Definitions

You should always check the lecture note and the textbook for the definition, and make sure the definitions are consistents.

1. We call $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ the coordinate vector of $v \in V$ with respect to the basis $(v_1, \cdots, v_n)$ of V if $v = a_1 v_1 + \cdots + a_n v_n$ for some $a_1, \cdots, a_n \in \mathbb{F}$. Denoted as $[v]_{v_1, \cdots, v_n}$

2. Let V, W be vector spaces over $\mathbb{F}$. Let $T : V \to W$ be a linear transformation. Let $\alpha = (v_1, \cdots, v_n)$ and $\beta = (w_1, \cdots, w_m)$ be ordered basis of V and W respectively. Then the matrix representation of T with respect to α and β is

$$[T]^{v_1, \cdots, v_n}_{w_1, \cdots, w_m} = \begin{bmatrix} \\ [T(v_1)]_{w_1, \cdots, w_m} & \cdots & [T(v_n)]_{w_1, \cdots, w_m} \\ \\ \end{bmatrix}$$

Remark. *When the question asks you the find the matrix of a given linear transformation $T : V \to W$ with respected to some bases as we have seen in the assignment, you are NOT asked to find a matrix A, such that $T(v) = Av$. You are asked to find the matrix representation as defined. While sometimes it is true that the matrix representation, let's call it A, will result $T(v) = Av$ for all $v \in V$, this is not always true. The Av is a matrix multiplication and we have seen vectors that are not vectors in $\mathbb{R}^n$.*

Examples: Coordinate vector

1. We will start with the familiar one $\mathbb{R}^n$. In next few questions, consider the vector space $V = \mathbb{R}_4$ with the usual operations.

   (a) Circle the elements which belong the vector space V

   $$(1,0,0,0)^T \qquad 1 \qquad (2,9,0,5)^T \qquad (1,i,0,0)^T \qquad (3,4,0)^T \qquad 5\cdot(1,4,0,0)^T$$

   $$1+x \qquad \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad (1,x,x^2,x^3)^T \qquad 3+x^2+x^3 \qquad 4x+i \qquad \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$$

   (b) What is the standard (ordered) basis for V? Call it α.

   (c) We know that α spans the vector space because it is a basis. For the elements above that are in the vector space, write those as linear combinations of the vectors in the basis. Find $[v]_\alpha$

   (d) Known $\beta = ((1,1,1,1)^T, (1,1,1,0)^T, (1,1,0,0)^T, (1,0,0,0)^T)$ is an ordered basis of V. Repeat the previous subquestion with β.

2. In the next few questions, consider the vector space $V = \mathbb{P}_3(\mathbb{R})$ with the usual operations.

   (a) Circle the elements which belong the vector space V

   $$(1,0,0,0)^T \qquad 1 \qquad (2,9,0,5)^T \qquad (1,i,0,0)^T \qquad (3,4,0)^T \qquad 5\cdot(1,4,0,0)^T$$

   $$1+x \qquad \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix} \qquad (1,x,x^2,x^3)^T \qquad 3+x^2+x^3 \qquad 4x+i \qquad \begin{pmatrix}1 & 4\\ 0 & 1\end{pmatrix}$$

   (b) What is the standard (ordered) basis for V? Call it α.

   (c) We know that α spans the vector space because it is a basis. For the elements above that are in the vector space, write those as linear combinations of the vectors in the basis. Find $[v]_\alpha$

   (d) One can check that $\beta = (2, 1+x, 1+x+x^2, 1+x+x^2+x^3)$ is an ordered basis of V. Repeat the previous subquestion with β.

3. In the next few questions, consider the vector space $V = M_n(\mathbb{R})$ with the usual operations.

   (a) Circle the elements which belong the vector space V

   $$(1,0,0,0)^T \quad 1 \quad (2,9,0,5)^T \quad (1,i,0,0)^T \quad (3,4,0)^T \quad 5\cdot(1,4,0,0)^T$$

   $$1+x \quad \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix} \quad (1,x,x^2,x^3)^T \quad 3+x^2+x^3 \quad 4x+i \quad \begin{pmatrix}1 & 4\\ 0 & 1\end{pmatrix}$$

   (b) What is the standard (ordered) basis for V? Call it α.

   (c) We know that α spans the vector space because it is a basis. For the elements above that are in the vector space, write those as linear combinations of the vectors in the basis. Find $[v]_\alpha$

   (d) One can check that $\beta = (\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix})$ is an ordered basis of V. Repeat the previous subquestion with β.

Examples: Matrix representation of linear transformation

1. In the next few questions, consider the following linear transformation.

$$T : V \to W \text{ by } T(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}) = \begin{pmatrix} v_1 + 2v_2 \\ v_2 \\ v_1 \end{pmatrix}$$

where $V = \mathbb{R}^2, W = \mathbb{R}^3$

(a) Compute $T(\begin{pmatrix} 2 \\ 3 \end{pmatrix})$

(b) Let α and β be the standard bases of V and W respectively. Find the matrix representation $[T]_\beta^\alpha$ by our definition and notation.

(c) Let $\gamma = (\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix})$ which is another ordered basis of V. Find the matrix representation $[T]_\beta^\gamma$ by our definition and notation.

(d) Let $\delta = (\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix})$ which is another ordered basis of W. Find the matrix representation $[T]_\delta^\alpha$ by our definition and notation.

2. In the next few questions, consider the following linear transformation.

$$T : V \to W \text{ by } T(f) = \begin{pmatrix} f'(2) \\ f(0) \end{pmatrix}$$

where $V = \mathbb{P}_2(\mathbb{R}), W = \mathbb{R}^2$

(a) Compute $T(1 + x)$ and $T(x^2 + x^3)$

(b) Let α and β be the standard bases of V and W respectively. Find the matrix representation $[T]^{\alpha}_{\beta}$ by our definition and notation.

(c) Let $\gamma = (1, 1 + x, 1 + x + x^2)$ which is another ordered basis of V. Find the matrix representation $[T]^{\gamma}_{\beta}$ by our definition and notation.

(d) Let $\delta = (\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix})$ which is another ordered basis of W. Find the matrix representation $[T]^{\gamma}_{\delta}$ by our definition and notation.