Introduction

Have you ever look at a question and have no idea how to start a proof? In this section, we will fucus on how to start a proof and how to write a proof with logical reasoning. A key question that you should always ask yourself before writing anything down in the proof is whether it makes sense. The readers are assumed to be familiar with the basic math logic. The purpose of this section is to improve the proof writing skill.

Contents

1 How to start a proof

1.1 Start from the assumption

An argument is valid if when all the premises are true then the conclusion is true. When writing a proof, we are usually proving that an argument is valid. In mathematics, we sometimes call the premises of an argument the assumption when writing a proof. Here is an example in linear algebra.

$$\text{If } a_1v_1 + ... + a_nv_n = 0, \text{then } a_1 = ... = a_n = 0$$

The premise of this argument is $a_1v_1 + ... + a_nv_n = 0$ and the conclusion is $a_1 = ... = a_n = 0$. When writing a proof, here is how we should start the proof:

Proof. Assume $a_1v_1 + ... + a_nv_n = 0$, ... □

1.1.1 The premises may not be that explicit

Here is an example which is not explicitly an if-then statement.

A normal operator with real eigenvalues is self-adjoint.

To prove this argument, we want to show that any normal operator with real eigenvalues is self-adjoint. Therefore we should start with an arbitrary normal operator that satisfies the required condition. When writing a proof, here is how we should start the proof:

Proof. Let N be a normal operator with real eigenvalues, ... □

1.1.2 Sometimes we might not have to write down the assumption

While there are always some preconditions for a proof to make sense, we dont have to mention all of those. Here is an example of set equality:

$$\mathbb{P}_2(\mathbb{R}) = \text{span}\{1, x, x^2\}$$

1.2 Remember the final goal of your proof

Once we started the proof, we should remind ourselves the final goal of the proof before going further, that is the conclusion of the argument. We usually write 'want to show' or simply 'WTS' after the assumption line to indicate the goal of the proof. For the above two examples, here are how we should continue the proofs.

Proof. Assume $a_1v_1 + ... + a_nv_n = 0$.
WTS: $a_1 = ... = a_n = 0$
... □

Proof. Let N be a normal operator with real eigenvalues.
WTS: N is self-adjoint.
... □

Up to this point, we should be able to start a proof for any mathematical statement. There are a few common formats of the statements that you should be familiar by the end of the course.

1.2.1 showing an if-then statement

- Example: If $A = A^*$, then $A + iI$ is invertible.

  Proof. Assume $A = A^*$.

  WTS: $A + iI$ is invertible

  ... □

- Example: A normal operator with unimodular eigenvalues is unitary.

Proof. Let N be a normal operator with unimodular eigenvalues.
WTS: N is unitary. $\square$

1.2.2 showing an iff statment

Proving an iff (if and only if) statement is equivalent to proving two if-then statements. For example,

A square matrix is invertible if and only if its determinant is nonzero.

is equivalent to

- If a square matrix is invertible, then its determinant is nonzero, and
- if the determinant of a square matrix is nonzero, then it is invertible.

and here is how we should start the proof:

Proof. Let A be a square matrix.
($\Rightarrow$) Assume A is invertible. WTS: The determinant of A is nonzero. ...
($\Leftarrow$) Assume the determinant of A is nonzero. WTS: A is invertible. ... $\square$

We sometimes use the arrows to indicate which if-then we are proving.

1.2.3 showing a set equality

In linear algebra, we often look at sets. To prove that two sets are equal, we need to know what it means to say that two sets are equal, that is the definition of set equality. With the definition, here is an example and how we should start the proof.

$$\mathbb{P}_2(\mathbb{R}) = \operatorname{span}\{1, x, x^2\}$$

Proof. .
($\subseteq$) Let $f \in \mathbb{P}_2(\mathbb{R})$. WTS: $f \in \operatorname{span}\{1, x, x^2\}$. ...
($\supseteq$) Let $f \in \operatorname{span}\{1, x, x^2\}$. WTS: $f \in \mathbb{P}_2(\mathbb{R})$. ... $\square$

Sometimes one direction of the proof is relatively easier, but you should always prove it because that is what the definition requires.

1.2.4 proof by definition

We often define mathematical terms which might be included as part of the statement. Therefore we have to check the definition to make sure that we understand what we want to prove. Here is an example:

A normal operator with real eigenvalues is self-adjoint.

Proof. Let N be a normal operator with real eigenvalues.
WTS: N is self-adjoint, that is we want to show that $N^* = N$. ... $\square$

This is just the beginning of the proof and if you are not sure about what N^* means, you should again check the definition.

1.3 Before we move on

Now you should be able to start the proof by indicating the assumption and WTS, and understand what the WTS wants. One can notice that we do not care about what further information is given in the assumption and how to prove the WTS yet at this stage.

1.4 Exercises

For each of the following statements, start the proof by writing down the assumption and WTS. Some of the statements may not be valid, but that is not what we are interested in this section.

1. Let V be a vector space and $T : V \to V$ be a linear transformation. Show that if $T^n = 0$ for some $n > 0$, then 0 is an eigenvalue, and it is the only eigenvalue.
2. Let M be an $m \times n$ real matrix such that the only solution to $Mx = 0$ is $x = 0 \in \mathbb{R}^n$. Prove that for any $y \in \mathbb{R}$, there exists $z \in \mathbb{R}^m$, such that $M^T z = y$
3. Let $T : V \to V$ be a linear transformation. Let E be a subspece of V such that $T(E) \subset E$. Prove that if T has no eigenvalues, then $\dim(E)$ is even.
4. Let E be a subspace of an inner product space V. Show that $(E^\perp)^\perp = E$

2 How to continue a proof

Recall that we indicate the assumption and WTS, and check the definition for what the WTS means at the beginning of the proof. To continue the proof, we will usually build connection between the assumption and WTS by checking the definitions together with our past experiences working on other problems. Here is an example of what kinds of information we can get from the assumption and WTS.

Prove that a normal operator with real eigenvalues is self-adjoint.

1. Assumption: A normal operator N with real eigenvalues.
   (a) What does it mean to say that N is normal? $N^*N = NN^*$
   (b) What does it mean to say that N has real eigenvalue? $Nv = \lambda v$?
   (c) How can we relate the normal definition to eigenvalue?
   (d) What is special about a normal operator in this context based on our past experience?
   (e) Diagonalization of a normal operator? What does it give us? $N = UDU^*$? What are U and D? How is that related to the assumption of real eigenvalues?
2. WTS: N is self-adjoint, that is $N^* = N$.

(a) Given the information from assumption, what might be useful to show our WTS?

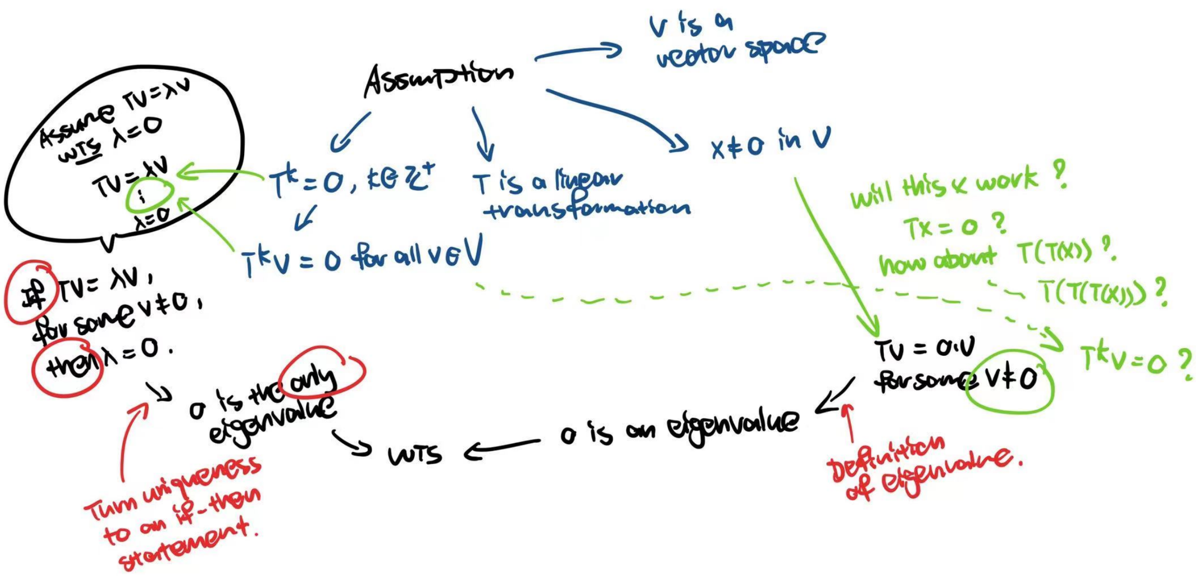
(b) If $Nv = \lambda v$, am I allowed to take the adjoint of both side? Does it give me any useful information? If yes, how. If not, go back and try a different approach.

(c) If $N = UDU^*$, then what is N^*? Real eigenvalues?

You can also draw a diagram to help you with building the connections until when you see a clear path. However, remember that we want to prove the argument by making the assumption to prove the conclusion in one direction with logical reasoning. That is, given the connection that you builded, check each steps before and after writing it down to make sure that your proof is logically correct.

2.1 Example

Example: Let V be a vector space and assume that there exists $x \neq 0$ in V. Let $T : V \to V$ be a linear transformation. Prove that if $T^k = 0$ for some positive integer k, then 0 is an eigenvalue of T and it is the only eigenvalue of T.
Intuition:



Proof:

Proof. Assume $T^k = 0$ for some positive integer k. Assume $x \neq 0$ in the vector space V.
WTS: 0 is an eigenvalue of T and 0 is the only eigenvalue of T.

1. WTS: 0 is an eigenvalue of T, that is $Tv = 0 \cdot v$ for some non-zero $v \in V$

   Known that $T^k = 0$, such that $T^k(x) = 0$, and there exists a smallest positive integer $s \leq k$ such that $T^s(x) = 0$. Since $x \neq 0$, therefore $1 \leq s \leq k$ and $0 \leq s - 1 < k$.

Since s is the smallest such positive integer, therefore $T^{s-1}(x) \neq 0$. Define $v := T^{s-1}(x) \in V$, then $T(v) = T^s(x) = 0 = 0 \cdot v$, where $v \neq 0$.

Therefore 0 is an eigenvalue of T.

2. WTS: 0 is the only eigenvalue of T, that is if $Tv = \lambda v$ for some $v \neq 0$, then $\lambda = 0$.

   Assume $Tv = \lambda v$ for some $v \neq 0$.

   WTS: $\lambda = 0$

   Notice $Tv = \lambda v$ and T is a linear transformation such that $T^2(v) = T(T(v)) = T(\lambda v) = \lambda T(v) = \lambda^2 v$. Repeat this k times, and we get that $T^k(v) = \lambda^k v$.

   Since $T^k = 0$, such that $0 = T^k(v) = \lambda^k v$ where $v \neq 0$. Therefore $\lambda^k = 0$ and hence $\lambda = 0$ as required.

$\square$

3 Check and learn from your proof

After finishing the proof, you should go back to the beginning, read through the proof and make sure that it is mathemtically and logically correct. At least when time is limited, you should check whether your final answer make sense.
Lastly, we always want to learn from our proof. Recall that one way to build the connection between assumption and WTS is by thinking about past experiences.