Examples: Linear system

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Example: Double root

$A = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}$

characteristic eqn: $\det(A - rI) = \begin{vmatrix} 1-r & -4 \\ 4 & -7-r \end{vmatrix} = 0$

$(1-r)(-7-r) + 16 = 0$

$r^2 + 6r + 9 = 0 \Rightarrow \boxed{r_1 = r_2 = -3}$ eigenvalue

Eigenvector, $(A - r)v = 0$

$$\begin{pmatrix} 1+3 & -4 \\ 4 & -7+3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$4v_1 - 4v_2 = 0$

pick $v_1 = 1$, then $v_2 = 1$, s.t. $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\boxed{r = -3.}$

Generalized Eigenvector: To solve $(A - rI)s = v$ ← eigenvector

generalized eigenvector

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$(A - rI)^2 s = 0$

eigen vector.

Find $(A - rI)v = 0$

$4s_1 - 4s_2 = 1$

pick $s_1 = \frac{1}{4}$, then $s_2 = 0$.

s.t. $s = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$

Therefore, the solution is given by

eigenvector generalized eigenvector

$X = c_1 x_1 + c_2 x_2$, where $x_1 = e^{rt} v$, $x_2 = e^{rt} t v + e^{rt} s$.

$$= c_1 \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}}_{x_1} + c_2 \underbrace{\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} e^{-3t} \right)}_{x_2}.$$

Example: complex roots.

$A = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}$. $r = \pm 3i$. will be the eigenvalues.

Take $r = 3i$.

solve $(A - r)v = 0$

$$\begin{pmatrix} 1-3i & 2 \\ -5 & -1-3i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$(1-3i)v_1 + 2v_2 = 0.$

pick $v_1 = 1$, then $v_2 = \frac{1}{2}(3i - 1) = \frac{3}{2}i - \frac{1}{2}$.

s.t. $v = \begin{pmatrix} 1 \\ \frac{3}{2}i - \frac{1}{2} \end{pmatrix}$

$$e^{rt} v = e^{3i} \begin{pmatrix} 1 \\ \frac{3}{2}i - \frac{1}{2} \end{pmatrix}$$

$r = 3i$.

$$= (\cos 3t + i \sin 3t) \begin{pmatrix} 1 \\ \frac{3}{2}i - \frac{1}{2} \end{pmatrix}$$

$r = a + ib.$

$a.$ $3.$

$$= \begin{pmatrix} \cos 3t + i \sin 3t \\ -\frac{1}{2}\cos 3t - \frac{3}{2}\sin 3t + \frac{3}{2}i\cos 3t - \frac{1}{2}i \sin 3t \end{pmatrix}$$

$e^a = 1$

$$= e^a = 1 \quad \underbrace{\begin{pmatrix} \cos 3t \\ -\frac{1}{2}\cos 3t - \frac{3}{2}\sin 3t \end{pmatrix}}_{\text{real}} + i \, e^a = 1 \underbrace{\begin{pmatrix} \sin 3t \\ \frac{3}{2}\cos 3t - \frac{1}{2}\sin 3t \end{pmatrix}}_{\text{Imaginary}}.$$

s.t.

$$\boxed{X = c_1 \begin{pmatrix} \cos 3t \\ -\frac{1}{2}\cos 3t - \frac{3}{2}\sin 3t \end{pmatrix} + c_2 \begin{pmatrix} \sin 3t \\ \frac{3}{2}\cos 3t - \frac{1}{2}\sin 3t \end{pmatrix}}$$

$e^a = 1$ $e^a = 1$