Matrix_rep_and_coordinate_vector

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ordered basis.

$T: V \to W$ be a L.T, where V, W finite dimensional,

$\alpha = (v_1, \dots, v_n)$ be a basis of V

$\beta = (w_1, \dots, w_m)$ be a basis of W

Coordinate vector (wrt basis)

$$[v]_\alpha = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \text{ if } v = a_1v_1 + \dots + a_nv_n \in V$$

if $\alpha' = (v_2, v_1, v_3, v_4, \dots, v_n)$

then with $v = a_1v_1 + \dots + a_nv_n$, $[v]_{\alpha'} = \begin{pmatrix} a_2 \\ a_1 \\ a_3 \\ \vdots \\ a_n \end{pmatrix}$

similarly for W.

Matrix representation of $T: V \to W$ wrt α, β.

$[T]_\alpha^\beta$ ← wrt W (for β), ← wrt V (for α)

$\alpha = \{v_1, \dots, v_n\}$

$\beta = \{w_1, \dots, w_m\}$.

$$[T]_\alpha^\beta = \underbrace{\begin{bmatrix} | & & | \\ [T(v_1)]_\beta & \cdots & [T(v_n)]_\beta \\ | & & | \end{bmatrix}}_{n \text{ columns}} \Bigg\} m \text{ rows}$$

with $[T(v_1)]_\beta = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

s.t $T(v_1) = b_1w_1 + \dots + b_mw_m$

n columns.

Question:

Given the above formula, $T: V \to W$ (α, β)

show
$$\underbrace{[T(v)]_\beta}_{\substack{\text{the coordinate} \\ \text{vector of } T(v) \text{ in } W \\ \text{wrt basis } \beta}} = \underbrace{\underbrace{[T]_\alpha^\beta}_{\substack{\text{matrix} \\ \text{rep of } T \\ \text{from } V \text{ to} \\ W \text{ wrt } \alpha \\ \text{and } \beta}} \underbrace{[v]_\alpha}_{\substack{\text{coordinate} \\ \text{vector of } v \text{ in } V \\ \text{wrt basis } \alpha}}}_{\text{matrix multiplication.}} \quad \text{for all } v \in V$$

$\alpha = \{v_1, \dots, v_n\}$ and $\beta = \{w_1, \dots, w_m\}$.

Proof: Let $v \in V$, s.t $v = a_1v_1 + \dots + a_nv_n$ by α being a basis.

s.t $[v]_\alpha = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, $a_1, \dots, a_n \in \mathbb{F}$ ⊛.

and known $T(v) = T(a_1v_1 + \dots + a_nv_n)$

$= a_1T(v_1) + \dots + a_nT(v_n)$ by linearity of T

Notice $T(v_i) \in W$ fixed,

and $T(v_i) = b_{1i}w_1 + \dots + b_{mi}w_m$ for some $b_{1i}, \dots, b_{mi} \in \mathbb{F}$

s.t $[T(v_i)]_\beta = \begin{pmatrix} b_{1i} \\ \vdots \\ b_{mi} \end{pmatrix}$ ⊛.

s.t $T(v) = a_1T(v_1) + \dots + a_nT(v_n)$

$= a_1(b_{11}w_1 + \dots + b_{m1}w_m) + \dots + a_n(b_{1n}w_1 + \dots + b_{mn}w_m)$

$= (a_1b_{11} + \dots + a_nb_{1n})w_1 + \dots + (a_1b_{m1} + a_nb_{mn})w_m$

$$[T(v)]_\beta = \begin{pmatrix} a_1b_{11} + \dots + a_nb_{1n} \\ \vdots \\ a_1b_{m1} + a_nb_{mn} \end{pmatrix} = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

$$= \begin{pmatrix} | & & | \\ [T(v_1)]_\beta & \cdots & [T(v_n)]_\beta \\ | & & | \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \text{ by ⊛}$$

$$= [T]_\alpha^\beta [v]_\alpha \text{ by ⊛} \quad \blacksquare$$

Thinking: Assume V, W over $\mathbb{R}$

dim n, $\alpha = \{v_1, \dots, v_n\}$ dim m, $\beta = \{w_1, \dots, w_m\}$

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, leftrightarrow, "L_\alpha"'] & W \arrow[d, leftrightarrow, "L_\beta"] \\
\mathbb{R}^n \arrow[r, "{[T]_\alpha^\beta}"'] & \mathbb{R}^m
\end{tikzcd}$$

Instead of talking about T

coordinat vector for each $v \in V$ wrt α

coordinat vector for each $w \in W$ wrt β

$[T]_\alpha^\beta$ matrix representation.

Example: $T: P_n(\mathbb{R}) \to P_n(\mathbb{R})$ defined as $T(f) = f'$

Let $\alpha = \{1, x, \dots, x^n\}$ be a basis of $P_n(\mathbb{R})$

let $f \in P_n(\mathbb{R})$, $f(x) = a_0 + a_1x + \dots + a_nx^n$,

s.t $[f]_\alpha = \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix}$

and $T(f) = f'$, $T(f)(x) = a_1 + \dots + na_nx^{n-1} + (0 \cdot x^n)$

s.t $[T(f)]_\alpha = \begin{pmatrix} a_1 \\ \vdots \\ na_n \\ 0 \end{pmatrix}$

and
$$[T]_\alpha^\alpha = \begin{pmatrix} | & & | \\ [T(1)]_\alpha & \cdots & [T(x^n)]_\alpha \\ | & & | \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & & n \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$T(1) = 0$ $T(x) = 1$ $T(x^2) = 2x$ $T(x^n) = nx^{n-1}$

More example, $\beta = \{1, 1+x, 1+x+x^2, \dots, 1+x+x^2+\dots+x^n\}$

$$[T]_\alpha^\beta = \begin{pmatrix} 0 & 1 & -2 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & & -n \\ \vdots & \vdots & \vdots & & n \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$T(1) = 0$ $T(x) = 1$

$T(x^2) = 2x = 2(1+x) - 2$

$= -2(1) + 2(1+x)$

$T(x^n) = nx^{n-1}$

$= n(1+x+\dots+x^{n-1}) - n(1+x+\dots+x^{n-2})$

$= -n(1+x+\dots+x^{n-2}) + n(1+x+\dots+x^{n-1})$

and

$$[T]_\beta^\alpha = \begin{pmatrix} | & | & & | \\ [T(1)]_\alpha & [T(1+x)]_\alpha & \cdots & [T(1+x+\dots+x^n)]_\alpha \\ | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} | & | & & | \\ [0]_\alpha & [1]_\alpha & \cdots & [1+2x+\dots+nx^{n-1}]_\alpha \\ | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 2 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & n \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

similarly, we can have $[T]_\beta^\beta$. $\blacksquare$