Problem List 1

(1) (5 points) Find critical points of the following function on $\mathbb{R}^2$ and analyze each point about whether it is a local/global minimum/maximum or saddle point

$$f(x) = 6x_1x_2 - x_1x_2^2 - x_1^2x_2.$$

(2) (10 points) Find the maximum of the function $F(x) = \sum\limits_{i=1}^{n} \log(\alpha_i + x_i)$ with $\alpha_i > 0, i = 1, ..., n$ subject to constraint $x_1 + ... + x_n = 1$. Prove that your solution is the global optimum.

(3) (15 points) Using method of Lagrange multipliers in optimization problem

$$\min\Big(-\prod_{i=1}^{n} x_i\Big) \qquad \text{s.t.} \qquad \sum_{i=1}^{n} x_i = 1, \qquad x_i \geq 0,$$

prove the Arithmetic-Mean Geometric-Mean inequality

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \left(\prod_{i=1}^{n} x_i\right)^{1/n}, \qquad x_i \geq 0.$$

(4) (15 points) Using method of Lagrange multipliers in optimization problem

$$\min_{x_i \in \mathbb{R}^n} \Big(-\det(x_1, ..., x_n)\Big), \qquad \text{s.t.} \qquad \|x_i\|^2 = 1,$$

prove Hadamard inequality

$$\det(x_1, ..., x_n) \leq \prod_{i=1}^{n} \|x_i\|.$$

(5) (10 points) Find a local minimum of the following function using the Gradient Descent method with the best step size starting from the point $x^0 = (1, 3)$

$$f(x) = x_1^3 + 4x_1^2 + 2x_1x_2 + \frac{5}{2}x_2^2.$$

(6) (20 points) Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable function and let $x \in \mathbb{R}^n$ be such that $\nabla f(x) \neq 0$. The approximation

$$f(x + h) = f(x) + \nabla f(x)^t h + o(h), \qquad \|h\|_2 \to 0$$

motivates the problem of finding best descent direction h on some subset $D \subset \mathbb{R}^n$:

$$\min_{h \in D} \nabla f(x)^t h.$$

Find optimal h if the set D is given by the constraint

a) $\|h\|_2 \leq 1$, b) $\|h\|_1 \leq 1$, c) $\|h\|_\infty \leq 1$,

d) $h^t Q h \leq 1$, where $Q \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is positive definite matrix.

(7) (5 points) Find a local minimum of the following function using the Newton Method starting from the point $x^0 = (2, 2)$

$$f(x) = 2x_1^2 + 9x_2^2 + 3x_1x_2 - 4x_1 - 3x_2 + 1.$$

(8) (10 points) Find a minimum on the set $X = \{2x_1 - 3x_2 = -6\} \subset \mathbb{R}^2$ of the following function using the Penalty Method with squared penalty function

$$f(x) = -8x_1^2 + 4x_1 - x_2^2 + 12x_2 - 3.$$

(9) (10 points) Solve the following LP problem using the simplex method

$$f(x) = x_1 + 3x_2 - x_3 \to \min,$$

$$x_1 + x_2 + x_3 \geq 3,$$

$$x_1 - 2x_2 \leq -2,$$

$$x_1 - 5x_2 - x_3 \geq -7,$$

$$x_1, x_2, x_3 \geq 0.$$