Econ Topic 3

Merton's Problem II/II: Lifetime Investment and Consumption in Continuous Time

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nickycanhelp@qq.com

We continue our discussion on Merton's problem and in this note we focus on an infinite-horizon investment and consumption problem where we face the same market as in the previous topic, namely, for $t \in [0, \infty)$,

$$dS_t = \mu S_t dt + \sigma S_t dW_t,\ S_0 = s_0 > 0, \tag{1}$$

$$dB_t = rB_t dt,\ B_0 = b_0 > 0, \tag{2}$$

where $\mu > r > 0$ and $\sigma > 0$ are given, and $W = (W_t)_{t\geq 0}$ is a standard Brownian motion.

We start with the initial wealth $X_0 = x_0 > 0$, and this time we want to decide both the lifetime investment strategy $\alpha = (\alpha_t)_{t\geq 0}$ and the consumption rule $c = (c_t)_{t\geq 0}$, where $\alpha_t \in \mathbb{R}$ and $c_t \in (0, 1)$[1] denote the proportion of wealth invested on the stock and consumed at time t, respectively. Our goal is to maximize the *expected lifetime aggregated utility of consumption* under the choice of processes α and c (see the problem $(\mathcal{P})$ below). For simplicity, we consider the power utility function, which is CRRA[2],

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma},\ x > 0, \tag{3}$$

for some $\gamma > 0$, $\gamma \neq 1$ that represents the investor's degree of risk aversion.

We first derive the dynamics of wealth process under our controls α and c, denoted by $X^{\alpha,c} = (X_t^{\alpha,c})_{t\in[0,T]}$. Note that dS_t/S_t represent the instant stock return at time t and according to the allocation rule we invest $\alpha_t X_t$ amount of money on the stock and $(1-\alpha_t)X_t$ depositing in the bank earning rate of return r at the same time. Moreover, we also consume $c_t X_t$ amount of money. Therefore, our wealth instant change dX_t is given by

$$\begin{aligned} dX_t^{\alpha,c} &= \alpha_t X_t^{\alpha,c} \frac{dS_t}{S_t} + (1-\alpha_t) X_t^{\alpha,c} \frac{dB_t}{B_t} - c_t X_t^{\alpha,c} \\ &= [\alpha_t \mu + (1-\alpha_t) r - c_t] X_t^{\alpha,c} dt + \alpha_t \sigma X_t^{\alpha,c} dW_t, \end{aligned} \tag{4}$$

slightly different compared with the wealth dynamics in the preceding pure-investment Merton's problem (see formula (5) in Econ Topic 2). In order to have a unique solution of the above controlled SDE (in other words, wealth process X要能够通过strategy α and c实现出来), we technically require that[3] our control variables α and c satisfy

$$\int_0^t |\alpha_t|^2 + |c_t| dt < +\infty\ \forall t \geq 0 \text{ a.s.} \tag{5}$$

[1] We require that the consumption is less than the current wealth.
[2] See Econ Topic 1 for more details if you are not familiar with this notion.
[3] 参考stochastic control基础知识讨论的material.

Obviously, the constant strategies $\alpha_t = a$ and $c_t = \zeta$ for all t for some $a \in \mathbb{R}$ and $\zeta \in (0,1)$ satisfy the condition (5). Therefore, our admissible control space is not empty. 意即我们不至于找不到任何strategy.

The lifetime investment and consumption problem is finally formulated as

$$(\mathcal{P}) \qquad \max_{\alpha,c} \ \mathbb{E}\left[\int_0^\infty e^{-\beta t} u(c_t X_t^{\alpha,c})\right]$$
$$\text{s.t.} \quad dX_t^{\alpha,c} = [\alpha_t \mu + (1-\alpha_t) r - c_t] X_t^{\alpha,c} dt + \alpha_t \sigma X_t^{\alpha,c} dW_t,$$

where β is the discount rate on the utility. This problem is a standard infinite-time stochastic control problem and in this case we can adopt dynamic programming (DP) to solve it. Let us define the value function starting at any state $X_0 = x > 0$ by

$$v(x) = \max_{\alpha,c} \ \mathbb{E}\left[\int_0^\infty e^{-\beta t} u(c_t X_t^{\alpha,c}) | X_0 = x\right]. \tag{6}$$

The corresponding HJB equation for the value function[4] of this problem is given by a partial differential equation (PDE)

$$\max_{a\in\mathbb{R}, c\geq 0} \left\{ b(x,a,c)\frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2(x,a,c)\frac{\partial^2 v}{\partial x^2} + u(cx)\right\} = \beta v, \tag{7}$$

where the coefficient functions are coming from (4),

$$b(x,a,c) = [a\mu + (1-a)r - c]x, \tag{8}$$
$$\sigma(x,a,c) = a\sigma x. \tag{9}$$

The PDE of (7) is again an example of the general Cauchy linear parabolic PDE[5], which owns a classical approach to solve. More specifically, we surmise that the solution of this problem is given by the form

$$v(x) = Ku(x) \tag{10}$$

for some nonnegative constant K.[6] By doing so, we switch from solving for $v(x)$ to solving for K. Note that

$$\frac{\partial v}{\partial x} = Ku'(x) = Kx^{-\gamma}, \ \frac{\partial^2 v}{\partial x^2} = Ku''(x) = -\gamma K x^{-\gamma-1}. \tag{11}$$

Hence, (7) is simplified to

$$K(1-\gamma)\max_{a\in\mathbb{R}}\left\{-\frac{1}{2}a^2\sigma^2\gamma + (\mu - r)a + r\right\} + \max_{c\geq 0}\left\{c^{1-\gamma} - (1-\gamma)cK\right\} - \beta K = 0. \tag{12}$$

[4]This is a standard procedure of solving stochastic control problem by DP. 参考stochastic control基础知识讨论的material

[5]经典的European option pricing formula derived from PDE approach中的PDE也是这种类型, 我们以后会在FinMath系列详细讲述.

[6]The nonnegativeness is because of the definition in (6), the value function should be nonnegative.

By assuming

$$0 < \beta - \rho < \gamma \tag{13}$$

where

$$\begin{aligned}\rho &= (1-\gamma)\max_{a\in\mathbb{R}}\left\{-\frac{1}{2}a^2\sigma^2\gamma + a(\mu - r) + r\right\} \\ &= \frac{(\mu-r)^2}{2\sigma^2}\frac{1-\gamma}{\gamma} + (1-\gamma)r,\end{aligned} \tag{14}$$

we obtain the optimal investment and consumption policies

$$c_t^* = \zeta^* = K^{-\frac{1}{\gamma}} = \frac{\beta-\rho}{\gamma} \in (0,1), \tag{15}$$

$$K = \left(\frac{\gamma}{\beta-\rho}\right)^{\gamma} > 0, \tag{16}$$

and

$$\alpha_t^* = a^* = \frac{\mu - r}{\sigma^2\gamma},\ \forall t \geq 0. \tag{17}$$

Therefore, the value function is given by

$$v(x) = Ku(x) = \left(\frac{\gamma}{\beta-\rho}\right)^{\gamma}\frac{x^{1-\gamma}}{1-\gamma},\ x > 0. \tag{18}$$

In this problem, the optimal proportions of wealth on investment and consumption, α^* and c^*, are time-invariant and satisfy the condition (5). We observe that the larger the risk aversion γ, the smaller the proportion invested on the risky stock and consumed, which is quite intuitive.

◊ 至此, 我们求得了在同时考虑投资和消费的场景中、以lifetime消费水平效用最大化为目标的Merton's problem的最优策略. 为了数学的严谨性, 以下我们严格论述了上述HJB的解就是真正的value function, 不感兴趣的reader可以忽略不看 :)

To end this note, we discuss some technical details for $v(x)$ obtained in (18) to be the real value function defined in (6). Note that the wealth dynamics under the optimal strategies is a standard GBM,

$$dX_t^{\alpha^*,c^*} = [a^*\mu + (1-a^*)r - \zeta^*]X_t^{\alpha^*,c^*}dt + a^*\sigma X_t^{\alpha^*,c^*}dW_t,\ X_0^{\alpha^*,c^*} = x_0 > 0, \tag{19}$$

which has a unique solution

$$X_t^{\alpha^*,c^*} = x_0\exp\{[b^* - \frac{1}{2}(\sigma^*)^2]t + \sigma^* W_t\},\ \forall t \geq 0, \tag{20}$$

where

$$b^* = a^*\mu + (1-a^*)r - \zeta^*,\ \sigma^* = a^*\sigma. \tag{21}$$

The problem is indeed well-posed under α^* and c^*. To see this, we require that[7]

$$+\infty > \mathbb{E}\left[\int_0^\infty e^{-\beta t}|u(c_t^* X_t^{\alpha^*,c^*})|dt\right] = \int_0^\infty e^{-\beta t}(\zeta^*)^{1-\gamma}\mathbb{E}\left[\frac{(X_t^{\alpha^*,c^*})^{1-\gamma}}{1-\gamma}\right]dt \tag{22}$$

$$= \frac{(\zeta^* x)^{1-\gamma}}{1-\gamma}\int_0^\infty \exp\{[(1-\gamma)b^* - \frac{1}{2}(1-\gamma)(\sigma^*)^2 + \frac{1}{2}(1-\gamma)^2(\sigma^*)^2 - \beta]t\}dt, \tag{23}$$

which is equivalent to requiring that

$$0 > (1-\gamma)b^* - \frac{1}{2}\gamma(1-\gamma)(\sigma^*)^2 - \beta = \rho - (1-\gamma)\zeta^* - \beta, \tag{24}$$

which is indeed the case under the assumption (13). Moreover, by the verification theorem[8] we also need to guarantee that

$$\lim_{t\to\infty} \mathbb{E}[e^{-\beta t}v(X_t^{\alpha^*,c^*})] = 0. \tag{25}$$

To see this,

$$\mathbb{E}[e^{-\beta t}v(X_t^{\alpha^*,c^*})] = K\frac{x^{1-\gamma}}{1-\gamma}\exp\{[\rho - (1-\gamma)\zeta^* - \beta]t\} \to \infty,\ \text{as } t\to\infty, \tag{26}$$

since $\rho - (1-\gamma)\zeta^* - \beta < 0$ as shown in (24).

End.

[7] 当我们pose一个问题时，我们一般不希望我们的目标函数取值正无穷，otherwise问题就变得没意思了，所以此处我们require under the policy the objective $< +\infty$.
[8] 参考stochastic control基础知识讨论的material.