Learning Objective

Understand the definitions of convex sets and convex functions. Understand the proof structure of proving the convexity of a given set or function.

Definitions

1. A point $x \in \mathbb{R}^n$ is called **convex combination** of the points $x_1, x_2, ..., x_r$ in $\mathbb{R}^n$ if $x = \sum_{i=1}^{r} c_i x_i$ where $c_1, c_2, ..., c_r$ are some real numbers, such that $\sum_{i=1}^{r} c_i = 1$ and $c_i \geq 0, 1 \leq i \leq r$
2. A subset S of $\mathbb{R}^n$ is called **convex** if for any $x_1, x_2 \in S$, $x = \lambda x_1 + (1-\lambda)x_2 \in S$ for all $0 \leq \lambda \leq 1$.
3. A function f defined on a convex set S in $\mathbb{R}^n$ is called a **convex function** if

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

for all $0 \leq \lambda \leq 1$ and any $x_1, x_2 \in S$.

Four-phased method when writing a proof

How to Solve It (G.Polya, 1945) introduced a four-phased method to solve mathematical problems. It is very useful to follow the metod when writing a proof, so that you dont easily get lost at what you are trying to do.

1. Understanding: Assumption and Want-to-show directly by definition. Give yourself intuition by drawing graph if possible.
2. Devising a plan: Build the connection between the assumption and Want-to-show
3. Carrying out the plan: Use the assumption, follow the plan you have and use logical reasoning to prove (not just seeing) the Want-to-show.
4. Looking back: Have you proved what you need to prove? Is there any lack of reasoning?

Set notation

There are always two questions to ask when looking at a set. What does the elements look like and what conditions have to be satisfies by the elements? Write the following sets using set notation.

1. The set of all solutions to $Ax \leq b$
2. The graph of $x + y \leq 2$

Questions

Prove the following statements and be precise what your assumption and want-to-show are.

1. Show that the set of all solutions to $Ax \leq b$, if it is nonempty, is a convex set.

2. Show that the graph of $x + y \leq 2$ is a convex set.

3. Show that the open interval $(0, 1) \subset \mathbb{R}$ is a convex set.