Crossing-Preserving Geodesic Tracking on Spherical Images

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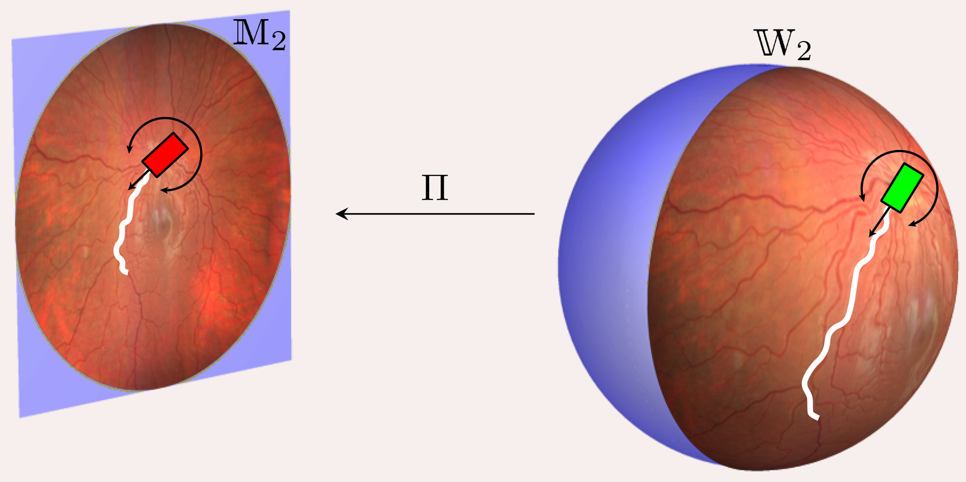
Introduction

The retinal vasculature is representative of the vasculature throughout the human body. Consequently, automating the tracking of the entire vasculature in retinal images is beneficial. As the eye is a sphere, we perform crossing-preserving geodesic tracking on spherical images instead of on their planar projections.



Tracking on Planar Images

To minimize tracking mistakes at crossings, we lift the image (via cake-wavelet ψ) to the homogeneous space of planar positions and orientations $\mathbb{M}_2 := \mathbb{R}^2 \rtimes S^1$, which is diffeomorphic to the roto-translation group $\mathrm{SE}(2) := \mathbb{R}^2 \times \mathrm{SO}(2)$.

We use the left-invariant frame $\{\mathcal{A}_i\}_{i=1}^3$, with the corresponding dual frame $\{\omega^j\}_{j=1}^3$, where $\omega^i \otimes \omega^i$ measures forward, lateral and angular movement for $i = 1, 2, 3$ respectively.

Definition 1 (Horizontality on $\mathbb{M}_2$) *On $\mathbb{M}_2$, we choose the distribution $\Delta^{\mathbb{M}_2} := \mathrm{span}\{\mathcal{A}_1, \mathcal{A}_3\} \subset T\mathbb{M}_2$. Then a smooth curve $\gamma : \mathbb{R} \to \mathbb{M}_2$ is said to be* horizontal *if $\dot{\gamma}(t) \in \Delta^{\mathbb{M}_2}_{\gamma(t)}$ for all t.*

The orientation score is such that curves are lifted to horizontal curves in $\mathbb{M}_2$.
We construct the Reeds-Shepp car models:

$$\mathcal{G}_{\mathbf{p}} = \begin{cases} C^2(\mathbf{p})\left(\xi^2_{\mathbb{M}_2}\omega^1 \otimes \omega^1 + \omega^3 \otimes \omega^3\right)|_{\mathbf{p}}, & \text{on } \Delta^{\mathbb{M}_2}_{\mathbf{p}} \times \Delta^{\mathbb{M}_2}_{\mathbf{p}}, \\ +\infty, & \text{else, and} \end{cases} \tag{1}$$

$$|\mathcal{F}(\mathbf{p}, \cdot)|^2 = \begin{cases} C^2(\mathbf{p})\left(\xi^2_{\mathbb{M}_2}\omega^1 \otimes \omega^1 + \omega^3 \otimes \omega^3\right)|_{\mathbf{p}}, & \text{on } \Delta^{\mathbb{M}_2,+}_{\mathbf{p}}, \\ +\infty, & \text{else,} \end{cases} \tag{2}$$

where $C : \mathbb{M}_2 \to \mathbb{R}_+$ positive cost function, $\xi_{\mathbb{M}_2}$ stiffness parameter, and $\Delta^{\mathbb{M}_2,+}_{\mathbf{p}} := \{\dot{\mathbf{p}} \in \Delta^{\mathbb{M}_2}_{\mathbf{p}} | \omega^1(\dot{\mathbf{p}}) \geq 0\}$.
We use these models to define the *geodesic distances*:

$$d_{\mathcal{G}}(\mathbf{p}, \mathbf{q}) = \inf_{\gamma \in \Gamma} \int_0^1 \sqrt{\mathcal{G}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}\,\mathrm{d}t, \quad d_{\mathcal{F}}(\mathbf{p}, \mathbf{q}) = \inf_{\gamma \in \Gamma} \int_0^1 \mathcal{F}(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}t, \tag{3}$$

where $\Gamma := \{\gamma : [0, 1] \to \mathbb{M}_2 \text{ piecewise } C^1 \mid \gamma(0) = \mathbf{p}, \gamma(1) = \mathbf{q}\}$.

Tracking on Spherical Images

To minimize tracking mistakes at crossings, we lift the image to the space of spherical positions and orientations $\mathbb{W}_2 := S^2 \times S^1 \cong \mathrm{SO}(3)$. We use the left-invariant frame $\{\mathcal{B}_i\}_{i=1}^3$, with the corresponding dual frame $\{\nu^j\}_{j=1}^3$.

Definition 2 (Horizontality on $\mathbb{W}_2$) *On $\mathbb{W}_2$, we choose the distribution $\Delta^{\mathbb{W}_2} := \mathrm{span}\{\mathcal{B}_1, \mathcal{B}_3\} \subset T\mathbb{W}_2$. Then a smooth curve $\delta : \mathbb{R} \to \mathbb{W}_2$ is said to be* horizontal *if $\dot{\delta}(t) \in \Delta^{\mathbb{W}_2}_{\delta(t)}$ for all t.*

The choice of distribution is such that the intuitive relationship between the left-invariant frame and the orientation score carries over from $\mathbb{M}_2$ to $\mathbb{W}_2$. In the theorem, we make the relation between the coordinates on both manifolds explicit.
We construct the Reeds-Shepp car models:

$$\mathcal{G}_{\mathbf{q}} = \begin{cases} C^2(\mathbf{q})\left(\xi^2_{\mathbb{W}_2}\nu^1 \otimes \nu^1 + \nu^3 \otimes \nu^3\right)|_{\mathbf{q}}, & \text{on } \Delta^{\mathbb{W}_2}_{\mathbf{p}} \times \Delta^{\mathbb{W}_2}_{\mathbf{q}}, \\ +\infty, & \text{else, and} \end{cases} \tag{4}$$

$$|\mathcal{F}(\mathbf{q}, \cdot)|^2 = \begin{cases} C^2(\mathbf{q})\left(\xi^2_{\mathbb{W}_2}\nu^1 \otimes \nu^1 + \nu^3 \otimes \nu^3\right)|_{\mathbf{q}}, & \text{on } \Delta^{\mathbb{W}_2,+}_{\mathbf{q}}, \\ +\infty, & \text{else,} \end{cases} \tag{5}$$

where $C : \mathbb{W}_2 \to \mathbb{R}_+$ positive cost function, $\xi_{\mathbb{W}_2}$ stiffness parameter, and $\Delta^{\mathbb{W}_2,+}_{\mathbf{q}} := \{\dot{\mathbf{q}} \in \Delta^{\mathbb{W}_2}_{\mathbf{q}} | \nu^1(\dot{\mathbf{q}}) \geq 0\}$.

Horizontal Curves to Horizontal Curves

Theorem 1 *Let $\Pi : D(\Pi) \to \mathbb{M}_2$, with $D(\Pi) := D(\pi) \times S^1$, such that a) π is the spatial projection of Π, and b) for any* horizontal *curve δ on $\mathbb{W}_2$ the curve $\gamma := \Pi \circ \delta$ is* horizontal *on $\mathbb{M}_2$. Then Π is uniquely defined and is given by*

$$\Pi(\alpha, \beta, \phi) = (\pi(\alpha, \beta), \arg(\dot{\pi}^1(\alpha, \beta) + i\,\dot{\pi}^2(\alpha, \beta))) \tag{6}$$

with components

$$\dot{\pi}^i(\alpha, \beta) := \frac{\partial \pi^i}{\partial \alpha}(\alpha, \beta)\cos(\phi) + \frac{\partial \pi^i}{\partial \beta}(\alpha, \beta)\frac{\sin(\phi)}{\cos(\alpha)}.$$

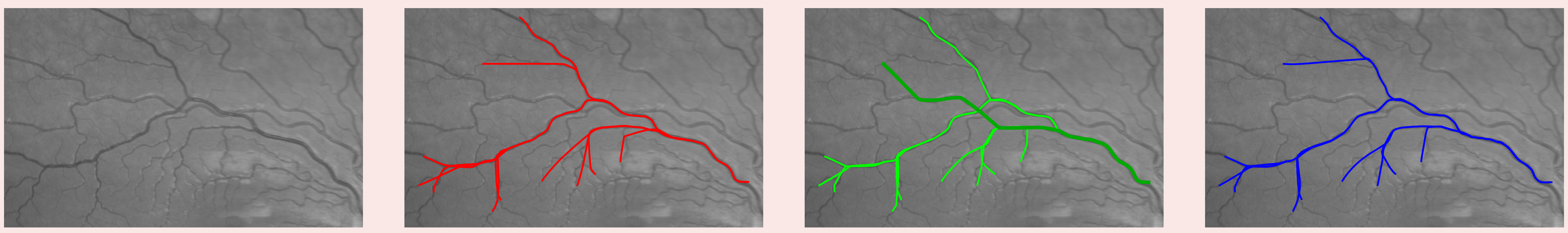
Calculating Distances and Experimental Results

Let $\mathcal{M} \in \{\mathbb{M}_2, \mathbb{W}_2\}$, and $\mathcal{F}$ the Finsler function in eqs. (2) and (5). We compute the distance map to some reference point $\mathbf{p}_0 \in \mathcal{M}$: $W(\mathbf{p}) := d_{\mathcal{F}}(\mathbf{p}_0, \mathbf{p})$. It is known that W is the (viscosity) solution of the eikonal PDE

$$\begin{cases} \mathcal{F}^*(\mathbf{p}, \mathrm{d}W(\mathbf{p})) = 1, & \text{on } \mathcal{M} \setminus \{\mathbf{p}_0\}, \\ W(\mathbf{p}_0) = 0, \end{cases} \tag{7}$$

where $\mathcal{F}^*$ is the dual Finsler function.
We can solve eq. (7) efficiently using Fast Marching techniques, though for technical reasons this requires sub-Riemannian and Finslerian metrics to be approximated by highly anisotropic metrics. To get exact sub-Riemannian and Finslerian constraints, we follow a finite difference approach in approximating W up to accuracy ϵ (grid-size).

Experimental Results



(i) Underlying Image (ii) Tracking in $\mathbb{M}_2$ (iii) Tracking in $\mathbb{W}_2$ with cost function on $\mathbb{R}^2$ (iv) Tracking in $\mathbb{W}_2$ with crossing-preserving cost function on $\mathbb{W}_2$

Conclusion

We developed a new cusp-free, crossing-preserving geodesic tracking model in the space of spherical positions and orientations. We clarified how to relate the space of *planar* and *spherical* positions and orientations. The results of crossing-preserving tracking in $\mathbb{W}_2$ show clear advantages over non-crossing-preserving tracking in $\mathbb{W}_2$ and are similar to crossing-preserving tracking in $\mathbb{M}_2$. All distance maps are computed with fast, simple, and accurate GPU-code, available via `github.com/finnsherry/IterativeEikonal`.