Lineær algebra – 3
Egenværdiproblemer og lineære transformationer

Lineær algebra og dynamiske systemer

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Linear transformations

A linear transformation is a mapping F from vector space X to space Y

$$\bar{x} \in X \qquad \bar{y} = F(\bar{x}) \qquad \bar{y} \in Y$$

"image"

$$\bar{\bar{A}}\,\bar{x} = \bar{y}$$

$$\bar{x} = x_1\bar{e}_1 + \dots x_n\bar{e}_n \qquad F(\bar{x}) = x_1F(\bar{e}_1) + \dots x_nF(\bar{e}_n)$$

$\bar{e}_i$ $(n\times 1)$ basis $\qquad$ $\bar{\varepsilon}_i = F(\bar{e}_i)$ $(m\times 1)$ basis

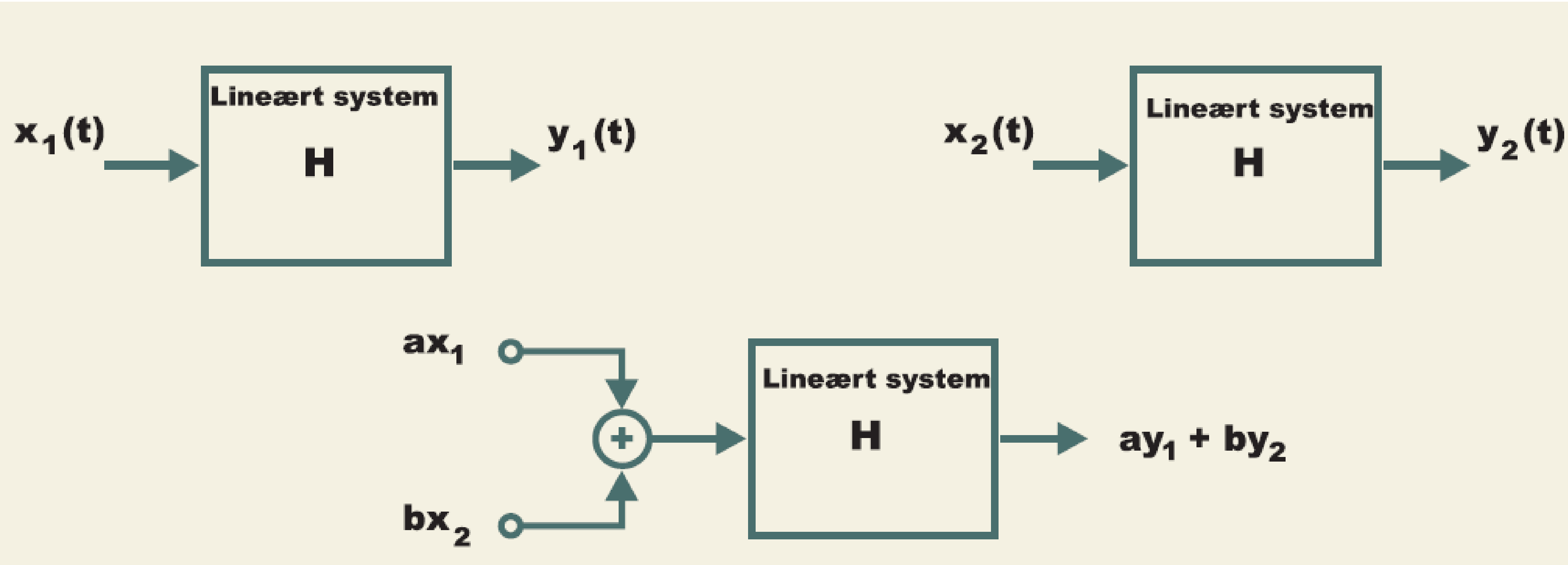
Linear transformation (cont.)

On the left we have $\bar{x}$ specified in terms of the basis $\bar{e}_i$, e.g. standard basis, in the vector space X, and on the right on the basis $\bar{\varepsilon}_i$ in vector space Y given by the columns of $\bar{\bar{A}}$.

We replace one object of study, $\bar{x}$, with another, $\bar{y}$, on another basis

Linear systems (general)

Superposition and Homogeneity
(proportionality)



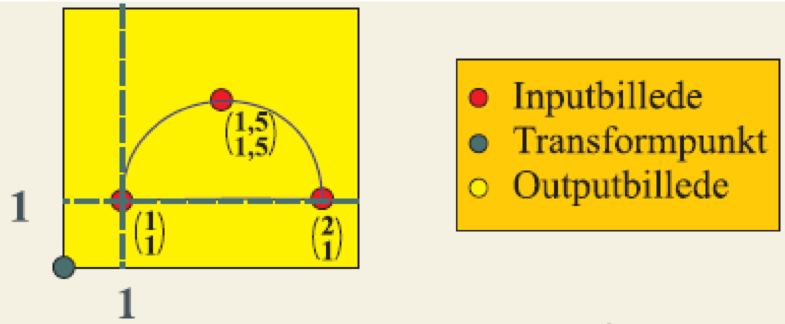
Linear transformations

If $\bar{\bar{A}}$ is square $(n \times n)$ and non-singular, $\bar{\bar{A}}^{-1}$ is defined, thus

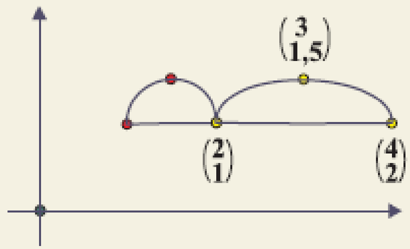
$$\bar{x} = F^{-1}(\bar{y}) = \bar{\bar{A}}^{-1}\bar{y}$$

An orthogonal $(n \times n)$ matrix is one such, $\bar{\bar{A}}^{-1} = \bar{\bar{A}}^{T}$, i.e. a rotation in 2D and 3D Euclidean space. It preserves the inner product (length).

Example linear transformations

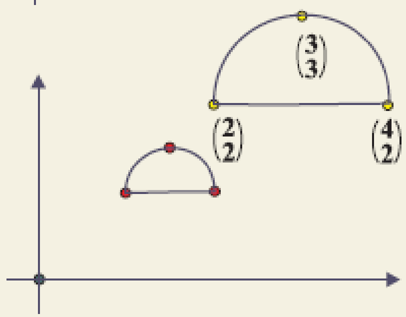


$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow$$



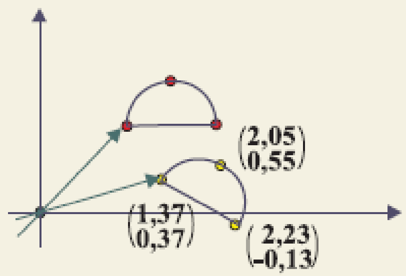
Stræk 2 gange i x-retningen

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \longrightarrow$$



Skaler 2 gange
(2 gange i x og 2 gange i y)

$$\frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix} \longrightarrow$$



Drej -30° (omkring trans-
formpunktet)

Homogeneous coordinates

projective coordinates

($ah = h$, a scalar)

9.1.5 Combining the Transformations

The four transformations can be combined in all kinds of different ways by multiplying the matrices in different orders, yielding a number of different transformations. One is shown in figure 9.1(f). Instead of defining the scale factors, the shearing factors and the rotation angle, it is common to merge these three transformation to one matrix. The combination of the four transformations is therefore defined as:

$$\begin{aligned} x' &= a_1 \cdot x + a_2 \cdot y + a_3 \\ y' &= b_1 \cdot x + b_2 \cdot y + b_3 \end{aligned} \Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a_3 \\ b_3 \end{bmatrix} \tag{9.8}$$

and this is the affine transformation. Below the relationships between equation 9.8 and the four above mentioned transformations are listed.

	a_1	a_2	a_3	b_1	b_2	b_3
Translation	1	0	Δ_x	0	1	Δ_y
Scaling	S_x	0	0	0	S_y	0
Rotation	$\cos\theta$	$-\sin\theta$	0	$\sin\theta$	$\cos\theta$	0
Shearing	1	B_x	0	B_y	1	0

Often *homogeneous coordinates* are used when implementing the transformation since they make further calculations faster. In homogeneous coordinates, the affine transformation becomes:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{9.9}$$

where $a_3 = \Delta x$ and $b_3 = \Delta y$.

"Introduction to Video and Image Processing", Thomas B. Moeslund, Springer, 2012, ISBN13: 9781447125020