Definition

You should always check the lecture note and the textbook for the definition, and make sure the definitions are consistents.

1. A **vector space** is a set V together with two operations

   (a) $+ : V \times V \to V$

   (b) $\cdot : \mathbb{F} \times V \to V$

   such that

   (1) $\forall v, w \in V, v + w = w + v$

   (2) $\forall u, v, w \in V, (u + v) + w = u + (v + w)$

   (3) $\exists 0 \in V, \forall v \in V, 0 + v = v$

   (4) $\forall v \in V, \exists w \in V$ such that $v + w = 0$. Denote w by $-v$

   (5) $\forall a \in \mathbb{F}, \forall u, v \in V, a \cdot (u + v) = a \cdot u + a \cdot v$

   (6) $\forall a, b \in \mathbb{F}, \forall u \in V, (a + b) \cdot v = a \cdot v + b \cdot v$

   (7) $\forall a, b \in \mathbb{F}, \forall v \in V, a \cdot (b \cdot v) = (ab) \cdot v$

   (8) $\forall v \in V, 1 \cdot v = v$

   If $\mathbb{F} = \mathbb{R}$, we call V together with the operations the **real vector space**.

Questions

1. Consider $\mathbb{R}_{>0} := \{a \in \mathbb{R} : a > 0\}$ equipped with the following operations:

   - $\forall a, b \in \mathbb{R}_{>0}, a \boxplus b = ab$, where ab is the usual multiplication in $\mathbb{R}$
   - $\forall a \in \mathbb{R}_{>0}, \forall k \in \mathbb{R}, k \boxtimes a = a^k$

   Show that $\mathbb{R}_{>0}$ with the above operations is a vector space over $\mathbb{R}$.

   Proof. (a) Check the operations. In other words, check that the operations are well-defined functions with the corresponding domain and codomain.

(b) $\forall v, w \in V, v + w = w + v$

(c) $\forall u, v, w \in V, (u + v) + w = u + (v + w)$

(d) $\exists 0 \in V, \forall v \in V, 0 + v = v$

(e) $\forall v \in V, \exists w \in V$ such that $v + w = 0$. Denote w by $-v$

(f) $\forall a \in \mathbb{F}, \forall u, v \in V, a \cdot (u + v) = a \cdot u + a \cdot v$

(g) $\forall a, b \in \mathbb{F}, \forall u \in V, (a + b) \cdot v = a \cdot v + b \cdot v$

(h) $\forall a, b \in \mathbb{F}, \forall v \in V, a \cdot (b \cdot v) = (ab) \cdot v$

(i) $\forall v \in V, 1 \cdot v = v$

$\square$

2. Let V be a vector space.

   (a) Show that the zero vector is unique.

   (b) Show that $c \cdot 0 = 0$

   (c) Let $v \in V$. Show that if $c \cdot v = 0$, then $c = 0$ or $v = 0$. You may use the fact that the multiplicative inverse of a non-zero scalar exists in a field.

3. Let $V = \mathbb{R}^n$ and A, B be $n \times n$ matrices. Show that $Y = W \cap U$ where

   - $Z := \{v \in V : ABv = 0\}$
   - $Y := \{v \in V : v = Bw \text{ for some } w \in Z\}$
   - $W := \{v \in V : Av = 0\}$
   - $U := \{v \in V : v = Bw \text{ for some } w \in V\}$

   Recall, we say that two sets A_1 and A_2 are equal and denote $A_1 = A_2$ if $A_1 \subseteq A_2$ and $A_2 \subseteq A_1$. Also we say $A_1 \subseteq A_2$ if $\forall v \in A_1, v \in A_2$.