## Definition

You should always check the lecture note and the textbook for the definition.

1. A linear transformation is a map $T : V \to W$ where $V, W$ are vector spaces over $\mathbb{F}$ such that
   - $\forall v_1, v_2 \in V, T(v_1 + v_2) = T(v_1) + T(v_2)$
   - $\forall v \in V, \forall k \in \mathbb{F}, T(k \cdot v) = k \cdot T(v)$
2. A linear transformation $T : V \to W$ is injective if
$$\forall v_1, v_2 \in V, T(v_1) = T(v_2) \implies v_1 = v_2$$
3. A linear transformation $T : V \to W$ is surjective if
$$\forall w \in W, \exists v \in V, T(v) = w$$

## Quick discussion

1. The $+$ and $\cdot$ can be different in the definition of linear transformation, why?
2. We have (when) seen a similar statement to the definition of injective, which said $\forall v_1, v_2 \in V, v_1 = v_2 \implies T(v_1) = T(v_2)$ Briefly describe what this statement said to the given linear transformation $T$.

## Questions

1. Let $V = \mathbb{R}$ with the usual operation and $W = \mathbb{R}^+$ with the operations that we have seen last time, check whether or not that $T : V \to W$ defined as $T(x) = e^x$ is a linear transformation.

2. Let $V = \mathbb{R}^2$ and $W = \mathbb{P}_3(\mathbb{R})$. Let $T : V \to W$ be a linear transformation, such that $T((1,1)) = x + x^2$ and $T((3,0)) = 2 + x + x^3$. Find an explicit formula for $T((v_1, v_2))$ given $(v_1, v_2) \in V$. Check whether or not $T$ is injective, and check whether or not $T$ is surjective.

3. Let $A$ be an $3 \times 3$ matrix with real entries. Check whether or not the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $T(x) = Ax$ is injective, if not give a counter example.

4. Let $A$ be an invertible $3 \times 3$ matrix with real entries. Check whether or not the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $T(x) = Ax$ is injective, if not give a counter example.

5. Let $A$ be an $3 \times 3$ matrix with real entries. Check whether or not the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $T(x) = Ax$ is surjective, if not give a counter example.

6. Let $A$ be an invertible $3 \times 3$ matrix with real entries. Check whether or not the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $T(x) = Ax$ is surjective, if not give a counter example.

7. Let $T : \mathbb{R} \to \mathbb{R}$ be a linear transformation. Prove that there exists $a \in \mathbb{R}$, such that $\forall x \in \mathbb{R}, T(x) = ax$

8. Prove that $T : \mathbb{R} \to \mathbb{R}$ given by $T(a) = a \cdot a$ is not a linear transformation. Explain why this is not a contradiction to the previous question.