Example_vector_space_field_and_basis_annotated

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Example_vector_space...

Definitions

You should always check the lecture note and the textbook for the definition, and make sure the definitions are consistents.

1. A **vector space** is a set V together with two operations

   (a) $+ : V \times V \to V$
   (b) $\cdot : \mathbb{F} \times V \to V$

   such that

   (1) $\forall v, w \in V, v + w = w + v$
   (2) $\forall u, v, w \in V, (u + v) + w = u + (v + w)$
   (3) $\exists 0 \in V, \forall v \in V, 0 + v = v$
   (4) $\forall v \in V, \exists w \in V$ such that $v + w = 0$. Denote w by $-v$
   (5) $\forall a \in \mathbb{F}, \forall u, v \in V, a \cdot (u + v) = a \cdot u + a \cdot v$
   (6) $\forall a, b \in \mathbb{F}, \forall u \in V, (a + b) \cdot v = a \cdot v + b \cdot v$
   (7) $\forall a, b \in \mathbb{F}, \forall v \in V, a \cdot (b \cdot v) = (ab) \cdot v$
   (8) $\forall v \in V, 1 \cdot v = v$

   If $\mathbb{F} = \mathbb{R}$, we call V together with the operations the **real vector space**.
   If $\mathbb{F} = \mathbb{C}$, we call V together with the operations the **complex vector space**.

2. A set α is called a basis of the vector space V if α is linearly independent and $\text{span}(\alpha) = V$.
3. Let $\alpha = \{v_1, \cdots, v_n\}$ be a set of vectors in V, $\text{span}(\alpha) = \{a_1 v_1 + \cdots + a_n v_n | a_1, \cdots, a_n \in \mathbb{F}\}$
4. Let $\alpha = \{v_1, \cdots, v_n\}$ be a set of vectors in V, recall that it is called linearly independent if

$$a_1 v_1 + \cdots + a_n v_n = 0 \text{ for some } a_1, \cdots, a_n \in \mathbb{F} \implies a_1 = \cdots = a_n = 0$$

Remark. *Notice the definitions of linearly independent and span depends* $\mathbb{F}$.

1 Define $\mathbb{C}$ as real vector space

field

Given $\mathbb{C} = \{a + ib | a, b \in \mathbb{R}\}$. Let V be the vector space $\mathbb{C}$ over $\mathbb{R}$ with the following addition and scalar multiplication

- $(a_1 + ib_1) + (a_2 + ib_2) := (a_1 + a_2) + i(b_1 + b_2)$ for all $a_1 + ib_1, a_2 + ib_2 \in V$
- $k(a + ib) := (ka) + i(kb)$ for all $k \in \mathbb{R}, a + ib \in V$

Consider the following questions with the above definition.

1. Show that $i \notin \text{span}(\{1\})$, conclude that $\{1\}$ is not a basis of V.
2. Show that $\alpha = \{1, i\}$ is linearly independent.
3. Show that α is a spanning set of V.
4. Conclude that α is a basis of V.

Q1 Assume $i \in \text{span}(\{1\})$, s/t $i = k \cdot 1$ for some $k \in \mathbb{F} = \mathbb{R}$
known $k \cdot 1 = k(1) = k$ by def of $\cdot$ in V
s/t $i = 1$ which is a contradiction, hence $i \notin \text{span}(\{1\})$
since $i \in V$, but $i \notin \text{span}(\{1\})$, s/t $\{1\}$ is not a spanning set of V, and hence not a basis ∎

Q2 Assume $a_1 \cdot 1 + a_2 \cdot (i) = 0$ for some $a_1, a_2 \in \mathbb{F} = \mathbb{R}$
s/t $a_1 + ia_2 = 0$ where $a_1, a_2 \in \mathbb{R}$
hence $a_1 = 0$ and $a_2 = 0$ by definition of $\mathbb{C}$ because a complex number of the form $a+ib$, $a, b \in \mathbb{R}$ is equal to zero iff $a = b = 0$
s/t 1 and i are linearly independent ∎

Q3 WTS $\alpha = \{1, i\}$ is a spanning set of V, i.e. $\text{span}(\alpha) = V$
($\subseteq$) WTS $\text{span}(\alpha) \subseteq V$
let $x \in \text{span}(\alpha)$, then $x = a_1 \cdot 1 + a_2 \cdot i$ for some $a_1, a_2 \in \mathbb{R}$
$= a_1 + a_2 i \in V$ by def

($\supseteq$) WTS $V \subseteq \text{span}(\alpha)$

let $v \in V$, s/t $v = a + ib$ for some $a, b \in \mathbb{R}$
$= a \cdot 1 + b \cdot i$ by def of $\cdot$
which is a linear combination of $1, i$ with $a, b \in \mathbb{R}$
s/t $v \in \text{span}(\alpha)$
Therefore, $\{1, i\}$ is a spanning set of V ∎

Q4 since α is LI and $\text{span}(\alpha) = V$, s/t α is a basis of V ∎

This sol'n:
If $\mathbb{C}$ is treated as a real vector space, then $\alpha = \{1, i\}$ is a basis of $\mathbb{C}$,
so now for any LT $T: \mathbb{C} \to \mathbb{C}$
we have to be careful finding its matrix rep.

$$[T]_\alpha^\alpha = \begin{pmatrix} | & | \\ [T(1)]_\alpha & [T(i)]_\alpha \\ | & | \end{pmatrix}$$

which is a 2x2 matrix

2 Define $\mathbb{C}$ as complex vector space

$\mathbb{F}$

Given $\mathbb{C} = \{a + ib | a, b \in \mathbb{R}\}$. Let V be the vector space $\mathbb{C}$ over $\mathbb{C}$ with the following addition and scalar multiplication

- $(a_1 + ib_1) + (a_2 + ib_2) := (a_1 + a_2) + i(b_1 + b_2)$ for all $a_1 + ib_1, a_2 + ib_2 \in V$
- $k(a + ib) :=$ ~~$(ka) + i(kb)$ for all $k \in \mathbb{C}, a + ib \in V$~~ $= (k_1 a - k_2 b) + i(k_2 a + k_1 b)$ for all $k = k_1 + ik_2 \in \mathbb{C}$, and $a + ib \in V$

Consider the following questions with the above definition.

1. Show that $\{1, i\}$ is a linearly dependent set in V. Conclude that it is not a basis of V.
2. Show that $\beta = \{1\}$ is linearly independent.
3. Show that β is a spanning set of V.
4. Conclude that β is a basis of V.

Q1 notice $1 \cdot 1 + i \cdot i = 1 - 1$ by def of $\cdot$
$= 0$
s/t $1 \cdot 1 + i \cdot i = 0$ where $1, i \in \mathbb{C}$, in particular $1 \neq 0$

vectors

linear combination of $1, i$ over field $\mathbb{F}$ is equal to 0 with non-zero coeff.

Therefore $1, i$ are linearly dependent
s/t $\{1, i\}$ is not a basis of V ∎

Q2 Assume $\alpha \cdot 1 = 0$ for some $\alpha = a + ib \in \mathbb{C}$, $a, b \in \mathbb{R}$
notice $\alpha \cdot 1 = a + ib = 0$ for some $a, b \in \mathbb{R}$
s/t $a = b = 0$ by def of $\mathbb{C}$
hence $\alpha = a + ib = 0 + i0 = 0$ as required. ∎

Q3 WTS $\text{span}\,\beta = V$, where $\beta = \{1\}$
($\subseteq$) let $x \in \text{span}\,\beta$, $x = \alpha \cdot 1$ for some $\alpha = a + ib \in \mathbb{C}$, $a, b \in \mathbb{R}$
WTS $x \in V = \mathbb{C}$
$x = \alpha \cdot 1 = a + ib \in V$ b/c $a, b \in \mathbb{R}$

($\supseteq$) let $v \in V = \mathbb{C}$, s/t $v = a + ib$ for some $a, b \in \mathbb{R}$
$= (a + ib) \cdot 1$ by def of $\cdot$
$= \alpha \cdot 1$ for some $\alpha = a + ib \in \mathbb{C}$
s/t $v \in \text{span}(\{1\})$
Therefore, $\text{span}\,\beta = V$ ∎

Q4 known β is LI and $\text{span}\,\beta = V$
s/t β is a basis of V which is vector space $\mathbb{C}$ over $\mathbb{C}$ ∎

similarly, for $T: \mathbb{C} \to \mathbb{C}$
where $\mathbb{C}$ is VS over $\mathbb{C}$,

$$[T]_\beta^\beta = \begin{pmatrix} [T(1)]_\beta \end{pmatrix} \quad \text{b/c } \beta = \{1\}$$

which is a 1x1 matrix