

Definition

Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or \mathbb{C} . An inner product on V is a function $\langle, \rangle: V \times V \rightarrow \mathbb{F}$, such that it satisfies the following four properties:

1. Conjugate symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in V$
2. Linearity: $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle$ for all $x, y, z \in V, a, b \in \mathbb{F}$
3. Non-negativity: $\langle x, x \rangle \geq 0$ for all $x \in V$
4. Non-degeneracy: $\langle x, x \rangle = 0$ if and only if $x = 0$

A vector space V over \mathbb{F} together with an inner product is called an inner product space. Given an inner product space, We define the norm as $\|x\| = \sqrt{\langle x, x \rangle}$.

Questions

1. Prove that for vectors in a inner product space over $\mathbb{F} = \mathbb{R}$ or \mathbb{C}

$$\|x \pm y\|^2 = \|x\|^2 + \|y\|^2 \pm 2\operatorname{Re} \langle x, y \rangle$$

where $\operatorname{Re} z = \frac{1}{2}(z + \bar{z})$

2. Show that $\langle A, B \rangle = \text{trace}(A + B)$ is not an inner product on the space of real 2×2 matrices

3. Show that $\langle f, g \rangle = \int_0^1 f'(t) \overline{g(t)} dt$ is not an inner product on the space of polynomials.

4. Show that $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ is not an inner product on \mathbb{C}^n

5. Show that $\langle x, y \rangle = \sum_{i=1}^n x_i \overline{y_i}$ define an inner product on \mathbb{C}^n

6. Let v_1, \dots, v_n be a spanning set of an inner product space over $\mathbb{F} = \mathbb{R}$ or \mathbb{C} . Prove that
- (a) If $\langle x, v \rangle = 0$ for all $v \in V$, then $x = 0$
 - (b) If $\langle x, v_k \rangle = 0$ for all k , then $x = 0$
 - (c) If $\langle x, v_k \rangle = \langle y, v_k \rangle$ for all k , then $x = y$