Definitions

1. If f is a **scalar function** that is defined on a smooth curve C which is parametrized by $\alpha(t)$ with $a \leq t \leq b$, then the **line integral of f along C** is given by

$$\int_C f ds = \int_a^b (f \circ \alpha) \cdot |\alpha'(t)| dt$$

   We sometimes call this the **line integral of f along C with respect to arc length**

2. The following are called the **line integrals of f along C with respect to x and y** respectively

   - $\displaystyle\int_C f(x,y)dx = \int_a^b f(x(t),y(t))x'(t)dt$
   - $\displaystyle\int_C f(x,y)dy = \int_a^b f(x(t),y(t))y'(t)dt$

3. If F is a continuous **vector field** that is defined on a smooth curve C which is parametrized by $\alpha(t)$ with $a \leq t \leq b$, then the **line integral of F along C** is given by

$$\int_C F \cdot dr = \int_a^b (F \circ \alpha) \cdot \alpha'(t) dt$$

4. $\displaystyle\int_C F \cdot dr = \int_C P dx + Q dy$ if $F(x,y) = (P,Q)$

Questions

0.1 Line integral of scalar function f along smooth C

Compute the following line integrals

1. $\int_C 2x\,ds$, where C is the curve $y = 9 - x^2$ from $x = -1$ to $x = 2$

2. $\int_C y^2 - 10xy\,ds$, where C is the left half of the circle with radius 6

0.2 Line integral of scalar function f along piecewise-smooth curve C

Compute the following line integrals

1. $\int_C 2x ds$, where C is the line segment from $(1, 0)$ to $(0, 1)$, then followed by the circle of radius 1 from $(0, 1)$ to $(1, 0)$ counterclockwise.

2. $\int_C 2x ds$, where C is triangle with vertices $(0, 0), (1, 0), (0, 1)$, starting at $(0, 0)$ in the counter-clockwise direction.

0.3 Line integral of f with respect to x, y and z

Compute the following line integrals

1. $\int_C xdy - xydx$ where C is the circle of radius 1 from $(0, 1)$ to $(0, -1)$ in the clockwise direction.

2. $\int_C z^2dx + x^2dy + y^2dz$, where C is the line segment from $(1, 0, 0)$ to $(4, 1, 2)$

0.4 Line integral of vector field F along smooth C

Compute the following line integrals

1. $\int_C F \cdot dr$ where C is the line segment from $(1, 3)$ to $(4, 5)$, and $F(x, y) = (y^2, x - 2y)$

2. $\int_C F \cdot dr$ where C is the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ from the positive y-axis to the negative x-axis in the clockwise direction.

0.5 Some difficulty when computing a line integral by definition

Try to compute the following line integrals and what difficulty do you get? (You will learn a better method to solve it next week, called the fundamental theorem for line integrals)

1. $\int_C F \cdot dr$ where C is the curve given by $\alpha(t) = (e^t sint, e^t cost)$ with $0 \leq t \leq \pi$, and $F(x, y) = (3 + 2xy, x^2 - 3y^2)$ is a vector field.

Instead, use the following theorem to compute the line integral by first finding a suitable scalar function f. Recall the idea of convervative vector field we talked about.

Theorem 1. *Let C be a smooth curve parametrized by $\alpha(t)$, with $a \leq t \leq b$. Let f be a differentiable function of two variables whose gradient vector ∇f is continuous on C. then*

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$