Tutorial.7 Vector Field

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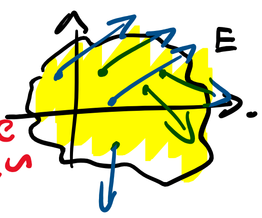
Definition: Let E be a subset of $\mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a function F that assigns to each point (x,y,z) in E a three-dimensional vector $F(x,y,z)$

Example: Electric field



$\mathbb{R}$ or $\mathbb{C}$-value function

Definition: Let f be a scalar function of three variables, then its gradient $\nabla f(x,y,z) = \begin{pmatrix} f_x(x,y,z) \\ f_y(x,y,z) \\ f_z(x,y,z) \end{pmatrix}$ is a vector field on $\mathbb{R}^3$ and is called gradient vector field.

Definition: A vector field F is called a conservative vector field if it is the gradient of some scalar function, ie $F = \nabla f$ where f is called the potential function of F

Question: Let $F(x,y,z) = (y, x+zy\sin z, z+y^2\cos z)$ be a conservative vector field. Find a potential function of F.

Sol: Find f, s.t. $F = \nabla f$

Start with the guess, $f(x,y,z)$, then $F = \nabla f$

$$\begin{pmatrix} f_x(x,y,z) \\ f_y(x,y,z) \\ f_z(x,y,z) \end{pmatrix} = \nabla f(x,y,z) = F(x,y,z) = \begin{pmatrix} y \\ x+zy\sin z \\ z+y^2\cos z \end{pmatrix} \quad ⊛.$$

(1) By $f_x(x,y,z) = y$, ⊛.

then $\boxed{f(x,y,z) = xy + h(y,z)}$ by integrating wrt x

(2) with ⊛, $f_y(x,y,z) = x + h_y(y,z)$

and known by ⊛, $f_y(x,y,z) = x + zy\sin z$

hence $h_y(y,z) = zy\sin z$

and $\boxed{h(y,z) = y^2\sin z + g(z)}$ by integrating wrt y

why not $g(x,z)$, b/c h is a function of y, z only

therefore, $\boxed{f(x,y,z) = xy + y^2\sin z + g(z)}$ ⊛.

(3) By ⊛, $f_z(x,y,z) = y^2\cos z + g'(z)$

and by ⊛, $f_z(x,y,z) = z + y^2\cos z$

ie $g'(z) = z \Rightarrow g(z) = \frac{1}{2}z^2 + C$, where $C \in \mathbb{R}$.

hence, $f(x,y,z) = xy + y^2\sin z + \frac{1}{2}z^2 + C$ will work for any $C \in \mathbb{R}$.

In particular, pick $C=0$,

$\boxed{f(x,y,z) = xy + y^2\sin z + \frac{1}{2}z^2}$, and $F = \nabla f$ ∎