Definitions and Useful Formulas:

1. **Partial Derivative**(14.6.1)
   If $z = f(x, y)$, then the partial derivatives f_x and f_y are defined as
   $$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$
   $$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

2. **Directional Derivative**(14.6.2)
   If $z = f(x, y)$, then the directional derivatives of f at (x_0,y_0) in the directional of a unit vector $\mathbf{u} = < a, b >$ is
   $$D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

3. **Directional and Partial Derivatives**(14.6.3)
   If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = < a, b >$ and
   $$D_u f(x, y) = f_x(x, y)a + f_y(x, y)b$$
   If the unit vector $\mathbf{u}$ makes angle θ with the positive x-axis, then write $\mathbf{u} =< cos\theta, sin\theta >$ and the formula in Theorem 3 becomes
   $$D_u f(x, y) = f_x(x, y)cos\theta + f_y(x, y)sin\theta$$

4. **Gradient**(14.6.8)
   If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by
   $$\nabla f(x, y) =< f_x(x, y), f_y(x, y) >= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$
   therefore, if f is differentiable as described in equation 3,
   $$D_u f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

5. **Maximizing the Directional Derivative**(14.6.15)
   Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_u f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

   Question: Why? (Hint: Equation4 and formula for dot product)

Intuition:

Recall that if z=f(x,y), then the partial derivatives f_x and f_y represent the rates of change of the function along the x and y-axis respectively. In other words, the partial derivative tells you how much the value of the function changes by inputting a bit more of x or y. In this tutorial, we introduce the directional derivative, that enables us to find the rate of change of a function of two or more variables in any direction.

1 Computing Directional Derivative

1. Using the definition: **Directional Derivative**(14.6.2)
   If $z = f(x,y)$, then the directional derivatives of f at (x_0,y_0) in the directional of a **unit** vector $\mathbf{u} = < a, b >$ is

   $$D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

   Find the directional derivative simply by computing the limit. If u is not a unit vector, make sure to normalize it before using.

2. Using the gradient: **Directional derivative and Gradient** (14.6.8)
   If f is differentiable function of x and y,

   $$D_u f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$$

3. **Question:** Find the directional derivative of $f(x,y) = xy^3 - x^2$ at the point $(1,2)$ in the direction of a unit vector $\mathbf{u} =< \frac{1}{2}, \frac{\sqrt{3}}{2} >$ using two different methods above and verify whether you end up getting the same answer.

4. **Thinking question:** Do we get different answer if u is not a unit vector? Consider the same question above with $u' =< 1, \sqrt{3} >$.
   (a) Is u' a unit vector? (How do you check if a vector is a unit vector?)
   (b) Does u' have the same direction as u?
   (c) What answer do you get using u'? Do you get the same answer as in the last question using u?

Make sure to normalize your direction vector

5. Suggested problems from textbook: **14.6:** 5,7,9,11,13,15,17,35

2 Differentiability, Partial and Directional Derivatives

1. **Definition of differentiable** (14.4.7)
   If z = f(x,y), then f is differentiable at (a,b) if Δz can be expressed in the form

   $$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

   where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$

2. **Partial Derivative and Differentiability**(14.4.8)
   If the partial derivatives f_x and f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b)

This is how you check differentiability using partial derivatives.

(1) The existence of partial derivatives does not imply differentiability.

(2) Equation 3 and 5 in the previous page have the condition of being differentiable.

(3) Now, you should be comfortable with question 5 (b) in test 2.

3 Finding the Fastest Rate of Change

1. **Maximizing the Directional Derivative**(14.6.15)
   Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_u f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when **u** has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

2. **Procedure:**
   (a) Show that f is differentiable function
   (b) Compute the gradient vector at the given point
   (c) According to theorem 14.6.15 (theorem5 in the previous page), what is the maximum value of the directional derivative and in which direction?

3. **Question:** Find the maximum rate of change of f at the given point and the direction in which it occurs. (Equation 5 in the previous page is useful)
   (a) $f(x,y) = 4y\sqrt{x}$ at (4,1)
   (b) $f(x,y) = sin(xy)$ at (1,0)

4. **Question:** Show that a differentiable function f decreases most rapidly at x in the direction opposite to the gradient vector, that is, in the direction of $-\nabla f(x)$. Use this result to find the direction in which the function $f(x,y) = x^4 y - x^2 y^3$ decreases fastest at the point (2,-3)

5. **Suggested problems from textbook: 14.6:** 23,25,27,29,31,33

4 Properties of Gradient

1. The definition of **Gradient**(14.6.8) as follows,
   If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by
   $$\nabla f(x,y) = < f_x(x,y), f_y(x,y) > = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$
2. Question(**14.6.37**): show that the operation of taking the gradient of a function has the given property. Assume that u and v are differentiable functions of x and y and that a, b are constants. (Hint: Using peoperties of derivatives)
   (a) $\nabla(au + bv) = a\nabla(u) + b\nabla(v)$
   (b) $\nabla(uv) = u\nabla(v) + v\nabla(u)$
   (c) $\nabla(\frac{u}{v}) = \frac{v\nabla(u) - u\nabla(v)}{v^2}$
   (d) $\nabla u^n = nu^{n-1}\nabla u$

5 Questions:

1. Assume that u and v are differentiable functions of x and y. Show that $\nabla(uv) = u\nabla(v) + v\nabla(u)$.
2. Find the maximum rate of change of $f(x,y) = 2y^2\sqrt{x}$ at the point $(x,y) = (4,1)$ and the direction given in the unit vector **u** in which it occurs.
3. Can you express the directional derivative of f at point (x_0, y_0) in the direction of a vector $u = < a, b >$ (not necessarily a unit vector) as a limit?

Note: Make sure you justify every step, by definition, any theorems, or simply just the definition of the operation, etc, to help you study and really understand the concepts. If you are using a theorem, make sure to check the condition before using.