This list of questions is for the students in MATB24, TUT0005 summer 2021 at the University of Toronto Scarborough. You should not use any of the facts in this document as a reference in the midterm test. Everything covered in this document has been talked about in the lecture, tutorial or the textbook. The test may cover materials that are not included in this document. You should refer to the professor's email for the coverage of the test. Good luck in the midterm!

1 Subspace

1. Let $V = \mathbb{R}^2$ and let $W = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 - x_2 = 0\}$. Show that W is a subspace of V.
2. Let W be a subspace of a vector space V. Let $y \in V$, and define the set $y + W = \{x \in V | x = y + w \text{ for some } w \in W\}$. Show that $y + W$ is a subspace of V if and only if $y \in W$.

2 Linear independent, span and basis

1. Let V be the real vector space consisting of all continuous function from $\mathbb{R}$ to $\mathbb{R}$. Show that $\{sin^2(x), cos^2(x)\} \subset V$ is linearly independent.
2. Let v_1, v_2 be vectors in a real vector space V. Show that $span(v_1, v_2) = span(v_1, 2v_1 + 3v_2)$ directly by the definition.
3. Show that if (v_1, v_2) is a basis of a real vector space V, then for any non-zero scalar k, $(kv_1, v_1 + v_2)$ is also a basis of V.
4. Let S_1, S_2 be subsets of a vector space V. Assume that $span(S_1) = V$ and $S_1 \subset span(S_2)$. Show that $V = span(S_2)$.

3 Matrices and determinant

1. Check whether $(1, 3, 9, 1)^T, (1, 0, 1, 2)^T, (2, 0, 0, 0)^T, (4, 1, 2, 0)^T$ are linear independent vectors in $\mathbb{R}^4$ directly by the definition and show all the computational details.
2. Compute the determinant of the matrix $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 6 \\ 4 & 8 & 9 \end{pmatrix}$ with only the properties of determinant that we discussed before.
3. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 \end{pmatrix}$ Check whether the following is true, otherwise give an explicit counter example to support your argument
   (a) The columns of A are linearly independent
   (b) The columns of A span $\mathbb{R}^4$

(c) The columns of B are linearly independent

(d) The columns of B span $\mathbb{R}^3$

In general, how can you check the above statements using the pivots of the reduced row echelon form of a matrix? Give yourself some intuition with the ideas of solving an argumented matrix.

4 Linear transformation, coordinate and matrix representation

1. Suppose T is a linear transformation from V to W. If A is a subspace of V, show that $T(A)$ is a subspace of W.

2. Prove that there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. Find $T(a,b)$ for any arbitrary $a, b \in \mathbb{R}$

3. Let $T : V \to W$ be a linear transformation. Show that $T(I)$ is linearly independent for any linearly independent finite subset I of V if and only if $ker(T) = \{0\}$.

4. Let V be a finite-dimensional vector space and W be a subspace of V. Show that there exists aa linear transformation $T : V \to W$ whose image is W and is identity on W (i.e. $\forall w \in W, T(w) = w$)

5. Let $T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{P}_2(\mathbb{R})$ be a linear transformation. Let $\alpha = (1 + x + x^2, x + x^2, x^2)$. Suppose that $[T]_\alpha^\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 0 & 3 & 2 \end{pmatrix}$. Find $T(1), T(x)$ and $T(x^2)$.

6. Let $T : V \to W$ be a linear transformation. Let α, β be bases of V and W. Show that if T is an invertible linear transformation, then the matrix $[T]_\beta^\alpha$ is invertible. (Following the definition and notation that we used in class)

7. Let V be a real vector space and $\alpha = (v_1, v_2, v_3)$ be a basis of V. Let $T : V \to V$ be a linear transformation such that $T(v_1) = v_1, T(v_2) = 2v_2, T(v_3) = -v_3$. (You can verify that T is a linear transformation). Find $[T]_\alpha^\alpha$. Let $\beta = (v_1 + v_2, v_2 + v_3, v_3 + v_1)$, find $[T]_\beta^\beta$.

8. Let V be an n-dimensional real vector space. Let λ be a nonzero real number. Define $T : V \to V$ by $T(v) = \lambda v$. Let α, β be ordered bases of V. Show that $[T]_\beta^\alpha = \lambda I_n$ if and only if $\alpha = \beta$. (Following the definition and notation that we used in class)

5 Eigenvalue, eigenvector and diagonalizable

1. Define $T : \mathbb{P}_3(\mathbb{R}) \to \mathbb{P}_3(\mathbb{R})$ by $T(p)(x) := 3p(x) + xp''(x)$. Check whether T is diagonalizable.

2. Find all values a, b, c such that $\begin{pmatrix} 1 & a & b \\ 0 & 3 & c \\ 0 & 0 & 3 \end{pmatrix} \in M_{3\times3}(\mathbb{R})$ is diagonalizable.