater or equal to the one of g! Aha!

In fact, what really matters to the existence of the limit is whether g or h takes priority.

For example, $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ does not exist, because as $(x,y) \to (0,0)$, the $x^2+y^4 \to 0$ faster than $xy^2 \to 0$, such that it seems like informally $\frac{xy^2}{x^2+y^4} \to \frac{k}{0}$ where $k \neq 0$, therefore limit does not exist.

<u>**Exercise:**</u> Check if you can see a similar intuition for those limits that do not exist?

What if limit exists?

Question: Show that $\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^4+y^4} = 0$

Useful Technique and Formula

1. **Squeeze Theorem:**
   If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ then $\lim_{x\to a} g(x) = L$
2. $(x-y)^2 = x^2 - 2xy + y^2 \geq 0$, hence $x^2 + y^2 \geq 2xy$

Additional Problems and Reminder:

1. Please work on the questions in the textbook chapter14.2 5-22.
2. MAT235 Quiz3 : October.13 Wednesday Tutorial; Notice as I mentioned before, you will only be quizzed on materials that we talked about in tutorial. The last two quizzes were not hard. In fact, the averages of our section are higher than many of the others.