Nick January 8,2020 **Tutorial 9b** MAT235 TUT0401

# Method of Lagrange Multipliers

(14.8) To find the maximum or minimum values of $f(x, y, z)$ (cost function) subject to the constraint of the form $g(x, y, z) = k$. Assume that these extreme values exist,

1. Verify if $g \neq 0$ on the surface $g(x, y, z) = k$. If the gradient vector is zero, you can not use Lagrange method.

2. Find all values of x,y,z, and $\lambda$ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

$$g(x, y, z) = k$$

   - $\lambda$ is called a Lagrange multiplier (with respect to the constraint g). If $\nabla g \neq 0$, then there exists a number $\lambda$ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ at the extreme point (x,y,z).
   - If you have multiple constraints, then you have more Lagrange multipliers.
   - Suppose now that we want to find the max and min values of a function $f(x, y, z)$ subject to two constraints $g(x, y, z) = k$ and $h(x, y, z) = c$. Assume the gradient vectors of g and h are not zero and not parallel(they are not in fact the same constraint), then there are numbers $\lambda$ and $\mu$, such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$ at the extreme point (x,y,z)

3. Evaluate f at all the points (x,y,z) that result from step 2. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

# Procedure

- Step0: Set up the Model
- Step1: Check the Condition of Lagrange Method
- Step2: Solve the System of Equations
- Step3: Evaluate the cost function at all points from step2

The method of Lagrange multipliers is to find the max or min of a cost function f subject to some constraints. In the next section, you will follow the procedure above to solve the problems using the method of Lagrange multipliers.

# 1 Problems:

***__**Volume Question:**__ The base of an aquarium(rectangular) with given volume $V$ is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass, find the dimensions of the aquarium that minimize the cost of the materials.

- What do you want to minimize? Other than the cost function, what function do you know that always have a fixed value? So, what are the cost function and constraints?
- In many real life problems, the variables have the condition of being positive. Did you miss any constraint in last step? Is $\vec{0}$ a possible solution according to your model?
- Now you have set up the model, can you use the Lagrange method to solve the question with this model? You need to make sure that the gradient vector of the constraint is not zero for any possible solution in your model. If you have a zero gradient, then you probably did something wrong.
- So, what is the system of equations that you should solve? You should be able to solve the problem using Lagrange method now. :)

__**Key points for this type of question:**__

- In many real life problems, the variables have the condition of being positive.
- You should be comfortable with the equations for computing volume of different regions, and be able to expand this type of questions to other shapes.
- The cost of glass and slate doesn't really matter, as long as you know the relations.

__*Volume and Area Problem:__ Find the maximum and minimum volumes of a rectangular box whose surface area is $1500cm^2$ and whose total edge length is $200cm$.
__*Volume and lengths Problem:__ Find the dimensions of a rectangular box of maximum volume such that the sum of the lengths of its 12 edges is a constant c.
***__**Function-Value Question:**__ Using Lagrange Multipliers to find the maximal and minimal directional derivatives of f at (0,0,0) in the direction $< a, b, c >$, where $a^2 + b^2 + c^2 = 1$.

$$f(x, y, z) = \begin{cases} \frac{xyz}{x^2+y^2+z^2} & (x, y, z) \neq (0, 0, 0) \\ 0 & (x, y, z) = (0, 0, 0) \end{cases}$$

__*Function Extreme values:__ Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$

__*Distance Problem:__ Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point (3,1,-1)

__*Multiple Constraints:__ Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$