Linear transformation, coordinate and matrix representation

Let $T : V \to W$ be a linear transformation. Show that $T(I)$ is linearly independent for any linearly independent finite subset I of V if and only if $ker(T) = \{0\}$.

Proof. The statement is an if and only if statement, so we will prove the following two directions.

($\Rightarrow$)

Assume that for any linearly independent finite subset I of V, $T(I)$ is also linearly independent. We want to show that $ker(T) = \{0\}$.

Since $T(0) = 0$ by T being a linear transformation, we know that $0 \in ker(T)$. Therefore $\{0\} \subseteq ker(T)$.

Let $v \in ker(T)$, then $T(v) = 0$. We want to show that $v = 0$.

Assume $v \neq 0$ for the sake of contradiction, then we know that $\{v\}$ is a linearly independent finite subset of V. Then by the assumption, $\{T(v)\}$ is linearly independent, but we know that $T(v) = 0$, such that $\{0\}$ is a linearly independent.

However, we know that a set containing the zero vector is linearly dependent, therefore we have a contradiction. As a result, $v = 0$ and hence $ker(T) \subseteq \{0\}$.

Therefore $ker(T) = \{0\}$

($\Leftarrow$)

Assume $ker(T) = \{0\}$. Let $I = \{v_1, ..., v_n\}$ be a finite linearly independent subset of V. We want to show that $T(I) = \{T(v_1), ..., T(v_n)\}$ is linealry independent. Assume $a_1T(v_1) + ... + a_nT(v_n) = 0$ for some $a_i \in \mathbb{F}$, then

$$\begin{aligned}
a_1T(v_1) + ... + a_nT(v_n) &= 0 \\
T(a_1v_1 + ... + a_nv_n) &= 0 \text{ by T being a linear transformation} \\
a_1v_1 + ... + a_nv_n &= 0 \text{ because } a_1v_1 + ... + a_nv_n \in ker(T) = \{0\} \\
a_1 = ... = a_n &= 0 \text{ by } I \text{ being linearly independent}
\end{aligned}$$

Therefore $T(I)$ is linearly independent.

$\square$