Example_using_linearly_independent_as_assumption_annotated

Sunday, May 30, 2021 10:03

Example_using_linearl…

Definition

Recall, let V be a vector space. We say that $\{v_1, \cdots, v_n\} \subset V$ is linearly independent if

$$a_1v_1 + \cdots + a_nv_n = 0 \text{ for some } a_1, \cdots, a_n \in \mathbb{F} \implies a_1 = \cdots = a_n = 0$$

Otherwise, we call them linearly dependent.

Remark. *Notice the definition of linearly independent is an implication, instead of an 'and' statement. Therefore, given a linearly independent set, we can not make any conclusion until the hypothesis of the implication is satisfied.*

Questions

1. Let V, W be vector spaces over $\mathbb{F}$. Let $T : V \to W$ be an invertible linear transformation. Let $I = \{v_1, \cdots, v_n\}$ be a linearly independent set of vectors in V. Prove that,

   (a) Show that $T(I) = \{T(v_1), \cdots, T(v_n)\}$ is also linearly independent

Assume T is invertible, st $\exists\, T^{-1}: W \to V$, st $T \circ T^{-1} = id_W$ and $T^{-1} \circ T = id_V$

Assume T is a LIT and I is LI

st $b_1v_1 + \cdots + b_nv_n = 0$ for some $b_1, \cdots, b_n \in \mathbb{F} \Rightarrow b_1 = \cdots = b_n = 0$ ⊛

WTS $T(I)$ is LI

Assume $a_1T(v_1) + \cdots + a_nT(v_n) = 0$ for some $a_1 \cdots a_n \in \mathbb{F}$

$\boxed{\text{WTS } a_1 = \cdots = a_n = 0}$

st $T(a_1v_1 + \cdots + a_nv_n) = 0$ by T being a LIT

and $T(a_1v_1 + \cdots + a_nv_n) = T(0)$ b/c $T(0) = 0$ given T is a LIT

$T^{-1}(T(a_1v_1 + \cdots + a_nv_n)) = T^{-1}(T(0))$

$id_V(a_1v_1 + \cdots + a_nv_n) = id_V(0)$

$a_1v_1 + \cdots + a_nv_n = 0$

$\boxed{a_1 = \cdots = a_n = 0 \text{ by } ⊛}$

This is when the hypothesis of ⊛ is satisfied! we can make some conclusion by ⊛ after this point.

Therefore, $T(I)$ is LI ∎

2. Let V, W be vector spaces over $\mathbb{F}$. Let $T : V \to W$ be a linear transformation. Assume that $S = \{T(v_1), \cdots, T(v_n)\}$ are linearly independent for some $v_1, \cdots, v_n \in \mathbb{F}$. Show that $v_1, \cdots, v_n$ are linearly independent.

Assume S is LI, T is a LIT

That is if $a_1T(v_1) + \cdots + a_nT(v_n) = 0$, then $a_1 = \cdots = a_n = 0$

WTS $v_1, \cdots, v_n$ are LI

Assume $b_1v_1 + \cdots + b_nv_n = 0$, WTS $b_1 = \cdots = b_n = 0$

$T(b_1v_1 + \cdots + b_nv_n) = T(0) = 0$ b/c $T(0) = 0$

$b_1T(v_1) + \cdots + b_nT(v_n) = 0$ by T being a LIT

then by S being LIT, $b_1 = \cdots = b_n = 0$ ∎