Econ Topic 2

Merton's Problem I/II: Portfolio Selection in Continuous Time

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Merton (1969) pioneered the portfolio selection problem in continuous time, although he may be more well-known on his work in option pricing which shared the Nobel prize with Scholes. The Merton's model we discuss here and next topic plays a very important role in economics and financial modelling, as it provides a beautiful and solvable example of how stochastic control techniques can be applied in economics and finance.

⚠ 本系列所用到的stochastic control基础知识, 例如基本的stochastic calculous内容, geometric Brownian motion (GBM), Itô formula, stochastic differential equation (SDE), dynamic programming (DP), Hamilton-Jacobi-Bellman (HJB) equation and etc will be covered in other materials. 如果你不是很清楚或者想系统学习, 请多多留意日后发布! 或者参考Pham (2009)以及其他相关textbook.

In this note, we focus on a finite-horizon continuous-time portfolio allocation problem where the market consists of one stock whose price is driven by GBM and one bank account with continuous compounding risk-free rate r. More precisely, the dynamics of assets' prices are given by, for $t \in [0, T]$ with $0 < T < +\infty$,

$$dS_t = \mu S_t dt + \sigma S_t dW_t,\ S_0 = s_0 > 0, \tag{1}$$
$$dB_t = rB_t dt,\ B_0 = b_0 > 0, \tag{2}$$

where $\mu > r > 0$ and $\sigma > 0$ are given, and $W = (W_t)_{t\in[0,T]}$ is a standard Brownian motion.

◊ 这里简短回顾下GBM (1)的解,如果你还记得的话,她来自Itô formula的应用:

$$S_t = S_0 \exp\left\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right\}. \tag{3}$$

We start with the initial wealth $X_0 = x_0 > 0$, and want to decide the optimal allocation process $\alpha = (\alpha_t)_{t\in[0,T]}$ where $\alpha_t \in \mathbb{R}$ denotes the proportion of wealth invested on the stock at time t,[1] so that our objective, the *expected utility*[2] of the terminal wealth X_T^α under the strategy α, is maximized (see the problem $(\mathcal{P})$ below.) For simplicity, we consider the power utility function

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma},\ x > 0, \tag{4}$$

for some $\gamma > 0$, $\gamma \neq 1$ that represents the investor's degree of risk aversion.

[1]We assume that shorting is allowed and any fractional shares invested on stock is permitted.
[2]When we choose the linear utility $u(x)$ the objective could become the expected terminal wealth itself.

We first derive the dynamics of wealth process under α, denoted by $X^\alpha = (X_t^\alpha)_{t\in[0,T]}$. Note that dS_t/S_t represent the instant stock return at time t and according to the allocation rule we invest $\alpha_t X_t$ amount of money on the stock and $(1-\alpha_t)X_t$ depositing in the bank earning rate of return r at the same time. Therefore, our wealth instant change dX_t is given by

$$\begin{aligned} dX_t^\alpha &= \alpha_t X_t^\alpha \frac{dS_t}{S_t} + (1-\alpha_t)X_t^\alpha \frac{dB_t}{B_t} \\ &= [\alpha_t\mu + (1-\alpha_t)r]X_t^\alpha dt + \alpha_t\sigma X_t^\alpha dW_t. \end{aligned} \tag{5}$$

The formula of type (5) is known as the *controlled* SDE. In order to have a unique solution of the controlled SDE (in other words, 我们期望的wealth process X要能够通过strategy α实现出来), we technically require that[3] our control variable α satisfies

$$\int_0^T \alpha_t^2 dt < +\infty, \text{ a.s.} \tag{6}$$

Obviously, the constant strategy $\alpha_t = a$ for all t for some $a \in \mathbb{R}$ satisfies the condition (6). Therefore, our admissible control space is not empty. 意即我们不至于找不到任何strategy.

The portfolio selection problem is finally formulated as

$$\begin{aligned} (\mathcal{P}) \qquad & \max_{\alpha} \quad \mathbb{E}[u(X_T^\alpha)] \\ & \text{s.t.} \quad dX_t^\alpha = [\alpha_t\mu + (1-\alpha_t)r]X_t^\alpha dt + \alpha_t\sigma X_t^\alpha dW_t. \end{aligned}$$

This problem is a standard stochastic control problem and in this case we can adopt dynamic programming (DP) to solve it. Let us define the value function starting at any time $t \in [0,T]$ with $X_t = x > 0$ by

$$v(t,x) = \max_{\alpha} \ \mathbb{E}[u(X_T^{t,\alpha})|X_t = x], \tag{7}$$

which represents the largest expected utility of final wealth we can obtain starting from (t,x). The corresponding HJB equation for the value function[4] of this problem is given by a partial differential equation (PDE)

$$\frac{\partial v}{\partial t} + \max_{a\in\mathbb{R}} \left\{ b(x,a)\frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2(x,a)\frac{\partial^2 v}{\partial x^2} \right\} = 0, \tag{8}$$

$$v(T,x) = u(x). \tag{9}$$

where the coefficient functions are coming from (5),

$$b(x,a) = [a\mu + (1-a)r]x, \tag{10}$$

$$\sigma(x,a) = a\sigma x. \tag{11}$$

[3]参考stochastic control基础知识讨论的material.

[4]This is a standard procedure of solving stochastic control problem by DP. 参考stochastic control基础知识讨论的material

The PDE of (8) with terminal condition (9) is actually an example of the general Cauchy linear parabolic PDE[5], which owns a classical approach to solve. More specifically, we surmise that the solution of this problem is given by the form

$$v(t,x) = \phi(t)u(x) \tag{12}$$

for some function $\phi(t)$ with $\phi(T) = 1$ (so that $v(T,x) = u(x)$ is fulfilled.) By doing so, we switch from solving for $v(t,x)$ to solving for $\phi(t)$. We then have

$$\frac{\partial v}{\partial t} = \phi'(t)u(x),\ \frac{\partial v}{\partial x} = \phi(t)u'(x) = \phi(t)x^{-\gamma},\ \frac{\partial^2 v}{\partial x^2} = \phi(t)u''(x) = -\gamma\phi(t)x^{-\gamma-1}. \tag{13}$$

And hence (8) is simplified to

$$\phi'(t)\frac{x^{1-\gamma}}{1-\gamma} + \max_{a\in\mathbb{R}}\left\{b(x,a)\phi(t)x^{-\gamma} + \frac{1}{2}\sigma^2(x,a)(-\gamma\phi(t)x^{-\gamma-1})\right\} = 0, \tag{14}$$

i.e., the ordinary differential equation (ODE)

$$\phi'(t) + \rho\phi(t) = 0,\ \text{with } \phi(T) = 1, \tag{15}$$

where

$$\begin{aligned}\rho &= (1-\gamma)\max_{a\in\mathbb{R}}\left\{-\frac{1}{2}a^2\sigma^2\gamma + a(\mu - r) + r\right\}\\ &= \frac{(\mu-r)^2}{2\sigma^2}\frac{1-\gamma}{\gamma} + (1-\gamma)r,\end{aligned} \tag{16}$$

with

$$\alpha_t^* = a^* = \frac{\mu - r}{\sigma^2\gamma},\ \forall t \in [0,T]. \tag{17}$$

Then the solution of ODE (15) is easily obtained as

$$\phi(t) = e^{\rho(T-t)}. \tag{18}$$

Therefore, the value function is given by

$$v(t,x) = \phi(t)u(x) = e^{\rho(T-t)}\frac{x^{1-\gamma}}{1-\gamma},\ t\in[0,T],\ x>0, \tag{19}$$

which is just the unique solution of HJB (8) and (9). The optimal strategy in this problem is to maintain the same proportion of wealth invested on the stock all the time, as given in (17), and we find that the larger the γ (the larger the risk aversion), the smaller the proportion invested on the risky stock, which is quite intuitive. The wealth would behave according to

$$dX_t^{\alpha^*} = [a^*\mu + (1-a^*)r]X_t^{\alpha^*}dt + a^*\sigma X_t^{\alpha^*}dW_t,\ X_0^{\alpha^*} = x_0, \tag{20}$$

[5]经典的European option pricing formula derived from PDE approach中的PDE也是这种类型, 我们以后会在FinMath系列详细讲述.

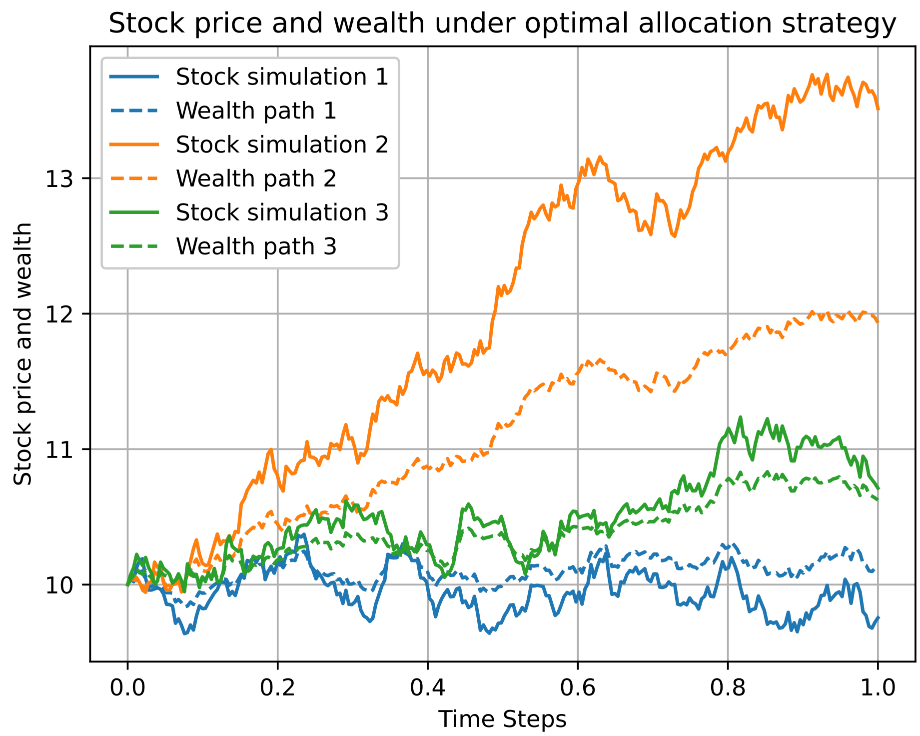

Figure 1: Simulations.

which is just a GBM, and it finally leads to the terminal wealth[6]

$$X_T^{\alpha^*} = x_0 \exp\{(a^*\mu + (1-a^*)r - \frac{1}{2}(a^*)^2\sigma^2)T + a^*\sigma W_T\}. \tag{21}$$

To end this topic, let us do some experimental analysis by simulating the market. The results in Figure 1 come from the parameters $\mu = \sigma = 0.1$, $r = 0.05$, $T = 1$, $x_0 = s_0 = 10$ and $\gamma = 10$. We can see that, given the optimal allocation as a constant proportion, the wealth process under this strategy fluctuates synchronously with the stock price movement. And we can imagine that when γ is smaller, the fluctuation of your wealth is more amplified. This shows that your risk attitude determines your wealth's "beta" w.r.t. the market.

◊ 总结下本节的Merton's model现实应用可能存在的问题:
(1) 现实中不只一个stock, 股价也不一定follow GBM, 即便follow GBM参数估计的准确性,市场风格的切换都在影响着模型的效果;
(2) 以上解析解形式依赖特定的utility形式, 也即属于CRRA type的power utility. 现实中让投资者决定自己的utility function是一个很抽象的过程, 相比于此, mean-variance criterion可能是投资者更好的选择. 幸运的是, 文献中也有经典的mean-variance portfolio selection model, see Zhou and Li (2000).

References

Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case. *The review of Economics and Statistics*, pages 247–257.

[6]This result can be easily obtained by referring to the template of stock price in (3).

Pham, H. (2009). *Continuous-time stochastic control and optimization with financial applications*, volume 61. Springer Science & Business Media.

Zhou, X. Y. and Li, D. (2000). Continuous-time mean-variance portfolio selection: A stochastic lq framework. *Applied Mathematics and Optimization*, 42:19–33.

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