Double Integrals over General Regions:

1. **Double Integral of f over rectangle R (15.1.5)**

$$\iint_R f(x,y)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

   if this limit exists.

2. **Double Integral of f over region D (15.2.2)**

$$\iint_D f(x,y)dA = \iint_R F(x,y)dA$$

   where

$$F(x,y) = \begin{cases} f(x,y) & if(x,y) \in D \\ 0 & if(x,y) \in R \setminus D \end{cases}$$

3. **Type I and II Region D (15.2.3/4/5)**
   If f is continuous on a region D such that,
   $D = (x,y)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$ **Type I**,
   and also $D = (x,y)|c \leq y \leq d, h_1(y) \leq x \leq h_2(y)$ **Type II**, then

$$\iint_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

1 Graphing and Intuition:

Recall, a strategy to solve a problem is to get intuition, set up model and do computation. For simplification, let's consider the question in three dimension only. Consider the following examples and give yourself intuitions why they work and/or what do they look like.

1. Circle centered at (a,b,c) with radius r: $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$
2. Cylinders: $x^2 + y^2 = r^2$; $x^2 + z^2 = r^2$; $y^2 + z^2 = r^2$
3. Tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$ and $z = 0$
4. The parabolic cylinder $y = x^2$

At the beginning stage, I highly recommend you to get the visualization of the region using the 3D graphing from GeoGebra.

2 Setup Integration Model:

According to the intuition that you get from last step, you should be able to set up the model now. One type of question is to find the volume of a given region. For example, find the volume of the region, under the plane $3x + 2y - z = 0$ and above the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

- One way is to consider the function z=f(x,y) where the domain D on the xy-plane by looking at the graph, so that the region is $\{(x, y, f(x, y)) : (x, y) \in D\}$.
- Algebraically, we can define the function f by considering the equations as constraints that (x,y,z) has to satisfy, then isolate z, if possible.

Question: Find the volume of the solid bounded by the cylinders $x^2+y^2 = 4$ and $y^2+z^2 = 4$.
Question: Evaluate the integral $\int_0^2 \int_{2x}^4 \frac{y}{y^3+1} dydx$

3 Computation:

The next step is to compute, once you have set up the model. There are a few strategies(you have known many of them) to smoothly(at least doable) compute the integration.

- **Substitution Rule:** If u = g(x) is a differentiable function whose range is an interval I and f is continuous on I, then $\int f(g(x))g'(x)dx = \int f(u)du$
- **Integration by parts:** $\int udv = uv - \int vdu$
- **Fubini's Theorem:**
  If f is continuous on the rectangle $R = \{(x, y) = |a \le x \le b, c \le y \le d\}$, then
$$\iint_R f(x, y)dA = \int_a^b \int_c^d f(x, y)dydx = \int_c^d \int_a^b f(x, y)dxdy$$

Notice: The range of the integration may vary when changing the order, if R not a rectangle.

4 Some properties of Double Integrals:

- If $D = D_1 \cup D_2$ and $D_1 \cap D_2 = \emptyset$, $\iint_D f(x, y)dA = \iint_{D_1} f(x, y)dA + \iint_{D_2} f(x, y)dA$
- $Area(D) := A(D) = \iint_D 1dA$
- If $m \le f(x, y) \le M$ for all (x,y) in D, then
$$mA(D) \le \iint_D f(x, y)dA \le MA(D)$$