



Data Intensive Systems (DIS) KBH-SW7 E25

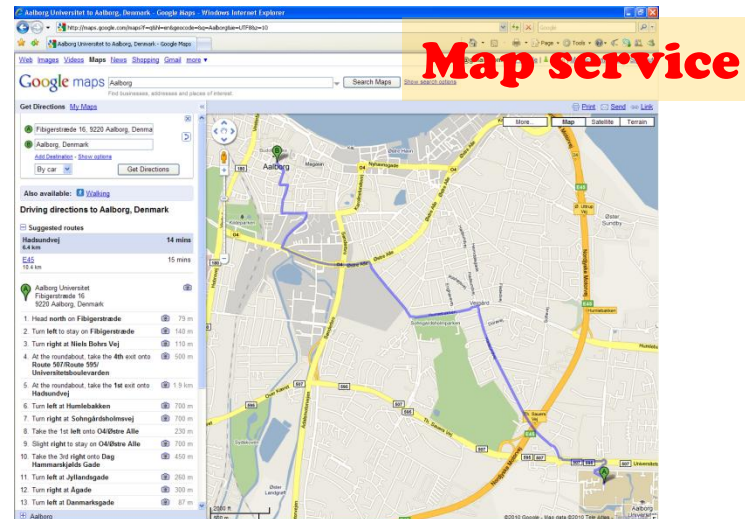
11. Spatial Data and R-tree

Agenda

- Introduction
 - Spatial data model
 - › Data types and spatial relationships
 - Spatial queries
- R-tree
- R-tree Variants

Motivation

► Application scenarios



Map service

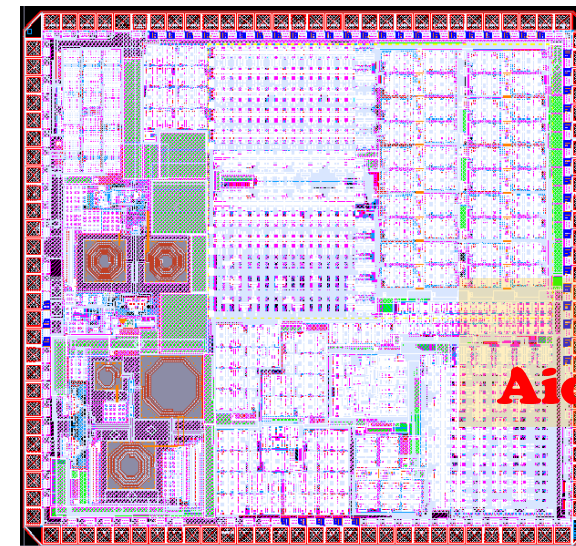


Buildings

► Data with locations

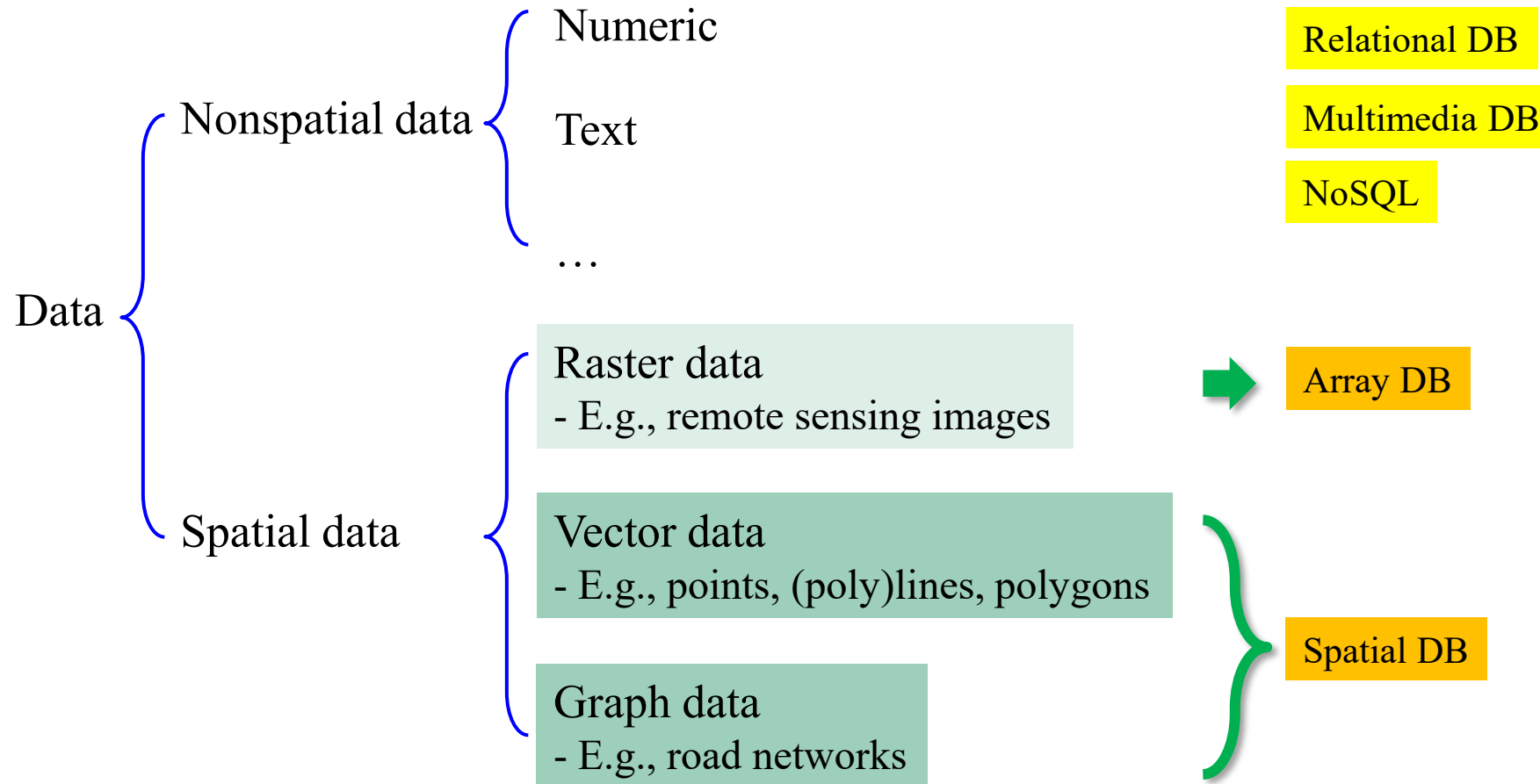
► Special kinds of data

Spatial Data



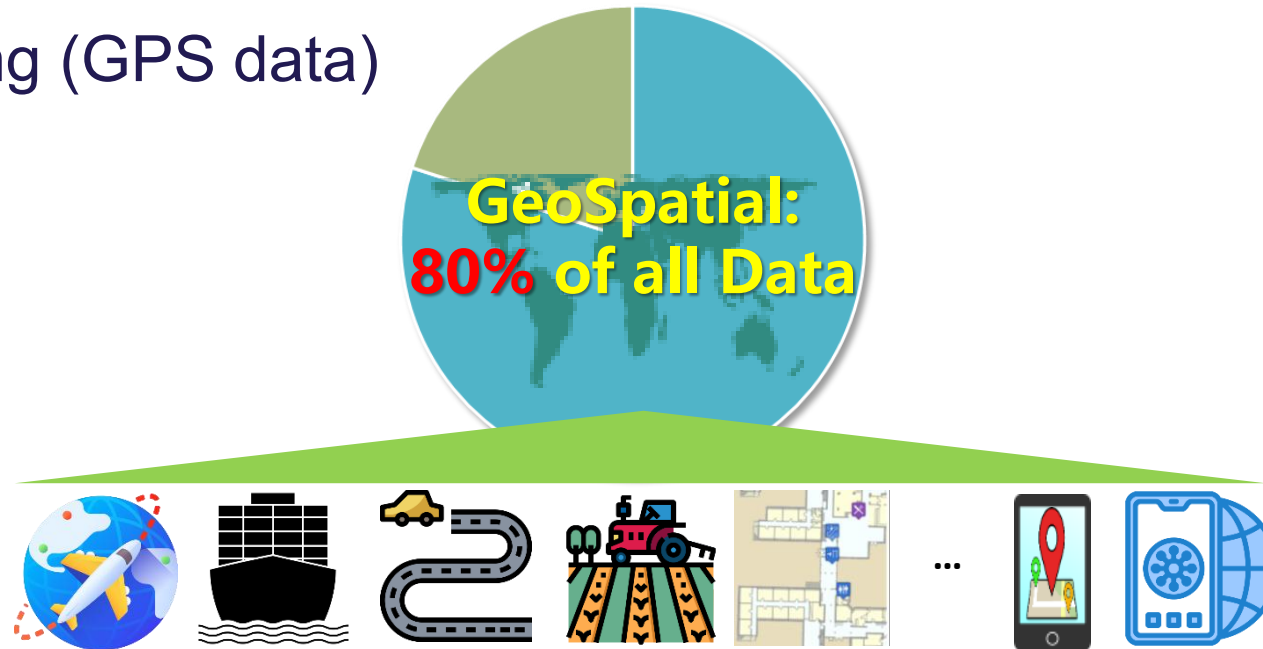
Computer Aided Design

Data Categorization



Big Spatial Data

- ▶ Air flight tracking
- ▶ Ship tracking (AIS data)
- ▶ Vehicle tracking (GPS data)
- ▶ ...

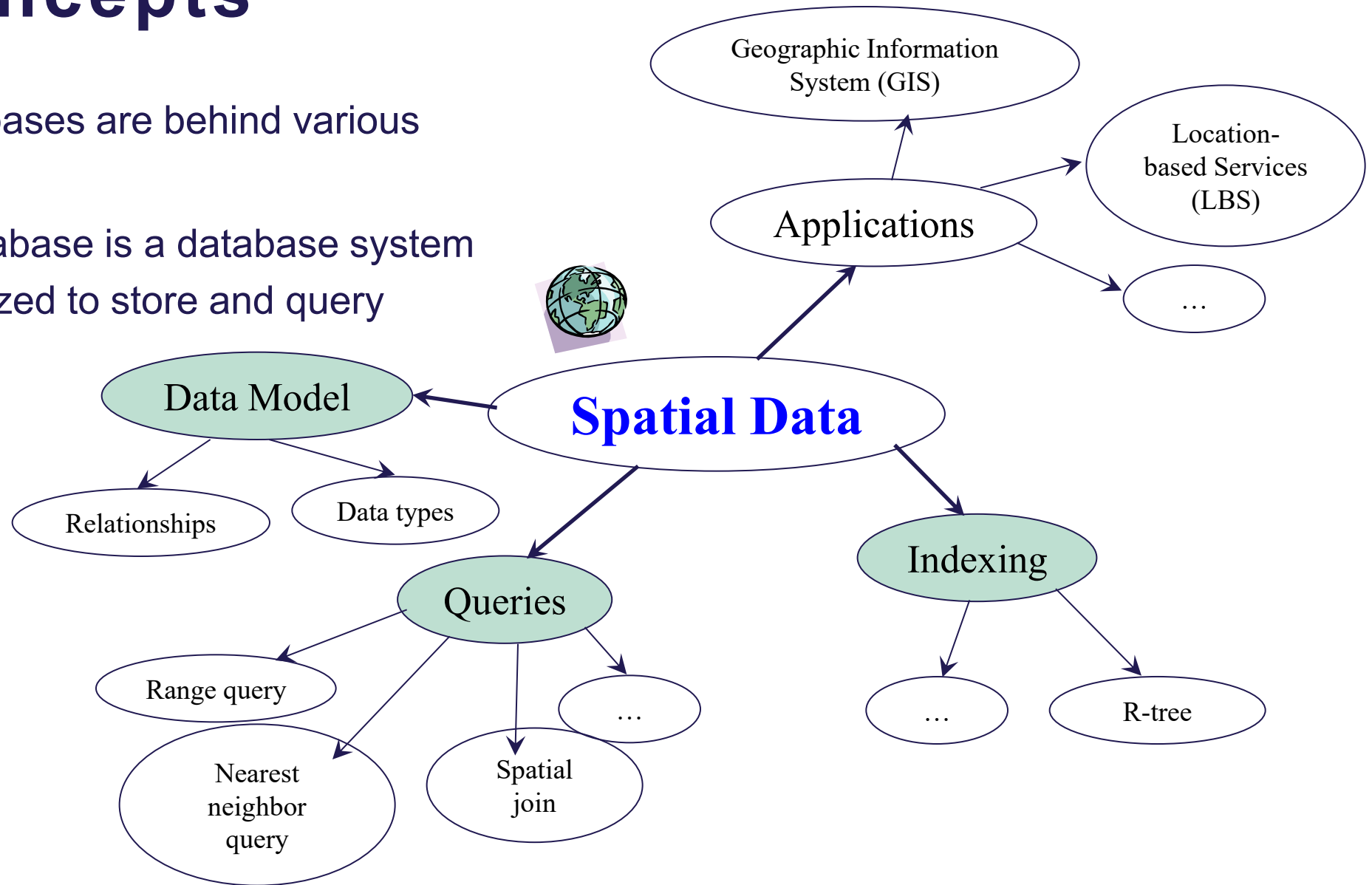


Spatial Data Management

- How to manage spatial data that is everywhere?
 - Modelling, storage, access and process
- Data files
 - No guarantee for logical and physical data independences
- Traditional databases
 - No capabilities for efficient spatial data management
- Spatial Databases
 - For representing, storing, querying and mining spatial data for multiple domains.
 - Advantages of *traditional databases* + *specializations* for spatial data

The Concepts

- Spatial databases are behind various applications.
- A spatial database is a database system that is optimized to store and query **spatial data**.



Spatial Data Management

- A **spatial object** has some spatial attributes that describe its location and geometry
 - Typically 2-dimensional or 3-dimensional data
- A **spatial relation** (dataset, or table) is a collection of spatial objects of the *same* type
 - E.g., rivers, cities, roads
- An example



Road segments of a region in CA



ID	Name	Type	Polyline
1	Boulevard	avenue	(10023,1094), (9034,1567), (9020,1610)
2	Leeds	highway	(4240,5910), (4129,6012), (3813,6129), (3602,6129)
...

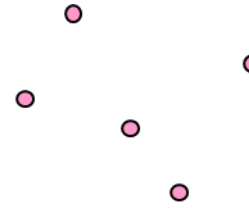
A spatial relation

A particular data type
supported by a spatial database

Fundamental Spatial Object Types

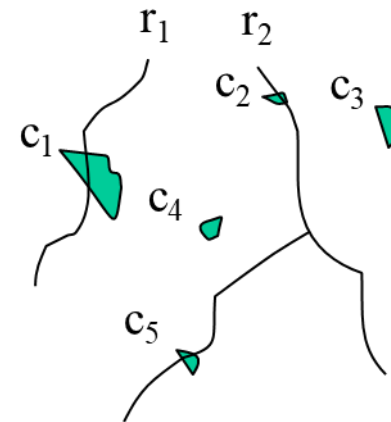
► Points

- Objects whose dimensionality is not important
- E.g., cities in a small-scale map
- E.g., gas stations



► Line segments, and polylines

- Objects whose lengths are of importance
- E.g., roads, rivers



► Polygons

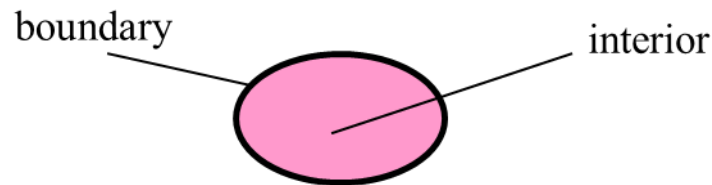
- Objects with extent
- E.g., forests, districts, a city area.

Spatial Relationships

- ▶ A **spatial relationship** links two objects according to their relative location and extent in space
 - Example: “My apartment is **close to** the little mermaid”
- ▶ To be distinguished from “spatial relation”, which means a dataset of spatial objects
- ▶ Types of spatial relationships
 - *topological* relationships
 - *distance* relationships
 - *directional* relationships

Topological Concepts

- Each object is characterized by the space it occupies in the universe.
 - a (finite or infinite) set of pixels
- Each object has a boundary and an interior
 - **boundary**: the set of pixels the object occupies, that are adjacent to at least one pixel not occupied by the object
 - **interior**: the set of pixels occupied by the object, which are not part of its boundary



- Some objects may not have interior
 - E.g., points, line segments

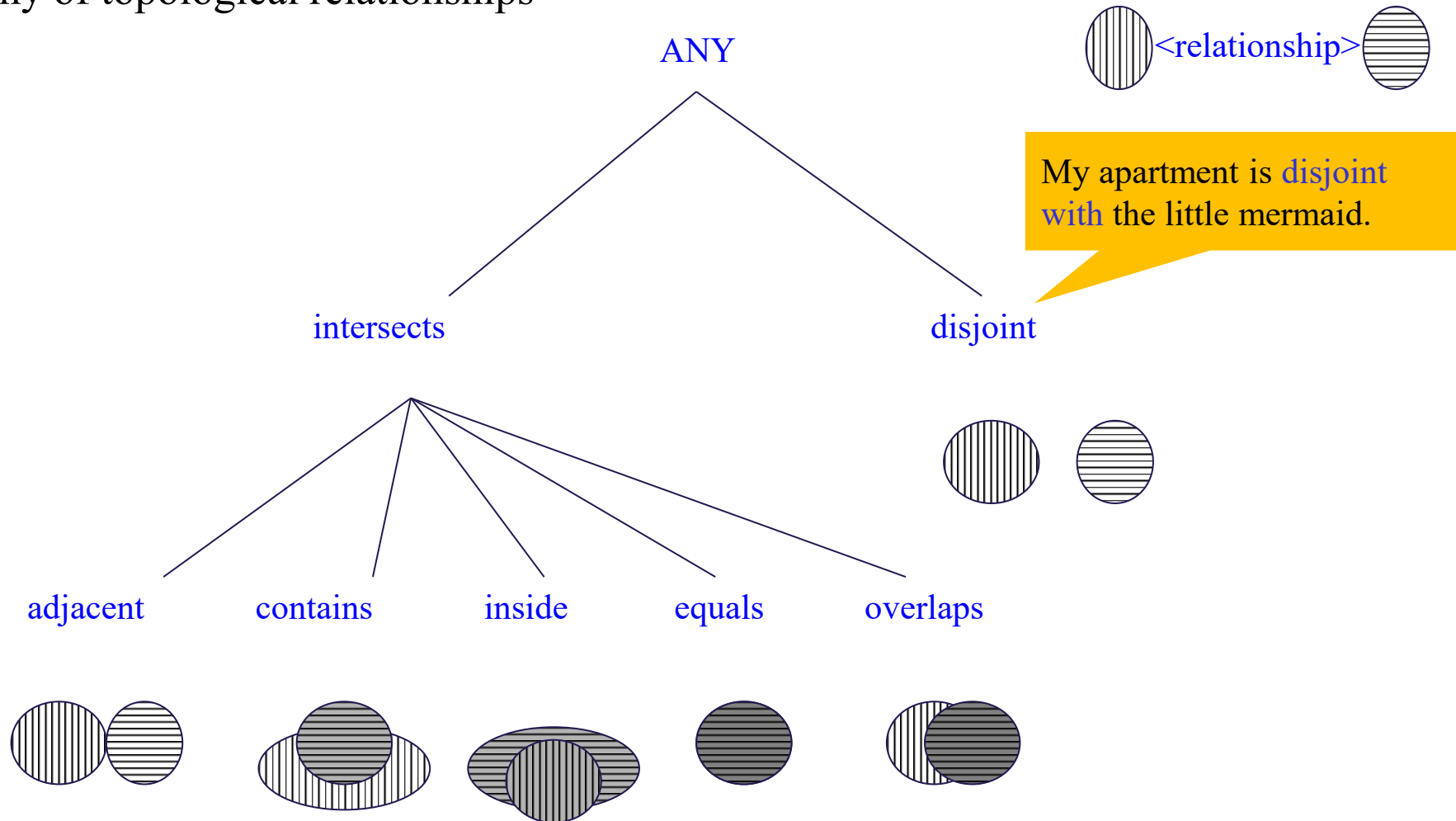
Topological Relationships

- A topological relationship between *two* objects is defined by a set of (set-based) relationships between their boundaries and interiors
 - E.g., o_1 is inside o_2 if $\text{interior}(o_1) \subset \text{interior}(o_2)$
- 7 types of topological relationships
 - *intersects* means any of *equals*, *inside*, *contains*, *adjacent*, *overlaps*
 - *intersects* $\Leftrightarrow \neg \text{disjoint}$

Topological relationship	equivalent boundary/interior relationships
$\text{disjoint}(o_1, o_2)$	$(\text{interior}(o_1) \cap \text{interior}(o_2) = \emptyset) \wedge (\text{boundary}(o_1) \cap \text{boundary}(o_2) = \emptyset)$
$\text{intersects}(o_1, o_2)$	$(\text{interior}(o_1) \cap \text{interior}(o_2) \neq \emptyset) \vee (\text{boundary}(o_1) \cap \text{boundary}(o_2) \neq \emptyset)$
$\text{equals}(o_1, o_2)$	$(\text{interior}(o_1) = \text{interior}(o_2)) \wedge (\text{boundary}(o_1) = \text{boundary}(o_2))$
$\text{inside}(o_1, o_2)$	$\text{interior}(o_1) \subset \text{interior}(o_2)$
$\text{contains}(o_1, o_2)$	$\text{interior}(o_2) \subset \text{interior}(o_1)$
$\text{adjacent}(o_1, o_2)$ (or $\text{meets}(o_1, o_2)$)	$(\text{interior}(o_1) \cap \text{interior}(o_2) = \emptyset) \wedge (\text{boundary}(o_1) \cap \text{boundary}(o_2) \neq \emptyset)$
$\text{overlaps}(o_1, o_2)$	$(\text{interior}(o_1) \cap \text{interior}(o_2) \neq \emptyset) \wedge (\exists p \in o_1 : p \notin \text{interior}(o_2) \wedge p \notin \text{boundary}(o_2))$ $\wedge (\exists p \in o_2 : p \notin \text{interior}(o_1) \wedge p \notin \text{boundary}(o_1))$

Topological Relationship Hierarchy

► Hierarchy of topological relationships

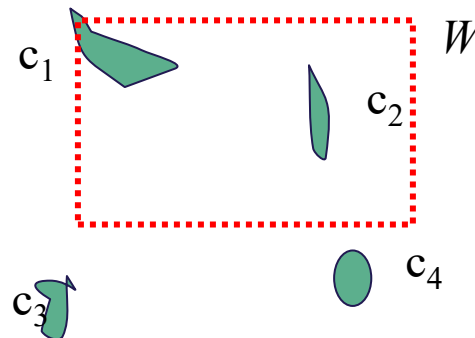
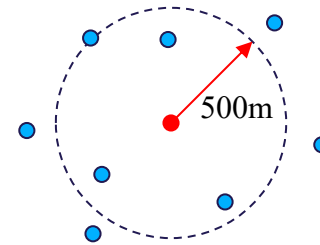


Spatial Queries

- A spatial query is a type of database query
 - Applied on one (or more) spatial relation/table(s)
 - To retrieve spatial objects (or combinations of spatial objects) that satisfy some user-specified spatial relationships
 - with a reference query object, or
 - between two spatial relations
- Typical spatial query types
 - Range query (*with a query object*)
 - Nearest neighbor query (*with a query object*)
 - Spatial join (*between two spatial relations*)

Range Query

- A.k.a spatial selection
- The query range can be a window or a circle (or a distance)
 - Window: top-left and bottom-right corners
 - Circle: a query point as the center, and a distance as the radius.
 - Different types of spatial relationships can be used.
- E.g., find all restaurants within *500 meters* from my apartment.
 - Depending on distance type, query result can differ.
 - E.g., in a city Manhattan distance makes more sense
- E.g., find all cities that *intersect* window W
 - *Result set*: $\{c_1, c_2\}$
 - If a different predicate is used, the query result could be different.
 - E.g., all cities contained in W



Spatial Query Processing

- The process of finding the result for a given spatial query.
- The process can be time-consuming if no appropriate data structure or algorithm is used.
- A *naïve* query processing approach: **sequential scan**
 - Given a spatial relation R and a query Q , check each spatial object o in R to see if o satisfies the query condition in Q .
 - The query result is the *complete* set of all such objects.
 - This approach is time-consuming as every single object is checked.
- To speed up spatial query processing, we need spatial access methods.

Problem Setting

- ▶ Large spatial dataset(s), stored on disk
 - points, lines, polygons, ...
- ▶ We want to process different spatial queries efficiently
 - Range, NN, spatial join
 - In query processing, we need to load data into memory from disk
 - Query processing cost: IOs >> CPU time
- Object clustering: Spatially close objects are stored also in the same or adjacent disk pages
 - This is to reduce IO cost in query processing
 - This is also preferred in relational databases, but what's special about spatial data?

What Is Special About Spatial?

- Dimensionality

- In the 2D or 3D space, There is no total ordering of objects that preserves spatial proximity

- Complex spatial extent

- The spatial extent adds to the complexity of clustering objects into disk pages imposed by dimensionality

- Implication

- Relational indexes (e.g. B+-trees) and query processing methods (e.g. sort-merge join, hash-join) are not readily applicable to spatial data
 - New, spatial access methods (SAMs) for spatial data have to be defined

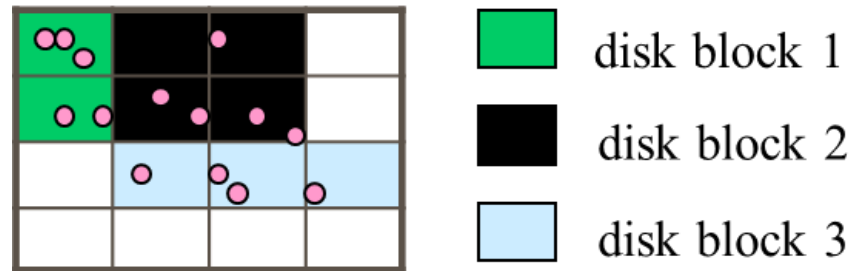
Problem of Spatial Access Methods

- ▶ How to index spatial data such that spatial queries can be answered correctly and efficiently?
 - **Correctly:** Query result should be the same as the counterpart sequential scan
 - **Efficiently:** Query processing should be fast, aiming at the minimization of the expected I/O cost
- ▶ Early SAMs index multidimensional points by dividing the space into *disjoint* partitions
 - Examples: the grid file, the k-d-B-tree
- Other SAMs partition data instead
 - E.g., R-tree

Point Access Methods

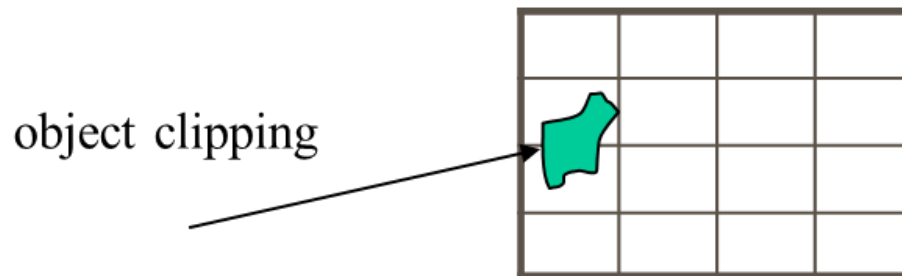
- ▶ **Point access methods** divide the space into **disjoint partitions** and group the points according to the regions they belong

- E.g., the grid-file



- ▶ **Drawback:** point access methods are not effective for extended objects

- Objects may be *clipped* into several parts → data redundancy and affects performance negatively



Space Partitioning SAMs

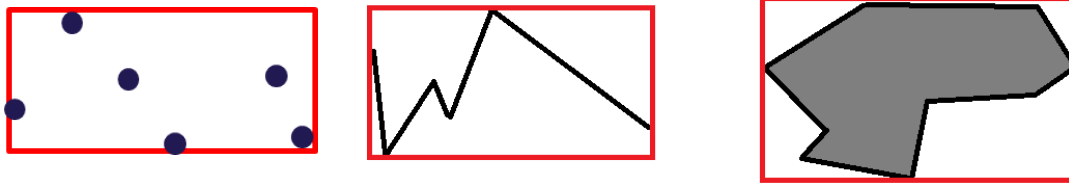
- ▶ Space partitioning methods can handle points well
- ▶ But they have difficulty in indexing other objects (e.g., polylines and polygons)
 - ▶ E.g., object clipping problem
- ▶ An alternative is to partition the data instead of the space
 - ▶ R-tree
 - › 'R' for Rectangle
 - › Based on MBR (minimum bounding rectangles)

Agenda

- Introduction
- R-tree
 - Basic idea and structure
 - Insertion and split
 - Deletion
- R-tree Variants

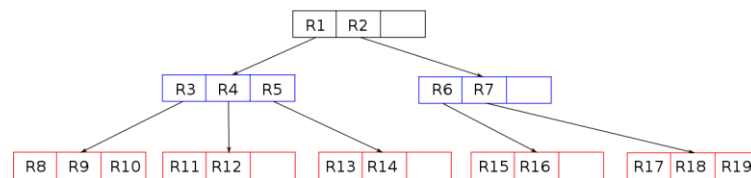
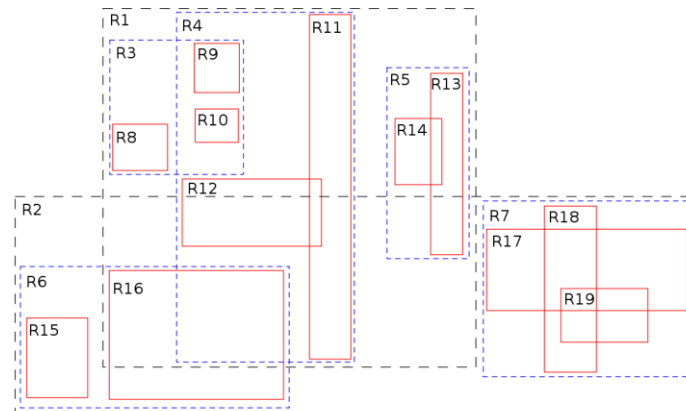
R-tree Concepts

► Minimum Bounding Rectangles (MBRs)



MBRs of points, a polyline and a polygon

► Tree Architecture

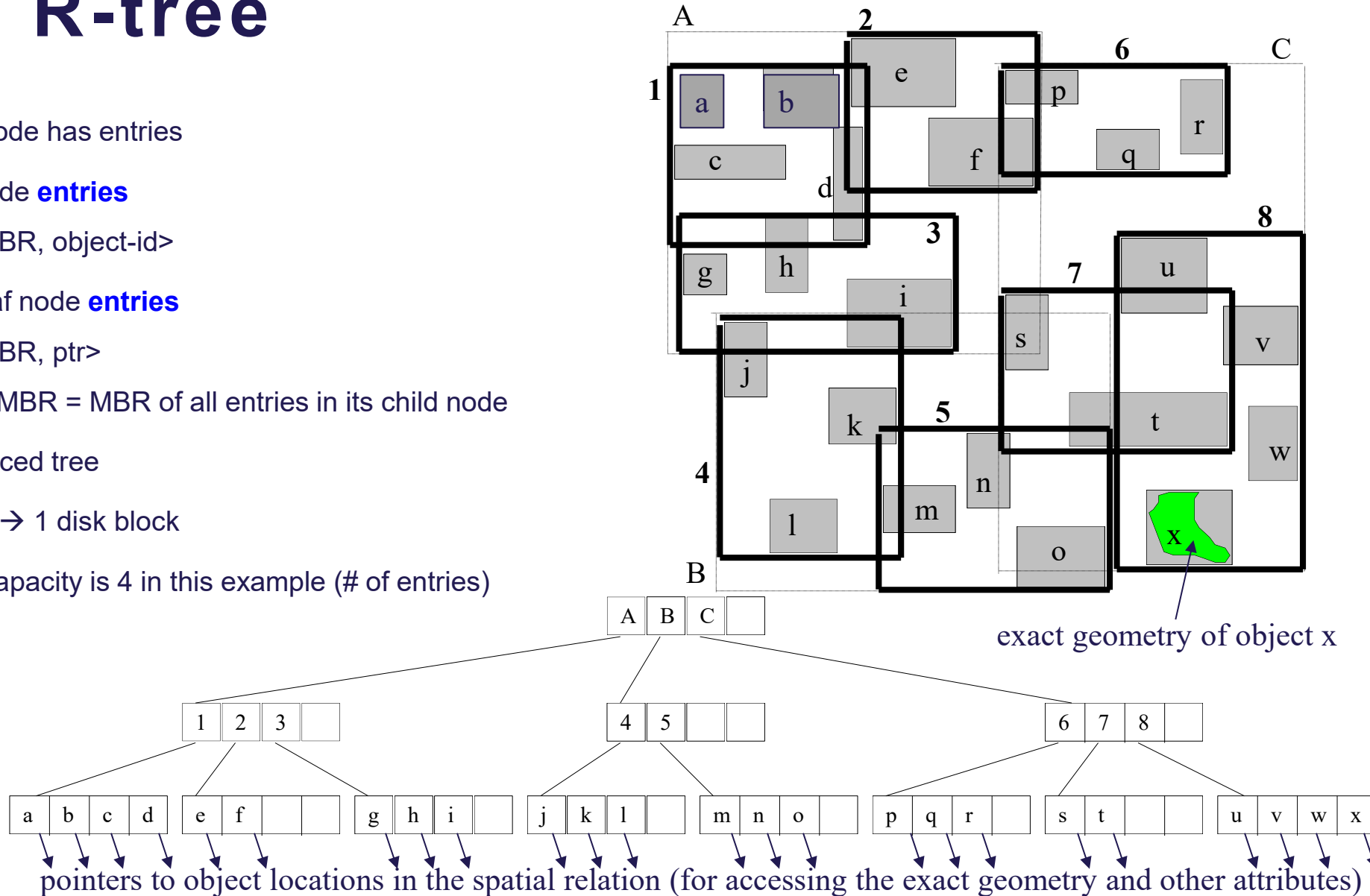


The R-tree

- Groups **object MBRs** to disk blocks hierarchically
- Each group of objects (a disk block) is a leaf of the tree
 - Objects in the same leaf node should be physically close together in one disk block
- The MBRs in a leaf node are grouped to form nodes at the next upper level
 - MBRs close together are grouped
- Grouping is recursively applied at each level until a single group (the root) is formed
- A generalization of the B⁺-tree

The R-tree

- Each node has entries
- Leaf node **entries**
 - <MBR, object-id>
- Non-leaf node **entries**
 - <MBR, ptr>
 - Its MBR = MBR of all entries in its child node
- A balanced tree
- 1 node → 1 disk block
- Node capacity is 4 in this example (# of entries)

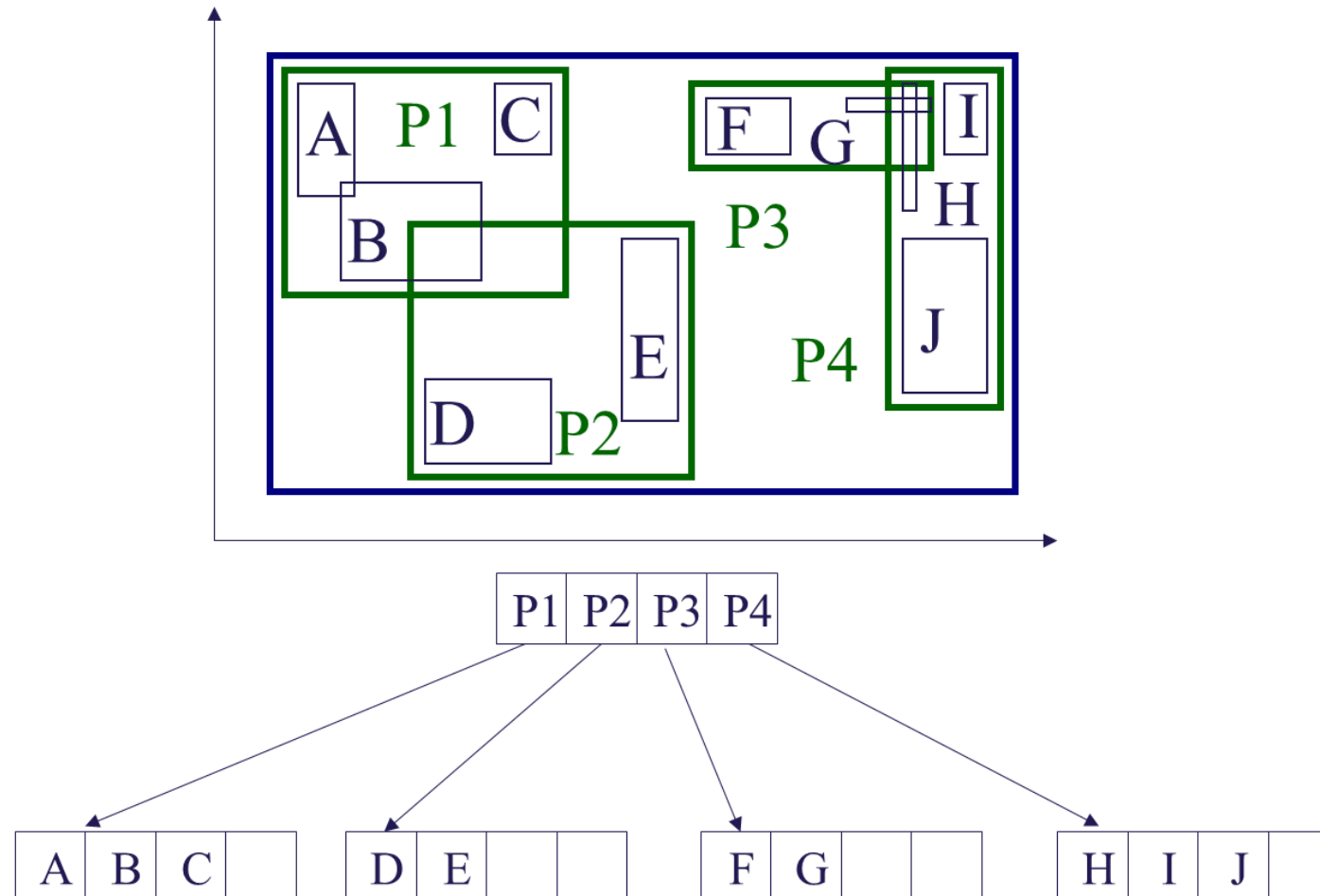


R-tree Properties

- A multi-way external memory tree
- Index nodes and data (leaf) nodes
- All leaf nodes are on the same level
- Every node contains between m ($\leq M/2$) and M entries.
 - Sometimes M is a.k.a fanout (F).
- The root node has at least 2 entries (children)
- Every parent node completely covers all its 'children' in terms of MBRs
- A child MBR may be covered by also its parent's sibling. However, it is stored under its parent.

Example

- Node capacity $M = 4$



R-tree Family

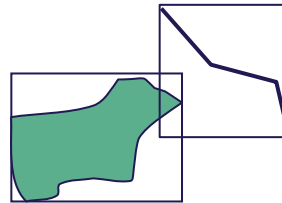
- R-tree
 - The fundamental one
- R*-tree
 - Optimized insertion and deletion algorithms
- Search algorithms are the same to R-tree and R*-tree
- R⁺-tree
 - Object splitting to avoid overlap between MBRs
 - Different algorithms are used compared to the former two

Two-step Spatial Query Processing

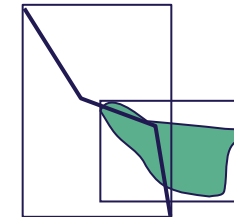
- Evaluating spatial relationships on geometric data is slow
- A spatial object is approximated by its MBR
- A spatial query is usually processed in two steps:
 - 1. **Filter step**: The MBR is tested against the query predicate. → **Fast!**
Often, a particular spatial index is used in this step.
 - 2. **Refinement step**: The exact geometry of objects that pass the filter step is tested for qualification → **Slow!**
- Example: spatial **intersection** joins between forests and rivers



filtered pair



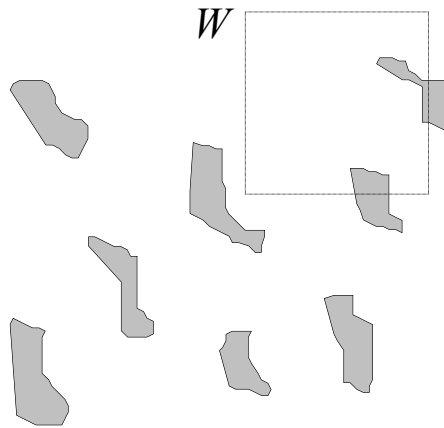
non-qualifying pair that passes the filter step (false hit or false positive)



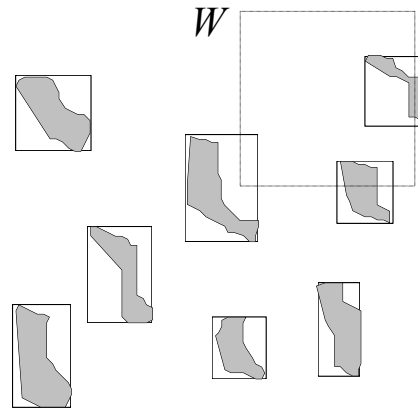
qualifying pair

Two-step Range Query Processing

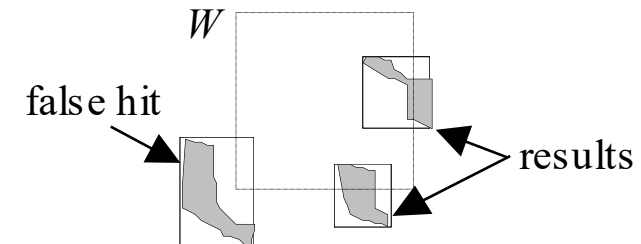
- Problem: Find all objects that intersect the query window W
- Comparison cost \approx number of examined vertices
- Suppose that each polygon has 100 vertices
 - Comparisons in Figure (a) $\approx 100 \cdot 8 = 800$
 - Comparisons in Figure (b) $\approx 4 \cdot 8 + 100 \cdot 3 = 332$



(a) objects and a query



(b) object MBRs



(c) candidates and results

Range Search via R-tree

- ▶ **Range_query**(query W , R-tree node n):

- ▶ If n is not a leaf node

- ▶ For each index entry e in n such that $e.MBR$ intersects W
 - ▶ visit node n' pointed by $e.ptr$
 - ▶ **Range_query**(W, n')

→ Filter

- ▶ If n is a leaf

- ▶ For each index entry e in n such that $e.MBR$ intersects W
 - ▶ visit object o pointed by $e.object-id$
 - ▶ test range query against exact geometry of o ; if o intersects W , report o

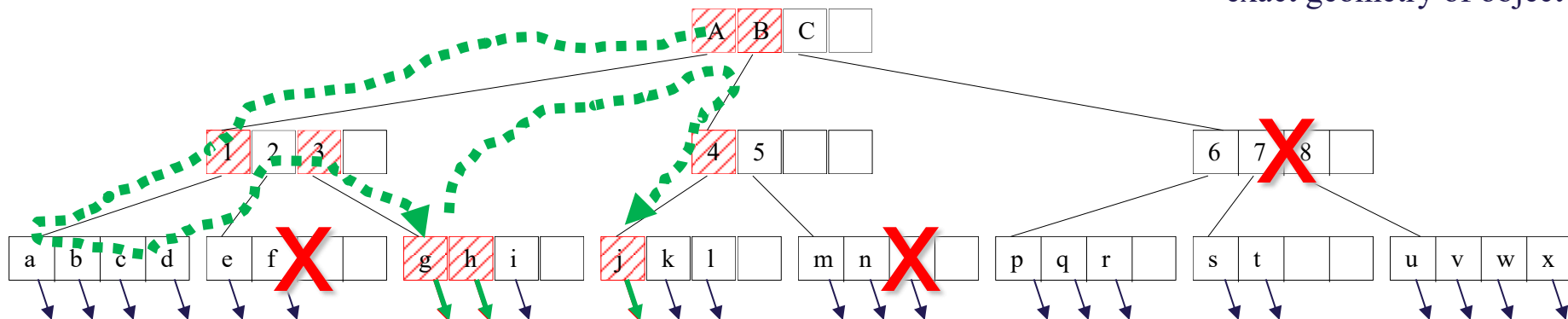
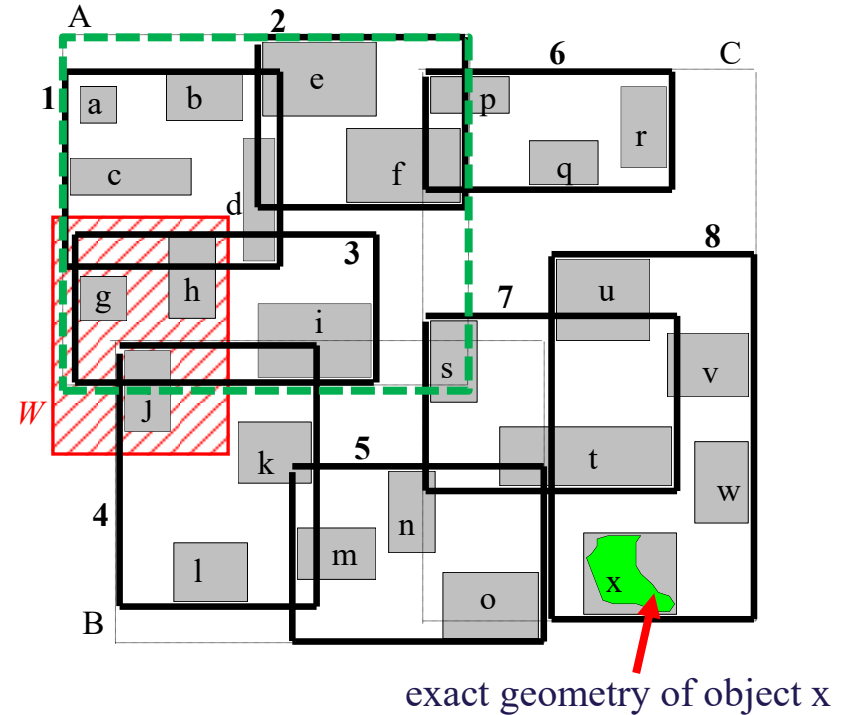
→ Refinement

- ▶ The recursive search starts from the R-tree root node
- ▶ May follow multiple paths during search

Range Search via R-tree

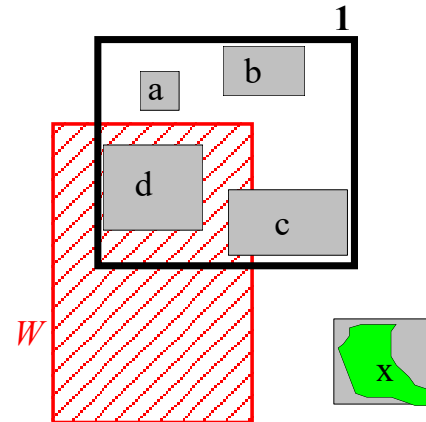
► W is a range query (window intersection query)

- Multiple search paths are involved.
- Many tree nodes (and thus objects) are pruned.
- MBRs are used for filter
- Exact geometries are fetched for refinement
 - E.g., object x



Range Search with Other Predicates

- The search algorithm considers the intersection predicate, i.e., finding all objects intersecting the query window W
- Modify the algorithm for other search predicates?
- E.g., find all objects **inside** W
 - Search predicate for the object o
 - › $\text{inside}(o, W)$
 - Search predicate for a non-leaf MBR R
 - › $\text{intersect}(R, W)$. Why?
 - Search predicate for an object's MBR R'
 - › $\text{inside}(R', W)$.



Agenda

- R-tree
 - Basic idea and structure
 - Insertion and split
 - Deletion
- R-tree Variants

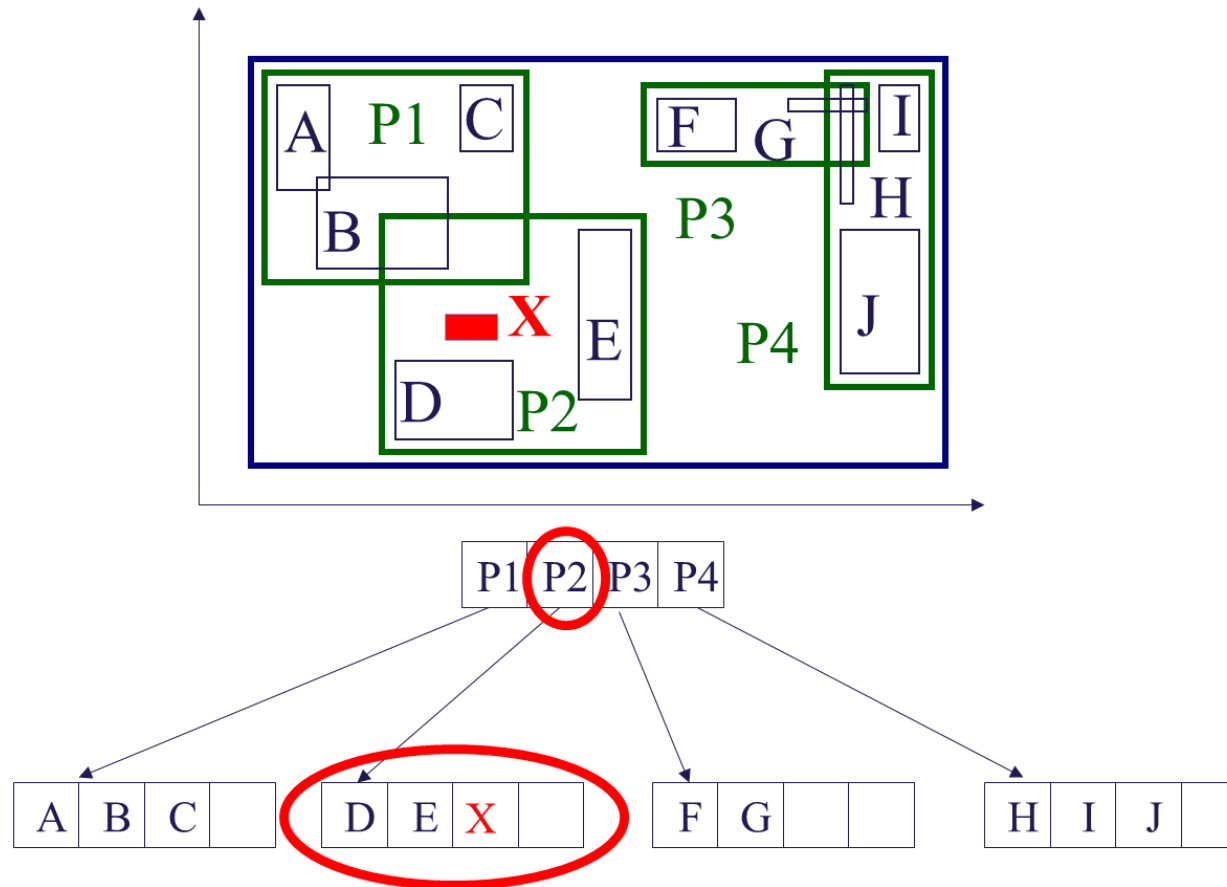
Insert Object o to R-tree

- Start at root and go down to “best-fit” leaf node L .
 - In each step downwards, go to the child whose MBR needs the least area enlargement to include o . Resolve ties by going to the smallest area child.
 - This is encapsulated in Algorithm **ChooseLeaf**
- If the best-fit leaf L has space, insert entry and stop. Otherwise, split L into $L1$ and $L2$.
 - Adjust entry for L in its parent so that the MBR now covers (only) $L1$.
 - Add an entry (in the parent node of L) for $L2$. (This could cause the parent node to recursively split.)
- Propagate changes upward by Algorithm AdjustTree.

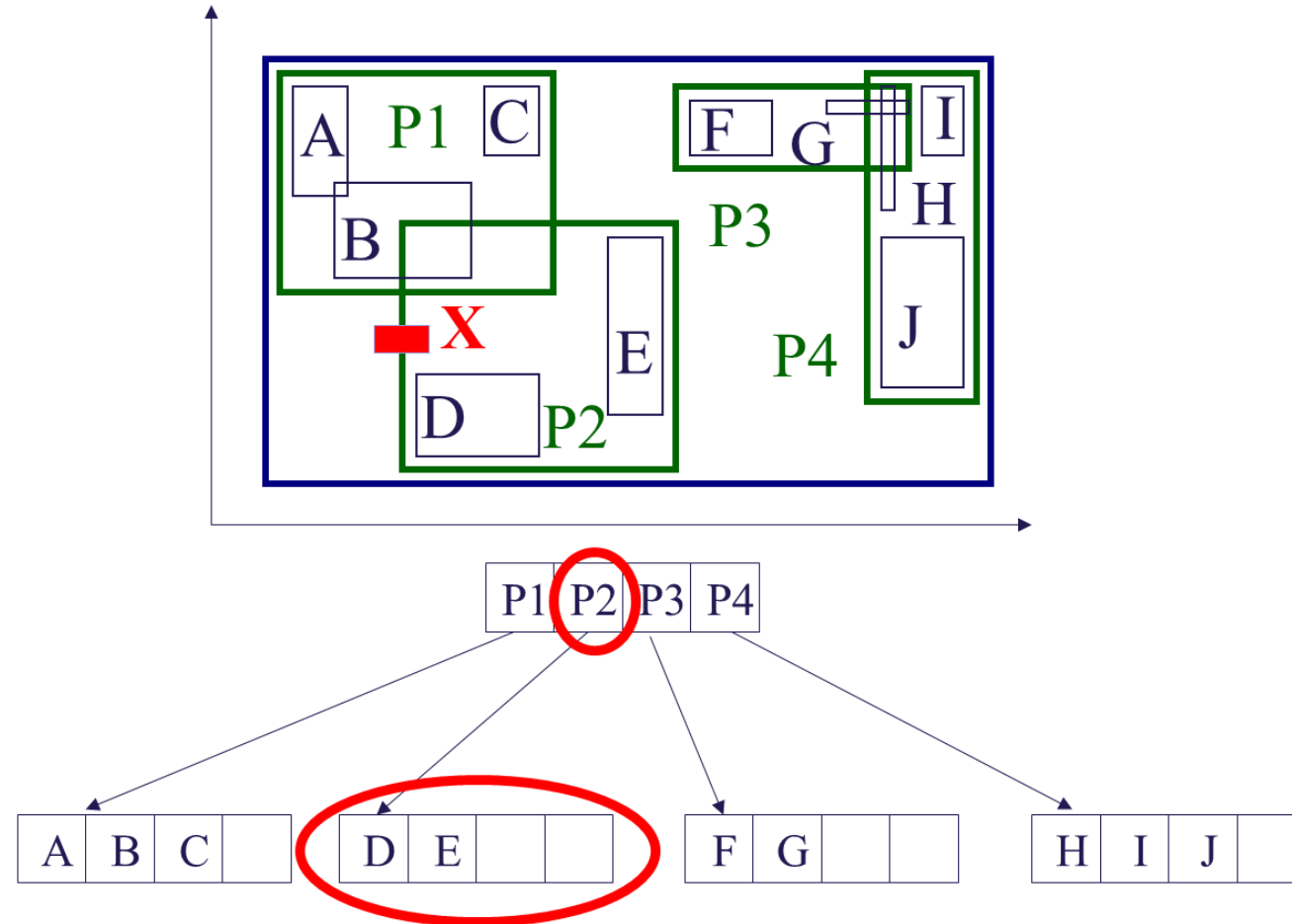
Antonin Guttman: R-Trees: A Dynamic Index Structure for Spatial Searching. ACM SIGMOD Conference 1984: 47-57

Insert: No Split, No Enlargement

- X is inserted to the child node that P2 points to.

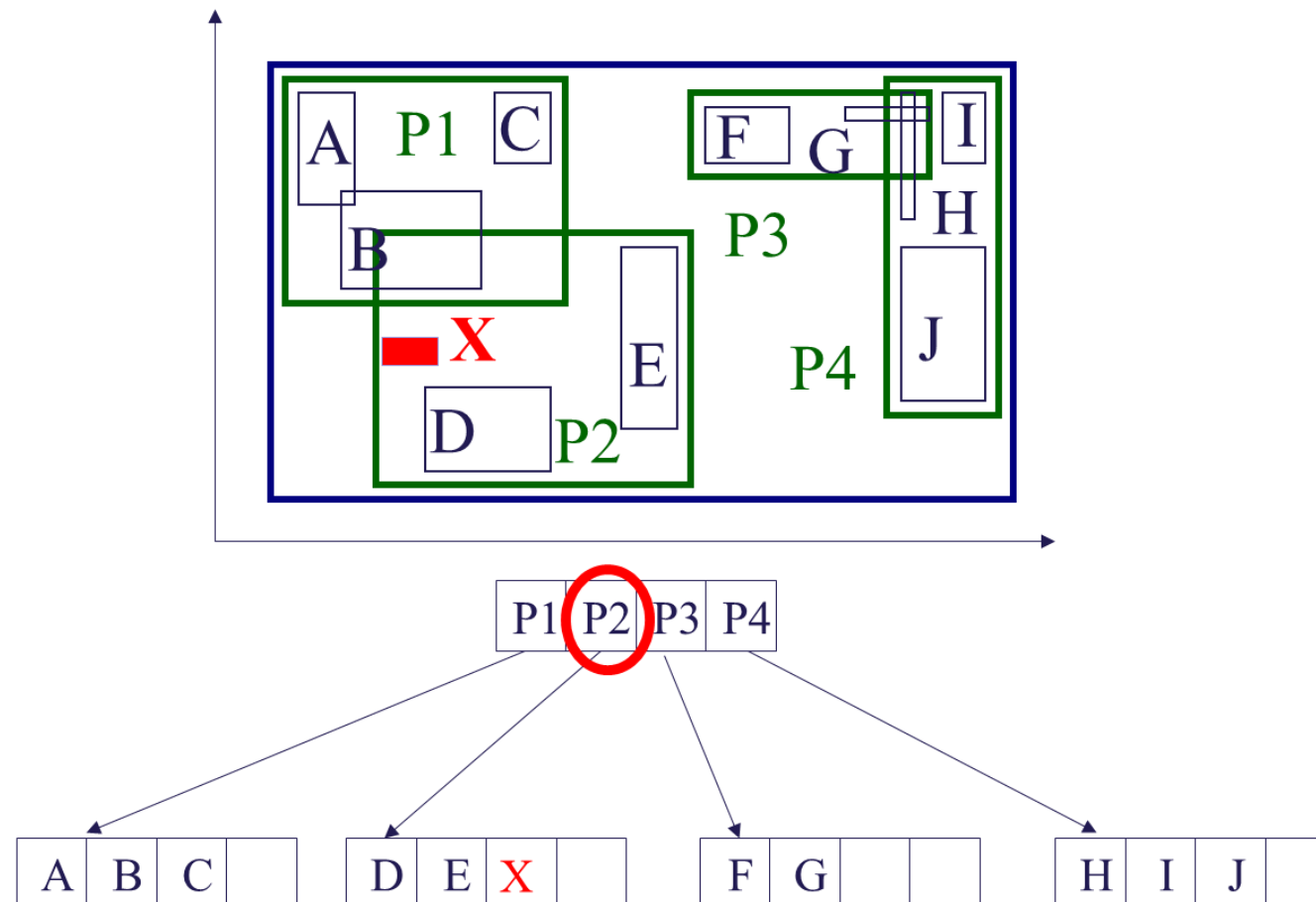


Insert: No Split, But Enlargement



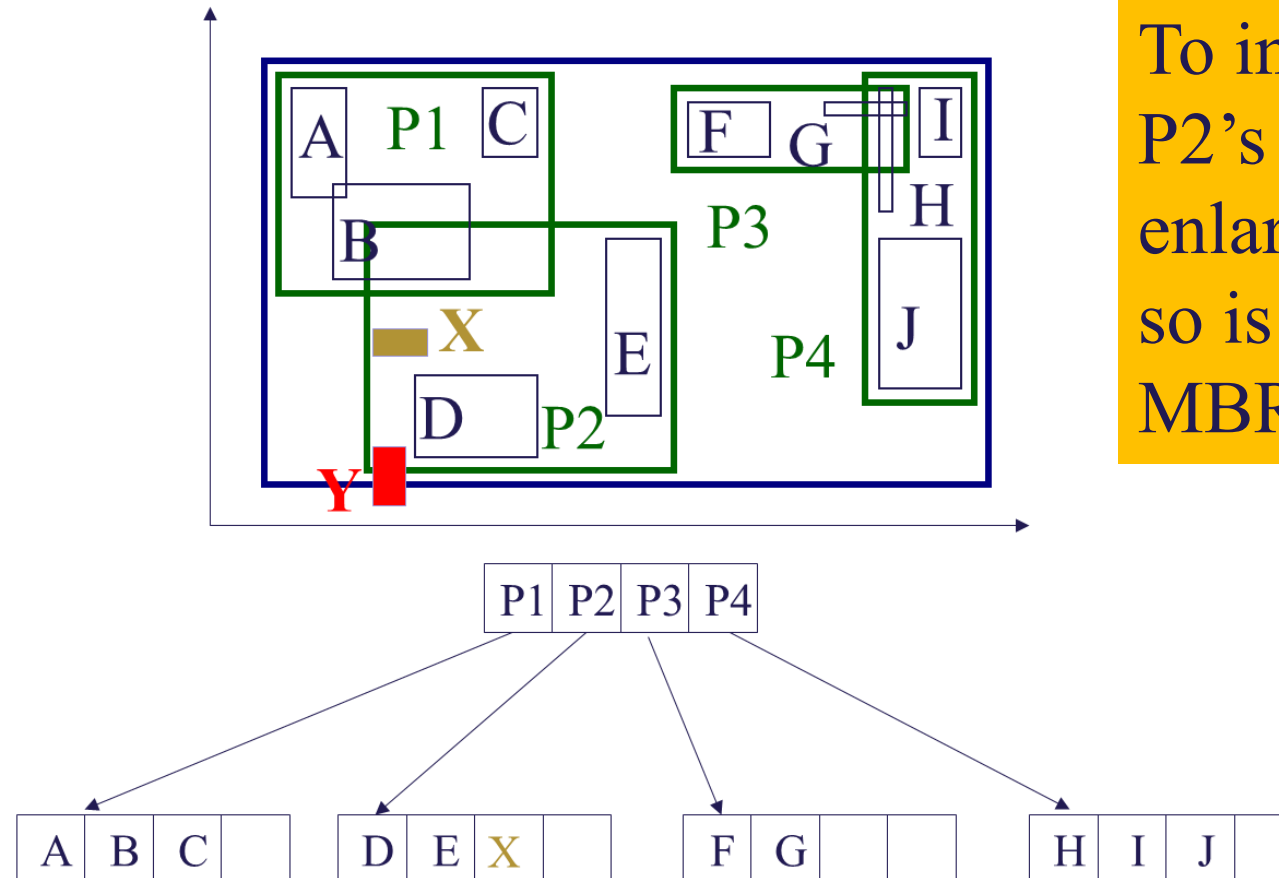
Insert: No Split, But Enlargement

- P2's MBR is enlarged.



Insert: No Split, But Enlargement

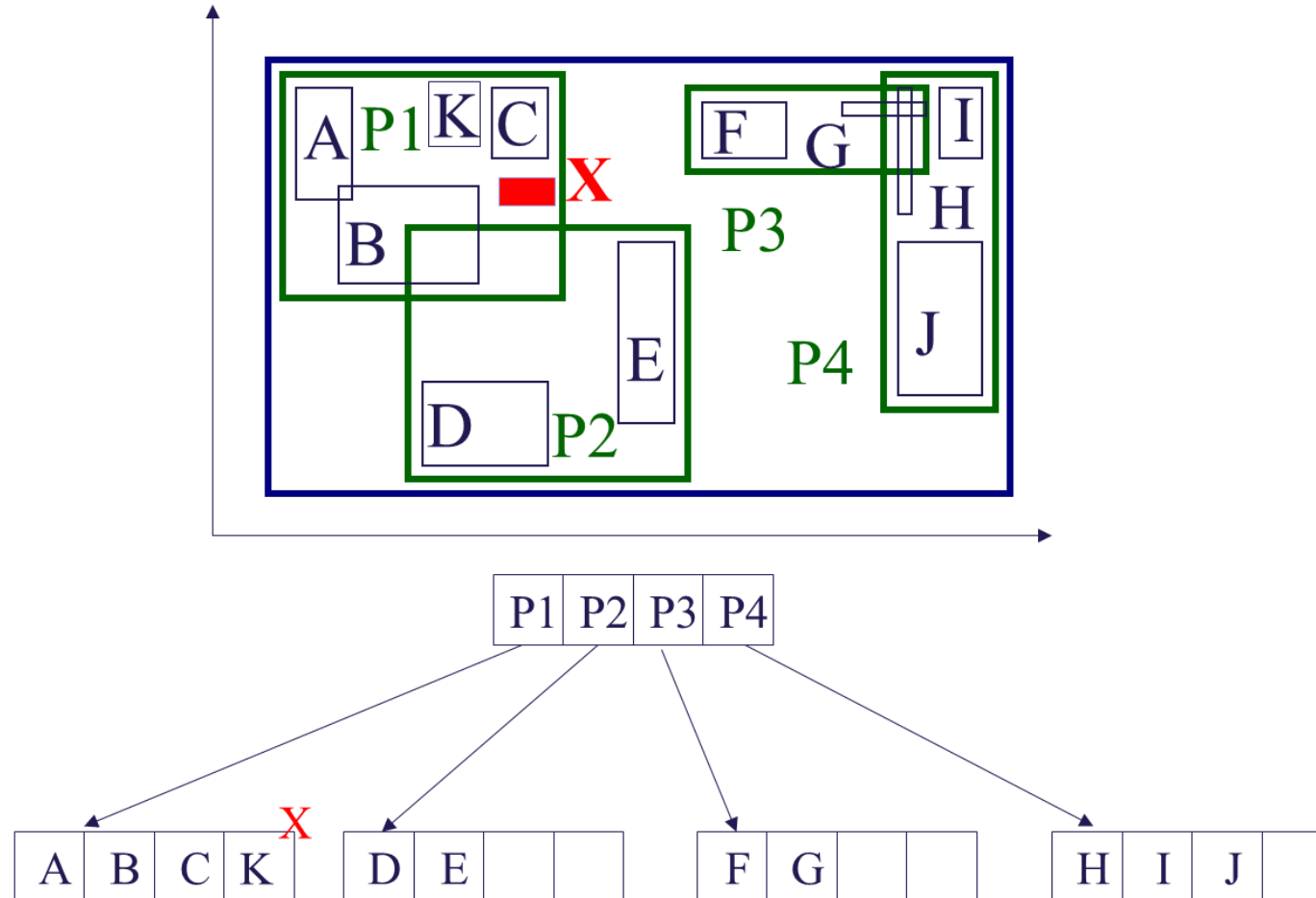
- ▶ Enlargement may propagate upward



To insert Y, P2's MBR is enlarged. And so is the overall MBR.

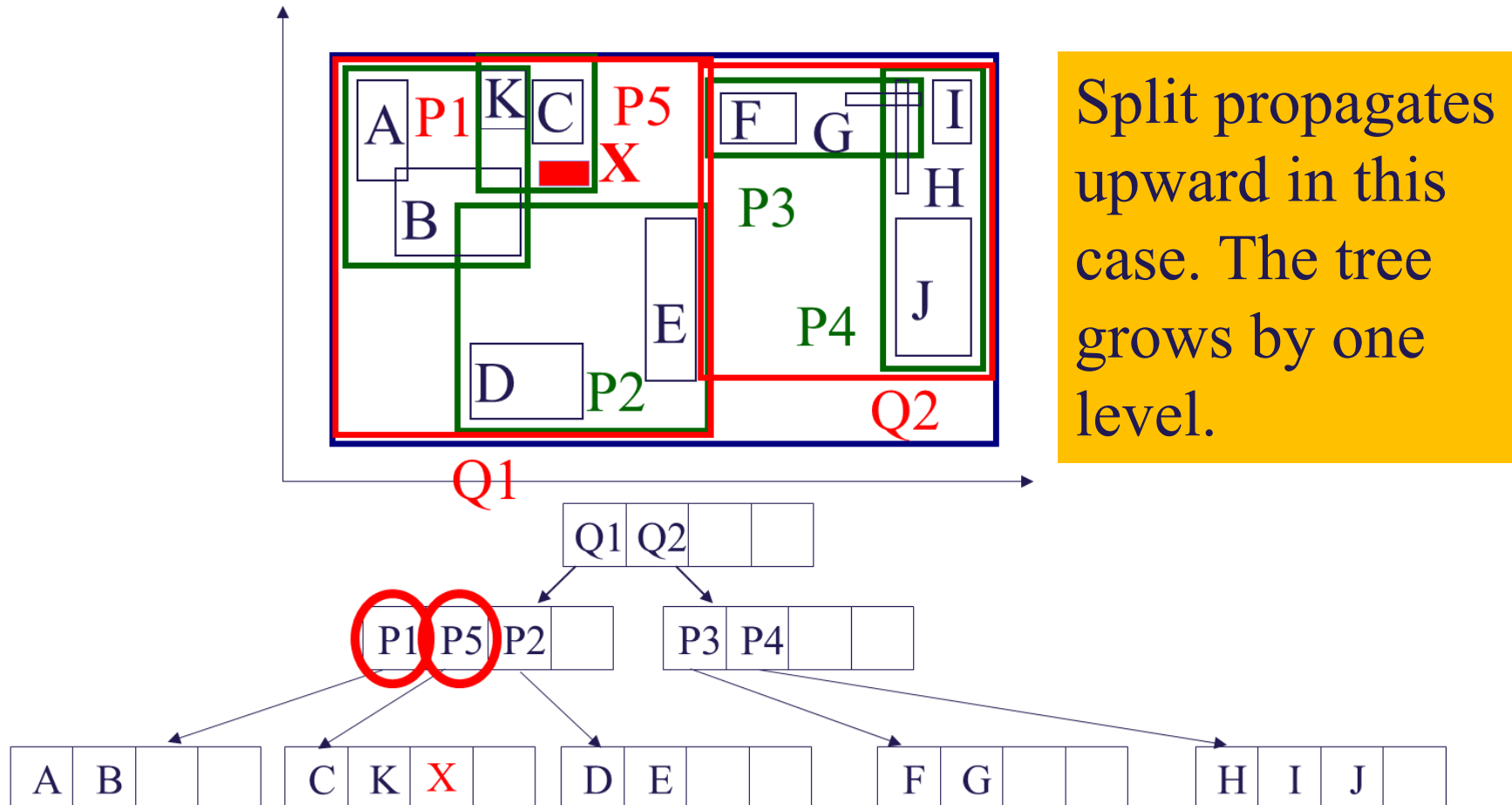
Insert: Split

- P1 is to be split when X is being inserted.



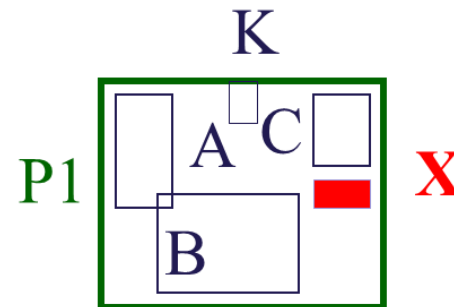
Insert: Split

- P1 is split to P1 and P5.



Node Splitting in Insertion

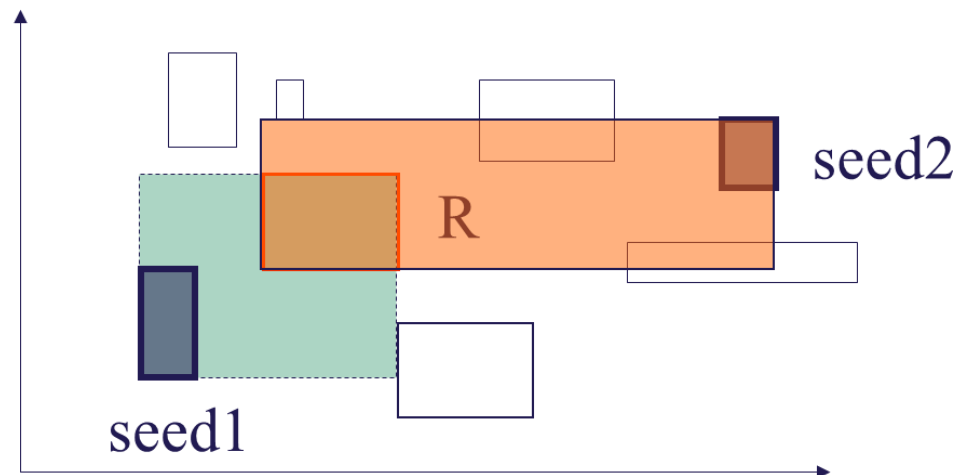
- ▶ The entries in node L plus the newly inserted entry must be (evenly) distributed between $L1$ and $L2$.
- ▶ The goal is to reduce likelihood of both $L1$ and $L2$ being searched on subsequent queries.
- ▶ **Idea:** Redistribute so as to minimize the sum of $L1$'s area and $L2$'s area.
- ▶ Split node $P1$: partition the MBRs into two groups.
 - ▶ Option 1: exponential split; $2^M - 1$ choices
 - ▶ Option 2: plane sweep, until 50% of MBRs
 - ▶ Option 3: quadratic split
 - ▶ Option 4: 'linear' split



Pick two seeds to
initiate two
groups.

R-trees: Split

- ▶ Pick two rectangles as 'seeds' and put them into two groups.
 - ▶ Option 3 and Option 4 differ only in this step.
 - › Next slide
- ▶ Assign each MBR 'R' to its 'closest' 'seed'.
 - ▶ 'closest': the smallest increase in area



How to Pick Seeds in Split?

► Option 3: Quadratic split

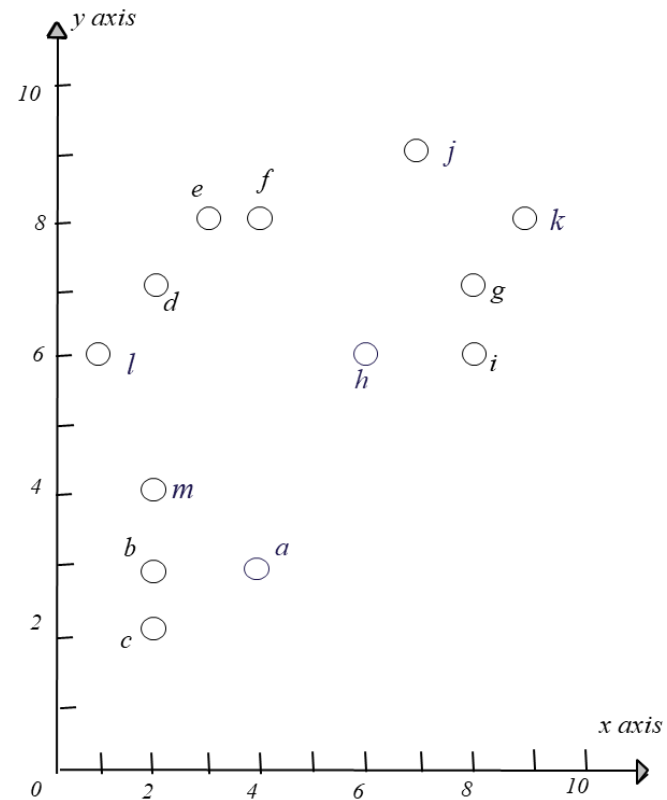
- For each pair E1 and E2, calculate
 - › The rectangle $J = \text{MBR}(E1, E2)$
 - › $d = \text{area}(J) - \text{area}(E1) - \text{area}(E2)$.
- Choose the pair with the largest d.

► Option 4: Linear split

- Along each dimension
 - › Find the entry with the *highest low* side, and the one with the *lowest high* side.
 - › Normalize the separation along the corresponding dimension.
- Choose the pair with the greatest normalized separation.

Exercise

- ▶ Create an R-tree for the following data points.
 - ▶ Each node contains at most 3 entries.
 - ▶ Insertions are in alphabetic order



Agenda

- R-tree
 - Basic idea and structure
 - Insertion and split
 - Deletion
- R-tree Variants

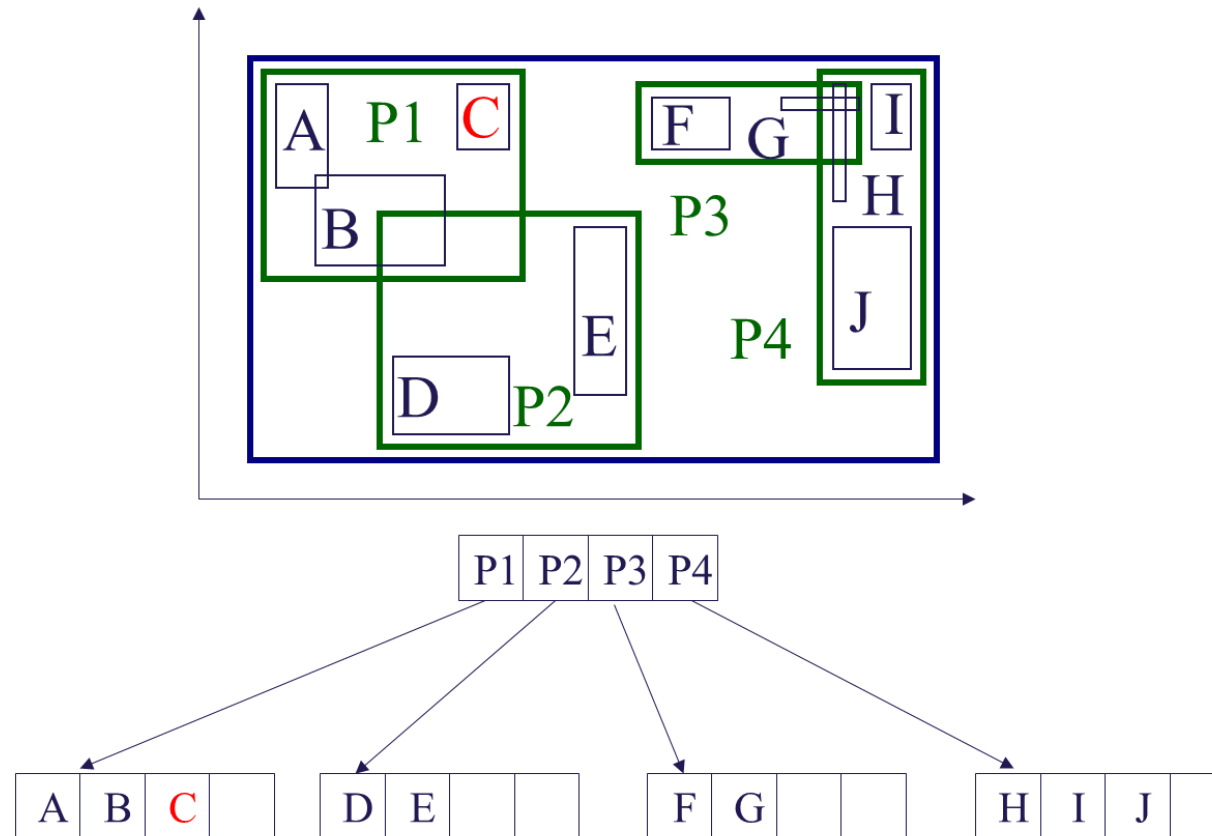
R-Trees: Deletion

- Find the leaf node N that contains the entry E
- Remove E from this node N
- If node N goes underflow:
 - Eliminate node N by removing its entry E_N from its parent node and add N to Q
- Otherwise, shrink N 's MBR accordingly
- Propagation upward
 - Reinsert all entries of nodes in Q into the tree using Algorithm **Insert**

Antonin Guttman: R-Trees: A Dynamic Index Structure for Spatial Searching. ACM SIGMOD Conference 1984: 47-57

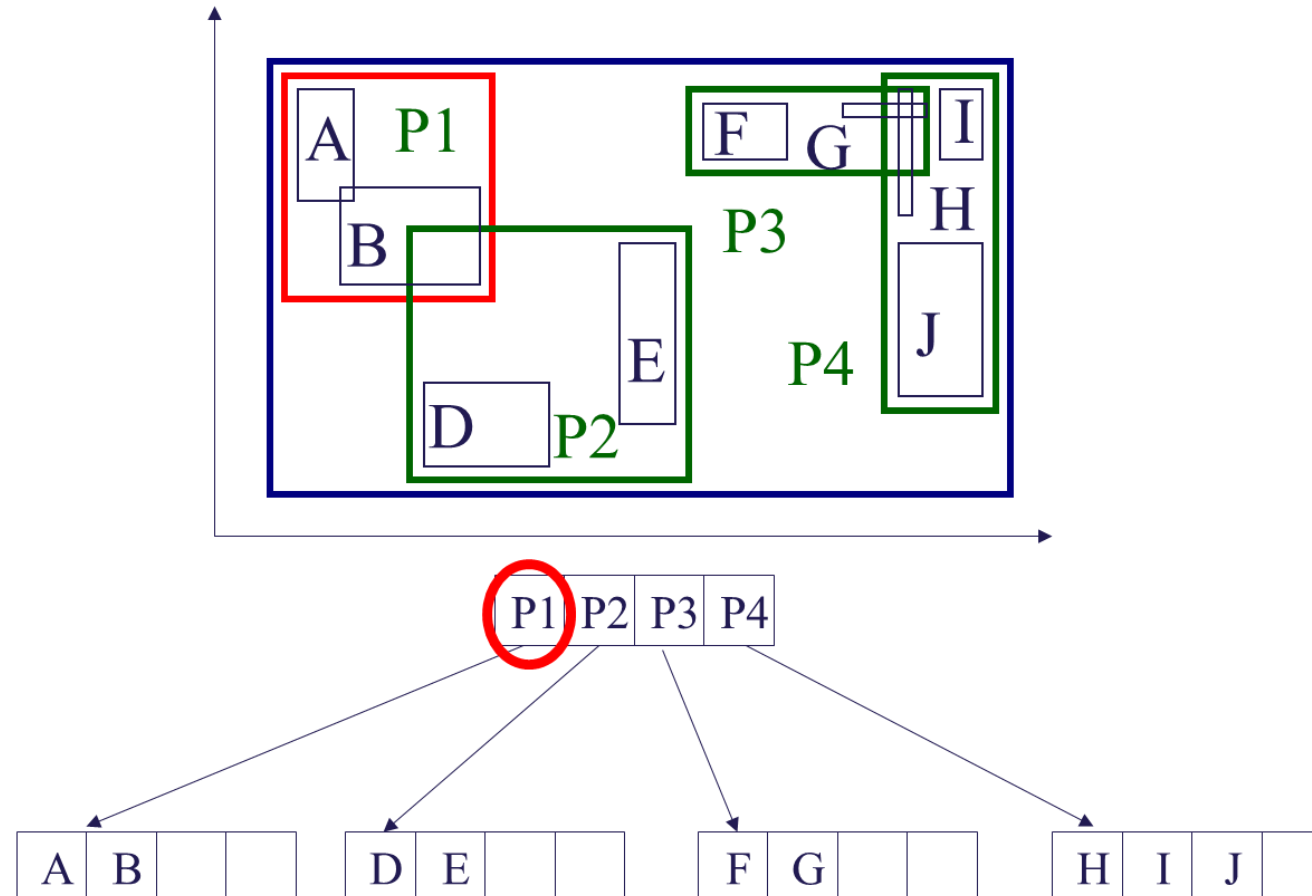
Delete: Shrink MBR

► To delete C



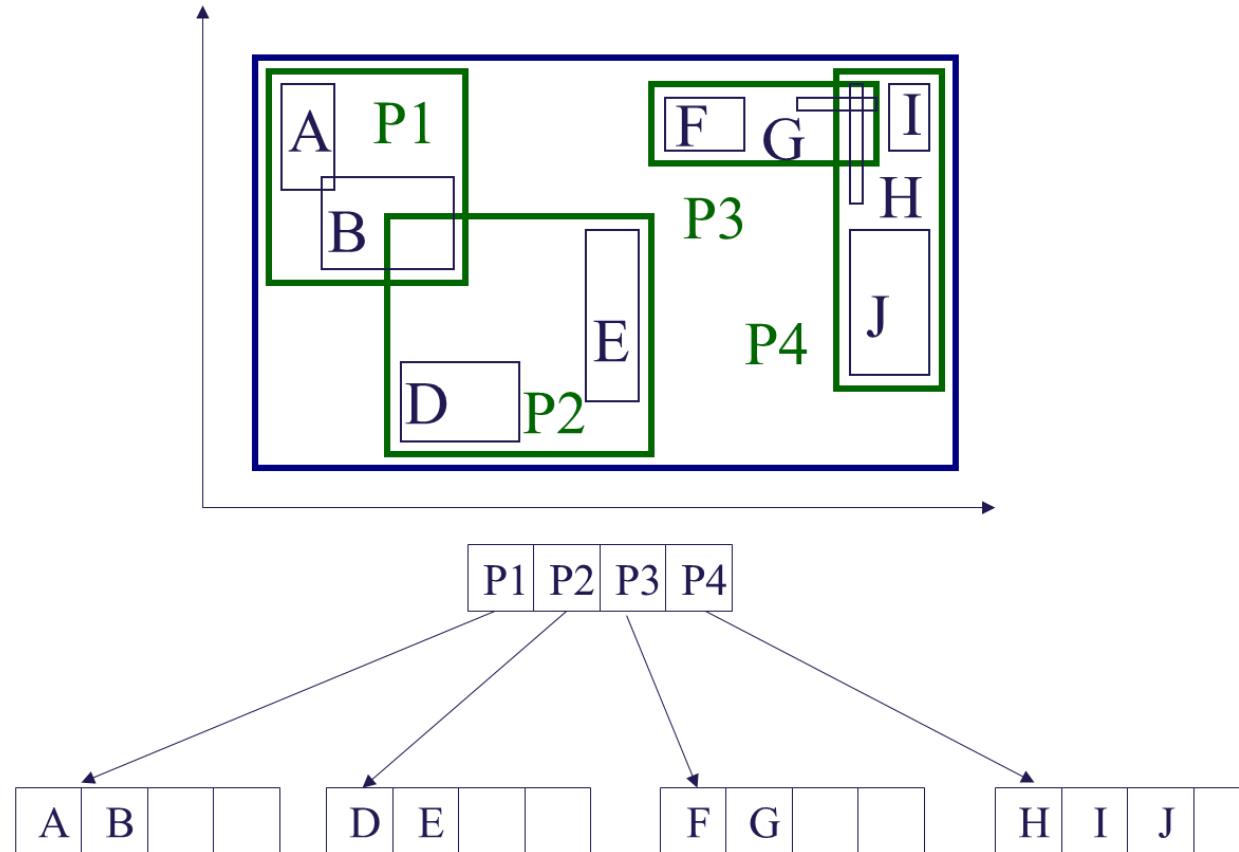
Delete: Shrink MBR, cont.

- After deleting C, P1's MBR has been shrunk.



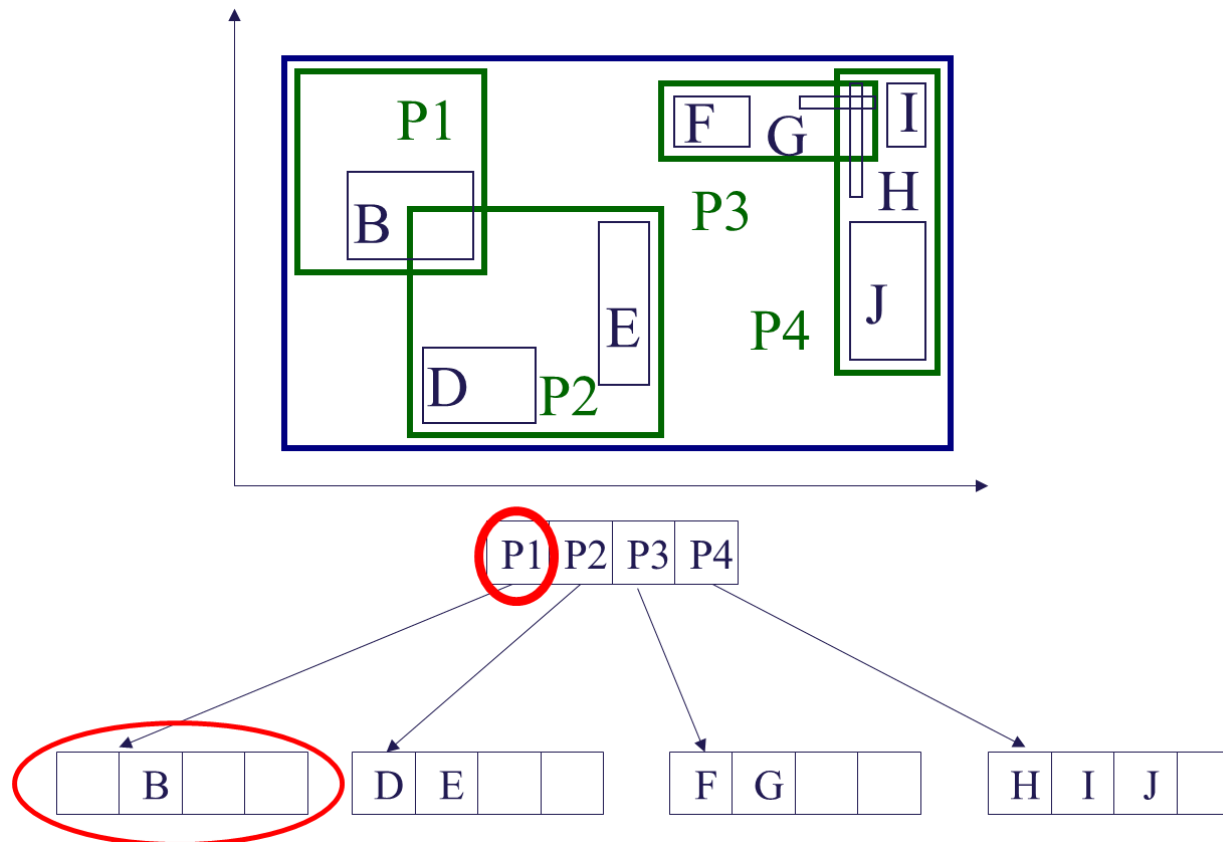
Delete: Node Underflow

► To delete A.



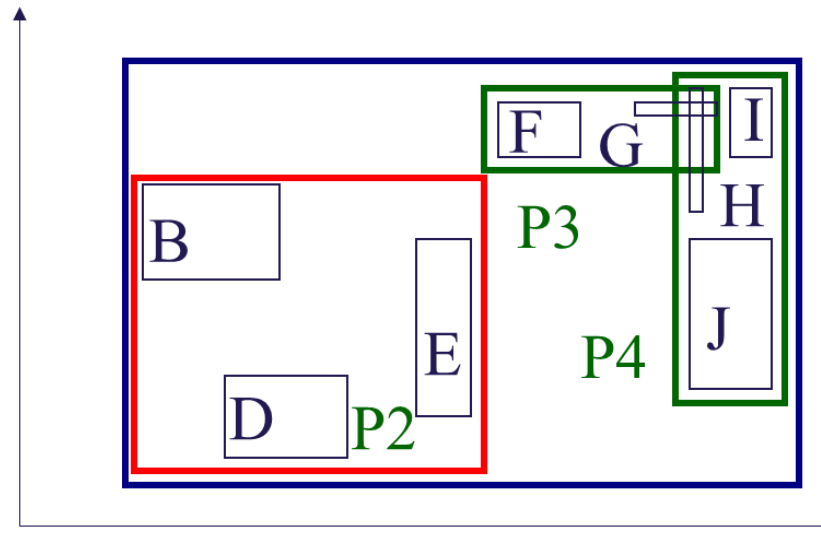
Delete: Node Underflow, cont.

- After deleting A, P1 will become underflow.

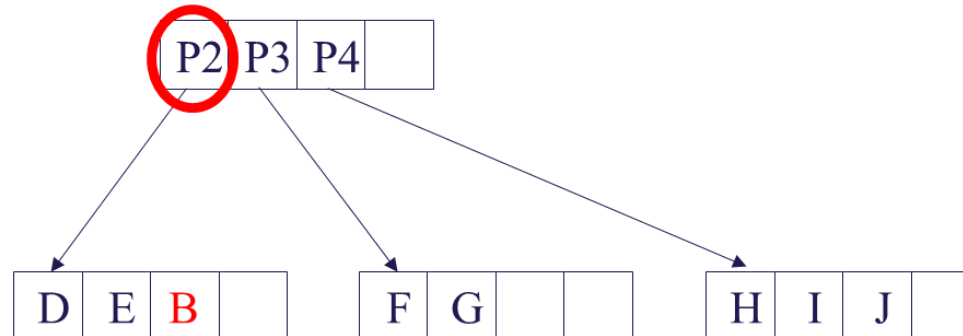


Delete: Node Underflow, cont.

- P1 is removed from root, and B is re-inserted (to P2).



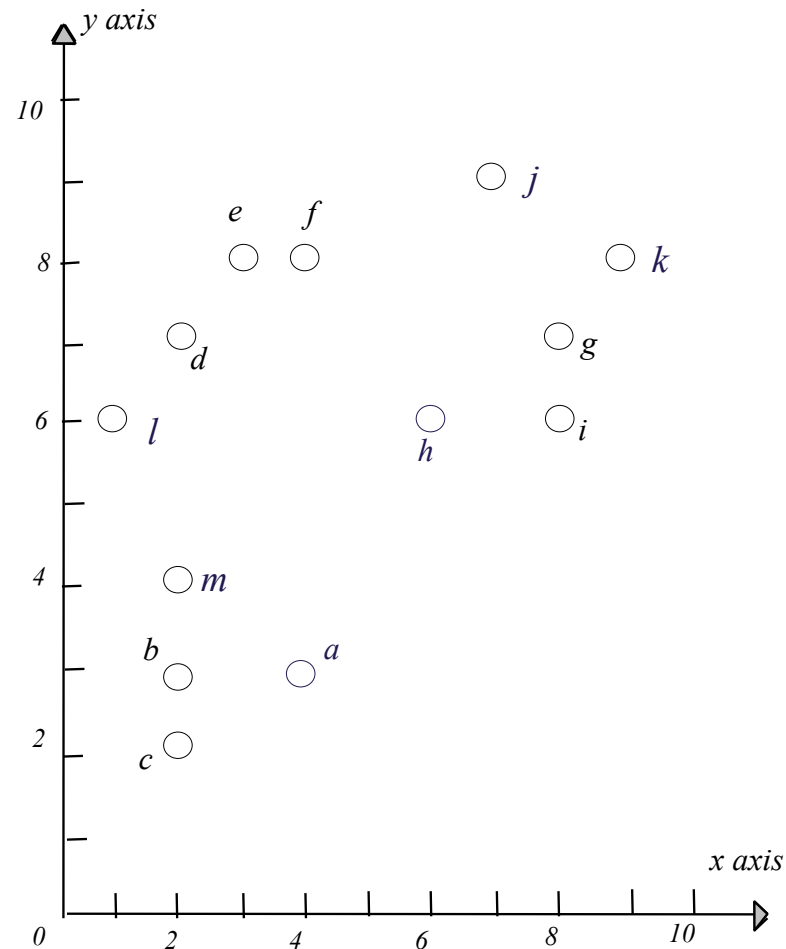
Note the overall MBR has also been shrunk.



Exercise

► On the R-tree you've created

- Delete point *d*.
- Delete point *m*.
- Delete point *a*.
- Delete point *c*.



Agenda

- R-tree
- R-tree Variants
 - R*-tree
 - Bulk loading

R-trees: Variations

► R*-tree

- Change the insertion, deletion algorithms
- More criteria to optimize in insertions and deletions
- *Forced re-insertion* instead of splitting for node overflows
 - › Remove some (e.g., 30%) entries and reinsert them into the tree.
 - › Could result in all reinserted entries fitting on some existing pages, avoiding a node split. Overall, node splits are reduced and deferred.

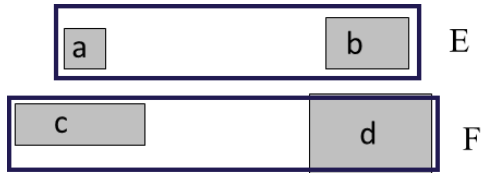
► Hilbert R-tree

- Uses the Hilbert-curve values to insert objects into the tree

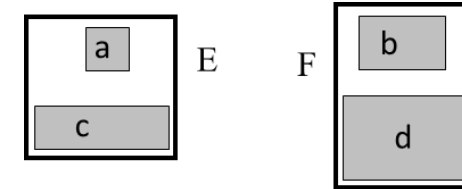
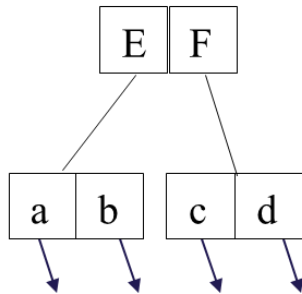
► R⁺-tree

- Avoids overlap by splitting and inserting an object into multiple leaves if necessary.
- Searches now may take a single path (or less paths) to a leaf, at cost of redundancy.

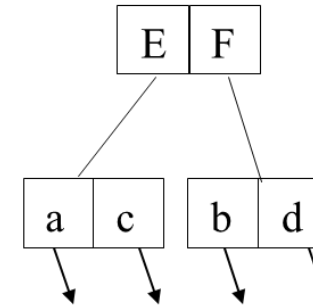
Which R-tree is better? Why?



Tree #1



Tree #2

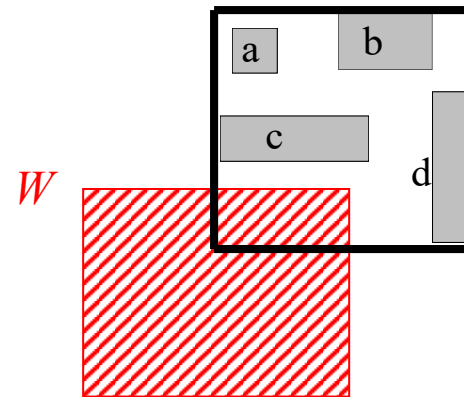
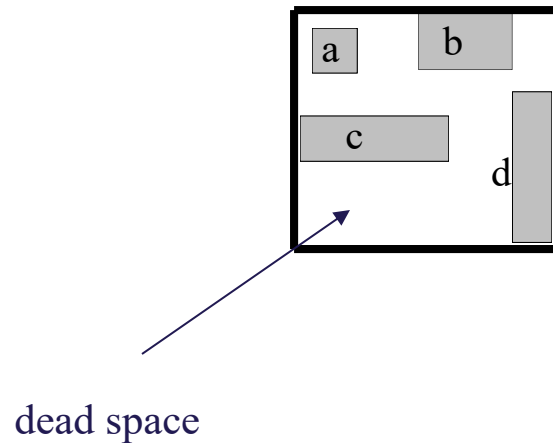


- ▶ A “good” tree should have
 - ▶ nodes with small MBRs
 - ▶ nodes with small MBR overlap
 - ▶ nodes that look like squares
 - ▶ nodes as full as possible

Sometimes it is impossible to optimize all these criteria at the same time!

R*-tree Optimization Criteria (1)

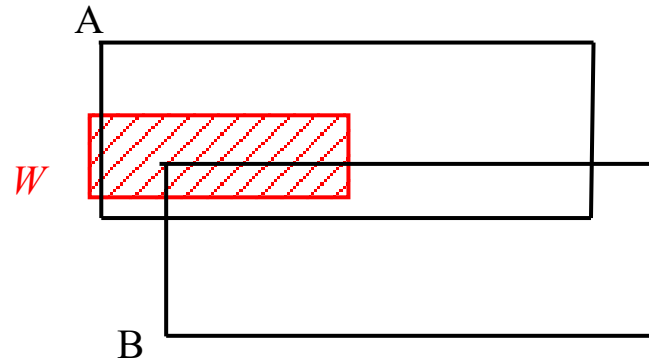
- ▶ Minimize the area covered by an index MBR
 - Small area means small dead space



window query intersects node MBR, but no object!

R*-tree Optimization Criteria (2)

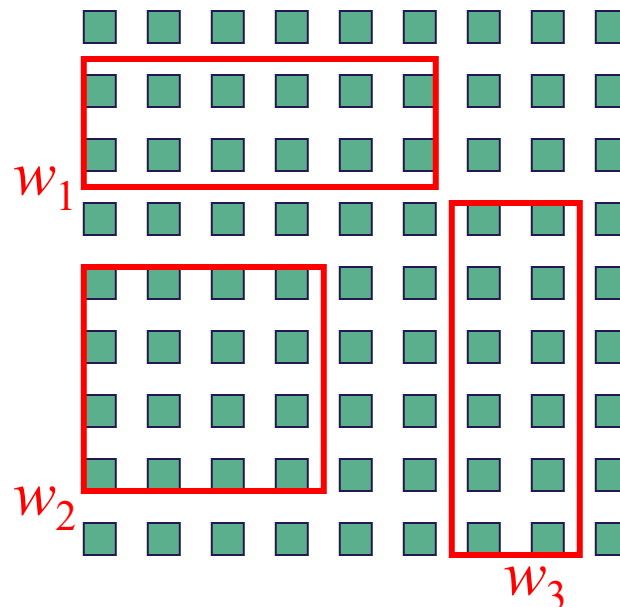
- ▶ Minimize overlap between node MBRs
 - Minimizes the number of traversed paths



Both nodes intersect query!

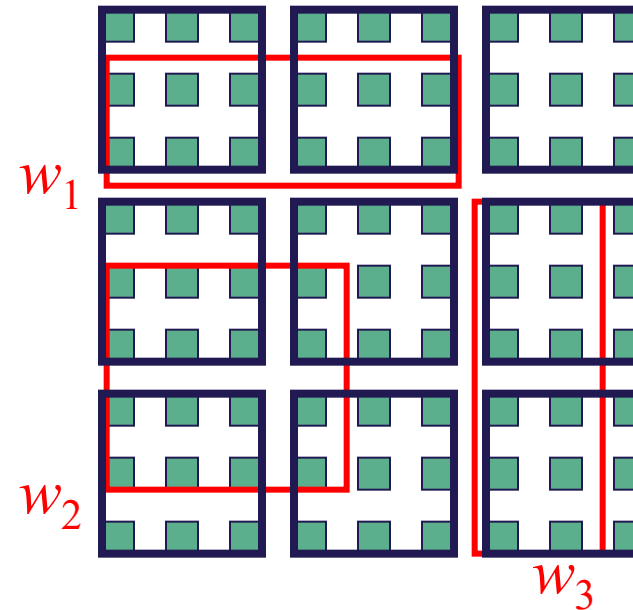
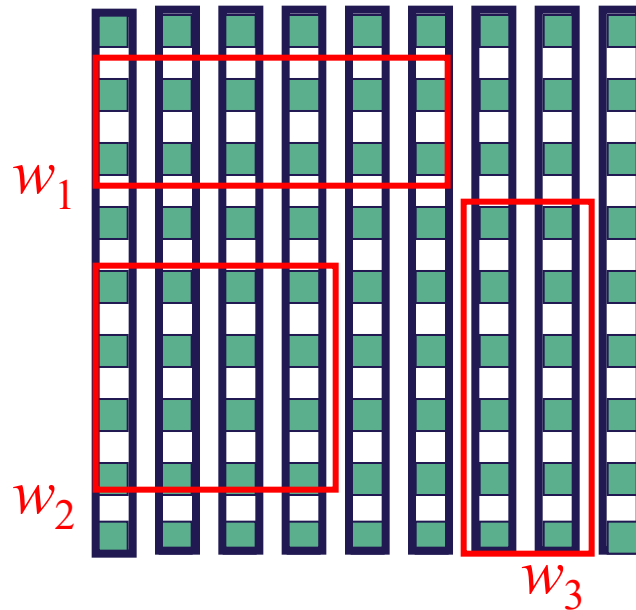
R*-tree Optimization Criteria (3)

- Minimize the margins of node MBRs
 - Margin is the sum of the lengths of the edges of a rectangle
 - Square-like nodes, smaller number of intersections for a random query, better structure
- Problem: group the rectangles into groups of 9, such that the expected number of group MBRs intersected by a (random) query is minimized



Effect of Margins Minimization

- Grouping 1 (minimal areas of group MBRs)
 - Bad grouping for queries w_1 , w_2
- Grouping 2 (minimization of margins)
 - Minimizes the expected number of groups touching a random query



R*-tree Optimization Criteria (4)

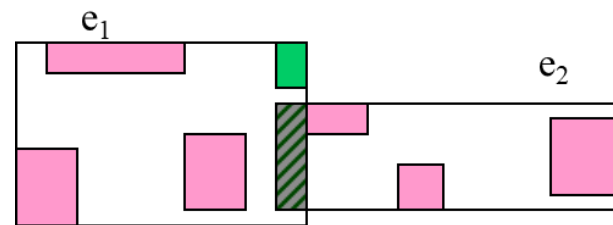
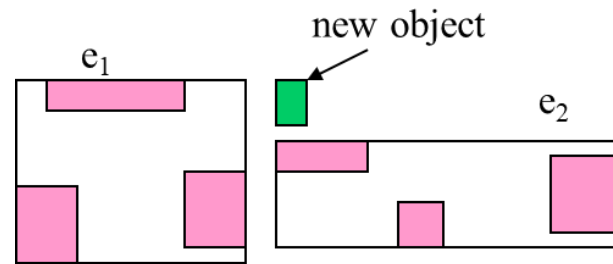
- Optimize the storage utilization
 - Nodes in tree should be filled as much as possible
 - Minimizes tree height and potentially decreases dead space
- For update-frequent scenarios, this criteria may not be desirable
- This criteria may be in conflict with the first optimization criteria
- Sometimes it is impossible to optimize all these criteria at the same time!

R*-tree Insertion Heuristics

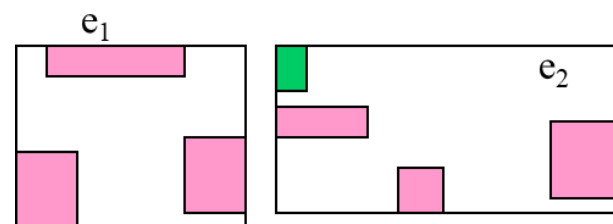
- ▶ When inserting a new entry e into the tree we follow one path from the root to a leaf
- ▶ The entry is then inserted to the leaf
- ▶ Issue: How to choose the path (subtree) from node N ?
 - ▶ If N 's child pointers point to leaves, choose the leaf node using the following criteria:
 - › Needs *the least overlap enlargement* to include the new entry
 - › Needs *the least area enlargement* to include the new entry
 - › Has *the smallest MBR area* to include the new entry
 - ▶ Else, choose the sub-node
 - › Needs *the least area enlargement* to include the new entry
 - › Has *the smallest MBR area* to include the new entry

R*-tree Insertion Heuristics, cont.

- The least MBR overlap enlargement after insertion
 - Break ties by choosing MBR with minimum area



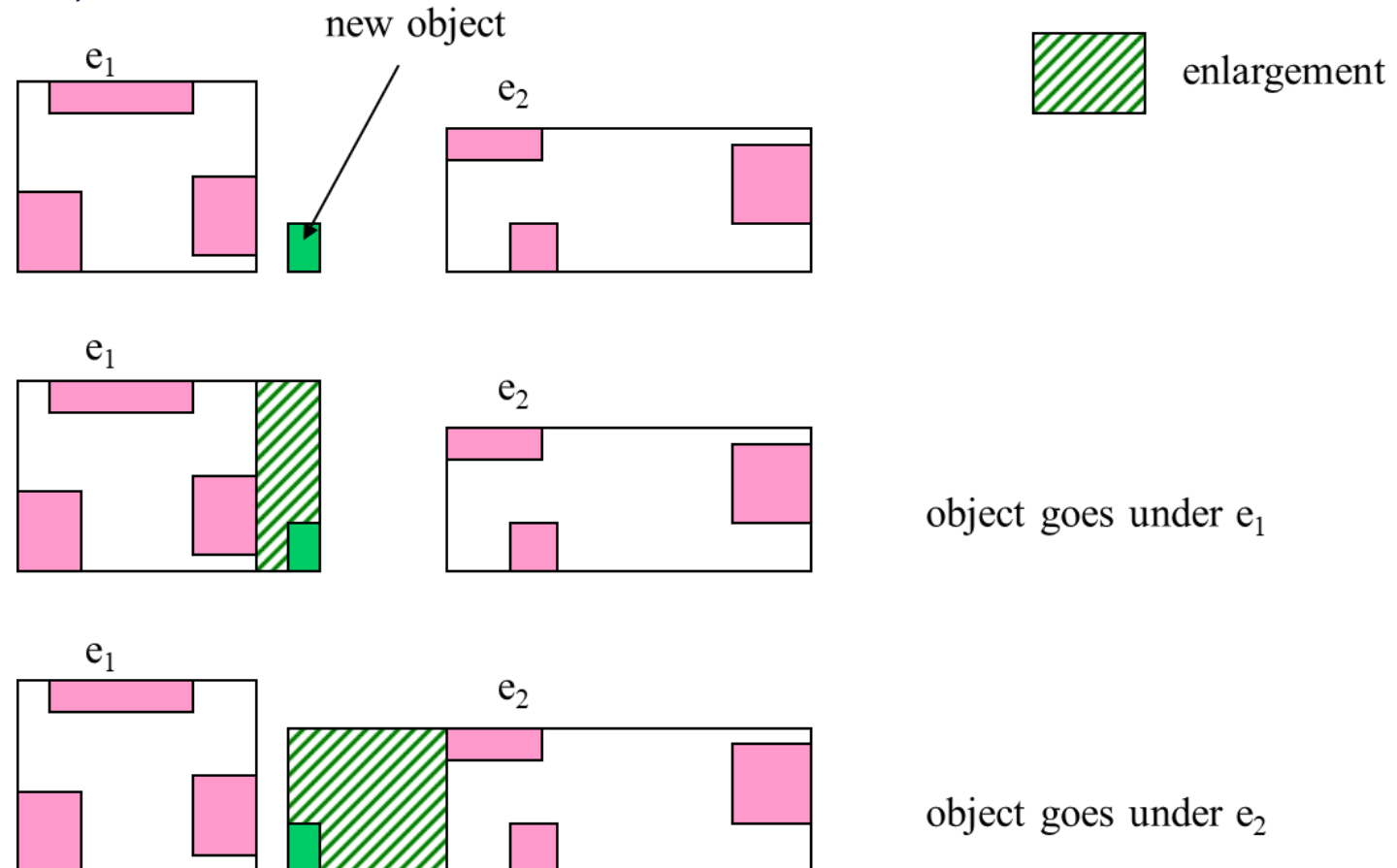
object goes under e_1



object goes under e_2

R*-tree Insertion Heuristics, cont.

- The least MBR enlargement after insertion
 - Used for internal (non-leaf) nodes



R*-tree Node Splitting

- If a node overflows we need to split it
- Issue: distribute (fast!) a set of rectangles into two nodes such that the areas, overlap, and margins are minimized.
 - Have to give weight on some optimization criteria (conflicting)
 - Distribution may not be even
- Sort the rectangles with respect to one axis (x or y) and find the best split distribution from the sorted lists

Determine the split axis

- For each axis (i.e., x and y axis)
 - Sum=0;
 - sort entries by the lower value, then by upper value
 - for each sorting (e.g. lower value)
 - for $k=m$ to $M+1-m$
 - place first k entries in group A, and the remaining ones in group B
 - $\text{Sum} = \text{Sum} + \text{margin}(A) + \text{margin}(B)$
- Choose axis with minimum Sum

Distribute entries along the axis

- ▶ Along the split axis, choose the distribution with minimum overlap
- ▶ If there are multiple groupings with minimal overlap choose $\langle A, B \rangle$ such that $\text{area}(A) + \text{area}(B)$ is minimized

Agenda

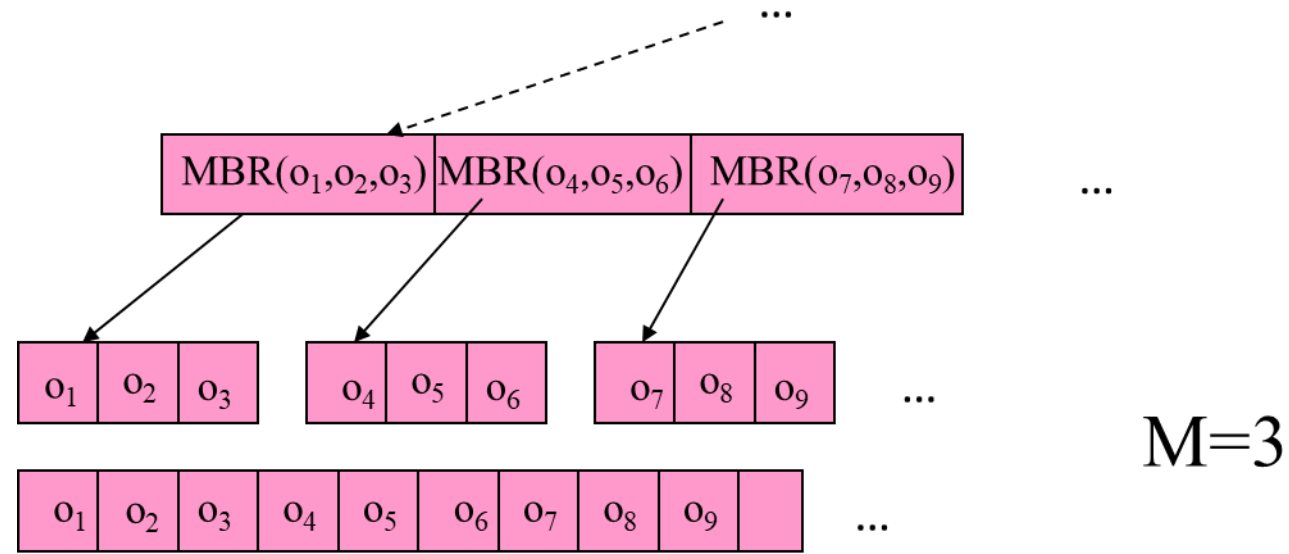
- ▶ R-tree
- ▶ R-tree Variants
 - ▶ R*-tree
 - ▶ Bulk loading

Bulking Loading R-tree

- Given a fixed set of spatial objects, how to create an “optimal” R-tree?
- Nearest-X (x-sorting)
 - Rectangles are sorted on the x-coordinate and nodes are created.
- Hilbert R-Tree
 - Uses the Hilbert value of the center of a rectangle to sort the leaf nodes and recursively builds the tree.
- Sort-Tile-Recursive (STR)

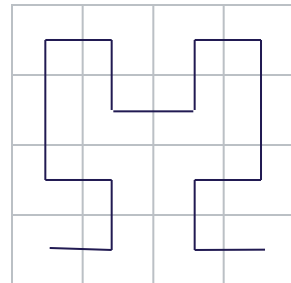
X-Sorting

- ▶ Sort using only one axis
 - ▶ sort rectangles using the x-coordinate of their center
 - ▶ pack M consecutive rectangles in leaf nodes
 - ▶ build tree bottom-up

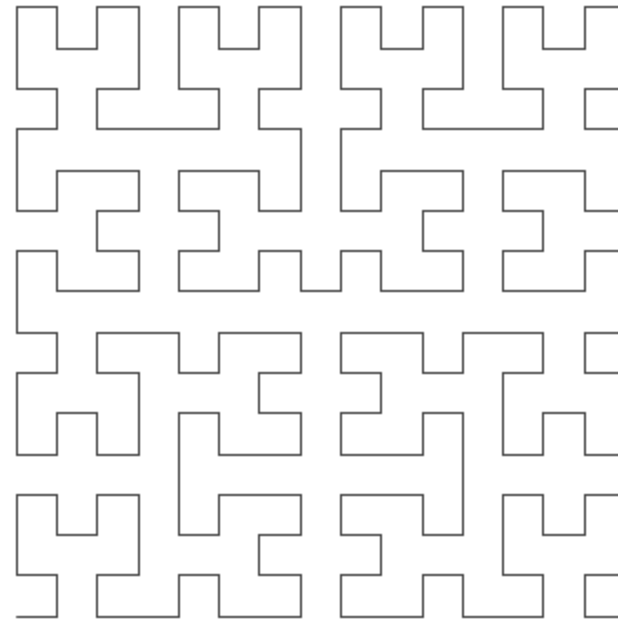


Hilbert R-Tree

- ▶ X-sorting results in leaf nodes that have long stripes as MBRs
- ▶ Hilbert R-Tree: use a space-filling curve to order the rectangles
 - ▶ much better structure, but still the nodes have large overlap

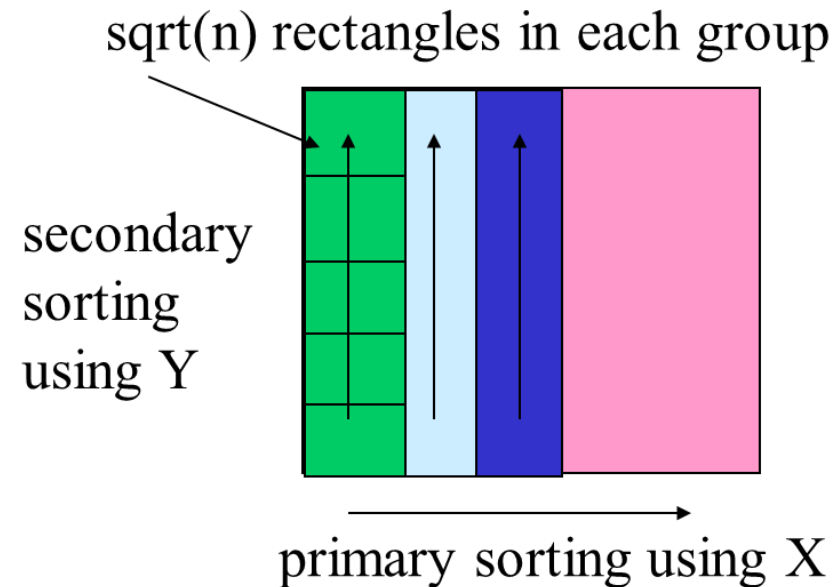


Hilbert-curve

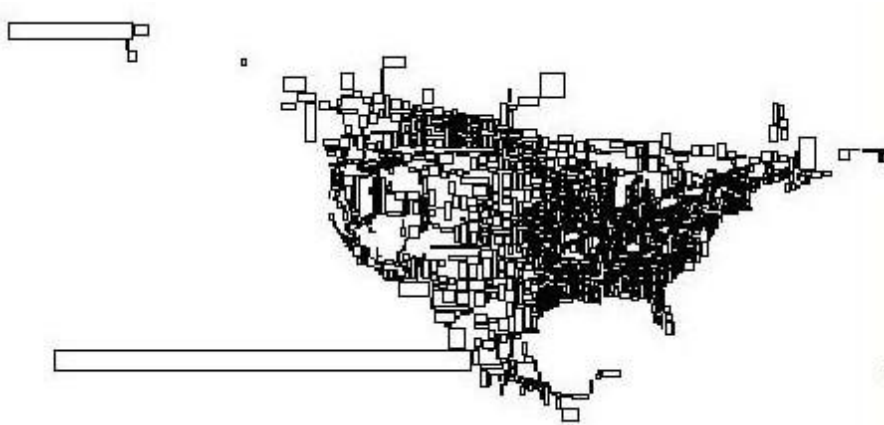


Sort-Tile-Recursive (STR)

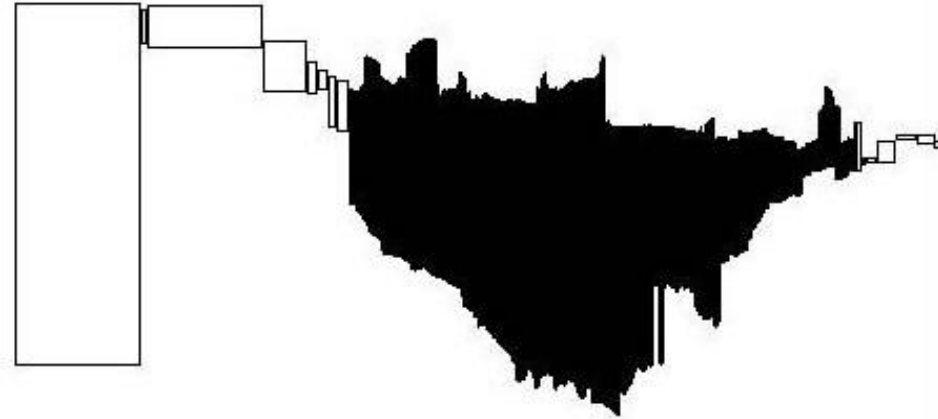
- ▶ Sort using one axis first, create \sqrt{n} tiles, and then groups of \sqrt{n} rectangles in each tile using the other axis ($n = \text{\#leaves} = \text{\#objects/capacity}$) [why \sqrt{n} ?]
- ▶ Usually better compared to other bulk-loading methods



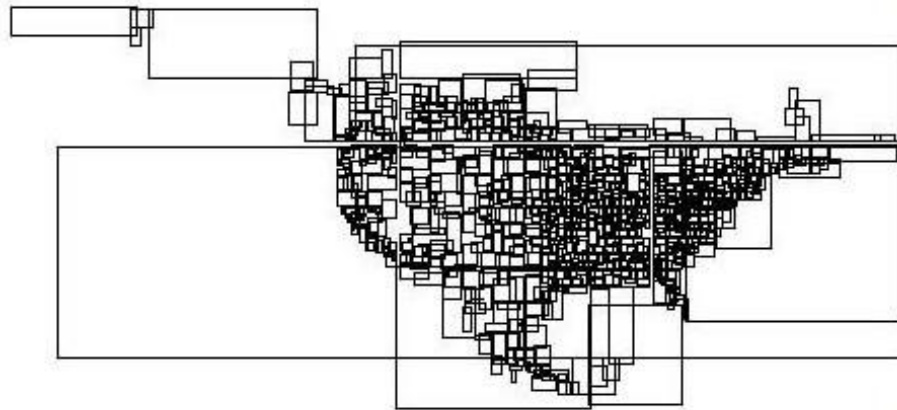
R-tree leaf nodes by different construction methods



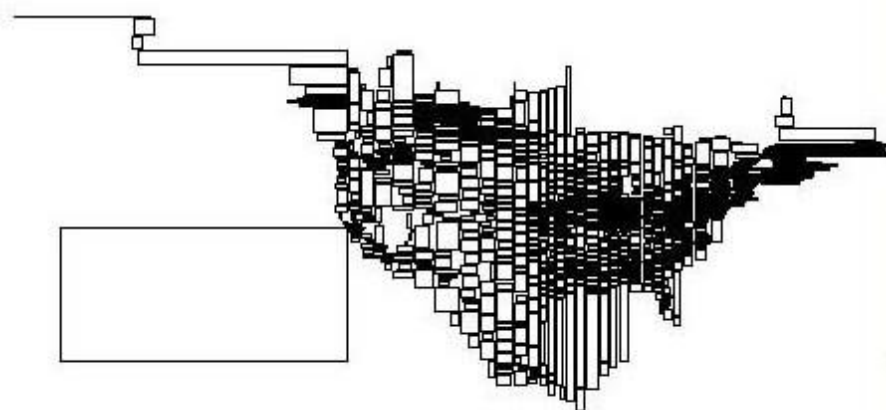
(a) R*-tree insertion



(b) x-sorting



(c) Hilbert sorting



(d) Sort-tile recursive

Summary

- Spatial data
 - Data types
 - Spatial relationships
 - Spatial queries
- R-tree
 - R-tree structure
 - R-tree insertions
 - R-tree deletions
- Variants
 - R*-tree optimization criteria
 - Bulk loading

Readings and Exercises

► Mandatory reading

- A. Silberschatz, H. F. Korth, S. Sudarshan: Database System Concepts (7th edition), McGraw-Hill.
 - › 8.4 Spatial Data
 - › 14.10 Indexing of Spatial and Temporal Data
- Antonin Guttman: R-Trees: A Dynamic Index Structure for Spatial Searching. SIGMOD Conference 1984: 47-57

► Further readings

- Norbert Beckmann, Hans-Peter Kriegel, Ralf Schneider, Bernhard Seeger: The R*-Tree: An Efficient and Robust Access Method for Points and Rectangles. SIGMOD Conference 1990: 322-331

► Exercises

- Those in the slides
- The document given in Moodle