Learning Objective

Understand the definitions and useful formulas to solve a Heat equation with the boundary condition of insulated ends.

Definition

The partial differential equation

$$u_t = ku_{xx}$$

is called the **heat equation** where k is a constant and $u(x,t)$ is a function of x, the position variable and t, the time variable. Suppose that we want to find the solution $u(x,t)$ on $0 < x < l$, with the given conditions:

- **Boundary conditions (Insulated ends):** $u_x(0,t) = 0$ and $u_x(l,t) = 0$
- **Initial condition** $u(x,0) = \phi(x)$

Using the method of separation of variables, the general solution is given by

$$u(x,t) = \sum_{n=0}^{\infty} A_n cos(\frac{n\pi x}{l}) e^{-(\frac{n\pi}{l})^2 kt}$$

where the coefficients are given by

$$A_0 = \frac{1}{l}\int_0^l \phi(x)dx$$

and for $n \neq 0$

$$A_n = \frac{2}{l}\int_0^l \phi(x) cos(\frac{n\pi x}{l})$$

Questions:

Find the solution to the heat equation on $0 \leq x \leq l$ with the boundary conditions $u_x(0,t) = 0$, $u_x(l,t) = 0$, and the initial conditions:

1. $\phi(x) = cos(\frac{2\pi x}{l})$
2. $\phi(x) = 1$
3. $\phi(x) = x$