POWER SERIES

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Today's agenda

- Power series
  - Convergence
  - Radius of convergence
  - Functions given by power series
- Taylor and McLaurin series

- Exercises!

Power series

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A **power series** *in powers of* $z - z_0$ is a series of the form

(1) $$\sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

where z is a complex variable, $a_0, a_1, \cdots$ are complex (or real) constants, called the **coefficients** of the series, and z_0 is a complex (or real) constant, called the **center** of the series. This generalizes real power series of calculus.

If $z_0 = 0$, we obtain as a particular case a *power series in powers of* z:

(2) $$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots .$$

Power series - Convergence

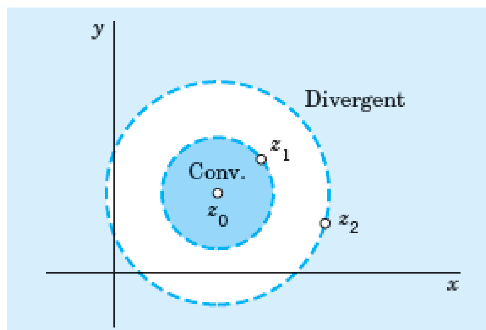
- If we fix z, all the concepts for series with constant terms apply.

Convergence of a Power Series

(a) *Every power series (1) converges at the center* z_0.

(b) *If* (1) *converges at a point* $z = z_1 \neq z_0$, *it converges absolutely for every* z *closer to* z_0 *than* z_1, *that is,* $|z - z_0| < |z_1 - z_0|$. See Fig. 365.

(c) *If* (1) *diverges at* $z = z_2$, *it diverges for every* z *farther away from* z_0 *than* z_2. See Fig. 365.

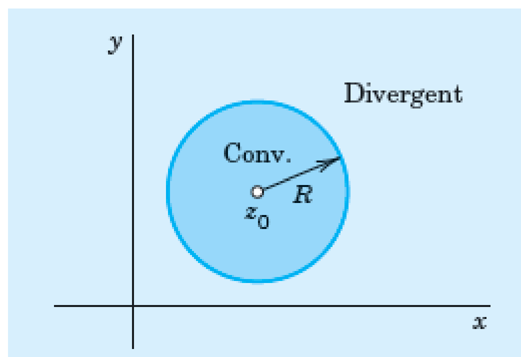


Power series – Radius of convergence

- We consider the ***smallest*** circle with center that includes all the points at which a given power series converges. Let *R* denote its radius. The circle

$$|z - z_0| = R$$

is the circle of convergence and R is the radius of convergence.



Power series – Radius of convergence

Radius of Convergence R

Suppose that the sequence $|a_{n+1}/a_n|$, $n = 1, 2, \cdots$, *converges with limit* L. *If* $L = 0$, *then* $R = \infty$; *that is, the power series* (1) *converges for all* z. *If* $L \neq 0$ (*hence* $L > 0$), *then*

(6) $$R = \frac{1}{L} = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$ **(Cauchy–Hadamard formula[1]).**

If $|a_{n+1}/a_n| \to \infty$, *then* $R = 0$ (*convergence only at the center* z_0).

Functions given by power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots \qquad (|z| < R).$$

- f(z) is represented by the power series.
- A function f(z) cannot be represented by two different power series with the same center → uniqueness of a power series representation
- A derived series of a power series is obtained by termwise differentiation.

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = a_1 + 2a_2 z + 3a_3 z^2 + \cdots .$$

- The derived series of a power series has the same radius of convergence as the original series.
- The integrated series of a power series has the same radius of convergence as the original series.

Taylor and Maclaurin series

- A Taylor series of a function f(z) is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \text{where} \qquad a_n = \frac{1}{n!} f^{(n)}(z_0)$$

- A Maclaurin series is a Taylor series with center $z_0=0$.

$$f(z) = f(z_0) + \frac{z - z_0}{1} f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0) + \cdots + \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) + R_n(z).$$

remainder of the Taylor series

Taylor and Maclaurin series

- Geometric series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$$

- Exponential function

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \cdots.$$

- Trigonometric function

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2} + \frac{z^4}{4} - + \cdots$$

- Logarithm

$$\operatorname{Ln}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - + \cdots$$