This list of questions is for the students in MATB24, TUT0005 summer 2021 at the University of Toronto Scarborough. You should not use any of the facts in this document as a reference in the final exam. Everything covered in this document has been talked about in the lecture, tutorial or the textbook. The test may cover materials that are not included in this document. You should refer to the professor's email for the coverage of the test. Good luck in the final!

Questions

1. Let $V = M_{n\times n}(\mathbb{R})$. Define $\langle A, B\rangle := \text{trace}(B^*A)$ for all $A, B \in V$. Show that it defines an inner product on V.

2. Suppose that $\langle,\rangle_1, \langle,\rangle_2$ are two inner products on V. Show that $\langle A, B\rangle := \langle A, B\rangle_1 + \langle A, B\rangle_2$ is also an inner product on V.

3. Let V be a real vector space. Let W be an inner product space over $\mathbb{R}$ with the inner product $\langle,\rangle_w$. Assume that $T : V \to W$ is a linear transformation. Show that $\langle x, y\rangle := \langle T(x), T(y)\rangle_w$ defines an inner product on V if and only if $\ker(T) = \{0\}$.

4. Let $V = \mathbb{R}^3$ with dot product. Let $v_1 = (1, 2, 0)^T, v_2 = (1, 0, 0)^T, v_3 = (1, 1, 1)^T$. Show that v_1, v_2, v_3 are linearly independent by the definition, then use the Gram-Schmidt process on thoses to obtain an orthonormal set.

5. Let V be a finite dimensional inner product space, and W be a subspace of V. Prove that if $x \notin W$, then there exusts $y \in V$ such that $y \in W^\perp$ and $\langle x, y\rangle \neq 0$.

6. Let $T : V \to V$ be a linear transformation. Prove that if $\langle T(x), y\rangle = 0$ for all $x, y \in V$, then T is the zero function.

7. Let $T : V \to V$ be a linear transformation. Prove that if $T^*T = 0$, then $T = 0$.

8. Let $T : V \to V$ be a linear transformation on the inner product space V. Prove that if T is self-adjoint, then $\langle T(x), x\rangle \in \mathbb{R}$ for all $x \in V$.

9. Let $T : V \to V$ be positive-definite. Show that $\langle T(x), x\rangle > 0$ for all $x \neq 0$.

10. Define $T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{R}$ by $T(f(x)) = f(2)$. Let $\alpha = \{1, x, x^2\}, \beta = \{1\}, \gamma = \{1, x, 1 + x^2\}, \delta = \{2\}$. Find $[T]^\alpha_\beta$ and $[T]^\gamma_\delta$, and find a matrix Q such that $[T]^\alpha_\beta = Q[T]^\gamma_\delta Q^{-1}$.

11. Let V be an n-dimensional real vector space. Let α be an ordered basis of V. Show that $T : V \to \mathbb{R}^n$ by $T(v) = [v]_\alpha$ which is the coordinate vector of v with respect to α is a linear transformation.

12. Let $A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \in M_2(\mathbb{C})$. Find the Ran$A$ and Ker$A$.

13. Prove that if an operator T on V is self-adjoint and nilpotent, then $T = 0$.