FinMath Topic

FinMath Exercise 1
One-period Trinomial Model with Exchange Rate

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nickycanhelp@qq.com

Exercise. *Suppose a British investor can at any time $n = 0, 1$:*

- *deposit £ in a British bank at the domestic interest rate $r \geq 0$;*
- *buy \$ at the exchange rate E_n (defined as the number of £ needed to buy \$1;*
- *deposit \$ in an American bank at the foreign interest rate $f \geq 0$;*
- *buy shares in an American stock valued at \$$S_n$.*

Suppose as usual borrowing and shorting are allowed. Denote by $d \in \mathbb{R}$ the amount of \$ and by $m \in \mathbb{R}$ the number of shares, bought at time $n = 0$. Assume that the values of (S, E) follow a trinomial model given below:

$\omega \in \Omega$	ω_1	ω_2	ω_3
$S_1(\omega)$	6	2	14
$E_1(\omega)$	6	4	2

with $S_0 = 4$ and $E_0 = 27/8$, and $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$.

(a) *Describe the formula of total wealth $V_1 := V_1^{x,d,m}$ (in £) at time $n = 1$ if initial wealth is $V_0 = x$ (in £) and what is the value (in £) of the American investmens (\$ plus stocks)?*
Assume from now on that $r = 1$ and $f = 1/2$.

(b) *Is this market model arbitrage-free? Is the market complete?*

(c) *Consider an option P on the stock, whose payoff (in \$) is given by*

$\omega \in \Omega$	ω_1	ω_2	ω_3
$P_1(\omega)$	4	8	10

What are the arbitrage-free prices of P (in both \$ and £)?

(d) *Consider the set $\mathcal{A}$ of all trading strategies that super-replicate the option P. Identify exactly the set of all $(x, d, m) \in \mathcal{A}$ with smallest initial capital, i.e., such that*

$$x = p := \min\{x' : (x', d, m) \in \mathcal{A}\}.$$

◊ 该题目考察的是extension of one-period binomial model以及no arbitrage和market completeness等概念, 只是放在汇率的框架下看起来会很绕. 另外我们要知道

$$\begin{cases} \text{how to compute risk-neutral probability} \\ \text{how to price option} \begin{cases} \text{risk-neutral pricing} \\ \text{replicated-portfolio method.} \end{cases} \end{cases}$$

Solution. For a British investor, there are two American investments in this market:

- exchange £ to \$ to buy American stock;
- exchange £ to \$ to deposit into American bank account.

(a) Suppose the investor buys m shares of American stock and deposit $\$d$ in the American bank at time $n = 0$, then the rest of his initial wealth $x - mS_0E_0 - dE_0$ goes into British bank. The value in £ of his American investments, considering the exchange rate E_1 at time $n = 1$, would be

$$mS_1E_1 + d(1+f)E_1. \tag{1}$$

And the British bank account would be $(x - mS_0E_0 - dE_0)(1 + r)$ at time $n = 1$. Therefore, the value in £ of his wealth at time $n = 1$ would be

$$\begin{aligned} V_1^{x,d,m} &= (x - mS_0E_0 - dE_0)(1+r) + mS_1E_1 + d(1+f)E_1 \\ &= \underbrace{x}_{V_0} + \underbrace{(x - mS_0E_0 - dE_0)r}_{\text{growth on British bank}} + \underbrace{m(S_1E_1 - S_0E_0)}_{\text{growth on American stock}} + \underbrace{d[(1+f)E_1 - E_0]}_{\text{growth on American bank}}. \end{aligned} \tag{2}$$

(b) Denote by q_i the risk-neutral probability of ω_i, $i = 1, 2, 3$. From the British investor's point of view, no arbitrage requires

$$\begin{cases} S_0E_0(1+r) = \mathbb{E}^{\mathbb{Q}}[S_1E_1] \\ E_0(1+r) = \mathbb{E}^{\mathbb{Q}}[E_1(1+f)]. \end{cases} \tag{3}$$

Therefore, we obtain a system of linear equations for q_i

$$\begin{cases} 36q_1 + 24q_2 + 12q_3 = 27 \\ 36q_1 + 8q_2 + 28q_3 = 27 \\ q_1 + q_2 + q_3 = 1, \ q_i > 0. \end{cases} \tag{4}$$

Solve and get $q_1 = 1/2$ and $q_2 = q_3 = 1/4$. Since risk-neutral probabilities q_i exist and unique, we conclude that the market is no arbitrage and complete.

(c) To price P by risk-neutral pricing, we should have

$$P_0E_0(1+r) = \mathbb{E}^{\mathbb{Q}}[P_1E_1]. \tag{5}$$

Therefore, we obtain $P_0 = \$100/27$ and $P_0E_0 = £25/2$.

(d) Super-replicate P by the strategy (x, d, m) means

$$V_1^{(x,d,m)}(\omega_i) \geq P_1(\omega_i)E_1(\omega_i),\ \forall i. \tag{6}$$

When x is the smallest initial wealth to do this, according to the no arbitrage we have $x = P_0E_0 = 25/2$ and the inequalities in (6) should be bind from which we solve and obtain $m = 11/9$ and $d = -5/12$. Therefore, the required set should be

$$\{(\frac{25}{2}, -\frac{5}{12}, \frac{11}{9})\}. \tag{7}$$

◊ 此题注意，站在British investor的角度当她面临的是American investments的时候，random的exchange rate是要考虑到的. American stock毫无疑问是风险资产，American bank account在考虑到汇率的波动后对于British investor而言也变成风险资产了. 综上，British investor面对的是两个风险资产(都受汇率影响)和一个无风险资产(British bank account)，同时假设了trinomial model(即市场有三种形态)，最终转化为三个未知数(risk-neutral probabilities)和三个等式的线性方程组求解问题. 另外注意，如果(b)问的结果是非唯一risk-neutral probabilities时，那么(d)问的set可能不会是一个singleton.