Understanding-Connecting-Proving-Review

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"University math is so much different from what I have learned in high school." This is a sentence which was said by many universities students, including myself, who were studying Calculus and linear algebra back in their first years. Different from high school math which students can pass a test by memorizing a formula or algorithm to a problem, without truly understanding the question, university math does require a deeper understanding of the concept and justification of what you write on a test. This level of understanding and reasoning will have to be much higher when solving an abstract mathematical problem, which cannot be easily visualized most of the times, such as a problem in group theory or linear algebra. In this article, we will introduce a method, which I summarized as "understanding-Connecting-Proving-Review", to solve mathematical problems from "How to Solve It" by G. Polya.

"How to Solve It" introduces a method of solving and teaching to solve mathematical problems from the views of students, teachers and sometimes just anyone who is interested in the technique of teaching mathematics. From the views of students or anyone who is trying to solve a mathematical problem in general, Polya mentioned that it was extremely important to understand the question before starting, including what the unknown, data and conditions are. One of the biggest challenges that the students have found the most difficult is that "I don't know how to express my idea correctly", according to one of my previous experience in being a teaching assistant for linear algebra II course. It is often the case that students may have some ideas of how to solve the problem, because the teachers have solved, or they have seen a similar problem before. However, the answers using such intuition only will very likely result some mistakes in their argument or even false statements, because they may have incorrectly understood what the question was asking. "I thought it was correct because it sounded correct" or "I thought that was what it meant", other sentences said by the students. Polya suggests that when understanding the question, it is necessary to introduce suitable notations to express your unknown, data and conditions. I rephase this sentence as following the definitions, so people can claim that what they write is correct by the definition, instead of intuition which is often unreliable. Consider a sample question which said, "Find two linearly independent vectors in the vector space of the set of polynomials with degree less than or equal to 2, equipped with the usual operations", it is a math problem which is hard to visualize, in linear algebra. As a result, following Polya's method might be a good idea to start. The unknown which we sometimes refer to as our goal, here is the "two vectors" that satisfies the condition which says, "linearly independent", given the data "set of polynomials with degree less than or equal to 2, equipped with the usual operation", which we sometimes refer to as our given assumption. Our goal here is to find "two such vectors". One can give further explanations of what those terms mean using their knowledge of linear algebra, to make this question clearer in some sense. The benefit of this step is that students now can make sure they understand what the question is asking, have some references to support their arguments, and at the very least have something to start with, rather than staring at the question and thinking randomly. Once we have understood the question, the next step is to find a way to connect the given unknown, data and conditions together to create a proof, because our final goal is to solve the question.

Polya suggests devising a plan after truly understanding what the question is asking. I call this step "connecting". There are multiple ways to achieve such a plan. For example, one can remind him/herself whether they have seen a similar problem like this before. Besides, one can also dig into the definition of the unknown, data and conditions to see if there is a way to relate them. Consider the same example in the previous paragraph, one should ask what the definition of "two vectors" mean, given the data "in the vector space of the set of polynomials of degree less than or equal to 2 equipped with the usual operations". Another question like, "what is a vector?", can also be asked, in order to see the connection between the unknown and the data. Some examples of such "vectors", such as "1, 2, or x?" should come up after some thinking using their knowledge of linear algebra, which is an amazing progress toward our final goal of solving the problem. There is also a "condition" from our first step, which we need to make sure that our unknown satisfies. The condition says, "linearly independent", which can be again checked to see what the definition is using the knowledge in linear algebra. Lastly, give some trials on the examples and see if they satisfy the condition. It is very easy to get lost when finding a connection among those unknown, data and conditions, so it is important to keep in mind that the final goal is to solve the problem, which is to solve for the unknown. Once you have devised a plan and built some connection, it is now the time to carry out the plan and start the proof.

The third and fourth steps before we reach our final goal of solving the problem is to carry out the plan that you have from the last step. I call these "proving-review". At this stage, students should have good understands of the questions and some ideas to solve the problems. In step 2, students should have found the connections and have the plans to write the answer. "The plan gives a general outline", say by Polya, and in the third step, "we have to convince ourselves that the details fit into the outline.” In the previous example, assume that the students have found some examples of “vectors” from the data that satisfies the condition of being “linearly independen”. When writing an answer, the students should be able to express and claim that they have found the desired unknown using correct notation and mathematical language, by finding a way to put everything together, so people who are reading the only solutions can understand. When writing down the solution, Polya suggests that we should be very careful in distinguishing between “seeing” and “proving”. In step 4, students should look back at their answers and make sure that they are able to stand for their answers with some references to support their argument, when people are questioning the students’ work. Students should make sure that their proof in step 3 is a valid proof, but just seeing. In the previous example, if a student thinks his/her answer is correct, they should have proven why the two vectors that they present are linearly independent, in his/her answer. A question that a student should always ask him/herself is that whether or not his/her solution is lack of reasoning, and how much details they should include to make sure. One can think of this last step as building a tower. You know the type of tower you are asked to build and the materials you are given in the first step of “understanding”, and you have a plan of how to build the tower using the materials in the second step of “connecting”, the third step will be to actually build the tower, and make sure that the tower is stable in the fourth step. When building a tower, people will want to make sure the upper floors are stable and supported by the lower floors, otherwise the tower will fall, and it is not a good tower. The idea is very similar when writing a proof or doing computations. We always want to make sure every argument that is made, is supported by the previous correct arguments, otherwise such a proof is untenable and will sometimes lead to a false statement, which cause the whole answer to be invalid. Once we have confirmed the validity of one answer, there is another way to convince ourselves that the argument is correct by trying to find another way to derive the result differently. The question did not end even when it is finished and proven. Lastly, once we have confirmed that the conclusion is correct followed from the proofs, we should always extend this by checking whether we can apply the result to another questions.

The four steps method “understanding-connecting-proving-Review” is a very useful strategy, but at the same time smoothly going through the steps might be hard, as mentioned by Polya. It is very common to get stuck and have no ideas how to move on in any of the stage. Teachers at this stage will play important roles in helping the students to learn. Polya mentioned that teachers should not be solving the problems for the students, but instead leading them to solve the problem using the method that was introduced. It is a good idea to ask questions instead of telling the students what to do, so that the students can think and remember what to do next time when they are stuck in any of the stage. Teachers can also remind them the method when the students forget. In the previous example, instead of saying “the vectors you want to find should be the form of polynomials of degree less than or equal to 2”, the teacher can ask “What should the vectors look like?” or “What is your vector space?”, which will force the students to think and look at the definitions. In step 3 - “proving”, it is very common for students to miss some reasonings and, see but don’t know how to prove their arguments. In this case, Polya suggests that the teacher give the students hint and go through the reasonings by asking the students to prove what they wrote. Following this method of teaching, Polya says that the students will get more confident in solving mathematical problems because they know what the question is asking in step 1, they know how to find the connection in step 2, and most importantly they make sure their answer is correct without any lack of reasoning in step 3.

“I have no idea how to start?” is a sentence which is usually said by the students when trying to solve a math problem. Using the method that I summarized as “understanding-connecting-proving” introduced by Polya, students should have something to start which is to understand what the question is asking, and at the very least one can even focus on every single words and symbols in the questions. There is always a question to ask when we have a math problem to solve, “what is the definition?”.

Reference

Polya, G. & Conway, J. H. (2014). Part I: In the Classroom; Part II: How to Solve It. In How to solve it: A new aspect of mathematical method (pp. 1-33). Princeton, NJ, US: Princeton University Press.