1 Prerequisites

The probability that the material lung tissue is assigned to a voxel is p_l. A particle traversing this voxelised geometry will cross n_z voxels. The probability $F(k)$ that k voxels consist of lung tissue is given by the Bernoulli distribution:

$$F(k) = \binom{n_z}{k} \cdot p_l^k \cdot (1-p_l)^{n_z-k} \tag{1}$$

Following the Moivre–Laplace central limit theorem this distribution can for large n_z be approximated to a sufficient degree of accuracy by a normal distribution:

$$F(k) = \frac{1}{\sqrt{2\pi\sigma_{nl}^2}} \cdot \exp\left(-\frac{(k-\mu_{nl})^2}{2\sigma_{nl}^2}\right) \tag{2}$$

where the expectation value μ_{nl} and the width σ_{nl} are given by:

$$\mu_{nl} = n_z \cdot p_l \qquad \sigma_{nl} = \sqrt{n_z \cdot p_l \cdot (1-p_l)} \tag{3}$$

Hence, a particle traversing the voxelised geometry at one of $n_x \times n_y$ possible paths will cross on average μ_{nl} voxels of lung tissue with a standard deviation of σ_{nl} voxels. From this distribution the function $F(t'|t,\sigma_t)$ can be derived giving the probability that the path a particle takes through the voxelised geometry has the water-equivalent thickness t':

$$F(t'|t,\sigma_t) = \frac{1}{\sqrt{2\pi\sigma_t^2}} \cdot \exp\left(-\frac{(t'-t)^2}{2\sigma_t^2}\right) \tag{4}$$

with t being the mean water-equivalent thickness. [1]

2 Stopping Power by Range-Energy Relationship

Definition of the Stopping Power:

$$S = -\frac{\mathrm{d}E}{\mathrm{d}x} \tag{5}$$

Energy-range relationship (Bragg-Kleeman rule):

$$(R_0 - z) = \alpha E(z)^p \tag{6}$$

p parameter has very slight energy dependence, but will be approximated as a constant.

The water equivalent thickness of a voxalized heterogeneous target is calculated using the ratio of the mean Stopping Power of water and of the targets constituents, using $\overline{S}_m$ for a given material and $\overline{S}_a$ for another material (usually air):

$$t_{\mathrm{H_2O}} = D \cdot \frac{\overline{S}_{mean}}{\overline{S}_{\mathrm{H_2O}}} \tag{7}$$

The mean of the mean stopping power $\overline{S}_{mean}$, here defined as the mean stopping power of lung $\overline{S}_l$, can be calculated by using a linear combination of its constituents (Eq. 8) [2] and by assuming the density ratio of lung ($\rho_{mean} = 0.26\,\frac{\mathrm{g}}{\mathrm{cm}^3}$) and water ($\rho_{\mathrm{H_2O}} = 1\,\frac{\mathrm{g}}{\mathrm{cm}^3}$ is equal to their mean stopping power ratio.

$$\overline{S}_{mean} = p_m \overline{S}_m + (1 - p_m)\overline{S}_a \tag{8}$$

$$\frac{\rho_{mean}}{\rho_{\mathrm{H_2O}}} = \frac{\overline{S}_{mean}}{\overline{S}_{\mathrm{H_2O}}} \rightarrow \overline{S}_{mean} = \frac{\rho_{mean}}{\rho_{\mathrm{H_2O}}}\overline{S}_{\mathrm{H_2O}} \tag{9}$$

The fill probability p_m follows from inserting Equation 9 into 8:

$$\frac{\rho_{mean}}{\rho_{\mathrm{H_2O}}}\overline{S}_{\mathrm{H_2O}} = p_m \overline{S}_m + (1 - p_m)\overline{S}_a \tag{10}$$

$$\rightarrow p_m = \frac{\frac{\rho_{mean}}{\rho_{\mathrm{H_2O}}}\overline{S}_{\mathrm{H_2O}} - \overline{S}_a}{\overline{S}_m - \overline{S}_a} \tag{11}$$

Mean Stopping power definition and inserting Equation 6 and 5, assuming a thick target [3]:

$$\overline{S} = \frac{\int_{E_i}^{E_f} S\mathrm{d}E}{\int_{E_i}^{E_f} \mathrm{d}E} \tag{12}$$

$$= \frac{\int_{E_i}^{E_f} (E^{1-p}(\alpha \cdot p)^{-1})\mathrm{d}E}{\int_{E_i}^{E_f} \mathrm{d}E} \tag{13}$$

$$= \frac{\frac{1}{2-p}(\alpha \cdot p)^{-1} \cdot (E_f^{2-p} - E_i^{2-p})}{E_f - E_i} \tag{14}$$

$$= \frac{(E_f^{2-p} - E_i^{2-p})}{(2-p)(\alpha \cdot p)(E_f - E_i)} \tag{15}$$

E_f after traversing a target of size D out of material *mat* is calculated using Equation 6:

$$E_{f,m} = E_m(D) = \left(\frac{R_{0,m} - D}{\alpha_m}\right)^{\frac{1}{p_m}} = \left(\frac{\alpha_m E_i^{p_m} - D}{\alpha_m}\right)^{\frac{1}{p_m}} \tag{16}$$

Inserting the mean stopping power assumption of Equation 9 into Equation 7 yields:

$$t_{\mathrm{H_2O}} = D\frac{\rho_{mean}}{\rho_{\mathrm{H_2O}}} \tag{17}$$

The WET variance is calculated by subtracting the variance for air from lung and using Equation 3 for the standard deviation of the binomial distribution:

$$\sigma_t^2 = \left(\sigma_{nm} \cdot d\frac{\overline{S}_m}{\overline{S}_{\mathrm{H_2O}}} - \sigma_{na} \cdot d\frac{\overline{S}_a}{\overline{S}_{\mathrm{H_2O}}}\right)^2 \tag{18}$$

$$\text{binomial dist .: } \sigma_{nm} = \sigma_{na} \to \; = (d \cdot \sigma_{nm})^2 \frac{(\overline{S}_m - \overline{S}_a)^2}{\overline{S}_{\mathrm{H_2O}}^2} \tag{19}$$

$$= d^2 \cdot n_z \cdot p_m \cdot (1 - p_m) \cdot \frac{(\overline{S}_m - \overline{S}_a)^2}{\overline{S}_{\mathrm{H_2O}}^2} \tag{20}$$

$$= \frac{D}{d} \cdot p_m \cdot (1 - p_m) \cdot d^2 \cdot \frac{(\overline{S}_m - \overline{S}_a)^2}{\overline{S}_{\mathrm{H_2O}}^2} \qquad \text{cf. Eq. (6) of [1]} \tag{21}$$

The modulation power follows from Equation 7 and 18:

$$P_{\text{mod}} := \frac{\sigma_t^2}{t} = \frac{\rho_{\text{H}_2\text{O}}}{\rho_{mean}} \cdot d \cdot p_m \cdot (1-p_m) \cdot \frac{(\overline{S}_m - \overline{S}_a)^2}{\overline{S}^2_{\text{H}_2\text{O}}} \tag{22}$$

$$\text{inserting } p_m \rightarrow \; = \frac{\rho_{\text{H}_2\text{O}}}{\rho_{mean}} \cdot d \cdot \frac{\frac{\rho_{mean}}{\rho_{\text{H}_2\text{O}}}\overline{S}_{\text{H}_2\text{O}} - \overline{S}_a}{\overline{S}_m - \overline{S}_a} \cdot \left(1 - \frac{\frac{\rho_{mean}}{\rho_{\text{H}_2\text{O}}}\overline{S}_{\text{H}_2\text{O}} - \overline{S}_a}{\overline{S}_m - \overline{S}_a}\right) \cdot \frac{(\overline{S}_m - \overline{S}_a)^2}{\overline{S}^2_{\text{H}_2\text{O}}} \tag{23}$$

$$\text{approximation } \overline{S}_a \approx 0 \rightarrow \; \approx d \cdot \left(1 - \frac{\frac{\rho_{mean}}{\rho_{\text{H}_2\text{O}}}\overline{S}_{\text{H}_2\text{O}}}{\overline{S}_m}\right) \cdot \frac{\overline{S}_m}{\overline{S}_{\text{H}_2\text{O}}} \tag{24}$$

$$\approx d \cdot \left(\frac{\overline{S}_m}{\overline{S}_{\text{H}_2\text{O}}} - \frac{\rho_{mean}}{\rho_{\text{H}_2\text{O}}}\right) \qquad \text{cf. Eq. (8) of [1] with } \left(p_l \approx \frac{\rho_{mean}}{\rho_l}\right) \tag{25}$$

$$\text{inserting Eq. 12 } \rightarrow \; \approx d \cdot \left(\frac{(E_{f,m}^{2-p_m} - E_i^{2-p_m})}{(E_{f,\text{H}_2\text{O}}^{2-p_{\text{H}_2\text{O}}} - E_i^{2-p_{\text{H}_2\text{O}}})} \cdot \frac{(2-p_{\text{H}_2\text{O}})(\alpha_{\text{H}_2\text{O}} \cdot p_{\text{H}_2\text{O}})(E_{f,\text{H}_2\text{O}} - E_i)}{(2-p_m)(\alpha_m \cdot p_m)(E_{f,m} - E_i)} - \frac{\rho_{mean}}{\rho_{\text{H}_2\text{O}}}\right) \tag{26}$$

$$\text{inserting Eq. 16} \rightarrow \; \approx d \cdot \left(\frac{\left(\left(\frac{\alpha_m E_i^{p_m} - D}{\alpha_m}\right)^{\frac{2}{p_m}-1} - E_i^{2-p_m}\right)}{\left(\left(\frac{\alpha_{\text{H}20} E_i^{p_{\text{H}20}} - D}{\alpha_{\text{H}20}}\right)^{\frac{2}{p_{\text{H}20}}-1} - E_i^{2-p_{\text{H}_2\text{O}}}\right)}\right. \tag{27}$$

$$\left. \cdot \frac{(2-p_{\text{H}_2\text{O}})(\alpha_{\text{H}_2\text{O}} \cdot p_{\text{H}_2\text{O}})\left(\left(\frac{\alpha_{\text{H}_2\text{O}} E_i^{p_{\text{H}_2\text{O}}} - D}{\alpha_{\text{H}_2\text{O}}}\right)^{\frac{1}{p_{\text{H}_2\text{O}}}} - E_i\right)}{(2-p_m)(\alpha_m \cdot p_m)\left(\left(\frac{\alpha_m E_i^{p_m} - D}{\alpha_m}\right)^{\frac{1}{p_m}} - E_i\right)} - \frac{\rho_{mean}}{\rho_{\text{H}_2\text{O}}}\right) \tag{28}$$

The given modulation power is depended on the initial Energy of a particle or beam E_i, the structure constant d and the targets thickness D. Moreover, material parameters of the rang-energy relationship are used as well as the density ratio of the modulated heterogeneous target (in this case lung). The Equation holds for approximating targets with low Z, such that the stopping power ratio between the target and water can be approximated by their density ratio. It should also be noted that the used mean stopping power approximates the whole target, resulting in a linear dependence between depth and stopping power, even though the stopping power is energy depended and thus does not scale linearly with depth. Hence, the stopping power is always overestimated but never underestimated. However, for high-energy particles the total energy loss in the target is relatively small, thereby resulting only in a small overestimation.

3 Approximate Stopping Power Ratio Model

Bethe-Bloch Equation of the mass stopping power:

$$\hat{S} = -\frac{1}{\rho}\frac{\mathrm{d}E}{\mathrm{d}x} = Kz^2\frac{Z}{A}\frac{1}{\beta^2}\left(\frac{1}{2}\ln\left(\frac{2m_ec^2\beta^2\gamma^2W_{\max}}{I^2}\right) - \beta^2 - \frac{\delta(\beta\gamma)}{2})\right), \tag{29}$$

where $W_{\max}$ is the maximum energy transfer to an electron in a single collision as

$$W_{\max} = \frac{2m_ec^2\beta^2\gamma^2}{1+2\gamma\frac{m_e}{M}+(\frac{m_e}{M})^2}. \tag{30}$$

A summary of all variables is shown in Table 1.

Symbol	Definition	Value or units
m_ec^2	electron energy	0.510998950 MeV
M	incident particle mass	$\frac{\mathrm{MeV}}{\mathrm{c}^2}$ (proton: 938.272 $\frac{\mathrm{MeV}}{\mathrm{c}^2}$)
z	charge number of incident particle	
Z	atomic number of absorber	
A	atomic mass of absorber	$\frac{\mathrm{g}}{\mathrm{mol}}$
K	Coefficient $4\pi N_A r_e^2 m_ec^2$	$0.307\,075\ \frac{\mathrm{MeV\,cm^2}}{\mathrm{mol}}$
r_e	classical electron radius $\frac{e^2}{4\pi\varepsilon_0 m_ec^2}$	$2.817\,940\,\mathrm{fm}$
I	mean excitation energy	eV
$\delta(\beta\gamma)$	density effect correction to ionization energy loss	
T	kinetic particle energy	$\frac{\mathrm{MeV}}{\mathrm{c}^2}$

Table 1: Summary of variables used in Bethe-Bloch equation.

γ and β from particles kinetic energy:

$$\gamma = \frac{m_0c^2+T}{m_0c^2} \tag{31}$$

$$\beta = \sqrt{1-\frac{1}{\gamma^2}} \tag{32}$$

The density effect correction term for nonconducters is given as

$$\delta(\beta\gamma) = \begin{cases} 2\ln(10)x-\overline{C}, \text{ if } x \geq x_1 \\ 2\ln(10)x-\overline{C}+a(x_1-x)^k, \text{ if } x_0 \leq x < x_1 \\ 0, \text{ if } x \leq x_0 \end{cases} \tag{33}$$

Here $x = \log_{10}(\beta\gamma)$ and $x_0, x_1, a, k, \overline{C}$ are the Sternheimer parameters, which are tabulated for the elements of interest in Table 2. No parameters for deflated lung tissue are available, H_2O's parameters are chosen as a substitute. Since the density effect correction only starts contributing at around 400 MeV of kinetic energy ($\beta\gamma \approx 1 \rightarrow x = 0 < x_0$), this has no impact as it vanishes for the energy region of interest ($<$ 250 MeV).

Material	a	k	X_0	X_1	I	ρ_0	$h\nu_p$	A	B	$\overline{C}$
PMMA	0.3996	0.2606	0.1824	2.2	74.0	1.190	23.09	0.08281	18.352	3.330
Air	0.2466	2.879	1.742	4.0	85.7	$1.205 * 10^{-3}$	0.7067	0.07664	18.058	10.595
H_2O	0.2065	3.007	0.2400	2.5	75.0	1.000	21.47	0.08523	18.325	3.502

Table 2: Density effect correction and Sternheimer parameters.

The mass stopping power ($\hat{S}$) of a mixture is given by the linear combination of its constituents with their mass fraction as the weighting factor w_i (Bragg's additivity rule [2]),. To simulate inflated lung tissue by any two material combination, their mixed mass stopping power has to be equal to that of inflated lung tissue. From this follows:

$$\hat{S}_{L,\text{model}} = w_m \hat{S}_m + (1 - w_m)\hat{S}_{air} \tag{34}$$

$$\rightarrow w_m = \frac{\hat{S}_{L,\text{model}} - \hat{S}_{air}}{\hat{S}_m - \hat{S}_{air}} = \frac{\frac{\hat{S}_{L,\text{model}}}{\hat{S}_{air}} - 1}{\frac{\hat{S}_m}{\hat{S}_{air}} - 1}, \tag{35}$$

where the mass stopping power of the inflated lung model is equated with the mass stopping power of lung tissue ($\hat{S}_{L,\text{model}} = \hat{S}_L$) given by Bragg's additivity rule using the constituents of lung tissue:

$$\hat{S}_L = w_{L,def.} \hat{S}_{L,def} + (1 - w_{L,def.})\hat{S}_{air} \tag{36}$$

$$\rightarrow w_m = w_{L,def.} \frac{\frac{\hat{S}_{L,\text{def.}}}{\hat{S}_{air}} - 1}{\frac{\hat{S}_m}{\hat{S}_{air}} - 1} \tag{37}$$

The mass fraction of lung tissue to air for inflated lung tissue is approximated by the density ratio $w_{L,def.} \approx \frac{\rho_L}{\rho_{L,def.}} = \frac{0.26\,\frac{\text{g}}{\text{cm}^3}}{1.05\,\frac{\text{g}}{\text{cm}^3}} = 0.248$.

The mass stopping power of deflated lung and air can be calculated by Bragg's additivity rule for (Z/A) using the mass fractions tabulated in Table 3:

$$\hat{S}_{L,def} = Kz^2 \left(\sum_{i \in \{H,C,N...\}} w_i \frac{Z_i}{A_i} \right) \frac{1}{\beta^2} \left(\frac{1}{2} \ln \left(\frac{2m_e c^2 \beta^2 \gamma^2 W_{\max}}{I^2} \right) - \beta^2 - \frac{\delta(\beta\gamma)}{2}) \right) \tag{38}$$

$$\hat{S}_{\text{air}} = Kz^2 \left(\sum_{i \in \{C,N,O,Ar\}} w_i \frac{Z_i}{A_i} \right) \frac{1}{\beta^2} \left(\frac{1}{2} \ln \left(\frac{2m_e c^2 \beta^2 \gamma^2 W_{\max}}{I^2} \right) - \beta^2 - \frac{\delta(\beta\gamma)}{2}) \right) \tag{39}$$

Mixture	Constituents	Mass fraction w_i	Z_i/A_i	$\sum_i w_i(Z_i/A_i)$
Lung	H	0.101278	0.9923	0.5496
	C	0.102310	0.4995	
	N	0.02865	0.4998	
	O	0.757072	0.5000	
	Na	0.001840	0.4785	
	Mg	0.000730	0.4937	
	P	0.0008	0.4843	
	S	0.002250	0.4990	
	Cl	0.002660	0.4795	
	K	0.001940	0.4860	
	Ca	0.000090	0.4990	
	Fe	0.000370	0.4656	
	Zn	0.000010	0.4589	
Air	C	0.000124	0.4995	0.4992
	N	0.755267	0.4998	
	O	0.231781	0.5000	
	Ar	0.012827	0.4506	

Table 3: Mass fractions used for Stopping power calculation of deflated lung with $\rho_{L,def.} = 1.05\,\frac{\mathrm{g}}{\mathrm{cm}^3}$, $I_{L,def.} = 75.3\,\mathrm{eV}$ and dry air (near sea level) $\rho_{air} = 1.204\,79\,\frac{\mathrm{kg}}{\mathrm{m}^3}$, $I_{air} = 85.7\,\mathrm{eV}$.

The modulation power can be calculated using the water equivalent thickness of Equation 7.

$$P_{\mathrm{mod}} := \frac{\sigma_t^2}{t} = \frac{S_{\mathrm{H_2O}}}{S_L} \cdot d \cdot w_m \cdot (1-w_m) \cdot \frac{(S_m - S_{air})^2}{S_{\mathrm{H_2O}}^2} \tag{40}$$

$$= d \cdot w_m \cdot (1-w_m) \cdot \frac{(S_m - S_{air})^2}{S_L \cdot S_{\mathrm{H_2O}}} \tag{41}$$

$$\approx d \cdot \frac{\frac{\hat{S}_L}{\hat{S}_a} - 1}{\frac{\hat{S}_m}{\hat{S}_a} - 1} \cdot \left(1 - \frac{\frac{\hat{S}_L}{\hat{S}_a} - 1}{\frac{\hat{S}_m}{\hat{S}_a} - 1}\right) \cdot \frac{S_m^2}{S_L \cdot S_{\mathrm{H_2O}}} \qquad | \; S_{air} \to 0 \tag{42}$$

$$= d \cdot \left(\frac{(\frac{\hat{S}_L}{\hat{S}_a} - 1)(\frac{\hat{S}_m}{\hat{S}_a} - \frac{\hat{S}_L}{\hat{S}_a})}{(\frac{\hat{S}_m}{\hat{S}_a} - 1)^2}\right) \cdot \frac{S_m^2}{S_L \cdot S_{\mathrm{H_2O}}} \tag{43}$$

$$= d \cdot \left(\frac{(\frac{\hat{S}_L}{\hat{S}_a} - 1)(\frac{\hat{S}_m}{\hat{S}_a} - \frac{\hat{S}_L}{\hat{S}_a})}{(\frac{\hat{S}_m}{\hat{S}_a} - 1)^2}\right) \cdot \frac{S_m^2}{S_L \cdot S_{\mathrm{H_2O}}} \tag{44}$$

To calculate the general stopping power ratio $\frac{S_i}{S_j}$, it is assumed that the density effect corrections vanish (i.e. approximately $\beta\gamma < 1 \hat{=} T < 400\,\mathrm{MeV}$ for most materials). From this follows:

$$\frac{S_i}{S_j} = \frac{\rho_i K z^2 \frac{Z_i}{A_i}\frac{1}{\beta^2}\left(\frac{1}{2}\ln\left(\frac{2m_ec^2\beta^2\gamma^2 W_{\max}}{I_i^2}\right) - \beta^2 - \frac{\delta_i(\beta\gamma)}{2}\right)}{\rho_j K z^2 \frac{Z_j}{A_j}\frac{1}{\beta^2}\left(\frac{1}{2}\ln\left(\frac{2m_ec^2\beta^2\gamma^2 W_{\max}}{I_j^2}\right) - \beta^2 - \frac{\delta_j(\beta\gamma)}{2}\right)} \tag{45}$$

$$\delta_{i,j}(\beta\gamma)\to 0 \quad = \frac{\rho_i}{\rho_j}\cdot\frac{\frac{Z_i}{A_i}\left(\ln\left(\frac{2m_ec^2\beta^2\gamma^2 W_{\max}}{I_i^2}\right) - 2\beta^2\right)}{\frac{Z_j}{A_j}\left(\ln\left(\frac{2m_ec^2\beta^2\gamma^2 W_{\max}}{I_j^2}\right) - 2\beta^2\right)} \tag{46}$$

Furthermore for protons the maximal energy transfer to an electron can be approximated by Equation 3. This is valid for $2\gamma m_e \ll M$[4], which holds for low-energy protons.

$$W_{\max} = 2m_ec^2\beta^2\gamma^2$$

Hence, the stopping power ratio can be expressed as:

$$\frac{S_i}{S_j} = \frac{\rho_i}{\rho_j}\frac{\frac{Z_i}{A_i}\left(\ln\left(\frac{4m_e^2c^4\beta^4\gamma^4}{I_i^2}\right) - 2\beta^2\right)}{\frac{Z_j}{A_j}\left(\ln\left(\frac{4m_e^2c^4\beta^4\gamma^4}{I_j^2}\right) - 2\beta^2\right)} \tag{47}$$

$$= \frac{\frac{Z_i}{A_i}\left(\ln\left(\frac{2m_ec^2\beta^2\gamma^2}{I_i}\right) - \beta^2\right)}{\frac{Z_j}{A_j}\left(\ln\left(\frac{2m_ec^2\beta^2\gamma^2}{I_j}\right) - \beta^2\right)} \tag{48}$$

$$= \frac{\frac{Z_i}{A_i}\left(\ln\left(\frac{2m_ec^2}{I_i}\right) + \ln(\frac{\beta^2}{1-\beta^2}) - \beta^2\right)}{\frac{Z_j}{A_j}\left(\ln\left(\frac{2m_ec^2}{I_j}\right) + \ln(\frac{\beta^2}{1-\beta^2}) - \beta^2\right)} \tag{49}$$

The energy dependence can be approximated by considering a first order Taylor Expansion of $\frac{A+x}{B+x}$ around 0, meaning at the 0 crossing of $\ln\left(\frac{\beta^2}{1-\beta^2}\right) - \beta^2$:

$$\frac{A+x}{B+x} = \frac{A+x}{B+x}|_{x=0} + (\frac{1}{B+x} - \frac{x+A}{(B+x)^2})|_{x=0}\cdot x + \mathrm{O}(x^2) \tag{50}$$

$$\approx \frac{A}{B}\left(1 + x\left(\frac{1}{A} - \frac{1}{B}\right)\right) \tag{51}$$

Approximating Equation 47 with Equation 51 yields:

$$\frac{S_i}{S_j} = \frac{\rho_i}{\rho_j}\frac{Z_i}{A_i}\left(\frac{Z_j}{A_j}\right)^{-1}\frac{\ln\left(\frac{2m_ec^2}{I_i}\right)}{\ln\left(\frac{2m_ec^2}{I_j}\right)}\left(1 + \left(\ln(\frac{\beta^2}{1-\beta^2}) - \beta^2\right)\left(\frac{1}{\ln\left(\frac{2m_ec^2}{I_i}\right)} - \frac{1}{\ln\left(\frac{2m_ec^2}{I_j}\right)}\right)\right) \tag{52}$$

The energy dependent term on the right side of Equation 52 has a negligible contribution to the stopping power ratio for approximately $0.05 < \beta < 0.95$ at similar mean excitation energies (cf. Figure 1). This also corresponds to the 0th order Taylor Expansion. From this follows the energy independent stopping power ratio, shown in Equation 53. In Figures 1 and 2 the density ratios deviate significantly from the stopping power ratios, even for PMMA which is usually considered similar to water regarding the mass stopping power.

$$\frac{S_i}{S_j} = \frac{\rho_i}{\rho_j}\frac{Z_i}{A_i}\left(\frac{Z_j}{A_j}\right)^{-1}\frac{\ln\left(\frac{2m_e c^2}{I_i}\right)}{\ln\left(\frac{2m_e c^2}{I_j}\right)} = \frac{\rho_i}{\rho_j}\frac{\hat{S}_i}{\hat{S}_j} \tag{53}$$

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(a) Stopping power ratio for H_2O and PMMA.

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(b) Zoom of stopping power ratio.

Figure 1: Comparision of stopping power ratio between H_2O/PMMA, where $I_{\mathrm{H_2O}} = 75\,\mathrm{eV}$ and $I_{PMMA} = 74\,\mathrm{eV}$. The approximation of Equation 53 holds for materials with a similar excitation energy. Here, Green represents Equation 49, orange is Equation 52, red is Equation 53, black is the density ratio and blue is the stopping power ratio using the database provided by PSTAR, which contains data from the ICRU.

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(a) Stopping power ratio for H_2O and and $PbWO_4$.

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(b) Zoom of stopping power ratio.

Figure 2: Comparision of stopping power ratio between H_2O/PMMA and $H_2O/PbWO_4$, where $I_{\mathrm{H_2O}} = 75\,\mathrm{eV}$ and $I_{\mathrm{PbWO_4}} = 600.7\,\mathrm{eV}$. Here, Green represents Equation 49, orange is Equation 52, red is Equation 53, black is the density ratio and blue is the stopping power ratio using the database provided by PSTAR and Geant4. A comparison mass stopping powers of Geant4 and PStar is shown in Appendix Figure 4, showing Geant4 gives accurate estimates.

From inserting Equation 53 into Equation 44 follows the energy independent modulation power:

$$P_{\text{mod}} = d \cdot \left(\frac{(\frac{\hat{S}_L}{\hat{S}_a} - 1)(\frac{\hat{S}_m}{\hat{S}_a} - \frac{\hat{S}_L}{\hat{S}_a})}{(\frac{\hat{S}_m}{\hat{S}_a} - 1)^2} \right) \cdot \frac{S_m^2}{S_L \cdot S_{\text{H}_2\text{O}}} \tag{54}$$

$$= d \cdot \frac{\rho_m^2}{\rho_L \cdot \rho_{\text{H}_2\text{O}}} \cdot \left(\frac{Z_m}{A_m}\right)^2 \left(\frac{Z_L}{A_L}\right)^{-1} \left(\frac{Z_{\text{H}_2\text{O}}}{A_{\text{H}_2\text{O}}}\right)^{-1} \frac{\ln\left(\frac{2m_ec^2}{I_m}\right)^2}{\ln\left(\frac{2m_ec^2}{I_L}\right) \cdot \ln\left(\frac{2m_ec^2}{I_{\text{H}_2\text{O}}}\right)} \tag{55}$$

$$\cdot \left(\frac{\left(\frac{Z_L}{A_L}\left(\frac{Z_a}{A_a}\right)^{-1} \frac{\ln\left(\frac{2m_ec^2}{I_L}\right)}{\ln\left(\frac{2m_ec^2}{I_a}\right)} - 1 \right)\left(\frac{Z_m}{A_m}\left(\frac{Z_a}{A_a}\right)^{-1} \frac{\ln\left(\frac{2m_ec^2}{I_m}\right)}{\ln\left(\frac{2m_ec^2}{I_a}\right)} - \frac{Z_L}{A_L}\left(\frac{Z_a}{A_a}\right)^{-1} \frac{\ln\left(\frac{2m_ec^2}{I_L}\right)}{\ln\left(\frac{2m_ec^2}{I_a}\right)} \right)}{\left(\frac{Z_m}{A_m}\left(\frac{Z_a}{A_a}\right)^{-1} \frac{\ln\left(\frac{2m_ec^2}{I_m}\right)}{\ln\left(\frac{2m_ec^2}{I_a}\right)} - 1 \right)^2} \right) \tag{56}$$

To compare this model to the density ratio model, a simulation is run with a heterogeneous target based on the models. The modulation power can then be extracted from the simulations using the resulting depth-dose distributions by using a Gaus convolution with a depth-dose distribution without a target. In this comparison the heterogeneous target is modeled via H_2O and air, resulting in Equation 56 simplifying to:

$$P_{\text{mod}} = d \cdot \frac{\rho_{\text{H}_2\text{O}}}{\rho_L} \cdot \left(\frac{Z_L}{A_L}\right)^{-1} \left(\frac{Z_{\text{H}_2\text{O}}}{A_{\text{H}_2\text{O}}}\right) \frac{\ln\left(\frac{2m_ec^2}{I_{\text{H}_2\text{O}}}\right)}{\ln\left(\frac{2m_ec^2}{I_L}\right)} \tag{57}$$

$$\cdot \left(\frac{\left(\frac{Z_L}{A_L}\left(\frac{Z_a}{A_a}\right)^{-1} \frac{\ln\left(\frac{2m_ec^2}{I_L}\right)}{\ln\left(\frac{2m_ec^2}{I_a}\right)} - 1 \right)\left(\frac{Z_{\text{H}_2\text{O}}}{A_{\text{H}_2\text{O}}}\left(\frac{Z_a}{A_a}\right)^{-1} \frac{\ln\left(\frac{2m_ec^2}{I_{\text{H}_2\text{O}}}\right)}{\ln\left(\frac{2m_ec^2}{I_a}\right)} - \frac{Z_L}{A_L}\left(\frac{Z_a}{A_a}\right)^{-1} \frac{\ln\left(\frac{2m_ec^2}{I_L}\right)}{\ln\left(\frac{2m_ec^2}{I_a}\right)} \right)}{\left(\frac{Z_{\text{H}_2\text{O}}}{A_{\text{H}_2\text{O}}}\left(\frac{Z_a}{A_a}\right)^{-1} \frac{\ln\left(\frac{2m_ec^2}{I_{\text{H}_2\text{O}}}\right)}{\ln\left(\frac{2m_ec^2}{I_a}\right)} - 1 \right)^2} \right) \tag{58}$$

The calculated parameters of both models are shown in Table 4.

Parameter	Density ratio	Stopping power ratio
Fill probability / %	0.2476	0.2251
Structure Constant / m	$\frac{P_{\text{mod}}}{0.7900}$	$\frac{P_{\text{mod}}}{0.7355}$

Table 4: Heterogeneous target model parameters.

The depth-dose distributions of both simulations are shown in Figure 3. The determined parameters are shown in Tables 5 and 6, showing an overall smaller relative and absolute deviation from the theoretical modulation power for the ASPR-model, while the density ratio model only sometimes gave better results.

It should be noted that this model only applies to materials with a higher or equal mass stopping power than lung tissue, because otherwise the mass stopping power of lung can not be calculated using Braggs additivity rule. For example $PbWO_4$ has a lower mass stopping power than lung tissue and air, even though it has a higher stopping power, resulting in a negative fill probability.

(a) Depth dose distributions of simulations using the density ratio model of H_2O and lung.

(b) Depth dose distributions of simulations using the approximated stopping power ratio model of H_2O and lung.

Figure 3: Geant4 simulations of the depth dose distribution of protons in water using heterogeneous targets with different modulation powers and thicknesses.

		Denstiy Ratio model			
$P_{\text{mod, theo.}}$ / µm	Thickness / mm	t / cm	σ_t / cm	P_{mod}/µm	Δ%
100	50	$1.244 \pm 6.75 \times 10^{-5}$	$0.091 \pm 3.70 \times 10^{-4}$	$66.190 \pm 5.40 \times 10^{-1}$	-33.81 ± 0.540
	100	$2.492 \pm 5.28 \times 10^{-5}$	$0.144 \pm 1.42 \times 10^{-4}$	$83.693 \pm 1.65 \times 10^{-1}$	-16.31 ± 0.165
	150	$3.737 \pm 4.92 \times 10^{-5}$	$0.180 \pm 1.11 \times 10^{-4}$	$87.177 \pm 1.07 \times 10^{-1}$	-12.82 ± 0.107
	200	$4.966 \pm 1.34 \times 10^{-4}$	$0.211 \pm 2.70 \times 10^{-4}$	$89.549 \pm 2.29 \times 10^{-1}$	-10.45 ± 0.229
200	50	$1.241 \pm 7.22 \times 10^{-5}$	$0.144 \pm 1.96 \times 10^{-4}$	$167.161 \pm 4.54 \times 10^{-1}$	-16.42 ± 0.227
	100	$2.487 \pm 6.01 \times 10^{-5}$	$0.210 \pm 1.21 \times 10^{-4}$	$177.627 \pm 2.04 \times 10^{-1}$	-11.19 ± 0.102
	150	$3.743 \pm 6.70 \times 10^{-5}$	$0.259 \pm 1.17 \times 10^{-4}$	$179.444 \pm 1.62 \times 10^{-1}$	-10.28 ± 0.081
	200	$4.978 \pm 1.11 \times 10^{-4}$	$0.304 \pm 1.76 \times 10^{-4}$	$186.073 \pm 2.16 \times 10^{-1}$	-6.96 ± 0.108
300	50	$1.243 \pm 6.33 \times 10^{-5}$	$0.179 \pm 1.43 \times 10^{-4}$	$256.466 \pm 4.12 \times 10^{-1}$	-14.51 ± 0.137
	100	$2.479 \pm 6.93 \times 10^{-5}$	$0.260 \pm 1.21 \times 10^{-4}$	$272.731 \pm 2.54 \times 10^{-1}$	-9.09 ± 0.085
	150	$3.729 \pm 7.48 \times 10^{-5}$	$0.317 \pm 1.17 \times 10^{-4}$	$270.147 \pm 1.99 \times 10^{-1}$	-9.95 ± 0.066
	200	$4.966 \pm 1.13 \times 10^{-4}$	$0.372 \pm 1.64 \times 10^{-4}$	$278.178 \pm 2.46 \times 10^{-1}$	-7.27 ± 0.082
400	50	$1.253 \pm 5.33 \times 10^{-5}$	$0.210 \pm 1.07 \times 10^{-4}$	$350.773 \pm 3.59 \times 10^{-1}$	-12.31 ± 0.090
	100	$2.481 \pm 1.08 \times 10^{-4}$	$0.301 \pm 1.74 \times 10^{-4}$	$364.335 \pm 4.21 \times 10^{-1}$	-8.92 ± 0.105
	150	$3.714 \pm 1.18 \times 10^{-4}$	$0.369 \pm 1.71 \times 10^{-4}$	$367.287 \pm 3.40 \times 10^{-1}$	-8.18 ± 0.085
	200	$4.968 \pm 1.08 \times 10^{-4}$	$0.428 \pm 1.47 \times 10^{-4}$	$368.260 \pm 2.53 \times 10^{-1}$	-7.94 ± 0.063
500	50	$1.247 \pm 1.09 \times 10^{-4}$	$0.234 \pm 2.03 \times 10^{-4}$	$441.028 \pm 7.64 \times 10^{-1}$	-11.79 ± 0.153
	100	$2.483 \pm 1.35 \times 10^{-4}$	$0.337 \pm 2.05 \times 10^{-4}$	$458.325 \pm 5.56 \times 10^{-1}$	-8.34 ± 0.111
	150	$3.720 \pm 1.02 \times 10^{-4}$	$0.410 \pm 1.42 \times 10^{-4}$	$452.922 \pm 3.13 \times 10^{-1}$	-9.42 ± 0.063
	200	$4.962 \pm 1.14 \times 10^{-4}$	$0.481 \pm 1.50 \times 10^{-4}$	$465.994 \pm 2.91 \times 10^{-1}$	-6.80 ± 0.058
600	50	$1.260 \pm 9.39 \times 10^{-5}$	$0.256 \pm 1.65 \times 10^{-4}$	$521.752 \pm 6.73 \times 10^{-1}$	-13.04 ± 0.112
	100	$2.485 \pm 1.49 \times 10^{-4}$	$0.366 \pm 2.17 \times 10^{-4}$	$539.547 \pm 6.39 \times 10^{-1}$	-10.08 ± 0.107
	150	$3.718 \pm 1.53 \times 10^{-4}$	$0.453 \pm 2.04 \times 10^{-4}$	$551.018 \pm 4.98 \times 10^{-1}$	-8.16 ± 0.083
	200	$4.952 \pm 1.37 \times 10^{-4}$	$0.525 \pm 1.75 \times 10^{-4}$	$557.469 \pm 3.72 \times 10^{-1}$	-7.09 ± 0.062
700	50	$1.240 \pm 1.78 \times 10^{-4}$	$0.283 \pm 2.95 \times 10^{-4}$	644.925 ± 1.35	-7.87 ± 0.192
	100	$2.490 \pm 1.93 \times 10^{-4}$	$0.401 \pm 2.70 \times 10^{-4}$	$645.249 \pm 8.71 \times 10^{-1}$	-7.82 ± 0.124
	150	$3.710 \pm 2.40 \times 10^{-4}$	$0.487 \pm 3.14 \times 10^{-4}$	$638.880 \pm 8.23 \times 10^{-1}$	-8.73 ± 0.118
	200	$4.963 \pm 1.93 \times 10^{-4}$	$0.562 \pm 2.42 \times 10^{-4}$	$636.077 \pm 5.48 \times 10^{-1}$	-9.13 ± 0.078
800	50	$1.231 \pm 1.96 \times 10^{-4}$	$0.299 \pm 3.16 \times 10^{-4}$	727.740 ± 1.54	-9.03 ± 0.192
	100	$2.504 \pm 2.16 \times 10^{-4}$	$0.428 \pm 2.95 \times 10^{-4}$	730.595 ± 1.01	-8.68 ± 0.126
	150	$3.728 \pm 2.14 \times 10^{-4}$	$0.519 \pm 2.75 \times 10^{-4}$	$723.710 \pm 7.65 \times 10^{-1}$	-9.54 ± 0.096
	200	$4.951 \pm 2.04 \times 10^{-4}$	$0.603 \pm 2.52 \times 10^{-4}$	$733.427 \pm 6.14 \times 10^{-1}$	-8.32 ± 0.077

Table 5: Parameters for the DR-model resulting from the fits of the depth dose distributions with using Gaus convolutions with linear interpolation splines between the data points. The average relative deviation from the theoretical value is 10.63% and the average absolute deviation is 42.695 µm.

		Approximated Stopping Power Ratio model			
$P_{\text{mod, theo.}}$ / µm	Thickness / mm	t / cm	σ_t / cm	P_{mod}/µm	Δ%
100	50	$1.132 \pm 4.18 \times 10^{-5}$	$0.101 \pm 1.29 \times 10^{-4}$	$90.130 \pm 2.30 \times 10^{-1}$	-9.87 ± 0.230
	100	$2.262 \pm 6.69 \times 10^{-5}$	$0.143 \pm 1.83 \times 10^{-4}$	$89.908 \pm 2.31 \times 10^{-1}$	-10.09 ± 0.231
	150	$3.391 \pm 1.22 \times 10^{-4}$	$0.179 \pm 2.75 \times 10^{-4}$	$94.898 \pm 2.91 \times 10^{-1}$	-5.10 ± 0.291
	200	$4.517 \pm 1.75 \times 10^{-4}$	$0.212 \pm 3.50 \times 10^{-4}$	$99.658 \pm 3.28 \times 10^{-1}$	-0.34 ± 0.328
200	50	$1.136 \pm 4.56 \times 10^{-5}$	$0.145 \pm 1.21 \times 10^{-4}$	$185.725 \pm 3.10 \times 10^{-1}$	-7.14 ± 0.155
	100	$2.264 \pm 7.02 \times 10^{-5}$	$0.213 \pm 1.40 \times 10^{-4}$	$199.616 \pm 2.63 \times 10^{-1}$	-0.19 ± 0.131
	150	$3.378 \pm 9.23 \times 10^{-5}$	$0.257 \pm 1.62 \times 10^{-4}$	$195.790 \pm 2.47 \times 10^{-1}$	-2.11 ± 0.124
	200	$4.522 \pm 1.44 \times 10^{-4}$	$0.303 \pm 2.31 \times 10^{-4}$	$203.075 \pm 3.09 \times 10^{-1}$	1.54 ± 0.155
300	50	$1.127 \pm 6.53 \times 10^{-5}$	$0.178 \pm 1.48 \times 10^{-4}$	$280.592 \pm 4.68 \times 10^{-1}$	-6.47 ± 0.156
	100	$2.249 \pm 1.00 \times 10^{-4}$	$0.260 \pm 1.75 \times 10^{-4}$	$299.573 \pm 4.05 \times 10^{-1}$	-0.14 ± 0.135
	150	$3.393 \pm 1.26 \times 10^{-4}$	$0.321 \pm 1.95 \times 10^{-4}$	$304.048 \pm 3.69 \times 10^{-1}$	1.35 ± 0.123
	200	$4.506 \pm 1.66 \times 10^{-4}$	$0.372 \pm 2.40 \times 10^{-4}$	$307.904 \pm 3.97 \times 10^{-1}$	2.63 ± 0.132
400	50	$1.131 \pm 8.49 \times 10^{-5}$	$0.211 \pm 1.70 \times 10^{-4}$	$393.329 \pm 6.35 \times 10^{-1}$	-1.67 ± 0.159
	100	$2.270 \pm 1.47 \times 10^{-4}$	$0.304 \pm 2.34 \times 10^{-4}$	$408.437 \pm 6.27 \times 10^{-1}$	2.11 ± 0.157
	150	$3.383 \pm 1.55 \times 10^{-4}$	$0.369 \pm 2.25 \times 10^{-4}$	$403.322 \pm 4.91 \times 10^{-1}$	0.83 ± 0.123
	200	$4.520 \pm 1.68 \times 10^{-4}$	$0.431 \pm 2.29 \times 10^{-4}$	$410.100 \pm 4.36 \times 10^{-1}$	2.53 ± 0.109
500	50	$1.130 \pm 1.27 \times 10^{-4}$	$0.233 \pm 2.37 \times 10^{-4}$	$482.549 \pm 9.79 \times 10^{-1}$	-3.49 ± 0.196
	100	$2.249 \pm 1.60 \times 10^{-4}$	$0.339 \pm 2.42 \times 10^{-4}$	$510.329 \pm 7.28 \times 10^{-1}$	2.07 ± 0.146
	150	$3.377 \pm 2.40 \times 10^{-4}$	$0.417 \pm 3.31 \times 10^{-4}$	$516.031 \pm 8.19 \times 10^{-1}$	3.21 ± 0.164
	200	$4.495 \pm 2.17 \times 10^{-4}$	$0.480 \pm 2.85 \times 10^{-4}$	$513.352 \pm 6.08 \times 10^{-1}$	2.67 ± 0.122
600	50	$1.137 \pm 1.57 \times 10^{-4}$	$0.259 \pm 2.75 \times 10^{-4}$	588.546 ± 1.25	-1.91 ± 0.208
	100	$2.252 \pm 1.97 \times 10^{-4}$	$0.368 \pm 2.87 \times 10^{-4}$	$599.885 \pm 9.35 \times 10^{-1}$	-0.02 ± 0.156
	150	$3.401 \pm 2.27 \times 10^{-4}$	$0.454 \pm 3.03 \times 10^{-4}$	$606.685 \pm 8.10 \times 10^{-1}$	1.11 ± 0.135
	200	$4.489 \pm 2.25 \times 10^{-4}$	$0.522 \pm 2.89 \times 10^{-4}$	$606.710 \pm 6.72 \times 10^{-1}$	1.12 ± 0.112
700	50	$1.146 \pm 1.96 \times 10^{-4}$	$0.285 \pm 3.24 \times 10^{-4}$	708.376 ± 1.61	1.20 ± 0.230
	100	$2.262 \pm 1.93 \times 10^{-4}$	$0.403 \pm 2.70 \times 10^{-4}$	$718.353 \pm 9.62 \times 10^{-1}$	2.62 ± 0.137
	150	$3.392 \pm 2.59 \times 10^{-4}$	$0.489 \pm 3.38 \times 10^{-4}$	$705.948 \pm 9.76 \times 10^{-1}$	0.85 ± 0.139
	200	$4.507 \pm 2.53 \times 10^{-4}$	$0.568 \pm 3.17 \times 10^{-4}$	$716.736 \pm 7.99 \times 10^{-1}$	2.39 ± 0.114
800	50	$1.129 \pm 2.25 \times 10^{-4}$	$0.306 \pm 3.60 \times 10^{-4}$	829.401 ± 1.95	3.68 ± 0.244
	100	$2.258 \pm 2.58 \times 10^{-4}$	$0.430 \pm 3.52 \times 10^{-4}$	818.239 ± 1.34	2.28 ± 0.167
	150	$3.375 \pm 3.14 \times 10^{-4}$	$0.520 \pm 4.03 \times 10^{-4}$	801.092 ± 1.24	0.14 ± 0.155
	200	$4.527 \pm 2.92 \times 10^{-4}$	$0.607 \pm 3.60 \times 10^{-4}$	$813.970 \pm 9.65 \times 10^{-1}$	1.75 ± 0.121

Table 6: Parameters for the ASPR-model resulting from the fits of the depth dose distributions with using Gaus convolutions with linear interpolation splines between the data points. The average relative deviation from the theoretical value is 2.644% and the average absolute deviation is 9.435 µm.

4 Bibliography

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5 Appendix

(a) Stopping Power of ICRU and Geant4 for H_2O.

(b) Stopping power ratio of ICRU and Geant4 data for H_2O, resulting in $\Delta \lesssim 1.5\%$.

Figure 4: Comparision of ICRU and Geant4 Stopping power data for H_2O.