Learning Objective

Understand the definitions of basic solutions and be able to use the idea of basic solution and extreme points to solve a given linear programming problem.

Brief summary

Here is a brief summary of the idea of using basic solution to solve a given LPP, and you should read the detailed proof in the textbook to understand the theory behind. Explicit examples on the conditions that are required in order to use the method would be discussed in the worksheets. You will see why the conditions are necessary in the worksheet.

- By theorem 1.8 in the textbook, we can find some extreme points by solving the corresponding argumented matrix with a focus on some linear independent columns of the matrix A.
- By theorem 1.9 in the textbook, given an extreme point, we can find some linear independent columns of A respectively, and extend it to a basis of $\mathbb{R}^m$ by corollary 1.1, or by what we have learned in linear algebra. Theorem 1.10, guarantees that the rest of the components of the given extreme point, other than those components that corresponde to the linearly independent columns of A that form a basis, must be zero.
- Given any set of linear independent columns of A that is a basis, there exists an unique solution to the system $A'x' = b$ as we have learned in linear algebra, where A' is the matrix consisting those independent columns, and x' is the variable respected to the corresponding columns.

  Let the values of the variables that are in x' be the values of the corresponding variables in x, set rest of the variables that are not in x' but in x zeroes, we call any such x a basic solution, which will be a solution of the original LPP in canonical form.
- Those basic solution that satisfy the constraints of the given LPP, is called a basic feasible solution. In fact, by construction, any basic solution would satisfy $Ax = b$, so it suffices to check whether those basic solutions satisfy $x \geq 0$.
- Combining all the above, we concluded (thm 1.11), for a linear programming problem in its canonical form, the set of all basic feasible solutions is equal to the set of all extreme points.
- Check whether the condition of extreme point theorem is satisfied, if yes, then the optimal solution exists at one of the extreme point, which is one of the basic feasible solution, and we know how to find it.

Method

Given the theory, we now know how to solve the given canonical form by finding all the basic feasible solutions, if the optimal solution exists and hence the optimal solution would be one of the extrme point by extreme point theorem.
Here are the steps

1. Convert the given LPP into canonical form and write it in matrix notation.
2. If $b \in \mathbb{R}^n$, find n linear independent columns of A, which will form a basis of $\mathbb{R}^n$, solve $A'x' = b$, where A' is the matrix with those columns, and x' is the variable corresponding to those columns. Then x will be basic solution by setting the values that match those in x', and other components zeroes.
3. Repeat the above step for all possible selections of linear independent columns. Orders do not matter, because they will result the same $A'x = b$ equivalently.
4. Once you have all the basic solutions, check the constraints $x \geq 0$ to only consider thoses basic feasible solutions.
5. Once you have all the basic feasible solutions, those are all the extreme points. If the optimal solution exists by the extreme point theorem, then one of those basic feasible solutions would be an optimal solution. Evaluate the optimal values at each to find the optimal solution after.

Example

This is an example from your midterm. One can solved this by drawing a graph, but given we have three variables, it could be hard to draw a nice graph. Instead, we can solve this using the idea of basic solutions.
Consider the following LPP

max $z = x_1 + 2x_2 - 3x_3$
subject to
$x_2 \leq 4$
$2x_1 + 3x_2 - 4x_3 \leq 24$
$x_1, x_2, x_3 \geq 0$

Proof. One can notice after drawing the feasible region and calculating the gradient, that the optimal solution exists , such that by the extreme point theorem (thm1.7), an optimal solution to the problem exists and occurs at an extreme point. By therem 1.11, the set of all basic feasible solution is equal to the set of all extreme points, such that we would solve this linear programming problem by finding all the basic feasible solutions.
In order to find the basic feasible solution, we must first convert the given LPP to canonical form and we get the following canonical form of the LPP.

max $z = x_1 + 2x_2 - 3x_3$
subject to
$x_2 + u = 4$
$2x_1 + 3x_2 - 4x_3 + v = 24$
$x_1, x_2, x_3, u, v \geq 0$

and in matrix notation,

max $z = c^T x$
subject to
$Ax = b$

$$x \geq 0$$

where $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ u \\ v \end{pmatrix}, c = \begin{pmatrix} 1 \\ 2 \\ -3 \\ 0 \\ 0 \end{pmatrix}, A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 2 & 3 & -4 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 4 \\ 24 \end{pmatrix}$

1. We start by finding the basic solution of this canonical LPP, which is to solve the argumented matrix $\left(A' \mid b\right)$ where A' is a 2x2 matrix consisting any two linear independent columns of A. Therefore we have the following possibilities.

   (a) $\left(\begin{array}{cc|c} 0 & 1 & 4 \\ 2 & 3 & 24 \end{array}\right)$ which gives $x_1 = 6, x_2 = 4$, such that $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ u \\ v \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

   (b) $\left(\begin{array}{cc|c} 0 & 1 & 4 \\ 2 & 0 & 24 \end{array}\right)$ which gives $x_1 = 12, u = 4$, such that $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ u \\ v \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 0 \\ 4 \\ 0 \end{pmatrix}$

   (c) $\left(\begin{array}{cc|c} 1 & 0 & 4 \\ 3 & -4 & 24 \end{array}\right)$ which gives $x_2 = 4, x_3 = -3$, such that $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ -3 \\ 0 \\ 0 \end{pmatrix}$

   (d) $\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 3 & 0 & 24 \end{array}\right)$ which gives $x_2 = 8, u = -4$, such that $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 0 \\ -4 \\ 0 \end{pmatrix}$

   (e) $\left(\begin{array}{cc|c} 1 & 0 & 4 \\ 3 & 1 & 24 \end{array}\right)$ which gives $x_2 = 4, v = 12$, such that $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 12 \end{pmatrix}$

   (f) $\left(\begin{array}{cc|c} 0 & 1 & 4 \\ -4 & 0 & 24 \end{array}\right)$ which gives $x_3 = -6, u = 4$, such that $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -6 \\ 4 \\ 0 \end{pmatrix}$

(g) $\left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 24 \end{array}\right)$ which gives $u = 4, v = 24$, such that $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 24 \end{pmatrix}$

This gives 7 basic solutions.

2. We are interested in the corresponding basic feasible solutions, which are the basic solutions that satisfy the constraints of the given LPP. By construction, those basic solutions already satisfy the condition that $Ax = b$, so it suffices to check whether we have $x \geq 0$. Therefore, we have the following feasible solutions.

(a) $x = \begin{pmatrix} 6 \\ 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ which gives $z = x_1 + 2x_2 - 3x_3 = 14$

(b) $x = \begin{pmatrix} 12 \\ 0 \\ 0 \\ 4 \\ 0 \end{pmatrix}$ which gives $z = x_1 + 2x_2 - 3x_3 = 12$

(c) $x = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 12 \end{pmatrix}$ which gives $z = x_1 + 2x_2 - 3x_3 = 8$

(d) $x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 24 \end{pmatrix}$ which gives $z = x_1 + 2x_2 - 3x_3 = 0$

Therefore the optimal solution is

$$x = \begin{pmatrix} 6 \\ 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and the optimal value is $z = 14$ $\square$

Additional practices

Solve the following LPP using the method that we discussed on the first page.

max $z = 2x_1 + 4x_2$

subject to

$5x_1 + 3x_2 + 5x_3 \leq 15$

$10x_1 + 8x_2 + 15x_3 \leq 40$

$x_1, x_2, x_3 \geq 0$

Make sure to check whether the solution exists, and if it exists, where should it be before doing the calculation. You will need the extreme point theorem to support your argument. Draw a graph if needed.

Verify that the feasible region is non-empty and bounded.