MATB44_TUT_2

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Exact Ordinary Differential Equations:

functions

IDEA: Consider the equation $M(x,y)dx + N(x,y)dy = 0$ $(M(x,y) + N(x,y)y' = 0)$
If the LHS can be written as $LHS = dF = \partial_x F dx + \partial_y F dy$, then we called the equation exact and have $F = C$.

$\partial_x F + \partial_y F \frac{dy}{dx}$

1. Verify exact equation: Assume $M = \partial_x F$ and $N = \partial_y F$, then $\boxed{\partial_y M = \partial_x N}$

   $\partial_{yx} F = \partial_{xy} F$

2. Find the function F:

   after, first

   - Find the x component of F first: Since $\partial_x F = M$, then $F(x,y) = \boxed{\int M dx + C(y)}$
   - Find the remaining y component after: $\partial_y(\int M dx + C(y)) = \partial_y F(x,y) = N$

3. Integrating Factors: The special case when the integrating factor depends on only one variable, i.e. $\mu(x)$ or $\mu(y)$

   $\mu(x,y)$

   - In general, $\mu(x,y)M(x,y) + \mu(x,y)N(x,y)y' = 0$ is exact, if and only if $\partial_y(\mu(x,y)M(x,y)) = \partial_x(\mu(x,y)N(x,y))$, and in other words, $M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$
   - $\mu(x)$ when $\frac{d\mu}{dx} = \frac{(M_y - N_x)}{N}\mu$ and $\frac{(M_y - N_x)}{N}$ is a function of x only
   - $\mu(y)$ when $\frac{d\mu}{dy} = \frac{(N_x - M_y)}{M}\mu$ and $\frac{(M_x - N_y)}{M}$ is a function of y only

   Once you found the integrating factor, multiply it on both sides of the original equation and you should have an exact equation now. :)

$\mu_y = \boxed{\frac{N_x - M_y}{M}}\mu$. y only

Question:

Find an integrating factor depending on y only and solve the equation.

$$y + (2xy - e^{-2y})y' = 0 \quad (1)$$

$y\,dx + (2xy - e^{-2y})dy = 0$

M, N

$M(x,y) = y$, $N(x,y) = 2xy - e^{-2y}$

$\partial_y M = 1$

$\partial_x N = 2y$, $\partial_y M \neq \partial_x N$, the original is not exact.

$\mu(y)$ be the integrating factor

M, N

$\boxed{\mu(y)y} + \boxed{\mu(y)(2xy - e^{-2y})}y' = 0$

$\partial_y(\mu(y)y) = \partial_x(\mu(y)2xy - \mu(y)e^{-2y})$

$\mu'(y)y + \mu(y) = 2y\mu(y)$

$\mu'(y)y = (2y-1)\mu(y)$.

$\int \frac{d\mu}{\mu} = \int \frac{2y-1}{y} dy$ $\quad \frac{2y-1}{y} = 2 - \frac{1}{y}$

$\log \mu = 2y - \log y + C$ $\quad C = 0$.

$\log \mu = 2y - \log y$.

$\mu = e^{2y - \log y} = e^{2y} \cdot e^{-\log y} = \frac{e^{2y}}{y}$

$\mu(y) = \frac{e^{2y}}{y}$ is an integrating factor

$\frac{e^{2y}}{y} \cdot y + \frac{e^{2y}}{y}(2xy - e^{-2y})y' = 0$

$e^{2y} + \left(\frac{e^{2y}}{y} \cdot 2xy - \frac{1}{y}\right)y' = 0$

$M = e^{2y}$, $N = e^{2y} \cdot 2x - \frac{1}{y}$

$\partial_x F = M \Rightarrow \boxed{F = \int M dx = \int e^{2y} dx = xe^{2y} + c(y)}$

$\partial_y F = 2xe^{2y} + c'(y) = N = 2xe^{2y} - \frac{1}{y}$

$c'(y) = -\frac{1}{y} \Rightarrow \boxed{c(y) = -\log(y) + C}$

$F = xe^{2y} - \log(y) + C$

$dF = 0 \Leftrightarrow F = C$

$\Rightarrow$. $xe^{2y} - \log(y) = C$ implicit formula.

$y = f(x)$ $\leftarrow$ explicit ■