MATB44 TUT0005: Final Review

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This document is for the students in MATB44, TUT0005 fall 2020 at the University of Toronto Scarborough. You should not use this document as your reference in the final exams. Everything covered in this document have been talked about in the lectures or in the textbook. The purpose of this document is for students to do more practices at various types of questions they have seen in class, mainly focusing on the materials covered after the midterm.

Contents

1 Method of Variation of Parameters

1.1 Algorithm

To solve a second order non-homogeneous differential equation of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

Note: If the equation is not in this standard form, you will need to convert it into standard form before using this method.

We first solve the corresponding homogeneous equation $y'' + p(t)y' + q(t)y = 0$ first and get the general solution to the homogeneous equation, $c_1y_1 + c_2y_2$. Refer to the midterm review file for algorithms for solving homogeneous equations.

Then we solve the original non-homogeneous equation as follow:

- Set $y = u_1y_1 + u_2y_2$ where u_1, u_2 are unknown functions of t.

- Write the **Lagrange system**

  $$u_1'y_1 + u_2'y_2 = 0$$

  $$u_1'y_1' + u_2'y_2' = g(t)$$

  Solve the Lagrange system for u_1, u_2 using the same idea as solving the system of equations with two unknowns. Keep the arbitrary constants c_1, c_2 when solving the integrals in the expression of u_1 and u_2.

- Then the general solution of the given non-homogeneous equation is given by

  $$y = u_1y_1 + u_2y_2$$

1.2 Practice problems

Use the method of parameters to find the general solutions of the given differential equations.

1. $y'' - 5y' + 6y = 2e^t$
2. $y'' + y = tan(2t)$
3. $y'' + y = tan(t) + e^{3t} - 1$

2 System of Linear Equations with Constant Coefficients

Instead of going over the general algorithm of solving system of linear equations with constant coefficients, we will go over some examples. The basic idea is to find the eigenvalues of the matrix, find the eigenvectors or the generalized eigenvectors, and form the fundamental set of solutions.

2.1 Finding Eigenvalues, eigenvectors and generalized eigenvector of a matrix A

1. To find the eigenvalues r of an $n \times n$ matrix A, we solve the characteristic equation $det(A - rI) = 0$ for r.

2. To find the eigenvectors v corresponding to eigenvalue r, we solve the equation $(A - rI)v = 0$ for a **non-zero** r. Equivalently solve the equation $Av = rv$.

3. To find the generalized eigenvector s corresponding to eigenvalue r and the eigenvector v, we solve the equation $(A - rI)s = v$ for a **non-zero** s. Equivalently solve the equation $As = rs + v$.

2.2 Real Distinct Eigenvalues

In general, you will be able to find "enough" eigenvectors with the corresponding eigenvalues to form the fundamental set of solution.

1. Let's start with a 2×2 matrix. $x' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} x$

2. How about a 3×3 matrix, $x' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} x$

2.3 Real Repeated Eigenvalues

When there are no "enough" eigenvectors to form the fundamental set of solution due to the repeated eigenvalues, we need to find the generalized eigenvectors. Then $x_1 = e^{rt}v$ and $x_2 = e^{rt}tv + e^{rt}s$ are particular solutions of the system of equations, where r is the repeated eigenvalue, v is the eigenvector, and s is the generalized eigenvector.

1. Let's start with a 2×2 matrix, $x' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} x$

2. How about a 3×3 matrix, $x' = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} x$

   Note: There is a repeated eigenvalue in this case, but you will have "enough" eigenvectors.

3. How about another example, $x' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} x$

4. With initial condition, $x' = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix} x$, $x(0) = \begin{pmatrix} -1 \\ 2 \\ 30 \end{pmatrix}$

2.4 Complex Eigenvalues

The basic idea is to seperate the real and imaginary parts to form the fundamental solutions as we have seen before.

1. Let's start with a 2×2 matrix. $x' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} x$

2. How about a 3×3 matrix. $x' = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} x$

2.5 Lagrange Method to Solve Non-homogeneous System

The idea of solving a non-homogeneous system using the Lagrange method is similar to the idea of solving non-homogeneous equation using the method of variation of parameters. Given the equation $x' = Ax + b$ The steps are:

1. Solve the corresponding homogeneous system, such that the general solution of the homogeneous system is of the form $x = c_1 x_1 + c_2 x_2$.

2. Set up the Lagrange system as follows:

$$u_1' x_1 + u_2' x_2 = b$$

   then solve the Lagrage system for u_1, u_2 which are functions of t.

3. The general solution of the non-homogeneous system is given by

$$x = u_1 x_1 + u_2 x_2$$

Try the above Lagrange method for the following question.

$$x' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x + \begin{pmatrix} e^t \\ t \end{pmatrix}$$

3 Series Solution of Second-order Linear Equations

To solve a general second-order linear differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

by finding two independent series solutions around $x = 0$.
We start with an assumption of the solution

$$y = x^r \sum_{n=0}^{\infty} a_n(r)x^n$$

for some integer r and assume $a_0 = 1$.
Solve for y by pluging into the original equation and determine the followings:

1. The value of r, by looking at the lowest power term and solve the indicial equation for r.

2. The recurrence relation for the coefficients a_n using the idea that "If a series is equal to 0, then each of the term is equal to 0". Notice a_n depends the choice of r.

3. The series convergence domain of the series y by computing

$$\lim_{n\to\infty} \left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right|$$

This only give one particular series solution of the equation and there are different cases for the values of r to find anohter independent series solution.

Frobenius Algorithm:

Let r_1, r_2 be the roots of the associated indicial equation with $r_1 > r_2$.

1. If r_1, r_2 are different, and $r_1 - r_2$ is not an integer, then the solitions are given by
   (a) $y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1)x^n$
   (b) $y_2(x) = x^{r_2} \sum_{n=0}^{\infty} a_n(r_2)x^n$

2. For equal root $r = r_1 = r_2$ (textbook 5.6), the solutions are given by
   (a) $y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1)x^n$

(b) $y_2(x) = \partial_r y_1(r,x)|_{r=r_1} = y_1(r_1,x)log(x) + x^{r_1}\sum_{n=1}^{\infty} b_n(r_1)x^n$
where $b_n(r) = \partial_r a_n(r)$

3. If r_1, r_2 are different, and $r_1 - r_2$ is an integer, say $r_1 = r_2 + N$ (textbook 5.6 or week10-11 note), then the solutions are given by

   (a) $y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n(r_1)x^n$

   (b) $y_2(x) = a[logx]y_1(x) + x^{r_2}\sum_{n=1}^{\infty} c_n(r_2)x^n$
   where $a = \lim\limits_{r\to r_2}(r - r_2)a_N(r)$ and $c_n(r) = (\frac{d}{dr}(r - r_2)a_n(r))$

3.1 Practice problems

In the practice problems below, you will see the three differnt cases mentioned above.

1. Find two independent solution of the equation

$$(x+2)x^2y'' - xy' + (1+x)y = 0$$

   using power series solutions centered at $x_0 = 0$. Write the recurrence equations for the coefficients. Calculate the first few terms. Find the series convergence domain.

(Q1 continued)

2. Find two independent solution of the equation

$$x^2 y'' - xy' + (1 - x)y = 0$$

using power series solutions centered at $x_0 = 0$. Write the recurrence equations for the coefficients. Calculate the first few terms. Find the series convergence domain.

(Q2 continued)

3. Find two independent solution of the equation

$$xy'' + 4y' - xy = 0$$

using power series solutions centered at $x_0 = 0$. Write the recurrence equations for the coefficients. Calculate the first few terms. Find the series convergence domain.

(Q3 continued)