Useful facts

0.1 Self-adjoint

1. Theorem 2.2, chapter 6: Let $A = A^*$ be a self-adjoint (or sometimes called Hermitian) matrix. Then A is unitarily equivalent to a diagonal matrix with real entries, i.e. $A = UDU^*$ where U is a unitary matrix and D is a diagonal matrix with real entries. If A is a real matrix, then U can be chosen to be real.

2. Proposition 6.5, chapter 5: A matrix A is unitarily equivalent to a diagonal one if and only if it has an orthonormal basis of eigenvectors.

3. Recall that $U^{-1} = U^*$ for a unitary matrix U.

Remark. *For a given self-adjoint matrix A, we can orthogonally diagonalize it, such that $A = UDU^{-1}$ where U is the matrix with the orthonormal basis of eigenvectors as the columns. The matrix D is always real.*

0.2 Normal

1. A linear transformation $N : X \to X$ where X is an inner product space is called normal if $N^*N = NN^*$.

2. Theorem 2.4, chapter 6: Any normal matrix N in a complex vector space has an orthonormal basis of eigenvectors. In other words, such N can be represented as $N = UDU^*$ where U is a unitary matrix, and D is a diagonal matrix.

Remark. *For a given normal matrix A, we can orthogonally diagonalize it, such that $N = UDU^{-1}$ where U is the matrix with the orthonormal basis of eigenvectors as the columns. The matrices U and D may not be real, even if N is real.*

Questions

1. Orthogonally diagonalize the matrix $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$. Find all the square roots of A, i.e. find all the matrices B such that $B^2 = A$.

2. Show that the rotation matrix $R_\alpha = \begin{pmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{pmatrix}$ is normal.

3. Let A be $m \times n$ matrix. Prove that
   (a) A^*A is self-adjoint
   (b) All eigenvalues of A^*A are non-negative.
   (c) $A^*A + I$ is invertible. Hint: Show that $\ker(A^*A + I) = \{0\}$