Definition

You should always check the lecture note and the textbook for the definition.

1. Let V be a vector space. We say that $\{v_1, \cdots, v_n\} \subset V$ is linearly independent if

$$a_1 v_1 + \cdots + a_n v_n = 0 \text{ for some } a_1, \cdots, a_n \in \mathbb{F} \implies a_1 = \cdots = a_n = 0$$

   Otherwise, we call them linearly dependent.

2. A subset B of a vector space V is called a basis of V if

   (a) $span(B) = V$

   (b) B is linearly independent

Quick Discussion

Consider the vector space $F(\mathbb{R})$ which is the set of all functions from $\mathbb{R}$ to $\mathbb{R}$, over the field $\mathbb{R}$, equipped with the usual operation. Let $f_1, \cdots, f_n \in F(\mathbb{R})$, what does it mean to say that they are linearly independent by the definition? Recall that $f(x)$ and f are not the same thing.

An equivalent definition for linearly dependent is that we say $\{v_1, \cdots, v_n\} \subset V$ is linearly dependent if one of the vector can be written as the linear combination of the others, otherwise they are called linearly independent. Show that if $\{v_1, \cdots, v_n\} \subset V$ is linear dependent by the original definition, then it is linearly dependent by this equivalent definition.

Remark. *As an exercise, show the other direction. Conclude that the definitions are equivalent.*

Questions

Only a selection of questions will be discussed in the tutorial, but the sample solutions will be posted in the annotated note.

1. Consider the vector space $V = \mathbb{P}_2(\mathbb{R})$ with the usual operations. Let $B = \{1, 1+x, 1+x+x^2\}$.

   (a) Show that B is linearly independent directly using the definition.

   (b) Show that $span(B) = V$.

   (c) Conclude that B is a basis of V.

2. An intuitive idea of linearly dependent is that one vector is a scalar multiple of another vector. In fact, this is not correct in general. Let V be a vector space.

   (a) Let $v, w \in V$. Show that $\{v, w\}$ is linearly dependent if and only if v is a scalar multiple of w or w is a scalar multiple of v.

   (b) Give an explicit example to show that the following two statements are not equivalent. In another words, it is possible that one is true while the other is false.

   - v is a scalar multiple of w, i.e. $v = kw$ for some $k \in \mathbb{F}$
   - w is a scalar multiple of v, i.e. $w = kv$ for some $k \in \mathbb{F}$

   (c) Give an explicit example to show that a set can be linearly dependent, but none of the vectors can be a scalar multiple of another one vector.

3. Let $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ 4 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ be a set of vectors in $\mathbb{R}^4$. Find a largest possible subset of the given set, such that the subset is linearly independent. Check whether the subset is a basis, otherwise extend it to a basis.