Special Topic

Arbitrage in Lottery of Football Games

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1 Introduction

The most charming sport events during the end of the year 2022 would be probably nothing else but just the FIFA World Cup. Except for watching the competitive matches and cheering up for the goals, some fans may think about betting on the games and therefore choose to buy the football lotteries. By jumping into this pool, some people would like to express their supports and loyalty to their home teams, while others might just want to gamble and imagine to be rich overnight :)

⚠ 俗话说的好, "搏一搏, 单车变摩托", 但结局往往是"天台见". 珍爱生命, 远离赌博.

Gambling on average is losing money (otherwise how the casinos or the bookmakers earn money from you) if you are not one of the few lucky dogs. This phenomenon makes us (as the rational financial mathematics learners) investigate whether there exists "free lunch" or arbitrage in the football lottery. Let us first take a look at an odds (赔率) table provided by the China Sports Lottery on their webpage on 2022.12.07 (note that the odds data would actively update before the end of sales).

荷兰	VS	阿根廷
胜 3.20	平 2.82	负 2.12

Figure 1: Odds quoted in *sporttery.cn* for the coming match Netherlands v.s. Argentina

The above table summarizes how much you can get back if you bet one shot (2 RMB) on different results of the game:

- if you bet 2 RMB on "Netherlands win" and if it is really the case, then you will receive 3.2 RMB, otherwise ("Draw" or "Netherlands lose") you lose your money;
- if you bet 2 RMB on "Draw" and if it really happens, then you will receive 2.82 RMB, otherwise (Netherlands wins or loses) you lose your bet;

- if you bet 2 RMB on "Netherlands lose" (i.e., "Argentina win") and if it really occurs, then you will receive 2.12 RMB, otherwise ("Draw" or "Netherlands win") you lose your money.

Apparently, the smaller odds represent the larger chance that the corresponding result would happen "theoretically". For our example, people may think Argentina would probably win the game. However, before the 90 minutes end, everything is possible. Is there any betting strategy such that no matter which scenarios happen, we can always win some money (or at least do not lose the money)? This is what this note tries to answer.

2 The Arbitrage Theorem

Suppose there are m possible outcomes for an event and also n possible wagers based on this event. Let us denote by $r_i(j)$ the return on betting \$1 on the wager i, $i = 1, \ldots, n$, if the outcome j, $j = 1, \ldots, m$, occurs. If we choose a betting strategy $x = (x_1, \ldots, x_n)'$ where each x_i denotes the dollar amount we bet on the wager i, then the total profit we could earn under this strategy when a certain outcome is realized would be

$$P_x(j) = \sum_{i=1}^{n} [r_i(j)x_i - x_i] = \sum_{i=1}^{n} [r_i(j) - 1]x_i, \ j = 1, \ldots, m. \tag{1}$$

◊ 很好理解: 利润=收入-成本.

We now give the formal definition of arbitrage under this model.

Definition. *The betting strategy x is called an arbitrage opportunity if under x the profit satisfies $P_x(j) \geq 0$ for all $j = 1, \ldots, m$* and *$P_x(j) > 0$ for at least one j.*

◊ 套利, 首先要保证在任何情况下不亏钱($P_x(j) \geq 0 \, \forall j$), 其次要保证在某些情况下赚钱($P_x(j) > 0$ for at least one j).

Theorem (The no-arbitrage principle)**.** *That there is no arbitrage opportunity is equivalent to that there exists a probability vector $q = (q_1, \ldots, q_m)'$ (called risk-neutral probabilities or martingale measure) with $\sum_j q_j = 1$ and $q_j > 0 \, \forall j = 1, \ldots, m$, such that*

$$\sum_j q_j [r_i(j) - 1] = 0, \ \forall i, \ i = 1, \ldots, n. \tag{2}$$

◊ 公式(2)说的是, 在这样一组risk-neutral概率下, 任何wager的期望利润都应该是0 (空手套白狼是不存在的), 否则就有arbitrage了.

We do not prove the above theorem in this note. Instead, we adopt it directly to seek whether there is any arbitrage chance in our case of the football lottery. More precisely, we need the proposition below to help us examine our example.

Proposition. *Consider a special model where $m = n$ and a special type of return functions where*

$$r_i(j) = \begin{cases} o_i, \ j = i \\ 0, \ j \neq i. \end{cases} \tag{3}$$

Then there is no arbitrage if and only if all $\tilde{q}_i = 1/o_i$ form a valid probability vector.

Proof. If there is no arbitrage, then according to the no-arbitrage principle, there should exist a probability distribution $q = (q_1, \dots, q_n)'$ such that for every wager i, $i = 1, \dots, n$,

$$0 = \sum_{j=1}^{n} q_j[r_i(j) - 1] = q_i(o_i - 1) - \sum_{j \neq i} q_i = q_i o_i - 1, \tag{4}$$

then we must have $q_i = 1/o_i$. Therefore, if all $\tilde{q}_i = 1/o_i$ calculated from the odds table do not form a valid probability vector, then there exists arbitrage in this model. □

Suppose after you investigate some real lotteries, you are exited to find that the sum of reciprocal of odds does not equal to 1. The next question is how to derive explicitly the arbitrage strategy, namely, how to bet in order to capture the arbitrage profit? We will illustrate these in more details through the example in the next section. But before you celebrate, pay attention to the underlying assumptions inherent in the arbitrage theorem:

- shoring is allowed;
- fractional betting is available.

3 The Arbitrage in a Lottery Example

Let us use the above knowledge to study our lottery example at the beginning with $m = n = 3$. Let us define the events $\alpha = \{Netherlands\,win\}$, $\beta = \{Draw\}$, and $\gamma = \{Argentina\,win\}$. Figure 1 indicates the following return functions on three wagers $a =$ bet on Netherlands win, $b =$ bet on draw, $c =$ bet on Argentina win,

$$r_a(\cdot) = \begin{cases} 1.6, \ \alpha \text{ occurs} \\ 0, \ \beta \text{ occurs} \\ 0, \ \gamma \text{ occurs} \end{cases}, \ r_b(\cdot) = \begin{cases} 0, \ \alpha \text{ occurs} \\ 1.41, \ \beta \text{ occurs} \\ 0, \ \gamma \text{ occurs} \end{cases}, \ r_c(\cdot) = \begin{cases} 0, \ \alpha \text{ occurs} \\ 0, \ \beta \text{ occurs} \\ 1.06, \ \gamma \text{ occurs} \end{cases} \tag{5}$$

It is easy to see that $\tilde{q}_\alpha$, $\tilde{q}_\beta$, $\tilde{q}_\gamma > 0$, and

$$\tilde{q}_\alpha + \tilde{q}_\beta + \tilde{q}_\gamma = \frac{1}{1.6} + \frac{1}{1.41} + \frac{1}{1.06} \approx 2.28 \neq 1. \tag{6}$$

According to the proposition in Section 2, it seems there exists arbitrage opportunity in this lottery. Now let us try to find out the arbitrage strategy $x = (x_a, x_b, x_c)'$, namely, in light of the definition of arbitrage, we are going to first solve x from the condition that the profit under x cannot be negative for all circumstances,

$$\begin{cases} x_a(1.6 - 1) - x_b - x_c \geq 0 \\ -x_a + (1.41 - 1)x_b - x_c \geq 0 \\ -x_a - x_b + (1.06 - 1)x_c \geq 0. \end{cases} \tag{7}$$

Solving the above type of system of linear inequalities requires the *Fourier-Motzkin (FM) elimination algorithm*. We will not describe the general FM method here but instead we are going to solve (7) one-step by one-step in the following.

Step 1. Eliminate x_c from (7): we rearrange to obtain

$$\begin{aligned} x_c &\geq \tfrac{50}{3}x_a + \tfrac{50}{3}x_b \\ x_c &\leq \tfrac{3}{5}x_a - x_b \\ x_c &\leq -x_a + \tfrac{2}{5}x_b, \end{aligned} \tag{8}$$

namely,

$$\frac{50}{3}x_a + \frac{50}{3}x_b \leq x_c \leq \min\{\frac{3}{5}x_a - x_b, -x_a + \frac{2}{5}x_b\}. \tag{9}$$

Step 2. Eliminate x_b: from (8) we first form

$$\begin{aligned} \tfrac{50}{3}x_a + \tfrac{50}{3}x_b &\leq \tfrac{3}{5}x_a - x_b \\ \tfrac{50}{3}x_a + \tfrac{50}{3}x_b &\leq -x_a + \tfrac{2}{5}x_b, \end{aligned} \tag{10}$$

and after simplifications we obtain

$$\begin{aligned} x_b &\leq -\tfrac{241}{265}x_a \\ x_b &\leq -\tfrac{159}{244}x_a, \end{aligned} \tag{11}$$

namely,

$$x_b \leq \min\{-\frac{241}{265}x_a, -\frac{159}{244}x_a\}. \tag{12}$$

Step 2. Finally, x_a is arbitrarily chosen so that (12) and (9) are satisfied.

Suppose $x_a^* = 2$, i.e., we bet 2 RMB on the wager a to gamble Netherlands win, then based on (12), the "feasible" dollar amount on betting b should be $x_b \leq -1.82$. Let us then set $x_b^* = -4$ which means we *sell* two shots of wager b to others (instead of buying from the bookmaker). Finally, according to (9) the "feasible" x_c should lie in the region $[-33.33, -3.6]$, and for example we set $x_c^* = -4$. Then we can check the profit we obtain under our betting strategy $x^* = (x_a^*, x_b^*, x_b^*)'$,

$$\begin{cases} 0.6x_a^* - x_b^* - x_c^* = 9.2 > 0 \\ -x_a^* + 0.41x_b^* - x_c^* = 0.36 > 0 \\ -x_a^* - x_b^* + 0.06x_c^* = 1.76 > 0. \end{cases} \tag{13}$$

We see that no matter what is the terminal result of this match, by investing x^* on the lottery we can always make money (at least 0.36 RMB and at most 9.2 per x^* in this example).

The problem is that in order to achieve this arbitrage profit, we need to take short position. However, in reality you cannot act as a bookmaker to sell the lottery to others on a large scale. Therefore, in general we may discover some arbitrage opportunities in the football lotteries, taking over this arbitrage profit seems impossible in the real life.

⚠ 俗话说的好, 天下没有免费的午餐, 还是老老实实搬砖, 珍爱生命, 远离赌博.

Or to be compromised, you probably could promise your friends who are Messi's fans that they will get 424 RMB back if they give you 400 first and also Argentina wins. Then you could lock down for sure at least 36 RMB arbitrage-free profit if you succeed to do so.

⚠ 朋友圈兜售起来?

4 Conclusion

In this note, we investigate the arbitrage opportunity in a real-life football lottery by utilizing the rigorous definition of arbitrage, the no-arbitrage principle and the existence of risk-neutral probabilities (equivalent martingale measures).

We finally find that there always exist arbitrage opportunities *on paper*, but when we try to execute our arbitrage betting strategy, we always encounter restrictions, such that we are not allowed to sell the lottery to the public so that it is not easy to fulfil the short position if we have to do so as required in the betting strategy. However, if you could find any lottery where we need all long positions enough, then just do it:)

End.