Example_matrix_representation_and_coordinate_vector_annotated

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Example_matrix_repr...

Definitions

You should always check the lecture note and the textbook for the definition, and make sure the definitions are consistents.

1. We call $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ the coordinate vector of $v \in V$ with respect to the basis $(v_1, \cdots, v_n)$ of V if $v = a_1v_1 + \cdots + a_nv_n$ for some $a_1, \cdots, a_n \in \mathbb{F}$. Denoted as $[v]_{v_1,\cdots,v_n}$
2. Let V, W be vector spaces over $\mathbb{F}$. Let $T : V \to W$ be a linear transformation. Let $\alpha = (v_1, \cdots, v_n)$ and $\beta = (w_1, \cdots, w_m)$ be ordered basis of V and W respectively. Then the matrix representation of T with respect to α and β is

$$[T]^{v_1,\cdots,v_n}_{w_1,\cdots,w_m} = \begin{bmatrix} [T(v_1)]_{w_1,\cdots,w_m} & \cdots & [T(v_n)]_{w_1,\cdots,w_m} \end{bmatrix}$$

Remark. *When the question asks you the find the matrix of a given linear transformation $T : V \to W$ with respected to some bases as we have seen in the assignment, you are NOT asked to find a matrix A, such that $T(v) = Av$. You are asked to find the matrix representation as defined. While sometimes it is true that the matrix representation, let's call it A, will result $T(v) = Av$ for all $v \in V$, this is not always true. The Av is a matrix multiplication and we have seen vectors that are not vectors in $\mathbb{R}^n$.*

Examples: Coordinate vector

1. We will start with the familiar one $\mathbb{R}^n$. In next few questions, consider the vector space $V = \mathbb{R}_4$ with the usual operations.
   (a) Circle the elements which belong the vector space V

   ① $(1,0,0,0)^T$ 1 ② $(2,9,0,5)^T$ $(1,i,0,0)^T$ $(3,4,0)^T$ ③ $5\cdot(1,4,0,0)^T$

   $i \notin \mathbb{R}$ $\mathbb{R}^3$

   $\mathbb{P}_n(\mathbb{R}), n \geq 1$ → $1+x$ $\begin{pmatrix}1&0\\0&i\end{pmatrix}$ $(1,x,x^2,x^3)^T$ $3+x^2+x^3$ $4x+i$ $\begin{pmatrix}1&4\\0&1\end{pmatrix}$

   (b) What is the standard (ordered) basis for V? Call it α.
   (c) We know that α spans the vector space because it is a basis. For the elements above that are in the vector space, write those as linear combinations of the vectors in the basis. Find $[v]_\alpha$
   (d) Known $\beta = ((1,1,1,1)^T, (1,1,1,0)^T, (1,1,0,0)^T, (1,0,0,0)^T)$ is an ordered basis of V. Repeat the previous subquestion with β.

(b) $\alpha = \left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right)$

(c) ①: $\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = 1\cdot\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 0\cdot\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + 0\cdot\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + 0\cdot\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$, s.t. $\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right]_\alpha = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

②: $\begin{pmatrix}2\\9\\0\\5\end{pmatrix} = 2\cdot\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 9\cdot\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + 0\cdot\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + 5\cdot\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$, s.t. $\left[\begin{pmatrix}2\\9\\0\\5\end{pmatrix}\right]_\alpha = \begin{pmatrix}2\\9\\0\\5\end{pmatrix}$

③: $5\cdot\begin{pmatrix}1\\4\\0\\0\end{pmatrix} = \begin{pmatrix}5\\20\\0\\0\end{pmatrix} = 5\cdot\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 20\cdot\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + 0\cdot\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + 0\cdot\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$, s.t. $\left[5\begin{pmatrix}1\\4\\0\\0\end{pmatrix}\right]_\alpha = \begin{pmatrix}5\\20\\0\\0\end{pmatrix}$

(d) $\beta = \left(\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right)$

①: $\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = 0\cdot\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + 0\cdot\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + 0\cdot\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + 1\cdot\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, s.t. $\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right]_\beta = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

②: $\begin{pmatrix}2\\9\\0\\5\end{pmatrix} = 5\cdot\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + (-5)\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + 9\cdot\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + (-7)\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, s.t. $\left[\begin{pmatrix}2\\9\\0\\5\end{pmatrix}\right]_\beta = \begin{pmatrix}5\\-5\\9\\-7\end{pmatrix}$

③: $5\cdot\begin{pmatrix}1\\4\\0\\0\end{pmatrix} = \begin{pmatrix}5\\20\\0\\0\end{pmatrix} = 0\cdot\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + 0\cdot\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + 20\cdot\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + (-15)\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, s.t. $\left[5\begin{pmatrix}1\\4\\0\\0\end{pmatrix}\right]_\beta = \begin{pmatrix}0\\0\\20\\-15\end{pmatrix}$

2. In the next few questions, consider the vector space $V = \mathbb{P}_3(\mathbb{R})$ with the usual operations.
   (a) Circle the elements which belong the vector space V

   $(1,0,0,0)^T$ 1 $(2,9,0,5)^T$ $(1,i,0,0)^T$ $(3,4,0)^T$ $5\cdot(1,4,0,0)^T$

   ① $1+x$ $\begin{pmatrix}1&0\\0&i\end{pmatrix}$ $(1,x,x^2,x^3)^T$ ② $3+x^2+x^3$ $4x+i$ $\begin{pmatrix}1&4\\0&1\end{pmatrix}$

   $4x + i\cdot 1 \in \mathbb{P}_3(\mathbb{C})$

   (b) What is the standard (ordered) basis for V? Call it α.
   (c) We know that α spans the vector space because it is a basis. For the elements above that are in the vector space, write those as linear combinations of the vectors in the basis. Find $[v]_\alpha$
   (d) One can check that $\beta = (2, 1+x, 1+x+x^2, 1+x+x^2+x^3)$ is an ordered basis of V. Repeat the previous subquestion with β.

(b) $\alpha = (1, x, x^2, x^3)$

(c) ①: $1+x = 1\cdot1 + 1\cdot x + 0\cdot x^2 + 0\cdot x^3$, s.t. $[1+x]_\alpha = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}$

②: $3+x^2+x^3 = 3\cdot1 + 0\cdot x + 1\cdot x^2 + 1\cdot x^3$, s.t. $[3+x^2+x^3]_\alpha = \begin{pmatrix}3\\0\\1\\1\end{pmatrix}$

(d) $\beta = (2, 1+x, 1+x+x^2, 1+x+x^2+x^3)$

①: $1+x = 0\cdot(1+x+x^2+x^3) + 0\cdot(1+x+x^2) + 1\cdot(1+x) + 0\cdot(2)$

s.t. $[1+x]_\beta = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ The order of the elements in β matters!!

②: $3+x^2+x^3 = 1\cdot(1+x+x^2+x^3) + 0\cdot(1+x+x^2) + (-1)\cdot(1+x) + \frac{3}{2}(2)$

s.t. $[3+x^2+x^3]_\beta = \begin{pmatrix}\frac{3}{2}\\-1\\0\\1\end{pmatrix}$

3. In the next few questions, consider the vector space $V = \cancel{M_{2\times2}(\mathbb{R})}$ $M_2(\mathbb{R})$ with the usual operations.
   (a) Circle the elements which belong the vector space V

   $(1,0,0,0)^T$ 1 $(2,9,0,5)^T$ $(1,i,0,0)^T$ $(3,4,0)^T$ $5\cdot(1,4,0,0)^T$

   $M_2(\mathbb{C})$ → $1+x$ $\begin{pmatrix}1&0\\0&i\end{pmatrix}$ $(1,x,x^2,x^3)^T$ $3+x^2+x^3$ $4x+i$ $\begin{pmatrix}1&4\\0&1\end{pmatrix}$

   (b) What is the standard (ordered) basis for V? Call it α.
   (c) We know that α spans the vector space because it is a basis. For the elements above that are in the vector space, write those as linear combinations of the vectors in the basis. Find $[v]_\alpha$
   (d) One can check that $\beta = (\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}1&1\\0&0\end{pmatrix}, \begin{pmatrix}1&1\\1&0\end{pmatrix}, \begin{pmatrix}1&1\\1&1\end{pmatrix})$ is an ordered basis of V. Repeat the previous subquestion with β.

(b) $\alpha = \left(\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right)$

(c) Notice $\begin{pmatrix}1&4\\0&1\end{pmatrix} = 1\begin{pmatrix}1&0\\0&0\end{pmatrix} + 4\begin{pmatrix}0&1\\0&0\end{pmatrix} + 0\cdot\begin{pmatrix}0&0\\1&0\end{pmatrix} + 1\cdot\begin{pmatrix}0&0\\0&1\end{pmatrix}$, s.t. $\left[\begin{pmatrix}1&4\\0&1\end{pmatrix}\right]_\alpha = \begin{pmatrix}1\\4\\0\\1\end{pmatrix}$

(d) $\begin{pmatrix}1&4\\0&1\end{pmatrix} = 1\begin{pmatrix}1&1\\1&1\end{pmatrix} - 1\begin{pmatrix}1&1\\1&0\end{pmatrix} + 4\begin{pmatrix}1&1\\0&0\end{pmatrix} - 3\begin{pmatrix}1&0\\0&0\end{pmatrix}$

s.t. $\left[\begin{pmatrix}1&4\\0&1\end{pmatrix}\right]_\beta = \begin{pmatrix}-3\\4\\-1\\1\end{pmatrix}$ ← Remember order matters!!

Examples: Matrix representation of linear transformation

1. In the next few questions, consider the following linear transformation.

$$T : V \to W \text{ by } T(\begin{pmatrix}v_1\\v_2\end{pmatrix}) = \begin{pmatrix}v_1+2v_2\\v_2\\v_1\end{pmatrix}$$

   where $V = \mathbb{R}^2, W = \mathbb{R}^3$
   (a) Compute $T(\begin{pmatrix}2\\3\end{pmatrix})$ $T\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}2+2\cdot3\\3\\2\end{pmatrix} = \begin{pmatrix}8\\3\\2\end{pmatrix}$
   (b) Let α and β be the standard bases of V and W respectively. Find the matrix representation $[T]^\alpha_\beta$ by our definition and notation.
   (c) Let $\gamma = (\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix})$ which is another ordered basis of V. Find the matrix representation $[T]^\gamma_\beta$ by our definition and notation.
   (d) Let $\delta = (\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix})$ which is another ordered basis of W. Find the matrix representation $[T]^\alpha_\delta$ by our definition and notation.

Notice $T\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}v_1+2v_2\\v_2\\v_1\end{pmatrix} = \begin{pmatrix}1&2\\0&1\\1&0\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}$

Even though this is true for this given $T: V \to W$, this is not how we find the matrix representation.

sometimes $[T]^\alpha_\beta$ will satisfy $T(v) = [T]^\alpha_\beta v$ but this is not saying → if $T(v) = Av$, then $[T]^\alpha_\beta = A$

(b) $\alpha = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)$, $\beta = \left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right)$

$[T]^\alpha_\beta = \left(\left[T\begin{pmatrix}1\\0\end{pmatrix}\right]_\beta \; \left[T\begin{pmatrix}0\\1\end{pmatrix}\right]_\beta\right) = \left(\left[\begin{pmatrix}1\\0\\1\end{pmatrix}\right]_\beta \; \left[\begin{pmatrix}2\\1\\0\end{pmatrix}\right]_\beta\right) = \begin{pmatrix}1&2\\0&1\\1&0\end{pmatrix}$

(c) $\gamma = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right)$ is an ordered basis of V

$[T]^\gamma_\beta = \left(\left[T\begin{pmatrix}1\\0\end{pmatrix}\right]_\beta \; \left[T\begin{pmatrix}1\\1\end{pmatrix}\right]_\beta\right) = \left(\left[\begin{pmatrix}1\\0\\1\end{pmatrix}\right]_\beta \; \left[\begin{pmatrix}3\\1\\1\end{pmatrix}\right]_\beta\right) = \begin{pmatrix}1&3\\0&1\\1&1\end{pmatrix}$

(d) $\delta = \left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right)$ is an ordered basis of W

$[T]^\alpha_\delta = \left(\left[T\begin{pmatrix}1\\0\end{pmatrix}\right]_\delta \; \left[T\begin{pmatrix}0\\1\end{pmatrix}\right]_\delta\right) = \left(\left[\begin{pmatrix}1\\0\\1\end{pmatrix}\right]_\delta \; \left[\begin{pmatrix}2\\1\\0\end{pmatrix}\right]_\delta\right) = \begin{pmatrix}1&1\\-1&1\\1&0\end{pmatrix}$

2. In the next few questions, consider the following linear transformation.

   Further practice: Show that T is actually a L.T.

$$T : V \to W \text{ by } T(f) = \begin{pmatrix}f'(2)\\f(0)\end{pmatrix}$$

   where $V = \mathbb{P}_2(\mathbb{R}), W = \mathbb{R}^2$
   (a) Compute $T(1+x)$ and $T(x^2+x^3)$
   (b) Let α and β be the standard bases of V and W respectively. Find the matrix representation $[T]^\alpha_\beta$ by our definition and notation.
   (c) Let $\gamma = (1, 1+x, 1+x+x^2)$ which is another ordered basis of V. Find the matrix representation $[T]^\gamma_\beta$ by our definition and notation.
   (d) Let $\delta = (\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix})$ which is another ordered basis of W. Find the matrix representation $[T]^\gamma_\delta$ by our definition and notation.

$T(1+x) = \begin{pmatrix}1|_{x=2}\\1+x|_{x=0}\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$, $T(x^2+x^3) = \begin{pmatrix}2x+3x^2|_{x=2}\\x^2+x^3|_{x=0}\end{pmatrix} = \begin{pmatrix}4+3\cdot4\\0\end{pmatrix} = \begin{pmatrix}16\\0\end{pmatrix}$

(b) $\alpha = (1, x, x^2)$, $\beta = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)$

$[T]^\alpha_\beta = \left([T(1)]_\beta \; [T(x)]_\beta \; [T(x^2)]_\beta\right) = \left(\left[\begin{pmatrix}0\\1\end{pmatrix}\right]_\beta \; \left[\begin{pmatrix}1\\0\end{pmatrix}\right]_\beta \; \left[\begin{pmatrix}4\\0\end{pmatrix}\right]_\beta\right) = \begin{pmatrix}0&1&4\\1&0&0\end{pmatrix}$

(c) $\gamma = (1, 1+x, 1+x+x^2)$

$[T]^\gamma_\beta = \left([T(1)]_\beta \; [T(1+x)]_\beta \; [T(1+x+x^2)]_\beta\right) = \left(\left[\begin{pmatrix}0\\1\end{pmatrix}\right]_\beta \; \left[\begin{pmatrix}1\\1\end{pmatrix}\right]_\beta \; \left[\begin{pmatrix}5\\1\end{pmatrix}\right]_\beta\right) = \begin{pmatrix}0&1&5\\1&1&1\end{pmatrix}$

(d) $\delta = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right)$

$[T]^\gamma_\delta = \left([T(1)]_\delta \; [T(1+x)]_\delta \; [T(1+x+x^2)]_\delta\right) = \left(\left[\begin{pmatrix}0\\1\end{pmatrix}\right]_\delta \; \left[\begin{pmatrix}1\\1\end{pmatrix}\right]_\delta \; \left[\begin{pmatrix}5\\1\end{pmatrix}\right]_\delta\right) = \begin{pmatrix}-1&0&4\\1&1&1\end{pmatrix}$