MATB44_TUT_3

Tuesday, September 29, 2020 11:02 AM

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First order ODE

$cf(t)$

$f(t)+C$.

Abel's Theorem and Wronskian

- **Wronskian**: If y_1 and y_2 are solutions to a given second order equation $L[y] = y'' + p(t)y' + q(t)y$, then the Wronskian of the solutions y_1 and y_2 is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t) \neq 0$$

$\begin{pmatrix} y_1 \\ y_1' \end{pmatrix}$ and $\begin{pmatrix} y_2 \\ y_2' \end{pmatrix}$ are linearly independent

function of t but the Wronskian is either always zero or never zero

then y_1 and y_2 form a fundamental set of solution if and only if their Wronskian is nonzero.

$\hookrightarrow y = C_1y_1 + C_2y_2$, where C_1, C_2 are constant

- **Abel's Theorem**: If y_1 and y_2 are solutions of the second-order linear differential equation

$$\boxed{L[y] = y'' + p(t)y' + q(t)y = 0}$$ standard form

where p and q are continuous on an open interval I, then the Wronskian $W[y_1, y_2](t)$ is given by

$$W[y_1, y_2](t) = c \times exp(-\int p(t)dt) = Ce^{-\int p(t)dt}$$

where c is a certain constant that depends on y_1 and y_2, but not on t. Further, $W[y_1, y_2](t)$ either is zero for all t in I (if $c = 0$) or else is never zero in I (if $c \neq 0$)

Question:

Equation $2t^2y'' + 3ty' - y = 0$ has a solution $y_1 = \frac{1}{t}$. Find a fundamental pair of solutions.

$\to y_1' = -\frac{1}{t^2}$

$$y'' + \underbrace{\boxed{\tfrac{3}{2t}}}_{p(t)}y' - \frac{1}{2t^2}y = 0 \quad \text{standard form}$$

pick $k=0$

Apply Abel's Theorem, $W[y_1,y_2](t) = Ce^{-\int \frac{3}{2t}dt} = Ce^{-\frac{3}{2}\log t + k} = \frac{C}{(t)^{\frac{3}{2}}}$

pick $C=1$, $\boxed{W[y_1,y_2](t) = \frac{1}{t^{\frac{3}{2}}}}$

$$W[y_1,y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1' = \frac{y_2'}{t} + \frac{y_2}{t^2}$$

s.t. $\frac{y_2'}{t} + \frac{y_2}{t^2} = \frac{1}{t^{3/2}}$

$$\boxed{y_2' + \frac{y_2}{t} = \frac{1}{t^{1/2}}}.$$ First order, linear eqn of y_2.

$\mu(t)$ be the integrating factor,

s.t. $(\mu y)' = \mu LHS = \mu y' + \mu\frac{1}{t}y = \mu RHS$.

$(\mu y)' = \mu'y + \mu y' = \mu y' + \mu\frac{1}{t}y$

$\mu'y = \mu\frac{1}{t}y$

$\frac{d\mu}{dt} = \frac{\mu}{t}$

$\int\frac{d\mu}{\mu} = \int\frac{1}{t}dt$ separable eqn.

$\log\mu = \log t + C$

$\mu = e^{\log t + C} = e^{\log t}\cdot\boxed{e^C}$

$= Ct$, $\boxed{C = e^C}$

$\boxed{\frac{C}{t}} \neq C$

(*)

pick $C=1$, $\mu = t$

$$\boxed{ty' + y = t\cdot\frac{1}{t^{\frac{1}{2}}} = t^{\frac{1}{2}}}$$

$(ty)'$

$\boxed{(\mu y)'} = t^{\frac{1}{2}}$

$(ty)' = t^{\frac{1}{2}}$

$ty = \int t^{\frac{1}{2}}dt = \frac{2}{3}t^{\frac{3}{2}} + k$

$$\boxed{y = \frac{2}{3}t^{\frac{1}{2}} + \boxed{\frac{k}{t}}}$$ is the general solution for y_2, given that $C=1$ in the wronskian

Pick $k=0$, $y_2 = \frac{2}{3}t^{\frac{1}{2}}$

s.t. $y_1 = \frac{1}{t}$ and $y_2 = \frac{2}{3}t^{\frac{1}{2}}$ will form a fundamental set of solution to given 2nd order ODE.