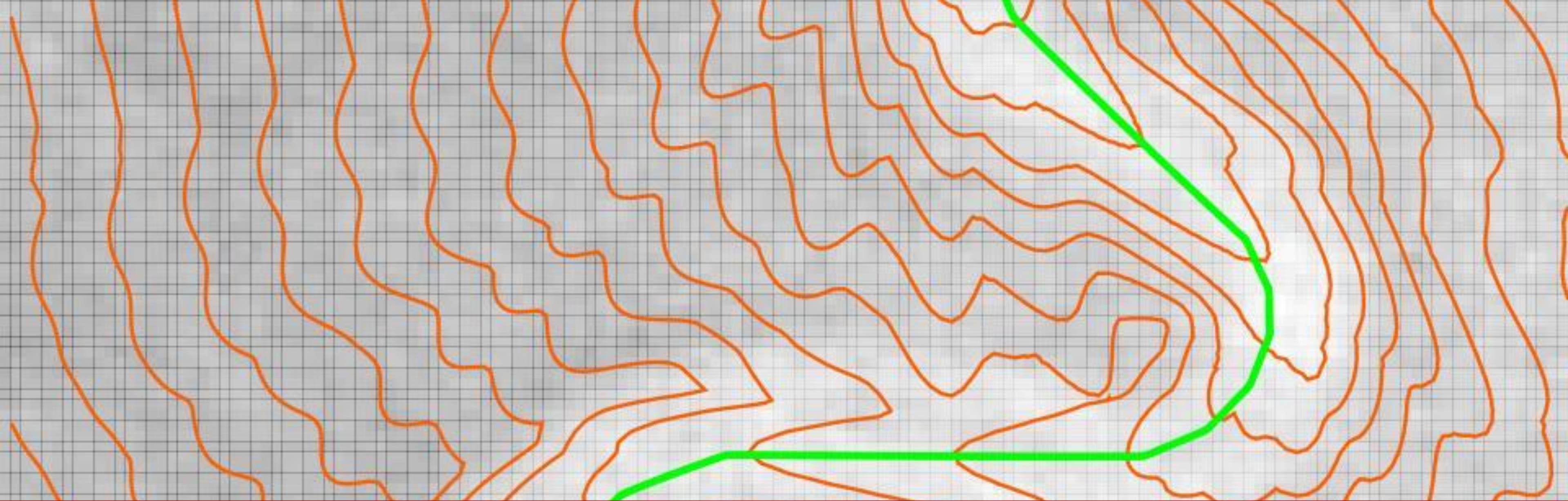

Geodesic Tracking via Data-Driven Geometry for Vascular Tree Tracking

GEOMETRY AND MACHINE LEARNING 2024

Nicky J. van den Berg

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Bart Smets (TU/e), Gautam Pai (TU/e), Jean-Marie Mirebeau (Paris-Saclay)

Mathematics & Computer Science

TU/e EINDHOVEN UNIVERSITY OF TECHNOLOGY

Motivation & Background

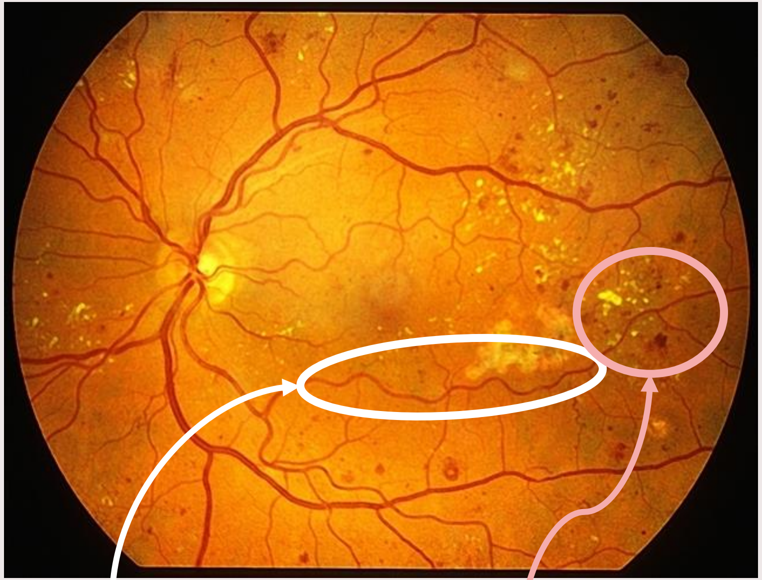
Data-driven frame for improved vessel tracking

Results

Conclusion

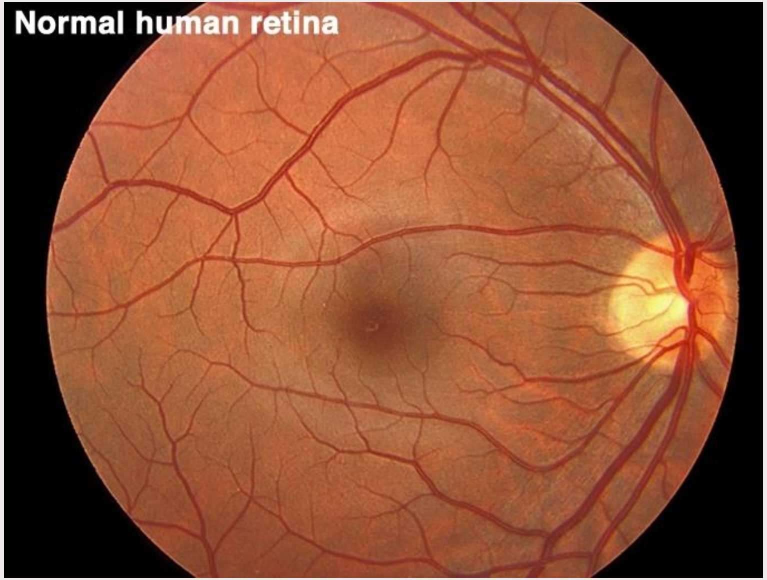
Retina Images

Retina of diabetic patient



Tortuous vessels
Hemorrhage and micro-aneurysms

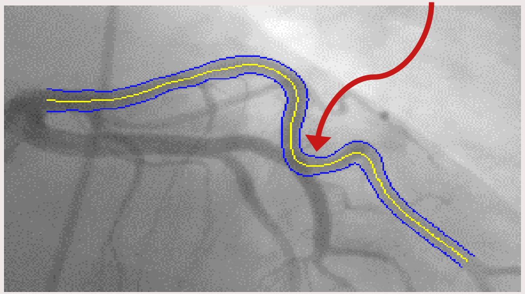
Healthy retina



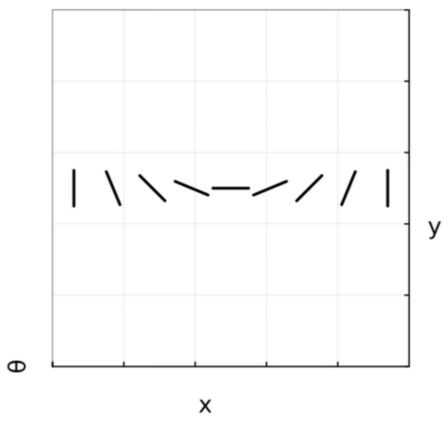
Tracking in Orientation Scores

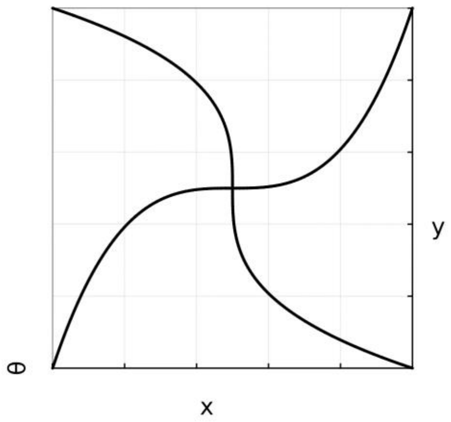
Track vessel from one of the red arrows to the blue arrow

Jumps to another vessel



Tracking in Orientation Scores



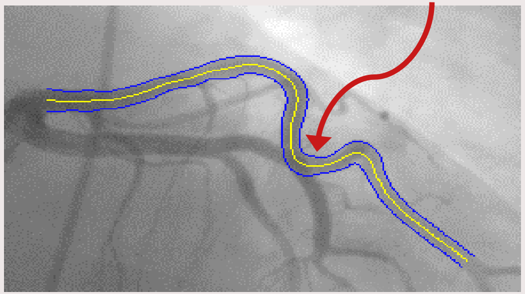


 Animations by Gijs Bellaard

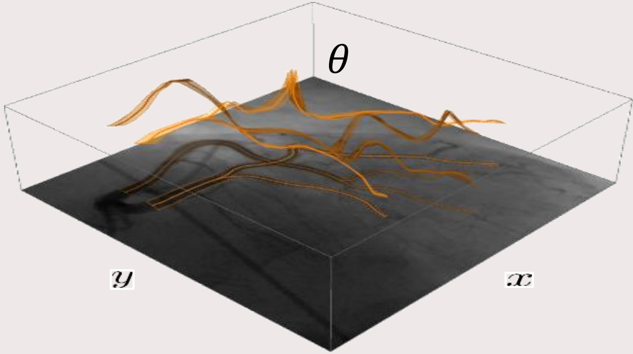
Tracking in Orientation Scores

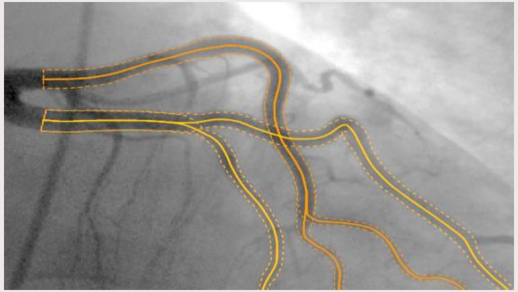
Track vessel from one of the red arrows to the blue arrow

Jumps to another vessel



1. Create a **3D image representation** based on the 2D image (in space of positions and orientations)
2. Perform **tracking** on 3D image (in orientation scores)
3. Project the tracking results onto the 2D image





Orientation Scores

Space of positions and orientations: $\mathbb{M}_2 := \mathbb{R}^2 \rtimes S^1$

where $S^1 \equiv \mathbb{R}/(2\pi\mathbb{Z})$

$(\mathbf{x}, \theta) \in \mathbb{M}_2$ where $\mathbf{x} \in \mathbb{R}^2$ and $\theta \in S^1$

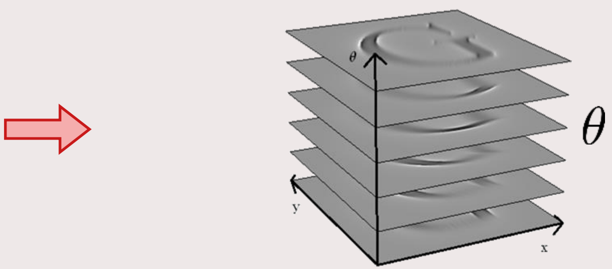
Group product given by

$$(\mathbf{x}_1, R_{\theta_1}) \cdot (\mathbf{x}_2, R_{\theta_2}) = (\mathbf{x}_1 + R_{\theta_1}\mathbf{x}_2, R_{\theta_1} R_{\theta_2})$$

Invertible orientation score transform

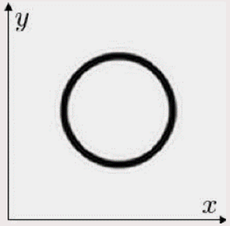
$$W_\psi f(\mathbf{x}, \theta) = \int_{\mathbb{R}^2} \overline{\psi\left(R_\theta^{-1}(\mathbf{y} - \mathbf{x})\right)} f(\mathbf{y}) \mathrm{d}\mathbf{y}$$

Convolution with rotating and translating wavelet ψ

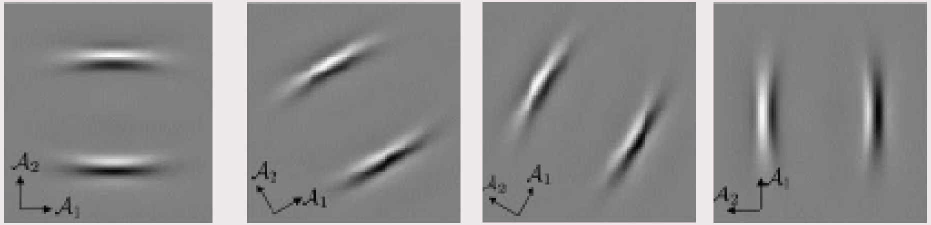


Left Invariant Frame and Metric Tensor Field Model

Image



Left Invariant Derivatives in Orientation Scores



Left invariant frame

$$\begin{cases} \mathcal{A}_1 = \cos\theta\ \partial_x + \sin\theta\ \partial_y \\ \mathcal{A}_2 = -\sin\theta\ \partial_x + \cos\theta\ \partial_y \\ \mathcal{A}_3 = \partial_\theta \end{cases}$$

Dual left invariant frame

$$\begin{cases} \omega^1 = \cos\theta\ \mathrm{d}x + \sin\theta\ \mathrm{d}y \\ \omega^2 = -\sin\theta\ \mathrm{d}x + \cos\theta\ \mathrm{d}y \\ \omega^3 = \mathrm{d}\theta \end{cases}$$

Relation frame and its dual

$$\langle \omega^i, \mathcal{A}_j \rangle = \delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Vessel Tracking

Objective: Find shortest path between point $\mathbf{p}_S$ and $\mathbf{p}$ in $\mathbb{M}_2$

Find minimizing curve γ with distance

$$d_{\mathcal{G}}(\mathbf{p}_S, \mathbf{p}) := \min_{\substack{\gamma \in \mathrm{Lip}([0,1], \mathbb{M}_2) \\ \gamma(0) = \mathbf{p}_S \\ \gamma(1) = \mathbf{p}}} \int_0^1 \|\dot{\gamma}(t)\| \, \mathrm{d}t$$

induced by metric tensor field $\mathcal{G}$

$$\|\dot{\gamma}(t)\|^2 = \mathcal{G}_{\gamma(t)}\left(\dot{\gamma}(t), \dot{\gamma}(t)\right)$$

Vectors aligned to vessel structure have smaller norms compared to random other vectors

Computing shortest curves

1. Compute a viscosity solution U to the eikonal PDE

$$\begin{cases} \|\mathrm{grad}\ U\| = 1 \\ U(\mathbf{p}_S) = 0 \end{cases}$$

U denotes the distance to the point $\mathbf{p}_S$

$$U(\cdot) = d_{\mathcal{G}}(\mathbf{p}_S, \cdot)$$

Computed U using Fast Marching approach

2. Compute the shortest path γ using steepest descent on U starting from end point $\mathbf{p}$

Wavefronts

Compute a viscosity solution U to the eikonal PDE

$$\begin{cases} \|\text{grad } U\| = 1 \\ U(\mathbf{p}_S) = 0 \end{cases}$$

U denotes the distance to the point $\mathbf{p}_S$

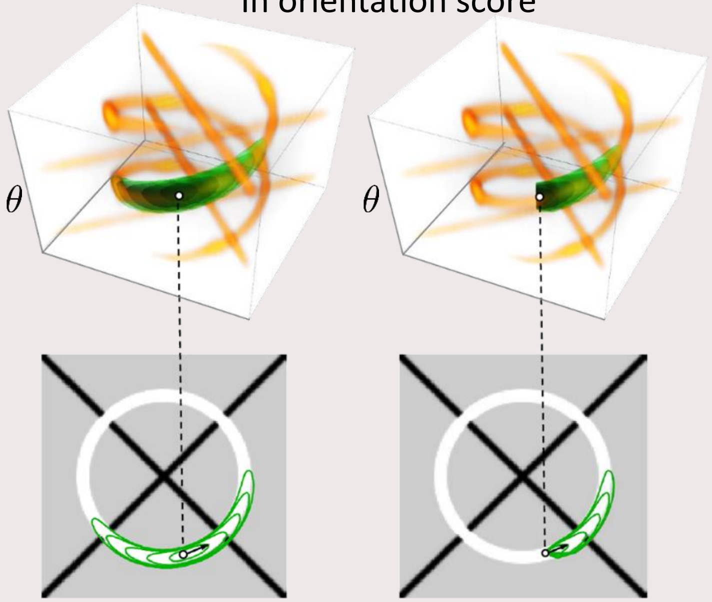
$$U(\cdot) = d_{\mathcal{G}}(\mathbf{p}_S, \cdot)$$

Interpretation: Level sets $\leftrightarrow$ Wavefronts propagated by viscous fluid

Important: wavefronts do not pass through each other!

Visualization: projected level curves by **minimizing** over orientations

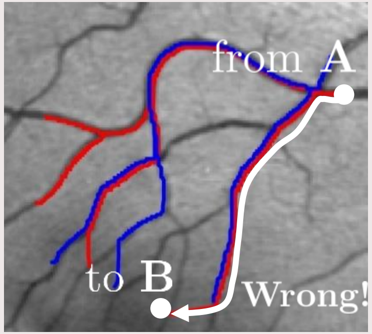
In orientation score

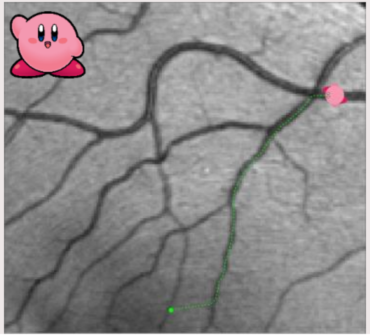


On input image

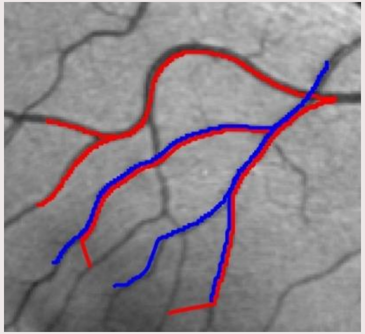
Left Invariant Metric Tensor Field Model

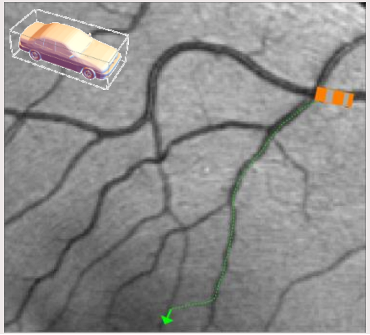
Standard Riemannian shortest paths in $\mathbb{R}^2$



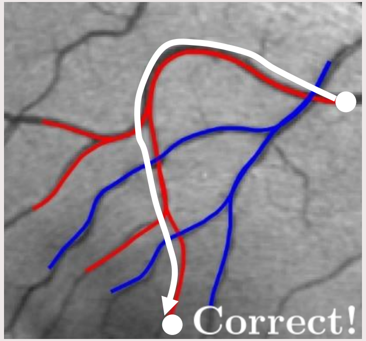


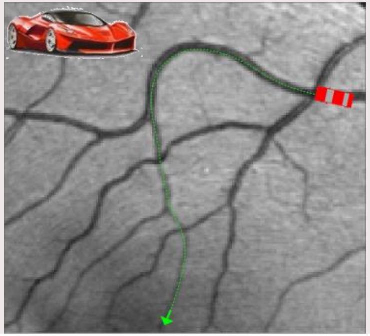
Riemannian (Isotropic) shortest paths in $SE(2)$





Sub-Riemannian shortest paths in $SE(2)$

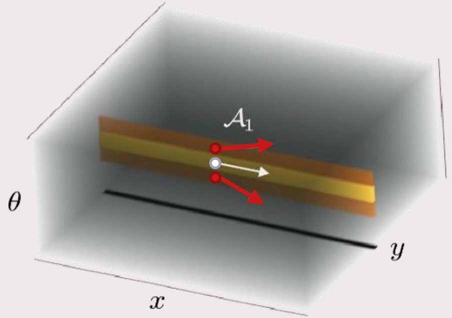




Orientation Scores and Left Invariant Frame

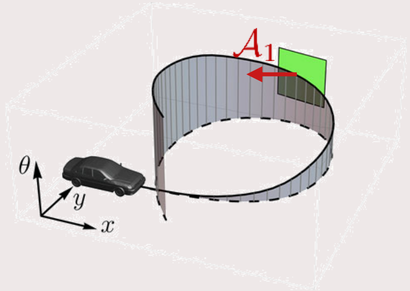
Misalignment due to orientation uncertainty and sampling

No perfect alignment of $\mathcal{A}_1$ and line structure



Possibly steering in wrong direction

Curvature

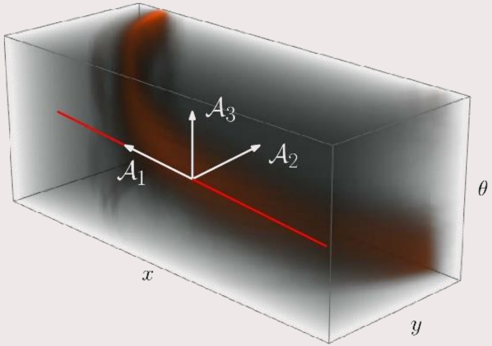


Only spatial alignment

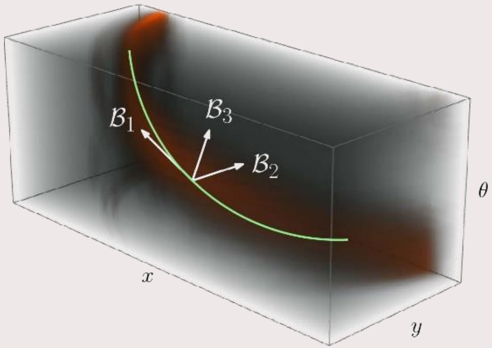
Solution to Short-Comings: New Geometrical Model

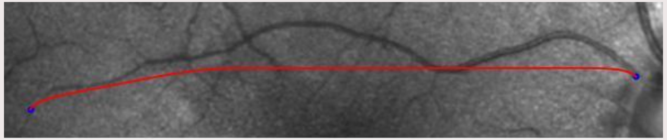
- Old model: diagonal with left invariant frame
- New model: **not** diagonal with left invariant frame
  - Metric tensor field locally aligned with lifted blood vessel
  - Also known as Data-Driven Left Invariant Frame
  - Fitted to image data

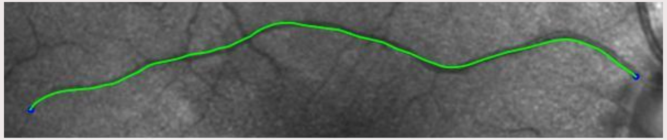
Left invariant frame



Data-driven left invariant frame







Motivation & Background

Data-driven frame for improved vessel tracking

Results

Conclusion

Equivariance

Metric tensor field is data-driven left invariant if

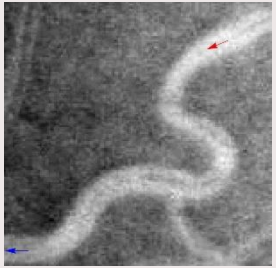
roto-translation of the input image

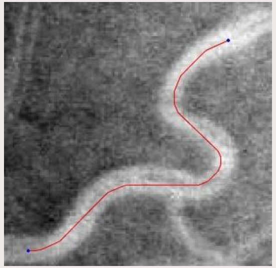
yields

geodesics that are rotated and translated accordingly.

In other words:

$$\mathcal{G}^{U}_{\mathbf{p}}(\dot{\mathbf{p}}, \dot{\mathbf{p}}) = \mathcal{G}^{\mathcal{L}_g U}_{g\mathbf{p}}((L_g)_* \dot{\mathbf{p}}, (L_g)_* \dot{\mathbf{p}})$$





Metric Tensor Fields

$$\mathcal{G}_{\mathbf{p}}\left(\dot{\mathbf{p}},\dot{\mathbf{p}}\right) = \mathcal{C}^2\left(\xi^2\left(\omega^1\otimes\omega^1 + \zeta^{-2}\omega^2\otimes\omega^2\right) + \omega^3\otimes\omega^3\right)\left(\dot{\mathbf{p}},\dot{\mathbf{p}}\right)$$

Cost function	$\mathcal{C}(\mathbf{p}) \in (\delta, 1]$
Weighting parameter	ξ
Anisotropy parameter	$\zeta \in (0,1]$
Forward gear parameter	$\epsilon \in [0,1)$

Dual left invariant frame

$\{\omega^1, \omega^2, \omega^3\}$

$(a)_- = \min\{0, a\}$

$$\mathcal{F}\left(\mathbf{p},\dot{\mathbf{p}}\right)^2 = \mathcal{C}^2\left(\xi^2\left(\omega^1\otimes\omega^1 + \left(\epsilon^{-2}-1\right)\left(\omega^1\otimes\omega^1\right)_- + \zeta^{-2}\omega^2\otimes\omega^2\right) + \omega^3\otimes\omega^3\right)\left(\dot{\mathbf{p}},\dot{\mathbf{p}}\right)$$

Metric Tensor Fields

$$\mathcal{G}_{\mathbf{p}}(\dot{\mathbf{p}},\dot{\mathbf{p}}) = \mathcal{C}^2\left(\xi^2\left(\omega^1\otimes\omega^1 + \zeta^{-2}\omega^2\otimes\omega^2\right) + \omega^3\otimes\omega^3\right)(\dot{\mathbf{p}},\dot{\mathbf{p}})$$

Dual left invariant frame

$\{\omega^1,\omega^2,\omega^3\}$

$$\mathcal{F}(\mathbf{p},\dot{\mathbf{p}})^2 = \mathcal{C}^2\left(\xi^2\left(\omega^1\otimes\omega^1 + \left(\epsilon^{-2}-1\right)\left(\omega^1\otimes\omega^1\right)_- + \zeta^{-2}\omega^2\otimes\omega^2\right) + \omega^3\otimes\omega^3\right)(\dot{\mathbf{p}},\dot{\mathbf{p}})$$

$$(a)_- = \min\{0,a\}$$

$$\mathcal{G}^U_{\mathbf{p}}(\dot{\mathbf{p}},\dot{\mathbf{p}}) = \mathcal{G}_{\mathbf{p}}(\dot{\mathbf{p}},\dot{\mathbf{p}}) + \mathcal{C}^2\lambda\frac{\left\| HU|_{\mathbf{p}}(\cdot,\dot{\mathbf{p}})\right\|_*^2}{\max_{\dot{\mathbf{q}}}\left\| HU|_{\mathbf{p}}(\cdot,\dot{\mathbf{q}})\right\|_*^2} = \mathcal{C}^2\left(a_1\omega^1_U\otimes\omega^1_U + a_2\omega^2_U\otimes\omega^2_U + a_3\omega^3_U\otimes\omega^3_U\right)(\dot{\mathbf{p}},\dot{\mathbf{p}})$$

Cost function $\mathcal{C}(\mathbf{p})\in(\delta,1]$

Curvature parameter $\lambda\in(0,\infty)$

Dual data-driven frame

$\{\omega^1_U,\omega^2_U,\omega^3_U\}$

$$\left|\mathcal{F}^U(\mathbf{p},\dot{\mathbf{p}})\right|^2 = \left|\mathcal{F}(\mathbf{p},\dot{\mathbf{p}})\right|^2 + \mathcal{C}^2\lambda\frac{\left\| HU|_{\mathbf{p}}(\cdot,\dot{\mathbf{p}})\right\|_*^2}{\max_{\dot{\mathbf{q}}}\left\| HU|_{\mathbf{p}}(\cdot,\dot{\mathbf{q}})\right\|_*^2}$$

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Metric Tensor Fields

$$\mathcal{G}_{\mathbf{p}}(\dot{\mathbf{p}},\dot{\mathbf{p}}) = \mathcal{C}^2\left(\xi^2\left(\omega^1\otimes\omega^1+\zeta^{-2}\omega^2\otimes\omega^2\right)+\omega^3\otimes\omega^3\right)(\dot{\mathbf{p}},\dot{\mathbf{p}})$$

Dual left invariant frame

$\{\omega^1,\omega^2,\omega^3\}$

$$\mathcal{F}(\mathbf{p},\dot{\mathbf{p}})^2 = \mathcal{C}^2\left(\xi^2\left(\omega^1\otimes\omega^1+\left(\epsilon^{-2}-1\right)\left(\omega^1\otimes\omega^1\right)_-+\zeta^{-2}\omega^2\otimes\omega^2\right)+\omega^3\otimes\omega^3\right)(\dot{\mathbf{p}},\dot{\mathbf{p}})$$

$$(a)_- = \min\{0,a\}$$

$$\mathcal{G}^U_{\mathbf{p}}(\dot{\mathbf{p}},\dot{\mathbf{p}}) = \mathcal{G}_{\mathbf{p}}(\dot{\mathbf{p}},\dot{\mathbf{p}})+\mathcal{C}^2\lambda\frac{\left\|HU|_{\mathbf{p}}(\cdot,\dot{\mathbf{p}})\right\|_*^2}{\max_{\dot{\mathbf{q}}}\left\|HU|_{\mathbf{p}}(\cdot,\dot{\mathbf{q}})\right\|_*^2} = \mathcal{C}^2\left(a_1\omega_U^1\otimes\omega_U^1+a_2\omega_U^2\otimes\omega_U^2+a_3\omega_U^3\otimes\omega_U^3\right)(\dot{\mathbf{p}},\dot{\mathbf{p}})$$

Dual data-driven frame

$\{\omega_U^1,\omega_U^2,\omega_U^3\}$

$$\left|\mathcal{F}^U(\mathbf{p},\dot{\mathbf{p}})\right|^2 = \left|\mathcal{F}(\mathbf{p},\dot{\mathbf{p}})\right|^2+\mathcal{C}^2\lambda\frac{\left\|HU|_{\mathbf{p}}(\cdot,\dot{\mathbf{p}})\right\|_*^2}{\max_{\dot{\mathbf{q}}}\left\|HU|_{\mathbf{p}}(\cdot,\dot{\mathbf{q}})\right\|_*^2}$$

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Straight vs. short curves for Left Invariant Framework

Left Invariant

Frame

$$\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$$

with dual

$$\{\omega^1, \omega^2, \omega^3\}$$

Connection:

$$\nabla^{[+]} = \sum_{i,k=1}^{3} \left(\omega^i \otimes (\mathcal{A}_i \circ \omega^k) + \sum_{j=1}^{3} (\omega^i \otimes \omega^j)\, c_{ij}^k \right) \mathcal{A}_k$$

Straight curves – parallel velocity

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0$$

Short curves – parallel momentum

$$\begin{cases} (\nabla)^*_{\dot{\gamma}}\, \lambda = 0 \\ \mathcal{G}\dot{\gamma} = \lambda \end{cases}$$

Duits, R., Smets, B.M.N., Wemmenhove, A.J., Portegies, J.W., Bekkers, E.J. (2023). Recent Geometric Flows in Multi-orientation Image Processing via a Cartan Connection. (https://doi.org/10.1007/978-3-030-98661-2_101)

Straight vs. short curves for Data-Driven Left Invariant Framework

Left Invariant

Frame

$$\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$$

with dual

$$\{\omega^1, \omega^2, \omega^3\}$$

Connection:

$$\nabla^{[+]} = \sum_{i,k=1}^{3} \left(\omega^i \otimes (\mathcal{A}_i \circ \omega^k) + \sum_{j=1}^{3} (\omega^i \otimes \omega^j)\, c_{ij}^k \right) \mathcal{A}_k$$

Straight curves – parallel velocity

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0$$

Data-Driven Left Invariant

Frame

$$\{\mathcal{A}_1^U, \mathcal{A}_2^U, \mathcal{A}_3^U\}$$

with dual

$$\{\omega_U^1, \omega_U^2, \omega_U^3\}$$

$$[\mathcal{A}_i^U, \mathcal{A}_j^U] = \tilde{c}_{ij}^k(\cdot)\, \mathcal{A}_k^U$$

Connection:

$$\nabla^U = \sum_{i,k=1}^{3} \left(\omega_U^i \otimes (\mathcal{A}_i^U \circ \omega_U^k) + \sum_{j=1}^{3} \left(\omega_U^i \otimes \omega_U^j\right) \tilde{c}_{ij}^k \right) \mathcal{A}_k^U$$

Short curves – parallel momentum

$$\begin{cases} (\nabla)_{\dot{\gamma}}^{*} \lambda = 0 \\ \mathcal{G}\dot{\gamma} = \lambda \end{cases}$$

New theorem

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Calculating Geodesics via Steepest Descent on the Distance Map

Theorem 1 *The shortest curve* $\gamma : [0,1] \to \mathbb{M}_2$ *with* $\gamma(0) = \mathbf{p}$ *and* $\gamma(1) = \mathbf{p}_0$ *can be computed by steepest descent tracking on distance map* $W(\mathbf{p}) = d_{\mathcal{F}^U}(\mathbf{p}, \mathbf{p}_0)$

$$\gamma(t) := \gamma^U_{\mathbf{p},\mathbf{p}_0}(t) = Exp_{\mathbf{p}}(t\, v(W)), \qquad t \in [0,1], \tag{1}$$

where Exp integrates the following vector field on $\mathbb{M}_2$*:* $v(W) := -W(\mathbf{p}) \nabla_{\mathcal{F}^U} W$ *and where* W *is the viscosity solution of the eikonal PDE system*

$$\begin{cases} \mathcal{F}^*_U(\mathbf{p}, \mathrm{d}W(\mathbf{p})) = 1 & \mathbf{p} \in \mathbb{M}_2, \\ W(\mathbf{p}_0) = 0, \end{cases} \tag{2}$$

assuming $\mathbf{p}$ *is neither a 1st Maxwell-point nor a conjugate point, with dual Finsler function* $\mathcal{F}^*_U(\mathbf{p}, \hat{\mathbf{p}}) := \max\{\langle \hat{\mathbf{p}}, \dot{\mathbf{p}} \rangle \mid \dot{\mathbf{p}} \in T_{\mathbf{p}}(\mathbb{M}_2) \text{ with } \mathcal{F}^U(\mathbf{p}, \dot{\mathbf{p}}) \leq 1\}$*. As* $v(W)$ *is data-driven left invariant, the geodesics carry the symmetry*

$$\gamma^{\mathcal{L}_g U}_{g\cdot\mathbf{p}, g\cdot\mathbf{p}_0}(t) = g\, \gamma^U_{\mathbf{p},\mathbf{p}_0}(t) \text{ for all } g \in SE(2), \mathbf{p}, \mathbf{p}_0 \in \mathbb{M}_2, t \in [0,1]. \tag{3}$$

New theorem

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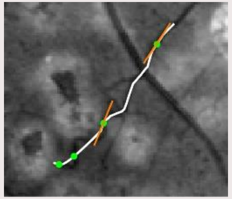
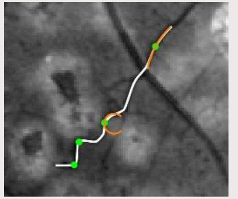
Straight vs. short curves

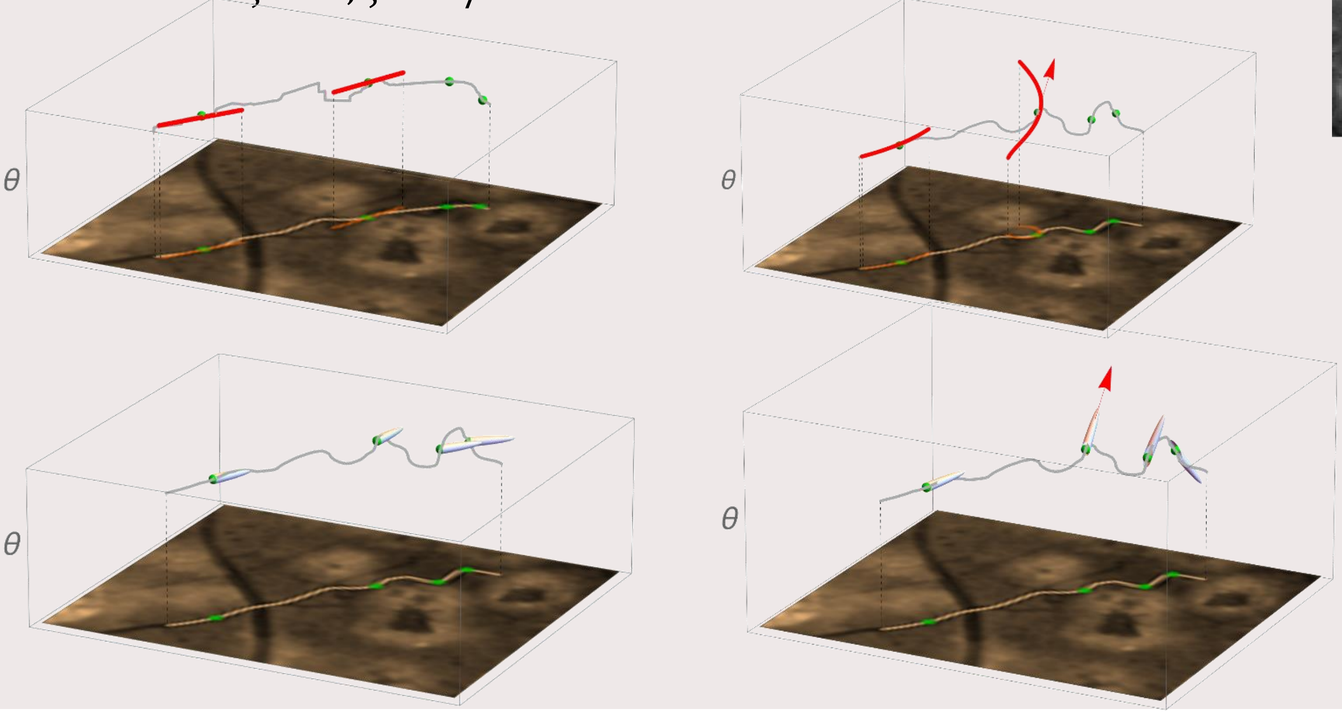


Left invariant frame
with $\xi = 1, \zeta = 1/8$

Data-driven left invariant frame
with $\lambda = 100$



Exponential curves
– Straight lines



Control sets

$$B_{\mathcal{G}}(\boldsymbol{p}) = \{\dot{\boldsymbol{p}} \in T_{\boldsymbol{p}}(\mathbb{M}_2) \mid \mathcal{G}_{\boldsymbol{p}}(\dot{\boldsymbol{p}}, \dot{\boldsymbol{p}}) \leq 1\}$$

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Motivation & Background

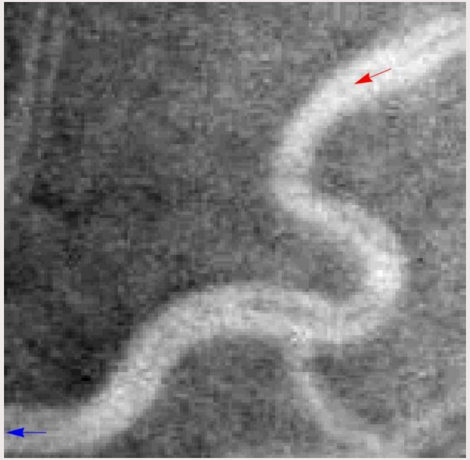
Data-driven frame for improved vessel tracking

Results

Conclusion

Influence Data-Driven Left Invariant Frame

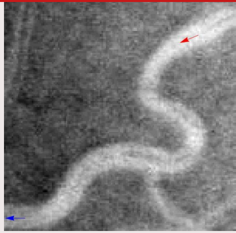
$N_\theta = 16$



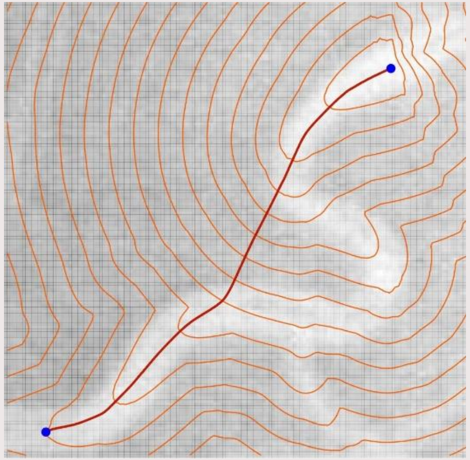
Experimental Setting:
Track from red to blue arrow

Influence Data-Driven Left Invariant Frame

$N_\theta = 16$

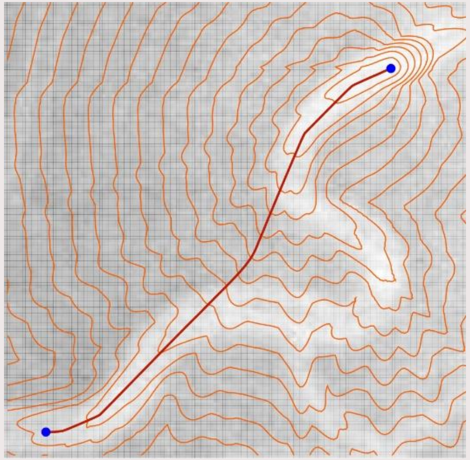


Isotropic left invariant frame, $\zeta = 1$



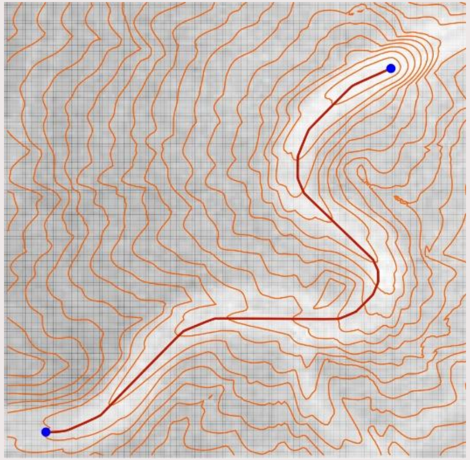
Wavefronts do not follow vessel structure

Anisotropic left invariant frame, $\zeta = 0.125$



Wavefronts do not follow vessel structure

Data-driven left invariant frame, $\zeta = 0.125$, $\lambda = 100$



Wavefronts follow vessel structure

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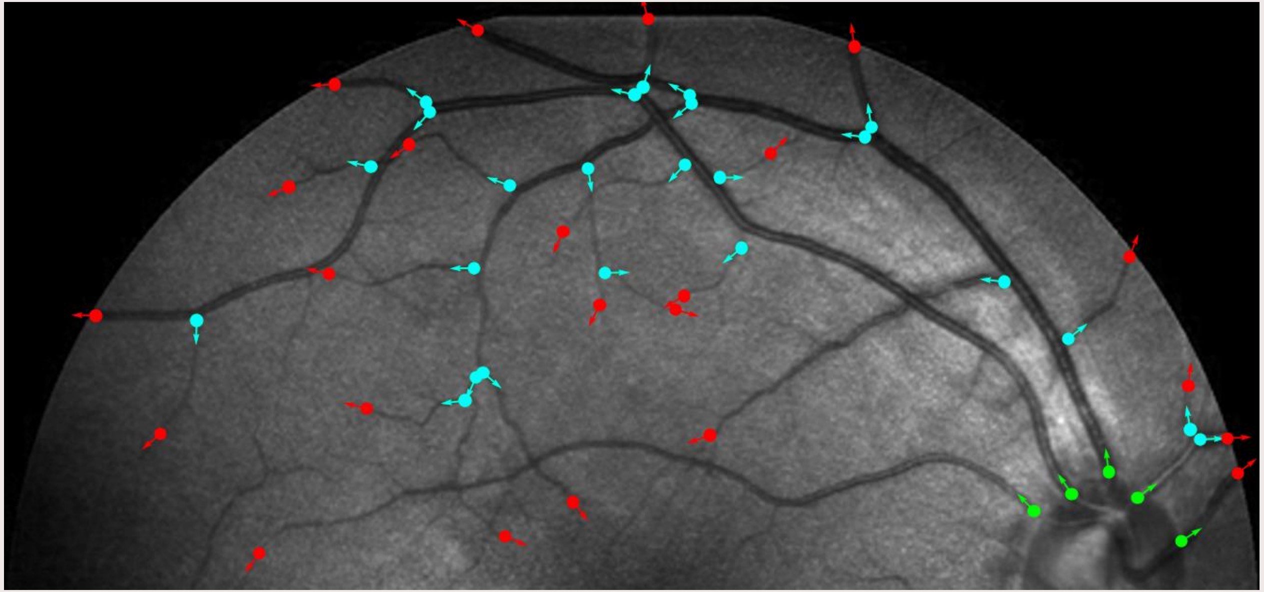
Tracking Vessels in Vessel Tree



Experimental setting

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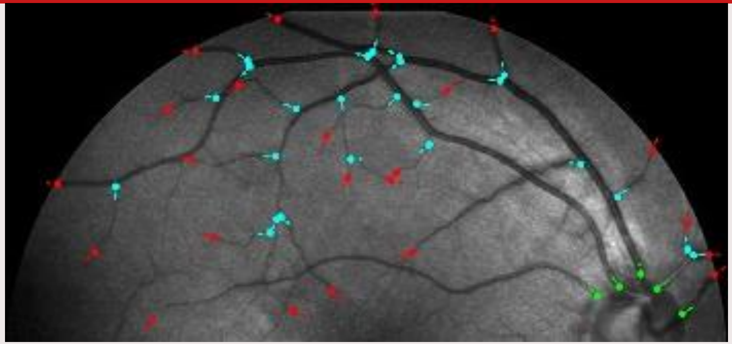
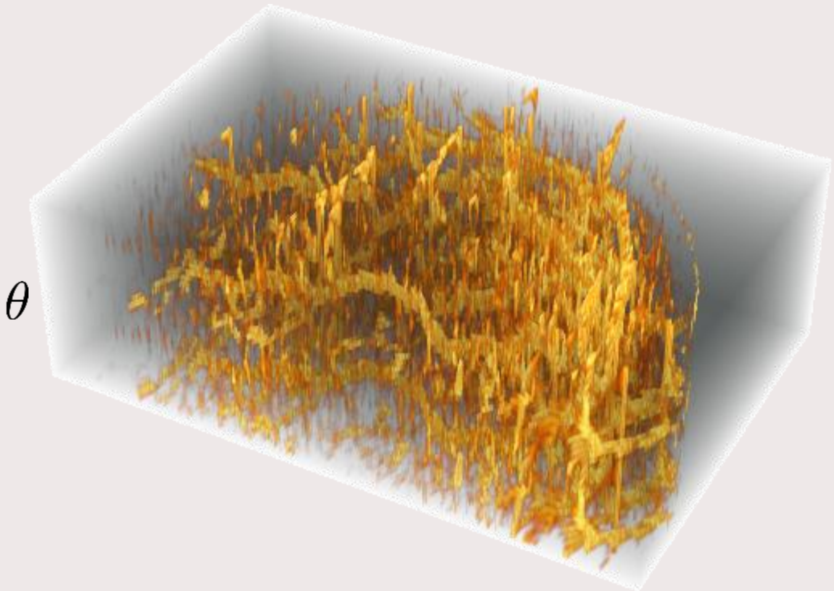
Tracking Vessels in Vessel Tree in Two Steps



Projection
cost function

3D visualization
cost function



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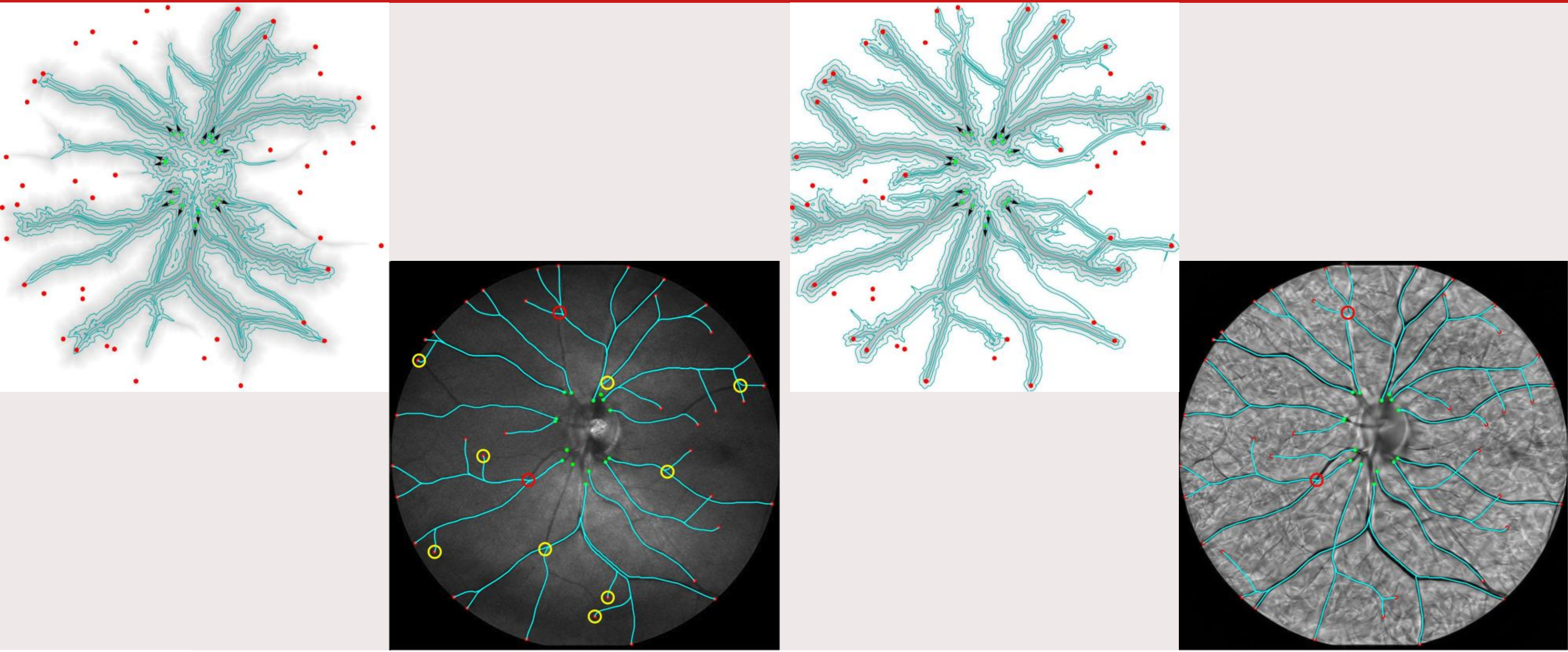
Geodesics Tracking of Vascular Trees



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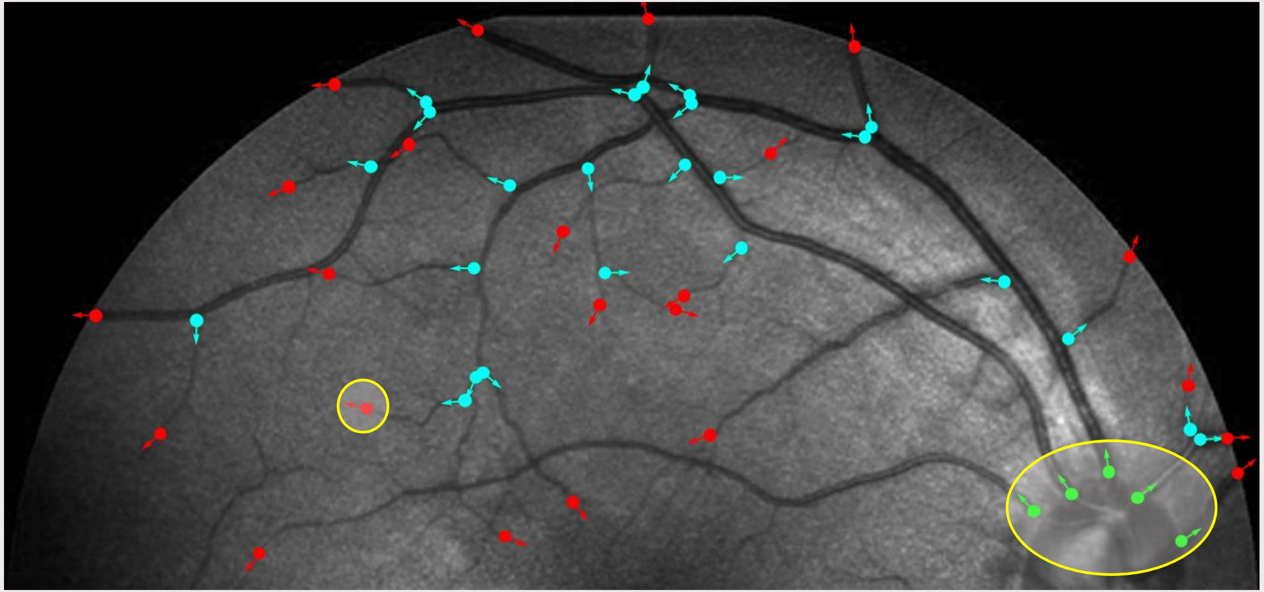
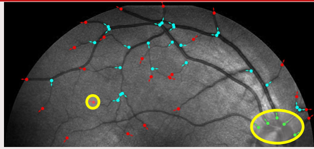
Tracking Vessels in Vessel Tree



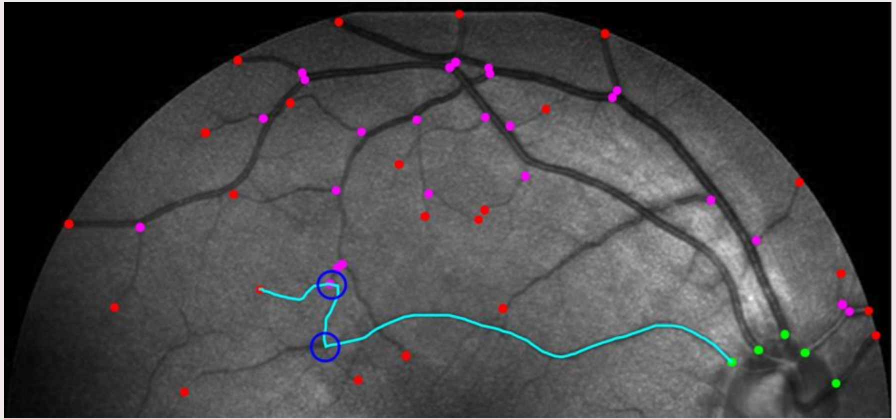
Experimental setting

N.J. van den Berg, B.M.N. Smets, G. Pai, J.M. Mirebeau, R. Duits: Geodesic Tracking via New Data-driven Connections of Cartan Type for Vascular Tree Tracking (JMIV 2024)

Tracking Vessels in Vessel Tree

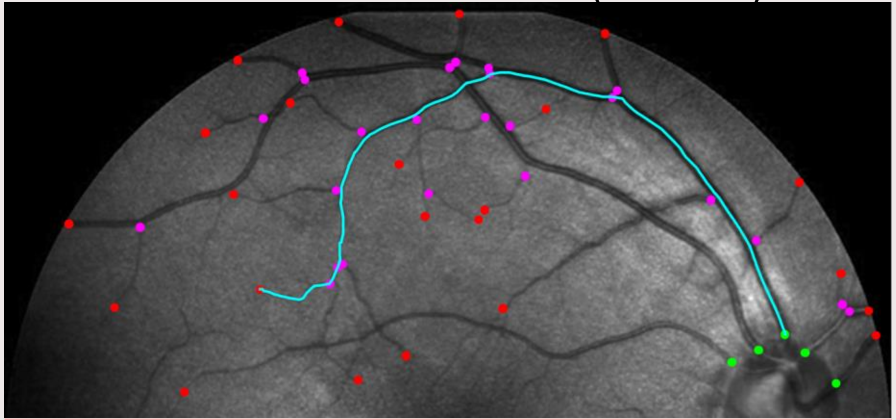


Data-driven left invariant frame
$\lambda = 100$



Occasionally runs into trouble at crossings

Solution: Mixed data-driven left invariant frame: Left invariant frame at crossings, data-driven left invariant frame otherwise ($\lambda = 100$)



Shows correct behavior at crossings

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Motivation & Background

Data-driven frame for improved vessel tracking

Results

Conclusion

Conclusion

- Model corresponding with the data-driven frame, relying on data-driven Cartan connection
    1. Better adaptation for curvature
- Theorem:
    1. Geodesics (parallel momentum w.r.t data-driven Cartan-connection) can still be found with steepest descent
    2. Geodesics have appropriate equivariant behavior
- Tracking of complete vascular tree from a single distance map gives good results

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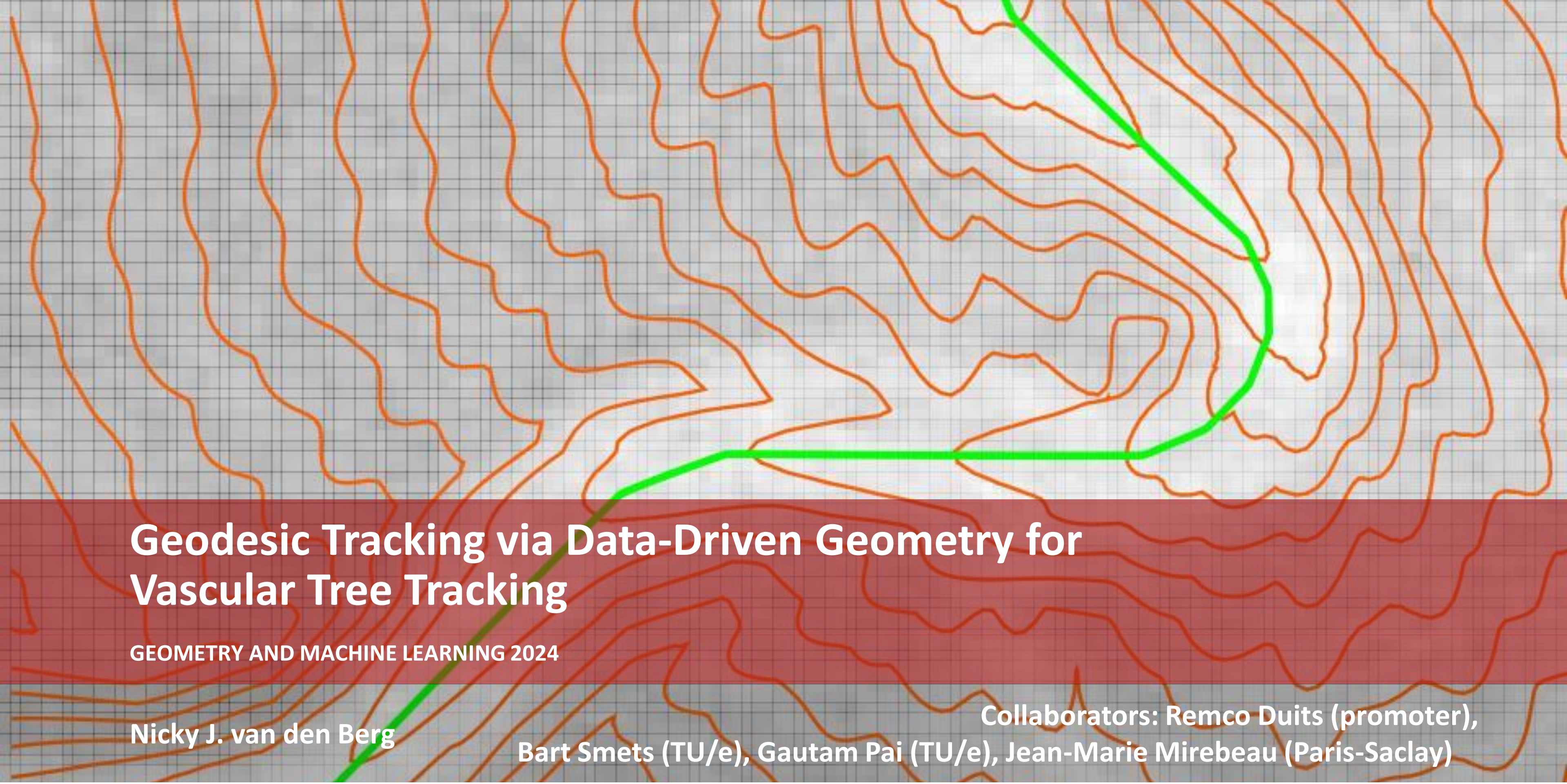

Geodesic Tracking via Data-Driven Geometry for Vascular Tree Tracking

GEOMETRY AND MACHINE LEARNING 2024

Nicky J. van den Berg

Collaborators: Remco Duits (promoter), Bart Smets (TU/e), Gautam Pai (TU/e), Jean-Marie Mirebeau (Paris-Saclay)

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