Learning Objective

Understand the method of separation of variables to solve the general solution of a heat equation.

Definition

The partial differential equation

$$u_t = ku_{xx}$$

is called the **heat equation** where k is a constant and $u(x,t)$ is a function of x, the position variable and t, the time variable. Suppose that we want to find the solution $u(x,t)$ on $0 < x < l$, with the given boundary conditions and initial condition.

Method of Separation of Variables

1. Start with the guess of the solution $u(x,t) = X(x)T(t)$, such that $u_t(x,t) = X(x)T'(t)$ and $u_{xx}(x,t) = X''(x)T(x)$. Therefore the original PDE becomes $XT' = kX''T$, and

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda$$

for some $\lambda > 0$. This gives a first and a second order ODE with constant coefficients which we know the general solutions.

2. For a heat equation, regardless the boundary condition and initial condition, we know that the method of separation of variables give

$$T(t) = Ae^{-\lambda kt}, X(x) = Ccos(\sqrt{\lambda}x) + Dsin(\sqrt{\lambda}x)$$

for some constants A, C and D, and $\lambda > 0$.

3. Consider the boundary conditions on the two ends with the equations above, resulting some restrictions of the constants. Gives the general solution of the PDE.

Questions

1. Fixed Temperature: Find the solution to the heat equation on $0 < x < l$ with $u(0,t) = 0, u_x(l,t) = 0$ and $u(x,0) = \phi(x)$

2. Mixed Boundary Condition: Find the solution to the heat equation on $0 < x < l$ with $u(0,t) = 0, u_x(l,t) = 0$ and $u(x,0) = \phi(x)$