Lineær algebra – 1
Introduction

Lineær algebra og dynamiske systemer

Troels B. Sørensen /Gilberto Berardinelli

Welcome

- **Two course lecturers**
  - **Troels B. Sørensen (first part),** tbs@es.aau.dk
  - **Gilberto Berardinelli (second part),** gb@es.aau.dk

- **Course information**
  - **Curricula https://moduler.aau.dk/course/2023-2024/ESNESDB4K2?lang=en-GB**
  - **Lecture information – reading, exercises, slides, .. on Moodle course page**

- **Two lectures per week**
  - **For about five weeks**

Literature

- **Books**
  - **"Advanced Engineering Mathematics", 10th ed., 2011 af Erwin Kreyszig, ISBN: 978-0-470-64613-7 (part 1)**
  - **"Discrete-Time Signal Processing", 2nd (Prenctice-Hall, Inc. 1999) or 3rd edition (Pearson 2014), A. V. Oppenheim and R. W. Schafer (part 2)**
  - **Older versions will do as well, but you need to ”translate” - topics are quite standard in engineering**

Course plan

- **Linear Algebra (about 50% of the course)** Tvels
  - **Matrix basics, determinants, rank, linear independence, Gauss elimination**
  - **Linear systems of equations, vector spaces and basis vectors, linear transformations, some numerical considerations**
  - **Eigenvalue problems, vector bases, numerical methods**
  - **Taylor and MacLaurin series** (Gilberto)
  - **Systems of differential equations, eigenvalue problems, stability**
  - **Systems of differential equations, linearization**

- **Discrete LTI systems (about 50% of the course)** Gilberto
  - **Continuous to discrete domain (link to previous courses)**
    - **Linearity and time invariance, impulse response and convolution, stability and causality**
  - **The z-transform**
    - **Definition and region of convergence (ROC), right- and left-sided sequences, ROC analysis**
  - **Inverse z_transform**
    - **Defintion and inspection method, partial fraction expansion**
  - **Transform analysis**
    - **Linear constant coefficient difference equations, stability and causality, inverse systems**

Some prerequisites

- **Calculus (ESD1):**
  - **complex numbers, first and second-order differential equations and Laplace as a transform**

- **Kredsløbsteori (ESD1):**
  - **application of first and second order differential equations and Laplace (with interpretation of time-frequency relation)**

- **Instrumentering, interfacekredsløb og dynamiske systemer (ESD2):**
  - **fundamentals on feedback (stability), basic linear dynamic systems**

Course execution and work load

- Classical lecture in class
  - Provides overview (hopefully), add supplementary information (a different view maybe), some examples
- Exercises in groups/individual
  - Important part of the curricula to apply theory and be confident with the topics
  - This is not (directly) exam preparation!
  - Advise to minimize the use of math programs
- Course load
  - A 5 ECTS course with expected 150 hours of effort (preparation for class, lecture in class, exercises in groups, follow up from lecture, preparation for the exam, exam attendance

Course exam

- A written three hour exam, start of June
  - The exam assignments will represent a subset of the course curricula – we sample
  - Allowed use of literature in physical and/or electronic format, personal notes, math programs – but …
  - Online access to digital exam (only!)

- No ”type assignments”
  - But, application of learned topic(s)
  - A typical exam exercise will combine different elements of what has been practized during exercise sessions
  - We will provide some example exercises during spring (a new course)

Exercises

- **Support how?**
  - **Pay a visit (in B1-101)?**
  - **Send an email?**
  - **?**
- **Solutions?**
  - **Will become available (to check) but we would like you to work ”from a blank sheet”, as preparation for the exam situation**

Matrix notation

$$\begin{array}{c} \text{— columns } n \text{ —} \\ \bar{\bar{A}} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \end{array} \begin{array}{c} | \\ \text{rows } m \\ | \end{array} = [a_{ij}]$$

scalar elements

$$\text{size}(\bar{\bar{A}}) = m \times n$$

Square matrices will be particularly interesting $m = n$

$\bar{\bar{A}}$ by column vector

$$\bar{\bar{A}} = [\bar{b}_1, \bar{b}_2, \cdots \bar{b}_n], \quad \bar{b}_i = \begin{bmatrix} b_{1i} \\ \vdots \\ b_{mi} \end{bmatrix}$$

Matrix notation

Matrix <u>algebra</u>: $\bar{y} = \bar{\bar{A}}\,\bar{x}$ "similar to numbers"

Matrix equality: $\bar{\bar{A}} = \bar{\bar{B}} \Longleftrightarrow [a_{ij}] = [b_{ij}]$
$\text{size}(\bar{\bar{A}}) = \text{size}(\bar{\bar{B}})$

Matrix transpose:

$$\bar{\bar{A}}^T = [a_{ij}]^T = [a_{ji}]$$

$\bar{\bar{A}}$ by row vector

$$\bar{\bar{A}} = [\bar{a}_1^T, \bar{a}_2^T, \ldots \bar{a}_m^T]^T, \quad \text{since } \bar{a}_i = (\bar{a}_i^T)^T$$
$$\bar{a}_i = [a_{i1}, \ldots a_{in}]$$

Special matrices (square)

Diagonal

$$\begin{bmatrix} a & & & \bar{\bar{0}} \\ & b & & \\ & & c & \\ \bar{\bar{0}} & & & \ddots \; z \end{bmatrix}$$

Triangular

lower/upper

$$\begin{bmatrix} 0 & \ddots & \bar{\bar{d}} \\ \vdots & & \ddots \\ 0 & \cdots & 0 \end{bmatrix}$$

Scalar

$$\begin{bmatrix} k & & & \bar{\bar{0}} \\ & k & & \\ & & \ddots & \\ \bar{\bar{0}} & & & k \end{bmatrix} = k \cdot \bar{\bar{I}}$$

Identity

$$\begin{bmatrix} 1 & & & \bar{\bar{0}} \\ & 1 & & \\ & & \ddots & \\ \bar{\bar{0}} & & & 1 \end{bmatrix} = \bar{\bar{I}}$$

Null (in various forms) $\bar{\bar{0}}$!

Special matrices (square)

Symmetric $\bar{\bar{A}}^T = \bar{\bar{A}}$

e.g. $\begin{bmatrix} 5 & -1 & 0 \\ -1 & 9 & 5 \\ 0 & 5 & 7 \end{bmatrix}$

Skew-symmetric $\bar{\bar{A}}^T = -\bar{\bar{A}}$

e.g. $\begin{bmatrix} 0 & -6 & 2 \\ 6 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}$

OBS! what's particular about it?

Special matrices (square)

Hankel

$$\begin{bmatrix} a & b & c & d & \cdots \\ b & c & d & e & \\ c & d & e & f & \\ d & e & f & & \\ \vdots & f & & & \end{bmatrix}$$

symmetric

"stacking" measurements for subsequent processing/estimation

Toeplitz

$$\begin{bmatrix} a & b & c & d & e & f & g & h \\ h & a & b & c & d & e & f & g \\ g & h & a & b & c & d & e & f \\ f & g & h & a & b & c & d & e \end{bmatrix}$$

"convolving" a signal through a system (not necessarily square)

Matrix algebra (addition)

Addition

$$\bar{\bar{A}} + \bar{\bar{B}} = \bar{\bar{B}} + \bar{\bar{A}}$$ commutative

$$(\bar{\bar{U}} + \bar{\bar{V}}) + \bar{\bar{W}} = \bar{\bar{U}} + (\bar{\bar{V}} + \bar{\bar{W}})$$ associative

$$\bar{\bar{A}} + \bar{\bar{0}} = \bar{\bar{A}}$$ null element

$$\bar{\bar{A}} + (-\bar{\bar{A}}) = \bar{\bar{0}}$$ negative (anti) element

Multiplication

$$\bar{\bar{A}}\,\bar{\bar{B}} \neq \bar{\bar{B}}\,\bar{\bar{A}}$$

can happen but in general not

Matrix algebra (multiplication)

$$\underset{m\times p}{\bar{\bar{A}}\,\bar{\bar{B}}} = \underset{m\times n}{\left[\bar{a}_1^T, \bar{a}_2^T, \ldots \bar{a}_m^T\right]^T} \underset{n\times p}{\left[\bar{b}_1, \bar{b}_2, \ldots \bar{b}_p\right]}$$

$$= \begin{array}{cc} & \text{k'th column} \\ \begin{bmatrix} \bar{a}_1\cdot\bar{b}_1 & \bar{a}_1\cdot\bar{b}_2 & \cdots & \bar{a}_1\cdot\bar{b}_p \\ \vdots & & \bar{a}_j\cdot\bar{b}_k & \vdots \\ \vdots & & & \vdots \\ \bar{a}_m\cdot\bar{b}_1 & \bar{a}_m\cdot\bar{b}_2 & \cdots & \bar{a}_m\cdot\bar{b}_p \end{bmatrix} & \leftarrow \text{j'th row} \end{array}$$

$$\bar{a}\cdot\bar{b} = \bar{a}^T\bar{b}$$ inner (dot) product

outer ($\otimes$) product

Matrix algebra (mult. cont.)

Since $\bar{\bar{A}}\bar{\bar{B}} \neq \bar{\bar{B}}\bar{\bar{A}}$ it means that

$\bar{\bar{B}} = \bar{\bar{C}} - \bar{\bar{D}}$

$$\bar{\bar{A}}\bar{\bar{B}} = \bar{\bar{0}} \not\Rightarrow \bar{\bar{A}} = 0, \bar{\bar{B}} = 0, \bar{\bar{B}}\bar{\bar{A}} = 0$$

$$\bar{\bar{A}}\bar{\bar{C}} = \bar{\bar{A}}\bar{\bar{D}} \not\Rightarrow \bar{\bar{C}} = \bar{\bar{D}}$$

DOES NOT !

! Multiplication order is important

pre- OR post-multiply

NOT THE SAME

! BUT $\bar{\bar{A}}\bar{\bar{I}} = \bar{\bar{I}}\bar{\bar{A}} = \bar{\bar{A}}$

Matrix algebra (mult. cont.)

$$\bar{\bar{A}}(\bar{\bar{B}}\bar{\bar{C}}) = (\bar{\bar{A}}\bar{\bar{B}})\bar{\bar{C}}$$ associative

$$(\bar{\bar{A}}+\bar{\bar{B}})\bar{\bar{C}} = \bar{\bar{A}}\bar{\bar{C}}+\bar{\bar{B}}\bar{\bar{C}}$$

distributive (wrt. addition)

$$\bar{\bar{C}}(\bar{\bar{A}}+\bar{\bar{B}}) = \bar{\bar{C}}\bar{\bar{A}}+\bar{\bar{C}}\bar{\bar{B}}$$

! observe order

We can say more about the matrix product when we know the matrix <u>rank</u> (later)

Matrix algebra (scalars)

$$c(\bar{\bar{A}}+\bar{\bar{B}}) = c\bar{\bar{A}} + c\bar{\bar{B}}$$

$$(c+k)\bar{\bar{A}} = c\bar{\bar{A}} + k\bar{\bar{A}}$$

distributive (vert. add./mult.)

$$c(k\bar{\bar{A}}) = (ck)\bar{\bar{A}}$$

$$(k\bar{\bar{A}})\bar{\bar{B}}) = k(\bar{\bar{A}}\bar{\bar{B}}) = \bar{\bar{A}}(k\bar{\bar{B}})$$

associative

$$1\bar{\bar{A}} = \bar{\bar{A}}$$

identity

Matrix algebra (transpose)

$$(\bar{\bar{A}} + \bar{\bar{B}})^T = \bar{\bar{A}}^T + \bar{\bar{B}}^T$$

$$(c\,\bar{\bar{A}})^T = c\,\bar{\bar{A}}^T$$

distributive (wrt. matrix add. scalar mult.)

$$(\bar{\bar{A}}\,\bar{\bar{B}})^T = \bar{\bar{B}}^T \bar{\bar{A}}^T$$

! NOT distributive (wrt. matr. mult.)

$$(A^n)^T = (A^T)^n$$

→ we will see later that a similar reverse order applies to matrix inverse

Matrix determinant $(n \times n)$

The determinant is defined for square $\bar{\bar{A}}$ $(n \times n)$

Row expansion (choose any row $j = 1, \dots n$)

$$\det(\bar{\bar{A}}) = a_{j1} C_{j1} + a_{j2} C_{j2} + \dots a_{jn} C_{jn}$$, where

the co-factor of a_{jk} is $C_{jk} = (-1)^{j+k} M_{jk}$

with M_{jk} the minor of a_{jk} (sub-determinant)

Column expansion (choose any column k)

$$\det(\bar{\bar{A}}) = a_{1k} C_{1k} + a_{2k} C_{2k} + \dots a_{nk} C_{nk}$$

Matrix determinant (Minor)

$$\begin{bmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & & & \vdots \\ \vdots & & a_{jk} & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

row j expan.

remove elements on red lines ---

--- and calculate subdeterminant, i.e. minor M_{jk} of order $n-1$

$$\begin{bmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & & & \vdots \\ \vdots & a_{jk} & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

column k expan.

Determinant (further)

$$\det(\bar{\bar{A}}) = \det(\bar{\bar{A}}^T)$$ as per co-factor expansion

and,

$$\det(\bar{\bar{A}}\bar{\bar{B}}) = \det(\bar{\bar{A}})\det(\bar{\bar{B}}) = \det(\bar{\bar{B}}\bar{\bar{A}})$$

despite $\bar{\bar{A}}\bar{\bar{B}} \neq \bar{\bar{B}}\bar{\bar{A}}$ in general

Matrix inverse (n x n)

Singular ("singulær") if $\det(\bar{\bar{A}}) = 0$

Non-singular ("regulær") if $\det(\bar{\bar{A}}) \neq 0$

If $\bar{\bar{A}}$ (n x n) is non-singular (why?):

$$\operatorname{inv}(\bar{\bar{A}}) = \bar{\bar{A}}^{-1} = \frac{1}{\det(\bar{\bar{A}})} [C_{jk}]^T = [C_{kj}]$$

! C_{jk} is the co-factor of a_{jk} placed in element kj position

Matrix inverse (mult.)

The inverse is that for which

$$\bar{\bar{A}}\,\bar{\bar{A}}^{-1} = \bar{\bar{A}}^{-1}\bar{\bar{A}} = \bar{\bar{I}}$$

from which we see by substitution

$$\bar{\bar{I}} = (\bar{\bar{A}}\,\bar{\bar{C}})(\bar{\bar{A}}\,\bar{\bar{C}})^{-1} \quad \text{pre-multiply } \bar{\bar{A}}^{-1}$$

$$\Downarrow$$

$$\bar{\bar{A}}^{-1}\bar{\bar{I}} = \underline{\bar{\bar{A}}^{-1}} = (\bar{\bar{A}}^{-1}\bar{\bar{A}})\,\bar{\bar{C}}\,(\bar{\bar{A}}\,\bar{\bar{C}})^{-1} = \underline{\bar{\bar{C}}\,(\bar{\bar{A}}\,\bar{\bar{C}})^{-1}}$$

$$\Downarrow$$

pre-multiply $\bar{\bar{C}}^{-1}$

$$\underline{\bar{\bar{C}}^{-1}\bar{\bar{A}}^{-1}} = (\bar{\bar{C}}^{-1}\bar{\bar{C}})(\bar{\bar{A}}\,\bar{\bar{C}})^{-1} = \underline{(\bar{\bar{A}}\,\bar{\bar{C}})^{-1}}$$

Matrix algebra (inverse)

$$\bar{\bar{I}}\,\bar{\bar{I}}^{-1} = \bar{\bar{I}} \iff \bar{\bar{I}}^{-1} = \bar{\bar{I}}$$

$$(\bar{\bar{A}}^{-1})^{-1} = \bar{\bar{A}}, \text{ since } \bar{\bar{A}}^{-1}(\bar{\bar{A}}^{-1})^{-1} = \bar{\bar{A}}^{-1}\bar{\bar{A}} = \bar{\bar{I}}$$

$$(\bar{\bar{A}}^{n})^{-1} = \bar{\bar{A}}^{-1}\bar{\bar{A}}^{-1}\cdots\bar{\bar{A}}^{-1} = (\bar{\bar{A}}^{-1})^{n}$$

cf. prev. slide

$$(\bar{\bar{A}}^{-1})^{T} = (\bar{\bar{A}}^{T})^{-1}, \text{ since } \bar{\bar{I}} = \bar{\bar{I}}^{T} = (\bar{\bar{A}}\bar{\bar{A}}^{-1})^{T} = (\bar{\bar{A}}^{-1})^{T}\bar{\bar{A}}^{T}$$

$$\det(\bar{\bar{I}}) = 1$$

$$\det(\bar{\bar{I}}) = \det(\bar{\bar{A}}\bar{\bar{A}}^{-1}) = \det(\bar{\bar{A}})\cdot\det(\bar{\bar{A}}^{-1})$$

$$\iff \det(\bar{\bar{A}}^{-1}) = 1/\det(\bar{\bar{A}})$$

Echelon form

A triangular matrix, such that:

- all 0's are below, left
- first $a'_{ij} \neq 0$ in each row is to the right of the $a'(\neq 0)$ in the row above AND has only 0's below it

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{mn} & \cdots & a_{mn} \end{bmatrix} \quad (=\bar{\bar{A}})$$

pivot – "the central element"

$$\begin{bmatrix} a'_{11} & \cdots & \cdots & a'_{1n} \\ 0 & \ddots & & \\ \vdots & 0 & a'_{ij} \cdots & a'_{in} \\ & & 0 \; 0 & a'_{pn} \\ 0 & \cdots & 0 \; 0 & 0 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \quad (=\bar{\bar{A}}')$$

row equivalent BUT different $\bar{\bar{A}}$

! first a'_{ij}'s are not necessarily on one diagonal

! there can be rows with all 0's

Echelon form - examples ?

✓ ✗ ?

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 7 & 5 & 3 \\ 0 & 1 & 0 & 0 & 3 & 1 & 9 \\ 0 & 0 & 5 & 0 & 8 & 0 & 1 \\ 0 & 0 & 0 & 1 & 7 & 7 & 7 \end{bmatrix}$$

✓ ✗ ?

$$\begin{bmatrix} 9 & 1 & 3 & 5 & 6 & 1 & 5 \\ 0 & 0 & 0 & 6 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

✓ ✗ ?

$$\begin{bmatrix} 0 & 1 & 0 & 9 & 6 & 3 \\ 0 & 5 & 1 & 6 & 9 & 3 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Reduced echelon form

A triangular matrix, such that:

- it fulfills the echelon form
- in addition, the first a_{ij} in each row is 1 with 0's below AND above

the pivot is 1

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Reduced echelon form – examples

✗
✓
?

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 7 & 5 & 3 \\ 0 & 1 & 0 & 0 & 3 & 1 & 9 \\ 0 & 0 & 1 & 0 & 8 & 0 & 1 \\ 0 & 0 & 0 & 1 & 7 & 7 & 7 \end{bmatrix}$$

✗
✓
?

$$\begin{bmatrix} 5 & 7 & 9 & 13 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Gauss elimination

Elementary row operations:

1) multiplication of a row by a constant $k \neq 0$
2) swapping rows
3) adding a weighted sum of rows to another row

Reduces a matrix to the row equivalent echelon form, and/or reduced echelon form

! these operations forms the basis of Gauss elimination

! can be done on rows OR columns

Gauss elimination – by matrix ops.

Operation — rows : EA (pre)

Operation — columns : AE (post)

Case of 3x3 for illustration

Multiply by a constant

$$\begin{array}{cc} E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{array}{l} \\ \leftarrow \text{this row} \\ \end{array} \\ \quad\quad\;\; \uparrow \text{this column} & \end{array}$$

OR

Swap rows / columns

$$\begin{array}{cc} E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} \leftarrow \text{these rows} \\ \\ \end{array} \\ \quad \text{these columns} & \end{array}$$

OR

Multiply and add

$$\begin{array}{cc} E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \leftarrow \text{this row} \end{array} \\ \quad\quad\;\; \uparrow \text{this column} & \end{array}$$

OR

Determinant (Gauss)

From the echelon form, we can calculate

$$\det(\bar{A}) = (-1)^r \cdot \prod_i p_i$$

where p_i are the (diagonal) pivots and r is the number of row interchanges.

! $\det(\bar{A}) \neq 0 \Rightarrow$ (diagonal) $p_i \neq 0$

Gauss elimination (det.)

1) Multiplikation by k $\Rightarrow$ $k \cdot \det(\bar{\bar{A}})$

OBS! $\det(k\bar{\bar{A}}) = k^n \det(\bar{\bar{A}})$

2) Swapping rows $\Rightarrow$ $-1 \cdot \det(\bar{\bar{A}})$

3) Adding a weighted sum $\Rightarrow$ no change

! Proportional rows/columns $\Rightarrow$
zero rows/columns $\Rightarrow$ $\det(\bar{\bar{A}}) = 0$

Linear independence

Given p vectors (n column $\bar{b}_i$'s or
m row $\bar{a}_i$'s)

$$c_1 \bar{v}_1 + c_2 \bar{v}_2 + \dots \dots c_p \bar{v}_p = \bar{0}$$

If only solution is for scalars $c_i = 0$, $1 \le i \le p$, then $\bar{v}_i$'s are linearly independent

"the essential vectors"

Linear independence (cont.)

In Gauss elimination we find the "essential vectors" – usually row vectors.

$\mathrm{rank}(\bar{\bar{A}})$

The maximum number of linearly independent vectors (row or column) is called the rank of $\bar{\bar{A}}$

! $\bar{\bar{A}} = \bar{\bar{0}} \Rightarrow \mathrm{rank}(\bar{\bar{A}}) = 0$! $\mathrm{rank}(\bar{\bar{A}}) = \mathrm{rank}(\bar{\bar{A}}^T)$

Matrix rank

Row equivalent matrices have the same rank! => we can find the rank by Gauss elimination, i.e. how many non-zero rows (or columns) result in the end.

! We need to check both rows and columns to get rank($\bar{A}$)

Rank implication

$$\bar{\bar{A}} = \begin{bmatrix} \bar{a}_1 \\ \vdots \\ \bar{a}_m \end{bmatrix}, \quad \bar{a}_i = [a_{i1}, \dots a_{in}]$$

n columns

If $\text{rang}(\bar{\bar{A}}) = p$, then:

$$c_1 \bar{a}_1' + c_2 \bar{a}_2' + \dots c_p \bar{a}_p' = \bar{0} \iff$$

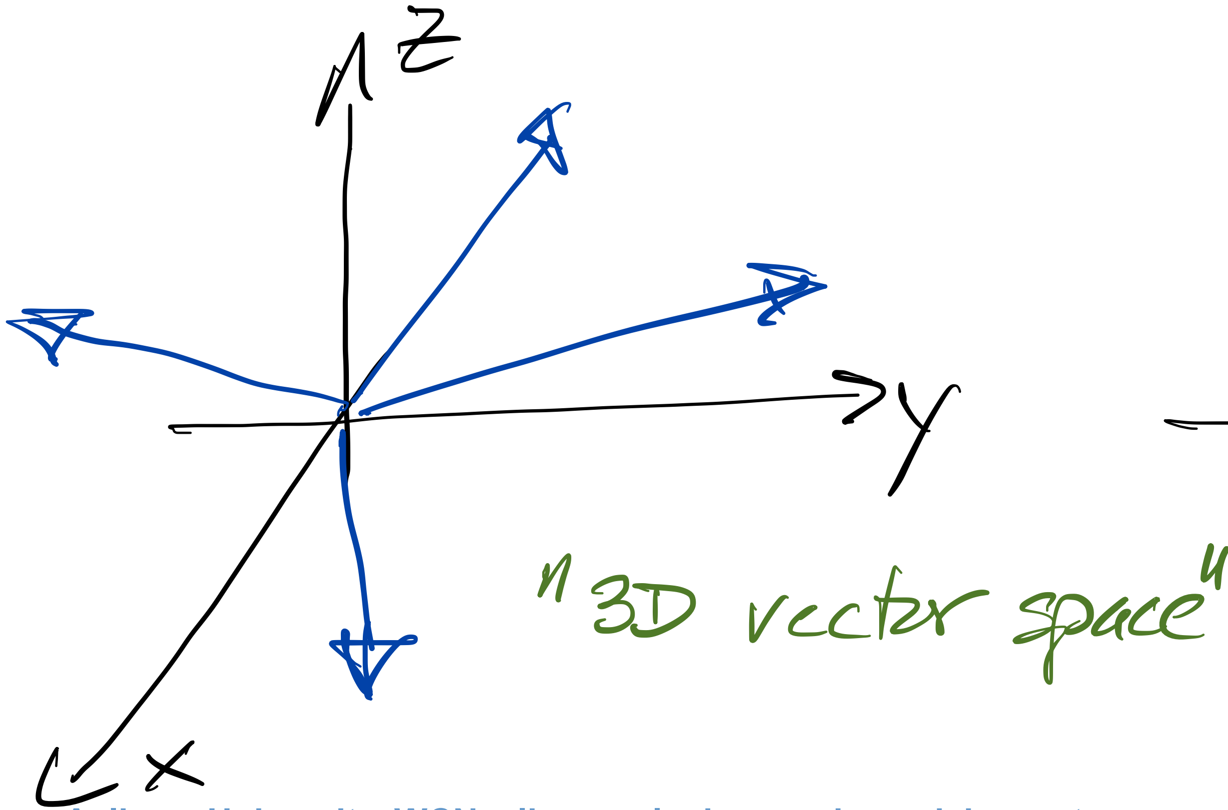
$$\forall c_i = 0, \quad 1 \leq i \leq p$$

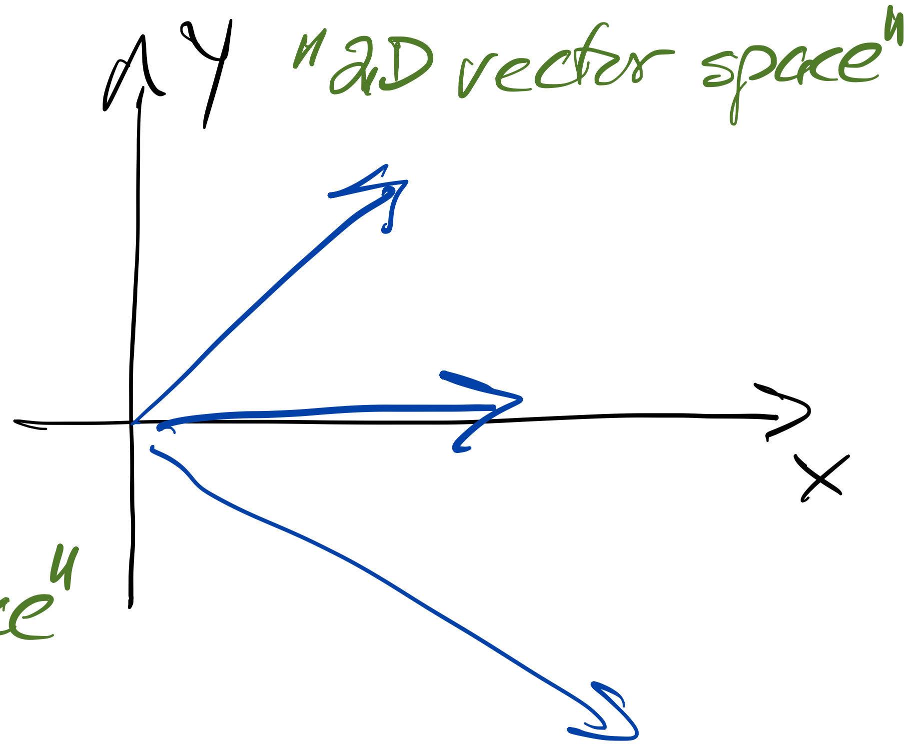
If $n < p$, since $\text{rank}(\bar{\bar{A}}) = \text{rank}(\bar{\bar{A}}^T)$

then $\text{rank}(\bar{\bar{A}}) \leq n < p \longrightarrow$

Rank implication (cont.)

p vectors with $n < p$ components are linearly dependent, always





Rank implication (cont.)

An $m \times n$ matrix $\bar{\bar{A}}$ has rank $r \iff \bar{\bar{A}}$ has an $r \times r$ submatrix $\bar{\bar{A}}_r$ with $\det(\bar{\bar{A}}_r) \neq 0$ and $\det(\bar{\bar{A}}_{r+1}) = 0$

$\bar{\bar{A}}$ is said to be rank deficient if $\text{rank}(\bar{\bar{A}}) < \min(m, n)$ (singular)

A square matrix $\bar{\bar{A}}$, $n \times n$, is full rank $\iff \text{rank}(\bar{\bar{A}}) = n$ and $\det(\bar{\bar{A}}) \neq 0$

Rank application (cont.)

$$\mathrm{inv}(\bar{\bar{A}}) = \bar{\bar{A}}^{-1} \quad (n \times n) \iff \det(\bar{\bar{A}}) \neq 0$$

$$\iff \underline{\mathrm{rank}(\bar{\bar{A}}) = n} \, . \quad \text{Also:}$$

$$\bar{\bar{A}}\,\bar{\bar{B}} = \bar{\bar{A}}\,\bar{\bar{C}} \iff \bar{\bar{B}} = \bar{\bar{C}}$$

$$\bar{\bar{A}}\,\bar{\bar{B}} = \bar{\bar{0}} \implies \bar{\bar{B}} = \bar{\bar{0}}$$

In general $\bar{\bar{A}}\,\bar{\bar{B}} \neq \bar{\bar{B}}\,\bar{\bar{A}}$