MATB42_TUT5_First_and_Second_order_ODE

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MATB42_TUT5_First_a...

Second order linear equations

When solving the general solution of the heat equation using the method of separation of variables, we have two linear equations $T' + \lambda kT = 0$ and $X'' + \lambda X = 0$ where $\lambda > 0$. Using the knowdledge of ordinary differential equations, we solved that

Differential eqn of 1-variable function

$$\boxed{T(t) = Ae^{-\lambda kt}}$$

and

$$\boxed{X(x) = Ccos(\sqrt{\lambda}x) + Dsin(\sqrt{\lambda}x)}$$

The details are as follow: To solve a second order differential equation $y(x)$

$$ay'' + by' + cy = 0$$ where a, b, c are constants

start with the guess $y = e^{rx}$ for some constant r. Plug into the differential equation, and ...

$$0 = ay'' + by' + cy = r^2ae^{rx} + rbe^{rx} + ce^{rx}$$

where $e^{rx} > 0$ for all x

$$0 = e^{rx}(ar^2 + br + c) \Rightarrow ar^2 + br + c = 0$$

Quadratic eqn of r which we know how to solve using quadratic formula.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

real, distinct

real, equal

complex

$\Downarrow$

$y = e^{rx}$ is a solution, given we solved r

1st derivative

Examples: $T' + \lambda kT = 0$, claim: $T(t) = Ae^{-\lambda kt}$

start with guess, $T(t) = e^{rt}$

$$re^{rt} + \lambda k e^{rt} = 0$$

$$r + \lambda k = 0 \Rightarrow r = -\lambda k$$

$$\Rightarrow T(t) = e^{-\lambda kt}$$

$\Rightarrow$ the general solution is $T(t) = Ae^{-\lambda kt}$, A is constant

Examples: $X'' + \lambda X = 0$, claim: $X(x) = C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$

2nd derivative!

start with guess, $X(x) = e^{rx}$

$$r^2e^{rx} + \lambda e^{rx} = 0$$

$$r^2 + \lambda = 0$$

$$r^2 = -\lambda < 0 \quad \text{b/c } \lambda > 0$$

$$r = \pm\sqrt{-\lambda} = \pm i\sqrt{\lambda}, \quad i = \sqrt{-1} \text{ or } i^2 = -1$$

when $r = i\sqrt{\lambda}$

$$X(x) = e^{rx} = e^{i\sqrt{\lambda}x}$$

$$= \cos(\sqrt{\lambda}x) + i\sin(\sqrt{\lambda}x)$$

Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$

since $\sin(\sqrt{\lambda}x)$, $\cos(\sqrt{\lambda}x)$ "independent" to each other

plug in eqn

$\Rightarrow$ both $X(x) = \cos(\sqrt{\lambda}x)$, $X(x) = \sin(\sqrt{\lambda}x)$ are solutions.

$\Rightarrow$ General solution is $X(x) = C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$ ☒