Задача 6.

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Пусть

$$
\mathbf{v} = \sum_{i=1}^{n} \xi_i(x)\partial_{x_i}, \quad \mathbf{u} = \sum_{i=1}^{n} \eta_i(x)\partial_{x_i},
$$
$$
\mathrm{pr}^{(1)}\mathbf{v}(L) = \dot{A}, \quad \mathrm{pr}^{(1)}\mathbf{u}(L) = \dot{B}.
$$

Тогда

$$
\begin{aligned}
[\mathbf{v},\mathbf{u}] &= \mathbf{v}\left(\sum_i \eta_i\partial_{x_i}\right) - \mathbf{u}\left(\sum_j \xi_j\partial_{x_j}\right) = \sum_j \xi_j\partial_{x_j}\left(\sum_i \eta_i\partial_{x_i}\right) - \sum_i \eta_i\partial_{x_i}\left(\sum_j \xi_j\partial_{x_j}\right) = \\
&= \sum_j \xi_j \sum_i (\partial_{x_j}\eta_i\partial_{x_i} + \eta_i\partial_{x_j}\partial_{x_i}) - \sum_i \eta_i \sum_j (\partial_{x_i}\xi_j\partial_{x_j} + \xi_j\partial_{x_i}\partial_{x_j}) = \\
&= \sum_i \sum_j \left(\xi_j\frac{\partial\eta_i}{\partial x_j} - \eta_j\frac{\partial\xi_i}{\partial x_j}\right)\partial_{x_i},
\end{aligned}
$$
$$
\mathrm{pr}^{(1)}[\mathbf{v},\mathbf{u}] = [\mathbf{v},\mathbf{u}] + \sum_i \sum_j \left(\dot{\xi}_j\frac{\partial\eta_i}{\partial x_j} + \xi_j\left(\frac{\partial\eta_i}{\partial x_j}\right)^{\cdot} - \dot{\eta}_j\frac{\partial\xi_i}{\partial x_j} - \eta_j\left(\frac{\partial\xi_i}{\partial x_j}\right)^{\cdot}\right)\partial_{\dot{x}_i}.
$$

Заметим, что

$$
\left(\frac{\partial\eta_i}{\partial x_j}\right)^{\cdot} = \sum_k \frac{\partial^2\eta_i}{\partial x_j\partial x_k}\dot{x}_k = \frac{\partial}{\partial x_j}\dot{\eta}_i.
$$

Отсюда

$$
\begin{aligned}
[\mathrm{pr}^{(1)}\mathbf{v},\mathrm{pr}^{(1)}\mathbf{u}] = \sum_i \Bigg(\xi_i \sum_j \left(\frac{\partial\eta_j}{\partial x_i}\partial_{x_j} + \eta_j\partial_{x_i}\partial_{x_j} + \frac{\partial\dot{\eta}_j}{\partial x_i}\partial_{\dot{x}_j} + \dot{\eta}_j\partial_{x_i}\partial_{x_j}\right) + \\
+ \dot{\xi}_i \sum_j \left(\eta_j\partial_{\dot{x}_i}\dot{j} + \frac{\partial\eta_j}{\partial x_i}\partial_{\dot{x}_j} + \dot{\eta}_j\partial_{\dot{x}_i}\partial_{\dot{x}_j}\right)\Bigg) - \sum_j (\ldots),
\end{aligned}
$$

и после взаимного уничтожения одинаковых слагаемых получаем

$$
\begin{aligned}
&[\mathrm{pr}^{(1)}\mathbf{v},\mathrm{pr}^{(1)}\mathbf{u}] = \\
&\quad = \sum_i \sum_j \left(\xi_j\frac{\partial\eta_i}{\partial x_j} - \eta_j\frac{\partial\xi_i}{\partial x_j}\right)\partial_{x_i} + \sum_i \sum_j \left(\xi_j\frac{\partial\dot{\eta}_i}{\partial x_j} + \dot{\xi}_j\frac{\partial\eta_i}{\partial x_j} - \eta_j\frac{\dot{\xi}_i}{x_j} - \dot{\eta}_j\frac{\partial\xi_i}{\partial x_j}\right)\partial_{\dot{x}_i} = \\
&\quad = \mathrm{pr}^{(1)}[\mathbf{u},\mathbf{v}].
\end{aligned}
$$

Нам достаточно доказать

$$
[\mathrm{pr}^{(1)}\mathbf{v},\mathrm{pr}^{(1)}\mathbf{u}](L) = \frac{d}{dt}\big(\mathrm{pr}^{(1)}\mathbf{v}(B) - \mathrm{pr}^{(1)}\mathbf{u}(A)\big).
$$

Заметим, что

$$\frac{\partial}{\partial x_i}\frac{dA}{dt} = \frac{\partial^2 A}{\partial t \partial x_i} + \sum_{\beta \geq 0}\sum_{\alpha} \frac{\partial^2 A}{\partial x_i \partial x_\alpha^{(\beta)}} x_\alpha^{(\beta+1)} = \frac{d}{dt}\frac{\partial A}{\partial x_i},$$

$$\frac{\partial}{\partial \dot{x}_i}\frac{dA}{dt} = \frac{\partial}{\partial \dot{x}_i}\left(\frac{\partial A}{\partial t} + \sum_{\alpha}\frac{\partial A}{\partial x_\alpha}\dot{x}_\alpha + \sum_{\beta \geq 1}\sum_{\alpha}\frac{\partial A}{\partial x_\alpha^{(\beta)}} x_\alpha^{(\beta+1)}\right) = \frac{d}{dt}\frac{\partial A}{\partial \dot{x}_i} + \frac{\partial A}{\partial x_i}.$$

Значит,

$$\begin{aligned}
&\frac{d}{dt}(\mathbf{v}(B) - \mathbf{u}(A)) = \\
&\quad = \sum_i \left(\dot{\xi}_i \frac{\partial B}{\partial x_i} + \xi_i \frac{\partial \dot{B}}{\partial x_i} + \dot{\xi}_i\left(\frac{\partial \dot{B}}{\partial \dot{x}_i} - \frac{\partial B}{\partial x_i}\right) - \dot{\eta}_i \frac{\partial A}{\partial x_i} - \eta_i \frac{\partial \dot{A}}{\partial x_i} - \dot{\eta}_i\left(\frac{\partial \dot{A}}{\partial \dot{x}_i} - \frac{\partial A}{\partial x_i}\right)\right) = \\
&\quad = \sum_i \left(\xi_i \sum_k \left(\frac{\partial \eta_k}{\partial x_i}\frac{\partial L}{\partial x_k} + \eta_k \frac{\partial^2 L}{\partial x_i \partial x_k} + \frac{\partial \dot{\eta}_k}{\partial x_i}\frac{\partial L}{\partial \dot{x}_k} + \dot{\eta}_k \frac{\partial^2 L}{\partial \dot{x}_k \partial x_i}\right) + \right. \\
&\quad \left. + \dot{\xi}_i \sum_k \left(\eta_k \frac{\partial^2 L}{\partial x_k \partial \dot{x}_i} + \frac{\partial \eta_k}{\partial x_i}\frac{\partial L}{\partial \dot{x}_k} + \dot{\eta}_k \frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_i}\right) - \dots\right),
\end{aligned}$$

и после взаимного уничтожения одинаковых слагаемых получаем

$$\begin{aligned}
\frac{d}{dt}(\mathbf{v}(B) - \mathbf{u}(A)) &= \sum_i \left(\xi_i \sum_k \left(\frac{\partial \eta_k}{\partial x_i}\frac{\partial L}{\partial x_k} + \frac{\partial \dot{\eta}_k}{\partial x_i}\frac{\partial L}{\partial \dot{x}_k}\right) - \eta_i \sum_k \left(\frac{\partial \xi_k}{\partial x_i}\frac{\partial L}{\partial x_k} + \frac{\partial \dot{\xi}_k}{\partial x_i}\frac{\partial L}{\partial \dot{x}_k}\right)\right) = \\
&= [\mathrm{pr}^{(1)}\mathbf{v}, \mathrm{pr}^{(1)}\mathbf{u}](L),
\end{aligned}$$

что и требовалось.