MAT337_TUT2

Tuesday, September 29, 2020 1:58 PM

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Strategy to Test Series Convergence

- If $lim_{n\to\infty} a_n \neq 0$, then the series diverges by the **Divergence theorem**
- If a series is a **p-series**, we know it converges if $p > 1$ and diverges otherwise $a_n = \frac{1}{n^p}$
- If a series is a **3.2.2:geometric series**, with $a_n = ar^n$, $a \neq 0$, then the series converges if $|r| < 1$ and diverges otherwise. In that case, $\boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}}$ when $|r|<1$
- If a series is similar to a p-series or a geometric series, you should consider a **3.2.3:comparison test** or a **3.2H:limit comparison test**.
- If the series has alternating signs $(-1)^n$, then the **3.2.6:alternating series test** will be helpful, if the terms are decreasing.
- If the terms a_n contain factorials or factorials and $n(th)$ powers, then try the **3.2I:ratio test**. $n!$ r^n
- If the terms contain $n(th)$ powers, the **3.2.4:root test** may be helpful $(e^n)^n$
- If the terms are positive and decreasing, the **3.2N:integral test** may be helpful if it can be easily integrated (when viewed as $f(x)$ where $f(n) = a_n$)
- 3.3.3 Proposition: An **absolutely convergent** series is convergent.

Questions:

- 3.1.A: Sum the series $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$
- $\boxed{3.1.B}$ Sum the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+3)(n+4)}$ $\boxed{\text{partial fraction, } \frac{12}{n(n+1)(n+3)(n+4)} = \frac{1}{n} - \frac{2}{n+1} + \frac{2}{n+3} - \frac{1}{n+4}.}$
- 3.2.P: Determine whether the following series converge or diverge.
  - $\sum_{n=2}^{\infty} \frac{3n}{n^3+1}$
  - $\sum_{n=1}^{\infty} sin(n\pi/4)$
  - $\sum_{n=1}^{\infty} \frac{n!}{n^n}$
  - $\sum_{n=1}^{\infty} \frac{n}{2^n}$
  - $\sum_{n=1}^{\infty} (-1)^n sin(1/n)$
  - $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$
  - $\sum_{n=1}^{\infty} e^{-n^2}$
  - $\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)^n$
  - $\sum_{n=2}^{\infty} \frac{1}{(logn)^n}$

1. sum the series $\boxed{\sum_{n=1}^{\infty} \frac{1}{n(n+2)}}$ converges, $\sum \frac{1}{n(n+2)} \le \sum \frac{1}{n^2}$ converges by p-series. $p = 2 > 1$

sln: $a_n = \frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$

for all n $= \frac{A(n+2)+Bn}{n(n+2)}$

$= \frac{\boxed{(A+B)n + 2A}}{n(n+2)}$

$\begin{cases} A+B = 0 \\ 2A = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{2} \\ B = -\frac{1}{2} \end{cases}$

s.t. $a_n = \frac{\frac{1}{2}}{n} + \frac{(-\frac{1}{2})}{n+2} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$

$\boxed{S_k} = \sum_{n=1}^{k} \frac{1}{n(n+2)} = \sum_{n=1}^{k} \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$

$= \frac{1}{2}\left[\left(\sum_{n=1}^{k} \frac{1}{n}\right) - \left(\sum_{n=1}^{k} \frac{1}{n+2}\right)\right]$ $\frac{1}{3}, \cdots \frac{1}{k+2}$

$= \frac{1}{2}\left[\left(\sum_{n=1}^{k} \frac{1}{n}\right) - \left(\sum_{n=3}^{k+2} \frac{1}{n}\right)\right]$ ☆

$= \frac{1}{2}\left[\frac{1}{1} + \frac{1}{2} + \sum_{n=3}^{k} \frac{1}{n} - \sum_{n=3}^{k} \frac{1}{n} - \frac{1}{k+1} - \frac{1}{k+2}\right]$

$= \boxed{\frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{k+1} - \frac{1}{k+2}\right)}$

$\boxed{\sum_{n=1}^{\infty} \frac{1}{n(n+1)}} = \lim_{k\to\infty} \boxed{S_k} = \lim_{k\to\infty} \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{k+1} - \frac{1}{k+2}\right)$

$= \frac{1}{2}\left(\frac{3}{2}\right) = \frac{3}{4}$ ∎

2. $\sum_{n=2}^{\infty} \frac{3n}{n^3+1}$. similar to $\frac{1}{n^2}$

① Using comparison test

$a_n = \frac{3n}{n^3+1} \le \frac{3n}{n^3} = \frac{3}{n^2}$ $\frac{1}{n^p}$, converges $p > 1$, diverge others.

$\sum_{n=2}^{\infty} \frac{3n}{n^3+1} \le \sum_{n=2}^{\infty} \frac{3}{n^2}$ converges by p-series.

② Ratio Test ✗

$a_n = \frac{3n}{n^3+1}$, $a_{n+1} = \frac{3(n+1)}{(n+1)^3+1}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3(n+1)}{(n+1)^3+1} \cdot \frac{n^3+1}{3n}$

$= \lim_{n\to\infty} \frac{(3n+3)(n^3+1)}{(n^3+3n^2+3n+1+1)(3n)}$

$= \lim_{n\to\infty} \frac{3n^4 + 3n + 3n^3 + 3}{3n^4 + 9n^3 + 9n^2 + 6n}$

$= \lim_{n\to\infty} \frac{3 + \frac{3}{n^3} + \frac{3}{n} + \frac{3}{n^4}}{3 + \frac{9}{n} + \frac{9}{n^2} + \frac{6}{n^3}}$

$= 1$ No conclusion

3. $\sum_{n=1}^{\infty} \frac{n}{2^n}$

$a_n = \frac{n}{2^n}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n}$

$= \lim_{n\to\infty} \frac{n+1}{n} \cdot \boxed{\frac{1}{2}}$

$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1} \cdot \frac{1}{2} = \frac{1}{2} < 1$

s.t. by ratio test, $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges

4. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt[n]{n}}$

① Maybe alternating series test.

$a_n = \frac{1}{\sqrt[n]{n}} = \frac{1}{n^{\frac{1}{n}}}$, $a_1 = \frac{1}{\sqrt[1]{1}} = 1$, $\boxed{a_2 = \frac{1}{\sqrt[2]{2}} = \frac{1}{\sqrt{2}}}$?

$\boxed{a_6 = \frac{1}{\sqrt[6]{6}}}$

$(a_6)^6 = \left(\frac{1}{\sqrt[6]{6}}\right)^6 = \frac{1}{6}$

$(a_2)^6 = \left(\frac{1}{\sqrt{2}}\right)^6 = \left[\left(\frac{1}{\sqrt{2}}\right)^2\right]^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8} < \frac{1}{6}$.

s.t. $a_2{}^6 < a_6{}^6$

since a_2, a_6 are positive,

s.t. $a_2 < a_6$.

Not decreasing, can't use alternating series test.

② Divergence theorem: If $\lim_{n\to\infty} a_n \neq 0$, then series diverges

$\boxed{a_n = \frac{(-1)^n}{\sqrt[n]{n}}.}$

$\boxed{\lim_{n\to\infty} \sqrt[n]{n}} = \lim_{n\to\infty} n^{\frac{1}{n}}$ $\boxed{\infty^0}$?

$= \boxed{\lim_{n\to\infty} e^{\log(n^{\frac{1}{n}})}} = \lim_{n\to\infty} e^{\frac{1}{n}(\log n)}$.

$\boxed{\lim_{n\to\infty} \frac{1}{n}(\log n)} = \lim_{n\to\infty} \frac{\log n}{n}$ $\to \infty$, $\to \infty$

$\stackrel{L.H.}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$

s.t. $\boxed{\lim_{n\to\infty} \sqrt[n]{n} = e^0 = 1}$ ☆

s.t. limit DNE, and hence in particular not zero

By divergence thm, $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt[n]{n}}$ diverges ∎

5. $\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)^n$

① Root test

$a_n = (\sqrt[n]{n} - 1)^n$

Root test, $\limsup_{n\to\infty} \sqrt[n]{a_n} = \limsup_{n\to\infty} (\sqrt[n]{n} - 1) = 1 - 1 = 0 < 1$, s.t. converges

② Ratio Test ?

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(\sqrt[n+1]{n+1} - 1)^{n+1}}{(\sqrt[n]{n} - 1)^n}$