This worksheet contains various examples of LPP with various situations that you might see when using the simplex method. The final answers are provided, so you can check the solution yourself, but the intermediate steps are more important than the final answer when doing the practices.

Questions

1. Solve the following LPP using the simplex method. Clearly indicate the canonical form, the current BFS at each stage, optimal solution and optimal cost.

   Note: Be precise with what the solution (with cost) from the simplex method is, and what the actual solution (with cost) to the orginal LPP is.

   min $z = -2x_1 - x_2$

   subject to

   $x_1 - x_2 \leq 2$

   $x_1 + x_2 \leq 6$

   $x_1, x_2 \geq 0$

   Answer. *Optimal solution* $x = (4, 2, 0, 0)$ *with optimal cost* $z = -10$

2. Solve the following LPP using the simplex method. Clearly indicate the canonical form, the current BFS at each stage, optimal solution and optimal cost. Explain why.

   max $z = 2x_1 + 3x_2 + x_3 + x_4$

   subject to

   $x_1 - x_2 - x_3 \leq 2$

   $-2x_1 + 5x_2 - 3x_3 - 3x_4 \leq 10$

   $2x_1 - 5x_2 + 3x_4 \leq 5$

   $x_1, x_2, x_3, x_4 \geq 0$

   Answer. *No optimal solution, the cost can be infinite.*

3. Solve the following LPP using the simplex method. Clearly indicate the canonical form, the current BFS at each stage, optimal solution and optimal cost.

   max $z = 200x_1 + 60x_2 + 206x_3$

   subject to

   $3x_1 + x_2 + 5x_3 \leq 8$

   $5x_1 + x_2 + 3x_3 \leq 5$

   $x_1, x_2, x_3 \geq 0$

   Answer. *Optimal solution* $x = (0, \frac{1}{2}, \frac{3}{2}, 0, 0)$ *with optimal cost* 339. *Be careful choosing the pivot entries when using the simplex method, find common denominator if needed.*

4. Consider the following complex tableaux when solving a LPP using simplex method.

$$\begin{array}{cccccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
1 & -1 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 2 & 0 & 1 & 0 & 3 \\
0 & 1 & 1 & 1 & 0 & 1 & 2 \\
\hline
0 & 0 & -14 & -3 & 0 & 0 & -28
\end{array}$$

(a) What is the current BFS given the simplex tableaux? What are the current cost?

(b) Looking at the tableaux, do we know the optimal solution and optimal cost now, why?

(c) If such optimal solution exists, can you find another optimal solution by introducing a new basic variable? (Hint: Introduce x_2, consider the x_2 column and convert the pivot as usual)

(d) If you can find two different optimal solutions from the previous questions, called them x and y, prove that any convex combination of x and y is also an optimal solution. (Hint: You will not be able to tell what A looks like by the tableaux (why?), but you could prove this in general by assuming some general form of the LPP in standard/canonical form.)

(e) You proved that any convex combination of x and y is also an optimal solution in the previous question, can any of these be an interior point of the feasible region? (Hint: Draw a graph for intuition and extreme point theorem)

5. In this question, you will explore the think about why we need the assumption $b \geq 0$ to start using the simplex method (not the general version) introduced in class, as well as the reason why we want to be careful when picking the pivots a_{ij}.

   (a) Find the largest possible ranges (without violating any of the constraint) for each of the x_1, x_2, x_3 given the following constraints, $2x_1 + 4x_2 - x_3 \leq 4$ and $x_1, x_2, x_3 \geq 0$.

   (b) Find the largest possible range for x_1 given the following constraints, $2x_1 + 4x_2 - x_3 \leq 4$, $4x_1 + 9x_2 - 5x_3 \leq 4$ and $x_1, x_2, x_3 \geq 0$. What have you noticed? and think about why we would want to find the entry with smallest ratio when choosing the pivot?

   (c) Find the largest possible range for x_1 given the constraints $-x_1 + 4x_2 - x_3 \leq 4$, $x_1, x_2, x_3 \geq 0$. Now add another constraint $4x_1 + 9x_2 - 5x_3 \leq 4$, what should the largest possible range for x_1 be, in particular without violating this new constraint. What have you noticed? and think about why we would want to find the non-negative entry when choosing the pivot?

   (d) Find the largest possible range for x_1 given $2x_1 + 4x_2 - x_3 \leq -4$, $x_1, x_2, x_3 \geq 0$. What have you noticed? While we still have a positive 2 as the coefficient of x_1 in that particular constraint with the smallest ratio (because there is only one such ratio), we would not be able to pick that as a pivot to find another BFS when using the simplex method with the same intepretation.

   However, try to do this using the method we introduced and see what the new so-called BFS is (this is not a BFS, but why)? This would warm you that we should avoid multiplying -1 on both side when doing row operations.

   That is one of the reason why we need the assumption $b \geq 0$, and you would learn another method in section 2.3 later to deal with this difficulty.