Vector Differential Calculus

Peng Mei
Department of Electronic Systems
Email: mei@es.aau.dk

AALBORG UNIVERSITY
DENMARK

Learning objectives:

- **Vector and scalar functions and their fields, derivates**;
- **Gradient of a scalar field, directional derivates;**
- **Divergence of a vector field;**
- **Curl of a vector field;**

Vector and scalar functions and their fields, derivates

Vector and scalar functions and their fields:

A vector function **v**, whose values are vectors:

$$\mathbf{v} = \mathbf{v}(P) = [v_1(P), v_2(P), v_3(P)] = v_1(P)\,\boldsymbol{i} + v_2(P)\,\boldsymbol{j} + v_3(P)\,\boldsymbol{k}$$

$$\mathbf{v} = (3x + 4y)\,\boldsymbol{i} + 2y\,\boldsymbol{j} + (2x + 5y + 3z^2)\,\boldsymbol{k}$$

A vector function defines a vector field in a domain of definition.

A scalar function *f*, whose values are scalars:

$$f = f(P)$$

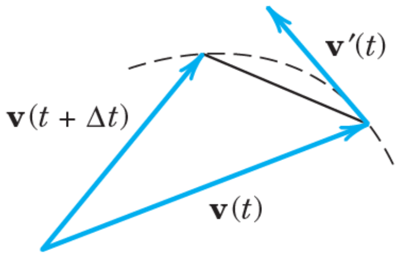
$$f(x, y, z) = 3x^2 + 2y + 4z^2$$

A scalar function defines a scalar field in that three-dimensional domain or surface or curve in space.

Derivatives of a Vector Function:

A vector function **v**(t) is said to be differentiable at a point t if the following limit exists:

$$\mathbf{v}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t}.$$



v'(t) is called the derivative of **v**(t).

In components with respect to a given Cartesian coordinate system:

$$\mathbf{v}'(t) = [v_1'(t), \quad v_2'(t), \quad v_3'(t)].$$

v'(t) is obtained by differentiating each component separately.

For instance: if $\mathbf{v} = [t, t^2, 0]$, then $\mathbf{v}' = [1, 2t, 0]$.

$$(c\mathbf{v})' = c\mathbf{v}'$$

$$(\mathbf{u} + \mathbf{v})' = \mathbf{u}' + \mathbf{v}'$$

How to prove!

$$(\mathbf{u} \cdot \mathbf{v})' = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}'$$

$$(\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'$$

$$(\mathbf{u} \quad \mathbf{v} \quad \mathbf{w})' = (\mathbf{u}' \quad \mathbf{v} \quad \mathbf{w}) + (\mathbf{u} \quad \mathbf{v}' \quad \mathbf{w}) + (\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}').$$

Partial Derivative of a Vector Function:

Suppose that the components of a vector function:

$$\mathbf{v} = [\upsilon_1, \upsilon_2, \upsilon_3] = \upsilon_1 \boldsymbol{i} + \upsilon_2\ \boldsymbol{j} + \upsilon_3\ \boldsymbol{k}$$

which is a differentiable function of n variables $t_1, \ldots, t_n$. Then the partial derivative of $\mathbf{v}$ with respect to t_m is denoted by:

$$\frac{\partial \mathbf{v}}{\partial t_m} = \frac{\partial v_1}{\partial t_m}\mathbf{i} + \frac{\partial v_2}{\partial t_m}\mathbf{j} + \frac{\partial v_3}{\partial t_m}\mathbf{k}.$$

Similarly, second partial derivatives are:

$$\frac{\partial^2 \mathbf{v}}{\partial t_l \partial t_m} = \frac{\partial^2 v_1}{\partial t_l \partial t_m}\mathbf{i} + \frac{\partial^2 v_2}{\partial t_l \partial t_m}\mathbf{j} + \frac{\partial^2 v_3}{\partial t_l \partial t_m}\mathbf{k},$$

For instance:

Let $\mathbf{r}(t_1, t_2) = a\cos t_1\,\mathbf{i} + a\sin t_1\,\mathbf{j} + t_2\,\mathbf{k}$. Then $\dfrac{\partial \mathbf{r}}{\partial t_1} = -a\sin t_1\,\mathbf{i} + a\cos t_1\,\mathbf{j}$ and $\dfrac{\partial \mathbf{r}}{\partial t_2} = \mathbf{k}$.

Common Derivatives

$$\frac{d}{dx}(x)=1$$

$$\frac{d}{dx}(\sin x)=\cos x$$

$$\frac{d}{dx}(\cos x)=-\sin x$$

$$\frac{d}{dx}(\tan x)=\sec^2 x$$

$$\frac{d}{dx}(\sec x)=\sec x\tan x$$

$$\frac{d}{dx}(\csc x)=-\csc x\cot x$$

$$\frac{d}{dx}(\cot x)=-\csc^2 x$$

$$\frac{d}{dx}(\sin^{-1} x)=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x)=-\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x)=\frac{1}{1+x^2}$$

$$\frac{d}{dx}(a^x)=a^x\ln(a)$$

$$\frac{d}{dx}(\mathbf{e}^x)=\mathbf{e}^x$$

$$\frac{d}{dx}(\ln(x))=\frac{1}{x},\quad x>0$$

$$\frac{d}{dx}(\ln|x|)=\frac{1}{x},\quad x\neq 0$$

$$\frac{d}{dx}(\log_a(x))=\frac{1}{x\ln a},\quad x>0$$

Practice exercise:

Find the first and second derivatives of $\mathbf{r} = [3 \cos 2t, 3 \sin 2t, 4t]$.

Find the first partial derivatives of $\mathbf{v_1} = [e^x \cos y, e^x \sin y]$ and $\mathbf{v_2} = [\cos x \cosh y, -\sin x \sinh y]$.

Gradient of a scalar field, directional derivates

We are given a scalar function $f(x, y, z)$ that is defined and differentiable in a domain in 3-space with Cartesian coordinates x, y, z.

$$\operatorname{grad} f = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Vector!

The gradient of $f(x, y, z)$ is itself a vector.

If $f(x,y,z) = 2y^3 + 4xz + 3x$, then grad $f = [4z + 3, 6y^2, 4x] = (4z + 3)\boldsymbol{i} + 6y^2\boldsymbol{j} + 4x\boldsymbol{k}$

∇f is suggested by the **differential operator** ∇.

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

Directional derivative:

$$\operatorname{grad} f = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

which gives the rates of change of $f(x, y, z)$ in the directions of the three coordinate axes.

How to find and calculate the rate of the change of f in an arbitrary direction in space?

$$\left(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}\right)\cdot\boldsymbol{i} = \nabla f \cdot \boldsymbol{i} = \frac{\partial f}{\partial x}$$

$$\left(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}\right)\cdot\boldsymbol{j} = \nabla f \cdot \boldsymbol{j} = \frac{\partial f}{\partial y}$$

$$\left(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}\right)\cdot\boldsymbol{k} = \nabla f \cdot \boldsymbol{k} = \frac{\partial f}{\partial z}$$

Calculate the directional derivative in an arbitrary direction, by taking the dot product of ∇f with a unit vector, $\vec{u}$, in the desired direction.

$$\boxed{D_{\vec{u}} f = \frac{\nabla f \cdot \vec{u}}{|\vec{u}|} = \nabla f \cdot \frac{\vec{u}}{|\vec{u}|}}$$

Practice exercise:

Find the directional derivative of $f(x, y, z) = 2x^2 + 3y^2 + z^2$ at P: $(2, 1, 3)$ in the direction of $\mathbf{a} = [1, 0, -2]$.

$$D_{\vec{u}}f = \frac{\nabla f \cdot \vec{u}}{|\vec{u}|} = \nabla f \cdot \frac{\vec{u}}{|\vec{u}|} = |\nabla f| \left|\frac{\vec{u}}{|\vec{u}|}\right| \cos\gamma$$

Gradient: maximum directional derivatives (direction of maximum increase)

Divergence of a Vector Field

Let **v** (x, y, z) be a differentiable vector function, where x, y, z are Cartesian coordinates , and let $\upsilon_1, \upsilon_2, \upsilon_3$ be the components of **v**. Then the function

$$\operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Scalar!

which is called the divergence of **v**.

For instance:

$$\mathbf{v} = [3xz,\ 2xy,\ -yz^2] = 3xz\mathbf{i} + 2xy\,\mathbf{j} - yz^2\mathbf{k},$$

$\upsilon_1 = 3xz$

$\upsilon_2 = 2xy$

$\upsilon_3 = -yz^2$

$$\operatorname{div} \mathbf{v} = 3z + 2x - 2yz.$$

$$\operatorname{div} \mathbf{v} = \nabla g \mathbf{v} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] g [\upsilon_1, \upsilon_2, \upsilon_3]$$

$$= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \bullet (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z},$$

Let $f(x, y, z)$ be a twice differentiable scalar function, then its gradient exists:

$$\mathbf{v} = \operatorname{grad} f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

If divergence is applied to **v**:

$$\operatorname{div} \mathbf{v} = \operatorname{div}(\operatorname{grad} f) = \boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.} \Longrightarrow \nabla^2 f \quad \boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$$

Laplacian Operator

Laplacian Operator in different coordinate systems:

In the Cartesian coordinate:

$$\nabla^2 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

In the Cylindrical coordinate:

$$\nabla^2 = \frac{1}{\rho}\frac{\partial^2}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

In the Spherical coordinate:

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

Practice exercise:

Find div $\mathbf{v}$ and its value at P.

$$\mathbf{v} = [x^2, 4y^2, 9z^2], \quad P: (-1, 0, \tfrac{1}{2}]$$

$$\mathbf{v} = [0, \cos xyz, \sin xyz], \quad P: (2, \tfrac{1}{2}\boldsymbol{\pi}, 0]$$

Calculate $\nabla^2 f$

$$f = \cos^2 x + \sin^2 y$$

$$f = e^{xyz}$$

Curl of a Vector Field

Let **v** $(x, y, z) = [\upsilon_1, \upsilon_2, \upsilon_3] = \upsilon_1\boldsymbol{i} + \upsilon_2\boldsymbol{j} + \upsilon_3\boldsymbol{k}$ be a differentiable vector function of the Cartesian coordinates x, y, z. Then the **curl** of the vector function **v** or of the vector field given by **v** is defined by the "symbolic" determinant:

$$\operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Vector!

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}.$$

For instance:

Let $\mathbf{v} = [yz, \quad 3zx, \quad z] = yz\mathbf{i} + 3zx\mathbf{j} + z\mathbf{k}$ with right-handed x, y, z.
$$\operatorname{curl} \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & 3zx & z \end{vmatrix} = -3x\mathbf{i} + y\mathbf{j} + (3z - z)\mathbf{k} = -3x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}.$$

The interrelation among Gradient (Grad), Divergence (Div), Curl:

Gradient fields are irrotational. If a continuously differentiable vector function is the gradient of a scalar function *f*, then its curl is the zero vector:

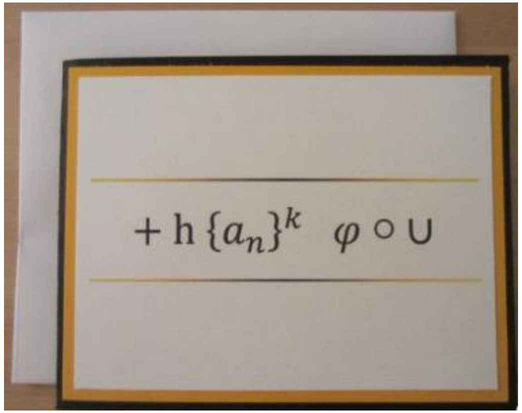
$$\operatorname{curl}(\operatorname{grad} f) = \mathbf{0}.$$

How to prove!

The divergence of the curl of a twice continuously differentiable vector function **v** is zero:

$$\operatorname{div}(\operatorname{curl} \mathbf{v}) = 0.$$

How to prove!


$+ \mathrm{h} \{a_n\}^k \quad \varphi \circ \cup$