MATB42_TUT3_Heat_Equation_insulated_ends

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MATB42_TUT3_Heat_E...

Learning Objective

Understand the definitions and useful formulas to solve a ~~wave~~ Heat equation with the boundary condition of insulated ends.

Definition

The partial differential equation

$$u_t = ku_{xx}$$

is called the **heat equation** where k is a constant and $u(x,t)$ is a function of x, the position variable and t, the time variable. Suppose that we want to find the solution $u(x,t)$ on $0 < x < l$, with the given conditions:

- **Boundary conditions (Insulated ends):** $u_x(0,t) = 0$ and $u_x(l,t) = 0$
- **Initial condition** $u(x,0) = \phi(x)$

Using the method of separation of variables, the general solution is given by

given this particular type of boundary condition.

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} A_n cos(\frac{n\pi x}{l}) e^{-(\frac{n\pi}{l})^2 kt}}$$

Given a different boundary condition, this can be completely different! see Tut 4.

where the coefficients are given by

$$\boxed{A_0 = \frac{1}{l}\int_0^l \phi(x)dx}$$

and for $n \neq 0$

$$\boxed{A_n = \frac{2}{l}\int_0^l \phi(x)cos(\frac{n\pi x}{l})dx}$$

Questions:

Find the solution to the heat equation on ~~$0 \le x \le l$~~ $0<x<l$ with the boundary conditions $u_x(0,t) = 0$, $u_x(l,t) = 0$, and the initial conditions:

1. $\phi(x) = cos(\frac{2\pi x}{l})$
2. $\phi(x) = 1$
3. $\phi(x) = x$

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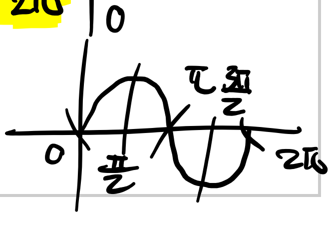
Given the boundary conditions of this heat equation, we known from lecture that $u(x,t) = \sum_{n=0}^{\infty} A_n \cos(\frac{n\pi x}{l}) e^{-(\frac{n\pi}{l})^2 kt}$

① Given the initial condition, $\phi(x) = \cos(\frac{2\pi x}{l})$

$$A_0 = \frac{1}{l}\int_0^l \phi(x)dx = \frac{1}{l}\int_0^l \cos(\frac{2\pi x}{l})dx = \frac{1}{l}\sin(\frac{2\pi x}{l}) \cdot \frac{l}{2\pi}\Big|_0^l$$

$$= \frac{1}{2\pi}\left(\sin(\frac{2\pi x}{l})\right)\Big|_0^l = \frac{1}{2\pi}\left[\sin(2\pi) - \sin(0)\right]$$



$$= \frac{1}{2\pi}[0-0] = 0$$

For $n \neq 0$,

$$A_n = \frac{2}{l}\int_0^l \phi(x)\cos(\frac{n\pi x}{l})dx = \frac{2}{l}\int_0^l \cos(\frac{2\pi x}{l})\cos(\frac{n\pi x}{l})dx$$

m

$$= \begin{cases} \frac{2}{l}\cdot\frac{l}{2} & \text{for } n=2 \\ \frac{2}{l}\cdot 0 & \text{for } n\neq 2 \end{cases} = \begin{cases} 1 & n=2 \\ 0 & n\neq 2 \end{cases}$$

$$\text{so } A_n = \begin{cases} 0 & n=0 \\ 1 & n=2 \\ 0 & n\neq 2 \end{cases}$$

② Given the initial condition $\phi(x) = 1$

$$A_0 = \frac{1}{l}\int_0^l \phi(x)dx = \frac{1}{l}\int_0^l 1\,dx = \frac{1}{l}(x)\Big|_0^l = \frac{1}{l}\cdot l - \frac{1}{l}\cdot 0$$

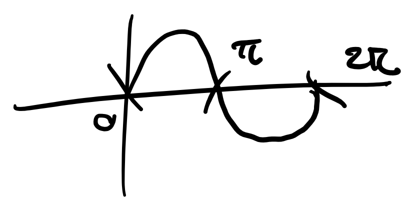
$$= 1 - 0 = 1$$

for $n \neq 0$,

$$A_n = \frac{2}{l}\int_0^l \phi(x)\cos(\frac{n\pi x}{l})dx = \frac{2}{l}\int_0^l \cos(\frac{n\pi x}{l})dx$$

$$= \frac{2}{l}\left[\sin(\frac{n\pi x}{l})\cdot\frac{l}{n\pi}\right]\Big|_0^l = \frac{2}{n\pi}\left[\sin(n\pi) - \sin(0)\right]$$

$$= \frac{2}{n\pi}[0-0] = 0$$



$$\text{so } A_n = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n\neq 0 \end{cases}$$

③ Given the initial condition $\phi(x) = x$

$$A_0 = \frac{1}{l}\int_0^l \phi(x)dx = \frac{1}{l}\int_0^l x\,dx = \frac{1}{l}\left(\frac{1}{2}x^2\right)\Big|_0^l$$

$$= \frac{1}{l}\left(\frac{1}{2}l^2 - \frac{1}{2}0^2\right) = \frac{l}{2}$$

for $n \neq 0$,

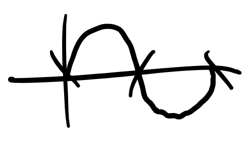
$$A_n = \frac{2}{l}\int_0^l \phi(x)\cos(\frac{n\pi x}{l})dx = \frac{2}{l}\int_0^l x\cos(\frac{n\pi x}{l})dx$$

Look at $\int_0^l x\cos(\frac{n\pi x}{l})dx$

using the integration by parts: ($\int u\,dv = uv - \int v\,du$)

$$\boxed{\begin{matrix} u = x & du = dx \\ dv = \cos(\frac{n\pi x}{l})dx & v = \sin(\frac{n\pi x}{l})\cdot\frac{l}{n\pi} \end{matrix}}$$

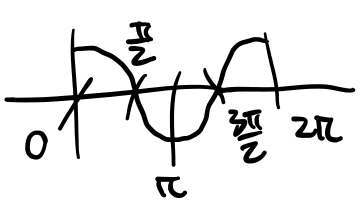
$$\int_0^l x\cos(\frac{n\pi x}{l})dx = \frac{l}{n\pi}x\sin(\frac{n\pi x}{l})\Big|_0^l - \int_0^l \frac{l}{n\pi}\sin(\frac{n\pi x}{l})dx$$



$$= \left[\frac{l}{n\pi}l\underbrace{\sin(n\pi)}_{=0} - \frac{l}{n\pi}\cdot 0\underbrace{\sin(0)}_{=0}\right] - \frac{l}{n\pi}\left[-\cos(\frac{n\pi x}{l})(\frac{l}{n\pi})\right]\Big|_0^l$$

$$= (0-0) + (\frac{l}{n\pi})^2\cos(\frac{n\pi x}{l})\Big|_0^l$$

$$= (\frac{l}{n\pi})^2\left[\cos(n\pi) - \cos(0)\right]$$



$$= \begin{cases} (\frac{l}{n\pi})^2[1-1] & \text{when n is even, } n\neq 0 \\ (\frac{l}{n\pi})^2[-1-1] & \text{when n is odd} \end{cases}$$

$$= \begin{cases} 0 & \text{if n is even, } n\neq 0 \\ -2(\frac{l}{n\pi})^2 & \text{if n is odd} \end{cases}$$

$$A_n = \frac{2}{l}\int_0^l \phi(x)\cos(\frac{n\pi x}{l})dx = \begin{cases} \frac{2}{l}\cdot 0 & \text{if n is even } n\neq 0 \\ \frac{-4l}{n^2\pi^2} & \text{if n is odd} \end{cases} = \begin{cases} 0 & \text{if n is even } n\neq 0 \\ \frac{-4l}{n^2\pi^2} & \text{if n is odd} \end{cases}$$

$$\text{so } A_n = \begin{cases} \frac{l}{2} & \text{if } n=0 \\ 0 & \text{if n is even, } n\neq 0 \\ \frac{-4l}{n^2\pi^2} & \text{if n is odd} \end{cases}$$