MATB61 TUT3/4 Final Review
Understanding Definition and Theory Behind

Nick Huang

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This document is for the students in MATB61, TUT0003 and TUT0004 winter 2021 at the University of Toronto Scarborough. You should not use this document as your reference in the final exam. Everything covered in this document have been talked about in the lectures or in the textbook. The purpose of this document is for students to do more practices at various types of questions that they have seen in class. Also some questions are designed for students to detect the mistakes that the questions have by the definitions of the concepts. This document may not covered all materials that will appear in the final exam.
Note: Make sure to review the lecture notes, tutorial notes and tutorial worksheets, which contains many valuable and classical questions that are not included in this document but you should be comfortable doing. This document focus on the materials after the midterm.

Contents

1 LPPs, Feasible Region and Convexity

1. Let $a \in \mathbb{R}^n$. Show that the function $f(x) = a^T x$ is a linear transformation.

2. Show that for any linear transformation $f : \mathbb{R}^n \to \mathbb{R}$, there exists a vector $a \in \mathbb{R}^n$ such that $f(x) = a^T x$ for all $x \in \mathbb{R}^n$.

3. Determine whether the following problem can be equivalent to a linear programming problem.

$$\max z = x + 2y$$
$$\text{subject to}$$
$$|x| - |y| \leq 1$$

4. Consider a LPP in its standard form with the usual notation. Show that if x_1, x_2 are feasible solutions to the LPP, and a, b are some non-negative constants such that $a + b = 1$, then $x = ax_1 + bx_2$ is also a feasible solution.

5. Consider a LPP in its standard form. Show that if x_1, x_2 are optimal solutions (not necessarily distinct) to the LPP, then any convex combination of x_1, x_2 is also an optimal solution.

6. Consider the following questions:

   (a) Is it true that the optimal solution will never occur in an interior point?

   (b) Find a sufficient condition such that the following statement is true. Prove it.
   If (sufficient condition), then the optimal solution will never occur in an interior point.

7. Answer the following true/false questions for arbitrary LPP, and give explicit examples if the statement is false.

   (a) Removing one of the constraint of a given LPP will always result a feasible region that is smaller than the original one.

   (b) If the feasible region is unbounded, then the LPP has no optimal solution and has infinite cost.

   (c) If an optimal solution exists, then every optimal solution must be an extreme point of the feasible region.

(d) It is impossible to have two distinct optimal cost.

(e) It is impossible to have two distinct optimal solution.

(f) The number of optimal solution is always finite.

(g) If the feasible region is empty, then the number of optimal solution is finite.

(h) There exists a LPP with exactly two distinct optimal solutions.

(i) If the feasible region is bounded, then the optimal solution must exists.

(j) The origin is always an extreme point.

2 Simplex Algorithm

1. True/false questions with justifications. Consider a linear programming problem with the usual notation in its standard form

$$\max z = c^T x$$
$$\text{subject to}$$
$$Ax \leq b$$
$$x \geq 0$$

where A is an $m \times n$ matrix, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$

(a) Once we convert the LPP to the canonical form by introducing the slack variable, introducing the slack variable as one of the basic variables when running the simplex algorithm will never result a change of the cost.

(b) The entries of a basic feasible solution must be positive.

(c) The two-phase method will always give us a BFS of the original LPP.

(d) The simplex algorithm will always give an optimal solution to the original LPP if the LPP has non-empty feasible region.

(e) There exists a LPP, such that all the basic solutions are basic feasible solutions.

2. Prove that if the auxiliary problem has an optimal solution with $x_0 \neq 0$ when using the two-phase method, then the orginal LPP is infeasible.

3 Duality

1. True/false questions with justifications. Consider a linear programming problem with the usual notation.

   (a) If the primal problem has n many decision variables, then the dual problem also has n many decision variables.

   (b) If the primal problem has an optimal solution at x with the cost z, then z is also an optimal solution to the dual problem with the same optimal cost.

   (c) If the primal has unbounded feasible region, then the dual problem must be infeasible.

   (d) One of the primal and dual must have a non-empty feasible region.

2. Prove that if a maximization primal problem has infinite cost, then the dual problem must be infeasible using the weak duality theorem.

4 Working with New Definition

1. Consider the following definition for the next question.

   Definition 1. *For a linear programming problem with feasible region $P \subseteq \mathbb{R}^n$, we say that d is a feasible direction at $x \in P$, if there exists $a \in \mathbb{R}^+$, such that $x + ad \in P$.*

   (a) Consider the following LPP in its canonical form.

   $$\begin{aligned} \max z &= c^T x \\ \text{subject to} & \\ Ax &= b \\ x &\geq 0 \end{aligned}$$

   where c, A, b are fixed and $c \in \mathbb{R}^n$. Prove that $d \in \mathbb{R}^n$ is a feasible direction if and only if $Ad = 0$ and $d_i \geq 0$ for any $i \in \{1, ..., n\}$ such that $x_i = 0$.