TUT3_Convex_set_and_Convex_function-1

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TUT3_Convex_set_and…

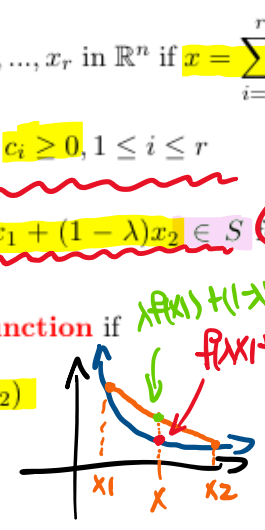
Nick Huang — **Convex Set and Convex Function** — MATB61 TUT03/04, Jan.28 2021 Week 3

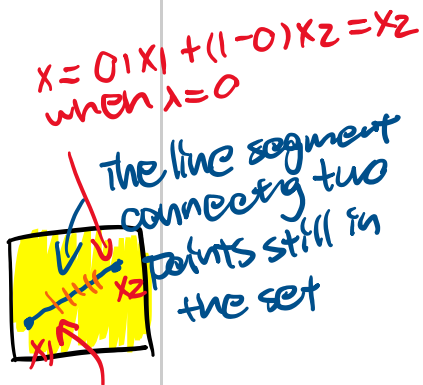
Learning Objective

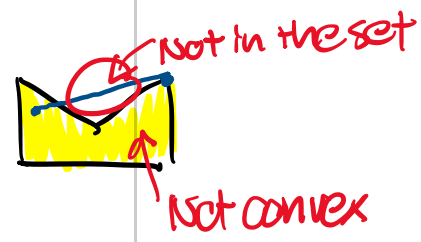
Understand the definitions of convex sets and convex functions. Understand the proof structure of proving the convexity of a given set or function.

Definitions

1. A point $x \in \mathbb{R}^n$ is called **convex combination** of the points $x_1, x_2, ..., x_r$ in $\mathbb{R}^n$ if $x = \sum_{i=1}^{r} c_i x_i$ where $c_1, c_2, ..., c_r$ are some real numbers, such that $\sum_{i=1}^{r} c_i = 1$ and $c_i \geq 0, 1 \leq i \leq r$
2. A subset S of $\mathbb{R}^n$ is called **convex** if for any $x_1, x_2 \in S$, $x = \lambda x_1 + (1-\lambda)x_2 \in S$ for all $0 \leq \lambda \leq 1$.
3. A function f defined on a convex set S in $\mathbb{R}^n$ is called a **convex function** if
$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$
for all $0 \leq \lambda \leq 1$ and any $x_1, x_2 \in S$.







Four-phased method when writing a proof

How to Solve It (G.Polya, 1945) introduced a four-phased method to solve mathematical problems. It is very useful to follow the metod when writing a proof, so that you dont easily get lost at what you are trying to do.

1. Understanding: Assumption and Want-to-show directly by definition. Give yourself intuition by drawing graph if possible.
2. Devising a plan: Build the connection between the assumption and Want-to-show
3. Carrying out the plan: Use the assumption, follow the plan you have and use logical reasoning to prove (not just seeing) the Want-to-show.
4. Looking back: Have you proved what you need to prove? Is there any lack of reasoning?

Set notation

There are always two questions to ask when looking at a set. What does the elements look like and what conditions have to be satisfies by the elements? Write the following sets using set notation.

1. The set of all solutions to $Ax \leq b$ $\{x \in \mathbb{R}^n \mid Ax \leq b\}$
2. The graph of $x + y \leq 2$ $\{(x,y) \in \mathbb{R}^2 \mid x+y \leq 2\}$

Note: Everything written in black pen below should be included in your proof.

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Questions

Prove the following statements and be precise what your assumption and want-to-show are.

1. Show that the set of all solutions to $Ax \leq b$, if it is nonempty, is a convex set.

this is possible b/c $S \neq \phi$

$Ax_1 \leq b$, $Ax_2 \leq b$

Let $S = \{x \in \mathbb{R}^n \mid Ax \leq b\} \neq \phi$, WTS S is convex

Let $x_1, x_2 \in S$, let $\lambda \in \mathbb{R}$ and assume $0 \leq \lambda \leq 1$ ← from the definition of convex set we start with arbitrary $x_1, x_2 \in S$ and arbitrary λ with condition $0 \leq \lambda \leq 1$

set $Ax_1 \leq b$ and $Ax_2 \leq b$ by definition of S

Let $x = \lambda x_1 + (1-\lambda)x_2 \in \mathbb{R}^n$, WTS $x \in S$, i.e. $Ax \leq b$

$\lambda, 1-\lambda$ are scalar, but remember that $AB \neq BA$ in general for matrix multiplication.

Notice $Ax = A(\lambda x_1 + (1-\lambda)x_2) = A\lambda x_1 + A(1-\lambda)x_2 = \lambda Ax_1 + (1-\lambda)Ax_2$

$\leq \lambda b + (1-\lambda)b = b$, s.t. $Ax \leq b$

known $0 \leq \lambda \leq 1$ and $Ax \leq b$, s.t. $b - Ax \geq 0$, then $\lambda(b - Ax) \geq 0$, $\lambda Ax_1 \leq \lambda b$ similarly for x_2 — hence $x \in S$ ■

Assumption of showing convex set

2. Show that the graph of $x + y \leq 2$ is a convex set.

this is what graph means in set notation

Let $S = \{(x,y) \in \mathbb{R}^2 \mid x+y \leq 2\}$

Let $(x_1,y_1), (x_2,y_2) \in S$, let $\lambda \in \mathbb{R}$, $0 \leq \lambda \leq 1$

then $(x,y) := \lambda(x_1,y_1) + (1-\lambda)(x_2,y_2) = (\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \in \mathbb{R}^2$

WTS of showing convex set

WTS $(x,y) \in S$, i.e. $x+y \leq 2$

$x+y = [\lambda x_1 + (1-\lambda)x_2] + [\lambda y_1 + (1-\lambda)y_2] = \lambda[x_1+y_1] + (1-\lambda)[x_2+y_2]$

$\leq 2\lambda + (1-\lambda)\cdot 2$ b/c $x_1+y_1 \leq 2$, $x_2+y_2 \leq 2$ since $(x_1,y_1), (x_2,y_2) \in S$, and $0 \leq \lambda \leq 1$

$= 2$ as required ■

3. Show that the open interval $(0,1) \subset \mathbb{R}$ is a convex set.

Let $S = (0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$

Let $x, y \in S$, let $\lambda \in \mathbb{R}$, $0 \leq \lambda \leq 1$

Define $z = \lambda x + (1-\lambda)y \in \mathbb{R}$ ← Assumption

WTS $0 < z < 1$ ← "want to show"

(1) $z = \lambda x + (1-\lambda)y < \lambda + 1 - \lambda$ by $x < 1$, $y < 1$ and $\lambda, 1-\lambda \geq 0$ and at least one of $\lambda, 1-\lambda$ is non-zero
$= 1$

s.t. $z < 1$

(2) $z = \lambda x + (1-\lambda)y > \lambda \cdot 0 + (1-\lambda)\cdot 0$ by $x > 0$, $y > 0$, $\lambda, 1-\lambda \geq 0$ and at least one of $\lambda, 1-\lambda$ is non-zero
$= 0$

This is important so we have strict inequality

s.t. $z > 0$

Therefore $0 < z < 1$, s.t. $z \in S$

Hence S is convex ■