MATB44_TUT_5

Tuesday, October 27, 2020 11:09

MATB44_TUT_5

Method of Variation of Parameters

To solve a second order non-homogeneous differential equation of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

We first solve the corresponding homogeneous equation $y'' + p(t)y' + q(t)y = 0$ first and get the general solution to the homogeneous equation, $c_1y_1 + c_2y_2$ where y_1, y_2 are solutions to homogeneous equation, eg homogenes eqn with constant coeff, Euler equation.

Then we solve the original non-homogeneous equation as follow:

- Set $y = u_1y_1 + u_2y_2$ where u_1, u_2 are unknown functions of t.
- Write the Lagrange system

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = g(t) \end{cases}$$

  Solve the Lagrange system for u_1, u_2 using the same idea as solving the system of equations with two unknowns. Keep the arbitrary constants c_1, c_2 when solving the integrals in the expression of u_1 and u_2.

  $u_1' = f(t) \Rightarrow u_1 = \int f(t)dt = h_1(t) + \boxed{c_1}$ $\quad u_2 = h_2(t) + c_2$

- Then the general solution of the given non-homogeneous equation is given by

$$\boxed{y = u_1y_1 + u_2y_2} = c_1y_1 + c_2y_2 + \underbrace{h_1(t)y_1(t) + h_2(t)y_2(t)}_{\text{particular solution}}$$

Notice: The reason why we introduce the method of variation of parameters is because this method applies to all forms of $g(t)$, while the method of undetermined coefficient only applies to those special cases of $g(t)$ that we introduced before.

Question

Solve using the variation of parameters

$$x^2y'' - 3xy' + 4y = x^2logx$$

the corresponding homogeneous eqn is $x^2y'' - 3xy' + 4y = 0$

This is an Euler equation.

let $y = x^r$ be the solution.

$x^2(r)(r-1)x^{r-2} - 3x \cdot rx^{r-1} + 4x^r = 0$

$(r)(r-1)x^r - 3rx^r + 4x^r = 0$

$r(r-1) - 3r + 4 = 0$,

$r^2 - r - 3r + 4 = 0$

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0 \Rightarrow r = 2$, repeated root for an Euler equation

set $y_1 = x^2$, $y_2 = x^2\log x$ are the solutions that form the fundametal set of solution to the homogenes equation

normalize the non-homogenous eqn. $y'' - 3x^{-1}y' + 4x^{-2}y = \log x$

set $y = u_1y_1 + u_2y_2$

the Lagrange system is

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = \boxed{\log x} \end{cases}$$

$y_1' = 2x$, $y_2' = 2x\log x + x^2 \cdot \frac{1}{x} = 2x\log x + x$

so

$$\begin{cases} u_1'x^2 + u_2'x^2\log x = 0 \\ u_1' \cdot 2x + u_2' \cdot (2x\log x + x) = \log x \quad ② \end{cases}$$

$\Rightarrow u_1' + u_2'\log x = 0 \Rightarrow \boxed{u_1' = -\log x\, u_2'}$ ①

sub ① into ②.

$-\log x\, u_2' \cdot 2x + u_2'(2x\log x) + u_2'x = \log x$

$u_2'x = \log x$.

$u_2' = \frac{\log x}{x}$

$u_2 = \int \frac{\log x}{x} dx$

$= \int u\,du$. $\quad u = \log x$, $du = \frac{1}{x}dx$

$= \frac{1}{2}u^2 + c_2$

$u_2 = \frac{1}{2}(\log x)^2 + c_2$

Recall, $u_1' = -\log x\, u_2'$

$= -\log x \cdot \frac{\log x}{x} = -\frac{(\log x)^2}{x}$

$u_1 = \int -\frac{(\log x)^2}{x} dx$

$= \int -u^2 du$ $\quad u = \log x$, $du = \frac{1}{x}dx$

$= -\frac{1}{3}u^3 + c_1$

$= -\frac{1}{3}(\log x)^3 + c_1$

Therefore the general solution is $y = u_1y_1 + u_2y_2$

$= \left(-\frac{1}{3}(\log x)^3 + c_1\right)x^2 + \left(\frac{1}{2}(\log x)^2 + c_2\right)x^2\log x$

$= c_1x^2 + c_2x^2\log x + \left(-\frac{1}{3}(\log x)^3x^2 + \frac{1}{2}(\log x)^3x^2\right)$

$= c_1x^2 + c_2x^2\log x + \frac{1}{6}(\log x)^3x^2$