Examples on Isomorphisms

Thursday, November 12, 2020 15:50

① Let $V = P_2(\mathbb{R})$, let $v_1 = t$, $v_2 = 1+t^2$, $v_3 = 1-t$ where $B = \{v_1, v_2, v_3\}$ is a basis of V

~~Describe explicitly~~ the coordinate isomorphism $V \cong \mathbb{R}^3$ given by B.

given arbitrary input, where do we map it to?

$V \to \mathbb{R}^n$

$[f(t)]_A$

change of basis matrix $\mathbb{R}^n \to \mathbb{R}^n$.

Pf: Let $f(t) = a + bt + ct^2 \in P_2(\mathbb{R})$

def $[f(t)]_B = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$

known $f(t) = k_1 v_1 + k_2 v_2 + k_3 v_3$ for some k_1, k_2, k_3 b/c B form a basis of V

$$a + bt + ct^2 = k_1 t + k_2(1+t^2) + k_3(1-t)$$
$$= (k_2+k_3) + (k_1 - k_3)t + k_2 t^2$$

s.t $\begin{cases} a = k_2 + k_3 & ① \\ b = k_1 - k_3 & ② \\ c = k_2 & ③ \end{cases}$

sub ③ into ①

$k_3 = a - c$ ④

sub ④ into ②, $b = k_1 - a + c \Rightarrow k_1 = b - c + a$

s.t $[f(t)]_B = \begin{pmatrix} b-c+a \\ c \\ a-c \end{pmatrix}$ ∎

② Let P be an $n \times n$ matrix. Prove the function $A \overset{T}{\mapsto} PAP^{-1}$ from $M_{n\times n}$ to $M_{n\times n}$ is an isomorphism.

Linear.

Invertible

(1) WTS T is linear

Let $A, B \in M_{n\times n}$, $a, b \in F$

$T(aA + bB) = P(aA+bB)P^{-1}$ def of T

$= aPAP^{-1} + bPBP^{-1}$ properties of matrix multiplication and b/c $a, b \in F$

$= aT(A) + bT(B)$

s.t T is a linear transformation.

(2) WTS T is invertible

① WTS T is 1-1

Assume $T(A) = T(B)$ WTS $A = B$

By assumption, $PAP^{-1} = PBP^{-1}$

$$P^{-1}PAP^{-1}P = P^{-1}PBP^{-1}P$$
$$IAI = IBI$$
$$A = B$$

Def of onto

$T: V \to W$

$\forall B \in W$, $\exists A \in V$, s.t $T(A) = B$

② WTS T is onto

let $B \in M_{n\times n}$, WTS $\exists A \in M_{n\times n}$, s.t $T(A) = B$

pick $A = (P^{-1}BP) \in M_{n\times n}$.

then $T(A) = P(P^{-1}BP)P^{-1} = IBI = B$

Therefore, T is an isomorphism ∎