WS1_Linear_Programming_Problem

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WS1_Linear _Program...

Learning Objective

Understand the structures of the linear programming problem in standard form and in canonical form. Be able to set up a linear programming model using suitable notation, and convert to different forms when needed. Understand how to reformulate a special type of piecewise linear programming problem into a standard linear programming problem.

Questions

1. For each of the following problems, determine whether or not it is a standard linear programming problem. If not, explain the reason and reformulate it to a standard linear programming problem using matrix notation, if it is not possible, explain why.

(a) min $z = 4x + 5y$
subject to
$-4x \le 5$
$x \ge 0$

NO, this is a min problem and y is unconstraint.

Let $y = y^+ - y^-$, where $y^+, y^- \ge 0$
$-\max -4x - 5y^+ + 5y^-$
subject to $-4x \le 5$
$x \ge 0, y^+ \ge 0, y^- \ge 0$

$\vec{x} = \begin{pmatrix} x \\ y^+ \\ y^- \end{pmatrix}$, $c = \begin{pmatrix} -4 \\ -5 \\ 5 \end{pmatrix}$

$A = (-4\ 0\ 0)$ $b = (5)$

matrix notation:
$-\max c^T x$
s.t. $Ax \le b$
$x \ge 0$

(b) max $z = 4x_1 + 5x_2 + 1$
subject to
$4x_1 \le 5$
$x_1 \ge 0,\ x_2 \ge 0,\ 1 \ge 0$

NO, 1 is not a variable. Let x_3 be a variable with constraint $x_3 = 1$

Reformulate as
max $z = 4x_1 + 5x_2 + x_3$
s.t. $4x_1 \le 5$
$x_3 \le 1$
$-x_3 \le -1$
$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ $b = \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}$ $c = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$

matrix notation
max $c^T x$
s.t. $Ax \le b$
$x \ge 0$

2. Reformulate the following problem into a canonical linear programming problem using matrix notation.

min $z = 4x + 5y$
subject to
$-4x \le 5$
$x \ge 0$

known from 1(a),
$-\max -4x - 5y^+ + 5y^-$
s.t. $-4x \le 5$,
$x \ge 0, y^+ \ge 0, y^- \ge 0$

Introduce slack variable u
s.t.
$-\max -4x - 5y^+ + 5y^-$
s.t. $-4x + u = 5$
$x \ge 0, y^+ \ge 0, y^- \ge 0, u \ge 0$

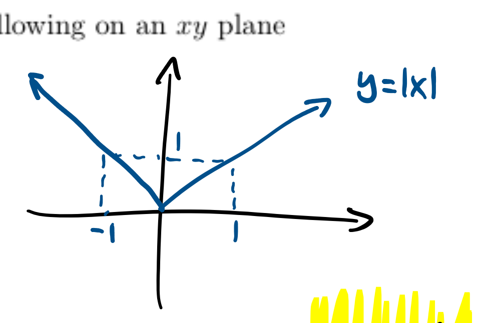
Matrix notation:
$\vec{x} = \begin{pmatrix} x \\ y^+ \\ y^- \\ u \end{pmatrix}$ $A = (-4\ 0\ 0\ 1)$

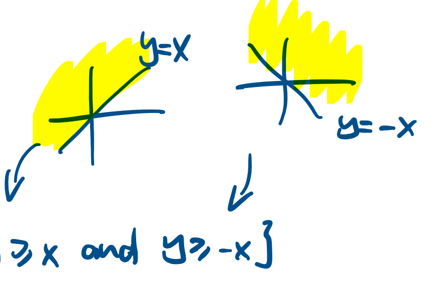
$c = \begin{pmatrix} -4 \\ -5 \\ 5 \\ 0 \end{pmatrix}$ $b = (5)$

s.t. $-\max c^T x$
s.t. $A\vec{x} = b$
$\vec{x} \ge 0$

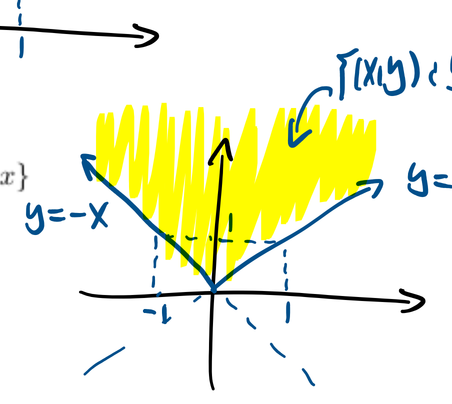
3. Sketch the following on an xy plane

(a) $y = |x|$



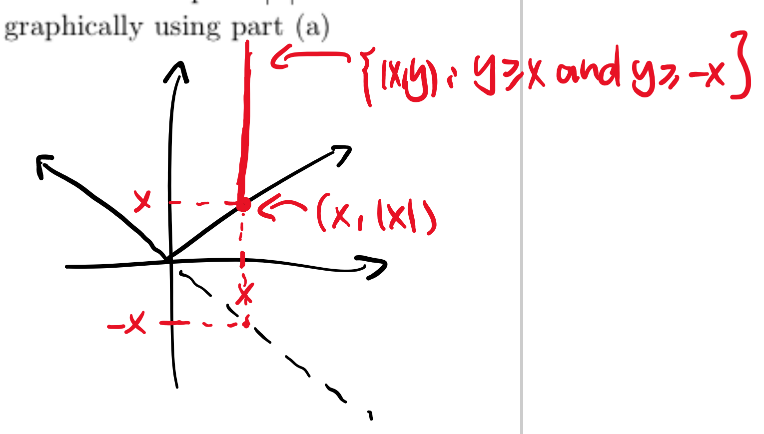


(b) $\{(x, y) : y \ge x \text{ and } y \ge -x\}$



(c) Let x be an arbitrary fixed real number, so $|x|$ is also a real number now. Express $|x|$ as an element of the set $\{y : y \ge x \text{ and } y \ge -x\}$, and explain it graphically using part (a) and (b).

For any fixed $x \in \mathbb{R}$,
$|x| = \min\{y : y \ge x \text{ and } y \ge -x\}$



4. Consider the following problem:

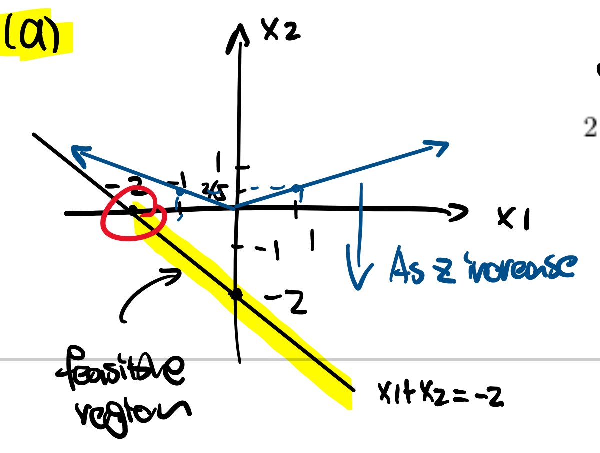
min $z = 2|x_1| - 5x_2$
subject to
$x_1 + x_2 = -2$
$x_2 \le 0$

(a) Without reformulating the problem, draw the feasible region and solve it directly using graphical method.

(b) Reformulate the problem into a standard linear programming problem using the idea from question 3(c). Draw the feasible region and solve it graphically.

(c) [redacted] Verify whether you got the same answer.

(a)



when $z = 0$,
$0 = 2|x_1| - 5x_2$
$x_2 = \frac{2}{5}|x_1|$
s.t. optimal solution is at $x_1 = -2$, $x_2 = 0$
and the optimal cost is $z = 2|x_1| - 5x_2 = 2\cdot|-2| - 5\cdot 0 = 4$

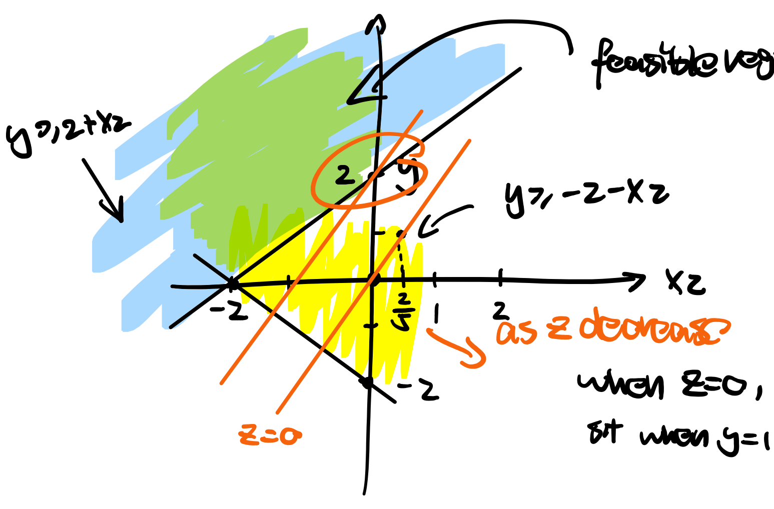
(b) $|x_1| = \min\{y : y \ge x_1 \text{ and } y \ge -x_1\}$
and since $x_1 \ge 0$, s.t. any such $y \ge x_1 \ge 0$
then the original question is equivalent to

min $2y - 5x_2$
s.t. $x_1 + x_2 = -2$
$y \ge x_1$
$y \ge -x_1$
$x_1, y \ge 0,\ x_2 \le 0$

Notice $x_1 + x_2 = -2$, s.t. $x_1 = -2 - x_2$
then the question is equivalent to

min $z = 2y - 5x_2$
s.t. $y \ge -2 - x_2$
$y \ge 2 + x_2$
$y \ge 0,\ x_2 \le 0$

The feasible region on the $x_2 y$ plane is



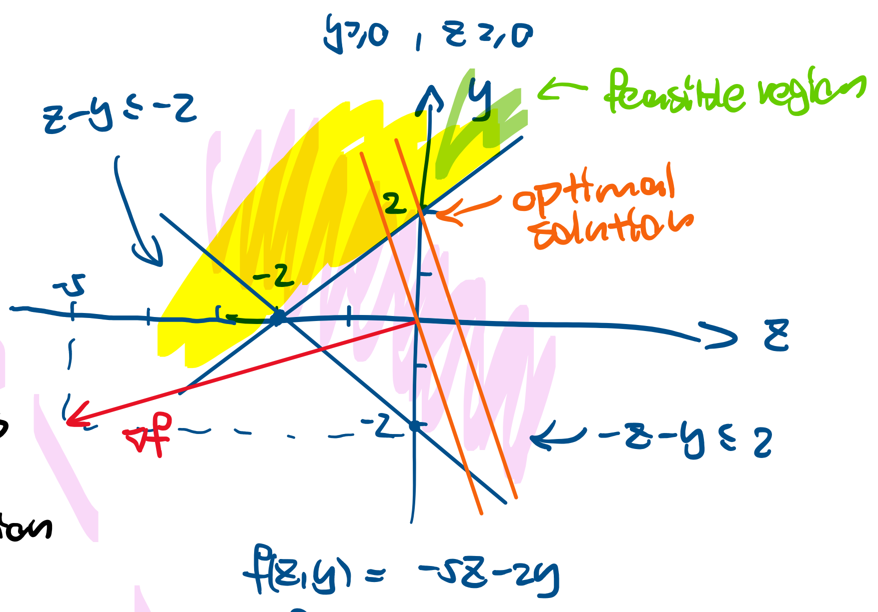
when $z = 0$, $0 = 2y - 5x_2$
s.t. when $y = 1$, $x_2 = \frac{2}{5}$

$(x_2, y) = (0, 2)$ is an optimal solution
and the optimal cost is $z = 2y - 5x_2 = 4$

(c) which is the same as what we got from (a)

Note, what I have done in 4(b) is not in standard form. Let's do that in standard form and see whether we get the same answer

→ let $z = -x_2$, and s.t. $z = -x_2 \ge 0$
$-\max -2y - 5z$
s.t. $z - y \le -2$
$-z - y \le 2$
$y \ge 0,\ z \ge 0$



$f(z, y) = -5z - 2y$
$\nabla f = (-5, -2)$
optimal solution at $(z, y) = (0, 2)$
the cost is $-(-2y - 5z) = -(-4) = 4$
we get the same answer!