TUT12 _orthogonally_diagonalize_printer_friendly_version

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TUT12_orthogonally_...

Similar: A is similar to B if $A = PBP^{-1}$ for some invertible P.

Useful facts

In particular, A is similar to D. ← $A = UDU^{-1}$, where D is a diagonal matrix and U is an invertible matrix with orthonormal vectors being the columns

$A^* = \bar{A}^T$

0.1 Self-adjoint

1. Theorem 2.2, chapter 6: Let $A = A^*$ be a self-adjoint (or sometimes called Hermitian) matrix. Then A is unitarily equivalent to a diagonal matrix with real entries, i.e. $A = UDU^*$ where U is a unitary matrix and D is a diagonal matrix with real entries. If A is a real matrix, then U can be chosen to be real.
2. Proposition 6.5, chapter 5: A matrix A is unitarily equivalent to a diagonal one if and only if it has an orthonormal basis of eigenvectors.
3. Recall that $U^{-1} = U^*$ for a unitary matrix U.

Remark. *For a given self-adjoint matrix A, we can orthogonally diagonalize it, such that $A = UDU^{-1}$ where U is the matrix with the orthonormal basis of eigenvectors as the columns. The matrix D is always real.*

0.2 Normal

1. A linear transformation $N : X \to X$ where X is an inner product space is called normal if $N^*N = NN^*$.
2. Theorem 2.4, chapter 6: Any normal matrix N in a complex vector space has an orthonormal basis of eigenvectors. In other words, such N can be represented as $N = UDU^*$ where U is a unitary matrix, and D is a diagonal matrix.

Remark. *For a given normal matrix A, we can orthogonally diagonalize it, such that $N = UDU^{-1}$ where U is the matrix with the orthonormal basis of eigenvectors as the columns. The matrices U and D may not be real, even if N is real.*

Questions

1. Orthogonally diagonalize the matrix $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$. Find all the square roots of A, i.e. find all the matrices B such that $B^2 = A$.

① Find the eigenvalues:

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, \quad \det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{pmatrix} = (3-\lambda)^2-4$$
$$= 9-6\lambda+\lambda^2-4 = \lambda^2-6\lambda+5$$
$$= (\lambda-1)(\lambda-5) = 0$$

s.t. eigenvalues are $\lambda=1$ and $\lambda=5$

② Notice $A^* = \bar{A}^T = \begin{pmatrix} \bar{3} & \bar{2} \\ \bar{2} & \bar{3} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} = A$, hence A is self-adjoint

③ Find the eigenvectors:

(1) $\lambda_1 = 1$: $\ker(A-\lambda_1 I) = \ker\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$

After normalization, pick $v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ where $\sqrt{2} = \left\|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\|$

(2) $\lambda_2 = 5$: $\ker(A-\lambda_2 I) = \ker\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

After normalization, pick $v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ where $\sqrt{2} = \left\|\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\|$

s.t. $A = UDU^*$ where $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

and $D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$

④ Find B, s.t. $B^2 = A$

- let B be a 2x2 matrix, s.t. $\boxed{B^2 = A = UDU^*}$

$U^*B^2U = U^*UDU^*U = IDI = D$

hence $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} = D = U^*B^2U = (U^*BU)^2$

s.t. $U^*BU = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm\sqrt{5} \end{pmatrix}$, hence $B = U\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm\sqrt{5} \end{pmatrix}U^*$

- verify them by yourself, $B^2 = \cdots = A$. $\blacksquare$

2. Show that the rotation matrix $R_\alpha = \begin{pmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{pmatrix}$ is normal.

N is normal if $N^*N = NN^*$

Let $\alpha \in \mathbb{R}$

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$$R_\alpha^* = \bar{R_\alpha}^T = \begin{pmatrix} \overline{\cos\alpha} & \overline{\sin\alpha} \\ -\overline{\sin\alpha} & \overline{\cos\alpha} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

$$R_\alpha R_\alpha^* = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$R_\alpha^* R_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = R_\alpha R_\alpha^*$$

s.t. R_α is normal. $\blacksquare$

Remark: Therefore, it is possible to orthogonally diagonalize R_α. (HW10)

3. Let A be $m \times n$ matrix. Prove that
   - (a) A^*A is self-adjoint
   - (b) All eigenvalues of A^*A are non-negative.
   - (c) $A^*A + I$ is invertible. Hint: Show that $\ker(A^*A + I) = \{0\}$

(a) $(A^*A)^* = \overline{(\bar{A}^T A)}^T = (\bar{\bar{A}}^T \bar{A})^T = (A^T\bar{A})^T$

$= \bar{A}^T(A^T)^T = \bar{A}^T A = A^*A$

or use the fact that $(AB)^* = B^*A^*$, and $(A^*A)^* = A^*(A^*)^* = A^*A$.

then A^*A is self-adjoint.

(b) let λ be an eigenvalue of A^*A corresponding an eigenvector $v \neq 0$

consider $\boxed{(A^*Av, v) = (Av, Av)}$ ☆ by the definition of adjoint.

≥ 0 by non-negativity of inner product.

and LHS $= (A^*Av, v) = (\lambda v, v)$ by assumption.

$= \lambda(v, v)$ by linearity of inner product.

s.t. $\lambda(v,v) \geq 0$ where $v \neq 0$, s.t. by non-degeneracy and non-negativity, $(v,v) > 0$

hence $\lambda \geq 0$ as required.

(c) WTS $\ker(A^*A+I) = \{0\}$

(⊆) let $x \in \ker(A^*A+I)$, WTS $x \in \{0\}$

consider $((A^*A+I)x, x) = (A^*Ax + x, x)$

$= (A^*Ax, x) + (x, x)$ by linearity.

$= (Ax, Ax) + (x, x)$ by def of adjoint.

and LHS $= ((A^*A+I)x, x)$

$= (0, x)$ b/c $x \in \ker(A^*A+I)$

zero vector

$= (0 \cdot x, x) = 0(x, x)$ by linearity.

zero scalar

$= 0$

hence $0 = (Ax, Ax) + (x, x) \geq (x, x)$ b/c $(Ax, Ax) \geq 0$ by non-negativity.

and $(x,x) \geq 0$ by non-negativity,

s.t. $0 \geq (x,x) \geq 0$, hence $(x,x) = 0$

s.t. $x = 0$ by non-degeneracy. hence $x \in \{0\}$.

(⊇) let $x \in \{0\}$, s.t. $x = 0$, WTS $x \in \ker(A^*A+I)$

$(A^*A+I)x = (A^*A+I)0 = 0$

hence $\ker(A^*A+I) = \{0\}$, s.t. A^*A+I is invertible. $\blacksquare$