Question 3: True or false? Justify your answer. That is to give an explicit counter example if you think the statement is false, or to prove that the statement if you think it is true. (5 points each subquestions)
In general,

- 1/5 for a correct answer of true or false without justification.
- 5/5 for a correct answer with correct justifications that demonstrate the correct understanding of the concepts
- -1 for each missing justifications or minor mistakes
- partial marks might be granted if you have shown some correct understanding of the concepts.

Here are the answers.

(a) False

(b) True

(c) False

(d) True

(e) True

(f) False

(g) False

Justifications as follow.

(a) The union of finite collections of convex sets still a convex set.

Proof. This statement is false and we will show that its negation is true by giving a counter example. Let $A = [0, 1], B = [2, 3]$. Notice the sets A and B are convex (one can check that they are by the definition of convex set) Let $C = A \cup B$ which is an union of finite collections of convex sets. We want to show that C is not convex. Pick $x = 1 \in C, y = 2 \in C, \lambda = \frac{1}{2} \in [0, 1]$, then

$$\lambda x + (1 - \lambda)y = \frac{1}{2} \cdot 1 + (1 - \frac{1}{2}) \cdot 2 = \frac{3}{2} \notin C$$

Therefore C is not convex as required, and hence the original statement is false. □

Remark. *A graphical proof is also valid with enough and correct justifications. Sketching a graph can be a good intuition to disprove this statement.*

(b) The intersection of finite collections of convex sets still a convex set.

Proof. This statement is true. Consider the following direct proof. Let $A_1, ..., A_n$ be a finite collection of convex sets, for some $n \in \mathbb{N}^+$, and A_i is convex for all $i = 1, \cdots, n$. We want to show that $C := \bigcap_{i=1}^{n} A_i = A_1 \cap \cdots \cap A_n$ is convex.

Let $x, y \in C$, let $\lambda \in [0, 1]$, we want to show that $\lambda x + (1 - \lambda)y \in C$.

Since $x \in C = \bigcap_{i=1}^{n} A_i$, such that $x \in A_i$ for all $i = 1, \cdots, n$ by the definition of intersection.

Similarly $y \in C = \bigcap_{i=1}^{n} A_i$, such that $y \in A_i$ for all $i = 1, \cdots, n$

For any $i = 1, \cdots, n$, A_i is convex and given $x, y \in A_i$, $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y \in A_i$ by the definition of convex set, for any $i = 1, \cdots, n$

Therefore $\lambda x + (1 - \lambda)y \in C$ by the definition of intersection, as required. $\square$

Remark. *A proof by induction is not necessary here. A direct proof is sufficient.*

(c) Let C, D be two convex sets in $\mathbb{R}^n$ that do not intersect. Then there always exists a non zero $\vec{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$, such that $\vec{a}^T x < b$ for all $x \in C$ and $a^T x > b$ for all $x \in D$.

Proof. This statement is false. If the original statement is true, then it is true for all n. We want to show that its negation is true by giving a counter example. Pick $n = 1$. Pick $C = (0, 1)$ and $D = [1, 2]$. (One can check that C, D are convex sets.) Notice C and D are convex sets that do not intersect. Consider the following proof by contradiction. Assume for the sake of contradiction that, there always exists a non zero $\vec{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$, such that $\vec{a}^T x < b$ for all $x \in C$ and $a^T x > b$ for all $x \in D$.

Since $n = 1$, equivalently there exists non-zero $a \in \mathbb{R}$, and there exists $b \in \mathbb{R}$, such that $ax < b$ for all $x \in (0, 1)$ and $ax > b$ for all $x \in [1, 2]$.

Consider the following proof by cases on $a \neq 0$

Case1: If $a > 0$, then $x < \frac{b}{a}$ for all $x \in (0, 1)$, and hence $1 \leq \frac{b}{a}$, otherwise we will have a contradiction to the universal condition (for all $x \in (0, 1)$) , and hence the original statement is false. Besides, $x > \frac{b}{a}$ for all $x \in [1, 2]$, and hence $\frac{b}{a} < 1$, otherwise we will have a contradiction to the universal condition (for all $x \in [1, 2]$). Together we have

$$1 \leq \frac{b}{a} < 1$$

and hence $1 < 1$ which is a contradiction. Therefore the original statement is false.

Case2: If $a < 0$, then $x > \frac{b}{a}$ for all $x \in (0, 1)$, and hence $1 > \frac{b}{a}$, otherwise we will have a contradiction to the universal condition (for all $x \in (0, 1)$) , and hence the original statement is false. Besides, $x < \frac{b}{a}$ for all $x \in [1, 2]$, and hence $\frac{b}{a} \geq 1$, otherwise we will have a contradiction to the universal condition (for all $x \in [1, 2]$). Together we have

$$1 \leq \frac{b}{a} < 1$$

and hence $1 < 1$ which is a contradiction. Therefore the original statement is false.

In conclusion, we have proven that the negation of the original statement is true, and hence the original statement is false. □

Remark. *A graphical proof is also valid with enough and correct justifications with the idea of hyperplane.*

(d) There exists a linear programming system such that its feasible region is defined by the following expression $\{(x, y) \in \mathbb{R}^2 | y \leq 4x, y \geq 0.5x\}$

Proof. This statement is true. Consider the following linear programming problem.

max $z = x + y$

subject to

$$y - 4x \leq 0$$

$$0.5x - y \leq 0$$

which has the feasible region as defined by the given expression. □

(e) There exists a linear programming system such that its feasible region is defined by the following expression $\{(x, y) \in \mathbb{R}^2 | y \geq 4x, y \leq 0.5x\}$

Proof. This statement is true. Consider the following linear programming problem.

max $z = x + y$

subject to

$$4x - y \leq 0$$

$$y - 0.5x \leq 0$$

which has the feasible region as defined by the given expression. □

Remark. *A linear programming system should contain the decision variables, Objective function and constraints. A key idea here is that the feasible region of a linear programming system is always convex.*

Remark. *One can also draw the feasible region as defined by the expression in $\mathbb{R}^2$.*

Remark. *Note that this question is not asking whether the feasible region is empty. It is possible that a linear programming problem has empty feasible region. The question is also not asking whether the optimal solution exists.*

(f) The number of extreme points of any convex set in $\mathbb{R}^n$ is finite.

Proof. This statement is false. Consider the counter example $C = \{\vec{x} \in \mathbb{R}^2 | ||\vec{x}|| \leq 1\}$ which is a convex set, but every points on its boundary is an extreme point, and there are infinitely many points on its boundary. □

Remark. *A graphical proof is also valid with enough and correct justifications.*

(g) If $A\vec{x} = \vec{b}$ defines a non-empty feasible region, then $A^T\vec{y} = 0, y^T b < 0$ may still define a non-empty feasible region.

Proof. This statement is false. Consider the following counter example in $\mathbb{R}$. Pick $A = I$, $b = \vec{0}$, then $A\vec{x} = \vec{b}$ defines the non-empty feasible region as follow:

$$\{\vec{x} \in \mathbb{R} | A\vec{x} = \vec{b}\} = \{\vec{x} \in \mathbb{R} | I\vec{x} = 0\} = \{0\}$$

Assume for the sake of contradiction that there exists some $y \in \mathbb{R}$, with $A^T\vec{y} = 0, y^T b < 0$. However $0 = A^T\vec{y} = I^T y = y$, such that if such y exists, then $y = 0$. Notice we also have the condition that $0 > y^T b = 0$ which is a contradiction.

Therefore such y does not exist, and hence $A^T\vec{y} = 0, y^T b < 0$ define a empty feasible region. In conclusion, the original statement is false. □

Remark. *In fact, for any $A \in M_{m\times n}(\mathbb{R})$, $b \in \mathbb{R}^m$. If both conditions define non-empty feasible regions, in other words, there exists $\vec{x} \in \mathbb{R}^n$, and there exists $\vec{y} \in \mathbb{R}^m$, such that $A\vec{x} = \vec{b}$ and $A^T\vec{y} = \vec{0}, y^T b < 0$. Then*

$$\begin{aligned} 0 > y^T b &= y^T A\vec{x} \text{ since } A^T\vec{x} = b \\ &= (A^T y)^T x \text{ by property of transpose} \\ &= \vec{0}^T\vec{x} = 0 \end{aligned}$$

we have a contradiction here. Therefore, the original statement is false.

Remark. *One can also refer to the Farkas's lemma as proven in class, with a slightly modification and a valid justification.*