1 Induced linear transformation from a matrix

Let's start by the definition

- Let A be an $m \times n$ matrix with real entries.
  The linear transformation $L_A : \mathbb{R}^n \to \mathbb{R}^m$ defined as

  $$L_A(x) = Ax \text{ for all } x \in \mathbb{R}^n$$

  is called the induced linear transformation of the matrix A.

1.1 Questions

1. Let A be an $m \times n$ matrix with real entries. Let $L_A : \mathbb{R}^n \to \mathbb{R}^m$ be the induced linear transformation of A.
   (a) Show that $Ker(L_A) = null(A)$, where recall $null(A) = \{x \in \mathbb{R}^n : Ax = 0\}$

   (b) Show that $Im(L_A) = range(A)$, where recall $range(A)$ is the column space of A, that is the span of the columns of A.

2 Induced linear transformation from a list of vectors

Let's start by the definition

- Let $S = (v_1, \cdots, v_n)$ be a list of vectors in the vector space V. The linear transformation $L_S : \mathbb{R}^n \to V$ defined as

$$L_S(x) = a_1 v_1 + \cdots + a_n v_n \text{ for all } x = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$$

  is called the induced linear transformation of S.

2.1 Questions

1. Let $S = (v_1, \cdots, v_n)$ be a list of vectors in V and let L_S be the induced linear transformation. Show that

   (a) $v_1, \cdots, v_n$ is linearly independent if and only if L_S is injective.

(b) $v_1, \cdots, v_n$ spans V if and only if L_S is surjective.

(c) Conclude that if $\alpha = (v_1, \cdots, v_n)$ is an ordered basis of V, then L_α is an isomorphism.

3 Coordinate vector and matrix representation

Let's start by the definition

- Let $\alpha = (v_1, \cdots, v_n)$ be an ordered basis of a vector space V over $\mathbb{R}$.
  The coordinate vector of $v \in V$ is
  $$[v]_\alpha = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$
  if $v = a_1 v_1 + \cdots + a_n v_n$ for $a_1, \cdots, a_n \in \mathbb{F}$
- Let V, W be vector spaces over $\mathbb{F}$. Assume $\alpha = (v_1, \cdots, v_n)$ and $\beta = (w_1, \cdots, w_m)$ are bases of V and W respectively. Let $T : V \to W$ be a linear transformation, then the matrix representation of T with respect to α and β is
  $$[T]_\alpha^\beta = \begin{bmatrix} [T(v_1)]_\beta & \cdots & [T(v_n)]_\beta \end{bmatrix}$$

3.1 Questions

1. Let V, W be vector spaces over $\mathbb{F}$ with bases $\alpha = (v_1, \cdots, v_n)$ and $\beta = (w_1, \cdots, w_m)$ respectively. Let $T : V \to W$ be a linear transformation. Show that for any $v \in V$,
   $$[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$$

2. Let V, W be vector spaces over $\mathbb{F}$ with bases $\alpha = (v_1, \cdots, v_n)$ and $\beta = (w_1, \cdots, w_m)$ respectively. Let $T : V \to W$ be a linear transformation. Let P be the matrix representation of T with respect to α and β. Let L_P, L_α, L_β be the induced linear transformations. Show that

$$T \circ L_\alpha = L_\beta \circ L_P$$

as linear transformation.

3. Let V, W be vector spaces over $\mathbb{F}$ with bases $\alpha = (v_1, \cdots, v_n)$ and $\beta = (w_1, \cdots, w_m)$ respectively. Let $T : V \to W$ be a linear transformation. Finish the diagram with the corresponding linear transformations. Assume that you are given $v \in V$.

$$\begin{array}{ccc} V & \qquad\qquad & W \\ & & \\ & & \\ \mathbb{R}^n & & \mathbb{R}^m \end{array}$$

More questions

Before we start the questions, recall some useful proposition.

- Proposition 5.8 from TCL3: Linearly independent vectors can be extended to a basis.
- Rank-Nullity Theorem: Let V, W be finite dimensional vector spaces. Let $T : V \to W$ be a linear transformation, then

$$dim(V) = dim(Ker(T)) + dim(Im(T))$$

Now we can start.

1. Let V be a finite-dimensional vector space and W be a subspace of V. Show that there exists a linear transformation $T : V \to V$ whose image is W and is identity on W (i.e. $\forall w \in W, T(w) = w$)

2. Let V be finite dimensional, and W be a subspace of V. Show that given any vector space U, there exists a linear transformation $T : V \to U$ with $Ker(T) = W$ if and only if $dim(U) \geq dim(V) - dim(W)$