Definitions

For a given $m \times n$ matrix A, what are the four Fundamental subspaces of A?

Here is an important theorem.

Theorem. *Let A be an $m \times n$ matrix, then*

1. $dim(KerA) + dim(RanA) = n$
2. $dim(KerA^T) + dim(RanA^T) = m$
3. $dim(RanA) = dim(RanA^T)$

Questions

1. A 54×37 matrix has rank 31. What are the dimensions of all 4 fundamental subspaces?

2. Prove that if $Ax = 0$ has unique solution, then the equation $A^T x = b$ has a solution for every right side b.

3. Write a matrix with the required property, or explain why no such matrix exists

   (a) Column space contains $(1, 0, 0)^T, (0, 0, 1)^T$, row space contains $(1, 1)^T$

   (b) Column space is spanned by $(1, 1, 1)^T$, nullspace is spanned by $(1, 1)^T$

   (c) Column space is $\mathbb{R}^4$, row space is $\mathbb{R}^3$.

4. Compute the rank and find bases of all four fundamental subspaces for the matrices

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$