- Line integrals and path independence of line integrals
- Double integrals
- Green's theorem in the plane

Vector Integral Calculus

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Learning objectives:

- **Surface integrals;**
- **Triple integrals;**
- **Divergence theorem of Gauss;**
- **Stokes's theorem;**

Surface integrals:

For a surface *S*:

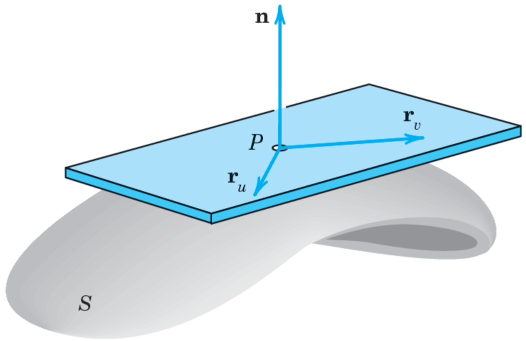
$$\mathbf{r}(u, v) = [x(u, v), y(u, v), z(u, v)] = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

(u, v) varies over a region *R* in the *uv*-plane. *S* has a normal vector:

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \qquad \text{and unit normal vector} \qquad \mathbf{n} = \frac{1}{|\mathbf{N}|}\mathbf{N}$$

Surface integral over *S* is defined by:

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, dA = \iint_R \mathbf{F}(\mathbf{r}(u, v)) \bullet \mathbf{N}(u, v)\, du\, dv.$$



$$\mathbf{n}\, dA = \mathbf{n}|\mathbf{N}|\, du\, dv = \mathbf{N}\, du\, dv.$$

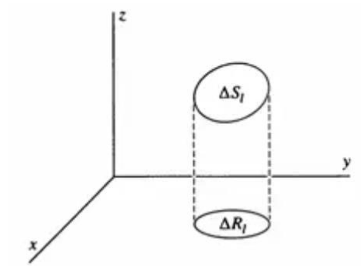
If the surface integral is written in components:

$\mathbf{F} = [F_1, F_2, F_3]$, $\boldsymbol{N} = [N_1, N_2, N_3]$, $\boldsymbol{n} = [\cos\alpha, \cos\beta, \cos\gamma]$. α, β, γ are the angles between $\boldsymbol{n}$ and the coordinate axes.

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, dA = \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, dA$$
$$= \iint_R (F_1 N_1 + F_2 N_2 + F_3 N_3)\, du\, dv.$$

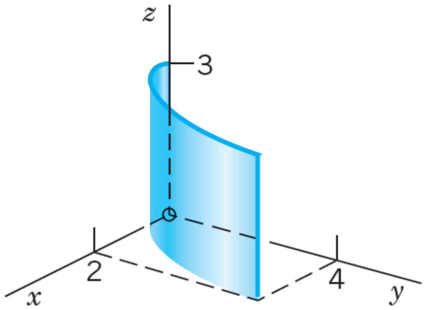
We also have that $\cos\alpha dA = dydz$, $\cos\beta dA = dzdx$, $\cos\gamma dA = dxdy$, then:

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, dA = \iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy).$$



Practice exercise:

Compute the flux of water through the parabolic cylinder S: $y = x^2$, $0 \leqq x \leqq 2$, $0 \leqq z \leqq 3$ (Fig. 245) if the velocity vector is $\mathbf{v} = \mathbf{F} = [3z^2, \quad 6, \quad 6xz]$, speed being measured in meters/sec. (Generally, $\mathbf{F} = \rho\mathbf{v}$, but water has the density $\rho = 1 \text{ g/cm}^3 = 1 \text{ ton/m}^3$.)



Triple integral:

In a triple integral, we integrate a function $f(x, y, z)$ over a bounded and three-dimensional region T.

$$\iiint_T f(x, y, z)\, dx\, dy\, dz \qquad \text{or by} \qquad \iiint_T f(x, y, z)\, dV.$$

Triple integrals can also be evaluated by three successive integrations.

If $g(x, y) \leqq z \leqq h(x, y)$

$$\iiint_T f(x,y,z)dxdydz = \iint_R \left[\int_{g(x,y)}^{h(x,y)} f(x,y,z)dz\right]dxdy$$

If $p(x) \le y \le q(x)$

$$\iint_R \left[\int_{g(x,y)}^{h(x,y)} f(x,y,z)dz\right]dxdy = \int_{x_1}^{x_2}\left[\int_{p(x)}^{q(x)}\left[\int_{g(x,y)}^{h(x,y)} f(x,y,z)dz\right]dy\right]dx$$

Divergence Theorem of Gauss:

Transformation between triple and surface integrals

Let T be a closed bounded region in space whose boundary is a piecewise smooth orientable surface S. *Let* $\mathbf{F}$ (x, y, z) *be a vector function that is continuous and has continuous first partial derivatives in some domain containing* T. *Then,*

$$\iiint_T \operatorname{div} \mathbf{F} \, dV = \iint_S \mathbf{F} \bullet \mathbf{n} \, dA.$$

In components of $\mathbf{F} = [F_1, F_2, F_3]$ and of the outer unit normal vector $\boldsymbol{n} = [\cos\alpha, \cos\beta, \cos\gamma]$

$$\operatorname{div} \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

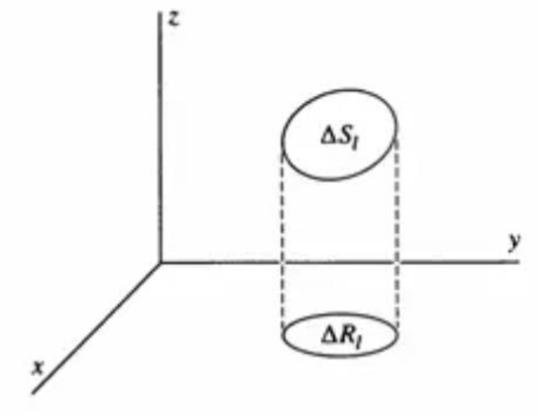
$$\iiint_T \operatorname{div} \mathbf{F} \, dV = \iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\, dy\, dz$$

$$\iint_S \mathbf{F} \bullet \mathbf{n} \, dA = \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, dA$$

$$\iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, dA$$

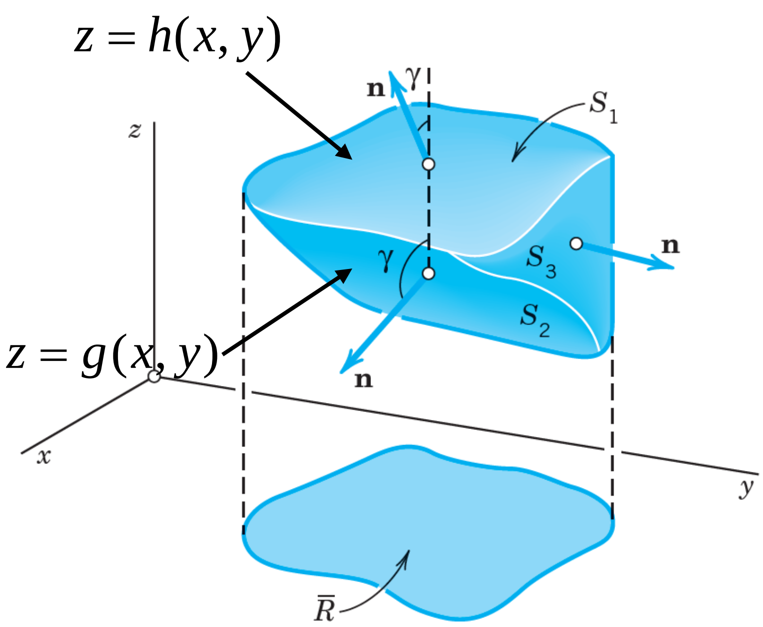
$$\cos\alpha\, dA = dy dz$$
$$\cos\beta\, dA = dz dx$$
$$\cos\gamma\, dA = dx dy$$

$$\iint_R (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy)$$



$$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\, dy\, dz = \iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy)$$

Prove the Divergence Theorem ?

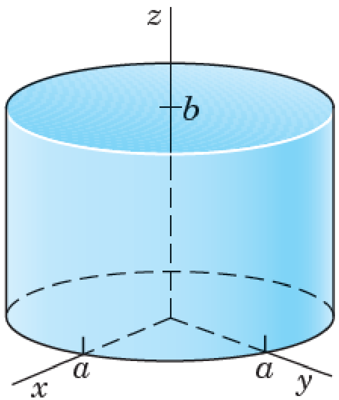


Practice exercise:

$$I = \iint_S (x^3\, dy\, dz + x^2 y\, dz\, dx + x^2 z\, dx\, dy)$$

$$x^2 + y^2 = a^2 \; (0 \leqq z \leqq b)$$

$$z = 0 \text{ and } z = b \; (x^2 + y^2 \leqq a^2)$$



$$\iiint_V f(x,y,z)dV = \iiint_V f(\rho\cos\varphi, \rho\sin\varphi, z)\rho d\rho d\phi dz$$

$$= \iiint_V f(r\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi) r^2 \sin\varphi dr d\varphi d\theta$$

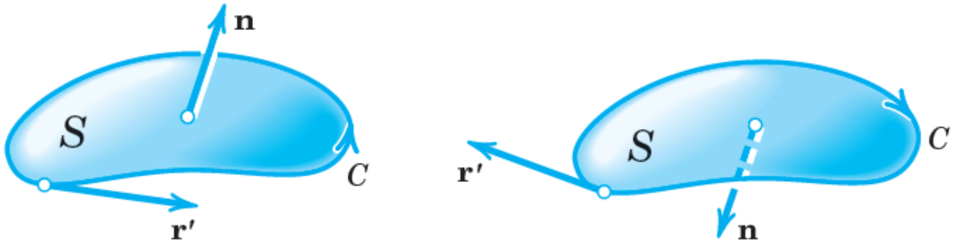
Stokes's Theorem:

Transformation between surface and line integrals

Let S be a piecewise smooth oriented surface in space and let the boundary of S be a piecewise smooth simple closed curved C. Let $\mathbf{F}$ (x, y, z) be a continuous vector function that has continuous first partial derivatives in a domain in space containing S. Then

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dA = \oint_C \mathbf{F} \cdot \mathbf{r}'(s) \, ds.$$

n is a unit normal vector of S, $\boldsymbol{r}' = d\mathbf{r}/ds$ is the unit tangent and s is the arc length of C.



$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$\mathbf{F} = [F_1, F_2, F_3]$, $\mathbf{N} = [N_1, N_2, N_3]$, $\mathbf{n}\, dA = \mathbf{N}\, du dv$, R is the region with boundary curve in the uv-*plane corresponding to S represented by* $\mathbf{r}\,(u,v)$.

$$\iint_S (\operatorname{curl} \mathbf{F}) \bullet \mathbf{n}\, dA$$

$$\Downarrow$$

$$\iint_R \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) N_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) N_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) N_3 \right] du\, dv$$

F = [F_1, F_2, F_3], ***r'*** *ds* = [*dx*, *dy*, *dz*], and *R* is the region with boundary curve in the u*v*-*plane corresponding to S represented by* **r** (*u*,*v*).

$$\oint_C \mathbf{F} \bullet \mathbf{r}'(s)\, ds = \oint_{\overline{C}} (F_1\, dx + F_2\, dy + F_3\, dz)$$

$$\iint_R \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)N_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)N_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)N_3\right] du\, dv$$

$$= \oint_{\overline{C}} (F_1\, dx + F_2\, dy + F_3\, dz).$$

A special case here:

Let $\mathbf{F} = [F_1, F_2] = F_1\mathbf{i} + F_2\mathbf{j}$ be a vector function that is continuous differentiable in a domain in the xy-plane containing a simply connected bounded closed region S whose boundary C is a piecewise smooth simple closed curve,

$$(\text{curl } \mathbf{F}) \bullet \mathbf{n} = (\text{curl } \mathbf{F}) \bullet \mathbf{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \oint_C (F_1\, dx + F_2\, dy).$$

Green's theorem: $$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy = \oint_C (F_1\, dx + F_2\, dy).$$

Evaluation of a line integral by Stokes's theorem:

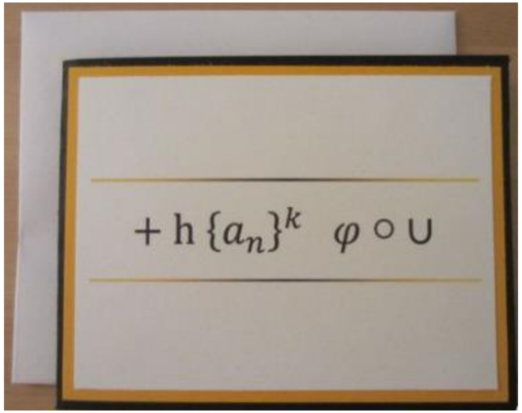
Evaluate $\int_C \mathbf{F} \cdot \mathbf{r}' ds$, where C is the circle $x^2 + y^2 = 4$, $z = -3$, oriented counterclockwise as seen by a person standing at the origin, and, with respect to right-handed Cartesian coordinates,

$$\mathbf{F} = [y, \quad xz^3, \quad -zy^3] = y\mathbf{i} + xz^3\mathbf{j} - zy^3\mathbf{k}.$$

$$(\text{curl } \mathbf{F}) \cdot \mathbf{n} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -27 - 1 = -28.$$

Then, the line integral is -28 × 4π = -112 π.

If solve it directly?


$+ \mathrm{h} \{a_n\}^k \quad \varphi \circ \cup$