Intuition:

Intuitively, recall from first year calculus, given a function y=f(x), we can take higher-order derivatives given its derivative is still differentiable. For example, consider the function $y = f(x) = x^5$, we have $f'(x) = 5x^4$. Notice that f'(x) is a function, however, given input x=a, f'(a) is a value. Since f(x) is a function, therefore we can also take its derivative. $f''(x) = 20x^3$, $f^{(3)}(x) = f'''(x) = 60x^2$, $f^{(4)}(x) = 120x$, $f^{(5)}(x) = 120$, $f^{(6)}(x) = 0$, $f^{(7)}(x) = 0$, ... In this worksheet, I make some extra problems for you to think about, in order to solve the following:

Question:

Suppose that z=f(x,y) has continuous second order partial derivatives and $x = s^2 - t^2$, $y = 2st$. Use the chain rule to find the function g(s,t) satisfying the following equation:

$$\frac{\partial^2 z}{\partial s^2} + \frac{\partial^2 z}{\partial t^2} = g(s,t)(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2})$$

Preparation:

1. Consider the function $z = f(y) = y^2 + 2y, y = g(x) = cos(x)$. What is dz/dx?
2. Consider the function $z = f(y), y = g(x)$. What is the general formula for dz/dx
3. Consider the function $z = f(y) = y^2 + 2y, y = g(x) = cos(x)$. What is d^2z/dx^2?
4. Consider the function $z = f(y), y = g(x)$. What is the general formula for d^2z/dx^2
5. Consider the function $x = s^2 - t^2$. What is $\partial x/\partial s$? What is $\partial x/\partial t$?
6. Consider the function $y = 2st$. What is $\partial y/\partial s$? What is $\partial y/\partial t$?
7. Consider the function $z = f(x,y) = x + 3x^3y^2$, $x = t$, $y = t^4$. Can you find a function such that $z = g(t)$? Now, given $g(t)$, can you find $g'(t)$? Now, instead of using g(t), can you find dz/dt by using f, x(t), y(t), and chain rule? Is your answers (dz/dt and $g'(t)$) the same, after doing all the simplification?
8. Given two functions, f(t) and g(t). What is the derivative of the function $f(t)g(t)$, by product rule? Given the equation $f(s,t) = [\frac{\partial z}{\partial x}(s,t)]2s + [\frac{\partial z}{\partial y}(s,t)]2t$, what is $\frac{\partial f}{\partial s}$? Notice that, $\frac{\partial z}{\partial x}$ is a function of s and t. $\frac{\partial^2 z}{\partial s \partial x} = \frac{\partial}{\partial s}(\frac{\partial z}{\partial x})$, which is the partial derivative of the function $\frac{\partial z}{\partial x}$ with respect to s.
9. You should now be ready to solve our main question above. Be prepared to get a very long formula, however, some of the terms will cancel out very soon. I suggest you to start by finding $\frac{\partial^2 z}{\partial s^2}$ and $\frac{\partial^2 z}{\partial t^2}$.