Definition

You should always check the lecture note and the textbook for the definition, and make sure the definitions are consistents.

1. A **vector space** is a set V together with two operations

   (a) $+ : V \times V \to V$
   (b) $\cdot : \mathbb{F} \times V \to V$

   such that

   (1) $\forall v, w \in V, v + w = w + v$
   (2) $\forall u, v, w \in V, (u + v) + w = u + (v + w)$
   (3) $\exists 0 \in V, \forall v \in V, 0 + v = v$
   (4) $\forall v \in V, \exists w \in V$ such that $v + w = 0$. Denote w by $-v$
   (5) $\forall v \in V, 1 \cdot v = v$
   (6) $\forall a, b \in \mathbb{F}, \forall v \in V, a \cdot (b \cdot v) = (ab) \cdot v$
   (7) $\forall a \in \mathbb{F}, \forall u, v \in V, a \cdot (u + v) = a \cdot u + a \cdot v$
   (8) $\forall a, b \in \mathbb{F}, \forall u \in V, (a + b) \cdot v = a \cdot v + b \cdot v$

   If $\mathbb{F} = \mathbb{R}$, we call V together with the operations the **real vector space**. If $\mathbb{F} = \mathbb{C}$, we call V together with the operatios the **complex vector space**.

   Remark. *Intuitively thinking, given a vector space V together with the operations $+$ and $\cdot$. V specifies the vectors that we are working with. The operations tell you how you can "add" two vectors in V and how you can "multiply" an element in V with a scalar in the field $\mathbb{F}$ as given in the definition of the operation. However the definition of vector space did not tell you how to multiply two vectors, so we should not assume this.*

Discussion

In each tutorial, we will have a discussion and go over sample questions in four subsections divided by trying to follow the four-phase method introduced by Polya.

0.1 Thinking about the definition

1. There are in fact four operations within the axioms of the definition of vector space, two ($+$ and $\cdot$) explicitly stated in the definition and two from the field $\mathbb{F}$. Look at each axioms and tell what the operations are corresponding to.

2. What is the 1 in the fifth axiom and how can you know this?

3. What is the 0 in the fourth axiom and how do you know such 0 exists?

0.2 Example

1. As we have seen in class, for $n \in \mathbb{N}^+$, $\mathbb{R}^n$ together with the usual operations is a real vector space. What are the 0 and the 1 in the definition for this particular vector space?

   Remark. *Try to give a real vector space with the operations other than the usual ones. Dont think about the complicated one, try to make an easy one.*

2. As we have seen in class, $\mathbb{P}_n(\mathbb{R})$ with the usual operations and $n \in \mathbb{N}^+$ is a vector space. Show that the axioms (3),(4) and (5) are true for this vector space by definitions. Include the details and justify every steps.

   - axiom(3): $\exists 0 \in V, \forall v \in V, 0 + v = v$
   - axiom(4): $\forall v \in V, \exists w \in V$ such that $v + w = 0$. Denote w by $-v$
   - axion(5): $\forall v \in V, 1 \cdot v = v$

   Remark. *Can you describe what $\mathbb{P}_n(\mathbb{R})$ is, instead of using the set notation?*

3. Let $n \in \mathbb{N} \cup \{0\}$, let V be the set of polynomial of degree n with coefficients in $\mathbb{R}$ with the operations similar to $\mathbb{P}_n(\mathbb{R})$. Write V using the set notation. Check whether or not V is a vector space.

0.3 More questions

1. Let V be a vector space. Prove the following statements.

   (a) The zero vector 0 is unique.

   Remark. *By proving this statement, you should learn how you could approacch to prove the uniqueness of an element.*

   Further practice. *Let V be a vector space. Prove that $\forall v \in V, -v$ is unique.*

   (b) $\forall v \in V, 0 \cdot v = 0$. Before proving the statement, check what the zeores are representing in this statement, and how do you know this?

   Remark. *By proving this statement, you should recognize why it is important to include the details and justifications in your proof, otherwise the proof can be quiet confusing.*

   Further practice. *Let V be a vector space. Prove that $\forall v \in V, \forall a \in \mathbb{F}, (-a) \cdot v = -(a \cdot x)$. Similar to the sample question, check what the negative signs are representing in this statement, and be careful with the operations before you start proving the statement.*

(c) Let V be a real vector space. Prove that if $v \in V, a \in \mathbb{R}$ and $a \cdot v = 0$, then either $a = 0$ or $v = 0$. Answer the following subquestions before proving the statement.

i. What are the hypothesis and conclusion?

ii. What are the zeroes representing in the statement?

iii. Look at the following proof and whether the following argument is valid?

Proof. Let $v \in V, a \in \mathbb{F}$.
If $a = 0$, then $a \cdot v = 0 \cdot v = 0$ as proven before.
If $v = 0$, then $a \cdot v = a \cdot 0 = 0$ as proven before.
Therefore we must have $a = 0$ or $v = 0$. $\square$

iv. Look at the following proof and whether the following argument is valid?

Proof. Let $v \in V, a \in \mathbb{F}$, assume $a \cdot v = 0$.
If $a \neq 0, v \neq 0$, then it is impossible to have $a \cdot v = 0$.
Therefore we must have $a = 0$ or $v = 0$. $\square$

v. Prove the original statement.

Remark. *By proving this statement, you should recognize why it is important to check every steps in your proof and make sure the math logic is correct. You will also learn how to do a proof by cases.*

0.4 Extend and review

Looking at the remarks from the previous questions, you might have learned something from each sample questions. After writing a proof, it is important to check whether the proof is correct and to identify the keys for why the proof works. However in general, there are a few things that you should keep in mind now.

1. Always ask the definitions. In particular, whenever we talk about addition and scalar multiplication, think about what those mean in the given content.
2. Intuition is good, but your proof should be more formal than the intuition.
3. Justify your steps to not confuse the audiences who will read your proof.
4. Review the math logic and various proof techniques. I will keep reminding you this.