Intuition:

By this point, we should be familiar with partial derivatives in higher dimension. Now the question is, how should we differentiate a composite function. For example, if y=f(x) and x=g(t), what is $\frac{dy}{dt}$? We know how to do this using chain rule from first year calculus. If z=f(x,y), x=g(t) and y=h(t), then z is a function of t. What is $\frac{dz}{dt}$? This leads to our discussion of chain-rule in higher dimension.

Useful Formulas:

1. **The Chain Rule(General Version):**
   Suppose that u is a differentiable function of the n variables x_1, x_2, ... ,x_n and each x_j is a differentiable function of the m variables t_1, t_2, ..., t_m. Then u is a function of t_1, t_2, ... t_m and
   $$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + ... + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

2. **Implicit Function Theorem:**
   If F is defined on a disk containing (a,b), where $F(a,b) = 0$, $F_y(a,b) \neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x,y) = 0$ defines y as a function of x near the point (a,b) and the derivative of this function is given by
   $$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

3. Suppose that z is given implicitly as a function z=f(x,y) by an equation of the form F(x,y,z)=0. This means that F(x,y,f(x,y))=0 for all (x,y) in the domain of f, then
   $$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$
   $$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Questions:

1. Use the Chain Rule to find dz/dt: $z = xy^3 - x^2y$, $x = t^2 + 1$, $y = t^2 - 1$

2. Use the Chain Rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$: $z = (x - y)^5$, $x = s^2t$, $y = st^2$

3. Suppose that z=f(x,y) has continuous second order partial derivatives and $x = s^2 - t^2$, $y = 2st$. Use the chain rule to find the function g(s,t) satisfying the following equation:
$$\frac{\partial^2 z}{\partial s^2} + \frac{\partial^2 z}{\partial t^2} = g(s,t)(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2})$$

4. Let $p(t) = f(g(t), h(t))$, where f is differentiable, $g(2) = 4$, $g'(2) = -3$, $h(2) = 5$, $h'(2) = 6$, $f_x(4,5) = 2$, $f_y(4,5) = 8$. Find $p'(2)$

5. Find the equations of all planes that pass through the points (1,1,1) and (2,0,1) and also are tangent to the surface $\frac{x^2}{2} + \frac{y^2}{2} + z^2 = 1$
   (a) Let $F(x,y,z) = \frac{x^2}{2} + \frac{y^2}{2} + z^2$. What is F_x, F_y, and F_z
   (b) Notice the surface $\frac{x^2}{2} + \frac{y^2}{2} + z^2 = 1$ is the level surface of the function $F(x,y,z)$. What is the equation of the tangent plane to the level surface at (a,b,c)?
   (c) Can you simplify the equation, given that the point (a,b,c) is on the surface(i.e. satisfies the equation)?
   (d) Deduce an equation of a,b,c, given the conditions the points on the plane passed through the points(1,1,1) and (2,0,1)