TUT11_adjoint_isometry_and_unitary

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TUT11_adjoint_isomet...

1 Adjoint of a linear transformation

- Definition: Let V, W be inner product space. Let $A : V \to W$ be an operator, then the adjoint of A is the operator $A^* : W \to V$ such that

$$(Ax, y) = (x, A^*y) \qquad \forall x \in V, y \in W$$

- Definition: For a $m \times n$ matrix A, its adjoint $A^* := \overline{A^T}$. Given the definition, it satisfies the property

$$(Ax, y) = (x, A^*y) \qquad \forall x \in \mathbb{C}^n, y \in \mathbb{C}^m$$

Question

Let A be an $m \times n$ matrix. Show that $\text{Ker}A = \text{Ker}(A^*A)$

($\subseteq$) let $x \in \ker(A)$, s.t. $Ax = 0$

WTS $x \in \ker(A^*A)$

$(A^*A)(x) = A^*(Ax) = A^*(0) = 0$

s.t. $x \in \ker(A^*A)$

($\supseteq$) let $x \in \ker(A^*A)$, s.t. $(A^*A)x = 0$

WTS $x \in \ker(A)$

inner product defined on the IPS.

$\|Ax\|^2 = (Ax, Ax)$ by def of norm.

$= (x, A^*Ax)$ by definition of adjoint.

$= (x, 0)$ by assumption.

$= \overline{(0, x)}$ by conjugate symmetry.

zero vector

$= \overline{(0x, x)} = \overline{0(x,x)}$ by linearity wrt first component.

zero scalar

$= \bar{0}\,\overline{(x,x)} = 0\,\overline{(x,x)} = 0$

s.t. $Ax = 0$ by non-degeneracy.

hence $x \in \ker(A)$ ∎

2 Isometries and unitary operators

- Definition: An operator $U : X \to Y$ is called an isometry, if it preserves the norm. That is

$$\|Ux\| = \|x\| \qquad \forall x \in X$$

Some useful facts about isometry:

1. An operator $U : X \to Y$ is an isometry if and only if

$$(x, y) = (Ux, Uy) \qquad \forall x, y \in X$$

2. An operator $U : X \to Y$ is an isometry if and only if $U^*U = I$

That implies U is left-invertible.

- Definition: An isometry $U : X \to Y$ is a unitary operator if it is invertible.
Some useful facts about unitary operator:

1. If $U : X \to Y$ is a unitary operator, then $U^{-1} = U^*$
2. An isometry $U : X \to Y$ is a unitary operator if and only if $\dim X = \dim Y$

Questions

1. Show that a product of unitary matrices is unitary.

let A, B be unitary matrices, then A, B must be square matrices.

since for example, $A: \mathbb{R}^n \to \mathbb{R}^m$, then $n = \dim(\mathbb{R}^n) = \dim(\mathbb{R}^m) = m$

similarly to the matrix B. s.t. AB is also a square matrix, ie $AB: \mathbb{R}^n \to \mathbb{R}^n$. (※)

It suffices to show AB is an isometry.

A, B being an unitary operators.

$(AB)^*AB = \overline{(AB)^T}AB = \overline{B^T}\,\overline{A^T}AB = B^*A^*AB = B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$

s.t. AB is an isometry

hence by (※), AB is a unitary operator ∎

2. Let $U : X \to X$ be a linear transformation on a finite-dimensional inner product space. Prove that if $\|Ux\| = \|x\|$ for all $x \in X$, then U is unitary.

Assume $\|Ux\| = \|x\|$ for all $x \in X$,

then U is isometry by the definition.

Notice dimensions of the domain and codomain of U are the same.

s.t. given U is isometry, U is unitary. ∎

3. Let A, B be $n \times n$ matrices. Show that trace(AB) =trace(BA). Conclude that similar matrices have the same trace.

① let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix}$

Notice $AB = \begin{pmatrix} \sum_{i=1}^{n} a_{1i}b_{i1} & & * \\ & \ddots & \\ * & & \sum_{i=1}^{n} a_{ni}b_{in} \end{pmatrix}$ and $BA = \begin{pmatrix} \sum_{i=1}^{n} b_{1i}a_{i1} & & * \\ & \ddots & \\ * & & \sum_{i=1}^{n} b_{ni}a_{in} \end{pmatrix}$

then trace$(AB) = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} a_{ji}b_{ij}\right)$

$= \sum_{j=1}^{n}(a_{j1}b_{1j} + \cdots + a_{jn}b_{nj})$

$= \underbrace{(a_{11}b_{11} + \cdots + a_{1n}b_{n1})}_{j=1} + \cdots + \underbrace{(a_{n1}b_{1n} + \cdots + a_{nn}b_{nn})}_{j=n}$.

trace$(BA) = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} b_{ji}a_{ij}\right)$

$= \sum_{j=1}^{n}(b_{j1}a_{1j} + \cdots + b_{jn}a_{nj})$

$= \underbrace{(b_{11}a_{11} + \cdots + b_{1n}a_{n1})}_{j=1} + \cdots + \underbrace{(b_{n1}a_{1n} + \cdots + b_{nn}a_{nn})}_{j=n}$.

$= (b_{11}a_{11} + \cdots + b_{n1}a_{1n}) + \cdots + (b_{1n}a_{n1} + \cdots + b_{nn}a_{nn})$

$=$ trace(AB)

②. Assume A, B are similar matrices,

then $A = PBP^{-1}$ for some invertible P

trace(A) = trace(PBP^{-1})

= trace$(P(BP^{-1}))$ = trace$((BP^{-1})P)$ by ①

= trace$(BP^{-1}P)$ = trace(BI) = trace(B). ∎