WS3_Convex_set_given_assumption

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WS3_Convex_set_give...

Learning Objective

This worksheets will guide you to write a proof to the convexity of a given set, using given useful assumptions, following the four phases method. You will be more confortable and gain confidence writing a proof by using the method.

Definition

Recall from linear algebra, a function mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ is called a **linear transformation** if $f(au + bv) = af(u) + bf(v)$ for any $u, v \in \mathbb{R}^n$ and any $a, b \in \mathbb{R}$.

Question

let's assume also that S is nonempty, otherwise $f(S) = \phi$ is convex trivially.

Prove that if S is a convex set in $\mathbb{R}^n$ and f is a linear transformation mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, then $f(S) = \{f(v) | v \in S\}$ is a convex set. The following subquestions will guide you to prove the statement. If you are comfortable with proving the statement, then you should be able to answer the subquestions as well.

1. What are you trying to show in the question? What does it exactly mean by definition? (i.e. you should be able to find a reference to the meaning in the textbook and/or lecture note).

   we want to show that $f(S)$ is a convex set,
   and by definition, that means,
   $\forall u_1, u_2 \in f(S), \forall \lambda \in [0,1], \lambda u_1 + (1-\lambda) u_2 \in f(S)$

2. What are the given assumptions of the question? (i.e. the "if" statement) What does each of those assumption mean by definition? (again, you should be able to find reference)

   - we are given that f is a linear transformation.
     s.t $f(av_1 + bv_2) = af(v_1) + bf(v_2)$ for all $a, b \in \mathbb{R}$
     and for all $v_1, v_2 \in S$.
   - Also S is a convex set, i.e $\forall v_1, v_2 \in S, \forall \lambda \in [0,1]$
     we have $\lambda v_1 + (1-\lambda) v_2 \in S$

3. Look at the following proof and determine whether it is a valid proof followed from the definition? If not, point out the mistakes that it has.

   Let $S = [0, 1]$

   WTS: S is a convex set.

   Notice for $x_1 = 0, x_2 = 1$ which are clearly in S, let $\lambda \in (0, 1)$ be arbitrary,

   then $x := \lambda x_1 + (1-\lambda)x_2$ is also in the set S, because $0 < x < 1$ given $x_1 = 0, x_2 = 1, \lambda \in (0, 1)$.

   Therefore any points in between x_1 and x_2 is also in the set S, and hence S is a convex set.

   - NO, x_1 and x_2 are not arbitrary.
     Also $\lambda \in [0,1]$ instead of $(0,1)$
   - This argument did not prove S being convex by the definition of convex set.

4. The previous questions should remind you the importance of picking arbitrary variables according to the definition of convex set. Now, look at question 2 again, what should you write at the beginning of your proof as the "let" statement, before actually start proving?

   let $u_1, u_2 \in f(S)$, let $\lambda \in [0,1]$

5. Look at the "want to show" now, what does it mean to say that $f(v) \in f(S)$? You should introduce quantifier to describe some of the variable. The following examples may give you some idea about the set notation $f(S)$.

   Example: $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$ Let $S = (-2, 2)$ which is a convex set.

   - $1 \in f(S)$, because $1 = 1^2 = f(1) = f(v)$ where $v = 1 \in S$
   - $1 \in f(S)$, because $1 = (-1)^2 = f(-1) = f(v)$ where $v = -1 \in S$
   - $4 \notin f(S)$, because $4 \neq f(v)$ for all $v \in S$. In particular, we know that if $4 = f(v) = v^2$, we must have $v = 2$ or -2, but $2, -2 \notin S$

   Now what does it mean to say $y \in f(S)$? You answer should include what y "looks like" and what "condition" it must be satisfied? Define all the variables and use the quantifier ("for all" or "some"?)

   This can possibly be not unique.

   $y \in f(S)$, if $\exists x \in S$, s.t $f(x) = y$.

6. Write your proof now to the question, starting from the let statement. If you are unsure how to move on, look at the "want to show" at the stage where you are unsure, and use all the given assumptions. The definition of linear transformation is very useful in making your assumption look like the "want to show". Remember, it is always a good idea to ask yourself, what you are trying to show at any stage of your proof (they might not be the same, as you go deeper into the original "want to show").

7. Is your proof lack of reasoning? Did you mention the reasoning in the important steps?

Q6 let $u_1, u_2 \in f(S)$, let $\lambda \in [0,1]$
since $u_1 \in f(S)$, s.t $u_1 = f(v_1)$ for some $v_1 \in S$ (*).
and $u_2 \in f(S)$, s.t $u_2 = f(v_2)$ for some $v_2 \in S$
WTS $\lambda u_1 + (1-\lambda) u_2 \in f(S)$

$$\lambda u_1 + (1-\lambda) u_2 = \lambda f(v_1) + (1-\lambda) f(v_2) \text{ by } (*)$$
$$= f(\lambda v_1 + (1-\lambda) v_2) \text{ by } f \text{ being a linear transformation}$$

where $v_1, v_2 \in S$, $\lambda \in [0,1]$
s.t $\lambda v_1 + (1-\lambda) v_2 \in S$ by S being convex, s.t $\lambda u_1 + (1-\lambda) u_2 = f(v)$ for some $v = \lambda v_1 + (1-\lambda) v_2 \in S$
s.t $\lambda u_1 + (1-\lambda) u_2 \in f(S)$ by def of $f(S)$
hence $f(S)$ is convex by def of convex set ∎

More questions:

The following problems can be solved using the same idea as above. You should be able to solve those problems by simply asking yourself what the definitions of the assumptions are. Answer the following questions using the definions that you can refer to the lecture notes and/or textbooks, don't answer them using intuitive words only.
For example, try not to answer the first question below by on ly saying "a subspace is closed under those operations, so will surely closed under convex combination", but use what it means by "closed under those operations" and "closed under convex combination"

1. Prove that a subspace of $\mathbb{R}^n$ is a convex set directly using the definition of subspace and convex set.

   let A be a subspace of $\mathbb{R}^n$, s.t $A \neq \phi$, let $x, y \in A$, let $\lambda \in [0,1]$
   then $\lambda x + (1-\lambda) y \in A$ by A being a subspace, s.t it is closed under addition and scalar multiplication.
   s.t A is convex by def ∎

2. Let V, W be subspaces of $\mathbb{R}^n$. Prove that $V \cap W$ is a convex set directly using the definition of subspace, convex set and intersection of two sets. what does $V \cap W$ mean in set notation?

   let V, W be subspaces, ① If $V \cap W = \phi$, then trivially $V \cap W$ is convex by def
   ② Assume $V \cap W \neq \phi$,
   let $x, y \in V \cap W$, let $\lambda \in [0,1]$
   then $x \in V$ and $y \in V$, s.t $\lambda x + (1-\lambda) y \in V$ by V being a subspace
   Also $x \in W$ and $y \in W$, s.t $\lambda x + (1-y) y \in W$ by W being a subspace.
   s.t $\lambda x + (1-\lambda) y \in V \cap W$ by def of intersection of two sets.
   then $V \cap W$ is convex ∎

3. Prove that an anbitrary hyperplane H is a convex set directly using the definition of hyperplane (Make sure that your hyperplane is arbitrary when defining).

   Let $a \in \mathbb{R}^n, c \in \mathbb{R}$, let $H = \{x \in \mathbb{R}^n \mid a^T x = c\}$
   let $x_1, x_2 \in H$, let $\lambda \in [0,1]$
   WTS $\lambda x_1 + (1-\lambda) x_2 \in H$
   known $x_1, x_2 \in H$, s.t $x_1, x_2 \in \mathbb{R}^n$; $a^T x_1 = c$, $a^T x_2 = c$ (*)
   then $\lambda x_1 + (1-\lambda) x_2 \in \mathbb{R}^n$

   $$a^T(\lambda x_1 + (1-\lambda) x_2) = \lambda a^T x_1 + (1-\lambda) a^T x_2 \text{ by } \lambda \in \mathbb{R}$$
   $$= \lambda c + (1-\lambda) c \text{ by } (*)$$
   $$= c$$

   s.t $\lambda x_1 + (1-\lambda) x_2 \in H$, hence H is convex ∎