Definition

You should always check the lecture note and the textbook for the definition.

1. A linear transformation is a map $T : V \to W$ where V, W are vector spaces over $\mathbb{F}$ such that

   - $\forall v_1, v_2 \in V, T(v_1 + v_2) = T(v_1) + T(v_2)$
   - $\forall v \in V, \forall k \in \mathbb{F}, T(k \cdot v) = k \cdot T(v)$

2. A linear transformation $T : V \to W$ is injective if

$$\forall v_1, v_2 \in V, T(v_1) = T(v_2) \implies v_1 = v_2$$

3. A linear transformation $T : V \to W$ is surjective if

$$\forall w \in W, \exists v \in V, T(v) = w$$

Quick discussion

1. The $+$ and $\cdot$ can be different in the definition of linear transformation, why?

2. We have (when) seen a similar statement to the definition of injective, which said $\forall v_1, v_2 \in V, v_1 = v_2 \implies T(v_1) = T(v_2)$ Briefly describe what this statement said to the given linear transformation T.

Questions

1. Let $V = \mathbb{R}$ with the usual operation and $W = \mathbb{R}^+$ with the operations that we have seen last time, check whether or not that $T : V \to W$ defined as $T(x) = e^x$ is a linear transformation.

2. Let $V = \mathbb{R}^2$ and $W = \mathbb{P}_3(\mathbb{R})$. Let $T : V \to W$ be a linear transformation, such that $T((1,1)) = x + x^2$ and $T((3,0)) = 2 + x + x^3$. Find an explicit formula for $T((v_1, v_2))$ given $(v_1, v_2) \in V$. Check whether or not T is injective, and check whether or not T is surjective.

3. Let A be an 3×3 matrix with real entries. Check whether or not the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $T(x) = Ax$ is injective, if not give a counter example.

4. Let A be an invertible 3×3 matrix with real entries. Check whether or not the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $T(x) = Ax$ is injective, if not give a counter example.

5. Let A be an 3×3 matrix with real entries. Check whether or not the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $T(x) = Ax$ is surjective, if not give a counter example.

6. Let A be an invertible 3×3 matrix with real entries. Check whether or not the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $T(x) = Ax$ is surjective, if not give a counter example.