MAT224 TUT0101/0102

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Coordinate vector and matrix representation

Recall the definitions and a nice diagram

- Let $\alpha = (v_1, \cdots, v_n)$ be an ordered basis of a vector space V over $\mathbb{R}$.

  The coordinate vector of $v \in V$ is
  $$[v]_\alpha = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$
  if $v = a_1 v_1 + \cdots + a_n v_n$ for $a_1, \cdots, a_n \in \mathbb{F}$

- Let V, W be vector spaces over $\mathbb{F}$. Assume $\alpha = (v_1, \cdots, v_n)$ and $\beta = (w_1, \cdots, w_m)$ are bases of V and W respectively. Let $T : V \to W$ be a linear transformation, then the matrix representation of T with respect to α and β is
  $$[T]_\alpha^\beta = \begin{bmatrix} [T(v_1)]_\beta & \cdots & [T(v_n)]_\beta \end{bmatrix}$$

- Let V, W be vector spaces over $\mathbb{F}$ with bases $\alpha = (v_1, \cdots, v_n)$ and $\beta = (w_1, \cdots, w_m)$ respectively. Let $T : V \to W$ be a linear transformation. Finish the diagram with the corresponding linear transformations. Assume that you are given $v \in V$.

$$\begin{array}{ccc} V & \qquad\qquad & W \\ & & \\ \mathbb{R}^n & & \mathbb{R}^m \end{array}$$

0.1 Discussions

1. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be a linear transformation. Let $\alpha = \{1 + x + x^2, x + x^2, x^2\}$. Suppose that

$$[T]_\alpha^\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 0 & 3 & 2 \end{pmatrix}$$

   Find $T(x)$

2. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be a linear transformation. Let $\alpha = \{1 + x + x^2, x + x^2, x^2\}$ and $\beta = \{1, x, x^2\}$. Suppose that

$$[T]_\alpha^\alpha = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

   (a) Find $[T]_\alpha^\beta$

   (b) If T is invertible, find $[T^{-1}]_\alpha^\alpha$. Use it to calculate $T^{-1}(ax^2 + bx + c)$

   (c) If T is invertible, find $[T^{-1}]_\alpha^\beta$ and $[T^{-1}]_\beta^\alpha$

3. Let V be a 3-dimensional vector space with a basis $\alpha = \{v_1, v_2, v_3\}$. Let $T : V \to V$ be a linear transformation such that $T(v_1) = v_1, T(v_2) = 2v_2$ and $T(v_3) = -v_3$.

   (a) Find $[T]_\alpha^\alpha$

   (b) Let $\beta = \{v_1 + v_2, v_2 + v_3, v_3 + v_1\}$. Find $[T]_\beta^\beta$

4. Let I_n be the $n \times n$ identity matrix. Let V be an n-dimensional vector space and λ be a nonzero real number. Define $T : V \to V$ by $T(v) = \lambda v$. Let α and β be ordered bases of V. Show that $[T]_\alpha^\beta = \lambda I_n$ if and only if $\alpha = \beta$.