List of problems

1

N.1 $\mathbb{R}^2$

analyze points for global min / local max

$$f(x) = f(x_1, x_2) = 6x_1x_2 - x_1x_2^2 - x_1^2x_2$$

$$\frac{\partial f}{\partial x_1} = 6x_2 - x_2^2 - 2x_1x_2 \qquad \frac{\partial f}{\partial x_2} = 6x_1 - 2x_1x_2 - x_1^2$$

$$\frac{\partial f}{\partial^2 x_1} = -2x_2 \qquad \frac{\partial f}{\partial^2 x_2} = -2x_1$$

$$\frac{\partial^2 f}{\partial x_1 x_2} = 6 - 2x_2 - 2x_1$$

$$\text{grad} f(x) = (6x_2 - x_2^2 - 2x_1x_2,\ 6x_1 - 2x_1x_2 - x_1^2) = 0$$

$$\begin{cases} 6x_2 - x_2^2 - 2x_1x_2 = 0 \\ 6x_1 - 2x_1x_2 - x_1^2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_2(6 - x_2 - 2x_1) = 0 \\ x_1(6 - 2x_2 - x_1) = 0 \end{cases}$$

$$\boxed{p_1 = (0, 0)}$$

i) Let $x_1 = 0,\ x_2 \neq 0 \Rightarrow 6 - x_2 = 0 \Rightarrow x_2 = 6$

$$\boxed{p_2 = (0, 6)}$$

ii) Let $x_2 = 0 \Rightarrow 6 - x_1 = 0 \Rightarrow x_1 = 6$

$$\boxed{p_3 = (6, 0)}$$

iii) $\begin{cases} 6 - x_2 - 2x_1 = 0 \\ 6 - 2x_2 - x_1 = 0 \end{cases} \Rightarrow \begin{cases} 6 - x_2 - 2(6 - 2x_2) = 0 \\ x_1 = 6 - 2x_2 \end{cases} \Rightarrow$

$\Rightarrow 6 - x_2 - 12 + 4x_2 = 0 \Rightarrow 3x_2 = 6 \Rightarrow x_2 = 2 \Rightarrow$

$\Rightarrow x_1 = 2$

$$\boxed{p_4 = (2, 2)}$$

<u>Hessian calc</u>

$$\text{Hesse}(f) = \det \begin{pmatrix} -2x_2 & 6 - 2x_2 - 2x_1 \\ 6 - 2x_2 - 2x_1 & -2x_1 \end{pmatrix}$$

$p_1 \Rightarrow \text{Hesse}(f(p_1)) = \det \begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix} = -12$, $\Delta_1 = 0$, $\Delta_2 < 0$

saddle point

$p_2 \Rightarrow \text{Hesse}(f(p_2)) = \det \begin{pmatrix} 0 & -6 \\ -6 & -12 \end{pmatrix} = -36$, $\Delta_1 = 0$, $\Delta_2 < 0$

saddle point

$p_3 \rightarrow \text{Hesse}(f(p_3)) = \det\begin{pmatrix} -12 & -6 \\ -6 & 0 \end{pmatrix} = -36, \quad \Delta_1 < 0$

$\Delta_2 < 0$

saddle point

$p_4 \rightarrow \text{Hesse}(f(p_4)) = \det\begin{pmatrix} -4 & -2 \\ -2 & -4 \end{pmatrix} = 12$

$\Delta_1 < 0$

$\Delta_2 > 0$

local max

$\blacksquare$

N2

$$F(x) = \sum_{i=1}^{n} \log(\alpha_i + x_i) \qquad \alpha_i > 0$$

constraint $x_1 + \dots + x_n = 1$

$g = x_1 + \dots + x_n - 1$

$$L(x, \lambda_0, \lambda_i) = \lambda_0 \sum \log(\alpha_i + x_i) + \lambda (x_1 + \dots + x_n - 1)$$

$$\text{grad} = 0 \Rightarrow \begin{cases} \lambda_0 \dfrac{1}{\alpha_1 + x_1} + \lambda = 0 \\ \vdots \\ \lambda_0 \dfrac{1}{\alpha_n + x_n} + \lambda = 0 \end{cases}$$

Let $\lambda_0 = 0 \Rightarrow \lambda = 0$ (because $\left\{\frac{\partial g}{\partial x_i}\right\}$ linear indep) $\Rightarrow \lambda_0 \neq 0 \Rightarrow$

$\Rightarrow$ Let $\lambda_0 = 1$

$$\Rightarrow \begin{cases} x_1 = \dfrac{1}{\lambda} - \alpha_1 \\ \vdots \\ x_n = \dfrac{1}{\lambda} - \alpha_n \end{cases} \longrightarrow x_1 + \dots + x_n = 1 \Rightarrow \left(\frac{1}{\lambda} - \alpha_1\right) + \dots + \left(\frac{1}{\lambda} - \alpha_n\right) = 1 \Rightarrow$$

$$\Rightarrow -\frac{n}{\lambda} - (\alpha_1 + \dots + \alpha_n) = 1 \Rightarrow$$

$$\Rightarrow -\frac{n}{\lambda} = 1 + (\alpha_1 + \dots + \alpha_n) \Rightarrow \lambda = -\frac{n}{1 + (\alpha_1 + \dots + \alpha_n)} \Rightarrow$$

$$\Rightarrow \begin{cases} x_1 = -\dfrac{1 + \alpha_1 + \dots + \alpha_n}{n} - \alpha_1 \\ \vdots \\ x_n = -\dfrac{1 + \alpha_1 + \dots + \alpha_n}{n} - \alpha_n \\ x_1 + \dots + x_n = 1 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -\dfrac{1 + \alpha_1 + \dots + \alpha_n}{n} - \alpha_1 \\ \vdots \\ -\dfrac{1 + \alpha_1 + \dots + \alpha_n}{n} - \alpha_n \end{pmatrix}$$

<u>Convexity</u>

$$\text{Hesse}\, L = \begin{vmatrix} \dfrac{-\lambda_0}{(\alpha_1 + x_1)^2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \left(\dfrac{-\lambda_0}{(\alpha_n + x_n)^2}\right) \end{vmatrix} =$$

$$= \frac{(-1)^n \lambda_0^n}{(\alpha_1+x_1)^2 \cdots (\alpha_n+x_n)^2} \Rightarrow h(x) = (\alpha_1+x_1)^2 \cdots (\alpha_n+x_n)^2 =$$

$$= \left(\frac{1+\alpha_1+\dots+\alpha_n}{n}\right)^2 \cdots \left(\frac{1+\alpha_1+\dots+\alpha_n}{n}\right)^2 =$$

$$= \left(\frac{1+\alpha_1+\dots+\alpha_n}{n}\right)^{2n} > 0$$

$(-1)^n \lambda_0^n = (-1)^n, \lambda_0 = 1 \Rightarrow$ if $n = 2k \Rightarrow$ Hesse$(L) > 0$

esle Hesse$(L) < 0$

$\Downarrow$

if $n = 2k \Rightarrow \hat{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -\frac{1+\alpha_1+\dots+\alpha_n}{n} - \alpha_1 \\ \vdots \\ -\frac{1+\alpha_1+\dots+\alpha_n}{n} - \alpha_n \end{pmatrix}$ - local opt

Let's proof that $\hat{x}$ - global opt

□ $x_i + \alpha_i > 0$ and $x_1 + \dots + x_n = 1 \Rightarrow$

$\Rightarrow$ every x_i has an upper border $\Rightarrow$

we have an open bounded set in $\mathbb{R}^n$.

In border one of our log's $\to -\infty$

by "inserting" border $\Rightarrow$ we have a compact

set $\Rightarrow$ by the Theorem if set is compact $\Rightarrow$ set

includes his optimums $\Rightarrow$

$\Rightarrow$ optimum can be achieved $\Rightarrow \hat{x}$ - global optimum ■

№3

$f(x) = -x_1 \cdot \ldots \cdot x_n$ $\qquad x_i \geq 0$

$g(x) = x_1 + \ldots + x_n - 1$

$L(x, 1, \lambda) = -x_1 \ldots x_n + \lambda(x_1 + \ldots + x_n - 1)$

$$\begin{cases} \dfrac{\partial L}{\partial x_1} = -x_2 \ldots x_{n-1} x_n + \lambda = 0 \\ \vdots \\ \dfrac{\partial L}{\partial x_n} = -x_1 x_2 \ldots x_{n-1} + \lambda = 0 \end{cases} \Rightarrow \begin{cases} x_n = \dfrac{\lambda}{x_2 \ldots x_{n-1}} \\ \vdots \\ x_2 = \dfrac{\lambda}{x_1 x_3 \ldots x_{n-2} x_n} \\ x_1 = \dfrac{\lambda}{x_2 x_3 \ldots x_{n-1}} \end{cases}$$

$x_2 \ldots x_n = x_1 x_3 \ldots x_n = \ldots = x_1 \ldots x_{n-1} \Rightarrow x_i = x_j$

$x_1^{n-1} = \lambda \Rightarrow x_1 = \sqrt[n-1]{\lambda}$ $\qquad n\sqrt[n-1]{\lambda} = 1 \Rightarrow$

$\Rightarrow \lambda = \left(\frac{1}{n}\right)^{n-1}$

$x_i^0 = \frac{1}{n}$ - min of f $\Rightarrow$

$\forall x \qquad f(x) \geq f(x^0)$

$-\prod_{i=1}^{n} x_i \geq -\left(\frac{1}{n}\right)^n \Rightarrow \prod_{i=1}^{n} x_i \leq \left(\frac{1}{n}\right)^n \Rightarrow$

$\Rightarrow \frac{1}{n} \geq \sqrt[n]{\prod_{i=1}^{n} x_i} \Rightarrow$

$\Rightarrow \frac{1}{n} \sum_{i=1}^{n} x_i \geq \sqrt[n]{\prod_{i=1}^{n} x_i}$

Proof that x_i^0 - min of function

Let H - Hesse Matrix, so

$$H(x^0) = \begin{pmatrix} 0 & -\left(\frac{1}{n}\right)^{n-2} & \cdots & -\left(\frac{1}{n}\right)^{n-2} \\ -\left(\frac{1}{n}\right)^{n-2} & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ -\left(\frac{1}{n}\right)^{n-2} & \cdots & 0 & 0 \end{pmatrix}$$

Let's see how H operate on vectors from contrated space. We want to prove that H is positive definite Matrix.

Let's take $\forall x \in \{x \mid \sum x_i = 0\}$ and do operation.

By the way we can spot that in determinant of H doesn't include $x_i x_j$, $i=j$ binoms $\Rightarrow x^t H x = -\left(\frac{1}{n}\right)^{n-2} \sum_{i \neq j} x_i x_j =$

$$= -\left(\frac{1}{n}\right)^{n-2} \sum_{i=1}^{n} -(x_1 + \dots + x_{i-1} + x_{i+1} + x_{i+2} + \dots + x_{n-1} + x_n)^2 =$$

$$= \left(\frac{1}{n}\right)^{n-2} \sum_{i=1}^{n} (x_1 + \dots + x_{i-1} + x_{i+1} + \dots + x_n)^2,$$ since components in product is positive it means that H is positive definite operator $\Rightarrow$ by the Theorem it means that x^0 is real minimum of function

$\blacksquare$

№5

$x^0 = (1, 3)$

$f(x) = x_1^3 + 4x_1^2 + 2x_1x_2 + \frac{5}{2}x_2^2$

<u>Gradient Descent</u>

$x^{k+1} = x^k - \alpha_k \nabla f(x^k)$

<u>Step size</u> chosen by $f(x^k - \alpha_k \nabla f(x^k)) = \min_{\alpha \in \mathbb{R}} f(x^k - \alpha \nabla f(x^k))$

$\|x - x^k\|_2 \to \min$

$\square$ $f(x^0) = 1 + 4 + 6 + \frac{5}{2} \cdot 3 = 11 + \frac{15}{2} = \frac{37}{2}$

1) $x^1 = (1, 3) + \alpha_k \nabla f(1, 3)$

$\nabla f = (3x_1^2 + 8x_1 + 2x_2, 2x_1 + 5x_2)$

$\nabla f(1, 3) = (17, 17)$

$\min_{\alpha \in \mathbb{R}} f((1, 3) - \alpha(17, 17)) =$

$= \min_{\alpha} (1 - 17\alpha, 3 - 17\alpha) =$

$= \min_{\alpha} \left((1 - 17\alpha)^3 + 4(1 - 17\alpha)^2 - 2(1 - 17\alpha)(3 - 17\alpha) + \frac{5}{2}(3 - 17\alpha)^2\right)$

$g(\alpha) = -\frac{1}{2}(9826\alpha^3 - 4335\alpha^2 + 612\alpha - 43)$

$g'(\alpha) = -\frac{1}{2}(29478\alpha^2 - 8670\alpha + 612) =$

$= -14739\alpha^2 + 4335\alpha + 306$

$g'(\alpha) = 0 \Rightarrow 14739\alpha^2 - 9335\alpha - 306 = 0 \Rightarrow$

$\Rightarrow (17\alpha + 1)(17\alpha - 6) = 0 \Rightarrow \begin{cases} \alpha = -\frac{1}{17} \\ \alpha = \frac{6}{17} \end{cases}$

min in $\alpha = \frac{2}{17} \Rightarrow$

$\Rightarrow x^1 = (1, 3) - \frac{2}{17}(17, 17)$

$x^1 = (1, 3) - (2, 2) = (-1, 1)$

$f(x^1) = -1 + 9 - 2 + \frac{5}{2} = 1 + \frac{5}{2} = \frac{7}{2}$

ii) $x^2 = x^1 - \alpha, \nabla f(x^1)$

$x^2 = (-1, 1) - \alpha, \nabla f(-1, 1)$

$\nabla f(-1, 1) = (3 - 8 + 2, -2 + 5) = (-3, 3)$

$\min_{\alpha \in \mathbb{R}} f((-1,1) - \alpha(-3,3)) =$

$= \min((-1 + 3\alpha)^3 + 9(-1 + 3\alpha)^2 + 2(-1 + 3\alpha)(1 - 3\alpha) + \frac{5}{2}(1 - 3\alpha)^2)$

$h(\alpha) = 27\alpha^3 + \frac{27}{2}\alpha^2 - 18\alpha + \frac{7}{2}$

$h'(\alpha) = 81\alpha^2 + 27\alpha - 18 = 0 \Rightarrow \begin{cases} \alpha = -\frac{2}{3} \\ \alpha = \frac{1}{3} \end{cases}$

min in $\alpha = \frac{1}{3} \Rightarrow x^2 = (-1, 1) - \frac{1}{3}(-3, 3) \Rightarrow$

$\Rightarrow x^2 = (0, 0)$

$f(x^2) = 0$, $\nabla f = 0 \Rightarrow \min f$ in point $x^2 = (0,0)$

$\blacksquare$

N 9

$$\min_{x_i \in \mathbb{R}^n} \left(-\det(x_1, \dots, x_n)\right) \qquad \|x_i\|^2 = 1$$

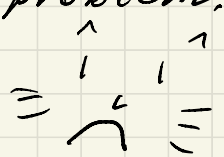
$$\|x_i\|^2 = 1 \Rightarrow x_{1i}^2 + \dots x_{ni}^2 = 1$$

$$\det(x_1, \dots, x_n) = \det\left(\|x_1\| \frac{x_1}{\|x_1\|}, \dots, \|x_n\| \frac{x_n}{\|x_n\|}\right) =$$

$$= \|x_1\| \dots \|x_n\| \underbrace{\det\left(\frac{x_1}{\|x_1\|}, \dots, \frac{x_n}{\|x_n\|}\right)}_{\leq 1 ?} \leq \|x_1\| \dots \|x_n\|$$

this chain of equation can be general idea to solve this problem, but I can't (I tried so much times) use minimization problem to proof that $\det\left(\frac{x_1}{\|x_1\|}, \dots, \frac{x_n}{\|x_n\|}\right) \leq 1$

Be happy if somebody told how use my idea to solve this problem.



Here
was
my tries
to
solve
problem
4

And

there

too

v7 using Newton Method

$x^0 = (2, 2)$ $\quad f(x) = 2x_1^2 + 9x_2^2 + 3x_1x_2 - 4x_1 - 3x_2 + 1$

$$x^{k+1} = x^k - H(f)(x^k)^{-1} \nabla f(x^k)$$

$$\text{grad} f = (4x_1 + 3x_2 - 4,\ 18x_2 + 3x_1 - 3)$$

$$\frac{d^2 f}{dx_1^2} = 4 \qquad \frac{d^2 f}{dx_2^2} = 18 \qquad \frac{d^2 f}{dx_1 x_2} = 3$$

$$H(f) = \begin{vmatrix} 4 & 3 \\ 3 & 18 \end{vmatrix} = 72 - 9 = 63$$

$$H^{-1}(f) = \frac{1}{63}\begin{pmatrix} 18 & -3 \\ -3 & 4 \end{pmatrix}$$

$$\nabla f(x^0) = (10, 39)$$

$$x^1 = x^0 - \left(H(f)^{-1} \nabla f(x^0)\right) = (2, 2) - \frac{1}{63}\begin{pmatrix} 18 & -3 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 10 \\ 39 \end{pmatrix} =$$

$$= (2, 2) - \frac{1}{63}\begin{pmatrix} 63 \\ 126 \end{pmatrix} = (1, 0)$$

$$\nabla f(x^1) = (0, 0)$$

$$f(x^1) = -1$$

in conclusion $H_f > 0$, $\nabla f(x^1) = (0, 0) \Rightarrow$

$\Rightarrow x^1$ - local min

$\blacksquare$

N8 using Penalty Method

$$X = \{ 2x_1 - 3x_2 = -6 \} \subset \mathbb{R}^2$$

$$f(x) = -8x_1^2 + 4x_1 - x_2^2 + 12x_2 - 3$$

<u>Penalty function</u>

$$F(x) = f(x) + r g^2(x)$$

$$F(x) = -8x_1^2 + 4x_1 - x_2^2 + 12x_2 - 3 + r(2x_1 - 3x_2 + 6)^2$$

$$\frac{dF}{dx_1} = -16x_1 + 4 + 4r(2x_1 - 3x_2 + 6) = 0$$

$$\frac{dF}{dx_2} = -2x_2 + 12 - 6r(2x_1 - 3x_2 + 6) = 0$$

$$\Rightarrow \begin{cases} -16x_1 + 4 + 8rx_1 - 12rx_2 + 24r = 0 \;/:4 \\ -2x_2 + 12 - 12rx_1 + 18rx_2 - 36r = 0 \;/:2 \end{cases}$$

$$\Rightarrow \begin{cases} -4x_1 + 1 + 2rx_1 - 3rx_2 + 6r = 0 \\ -x_1 + 6 - 6rx_1 + 9rx_2 - 18r = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} (-4 + 2r)x_1 - 3rx_2 + 6r + 1 = 0 \\ (-1 - 6r)x_1 + 9rx_2 + 6 - 18r = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x_1 = \dfrac{3rx_2 - 6r - 1}{2r - 4} \\ (-1-6r)\dfrac{3rx_2 - 6r - 1}{2r-4} + 9rx_2 + 6 - 18r = 0 \end{cases} \Rightarrow$$

$$x_2 = \frac{96r - 23}{39r} \Rightarrow \lim_{r\to+\infty} \frac{96r-23}{39r} = \lim_{r\to+\infty} \frac{96 - \cancel{\frac{23}{r}}^{\to 0}}{39} = \frac{96}{39} \Rightarrow$$

$$\Rightarrow \lim_{r\to+\infty} x_1 \overset{\text{by linear}}{=} \lim_{r\to+\infty} \frac{3r\cdot\frac{96}{39} - 6r - 1}{2r-4} = \lim_{r\to+\infty} \frac{\frac{96}{13} - 6 - \frac{1}{r}}{2 - \frac{4}{r}} = \frac{9}{13}$$

$$(x_1, x_2) = \left(\frac{9}{13}, \frac{32}{13}\right)$$

$$\frac{d^2F}{dx_1^2} = -16 + 8r \qquad \frac{d^2F}{dx_1x_2} = -12r$$

$$\frac{d^2F}{dx_2^2} = -2 + 18r$$

$$\text{Hesse}(F) = \begin{vmatrix} -16+8r & -12r \\ -12r & -2+18r \end{vmatrix} = (8r-16)(18r-2) + 144r =$$

$$= 16(9r^2 - 10r + 2) > 0 \iff r \in \left(-\infty, \frac{5-\sqrt{7}}{9}; \frac{5+\sqrt{7}}{9}, +\infty\right) \Rightarrow$$

$$\Rightarrow (x_1, x_2) = \left(\frac{9}{13}, \frac{32}{13}\right) \text{ - min of function } \blacksquare$$

N6

$f: \mathbb{R}^n \to \mathbb{R}$ $\qquad D \subset \mathbb{R}^n$

$x \in \mathbb{R}^n : \nabla f(x) \neq 0$

$\min_{h \in D} \nabla f^t(x)\, h \to$ rephase it means $t \cdot \frac{\partial f}{\partial x_1}(x) h_1 + \dots + \frac{\partial f}{\partial x_n}(x) h_n \xrightarrow{h \to \min}$

a) $\|h\|_2 \leq 1 \to \sqrt{\sum_i h_i^2} \leq 1 \to \sum_i h_i^2 \leq 1 \to$ Let $g(h) = \sum_i h_i^2 - 1$

$L = \sum_i \frac{\partial f}{\partial x_i}(x) h_i + \lambda g(h)$

$\frac{\partial L}{\partial h_i} = \frac{\partial f}{\partial x_i} + 2\lambda h_i = 0$ stationarity condition $\to$

$\to h_i = \frac{-\frac{\partial f}{\partial x_i}}{2\lambda} \to \sum_i \left(\frac{-\frac{\partial f}{\partial x_i}}{2\lambda} \right)^2 = 1 \to \sum_i \left(\frac{\partial f}{\partial x_i} \right)^2 = 4\lambda^2$

$2\lambda = \pm \|\nabla f(x)\|_2 \to h_i = \frac{\pm \frac{\partial f}{\partial x_i}}{\|\nabla f(x)\|_2}$

$$H(L) = \begin{pmatrix} 2\lambda & & 0 \\ & \ddots & \\ 0 & & 2\lambda \end{pmatrix} = \begin{pmatrix} \frac{\pm \|\nabla f(x)\|_2}{2} & & \\ & \ddots & \\ & & \pm \frac{\|\nabla f(x)\|_2}{2} \end{pmatrix}$$

since $\lambda = \frac{\|\nabla f(x)\|_2}{2}$ we have problems with KKT μ-value, so we can put $\mu = -\lambda$ and we have no problems with $\lambda = \frac{-\|\nabla f(x)\|_2}{2}$ cause $\mu = \frac{\|\nabla f(x)\|_2}{2}$, $H(L) > 0$

in conclusion we have a min in $h = \frac{-\nabla f(x)}{\|\nabla f(x)\|_2}$

$\blacksquare$

b) $\|h\|_1 \leq 1 \rightarrow |h_1| + \ldots + |h_n| \leq 1$

Note that we have a linear solution and can analyse it as follows:

We have an minimization problem, so we can interpret like finding maximum, like min = -max.

Note that the hugest value will be obtained by substitution 1 in the biggest coef, we can say that it will be in i-st place, so our min can be found by substitution $h = (0, \ldots, 0, \operatorname{sgn}(-\frac{\partial f}{\partial x_i}), \ldots, 0)$, gradient is non-zero, $h \neq 0$, it ends our proof

$\blacksquare$

c) $\|h\|_\infty \leq 1 \Rightarrow \max(x_i) \leq 1$

Similarly we can interpret our problem like found $-\sum_i |f(x_i)|$ by using similar ideas from (b) we can say that our min will be in

$$h = \left(\text{sign}\left(-\frac{\partial f}{\partial x_1}\right), \dots, \text{sign}\left(-\frac{\partial f}{\partial x_n}\right)\right)$$

$\blacksquare$

d) $h^t Q h \leq 1$, $Q \in Mat_{n\times n}(\mathbb{R})$ - positive definite

$h^t Q h = h_i q^i_j h^j$, since Q - pos def $\Rightarrow h^t Q h > 0, \forall h \neq 0$

$$\frac{\partial h_i q^i_j h^j}{\partial h_i} = h_j q^j_i$$

(I'll use Einstein's signs for matrix sum's)

So, if $L = \nabla f^t(x) h + \lambda(h^t Q h - 1) \Rightarrow$

$\Rightarrow \frac{\partial L}{\partial h_i} = \frac{\partial f}{\partial x_i} + \lambda h_j q^j_i$, $\frac{\partial L}{\partial h_i} = 0$ (Stationarity) $\Rightarrow$

$\Rightarrow \frac{\partial f}{\partial x_i} = -\lambda h_j q^j_i$

<u>By complementary slackness</u>

$\sum_i h_i \frac{\partial f}{\partial x_i} = -\lambda h_i q^i_j h^j = -\lambda \Rightarrow \lambda = -\sum_j h_j \frac{\partial f}{\partial x_j}$

moreover since $h_j q^j_i = \frac{-\frac{\partial f}{\partial x_i}}{\lambda} \Rightarrow$ by sum-ing all of equations we can spot that $Qh = -\frac{\nabla f}{\lambda} \Rightarrow$

$\Rightarrow h = -Q^{-1}\frac{\nabla f}{\lambda}$ $\quad \lambda^2 = \sum_{ij} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} q^{-1}_{ij} = \nabla f^t Q^{-1} \nabla f > 0$

$$\lambda^2 = \nabla f^t Q \nabla f \rightarrow \lambda = \sqrt{\nabla f^t Q^{-1} \nabla f} > 0$$

Dual feasibility condition

in conclusion $h_i = \dfrac{-\sum_j \frac{\partial f}{\partial x_j} q_{ij}^{-1}}{\sqrt{\nabla f^{-1} Q^{-1} \nabla f}}$

$\blacksquare$

№9

$$f(x) = x_1 + 3x_2 - x_3 \to \min$$

$x_1 + x_2 + x_3 \geq 3$ $\qquad -x_1 - x_2 - x_3 + x_4 = -3$

$x_1 - 2x_2 \leq -2$ $\qquad x_1 - 2x_2 + x_5 = -2$

$x_1 - 5x_2 - x_3 \geq -7$ $\qquad -x_1 + 5x_2 + x_3 + x_6 = 7$

$x_1, x_2, x_3 \geq 0$ $\qquad x_i \geq 0$

make the original less or equal $\Rightarrow$ and do 1-st step for simplex method

$$P_0 = (0, 0, 0, -3, -2, 7)$$

	b	x_1	x_2	x_3	x_4	x_5	x_6	Ratio
Obj	0	1	3	-1	0	0	0	
x_4	3	1	1	1	1	0	0	3
x_5	-2	1	-2	1	0	1	0	-2 minimal ratio
x_6	-7	1	-5	-3	0	0	1	$\frac{7}{3}$

$$P_1 = (0, 1, 0, 2, 0, -2)$$

	b	x_1	x_2	x_3	x_4	x_5	x_6	Ratio
Obj	-3	$\frac{5}{2}$	0	-1	0	$\frac{3}{2}$	0	
x_4	-2	$-\frac{3}{2}$	0	-1	1	$-\frac{1}{2}$	0	2
x_2	-2	1	-2	0	0	1	0	$+\infty$
x_6	2	$\frac{3}{2}$	0	1	0	$\frac{5}{2}$	1	2

we have two of min ratio I'll choose the first one

$$P_2 = (0, 1, 2, 0, 0, 0)$$

	b	x_1	x_2	x_3	x_4	x_5	x_6	Ratio
Obj	-1	4	0	0	-1	2	0	
x_3	-2	$-\frac{3}{2}$	0	-1	1	$-\frac{1}{2}$	0	$\frac{4}{3}$
x_2	-2	1	-2	0	0	1	0	-2
x_6	0	0	0	0	1	2	1	$\frac{0}{0}$

$x_1 = 0 \geq 0$
$x_2 = 1 \geq 0$
$x_3 = 2 \geq 0$
$x_4 = 0 \geq 0$
$x_5 = 0 \geq 0$
$x_6 = 0 \geq 0$

and this point satisfy all of conditions $\Rightarrow \hat{P}_2 = (0, 1, 2)$ is our min

$$f(\hat{P}_2) = 0 + 3 \cdot 1 - 2 = 1$$

since all of original basis values equals zero we can't do something with other values to minimize our function and it ends proof

sorry for my bad English, I'll try to show my best next time