1 Adjoint of a linear transformation

- <u>Definition:</u> Let V, W be inner product space. Let $A : V \to W$ be an operator, then the adjoint of A is the operator $A^* : W \to V$ such that

$$(Ax, y) = (x, A^*y) \qquad \forall x \in V, y \in W$$

- <u>Definition:</u> For a $m \times n$ matrix A, its adjoint $A^* := \overline{A^T}$. Given the definition, it satisfies the property

$$(Ax, y) = (x, A^*y) \qquad \forall x \in \mathbb{C}^n, y \in \mathbb{C}^m$$

Question

Let A be an $m \times n$ matrix. Show that $\mathrm{Ker}A = \mathrm{Ker}(A^*A)$

2 Isometries and unitary operators

- Definition: An operator $U : X \to Y$ is called an isometry, if it preserves the norm. That is

$$\|Ux\| = \|x\| \qquad \forall x \in X$$

Some useful facts about isometry:

1. An operator $U : X \to Y$ is an isometry if and only if

$$(x, y) = (Ux, Uy) \qquad \forall x, y \in X$$

2. An operator $U : X \to Y$ is an isometry if and only if $U^*U = I$

- Definition: An isometry $U : X \to Y$ is a unitary operator if it is invertible.
Some useful facts about unitary operator:

1. If $U : X \to Y$ is a unitary operator, then $U^{-1} = U^*$
2. An isometry $U : X \to Y$ is a unitary operator if and only if $\dim X = \dim Y$

Questions

1. Show that a product of unitary matrices is unitary.

2. Let $U : X \to X$ be a linear transformation on a finite-dimensional inner product space. Prove that if $\|Ux\| = \|x\|$ for all $x \in X$, then U is unitary.

3. Let A, B be $n \times n$ matrices. Show that trace(AB) =trace(BA). Conclude that similar matrices have the same trace.