Experiments

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1 Preliminaries

Definition 1. ***(submodular function)*** *V denotes the ground set. Given a set function $f : 2^V \to \mathbb{R}$, we say f is submodular function iff*

$$f(X \cup \{e\}) - f(X) \geq f(Y \cup \{e\}) - f(Y)$$

for any $X \subseteq Y \subseteq V$ and $e \in V \setminus Y$.

Definition 2. ***(multi-objective submodular optimization)*** *V denotes the ground set. Given d submodular functions $f_1, f_2, \cdots, f_d$ ($f_i : 2^V \to \mathbb{R}$ for $1 \leq i \leq d$), find a subset $X \subseteq V$ that maximize each submodular function as much as possible simultaneously, i.e.*

$$\max_{X \subseteq V}(f_1(X), \cdots, f_d(X))$$

For multi-objective submodular optimization problem, we usually find a solution set S who contains k solution, i.e. $S = \{X_1, \cdots, X_k\}$ where X_i ($1 \leq i \leq k$) represents a solution. We use the following regret ratio to measure the quality of a solution set.

Definition 3. ***(regret ratio for single objective)*** *Given the submodular function $f : 2^V \to \mathbb{R}$, the solution set S, the feasible solution set $\mathcal{C}$ (i.e. all possible solutions), the regret ratio is defined as follows:*

$$\mathrm{rr}_{f,\mathcal{C}}(S) = \frac{\max_{X \in \mathcal{C}} f(X) - \max_{X \in S} f(X)}{\max_{X \in \mathcal{C}} f(X)} \tag{1}$$

$$= 1 - \frac{\max_{X \in S} f(X)}{\max_{X \in \mathcal{C}} f(X)}. \tag{2}$$

Intuitively, regret ratio for single objective measures the distance between optimal solution in solution set S and the optimal solution in feasible solution set $\mathcal{C}$., the smaller the better.

Definition 4. ***(regret ratio for multi-objective)**[] Given d submodular function $f_1, \cdots, f_d$, a weight vector $\boldsymbol{w}$, the solution set S, the feasible solution set $\mathcal{C}$, the regret ratio for multi-objective functions is as follows:*

$$rr_{f_1,\cdots,f_d,\mathcal{C}}(S, \boldsymbol{w}) = rr_{f^{\boldsymbol{w}},\mathcal{C}}(S) = \frac{\max_{X \in \mathcal{C}} f^{\boldsymbol{w}}(X) - \max_{X \in S} f^{\boldsymbol{w}}(X)}{\max_{X \in \mathcal{C}} f^{\boldsymbol{w}}(X)}, \tag{3}$$

where $f^{\boldsymbol{w}}(X) = \sum_{i=1}^{d} \boldsymbol{w}_i f_i(X)$

Next, we define our problem.

Definition 5. ***(problem)****[] Given* d *submodular function* $f_1, \cdots, f_d$*, the feasible solution set* $\mathcal{C}$ *and a non-negative integer* k*, to find a solution set* $S \subseteq \mathcal{C}$ *with size at most* k *to minimize the maximum regret ratio w.r.t.* $f_1, \cdots, f_d$ *under any positive normal weight vector. i.e.,*

$$\arg \min_{S \subseteq \mathcal{C}, |S| \le k} \max_{\boldsymbol{w} \in \mathbb{R}_+^d} \mathrm{rr}_{f^{\boldsymbol{w}}, \mathcal{C}}(S) \tag{4}$$

Intuitively, our aim is to choose a solution set S who containing at most k solutions to approximate the Pareto Front of the feasible solution set $\mathcal{C}$, the regret smaller, the better.

2 Multi-objective Weighted Max-Cut

Given an undirected graph $G = (V, E)$, the weighted Max-Cut problem is to find a subset $X \subseteq V$ maximizing the weighted sum of edges connecting X and $V \setminus X$, i.e.,

$$\arg \max_{X \subseteq V} \sum_{e \in (X, V \setminus X)} w(e), \tag{5}$$

where $(X, V \setminus X)$ denotes the set of edges whose two vertices are in X and $V \setminus X$, respectively. The objective function is non-monotone submodular. For multi-objective weighted Max-Cut, each edge e has a weight vector $\boldsymbol{w}(e) = [w_1(e), w_2(e), \ldots, w_d(e)]^{\mathrm{T}}$, and there are d objectives $f_1, f_2, \ldots, f_d$, where $\forall i : f_i(X) = \sum_{e \in (X, V \setminus X)} w_i(e)$. The goal is to maximize these d objectives simultaneously with the feasible solution set $\mathcal{C} = 2^V$ (i.e. the feasible solution set $\mathcal{C}$ contains of subsets of V, so the size of $\mathcal{C}$ is $2^{|V|}$).

3 Experiment details

The students need to use **NSGA-II** algorithm and **MOEA** algorithm to solve the Multi-objective Weighted Max-Cut problem. **NSGA-II** algorithm and **MOEA** algorithm return a solution set S, the maximal regret ratio (**Definition 4**) among all weights is used to quantify the quality of the solution set S, i.e. the quality of the solution set S is $\max_{\boldsymbol{w} \in \mathbb{R}_+^d} rr_{f^{\boldsymbol{w}}, \mathcal{C}}(S)$. we limit the size of S as 20, i.e. $k = 20$ in **Definition 5**. All experiments are concluded on $d = 2, 3, 4, 5$ respectively. We provide all the experiment data as follows.

1. 'graph.txt' is the undirected graph data file. Each line of 'graph.txt' is like u v, which indicates node u and node v are adjacent.

2. the folders '2d','3d','4d','5d' contain the weight matrices of each edge for d=2,3,4,5 respectively. We use '2d' to be example. It contains 'objective1_weight_matrix.txt' and 'objective2_weight_matrix.txt'. 'objective1_weight_matrix.txt' is the weight matrix corresponding to the first weight of each edge. 'objective2_weight_matrix.txt' is the

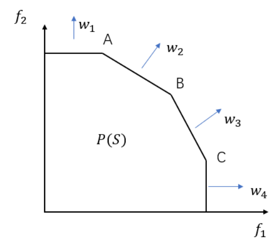

Figure 1:

weight matrix corresponding to the second weight of each edge. Let $M1$ be the matrix 'objective1_weight_matrix.txt' and $M2$ be the matrix 'objective2_weight_matrix.txt', the weight vector of edge $< u, v >$ is a two dimension vector $[M1[u, v], M2[u, v]]$.

Note that for the solution set S output by an algorithm, it is hard to directly compute the regret ratio $\mathrm{rr}_{f_1,f_2,\dots,f_d,\mathcal{C}}(S) = \max_{\boldsymbol{w}\in\mathbb{R}^d_+} \mathrm{rr}_{f^{\boldsymbol{w}},\mathcal{C}}(S)$, as all non-negative weight vectors have to be considered. We have provided a feasible way to compute it. Let $C_{\boldsymbol{f}}(S)$ denote the convex hull of $\{\boldsymbol{f}(X) \mid X \in S\}$ where $\boldsymbol{f}(X) = [f_1(X), \cdots, f_d(X)]$, and let $P(S) = \{\boldsymbol{x} \in \mathbb{R}^d_+ \mid \exists \boldsymbol{y} \in C_{\boldsymbol{f}}(S) : \boldsymbol{x} \leq \boldsymbol{y}\}$, where $\boldsymbol{x} \leq \boldsymbol{y}$ means $\forall i : x_i \leq y_i$. They proved that $\mathrm{rr}_{f_1,f_2,\dots,f_d,\mathcal{C}}(S) = \max_{\boldsymbol{w}} \mathrm{rr}_{f^{\boldsymbol{w}},\mathcal{C}}(S)$, where $\boldsymbol{w}$ runs over the non-negative unit normal vectors of all frontiers of $P(S)$. After the simplicity, We only need to consider a few weight vectors we compute the regret ratio. We give a example by Figure 1. Consider the case $d = 2$ and solution set S consider three points A, B, C. Then $C_{\boldsymbol{f}}(S)$ is the triangle Δ_{ABC} and $P(S)$ is the area marked in Figure 1. In order to compute the regret ratio of set S: $rr_{f_1,f_2,\mathcal{C}}(S)$, we don't need to consider all $\boldsymbol{w} \in \mathbb{R}^d_+$. We just compute $rr_{f_1,f_2,\mathcal{C}}(S, \boldsymbol{w}_1)$, $rr_{f_1,f_2,\mathcal{C}}(S, \boldsymbol{w}_2)$, $rr_{f_1,f_2,\mathcal{C}}(S, \boldsymbol{w}_3)$, $rr_{f_1,f_2,\mathcal{C}}(S, \boldsymbol{w}_4)$ where $\boldsymbol{w}_1, \cdots, \boldsymbol{w}_4$ are non-negative normal vector of the corresponding facets. The maximal value among $rr_{f_1,f_2,\mathcal{C}}(S, \boldsymbol{w}_1)$, $rr_{f_1,f_2,\mathcal{C}}(S, \boldsymbol{w}_2)$, $rr_{f_1,f_2,\mathcal{C}}(S, \boldsymbol{w}_3)$, $rr_{f_1,f_2,\mathcal{C}}(S, \boldsymbol{w}_4)$ is the final regret ratio of solution set S.

To compute the denominator $\max_{X\in\mathcal{C}} f^{\boldsymbol{w}}(X)$ is NP-hard when computing the regret, we will provide the value. So in your implementation, you can omit this work.

References