Inner Product

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Function.

Definition: An inner product on a vector space V over $\mathbb{F} = \mathbb{C}, \mathbb{R}$, is a function $\langle -,- \rangle : V \times V \to \mathbb{F}$ satisfying:

① Positive definite: $\forall v \in V$, $\langle v, v \rangle \geq 0$
and $\langle v, v \rangle = 0$ iff $v = 0$

② Conjugate symmetry: $\forall v, w \in V$,
$\langle v, w \rangle = \overline{\langle w, v \rangle}$ $\quad \overline{(a+ib)} = a - ib$, in $\mathbb{C}$

③ Linearity (on the first component),
$\forall w \in V$, $v \overset{f_w}{\mapsto} \langle v, w \rangle$ is a linear map $V \to \mathbb{F}$
ie $\forall w \in V$, $\forall v_1, v_2 \in V$, $\forall a, b \in \mathbb{F}$
$\langle av_1 + bv_2, w \rangle = a\langle v_1, w \rangle + b\langle v_2, w \rangle$.

- $(V, \langle -,- \rangle)$ is called the inner-product space given that $\langle -,- \rangle$ is an inner product of the vector space V.

Examples:

① V = space of continuous functions from $[0,1]$ to $\mathbb{R}$ with the usual operations.
$\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt$.

① Positive definite:
let $f \in V$, wts $\langle f, f \rangle \geq 0$, and $\langle f, f \rangle = 0$ iff $f = 0$. ($0 \in \mathbb{R}$.) (constant zero function)
$\langle f, f \rangle = \int_0^1 f(t)^2\,dt \geq 0$ b/c $f(t)^2 \geq 0$ on $[0,1]$.
moreover, $\int_0^1 f(t)^2\,dt = 0 \Leftrightarrow f(t) \equiv 0$.
b/c $f(t)$ is continuous on $[0,1]$ and $f(t)^2 \geq 0$.

② Conjugate symmetry:
let $f, g \in V$, wts $\langle f, g \rangle = \overline{\langle g, f \rangle}$.
$\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt = \int_0^1 g(t)f(t)\,dt = \overline{\int_0^1 g(t)f(t)\,dt}$ b/c $\int_0^1 g(t)f(t)\,dt \in \mathbb{R}$
$= \langle g, f \rangle$

③ Linearity:
let $g \in V$, let $f_1, f_2 \in V$, let $a, b \in \mathbb{F} = \mathbb{R}$.
$\langle af_1 + bf_2, g \rangle = \int_0^1 (af_1 + bf_2)(t)\,g(t)\,dt$
$= \int_0^1 (af_1(t) + bf_2(t))g(t)\,dt$
$= \int_0^1 af_1(t)g(t) + bf_2(t)g(t)\,dt$
$= a\int_0^1 f_1(t)g(t)\,dt + b\int_0^1 f_2(t)g(t)\,dt$
$= a\langle f_1, g \rangle + b\langle f_2, g \rangle$.
Therefore, $\langle , \rangle$ is an inner product on V ■

② V = space of polynomials in variable t of deg ≤ 3.
$\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt$.
Pf: V is a subspace of the vector space in example 1,
s.t. all conditions follows immediately by ①. ■

③ Non-example,
V = the space of infinite sequences $a_1, a_2, \ldots$
$\langle f, g \rangle = \sum_i a_i b_i$
Pf: This is not a well-defined function,
consider $f = (1, 1, \ldots)$, $g = (1, 1, \ldots)$.
notice $\langle f, g \rangle = \sum_i 1 \cdot 1 = \sum_i 1 = \infty$ ← symbol, saying that this series diverges.
but $\langle f, g \rangle = \infty \notin \mathbb{F}$
s.t. $\langle -,- \rangle$ is not a well-defined function from $V \times V \to \mathbb{F}$

④. V = the space of infinite bounded sequences $a_1 a_2 \ldots$ in $\mathbb{R}$.
$\langle f, g \rangle = \sum_i \frac{a_i b_i}{2^i}$

Pf: ① This is well-defined, b/c. $|a_i| \leq M_a$, $|b_i| \leq M_b$, $M_a, M_b \in \mathbb{R}^+$.
s.t. $\langle f, g \rangle$ where $f = (a_1, a_2, \ldots)$, $g = (b_1, b_2, \ldots)$.
$= \sum_i \frac{|a_i b_i|}{2^i} \leq \sum_i \frac{M_a M_b}{2^i} = M_a M_b \sum_i \left(\frac{1}{2}\right)^i$
known. $\sum_i \left(\frac{1}{2}\right)^i$ converges by geometric series b/c $\left|\frac{1}{2}\right| < 1$
s.t. $\langle f, g \rangle$ also converges in $\mathbb{R}$ by absolute convergence test ← $\sum |a_i| < \infty \Rightarrow \sum a_i < \infty$.
comparison test ← $\sum a_n \leq \sum b_n < \infty$, where $a_n, b_n \geq 0$.

② Positive definite:
let $f \in V$, $\langle f, f \rangle = \sum_i \frac{a_i^2}{2^i} \geq 0$ b/c $a_i^2 \geq 0$.
and $\left(\sum_i \frac{a_i^2}{2^i} = 0\right)$ iff $(a_i = 0$ for all $i)$ iff $f = 0$ ← constant 0 sequence.

③ Conjugate symmetry.

④ Linearity: b/c $\sum_i \frac{(ka_{1i} + sa_{2i})b_i}{2^i} = k\frac{\sum a_{1i}b_i}{2^i} + s\frac{\sum a_{2i}b_i}{2^i}$