MATB42_TUT2 _Integration_technique_Integration_by_parts

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MATB42_TUT2_Integra...

Learning Objective

Understand how and when to apply the technique of integration by parts to solve a given integration. The 'how' part is easy given the algorithm, but the 'when' part is more abstract and you will get some sense of it after looking at various examples starting from the basic ones. You will also review some trigonometry identities when working on the examples.

Integration by parts

To solve an integration of the form $\int f(x)dx$, the technique of integration by parts might be helpful. The technique of integration by parts said:

$$\int udv = uv - \int vdu$$

Here is an example:

$$\int xe^x dx$$

Let

$$u = x \text{ and } dv = e^x dx$$

then

$$du = dx \text{ and } v = e^x$$

Using the technique of integration by parts, we have the following

$$\begin{aligned}\int xe^x dx &= \int udv \\ &= uv - \int vdu \\ &= xe^x - \int e^x dx \\ &= xe^x - e^x + c\end{aligned}$$

Another example that can be solved using integration by parts is

$$\int x^2 sin(x)cos(x)dx$$

Questions

1. $\int xe^{2x}dx$ This is a similar problem as the one given above. Try using the technique of integration by parts and solve the integral. Next, try to solve it again using the setup $u = e^x$ and $dv = xdx$, what is the difficulty here?

$u = x,\ dv = e^{2x}dx$
$du = dx,\ v = \frac{1}{2}e^{2x}$
$\int xe^{2x}dx = \int udv = uv - \int vdu$
$= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx$
$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$

If we try, $u = e^{2x}\ dv = xdx$
$du = 2e^{2x}dx,\ v = \frac{1}{2}x^2$
$\boxed{\int xe^{2x}dx} = \int udv = uv - \int vdu$
$= \frac{1}{2}x^2e^{2x} - \int \frac{1}{2}x^2 \cdot 2e^{2x}dx$
$= \frac{1}{2}x^2e^{2x} - \boxed{\int \boxed{x^2} e^{2x}dx}$

This is difficulty

This is still difficult!!

2. $\int x^2e^x dx$ This is a similar problem as the given example. Try using the technique of integration by parts and see what will happen. What is the difficulty here?

$u = x^2\ dv = e^x dx$
$du = 2xdx,\ v = e^x$
s.t. $\int x^2e^x dx = \int udv = uv - \int vdu$
$= x^2e^x - \int 2xe^x dx$
$= x^2e^x - 2\int xe^x dx$

Apply the method again to $\int xe^x dx$
$u = x\ dv = e^x dx$
$du = dx\ v = e^x$
$\int xe^x dx = \int udv = uv - \int vdu = xe^x - \int e^x dx = xe^x - e^x + C$
s.t. $\int x^2e^x dx = x^2e^x - 2xe^x + 2e^x + C$

3. **Thinking:** How can we solve the integral $\int x^n e^x dx$, where n is an arbitrary positive integer?

$\int x^n e^x dx$ should be solvable if we apply the method of integration by parts n times.

4. $\int xe^{x^2}dx$ You will not need the technique of integration by parts to solve this question. This question is very similar to question 1 above, but why would we not need the technique?

$\int xe^{x^2}dx = \boxed{\frac{1}{2}e^{x^2} + C}$

$\frac{d}{dx}$: $e^{x^2} \cdot 2x = 2xe^{x^2}$

$\int e^{x^2}xdx = \int e^u \cdot \frac{1}{2}du = \frac{1}{2}e^u + C$
$u = x^2$
$du = 2xdx$

Try using integration by parts,
$u = x,\ dv = e^{x^2}dx$
$du = dx,$
then $v = ?$

This will be difficult!

5. $\int e^{x^2}dx$ Similar question as last one, but without the x. Try using the technique of integration by parts to solve the question.

$u = e^{x^2}\ dv = dx$
$du = 2xe^{x^2}dx\ v = x$
$\int e^{x^2}dx = \int udv = uv - \int vdu = xe^{x^2} - \boxed{\int 2x^2e^{x^2}dx}$

This is difficult, It might not be the case we can always do this!

6. $\int xcos(x)dx$ Instead of e^x, we have $cos(x)$ now. What is similar when solving this using the technique?

$u = x,\ dv = cos(x)dx$
$du = dx,\ v = sin(x)$
$\int x cos(x)dx = \int udv = uv - \boxed{\int vdu}$
$= xsin(x) - \int sin(x)dx$
$= xsin(x) - (-cos(x)) + C = xsinx + cosx + C$

nice

nice

7. $\int xsin(x)cos(x)dx$ Which trigonometry identity might be useful here?

$sin(2x) = 2sin(x)cos(x)$
$\int xsin(x)cos(x)dx = \int \frac{1}{2}xsin(2x)dx$
$= \frac{1}{2}\int xsin(2x)dx = \frac{1}{2}\int udv = \frac{1}{2}\left(uv - \int vdu\right)$
$= \frac{1}{2}\left(-\frac{1}{2}xcos(2x) - \int -cos(2x) \cdot \frac{1}{2}dx\right) = \frac{1}{2}\left(-\frac{1}{2}xcos(2x) + \frac{1}{2} \cdot \frac{1}{2}sin(2x) + C\right)$
$= -\frac{1}{4}xcos(2x) + \frac{1}{8}sin(2x) + C$

$u = x,\ dv = sin(2x)dx$
$du = dx,\ v = -cos(2x) \cdot \frac{1}{2}$

8. so how can we solve $\int x^2 sin(x)cos(x)dx$

$\int x^2 sinxcosxdx = \int \frac{1}{2}x^2 sin(2x)dx$

Using Q7 and apply integration by parts twice!

9. Let try something else $\int x\sqrt{4 - x^2}dx$

$\int x\sqrt{4-x^2}dx = \int x(4-x^2)^{\frac{1}{2}}dx = -\frac{1}{3}\boxed{(4-x^2)^{\frac{3}{2}}} + C$

$\frac{d}{dx}(4-x^2)^{\frac{3}{2}} = \frac{3}{2}(4-x^2)^{\frac{1}{2}}(-2x) = -3x(4-x^2)^{\frac{1}{2}}$

or use substitution with $u = 4-x^2,\ du = -2xdx$

10. **Bonus:** Now how about $\int_0^2 x^3\sqrt{4 - x^2}dx$? Try to solve it using integration by parts following the similar idea. You will need to change the choice of u and v to make everything 'nice'. One can also solve this using integration by substitution with $x = 2sin(\theta), 0 \leq \theta \leq \frac{\pi}{2}$ and $u = cos(\theta)$ after.

① Using integration by parts
$u = x^2,\ dv = x\sqrt{4-x^2}dx$
$du = 2xdx,\ v = -\frac{1}{3}(4-x^2)^{\frac{3}{2}}$
$\int_0^2 x^3\sqrt{4-x^2}dx \left(= \int udv = uv - \int vdu\right)$
$= x^2\left(-\frac{1}{3}(4-x^2)^{\frac{3}{2}}\right)\Big|_0^2 - \int_0^2 -\frac{1}{3}(4-x^2)^{\frac{3}{2}} \cdot 2xdx$
$= 4\left(-\frac{1}{3}(4-4)^{\frac{3}{2}}\right) - 0^2\left(-\frac{1}{3}(4-0^2)^{\frac{3}{2}}\right) + \frac{2}{3}\int_0^2 x(4-x^2)^{\frac{3}{2}}dx$
$= 0 + \frac{2}{3}\left[(4-x^2)^{\frac{5}{2}}\left(-\frac{1}{5}\right)\right]\Big|_0^2$

$\frac{d}{dx}\left((4-x^2)^{\frac{5}{2}}\right) = \frac{5}{2}(4-x^2)^{\frac{3}{2}}(-2x) = -5x(4-x^2)^{\frac{3}{2}}$

$= \frac{2}{3}\left[(4-4)^{\frac{5}{2}}\left(-\frac{1}{5}\right) - (4)^{\frac{5}{2}}\left(-\frac{1}{5}\right)\right] = \frac{2}{3} \cdot 2^5 \cdot \frac{1}{5}$

② Use substitution $\boxed{x = 2sin\theta,\ 0 \leq \theta \leq \frac{\pi}{2},}$ $dx = 2cos\theta d\theta$

$\int_0^2 x^3\sqrt{4-x^2}dx = \int_0^{\frac{\pi}{2}} 8sin^3\theta\sqrt{4-4sin^2\theta} \cdot 2cos\theta d\theta$

$\sqrt{4(1-sin^2\theta)} = \sqrt{4cos^2\theta} = |2cos\theta| = 2cos\theta$ $\quad 0 \leq \theta \leq \frac{\pi}{2}$

$= \int_0^{\frac{\pi}{2}} 8sin^3\theta \cdot 2cos\theta \cdot 2cos\theta d\theta$
$= 32\int_0^{\frac{\pi}{2}} sin^3\theta cos^2\theta d\theta = 32\int_0^{\frac{\pi}{2}} sin\theta(1-cos^2\theta)cos^2\theta d\theta$

$sin^3\theta = sin\theta sin^2\theta = sin\theta(1-cos^2\theta)$

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$= 32\int_1^0 (1-u^2)u^2(-1)du$

use substitution
$\boxed{u = cos\theta,}\ du = -sin\theta d\theta$

$= 32\int_1^0 (u^2-1)u^2du = 32\int_1^0 u^4 - u^2du = 32\left(\frac{1}{5}u^5 - \frac{1}{3}u^3\right)\Big|_1^0$
$= -32\left(\frac{1}{5} - \frac{1}{3}\right) = 32\left(\frac{1}{3} - \frac{1}{5}\right)$

Further questions for practice

Here are some more practice questions which you should already know (at least with some idea) how to solve using the technique of integration by parts. The technique of integration by substitution is also very useful when combining with trigonometry functions, but we will not discuss here.

1. $\int x^2e^{-x}dx$
2. $\int x^3e^{x^2}dx$
3. $\int x^2cos(2x)dx$
4. $\int 2x^2cos^2xdx$