MATB44_TUT_6

Tuesday, November 3, 2020 11:08

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Idea:
Find the eigenvalues λ_i, and then find the corresponding eigenvectors ξ_i

If all distinct, $x = \sum_{i=1}^{n} c_i e^{\lambda_i t} \xi_i$

Systems of First-order Linear Equation

To solve a nxn system of the form $\mathbf{x}'=\mathbf{A}\mathbf{x}$. Assume that $\mathbf{A}$ is a real-valued matrix. Please refer back to the course note for the algorithms in general.

- To find the eigenvalues of A: Solve the characteristic equation $det(A - \cancel{\lambda} rI) = 0$
- To find the eigenvectors of A corresponding to the eigenvalue $\cancel{\lambda}$ r: Find one non-zero vector of $Ker(A - \cancel{\lambda} rI)$, or equivalently solve $A\xi = r\xi$ for ξ
- To find the generalized eigenvector of A corresponding to the eigenvalue λ and the eigenvector ξ: Solve $\xi + r\eta = A\eta$ for η

$A\xi - r\xi = 0$

$(A-rI)\xi = 0$

$Ker(A-rI)$

Question:

Solve the linear systems with constant coefficients

$$x' = \begin{pmatrix} 1 & 2 \\ -4.5 & 1 \end{pmatrix} x$$

System of first order linear eqn.

Let $A = \begin{pmatrix} 1 & 2 \\ -4.5 & 1 \end{pmatrix}$

① Find the eigenvalues: $\det(A - rI) = 0$

$$0 = \det(A - rI) = \det\begin{pmatrix} 1-r & 2 \\ -4.5 & 1-r \end{pmatrix} = (1-r)(1-r) - 2\cdot(-4.5)$$

$$= 1 - 2r + r^2 + 9 = r^2 - 2r + 10$$

By quadratic equation,

$$r = \frac{2 \pm \sqrt{4 - 40}}{2} = \frac{2 \pm \sqrt{-36}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

② Let's consider $r = 1+3i$ and find the corresponding eigenvector.

Find one non-zero vector in $\boxed{Ker(A-rI)}$

or equivalently, solve $\boxed{(A-rI)\xi = 0}$ for ξ

$$A - rI = \begin{pmatrix} 1 & 2 \\ -4.5 & 1 \end{pmatrix} - (1+3i)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -3i & 2 \\ -4.5 & -3i \end{pmatrix}$$

(1) Using linear algebra,

$$\left(\begin{array}{cc|c} -3i & 2 & 0 \\ -4.5 & -3i & 0 \end{array}\right) \sim$$

OR (2) Solve the system, $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$.

$$\begin{cases} -3i\xi_1 + 2\xi_2 = 0 \\ -4.5\xi_1 - 3i\xi_2 = 0 \end{cases}$$

Are the same!!

Notice $\begin{vmatrix} -3i & 2 \\ -4.5 & -3i \end{vmatrix} = (3i)^2 + 9 = -9 + 9 = 0$

so that the second equation is redundant.

so let's just consider $-3i\xi_1 + 2\xi_2 = 0$

when $\xi_1 = 1$ in particular,

we have $\xi_2 = \frac{3}{2}i$

s.t. $\begin{pmatrix} 1 \\ \frac{3}{2}i \end{pmatrix}$ is an eigenvector

③ Separate the real and imaginary parts:

$$e^{rt}\xi = e^{(1+3i)t}\begin{pmatrix} 1 \\ \frac{3}{2}i \end{pmatrix}$$

Notice $e^{rt} = e^{(1+3i)t} = e^t(\cos 3t + i\sin 3t)$ by Euler formula.

s.t. $e^{(1+3i)t}\begin{pmatrix} 1 \\ \frac{3}{2}i \end{pmatrix} = e^t(\cos 3t + i\sin 3t)\begin{pmatrix} 1 \\ \frac{3}{2}i \end{pmatrix}$

$$= \begin{pmatrix} e^t\cos 3t + i\sin 3t\, e^t \\ e^t(\frac{3}{2}\cos 3t)i - \frac{3}{2}\sin 3t\, e^t \end{pmatrix}$$

$$= e^t\begin{pmatrix} \cos 3t \\ -\frac{3}{2}\sin 3t \end{pmatrix} + i e^t\begin{pmatrix} \sin 3t \\ \frac{3}{2}\cos 3t \end{pmatrix}$$

so that

$$x = c_1 e^t\begin{pmatrix} \cos 3t \\ -\frac{3}{2}\sin 3t \end{pmatrix} + c_2 e^t\begin{pmatrix} \sin 3t \\ \frac{3}{2}\cos 3t \end{pmatrix} \blacksquare$$