Learning Objective

Understand how to sketch the feasible region in a two or three dimensional space. Understand the idea and the algorithm to solve a linear programming problem using the graphical method.

Definition

A vector $\mathbf{x} \in \mathbb{R}^n$ that satisfies the constraints of a given linear programming problem is called a **feasible solution** to the problem. A feasible solution that maximize (or minimize) given linear programming problem is called an **optimal solution**.

How to sketch a linear inequality

Here are the steps to sketch the graph of a linear inequility of two or three variables:

1. Sketch the linear equation with equility. This will give you a line or a plane.
2. Pick a particular point on either side of the line (or plane), check whether the point satisfies the given inequility.
3. If yes, the graph contains the side where the point lives. Otherwise, the graph contains the other side where the point does not live.

Consider the following examples for sketching the graphs of $x + y \leq 1$ and $x + y \geq 1$. A linear inequality with three variables is similar, but we usually have a plane in step 1 instead.

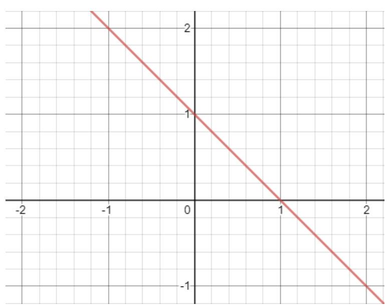

Figure 1: Sketch the linear equation $x + y = 1$ by joining the points $(1, 0)$ (when $y = 0$, get $x = 1$) and $(0, 1)$ (when $x = 0$, get $y = 1$)

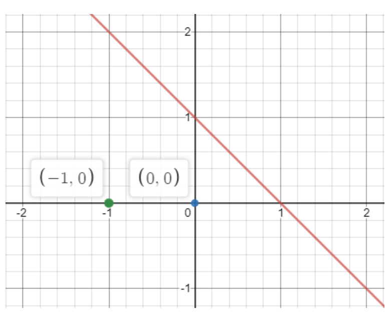

Figure 2: Find a point on either side of the line, both $(0, 0)$ and $(-1, 0)$ works here. The point on the line, such as (1,0) does not help.

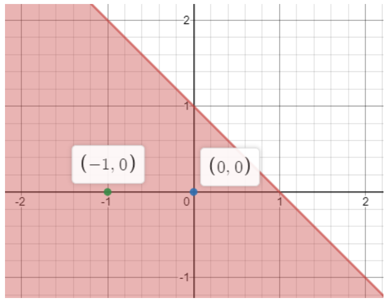

Figure 3: The points $(0,0)$ and $(-1,0)$ satisfy the inequility $x+y \leq 1$. Therefore this is the graph of $x+y \leq 1$.

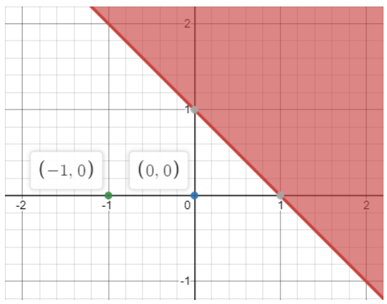

Figure 4: The points $(0,0)$ and $(-1,0)$ do notsatisfy the inequility $x+y \geq 1$. Therefore this is the graph of $x+y \geq 1$.

Solve a linear programming problem using Graphical Method

Level set of the objective function

For a given linear programming problem, the feasible region is the intersection of the graphs of all the constraints. Each point in the feasible region corresponds to a value when evaluated by the objective function (what you are trying to maximize or minimize).
For a given objective function $f(x_1, ..., x_n)$, define the level set

$$L_c(f) = \{(x_1, ..., x_n) \in \mathbb{R}^n | f(x_1, ..., x_n) = c\}$$

Intuitively saying, this is the set of all (not just the feasible ones) points such that they all have the same value when evaluated by the objective function f.
For a given objective function $f(x_1, ..., x_n)$, the gradient of f, $\nabla f = (\frac{\partial f}{\partial_{x_1}}, ..., \frac{\partial f}{\partial_{x_n}})$ points at the direction where the values of the objective function increases.

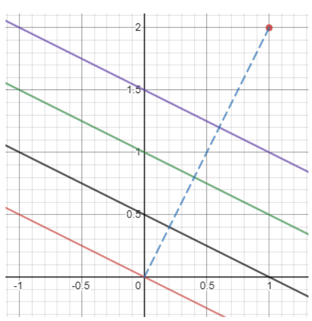

Figure 5: Level sets of $f(x,y) = x+2y$ with $c = 0,1,2,3$ and $\nabla f = (1,2)$

Graphical Method

We can use the graphical method to solve a linear programming problem by sketching the feasible region and the level sets, then identify where the optimal solution is by Looking at the level sets. Consider the following examples.

1. Find values x, y that will

   max $z = x + 2y$

   subject to

   $x + y \leq 1$

   $x \geq 0,\ y \geq 0$

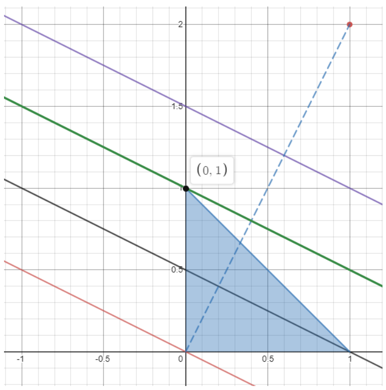

Figure 6: The values of the objective function increases in the direction of $(1, 2)$. Look at the level sets as the values increase (because we want to maximize the value), and notice that the point $(0, 1)$ is the "last" point before the level sets have no intersection with the feasible region. Therefore $(0, 1)$ is an optimal solution with value $z(0, 1) = 2$.

2. Find values x, y that will

   min $z = x + 2y$

   subject to

   $x + y \leq 1$

   $x \geq 0,\ y \geq 0$

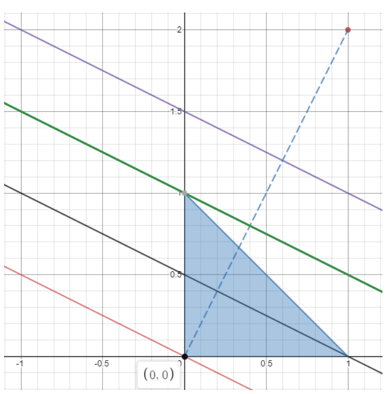

Figure 7: The values of the objective function decrease in the opposite direction of $(1, 2)$. Look at the level sets as the values decrease (because we want to minimize the value), and notice that the point $(0, 0)$ is the "last" point before the level sets have no intersection with the feasible region. Therefore $(0, 0)$ is an optimal solution with value $z(0, 0) = 0$.

Notice the feasible region is closed and bounded in $\mathbb{R}^2$ in the previous examples. Given the objective function is continuous, we know the maximum and minimum occur in the feasible region according to the extreme value theorem. What if the feasible region is not bounded?

1. Find values x, y that will

   max $z = x + 2y$

   subject to

   $x + y \geq 1$

   $x \geq 0,\ y \geq 0$

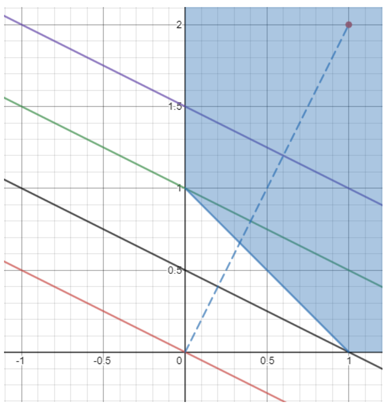

Figure 8: The values of the objective function increase in the direction of $(1, 2)$. Look at the level sets as the values increase (because we want to maximize the value), and notice that we can keep going further along the dashed line to find a feasible solution.

2. Find values x, y that will

   min $z = x + y$

   subject to

   $x + y \geq 1$

   $x \geq 0,\ y \geq 0$

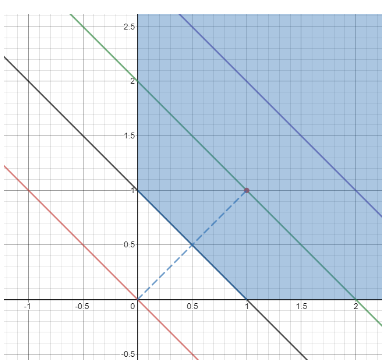

Figure 9: The values of the objective function decrease in the opposite direction of $(1, 1)$. Look at the level sets as the values decrease (because we want to minimize the value), and notice that the point $(1, 0)$ is a point before the level sets have no intersection with the feasible region. Therefore $(1, 0)$ is an optimal solution with value $z(1, 0) = 1$. Note that $(0, 1)$ is also an optimal solution. In fact, any points on the line segment joining $(1, 0)$ and $(0, 1)$ is an optimal solution according to the graphical method. Therefore the optimal solution is not always unique, and the number of optimal solutions can be infinite.