Orthogonal projection

Definition

For a vector v of an inner product space V with subspace E, the orthogonal projection of v onto the subspace E, denoted by $P_E v$ is a vector w, such that

1. $w \in E$
2. $v - w \perp E$

How to practically find the orthogonal projection?

Let V be an inner product space and let E be a subspace of V. Assume that they are finite-dimentional.

1. Known a basis of E, say $w_1, ..., w_n$
2. Construction an orthogonal basis of E by the Gram-Schmidt orthogonality algorithm, say $v_1, ..., v_n$, as follows
   (a) Define $v_1 = w_1$
   (b) For $i = 2, ..., n$, define $v_i = w_i - \sum_{j=1}^{i-1} \frac{(w_i, v_j)}{||v_j||^2} v_j$
3. Then the orthogonal projection $P_E v$ of a vector v is given by the formula

$$P_E v = \sum_{k=1}^{n} a_k v_k \text{ where } a_k = \frac{(v, v_k)}{||v_k||^2}$$

Example.1

Find the orthogonal projection of a vector $(1, 1, 1, 1)^T$ onto the subspace spanned by the vectors $v_1 = (1, 3, 1, 1)^T$, $v_2 = (2, -1, 1, 0)^T$ in $\mathbb{R}^4$ with the standard inner product.

Example.2

Find the orthogonal projection of a vector $(1,1,1,1)^T$ onto the subspace spanned by the vectors $w_1 = (3,2,2,1)^T$, $w_2 = (2,-1,1,0)^T$ in $\mathbb{R}^4$ with the standard inner product. Check that you get the same answer as in example.1

Example.3

Let an inner product on the space of polynomials be defined by $(f, g) = \int_{-1}^{1} f(t)\overline{g(t)}dt$. Apply the Gram-Schmidt orthogonalization to the system $1, t, t^2, t^3$. Let E be the subspace spanned by $1, t, t^2, t^3$. Find the orthogonal projection of $1 + t$ onto E.

Example.4

Let P be the orthogonal projection onto a subspace E of an inner product space V, say $\dim V = n$ and $\dim E = r$. Find the eigenvalues and eigenvectors of P. Find the algebraic and geometric multiplicities of each eigenvalue. Hint: Show that $P_E(P_E v) = P_E v$ and use the uniqueness of orthogonal projection proven by theorem 3.2 in the textbook.