Definitions

You should always check the lecture note and the textbook for the definition, and make sure the definitions are consistents.

1. We call $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ the coordinate vector of $v \in V$ with respect to the basis $(v_1, \cdots, v_n)$ of V if $v = a_1v_1 + \cdots + a_nv_n$ for some $a_1, \cdots, a_n \in \mathbb{F}$. Denoted as $[v]_{v_1,\cdots,v_n}$

   Remark. *Notice from the definition that*

   - *the coordinate vector depends on the choice of the basis*
   - *the order of the basis matters*
   - *the coordinate vector with respect to a fixed basis is unique. (why?)*

2. Let V, W be vector spaces over $\mathbb{F}$. A function $T : V \to W$ is a linear transformation if

   - $\forall v, u \in V, T(v + u) = T(v) + T(u)$
   - $\forall v \in V, \forall a \in \mathbb{F}, T(a \cdot v) = a \cdot T(v)$

3. Let V, W be vector spaces over $\mathbb{F}$. Let $T : V \to W$ be a linear transformation. Let $\alpha = (v_1, \cdots, v_n)$ and $\beta = (w_1, \cdots, w_m)$ be ordered basis of V and W respectively. Then the matrix representation of T with respect to α and β is

$$[T]^{v_1,\cdots,v_n}_{w_1,\cdots,w_m} = \begin{bmatrix} \\ [T(v_1)]_{w_1,\cdots,w_m} & \cdots & [T(v_n)]_{w_1,\cdots,w_m} \\ \\ \end{bmatrix}$$

   such that $\forall v \in V$

$$[T(v)]_{w_1,\cdots,w_m} = [T]^{v_1,\cdots,v_n}_{w_1,\cdots,w_m}[v]_{v_1,\cdots,v_n}$$

   Remark. *Why is the above equation true?*
   Assume $v = a_1v_1 + \cdots + a_nv_n$, then $T(v) = a_1T(v_1) + \cdots + a_nT(v_n)$ by linearity of T. Assume $T(v_i) = b_{1i}w_1 + \cdots + b_{mi}w_m$ for each $i = 1, \cdots, n$. Therefore,

$$\begin{aligned} T(v) =& a_1(b_{11}w_1 + \cdots + b_{m1}w_m) + \cdots + a_n(b_{1n}w_1 + \cdots + b_{mn}w_m) \\ =& (a_1b_{11} + \cdots + a_nb_{1n})w_1 + \cdots + (a_1b_{m1} + \cdots + a_nb_{mn})w_m \end{aligned}$$

$$[T]^{v_1,\cdots,v_n}_{w_1,\cdots,w_m} = \begin{bmatrix} \\ [T(v_1)]_{w_1,\cdots,w_m} & \cdots & [T(v_n)]_{w_1,\cdots,w_m} \\ \\ \end{bmatrix} = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \vdots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}$$

$$[v]_{v_1,\cdots,v_n} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, [T(v)]_{w_1,\cdots,w_m} = \begin{bmatrix} a_1b_{11} + \cdots + a_nb_{1n} \\ \vdots \\ a_1b_{m1} + \cdots + a_nb_{mn} \end{bmatrix}$$

   Now, you can check that the equation is true.

4. A linear transformation $T : V \to W$ is said to be invertible if there exists a linear transformation $S : W \to V$ such that $ST = I_V$ and $TS = I_W$. The transformation S is called an inverse of T. We have proven in class that such S if exists, is unique, so we can denote it as T^{-1}.
5. An invertible linear transformation $T : V \to W$ is called an isomorphism form V to W.
6. A matrix A is called invertible if there exists a matrix B such that $AB = BA = I$, and we call B the inverse of A.

Questions

1. Solve the following questions:
   (a) Let $T : V \to W$ be a linear transformation. Show that $T(0) = 0$.
   (b) Let $T : V \to W$ be a linear transformation. Assume $T(v_1), \cdots, T(v_n)$ are linearly independent for some $v_1, \cdots, v_n \in V$. Show that $v_1, \cdots, v_n$ are linearly independent.
   (c) Find a linear transformation $T : V \to W$. Such that $T(v_1), T(v_2), T(v_3)$ are linearly dependent for some linearly independent v_1, v_2, v_3.

2. Solve the following questions:

   (a) Let A be an $n \times n$ matrix with real entries. Show that if $\forall x \in \mathbb{R}^n, Ax = x$, then $A = I_n$.

   (b) Let $T : V \to W$ be an invertible linear transformation, where V, W are real vector spaces. Let $\alpha = (v_1, \cdots, v_n)$ and $\beta = (w_1, \cdots, w_n)$ be ordered bases of V and W respectively. Let A be the matrix representation of T with respect to α and β. Let B be the matrix representation of T^{-1} with respect to α and β. Show that A is inveritble and its inverse is B.

3. Give examples of 2×2 matrices such that

   (a) $A + B$ is not inveritble, but A and B are inveritble

   (b) $A + B$ is inveritble, but A and B are not inveritble

   (c) $A, B, A + B$ are all invertible.

4. Solve the following questions:

   (a) Let A be an $n \times m$ matrix. Assume A is invertible by our definition, prove that $n = m$.

   (b) Suppose $AB = 0$ where 0 is the matrix with all zero entries. Show that if B is not a matrix with all zero entries, then A is not invertible.

   (c) Let A and AB be invertible. Prove that B is invertible. Assume AB is well-defined. Before you start proving check to make sure any matrix multiplication that you are going to write make sense.