Definition

Recall, let V be a vector space. We say that $\{v_1, \cdots, v_n\} \subset V$ is linearly independent if

$$a_1v_1 + \cdots + a_nv_n = 0 \text{ for some } a_1, \cdots, a_n \in \mathbb{F} \implies a_1 = \cdots = a_n = 0$$

Otherwise, we call them linearly dependent.

Introduction

In this tutorial, we will work on a few problems related to linearly independent. As we discussed before, the definition of linearly independent is a logical implication with hypothesis and conclusion, instead of an 'and' statement with two conditions.

Questions

1. Let V, W be vector spaces over $\mathbb{F}$. Let $T : V \to W$ be an injective linear transformation. Let $I = \{v_1, \cdots, v_n\}$ be a linearly independent set of vectors in V. Prove that,

   (a) $T(0) = 0$

   (b) Show that $T(I) = \{T(v_1), \cdots, T(v_n)\}$ is also linearly independent

2. Let V, W be vector spaces over $\mathbb{F}$. Let $T : V \to W$ be a linear transformation. Assume that $S = \{T(v_1), \cdots, T(v_n)\}$ are linearly independent for some $v_1, \cdots, v_n \in \mathbb{F}$. Show that $v_1, \cdots, v_n$ are linearly independent.

3. Let V be a vector space. Assume that $S = \{v_1, \cdots, v_n\}$ be a set of linearly independent vectors in V. Assume $w \in V$ such that $w \notin span(S)$. Show that $A := S \cup \{w\}$ is also linearly independent

4. Let $C^\infty(\mathbb{R})$ be the vector space of functions $f : \mathbb{R} \to \mathbb{R}$ with derivatives of all orders. Let $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ defined by $T(f) = f - f''$.

   (a) What is $Ker(T)$?

   (b) Find a set of two linearly independent functions in $Ker(T)$. Check that the functions are in $Ker(T)$ and the functions are linearly independent.