Econ Topic 1
Utility Functions

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We make a bunch of decisions every single day: What should I eat for my lunch, 钵钵鸡 or 螺蛳粉; how should I allocate my lifetime consumption, and etc. The utility is a classical way to represent your preference or happiness. Mathematically, we use a utility function, $u(x)$, to describe your *degree of satisfaction* on $x \in \mathbb{X}$. For example, x could be number of hamburgers you eat in which $\mathbb{X} = \mathbb{N}_+$. Generally we require that the function $u(\cdot)$ is increasing to indicate that the decision maker is non-satiated (多多益善!), namely,

$$u'(x) > 0, \ \forall x \in \mathbb{X}. \tag{1}$$

In many real-life applications, we face situations under uncertainty. Consider a random variable X, for instance $X \geq 0$ could represent your future wealth or $X \in \mathbb{R}$ could be your investment gain or loss, we measure your "satisfaction" through **expected utility** $\mathbb{E}[u(X)]$, and we have the following simple theorem to specify the preference order between two random outcomes.

Theorem 1 (Expected Utility Theorem)**.** *The decision maker with utility function $u(\cdot)$ prefers random outcomes X to $\tilde{X}$ (we sometimes denote $X \succ \tilde{X}$) if and only if*

$$\mathbb{E}[u(X)] > \mathbb{E}[u(\tilde{X})]. \tag{2}$$

As a consequence, the decision maker will choose actions that maximize her expected utility, which is in line with the rational individual assumption in economics.

Proposition 1. *The utility function is unique up to affine transformation, namely, $u(\cdot)$ and $au(\cdot) + b$ with $a > 0$ and $b \in \mathbb{R}$ leads to the same preference.*

Proof. We use the linearity property of expectation, namely, when $\mathbb{E}[u(X)] > \mathbb{E}[u(\tilde{X})]$, we have $\mathbb{E}[au(X)+b] = a\mathbb{E}[u(X)]+b > a\mathbb{E}[u(\tilde{X})]+b = \mathbb{E}[au(\tilde{X})+b]$ when $a > 0$ and $b \in \mathbb{R}$. □

Consider a fair game whose random outcome X satisfies $\mathbb{E}[X] = 0$. For instance, let us toss a fair coin and we earn \$1 if head shows while lose \$1 when tail appears, then this is a fair game with

$$X = \begin{cases} 1, & \text{w.p. } \frac{1}{2} \\ -1, & \text{w.p. } \frac{1}{2}. \end{cases}$$

By comparing a fair game X with no gain or no loss (i.e., 0), we can mathematically define three types of decision maker.

Definition 1. *For a decision maker with utility function $u(\cdot)$ and a fair game with random outcome X, we say the decision maker is risk averse/risk neutral/risk seeking if*

$$\mathbb{E}[u(0)] > \mathbb{E}[u(X)] \ / \ \mathbb{E}[u(0)] = \mathbb{E}[u(X)] \ / \ \mathbb{E}[u(0)] < \mathbb{E}[u(X)]. \tag{3}$$

◊ 翻译一下就是: even for a fair game, risk averse不愿意冒险, risk neutral都行无所谓, risk seeking认为搏一搏单车变摩托.

We do have a convenient method to tell the risk attitude of a decision maker with utility function $u(\cdot)$ without everytime comparing a fair game with doing nothing. We just state here without proof.[1]

Proposition 2. *The decision maker with utility function $u(\cdot)$ is risk averse/risk neutral/risk seeking if*

$$u''(x) < 0 \;/\; u''(x) = 0 \;/\; u''(x) > 0, \tag{4}$$

corresponding to saying the utility function is concave/linear/convex.

The example of risk-neutral utility function is $u(x) = x$ for any $x \in \mathbb{R}$ while for the risk-seeking utility function we could have $u(x) = x^2$ for $x \geq 0$. Note that given the Proposition 1, there are in fact infinitely many such utility functions of any type. In economics, however, we are more interested in the risk-averse utility functions, as we often assume that a rational decision maker is risk-averse. Furthermore, among those risk-averse people, we distinguish their levels of risk aversion by two coefficients.

Definition 2. *For a risk-averse utility function $u(x)$, the coefficient of absolute risk aversion $\mathcal{A}(x)$ and the coefficient of relative risk aversion $\mathcal{R}(x)$ are defined respectively through*

$$\mathcal{A}(x) = -\frac{u''(x)}{u'(x)},\ \mathcal{R}(x) = -\frac{xu''(x)}{u'(x)} = x\mathcal{A}(x). \tag{5}$$

In the following, we discuss some commonly-used risk-averse utility functions. The power utility is often defined as, for any $\gamma > 0$, $\gamma \neq 1$,

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma},\ x > 0, \tag{6}$$

where γ represents the degree of risk aversion and the larger the γ, the larger the risk aversion. And γ is also the relative risk aversion of the utility function. To see this, we compute

$$\mathcal{R}(x) = -\frac{xu''(x)}{u'(x)} = -\frac{x \cdot (-\gamma x^{-\gamma-1})}{x^{-\gamma}} = \gamma. \tag{7}$$

Therefore, power utility is classified into the utility function with constant relative risk aversion (CRRA). Note that when $\gamma = 0$ in (6), we could get the linear utility. You may see other form of power utility like

$$u(x) = \frac{x^{\gamma}}{\gamma},\ x > 0, \tag{8}$$

for any $\gamma < 1$, $\gamma \neq 0$. But pay attention that in this formulation γ does not denote the degree of risk aversion.

Exercise 1. *Show that the square-root utility $u(x) = \sqrt{x}$, $x \geq 0$ is a special case of power utility.*

[1] The rigorous proof should link any random outcome X with a fair game and use Definition 1.

The logarithm utility function is simply given by $u(x) = \ln x,\ x > 0$, which in fact can be treated as a special case of power utility. To see this, we let $\gamma \to 1$ in the power utility of the form[2]

$$u(x) = \frac{x^{1-\gamma} - 1}{1-\gamma},\ x > 0, \tag{9}$$

and we get

$$\lim_{\gamma \to 1} \frac{x^{1-\gamma} - 1}{1-\gamma} \overset{\text{L'Hôpital rule on } \gamma}{=} \lim_{\gamma \to 1} \frac{-x^{1-\gamma} \ln x}{-1} = \ln x. \tag{10}$$

Therefore, a more general power utility family is given by

$$u(x) = \begin{cases} \frac{x^{1-\gamma}-1}{1-\gamma} & \gamma > 0,\ \gamma \neq 1 \\ \ln x & \gamma = 1 \end{cases},\ x > 0. \tag{11}$$

And the above form is sometimes called power-log utility. The exponential utility is defined as,[3] for any $\lambda > 0$,

$$u(x) = -e^{-\lambda x},\ x \in \mathbb{R}, \tag{12}$$

where λ represents the degree of risk aversion and the larger the λ, the larger the risk aversion. And λ is also the absolute risk aversion of the exponential utility. To see this, we compute

$$\mathcal{A}(x) = -\frac{u''(x)}{u'(x)} = -\frac{-\lambda^2 e^{-\lambda x}}{\lambda e^{-\lambda x}} = \lambda. \tag{13}$$

Therefore, the exponential utility is classified into the utility function with constant absolute risk aversion (CARA). The shapes of different utility functions are illustrated in Figure 1 below. The choice of utility functions is based on your specific needs in the model. For example, for $x > 0$ and unbounded utility, we could choose power type; for $x \in \mathbb{R}$ and bounded utility, we could choose exponential type, and etc.

The above utility functions can all be contained in a more general class called hyperbolic absolute risk aversion (HARA) utility, which is mathematically convenient for economics modelling. The HARA utility is defined as the form, for $a > 0$, $\gamma > 0$ and x such that $\frac{ax}{\gamma} + b > 0$,

$$u(x) = \frac{\gamma}{1-\gamma}[(\frac{ax}{\gamma} + b)^{1-\gamma} - 1], \tag{14}$$

and it is easy to verify that the risk tolerance $\mathcal{T}(x)$, defined as the reciprocal of absolute risk aversion $\mathcal{A}(x)$, is an affine function of x.

Exercise 2. *Show that the risk tolerance of HARA utility is given by*

$$\mathcal{T}(x) = \frac{x}{\gamma} + \frac{b}{a}. \tag{15}$$

[2]Note that according to Proposition 1 the forms (6) and (9) are equivalent.

[3]The minus sign in the exponential utility is used to make the function increasing and concave.

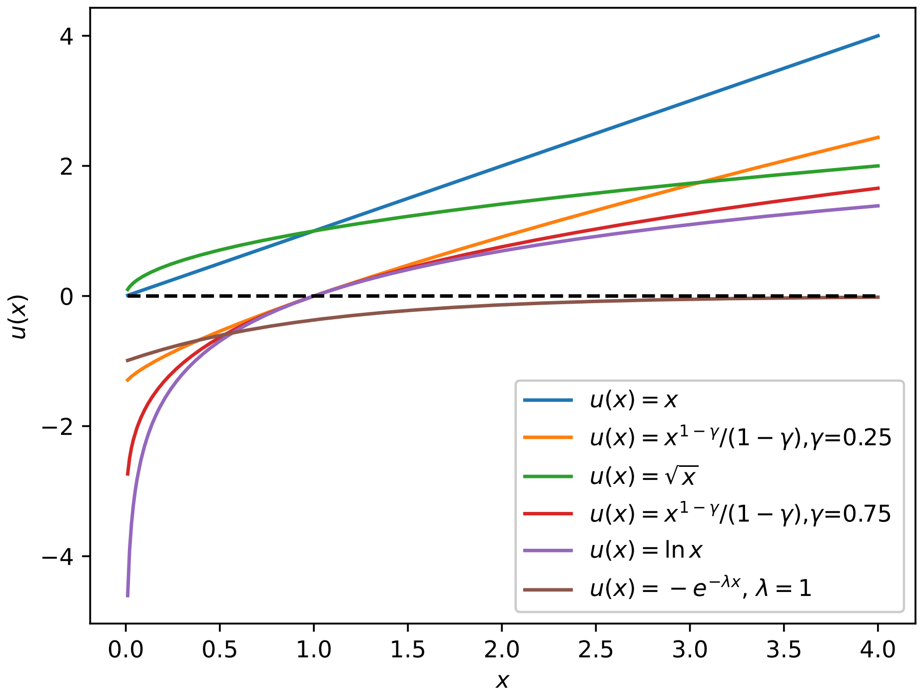

Figure 1: Utility functions of different types.

Exercise 3. *Show that in HARA utility,*

(i) when $a = \gamma$ and $b = 0$, it reduces to power utility;

(ii) when $a = \gamma$, $b = 0$ and $\gamma \to 1$, it reduces to log utility;

(iii) when $a = \lambda$, $b = 1$ and $\gamma \to +\infty$, it reduces to exponential utility.

End.