Definitions

You should always check the lecture note and the textbook for the definition, and make sure the definitions are consistents.

1. Let V be a vector space. A non-empty subset $W \subseteq V$ equipped with the same operations from V is called a subspace of V if
   (a) $\forall w \in W, \forall k \in \mathbb{F}, k \cdot w \in W$
   (b) $\forall w_1, w_2 \in W, w_1 + w_2 \in W$
2. Recall that $M^{\mathbb{R}}_{m\times n}$ which is the set of all $m \times n$ matrix with real entries together with the matrix addition and the usual scalar multiplication is a vector space.
3. $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
4. $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Discussions

0.1 The vector space of matrices

We will discuss the topic of subspace with the vector space of matrices $M^{\mathbb{R}}_{m\times n}$ as an example. For each of the following questions, check whether W is a subspace of V

1. Let $V = M^{\mathbb{R}}_{2\times 2}$ and $W = \{A \in V \mid A = A^T\}$ with the usual operations.

2. Let $V = M^{\mathbb{R}}_{2\times 2}$ and $W = \{A \in V \mid A \text{ is invertible}\}$ with the usual operations. Recall our definition of invertible matrix is the existence of inverse.

3. Let $V = M_{2\times 2}^{\mathbb{R}}$ and $W = \{A \in V \mid A_{12} = A_{21} = 0\}$ with the usual operations.

4. Let $V = M_{2\times 2}^{\mathbb{R}}$ and $W = \{A \in V \mid A_{ij} \geq 0\}$

0.2 Union and intersection of sets

Recall the following definitions, let A and B be two subspaces of a vector space V

1. $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
2. $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Prove or disprove the following statements.

1. $A \cap B$ is also a subspace of V

2. $A \cup B$ is also a subspace of V

3. Let $v \in V$ but $v \notin A$. Prove that if $x \in A$, then $x + v \notin A$

4. $A \cup B$ is a subspace if and only if $A \subseteq B$ or $B \subseteq A$