Definitions

You should always check the lecture note and the textbook for the definition, and make sure the definitions are consistents.

1. A **vector space** is a set V together with two operations

   (a) $+ : V \times V \to V$

   (b) $\cdot : \mathbb{F} \times V \to V$

   such that

   (1) $\forall v, w \in V, v + w = w + v$

   (2) $\forall u, v, w \in V, (u + v) + w = u + (v + w)$

   (3) $\exists 0 \in V, \forall v \in V, 0 + v = v$

   (4) $\forall v \in V, \exists w \in V$ such that $v + w = 0$. Denote w by $-v$

   (5) $\forall a \in \mathbb{F}, \forall u, v \in V, a \cdot (u + v) = a \cdot u + a \cdot v$

   (6) $\forall a, b \in \mathbb{F}, \forall u \in V, (a + b) \cdot v = a \cdot v + b \cdot v$

   (7) $\forall a, b \in \mathbb{F}, \forall v \in V, a \cdot (b \cdot v) = (ab) \cdot v$

   (8) $\forall v \in V, 1 \cdot v = v$

   If $\mathbb{F} = \mathbb{R}$, we call V together with the operations the **real vector space**.
   If $\mathbb{F} = \mathbb{C}$, we call V together with the operations the **complex vector space**.

2. A set α is called a basis of the vector space V if α is linearly independent and $\text{span}(\alpha) = V$.

3. Let $\alpha = \{v_1, \cdots, v_n\}$ be a set of vectors in V, $\text{span}(\alpha) = \{a_1 v_1 + \cdots + a_n v_n | a_1, \cdots, a_n \in \mathbb{F}\}$

4. Let $\alpha = \{v_1, \cdots, v_n\}$ be a set of vectors in V, recall that it is called linearly independent if

$$a_1 v_1 + \cdots + a_n v_n = 0 \text{ for some } a_1, \cdots, a_n \in \mathbb{F} \implies a_1 = \cdots = a_n = 0$$

Remark. *Notice the definitions of linearly independent and span depends* $\mathbb{F}$*.*

1 Define $\mathbb{C}$ as real vector space

Given $\mathbb{C} = \{a + ib | a, b \in \mathbb{R}\}$. Let V be the vector space $\mathbb{C}$ over $\mathbb{R}$ with the following addition and scalar multiplication

- $(a_1 + ib_1) + (a_2 + ib_2) := (a_1 + a_2) + i(b_1 + b_2)$ for all $a_1 + ib_1, a_2 + ib_2 \in V$
- $k(a + ib) := (ka) + i(kb)$ for all $k \in \mathbb{R}, a + ib \in V$

Consider the following questions with the above definition.

1. Show that $i \notin \text{span}(\{1\})$, conclude that $\{1\}$ is not a basis of V.
2. Show that $\alpha = \{1, i\}$ is linearly independent.
3. Show that α is a spanning set of V.
4. Conclude that α is a basis of V.

2 Define $\mathbb{C}$ as complex vector space

Given $\mathbb{C} = \{a + ib | a, b \in \mathbb{R}\}$. Let V be the vector space $\mathbb{C}$ over $\mathbb{C}$ with the following addition and scalar multiplication

- $(a_1 + ib_1) + (a_2 + ib_2) := (a_1 + a_2) + i(b_1 + b_2)$ for all $a_1 + ib_1, a_2 + ib_2 \in V$
- $k(a + ib) := (ka) + i(kb)$ for all $k \in \mathbb{C}, a + ib \in V$

Consider the following questions with the above definition.

1. Show that $\{1, i\}$ is a linearly dependent set in V. Conclude that it is not a basis of V.
2. Show that $\beta = \{1\}$ is linearly independent.
3. Show that β is a spanning set of V.
4. Conclude that β is a basis of V.