# Using the Simplex Method to Solve a LPP

There are multiple ways to set up a simplex tableaux to solve a given LPP using simplex method in general by picking different starting BFS (basic feasible solution) and using different notation. The method shown below follows the one that was introduced in lecture with the consistent notation.
Given a linear programming problem in its standard form

Max the **objective function** $z = c_1x_1 + ... + c_nx_n$
subject to the **Constraints**

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \leq b_1\\
&a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \leq b_2\\
&\qquad\qquad\vdots\\
&a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \leq b_m\\
&x_1, x_2, ..., x_n \geq 0
\end{aligned}$$

**Remark.** *Recall, what are the conditions for being a standard LPP? Why can we only consider LPP in standard form? What if we started with a general or canonical LPP?*

The standard LPP can be written in matrix notation as follows.
Recall, we can convert this to canonical form by introducing $m$ (the number of $\leq$ constraints) slack variables, called $x_{n+1}, ..., x_{n+m}$, and write the canonical LPP in matrix notation as usual.

Max $z = c^T x$
subject to
$Ax = b$
$x \geq 0$

**Remark.** *What should $A, c, x$ and $b$ look like given the above construction?*

Here are the steps to construct a simplex tableaux and solve the LPP using the simplex method. Make sure your LPP is in canonical form using matrix notation before you start.

1. Find the BFS $x = (0, \cdots, 0, x_{n+1}, \cdots, x_{n+m})$, where $x_{n+1}, \cdots, x_{n+m}$ are basic variables, and all other variables are non-basic variables, to start.

   **Remark.** *Using the definition from section 1.5 or last tutorial, given the canonical form in matrix notation, why is such $x$ always a BFS? Equivalently, what is such $x$ geometrically?*

2. Construct the tableaux as follows.

$$\left[\begin{array}{cc|c} 0 & A & b \\ \hline -1 & c^T & 0 \end{array}\right]$$

   with the following intepretations:

   - The top part with $A \mid b$ represents how the given BFS satisfies the constraint $Ax = b$ which can be used to solve for the BFS by solving the argumented matrix with the corresponding columns of $A$
   - The last row is called the objective row, where $c$ represents how the cost will change when changing to the corresponding variable as basic variable, and the bottom-right corner represents the **negative** of the current cost at the given BFS.

**Remark.** *Given the starting BFS in the previous step, why is the cost always 0 when we started? Check and convince yourself the intepretations above at the starting BFS.*

**Notation.** *Notice, you have* $\left[\begin{array}{ccc|c} 0 & A & I & b \\ \hline -1 & c^T & 0 & 0 \end{array}\right]$ *instead in the lecture note, where $A, b, c$ are the ones in the standard form LPP. Verify why the two tableauxes in different notations end up being the same tableaux given a fixed LPP? We will use the one with the canonical form notation below, but the processes are exactly the same to the one in the lecture note. You might see other ways of constructing the tableaux in other resources, but the ideas are the same.*

3. We are going to solve the LPP using the simplex method as follows.

   (a) Look at the left part of the objective row, and if there is a positive entry, go to next substep. If there is no positive entry, then the current BFS is the optimal solution and you may go to the next step by skipping all the substeps below.

   **Remark.** *What does it mean to say there is a positive entry using the intepretation? Why do we get the optimal solution if there is no positive entry?*

   (b) Choose one positive entry (it does not matter which one) from last substep, look at the column corresponding to $x_j$ above for a **positive** entry $a_{ij}$, such that $\frac{b_i}{a_{ij}}$ is the minimum such ratio among all possible $a_ij$, where $b_i$ is the corresponding $i$ entry of the top-right corner at the current BFS.

   **Remark.** *Recall that we want to be careful finding $a_{ij}$, because that intuitively represents how much change we can make to the new basic variable, and the ratios represent the amounts of changes that are allowed in order to still satisfies the corresponding constraint. Besides, recall that a non-positive $a_{ij}$ represents the fact that we can keep increasing $x_j$ without violating the corresponding constraint, but possibly violate other constraints. As a result, choosing a larger ratio, or non-positive $a_{ij}$ might violate other constraints, and hence we should choose the positive $a_{ij}$ with the smallest ratio. Write down some explicit constraints to think about the above intuitive explanations.*

   If there is no such positive $a_{ij}$ with such positive entry in the objective row, then the optimal cost is infinite, and there is no optimal solution.

   **Remark.** *Why is this the case using the previous remark? Why is the optimal cost positive infinite, not negative infinite, given the LPP? This should convince yourself why we wwant to make the standard LPP a maximization problem.*

   (c) Given that you have a good choice of $a_{ij}$, convert that column as the pivot column and the entry $a_{ij}$ as pivot using row operations.

   **Remark.** *Intuitively thinking, we moved from one BFS to another BFS with a higher cost during this process, such that we have a different set of basic variables and non-basic variables, and hence a new tableaux given the new current BFS. With the notion of basic direction, this idea can be further discussed geometrically. While most things in the tableaux change, the intepretations above work at the new BFS.*

   (d) We are now at the new BFS, go back to substep (a) until this loop ends at some point for any reason at any of the intermediate substeps.

4. Now you have stopped at an optimal solution or concluded that the LPP has no optimal solution. If there is such optimal solution, the corresponding bottom-right corner of the tableaux represents the **negative** of the optimal cost. We will solve the argumented matrix $A' \mid b$ where $A'$ is the matrix with the columns of the basic variables at the current BFS (though we are not sure what it looks like), for the optimal solution that occurs at that BFS, by setting all other non-basic variables zeores.

   **Remark.** *Why does the optimal solution, if exists, always exists at a BFS?*

## Questions

Last week, you see the following LPPs during the quizzes in one of the tutorial sections. Try to solve them using the simplex method and see whether you get the same answers. More practices will be given in the worksheet which will contain exammples with various situations that were discussed in the remarks above.

- TUT3: Solve the following linear programming problem.

  max $z = x_1 + 2x_2 - x_3$

  subject to

  $2x_1 - 6x_2 + 3x_3 \leq 12$

  $2x_2 + x_3 \leq 2$

  $x_1, x_2, x_3 \geq 0$

  **Answer.** *The optimal solution occurs at* $x = \begin{pmatrix} 9 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ *with optimal cost* $z = 11$

- TUT4: Solve the following linear programming problem.

  max $z = 2x_1 + x_2 - x_3$

  subject to

  $2x_1 - 3x_2 + 6x_3 \leq 12$

  $x_2 + 2x_3 \leq 2$

  $x_1, x_2, x_3 \geq 0$

  **Answer.** *The optimal solution occurs at* $x = \begin{pmatrix} 9 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ *with optimal cost* $z = 20$