MAT337_TUT1

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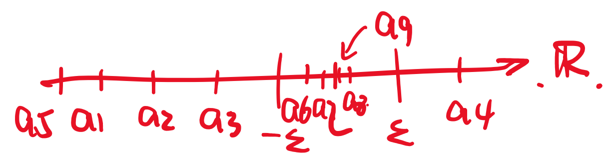
Nick, September.22,2020 **Limit of a Sequence** MAT337 TUT (TUE2-3)

D&D2.4: Limits of Sequences



Definition: Definition of the Limit of a Sequence

A real number L is the limit of a sequence of real numbers $(a_n)_{n=1}^{\infty}$ if for every $\epsilon > 0$, there is an integer $N = N(\epsilon) > 0$ such that

The "distance" between a_n and L is less than ε

$$|a_n - L| < \epsilon \text{ for all } n \geq N$$

We say that the sequence $(a_n)_{n=1}^{\infty}$ converges to L, and we write $\lim_{n\to\infty} a_n = L$

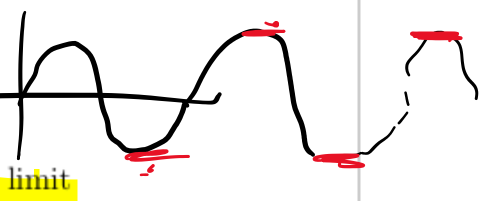
$\varepsilon > 0$. $|a_n - L|$

Goals:

- Understanding the definition of limits of sequences and understand the question
- Understand how to use the definition to prove our intuition and strategy to approach such question that is all about definition

Questions:

$\lim_{x\to\infty} \sin(x)$ DNE



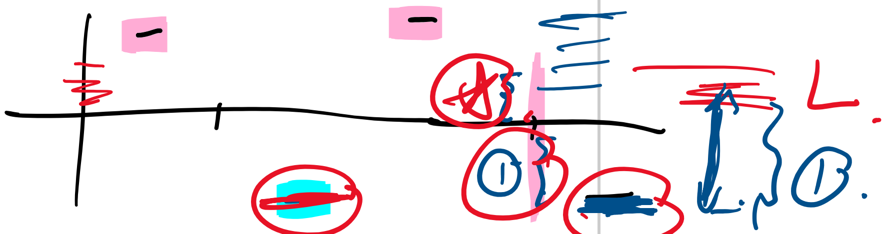
1. (D&D2.4B) Show that $\lim_{n\to\infty} sin(\frac{n\pi}{2})$ does not exist using the definition of limit
2. (D&D2.4C) Prove that if $a_n < b_n$ for $n \geq 1$, $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then $L \leq M$

Q1 $\sin(\frac{n\pi}{2})$, $n=1$, $\sin(\frac{\pi}{2}) = 1$

$n=2$, $\sin(\pi) = 0$. Intuition

$n=3$, $\sin(\frac{3\pi}{2}) = -1$

$\vdots$



Assume $\lim_{n\to\infty} \sin(\frac{n\pi}{2}) = L \in \mathbb{R}$,

By definition, $\forall \varepsilon > 0$, $\exists$ integer $N>0$, s.t $|\sin(\frac{n\pi}{2}) - L| < \varepsilon$ for all $n \geq N$

Let $\varepsilon = \frac{1}{2}$, there exists $N \in \mathbb{Z}$, s.t for all $n \geq N$, $|\sin(\frac{n\pi}{2}) - L| < \varepsilon$

consider the following cases:

① $L \geq 0$, pick $n = 4N+3 > N$, s.t we should have $|\sin\frac{n\pi}{2} - L| < \frac{1}{2}$

$\sin\frac{n\pi}{2} = \sin\frac{(4N+3)\pi}{2} = \sin(2N\pi + \frac{3}{2}\pi) = \sin(\frac{3}{2}\pi) = -1$

$|\sin\frac{n\pi}{2} - L| = |-1-L| = |1+L| = 1+L \geq 1$

$|\sin\frac{n\pi}{2} - L| < \frac{1}{2}$ contradiction

② $L < 0$, pick $n = 4N+1 > N$, s.t $|\sin\frac{n\pi}{2} - L| < \varepsilon = \frac{1}{2}$

$\sin\frac{n\pi}{2} = \sin\frac{(4N+1)\pi}{2} = \sin(2N\pi + \frac{1}{2}\pi) = \sin(\frac{1}{2}\pi) = 1$

$|\sin\frac{n\pi}{2} - L| = |1-L|$, since $L<0$, $-L>0$, $1-L>0$

$= 1-L \geq 1$

contradiction to $|\sin\frac{n\pi}{2} - L| < \frac{1}{2}$

$\lim_{n\to\infty} \sin(\frac{n\pi}{2})$ DNE ∎

Q2 Assume $a_n \leq b_n$ for all $n \geq 1$, let $L = \lim_{n\to\infty} a_n$, $M = \lim_{n\to\infty} b_n$,

WTS $L \leq M$

Equivalently, $L-\varepsilon < a_n < L+\varepsilon$

def $\begin{cases} \forall \varepsilon>0, \exists N_1 \in \mathbb{Z}^+, \text{ s.t } |a_n - L| < \varepsilon \text{ for all } n \geq N_1 \\ \forall \varepsilon>0, \exists N_2 \in \mathbb{Z}^+, \text{ s.t } |b_n - M| < \varepsilon \text{ for all } n \geq N_2 \end{cases}$

$M-\varepsilon < b_n < M+\varepsilon$

let $\varepsilon > 0$, $\exists N_1, N_2$ satisfy the above conditions respectively, let $N = \max(N_1, N_2) \in \mathbb{Z}$

$n \geq N_1$
$n \geq N_2$

Now consider $n \geq N$

$n \geq N \geq N_1$
$n \geq N \geq N_2$

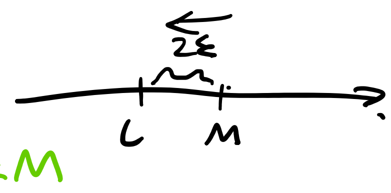
s.t $|a_n - L| < \varepsilon$ and $|b_n - M| < \varepsilon$ for all $n \geq N$

$L-\varepsilon < a_n < L+\varepsilon$ and $M-\varepsilon < b_n < M+\varepsilon$ for all $n \geq N$

since $a_n < b_n$ for all $n \geq 1$, in particular, for all $n \geq N$

$L - \varepsilon < a_n < b_n < M + \varepsilon$

$\boxed{L < M + 2\varepsilon}$, for all $\varepsilon > 0$



proof by contradiction again. $L \leq M$

$\boxed{L \leq M}$

Assume $L > M$, $L - M > 0$

Use $\varepsilon = \frac{L-M}{2} > 0$. $\varepsilon = L - M \Rightarrow L < 2L$

$L < M + 2 \cdot (\frac{L-M}{2}) = M + L - M = L$

i.e. $L < L$ contradiction

s.t $L \leq M$ as required ∎