Schrödinger Equation

We are interested in solving the unknown complex function $\Psi(x,t)$ on the finite domain $0 < x < l$, where x is the position variable and t is the time variable, to the Schrödinger Equation

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx}$$

where $\hbar, m$ are constants and $i \in \mathbb{C}$ such that $i^2 = -1$,
given the boundary conditions

$$\Psi(0,t) = 0 \qquad \Psi(l,t) = 0$$

and the initial condition

$$\Psi(x,0) = \phi(x)$$

where $\phi(x)$ is normalized, i.e. $\int_0^l |\phi(x)|^2 dx = 1$
Using the method of separation of variables with the guess that $\Psi(x,t) = X(x) \cdot T(t)$, we have

$$T(t) = Ae^{-\frac{iEt}{\hbar}} \qquad X(x) = Ccos(\sqrt{\frac{2mE}{\hbar^2}}x) + Dsin(\sqrt{\frac{2mE}{\hbar^2}}x)$$

where $E > 0$.
By considering the boundary conditions, the general solution is given by

$$\Psi(x,t) = \sum_{n=1}^{\infty} A_n sin(\frac{n\pi x}{l})e^{-\frac{iE_n t}{\hbar}}$$

where $E_n = \frac{n^2\pi^2\hbar^2}{2ml^2}$
and by considering the initial condition

$$A_n = \frac{2}{l}\int_0^l \phi(x)sin(\frac{n\pi x}{l})dx$$

Useful formulas

1. As usual, the orthogonal relation for fourier series.
2. $|\phi(x)|^2 = \overline{\phi(x)} \cdot \phi(x)$
3. Euler's formula: $e^{i\theta} = cos\theta + isin\theta$
4. $cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$
5. If $\phi(x)$ is not normalized, we can always multiply it with a constant N to normalize it, where the value of N can be found by solving the integral $\int_0^l |\phi'(x)|^2 dx = 1$, where $\phi'(x) = N\phi(x)$ is the normalized $\phi(x)$.

Questions

Define the **expected value** at time t to be

$$< x >= \int_0^l \overline{\Psi(x,t)} x \Psi(x,t) dx = \int_0^l x |\Psi(x,t)|^2 dx$$

1. Given $\phi(x)$, find the normalized constant $N \in \mathbb{R}$, such that $\phi(x)$ is normalized.
2. Given the normalized $\phi(x)$, find the solution $\Psi(x,t)$ to the Schrödinger Equation as discussed in the previous page. Find $|\Psi(x,t)|^2$.
3. Find $< x >$ at any arbitrary time t.

- $\phi(x) = N sin(\frac{n\pi x}{l})$
- $\phi(x) = N(sin(\frac{\pi x}{l}) + sin(\frac{2\pi x}{l}))$
- $\phi(x) = N(sin(\frac{\pi x}{l}) + e^{i\alpha} sin(\frac{2\pi x}{l}))$, where $\alpha \in \mathbb{R}$

0.1 $\phi(x) = N sin(\frac{n\pi x}{l})$

1. Find the normalized constant N
2. Find $\Psi(x,t)$ and $|\Psi(x,t)|^2$
3. Find $< x >$

0.2 $\phi(x) = N(sin(\frac{\pi x}{l}) + sin(\frac{2\pi x}{l}))$

1. Find the normalized constant N
2. Find $\Psi(x,t)$ and $|\Psi(x,t)|^2$
3. Find $< x >$

0.3 $\phi(x) = N(sin(\frac{\pi x}{l}) + e^{i\alpha} sin(\frac{2\pi x}{l}))$, where $\alpha = \frac{\pi}{2}$

1. Find the normalized constant N
2. Find $\Psi(x,t)$ and $|\Psi(x,t)|^2$
3. Find $< x >$

0.4 $\phi(x) = N(sin(\frac{\pi x}{l}) + e^{i\alpha} sin(\frac{2\pi x}{l}))$, where $\alpha = \pi$

1. Find the normalized constant N
2. Find $\Psi(x, t)$ and $|\Psi(x, t)|^2$
3. Find $< x >$