MATB61 TUT3/4 Midterm Review

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This document is for the students in MATB61, TUT0003 and TUT0004 winter 2021 at the University of Toronto Scarborough. You should not use this document as your reference in the midterm. Everything covered in this document have been talked about in the lectures or in the textbook. The purpose of this document is for students to do more practices at various types of questions that they have seen in class. Also some questions are designed for students to detect the mistakes that the questions have by the definitions of the concepts. This document may not covered all materials that will appear in the midterm.

Contents

1 Forms of Linear Programming Problem

1.1 Set up a LPP

We can set up a linear programming problem model to describe some given situations. When setting up the model, there are few keys that we have to include.

1. Decision variables: The unknowns that we are trying to find to maximize or minimize some value depending on the unknowns.
2. Objective function: The value that we are trying to maximize or minimize. Also indicate whether we are trying to maximize or minimize.
3. Constraints: Requirements related to the unknowns, as described in the situation.
4. Hidden constraints: Not explicitly mentioned constraints that appear due to the definition of the variables.
5. Verify whether the given situation is a linear programming problem

Sample Question:
Suppose that a company is thinking of buying and selling some products A and B. Each product A is originally \$5, the company is planning on selling each at \$11 after. Each product B is originally \$10, the company is planning on selling each at \$15 after. Suppose that the company only have \$100 budget for buying the products, and they can only buy once. Due to the storage, they can only store at most 20 products. The supplier of the products require that the difference between the number of products A and the number of products B should be no more than 5. Besides, according to the recent marketing survey, the company requires that the number of products A should be at least 30% of the number of product B to fit the demand. How many products A and products B should they buy in order to maximize the profit after when selling (supposing that they can sell all of them)? Set up a linear programming problem to describe the situation.

1.2 Standard LPP

Recall, a **linear programming problem in standard form** is as follow
Find values of the **decision variables** $x_1, x_2, ..., x_n$ that will
Max the **objective function** $z = c_1x_1 + ... + c_nx_n$
subject to the **Constraints**

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \leq b_1 \\
&a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \leq b_2 \\
&\qquad\qquad\qquad\vdots \\
&a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \leq b_m \\
&x_1, x_2, ..., x_n \geq 0
\end{aligned}
$$

There are few questioins we can ask when verifying a standard LPP

- Is the objective function a linear function of the decision variables?
- Does each decision variables x_i satisfies $x_i \geq 0$?
- Are the constraints, except for $x_i \geq 0$, all less-than-or-equal constraints?
- Is the LPP a maximization problem?

If the given LPP is not standard, then we can always convert it to a standard LPP (see tutorial 1 note for more details).
Sample Question:
Consider the sample question from section 1.1, verify whether or not that is a standard LPP, if not, convert it to a standard LPP.

1.3 LPP in Canonical Form

Recall, a **linear programming problem in canonical form** is as follow
Find values of the **decision variables** $x_1, x_2, ..., x_n$ that will
Max the **objective function** $z = c_1x_1 + ... + c_nx_n$
subject to the **Constraints**

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ &\vdots \\ &a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \\ &x_1, x_2, ..., x_n \geq 0 \end{aligned}$$

There are few questioins we can ask when verifying a canonical LPP

- Is the objective function a linear function of the decision variables?
- Does each decision variables x_i satisfies $x_i \geq 0$?
- Are the constraints (except $x_i \geq 0$), equality constraints? Introduce slack variables if needed.
- Is the LPP a maximization problem?
- Do the slack variables (if any) satisfy $x_i \geq 0$?

If the given LPP is not canonical, then we can always convert it to a canonical LPP (see tutorial 1 note for more details).
Sample Question:
Consider the sample question from section 1.1, verify whether or not that is a canonical LPP, if not, convert it to a canonical LPP.

2 Graphical Method to Solve a LPP

If there are only two or three decision variables, the graphical method might be useful in order to solve a given LPP. Here are the steps:

1. Draw the feasible region in the two or three dimensional spaces to see the possible choice of the solutions that satify the constraints of the given LPP.
2. Calculate the gradient and draw some level sets to see the direction in which the values of the objective function increase.
3. Given the direction of increasement, find the optimal solutions by looking at the feasible region. Find the solution according to the objective (max or min) of the LPP.
4. Given the optimal solution, calculate the corresponding optimal value.

More details about the graphical method can be found in tutorial 2 notes.
Sample question: Solve the standard LPP in section 1.2 using the graphical method.

3 Convex Set and Geometry of LPP

Recall, a subset S of $\mathbb{R}^n$ is called **convex** if
for any $x_1, x_2 \in S$, $x = \lambda x_1 + (1-\lambda)x_2 \in S$ for all $0 \leq \lambda \leq 1$.

3.1 Prove convexity of a set given intuition of a graph

A graph is not a proof of convexity of a set, but it will give you a good intuition in proving or disproving the convexity. When the sets are in higher dimensional space, a graph in two or three dimension can help your intuition, but you need to convert and talk about everything in the corresponding higher dimensional space in your proof.
Sample Questions: Verify whether the following sets are convex or not. Give a justification of your argument by the definition of convex set.

1. $S = \{(x, y) \in \mathbb{R}^2 | y \geq 0\}$

2. $S = \{(x, y) \in \mathbb{R}^2 | y \geq 0\} \backslash \{(x, y) \in \mathbb{R}^2 | y = 0 \text{ and } x \geq 0\}$

3. $S = \{(x, y) \in \mathbb{R}^2 | x \in (-\infty, 0], y \geq 0\}$

4. $S = \{\vec{x} \in \mathbb{R}^n | x_i \geq 0 \text{ for all } i = 1, ..., n\}$

5. $S = \{\vec{x} \in \mathbb{R}^n | |x_i| \leq 1 \text{ for all } i = 1, ..., n\}$

6. $S = \mathbb{R}^n \backslash \{\vec{x} \in \mathbb{R}^n | |x_i| < 1 \text{ for all } i = 1, ..., n\}$

3.2 Prove convexity of a set given definitions

Given useful definitions as the assumption, we can prove convexity of a set. It is very important to understand the assumption and the definition of the assumption in this type of proof. Sometime this type of proof can combine with the previous type for intuition.
Sample Questions: Prove or disprove the following statements.

1. $Cone(S)$ is a convex set for any set $S \subseteq \mathbb{R}^n$

2. Any arbitrary subspace of $\mathbb{R}^n$ is convex.

3. Any subset of a convex set in $\mathbb{R}^n$ is convex.

4. The set of all rational numbers between 0 and 4 inclusive is convex.

5. The open ball with radius $r > 0$ centered at some arbitrary $x_0 \in \mathbb{R}^n$ is convex.

6. The union of two arbitray open balls in $\mathbb{R}^2$ is convex.

7. The intersection of two arbitrary open balls in $\mathbb{R}^n$ is convex.

4 Extreme point and Extreme Point Theorem

Definition: A point u in a convex set S is called an **extreme point** of S if it is not an interior point of any line segment in S. That is, u is an extreme point of S if there are no distinct points x_1 and x_2 in S such that

$$u = \lambda x_1 + (1 - \lambda)x_2$$

for some $0 < \lambda < 1$

Extreme point theorem

Let S be the set of feasible solutions to a general linear programming problem.

1. If S is nonempty and bounded, then an optimal solution to the problem exists and occurs at an extreme point.
2. If S is nonempty and not bounded, then if an optimal solution to the problem exists, then an optimal solution occurs at an extreme point.
3. If an optimal solution to the problem does not exists, then either S is empty or S is unbounded.

Sample questions:

1. Show that if $Ax = b, x \geq 0$ has a solution, then $A^T y \geq 0, b^T y < 0$ has no solution.

2. Show that if the optimal value of the objective function of a linear programming problem is attained at several extreme points, then it is also attained at any convex combination of these extreme points.

3. Use the extreme point theorems to solve the following LPP

   - min $z = 5x - 3y$
     subject to
     $x + 2y \leq 4$
     $x + 3y \geq 6$
     $x, y \geq 0$

   - max $z = 2x + 3y$
     subject to
     $3x + y \leq 6$
     $x + y \leq 4$
     $x + 2y \leq 6$
     $x, y \geq 0$