MATB44_TUT_1

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MATB44_TUT_1-1

Intuition-Modelling-Computation

This is a useful technique to solve a computational math problem. For tutorial 1, we will focus on the development of the following skills:

- Intuition: Be able to identify linear, separable and homogeneous equations
- Modelling: Understanding the techniques to solve linear and separable ODEs
- Computation: Integration by parts and substitution

Ex $2t\,y' + \sin t\,y = \log t$

Non-example $2ty' + 4t^2y^2 = \sin t.$

Linear Equations and Separable Equations

- **Linear Equations and Integrating Factors:**

  $\frac{dy}{dt} = y'$ = derivative of y wrt t.

  Linear equation is of the form $P(t)\frac{dy}{dt} + Q(t)y = G(t)$

  IDEA: If we can make the $LHS = [f(t)y]'$ for some function f, then by integrating both sides with respect to t, we can solve for y. Recall, the product rule, $LHS = [f(t)y]' = f(t)\frac{dy}{dt} + f(t)'y$. Therefore, in other words, we have to verify if $f(t) = P(t)$, then we have $f'(t) = Q(t)$

  This is always possible by multiplying an suitable integrating factor $\mu(t)$ on both sides of the original linear equation.

  1. The equation is equivalent to $\frac{dy}{dt} + \frac{Q(t)}{P(t)}y = \frac{G(t)}{P(t)}$,
     i.e. $\frac{dy}{dt} + qy = g$, where $q = \frac{Q(t)}{P(t)}$, $g = \frac{G(t)}{P(t)}$
  2. Multiply unknown $\mu(t)$ on both sides, and we want $(\mu(t)y)' = \mu(t) \times LHS$
     Notice $(\mu y)' = \mu' y + \mu y'$ and $\mu \times LHS = \mu y' + \mu q y$
     In other words, we want to find an suitable μ such that $\frac{d\mu}{dt} = \mu' = \mu q$
  3. Finding a suitable μ is equivalent to solve $\int \frac{d\mu}{\mu} = \int q dt$ for μ
     μ is not unique, but you can pick the one that is most convenient to you.
  4. Multiply the μ on both sides and we will have $(\mu(t)y)' = \mu(t) \times LHS = \mu(t) \times RHS$
     So that $\mu(t)y = \int \mu(t)g(t)dt$, continue to solve for the **general solution** of y

- **Separable Equations:**

  Separable equation is of the form $M(x) + N(y)\frac{dy}{dx} = 0$

  We can solve this ODE by $\int M(x)dx = \int -N(y)dy$

Questions

1. Find the general solution of the equation, $y' = \frac{x^2+xy+y^2}{x^2} = 1 + \frac{y}{x} + (\frac{y}{x})^2 = 1 + v + v^2$

   Homogeneous Equations:

   Homogeneous equation is of the form $\frac{dy}{dx} = F(\frac{y}{x})$, where F is a function of $\frac{y}{x}$

   (a) Define new variable $v = \frac{y}{x}$, then $F(\frac{y}{x}) = \frac{dy}{dx} = \frac{d(vx)}{dx} = v + x\frac{dv}{dx}$

   (b) Now you have a separable ODE of $v(x)$

2. Solve the ODE, $(sint)y' + (cost)y = e^t$, $\boxed{y(1) = a}$, $0 < t < \pi$

ODE

Linear Equation

$P(t)\frac{dy}{dt} + Q(t)y = G(t)$

$(f(t)y)' = f'(t)y + f(t)y'$

$f(t)y = \int G(t)dt$

$y = \int G(t)dt \cdot \frac{1}{f(t)}$

Integrating factor $\mu(t)$

$\mu(t)\left(P(t)\frac{dy}{dt} + Q(t)y\right) = G(t)\mu(t)$

Separable Equation.

$M(y)\,y' + N(x) = 0$

Ex $4yy' + 2x^2 = 0$

$M(y)\frac{dy}{dx}dx + N(x) = 0$

$\int M(y)dy = \int -N(x)dx$

homogeneous equation

Q2 $(\sin t)y' + (\cos t)y = e^t$ linear, 1st order ODE

$\underbrace{\sin t}_{f(t)} \rightarrow \underbrace{\cos t}_{f'(t)}$

$(f(t)y)' = \boxed{(\sin(t)y)' = e^t}$

$\sin(t)y = \int e^t dt = e^t + C$

$y = \frac{e^t + C}{\sin(t)} = \frac{e^t}{\sin(t)} + \boxed{\frac{C}{\sin(t)}}$

$e^C := C$

$y(1) = a,$

$a = \frac{e^1}{\sin(1)} + \frac{C}{\sin(1)}$

$\sin(1)a = e^1 + C$

$C = \sin(1)a - e$

$y = \frac{e^t}{\sin(t)} + \frac{\sin(1)a - e}{\sin(t)}$

Q1 $y' = \frac{x^2+xy+y^2}{x^2}$

$y' = 1 + \frac{y}{x} + (\frac{y}{x})^2 = 1 + v + v^2$, $v = \frac{y}{x}$, $y = vx$

$y' = (vx)' = v'x + v\frac{d(x)}{dx} = \boxed{v'x + v}$

$v'x + v = 1 + v + v^2$

$\frac{dv}{dx}x = 1 + v^2$

$\int \frac{1}{1+v^2}\,dv = \int \frac{1}{x}dx$

$\boxed{\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}}$

$\arctan(v) = \log(x) + C$ implicit formula for v

$y = f(x)$ explicitly.

$y = \frac{y}{x} + y^2 + \log(y)$