Learning Objective

Understand the definitions of open sets and closed sets in $\mathbb{R}^n$. Use those language to talk about the convexity of sets in $\mathbb{R}^n$ in which the set can be hard to visualized.

Definitions

While there are equivalent definitions of open and closed in $\mathbb{R}^n$, we should only use the ones discussed in the lectures. The following definitions can be found in the lecture notes and/or the textbook.

1. A **open ball $B_r(v)$ centered at v with radius $r > 0$** in a set E is the set of all $x \in E$ such that $|x - v| < r$. In other words, $B_r(v) = \{x \in E : |x - v| < r\}$
2. A set U is called **open** if for all $x \in U$, there exists an open ball centered at x, that is contained in U. In other words, $\forall x \in U, \exists r > 0,$ such that $B_r(x) \subseteq U$
3. A set C is **closed** if it contains all its limit points. In other words, if $\lim\limits_{n\to\infty} x_n = x_0$ where $x_n \in C$ for all n, then $x_0 \in C$
4. The **interior** of a set C, $int(C) = \{x \in C : B_r(x) \subseteq C \text{ for some } r > 0\}$
5. The **boundary** ∂C of a set C is $\partial C = \{x \in \mathbb{R}^n : B_r(x) \cap C \neq \emptyset, B_r(x) \cap C^c \neq \emptyset \text{ for all } r > 0\}$. Equivalently, that is the set of all limit points of C that are not in the interior.
6. The **closure** $\bar{C}$ of a set C is the union of C and ∂C.

Some operations on sets. For some sets U_1, U_2, vector v and real number α

- $v + U_1 = \{v + u : u \in U_1\}$
- $\alpha \cdot U_1 = \{\alpha \cdot u : u \in U_1\}$
- $U_1 + U_2 = \{u_1 + u_2 : u_1 \in U_1, u_2 \in U_2\}$

Questions

Prove or disprove the following statements about sets in $\mathbb{R}^n$

1. $\forall \alpha \geq 0, \forall r > 0, \alpha \cdot B_r(0) = B_{r\alpha}(0)$.

2. $\forall \alpha > 0, \forall r > 0, \alpha \cdot B_r(0) = B_{r\alpha}(0)$.
   Note: We want to show a set equality, so there are two directions to show.

3. Show that $\forall x_0 \in \mathbb{R}^n, B_r(x_0) = x_0 + B_r(0)$

4. Show that for all $\alpha > 0, \forall r > 0, \forall x_0 \in \mathbb{R}^n$, $\alpha B_r(x_0) = B_{\alpha r}(x_0)$

5. Show that for all $\alpha > 0, \forall r > 0, \forall x_0 \in \mathbb{R}^n$, $\alpha B_r(x_0) = B_{\alpha r}(\alpha x_0)$

6. Show that the intersection of two open sets of $\mathbb{R}^n$ is also open.