Lineær algebra – 7
Nonlinear and inhomogeneous systems

Lineær algebra og dynamiske systemer

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Non linear systems

Consider a non-linear homogeneous system (1st order, $n=2$)

$$\bar{y}' = f(t, \bar{y}) = \bar{f}(\bar{y}) = \begin{bmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{bmatrix}$$

autonomous system

where f is non linear in y_1, y_2.

We are interested in the stability of the system at a critical point P_0

Linearization

Since P_0 is a critical point, per def. $f_1(0,0) = f_2(0,0) = 0$ and we can make an approximation at this point by retaining the first order (linear) terms (in $\bar{\bar{A}}$):

$$\bar{y}'\big|_{P_0} = \bar{f}(\bar{y}_{P_0}) + \bar{\bar{A}}(\bar{y} - \bar{y}_{P_0}) + \bar{h}(\bar{y})\big|_{P_0} \approx \bar{\bar{A}}\,\bar{y}$$

higher order terms

Linearization (cont.)

The linearization implies that h_1, h_2 terms should be small, with f_1, f_2 and their partial derivatives continuous in a neighbourhood of P_0.

If $\det(\bar{A}) \neq 0$ the stability is determined by the stability of the linearized system at P_0, except when $D = 0$ (equal eigenvalues) and $p = 0, q > 0$ (imaginary eigenval.)

Linearization procedure

non-linear system (model)

$$y^{(n)}(t) = f(y^{(n-1)}(t), \ldots, y(t), r(t))$$

stationary point (often $y^{(i)}(t) = 0$)

or generally, critical point

linearization (Taylor of nonlinear f)

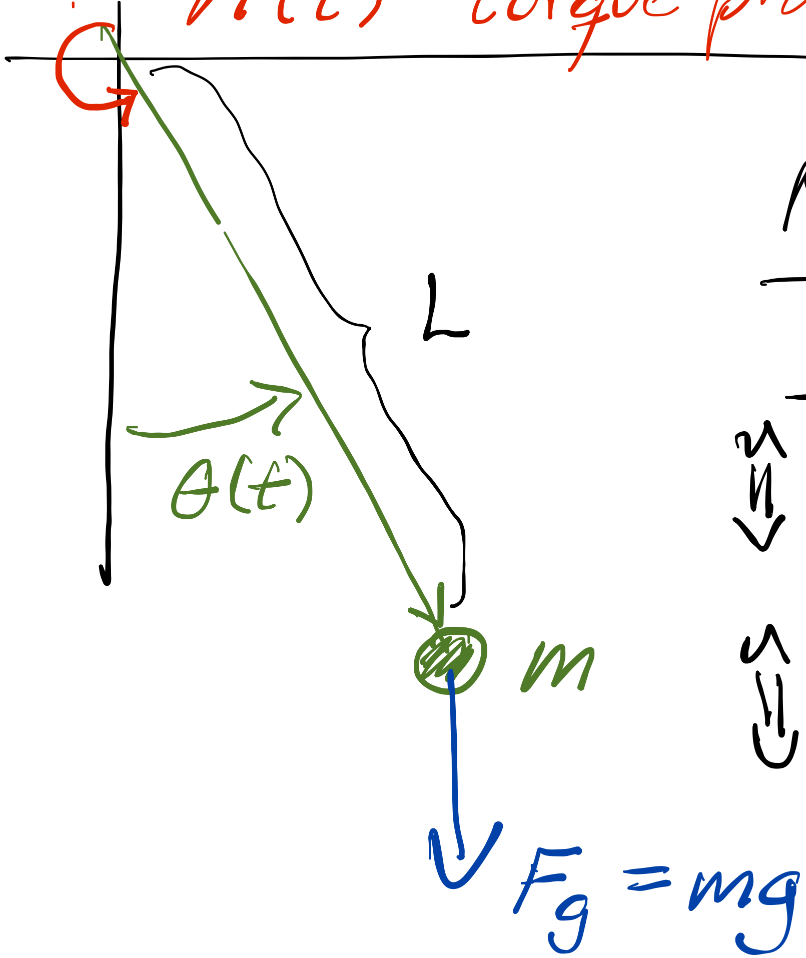
(new delta-variables)

linear model

(Laplace and transfer function)

Example (pendulum)

O $M(t)$ torque produced by motor at Origo



Newton's 2nd law (rotating)

$$\bar{I}\,\bar{\theta}''(t) = \sum \bar{\tau}_{ext}(t) \quad (\bar{\tau} = \bar{r} \times \bar{F})$$

$$\Updownarrow$$

$$mL^2\bar{\theta}'' = \bar{M} + \bar{L} \times \bar{F}_g$$

$$\Updownarrow$$

$$mL^2\theta'' = M - \underline{mgL\sin\theta}$$

Example pendulum (cont.)

Nonlinear system $\theta'' = \frac{1}{mL^2}M - \frac{g}{L}\sin\theta$

stationary point $\theta'' = \frac{1}{mL^2}M_0 - \frac{g}{L}\sin\theta_0 = 0$

thus $\theta_0 = \sin^{-1}\left(\frac{M_0}{mgL}\right)$

Linearization (around stationary point)

$$\theta(t) = \theta_0 + \Delta\theta(t)$$
$$\theta''(t) = 0 + \Delta\theta''(t)$$
$$M(t) = M_0 + \Delta M(t)$$

Linearization procedure

y $y^{(i)}$

Definition 4 *A differential equation is said to be nonlinear if any of its terms are nonlinear in the dependent variable, or any of the derivatives of the dependent variable with respect to the independent variable.*

t

Definition 5 *A system is said to be in a state of equilibrium if and only if its dependent variable is invariant with respect to time. In other words, all terms which are explicit functions of time, and all terms which are time-derivatives of the dependent variable must vanish.*

a critical point

to locate the equilibrium point(s) of a system, set all time-derivatives of the dependent variable to zero, and ignore all explicit functions of time. Then, solve the resulting nonlinear equation for the equilibrium value of the dependent variable. Once the equilibrium point has been determined, the nonlinear terms may then be linearized about this point, and substituted back into the original equation [redacted] If the equation [redacted] is linearized about an equilibrium point, any constant forcing terms will cancel out. If the equation [redacted] is linearized about any other point, there will still be some constant forcing terms in the equation. This makes sense if you realize that added force is required to hold a mechanism in any position other than an equilibrium position.

Example pendulum (cont.)

The nonlinear function: $\theta'' = f(\theta', \theta, t)$

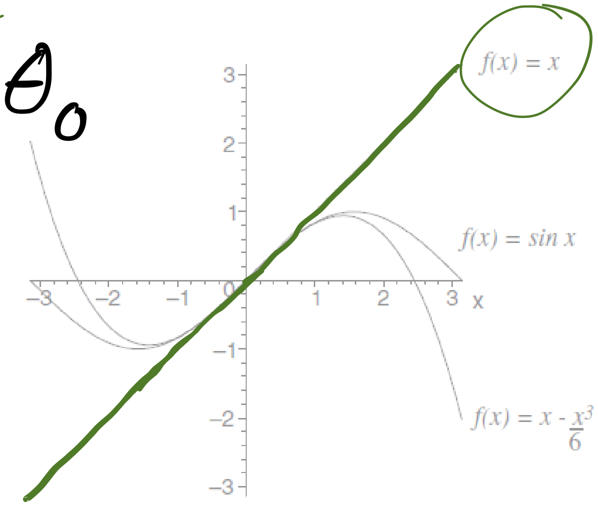
$$f(\theta) \approx f(\theta_0) + \frac{\partial f}{\partial \theta}\Big|_{\theta=\theta_0} \underbrace{(\theta - \theta_0)}_{\Delta\theta}$$

there's only one dependency here

$$\Delta\theta'' \approx \frac{1}{mL^2}(M_0 + \Delta M) - \frac{g}{L}\sin\theta_0 - \frac{g}{L}\cos\theta_0 \,\Delta\theta$$

@ stationary point

$$\Delta\theta'' = \frac{1}{mL^2}\Delta M - \frac{g}{L}\cos\theta_0 \,\Delta\theta$$



Example pendulum (cont.)

Compare to Kreyszig ($y_1 = \Delta\theta$, $y_2 = \Delta\theta'$)

$$y_1' = f_1(y_1, y_2) = y_2 \qquad \underline{\text{CASE } M=0}$$

$$y_2' = f_2(y_1, y_2) = -\frac{g}{L}\cos\theta_0 \cdot y_1$$

Critical points in nonlinear function

occur for $y_1' = y_2 = \theta' = 0$ and $y_2' = \theta'' =$

$$-\frac{g}{L}\sin\theta_0 = 0 \iff \theta_0 = n\pi,\ n \in \mathbb{Z}.$$

Example pendulum (cont.)

When we consider n even:

$$\bar{y}' = \bar{\bar{A}}\,\bar{y} = \begin{bmatrix} 0 & 1 \\ -g/L & 0 \end{bmatrix} \bar{y}$$

n even ∇ o

$\cos\theta_0 = 1$

We get

$$\left.\begin{aligned} p &= \operatorname{trace}(\bar{\bar{A}}) = 0 \\ q &= \det(\bar{\bar{A}}) = g/L > 0 \\ D &= p^2 - 4q = -4g/L < 0 \end{aligned}\right\}$$

Stable center

Example pendulum (cont.)

Looking at the points when n is odd we shift ($n=1$): $\tilde{\theta} = \theta - \theta_0$, $\theta_0 = \pi$

$\Delta\tilde{\theta}' = \Delta\theta' = y_2$ still $\Delta\theta$ but around $\theta_0 = \pi$

$\theta_0 = \pi$ gives $\cos\theta_0 = -1$, thus

$$\bar{y}' = \begin{bmatrix} 0 & 1 \\ g/L & 0 \end{bmatrix} \bar{y}$$

n odd !

unstable

saddle point

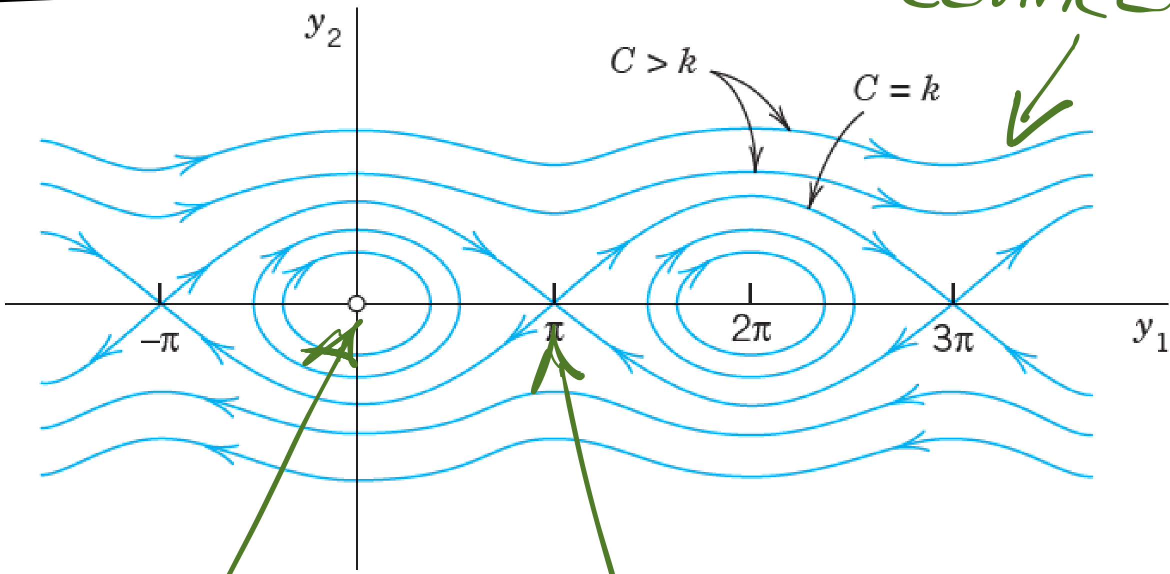
$p = 0$, $q = -g/L < 0$

Example pendulum (cont.)

Undamped case

$C \propto$ energy

$k = g/L$



"rotations" — connects critical points (we can fall in any of them)

center with stable osc.

$(C = -k)$

saddle with unstable osc.

$(C = k)$

Example pendulum (cont.)

With damping we get (c damping coefficient)

$c > 0$

$$\Delta\theta' = y_1' = f_1(y_1, y_2) = y_2$$

$$\Delta\theta'' = y_2' = f_2(y_1, y_2) = -g/L \cdot \cos\theta_0 \cdot y_1 - c\, y_2$$

Still, critical points at $(y_1', y_2') = (0, 0)$

<u>n even</u> $\cos\theta_0 = 1$

$$\bar{y}' = \begin{bmatrix} 0 & 1 \\ -g/L & -c \end{bmatrix} \bar{y}$$

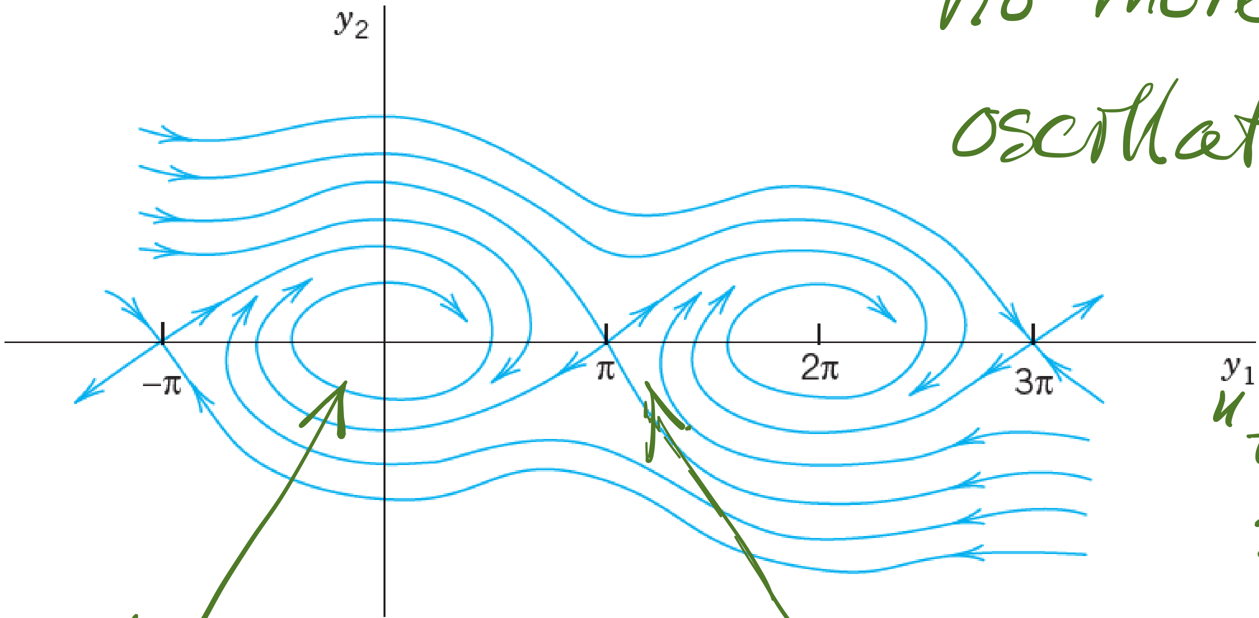
<u>n odd</u> $\cos\theta_0 = -1$

$$\bar{y}' = \begin{bmatrix} 0 & 1 \\ g/L & -c \end{bmatrix} \bar{y}$$

Example pendulum (cont.)

Damped case

no more steady oscillations



"turns into stable decr. osc."

spiral with damped (decreasing) oscillation

$p<0,\ q>0,\ D=c^2-4g/L$

saddle with unstable oscillations, still, but

$p<0,\ q<0$

Nonhomogeneous systems

Systems with a forcing function

$$\bar{y}'(t) = \bar{\bar{A}}(t)\,\bar{y}(t) + \bar{g}(t)$$

the general solution of which is

$$\bar{y}(t) = \bar{y}_h(t) + \bar{y}_p(t)$$

on interval $\bar{I}$: $\alpha < t < \beta$

Nonhomogeneous systems (cont.)

If $\bar{\bar{A}}(t) = \bar{\bar{A}}$ (constant coeff.)

$\hookrightarrow$ method of undetermined coef.
(special forcing functions)

If $\bar{\bar{A}}(t)$ and general $\bar{g}(t)$

$\hookrightarrow$ method of variation of parameters

Undetermined coeff. (y_p)

We check $\bar{g}(t)$ for standard form and apply exponential, polynomial, trigonometric function, combinations, ...

In case $\bar{g}$ contains $e^{\lambda t}$, λ being an eigenvalue of $\bar{\bar{A}}$ we try

$$\bar{y}_p = \bar{u}\, t\, e^{\lambda t} + \bar{v} e^{\lambda t}$$

standard modification

extra term!

Undetermined coeff.

... being a double eigenvalue of $\bar{\bar{A}}$, then

try $\bar{y}_p = \bar{u}t^2 e^{\lambda t} + \bar{v}te^{\lambda t} + \bar{w}e^{\lambda t}$

(similar to example of degenerate node)

Generally, $\bar{y}_p$ contains vector coefficients since $\bar{g}$ is a vector.

Example from Kreyszig

Variation of param. (y_p)

We need to know a (general) solution of the homogeneous system, $\bar{y}_h$

$$\bar{y}_h = c_1 \bar{y}_1 + \dots c_n \bar{y}_n$$

fundamental matrix

$$= [\bar{y}_1, \bar{y}_2, \dots \bar{y}_n]\, \bar{c} = \bar{\bar{Y}}(t)\, \bar{c}$$

We replace $\bar{c}$ (constant) by $\bar{u}(t)$

$$\underline{\bar{y}_p = \bar{\bar{Y}}(t)\, \bar{u}(t)}$$

$W(\bar{\bar{Y}}(t)) \neq 0$ on I
(Wronskian)

Variation of param. (cont.)

Substitution yields ($\bar{y}'(t) = \bar{\bar{A}}(t)\bar{y}(t) + \bar{g}(t)$)

$$\bar{y}_p' = \underline{\bar{\bar{Y}}'\bar{u} + \bar{\bar{Y}}\bar{u}' = \bar{\bar{A}}\bar{\bar{Y}}\bar{u} + \bar{g}}$$

But $\bar{\bar{Y}}' = \bar{\bar{A}}\bar{\bar{Y}}$, thus

$$\bar{\bar{Y}}\bar{u}' = \bar{g}$$

Since $\det(\bar{\bar{Y}}) \neq 0$ (Wronskian),

$\bar{\bar{Y}}^{-1}$ exists, therefore

$$\bar{\bar{Y}}^{-1}\bar{\bar{Y}}\bar{u}' = \bar{u}' = \bar{\bar{Y}}^{-1}\bar{g}$$

Variation of param. (cont.)

$$\bar{u}' = \bar{\bar{Y}}^{-1}\,\bar{g} \quad , \quad [t_0 \ldots t] \in \bar{I}$$

$$\Updownarrow$$

$$\bar{u}(t) = \int_{t_0}^{t} \bar{\bar{Y}}^{-1}(\tau)\,\bar{g}(\tau)\,d\tau$$

$$\bar{y}(t) = \bar{y}_h(t) + \bar{y}_p(t)$$

$$= \bar{\bar{Y}}(t)\left(\bar{c} + \int_{t_0}^{t} \bar{\bar{Y}}^{-1}(\tau)\,\bar{g}(\tau)\,d\tau\right)$$

integrate component-wise

Diagonalization

$$\bar{y}'(t) = \bar{\bar{A}}(t)\,\bar{y}(t) + \bar{g}(t)$$

e.g. normal matrix

If $\bar{\bar{A}}$ has an eigenbasis, then $\bar{\bar{D}} = \bar{\bar{X}}^{-1}\bar{\bar{A}}\,\bar{\bar{X}}$
where $\bar{\bar{D}} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots \lambda_n)$ and $\bar{\bar{X}}$ is a constant with eigenvectors of $\bar{\bar{A}}$, then defining $\bar{z} = \bar{\bar{X}}^{-1}\bar{y} \iff \bar{y} = \bar{\bar{X}}\,\bar{z}$ we get

$$\bar{y}' = \bar{\bar{X}}\,\bar{z}' = \bar{\bar{A}}\,\bar{\bar{X}}\,\bar{z} + \bar{g}$$

$\bar{\bar{X}}$ is a constant

$$\Updownarrow$$

$$\bar{\bar{X}}^{-1}\bar{\bar{X}}\,\bar{z}' = \bar{\bar{X}}^{-1}\bar{\bar{A}}\,\bar{\bar{X}}\,\bar{z} + \bar{\bar{X}}^{-1}\bar{g}$$

$$\Updownarrow$$

$$\bar{z}' = \bar{\bar{D}}\,\bar{z} + \bar{\bar{X}}^{-1}\bar{g}$$

Diagonalization (cont.)

or, with $\bar{h} = \bar{\bar{X}}^{-1}\bar{g}$,

$$\bar{z}'(t) = \bar{\bar{D}}\,\bar{z}(t) + \bar{h}(t)$$

$$z_j'(t) = \lambda_j z_j(t) + h_j(t), \quad 1 \le j \le n$$

first order linear ("Panserformel")

$$m_j(t) = \int p(t)\,dt = \int -\lambda_j\,dt = -\lambda_j t$$

$(\bar{y} = \bar{\bar{X}}\,\bar{z})$

$$z_j(t) = e^{\lambda_j t}\int e^{-\lambda_j t} h_j(t)\,dt + c_j e^{\lambda_j t}$$

Linear state space models

$$\dot{\bar{x}}(t) = \bar{\bar{A}}\,\bar{x}(t) + \bar{\bar{B}}\,u(t)$$

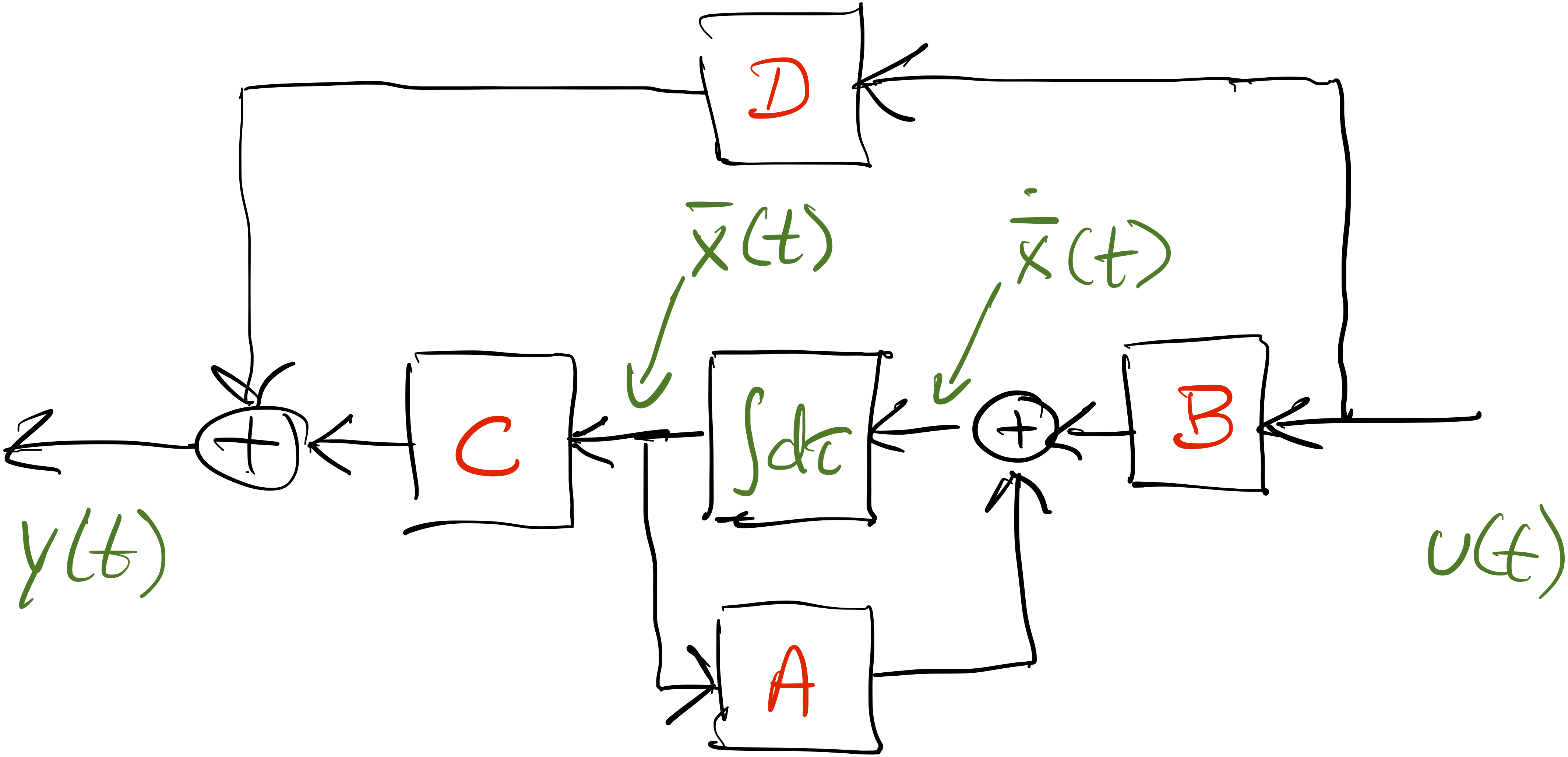
state   input

$$y(t) = \bar{C}\,\bar{x}(t) + D\,u(t)$$

output   (feedthrough)

Linear time-invariant (autonomous) first order vector differential equ. with initial condition $\bar{x}(t) = \bar{x}_0$.

State space (graphical)



Matrix exp function

$$\exp(\bar{\bar{A}}) = e^{\bar{\bar{A}}} = \sum_{n=0}^{\infty} \frac{1}{n!} \bar{\bar{A}}^n \quad \left(\frac{1}{0!}\bar{\bar{A}}^0 = \bar{\bar{I}}\right)_0^?$$

If $\bar{\bar{X}}$ is invertible (eigenbasis of $\bar{\bar{A}}$)

$$\bar{\bar{A}} = \bar{\bar{X}}^{-1} \bar{\bar{D}} \bar{\bar{X}}, \quad \bar{\bar{D}} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$

$$\bar{\bar{A}}^n = \bar{\bar{X}}^{-1} \bar{\bar{D}}^n \bar{\bar{X}}$$

$$\Downarrow$$

$$e^{\bar{\bar{A}}} = \bar{\bar{X}}^{-1} e^{\bar{\bar{D}}} \bar{\bar{X}}$$

Matrix exponential (cont.)

$$\bar{\bar{X}}(t) = \bar{\bar{A}}\, t \quad \Rightarrow$$

$$e^{\bar{\bar{X}}(t)} = e^{\bar{\bar{A}} t} = \sum_{n=0}^{\infty} \frac{\bar{\bar{A}}^n}{n!} t^n$$

$$\Downarrow$$

$$\frac{d e^{\bar{\bar{A}} t}}{dt} = \bar{\bar{A}}\, e^{\bar{\bar{A}} t} = e^{\bar{\bar{A}} t}\, \bar{\bar{A}} \quad \text{as usual}$$

Also,

$$e^{-\bar{\bar{X}}} e^{\bar{\bar{X}}} = \bar{\bar{X}}^0 = \bar{\bar{I}}$$

Solution for state vector

$$\dot{\bar{x}}(t) = \bar{\bar{A}}\,\bar{x}(t) + \bar{\bar{B}}\,u(t) \; ; \; \bar{y}(t) = \bar{\bar{C}}\,\bar{x}(t) + Du(t)$$

$$\Updownarrow$$

$$\underbrace{e^{-\bar{\bar{A}}t}\,\dot{\bar{x}}(t) - e^{-\bar{\bar{A}}t}\,\bar{\bar{A}}\,\bar{x}(t)}_{\text{chain rule}} = e^{-\bar{\bar{A}}t}\,\bar{\bar{B}}\,u(t)$$

$$\Updownarrow$$

$$\frac{d}{dt}\left(e^{-\bar{\bar{A}}t}\,\bar{x}(t)\right) = e^{-\bar{\bar{A}}t}\,\bar{\bar{B}}\,u(t)$$

$$\Updownarrow$$

$$\int_0^t d\left(e^{-\bar{\bar{A}}\tau}\,\bar{x}(\tau)\right) = \int_0^t e^{-\bar{\bar{A}}\tau}\,\bar{\bar{B}}\,u(\tau)\,d\tau$$

Solution for state vector (cont.)

$$\int_0^t d\left(e^{-\bar{\bar{A}}\tau}\bar{x}(\tau)\right) = e^{-\bar{\bar{A}}t}\bar{x}(t) - \bar{\bar{I}}\bar{x}_0$$

$$= \int_0^t e^{-\bar{\bar{A}}\tau}\bar{\bar{B}}u(\tau)d\tau$$

$\Updownarrow$

$$\bar{x}(t) = e^{\bar{\bar{A}}t}\left(\bar{x}_0 + \int_0^t e^{-\bar{\bar{A}}\tau}\bar{\bar{B}}u(\tau)d\tau\right)$$

$$= e^{\bar{\bar{A}}t}\bar{x}_0 + \int_0^t e^{\bar{\bar{A}}(t-\tau)}\bar{\bar{B}}u(\tau)d\tau$$

Impulse response

$$y(t) = \bar{c}\left(e^{\bar{\bar{A}}t}\bar{x}_0 + \int_0^t e^{\bar{\bar{A}}(t-\tau)}\bar{\bar{B}}u(\tau)d\tau\right) + Du(t)$$

The system's response is generally

$$y(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau$$

where $h(t)$ is the system's impulse response (to input $\delta(t)$):   "by inspection"

$$h(t) = \bar{c}\, e^{\bar{\bar{A}}t}\bar{\bar{B}} + D\,\delta(t), \quad t \geq 0$$

Numerically, we may approximate $e^{\bar{\bar{A}}t}$ and integrate (convolve)