Definition

Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. An inner product on V is a function $<,>: V \times V \to \mathbb{F}$, such that it satisfies the following four properties:

1. Conjugate symmetry: $< x, y >= \overline{< y, x >}$ for all $x, y \in V$
2. Linearity: $< ax + by, z >= a < x, z > +b < y, z >$ for all $x, y, z \in V$, $a, b \in \mathbb{F}$
3. Non-negativity: $< x, x > \geq 0$ for all $x \in V$
4. Non-degeneracy: $< x, x >= 0$ if and only if $x = 0$

A vector space V over $\mathbb{F}$ together with an inner product is called an inner product space. Given an inner product space, We define the norm as $||x|| = \sqrt{< x, x >}$.

Questions

1. Prove that for vectors in a inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$
$$||x \pm y||^2 = ||x||^2 + ||y||^2 \pm 2\text{Re} < x, y >$$
where Re $z = \frac{1}{2}(z + \bar{z})$
2. Show that $< A, B >= \text{trace}(A+B)$ is not an inner product on the space of real 2×2 matrices
3. Show that $< f, g >= \int_0^1 f'(t)\overline{g(t)}dt$ is not an inner product on the space of polynomials.
4. Show that $< x, y >= \displaystyle\sum_{i=1}^{n} x_i y_i$ is not an inner product on $\mathbb{C}^n$
5. Show that $< x, y >= \displaystyle\sum_{i=1}^{n} x_i \overline{y_i}$ define an inner product on $\mathbb{C}^n$
6. Let $v_1, ..., v_n$ be a spanning set of an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Prove that
   (a) If $< x, v >= 0$ for all $v \in V$, then $x = 0$
   (b) If $< x, v_k >= 0$ for all k, then $x = 0$
   (c) If $< x, v_k >=< y, v_k >$ for all k, then $x = y$