- Derivatives
- Gradient, directional derivatives
- Divergence
- Curl

Vector Integral Calculus

Peng Mei
Department of Electronic Systems
Email: mei@es.aau.dk

AALBORG UNIVERSITY
DENMARK

Learning objectives:

- **Line integrals and path independence of line integrals**;
- **Double integrals;**
- **Green's theorem in the plane;**
- **Surface integrals**

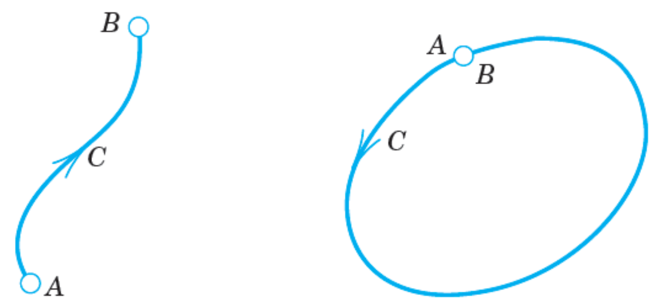
Line integral:

For a definite integral:

$$\int_a^b f(x)dx$$

Integrate the function $f(x)$, also known as the **integrand**, from $x = a$ along the x-axis to $x = b$.

In a line integral, we shall integrate a given function along a curve C in space or in the plane.



Unclosed path   Closed path

Example:

$$\int_1^2 x^2 dx$$

$$\int_1^2 x^2 dx = \frac{1}{3}x^3\Big|_1^2 = \frac{1}{3}(8-1) = \frac{7}{3}$$

$$\int_0^{\pi/2} \cos 2x dx$$

$$\int_0^{\pi/2} \cos 2x dx = \frac{1}{2}\sin 2x\Big|_0^{\pi/2} = 0 - 0 = 0$$

Vector line integral:

A line integral of a vector function **F(r)** over a curve C: **r**(*t*) is defined by:

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \qquad \mathbf{r}' = \frac{d\mathbf{r}}{dt}$$

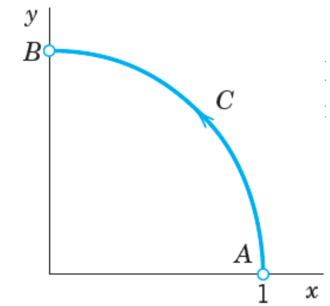
where: $\mathbf{r}(t) = [x(t), y(t), z(t)] = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad (a \leqq t \leqq b).$

If $\mathbf{F}(\mathbf{r}) = [F_1, F_2, F_3]$, $d\mathbf{r} = [dx, dy, dz]$, then

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C (F_1\, dx + F_2\, dy + F_3\, dz)$$

$$= \int_a^b (F_1 x' + F_2 y' + F_3 z')\, dt.$$

Example: line integral in the plane



If **F**(**r**) = [-*y*, -*xy*] = -*y****i*** - *xy****j***, *C* is the curve from *A* to *B*.

r(*t*) = [cos(*t*), sin(*t*)] = cos(*t*)***i*** + sin(*t*)***j***

$0 \le t \le \pi/2$

x(*t*) = cos(*t*), *y*(*t*) = sin(*t*)

F(**r**(*t*)) = [-sin(*t*), -cos(*t*)sin(*t*)] = -sin(*t*)***i*** - cos(*t*)sin(*t*)***j***

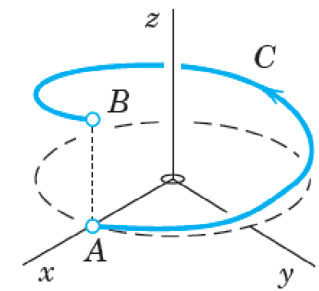
r'(*t*) = [-sin(*t*), cos(*t*)] = -sin(*t*)***i*** + cos(*t*)***j***

$$\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_0^{\pi/2} [-\sin t, -\cos t \sin t] \bullet [-\sin t, \cos t]\, dt = \int_0^{\pi/2} (\sin^2 t - \cos^2 t \sin t)\, dt$$

$$= \int_0^{\pi/2} \frac{1}{2}(1 - \cos 2t)\, dt - \int_1^0 u^2(-du) = \frac{\pi}{4} - 0 - \frac{1}{3} \approx 0.4521.$$

Example: line integral in space

If $\mathbf{F}(\mathbf{r}) = [z, x, y] = z\boldsymbol{i} + x\boldsymbol{j} + y\boldsymbol{k}$, C is the helix from A to B.



$\mathbf{r}(t) = [\cos(t), \sin(t), 3t] = \cos(t)\boldsymbol{i} + \sin(t)\boldsymbol{j} + 3t\boldsymbol{k}$

$x(t) = \cos(t),\ y(t) = \sin(t),\ z(t) = 3t$ $\qquad 0 \le t \le 2\pi$

$\mathbf{F}(\mathbf{r}(t)) = [3t, \cos(t), \sin(t)] = 3t\boldsymbol{i} + \cos(t)\boldsymbol{j} + \sin(t)\boldsymbol{k}$

$\mathbf{r}'(t) = [-\sin(t), \cos(t), 3] = -\sin(t)\boldsymbol{i} + \cos(t)\boldsymbol{j} + 3\boldsymbol{k}$

$$\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_0^{2\pi} (-3t \sin t + \cos^2 t + 3 \sin t)\, dt = 6\pi + \pi + 0 = 7\pi \approx 21.99.$$

Simple general properties of the line integral:

$$\int_C k\mathbf{F} \bullet d\mathbf{r} = k \int_C \mathbf{F} \bullet d\mathbf{r}$$

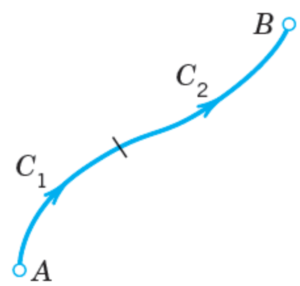
k is a constant.

$$\int_C (\mathbf{F} + \mathbf{G}) \bullet d\mathbf{r} = \int_C \mathbf{F} \bullet d\mathbf{r} + \int_C \mathbf{G} \bullet d\mathbf{r}$$

The orientation of C is the same in all three integrals.

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_{C_1} \mathbf{F} \bullet d\mathbf{r} + \int_{C_2} \mathbf{F} \bullet d\mathbf{r}$$

Path C is subdivided into two arcs C_1 and C_2 that have the same orientation as C.



Motivation of a vector line integral: work done by a force

The work W done by a constant force $\mathbf{F}$ in the displacement along a straight segment $\mathbf{d}$ is $W = \mathbf{F} \cdot \mathbf{d}$.

Let $\mathbf{F}(\mathbf{r})$ be a force, and a curve C: $\mathbf{r}(t)$ is the displacement of the force, then the work done by the force is:

$$W = \int_C \mathbf{F} \bullet d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{v}(t)\, dt.$$

According to Newton's second law, that is, force = mass × acceleration, then

$$\mathbf{F} = m\mathbf{v}'(t)$$

then:

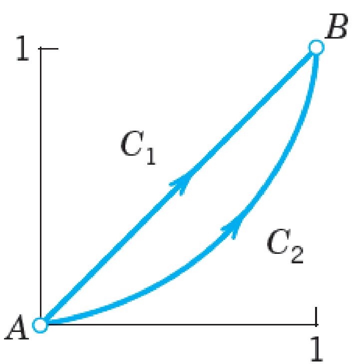
$$W = \int_a^b m\mathbf{v}' \bullet \mathbf{v}\, dt = \int_a^b m\left(\frac{\mathbf{v} \bullet \mathbf{v}}{2}\right)' dt = \frac{m}{2}|\mathbf{v}|^2 \Big|_{t=a}^{t=b}.$$

The work done equals to the gain in kinetic energy.

Path dependence:

$$\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt \qquad \mathbf{r}' = \frac{d\mathbf{r}}{dt}$$

The line integral generally depends not only on **F** and on the endpoints A and B of the path, but also on the path itself along which line the integral is taken.



If $\mathbf{F}(\mathbf{r}) = [0, xy, 0] = xy\boldsymbol{j}$,

C_1 : a straight segment, $\mathbf{r}_1(t) = [t, t, 0]$;

C_2 : a parabola, $\mathbf{r}_2(t) = [t, t^2, 0]$;

0 □ t □ 1

C_1 and C_2 have the same endpoints A and B.

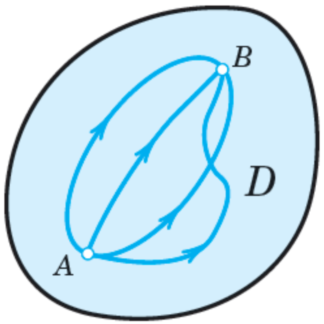
$$\mathbf{F}(\mathbf{r}_1(t)) \bullet \mathbf{r}_1'(t) = t^2, \longrightarrow \frac{1}{3}$$

$$\mathbf{F}(\mathbf{r}_2(t)) \bullet \mathbf{r}_2'(t) = 2t^4, \longrightarrow \frac{2}{5}$$

Path independence:

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$$

$$(d\mathbf{r} = [dx, \quad dy, \quad dz])$$



The integral has the same value for all paths in *D* that begins at *A* and end at *B*.

What kind of vector function **F(r)** should be to make the integral path independence?

- If **F(r)** = grad *f*, the integral would be path independence.

How to prove?

Prove:

$$\mathbf{F} = \operatorname{grad} f, \qquad \text{thus,} \qquad F_1 = \frac{\partial f}{\partial x}, \qquad F_2 = \frac{\partial f}{\partial y}, \qquad F_3 = \frac{\partial f}{\partial z}.$$

$$\begin{aligned}
\int_C (F_1\,dx + F_2\,dy + F_3\,dz) &= \int_C \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right) \\
&= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)dt \\
&= \int_a^b \frac{df}{dt}\,dt = f[x(t), y(t), z(t)]\Big|_{t=a}^{t=b} \\
&= f(x(b), y(b), z(b)) - f(x(a), y(a), z(a)) \\
&= f(B) - f(A).
\end{aligned}$$

$$\boxed{\int_A^B (F_1\,dx + F_2\,dy + F_3\,dz) = f(B) - f(A)} \qquad [\mathbf{F} = \operatorname{grad} f]$$

$$\int_a^b g(x)\,dx = G(x)\Big|_a^b = G(b) - G(a) \qquad [G'(x) = g(x)].$$

The value of the integral is only associated to the positions of *A* and *B*.

Example:

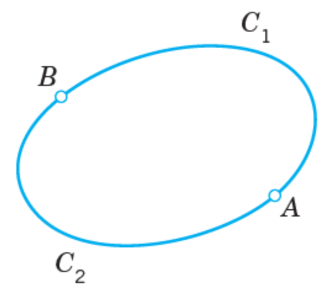
Show that the integral $\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C (2x\,dx + 2y\,dy + 4z\,dz)$ is path independent in any domain in space and find its value in the integration from A: (0, 0, 0) to B: (2, 2, 2).

We need to prove that $\mathbf{F} = [2x, 2y, 4z]$ is the gradient of a function f.

According to $F_1 = 2x$, $F_2 = 2y$, $F_3 = 4z$, we can find that $f(x, y, z) = x^2 + y^2 + 2z^2$, then

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C (2x\,dx + 2y\,dy + 4z\,dz) = f(B) - f(A) = f(2, 2, 2) - f(0, 0, 0) = 4 + 4 + 8 = 16$$

- The integral is path independent in a domain D if and only if its value around every closed path in D is zero.



If the integral is path independent, it means:

$$\begin{cases} \int_B^A \vec{F} \cdot d\vec{r} = f(A) - f(B) \\ \int_A^B \vec{F} \cdot d\vec{r} = f(B) - f(A) \end{cases} \Rightarrow \begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \int_B^A \vec{F} \cdot d\vec{r} + \int_A^B \vec{F} \cdot d\vec{r} \\ &= f(A) - f(B) + f(B) - f(A) \\ &= 0 \end{aligned}$$

Conversely, assume that the integral around any closed path C in D is zero, given any points A and B and any two curves C_1 and C_2 from A to B in D.

Due to the integral around the closed path C is zero, it means the integral from A to B along C_1 and the integral from A to B along C_2 must be equal.

Double integrals:

In a line integral, we integrate a function *f* (*x*) over a segment of the *x*-axis, while in a double integral, we integrate a function *f* (*x*, *y*) over a bounded region *R* in the *xy*-plane.

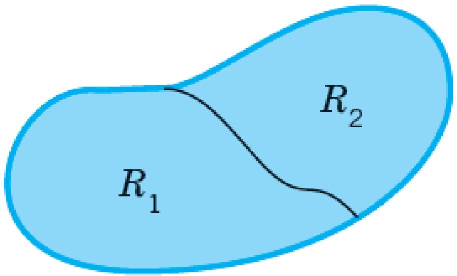
$$\iint_R f(x, y)\,dx\,dy \qquad \text{or} \qquad \iint_R f(x, y)\,dA.$$

Some simple general properties of the double integrals:

$$\iint_R kf\,dx\,dy = k\iint_R f\,dx\,dy$$

$$\iint_R (f + g)\,dx\,dy = \iint_R f\,dx\,dy + \iint_R g\,dx\,dy$$

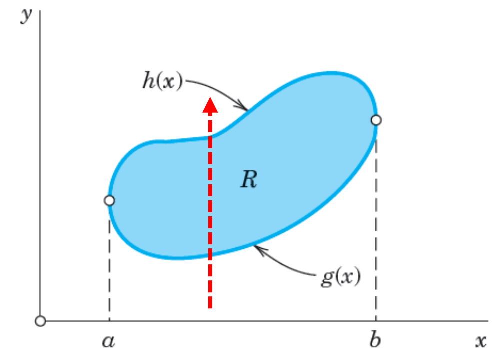
$$\iint_R f\,dx\,dy = \iint_{R_1} f\,dx\,dy + \iint_{R_2} f\,dx\,dy$$



Evaluation of double integrals by two successive integrations:

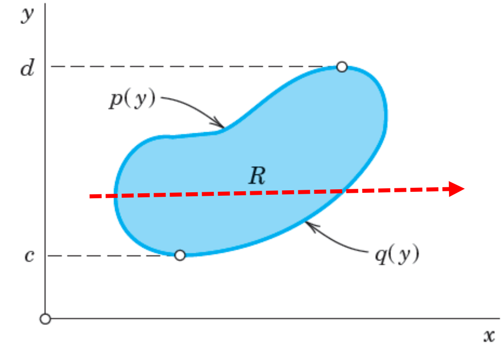
Double integrals over a region R may be evaluated by two successive integrations, we may integrate first over *y* and then over *x*:

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,y)\,dy \right] dx$$



Or we may integrate first over *x* and then over *y* :

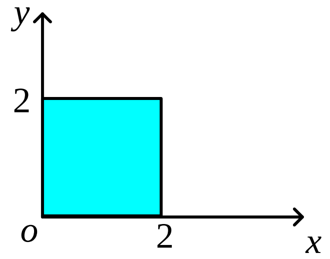
$$\iint_R f(x,y)\,dx\,dy = \int_c^d \left[\int_{p(y)}^{q(y)} f(x,y)\,dx \right] dy$$



Applications of double integrals

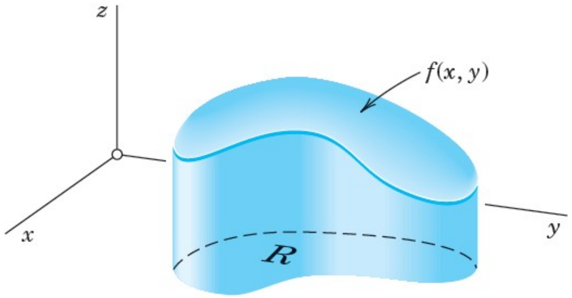
The area A of a region R in the xy-plane is given by the double integral:

$$A = \iint_R dx\,dy.$$



The **volume** V beneath the surface $z = f(x, y)$ (> 0) and above a region R in the xy-plane:

$$V = \iint_R f(x, y)\,dx\,dy$$



Green’s theorem in the plane

Transformation between double integrals and line integrals.

Let R be a closed bounded region in the xy-plane whose boundary C consists of finitely many smooth curves. Let $F_1(x, y)$ and $F_2(x, y)$ be functions that are continuous and have continuous partial derivatives $\partial F_1/\partial y$ and $\partial F_2/\partial x$ everywhere in some domain containing R, then

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint_C (F_1\,dx + F_2\,dy).$$

$$\operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}.$$

$$\iint_R (\operatorname{curl} \mathbf{F}) \bullet \mathbf{k}\,dx\,dy = \oint_C \mathbf{F} \bullet d\mathbf{r}.$$

Let's get used to Green's theorem from a specific example:

If $F_1 = y^2 - 7y$, $F_2 = 2xy + 2x$, R is an area enclosed by a closed curve C: $x^2 + y^2 = 1$.

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \iint_R [(2y + 2) - (2y - 7)]\,dx\,dy = 9\iint_R dx\,dy = \boxed{9\pi}$$

Let $\mathbf{r}(t) = [\cos t, \sin t]$, then $\mathbf{r}'(t) = [-\sin t, \cos t]$,

$$F_1 = y^2 - 7y = \sin^2 t - 7\sin t, \qquad F_2 = 2xy + 2x = 2\cos t \sin t + 2\cos t.$$

$$\oint_C (F_1 x' + F_2 y')\,dt = \int_0^{2\pi} [(\sin^2 t - 7\sin t)(-\sin t) + 2(\cos t \sin t + \cos t)(\cos t)]\,dt$$

$$= \int_0^{2\pi} (-\sin^3 t + 7\sin^2 t + 2\cos^2 t \sin t + 2\cos^2 t)\,dt$$

$$= 0 + 7\pi - 0 + 2\pi = \boxed{9\pi.}$$

Some applications of Green's Theorem

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy = \oint_C (F_1\, dx + F_2\, dy).$$

- **Area of a plane region as a line integral over the boundary**

If $F_1 = 0$, $F_2 = x$, and then $F_1 = -y$, $F_2 = 0$. this gives:

$$\iint_R dx\, dy = \oint_C x\, dy \qquad \text{and} \qquad \iint_R dx\, dy = -\oint_C y\, dx$$

The area A of R is:

$$A = \frac{1}{2}\oint_C (x\, dy - y\, dx)$$

➢ Area of a plane region in polar coordinates

Let r and θ be polar coordinates defined by $x = r\cos\theta$, $y = r\sin\theta$, then:

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta, \qquad dy = \sin\theta\, dr + r\cos\theta\, d\theta,$$

$$A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

The area A of R is:

$$A = \frac{1}{2}\oint_C r^2\, d\theta.$$

Surface integrals:

For a surface *S*:

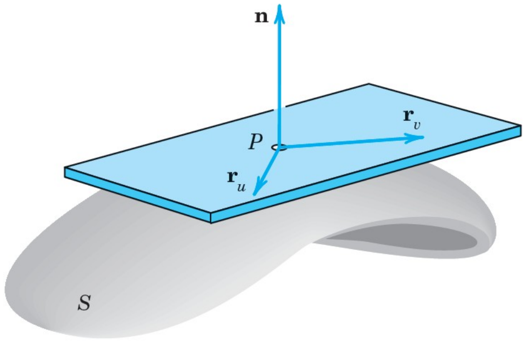
$$\mathbf{r}(u, v) = [x(u, v), y(u, v), z(u, v)] = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

(u, v) varies over a region *R* in the *uv*-plane. *S* has a normal vector:

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \qquad \text{and unit normal vector} \qquad \mathbf{n} = \frac{1}{|\mathbf{N}|}\mathbf{N}$$

Surface integral over *S* is defined by:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dA = \iint_R \mathbf{F}(\mathbf{r}(u, v)) \cdot \mathbf{N}(u, v)\, du\, dv.$$



$$\mathbf{n}\, dA = \mathbf{n}|\mathbf{N}|\, du\, dv = \mathbf{N}\, du\, dv.$$

If the surface integral is written in components:

$\mathbf{F} = [F_1, F_2, F_3]$, $\boldsymbol{N} = [N_1, N_2, N_3]$, $\boldsymbol{n} = [\cos\alpha, \cos\beta, \cos\gamma]$. α, β, γ are the angles between $\boldsymbol{n}$ and the coordinate axes.

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dA = \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, dA$$

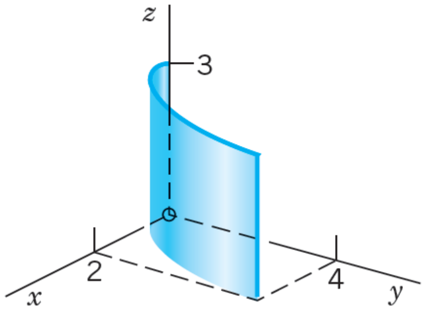
$$= \iint_R (F_1 N_1 + F_2 N_2 + F_3 N_3)\, du\, dv.$$

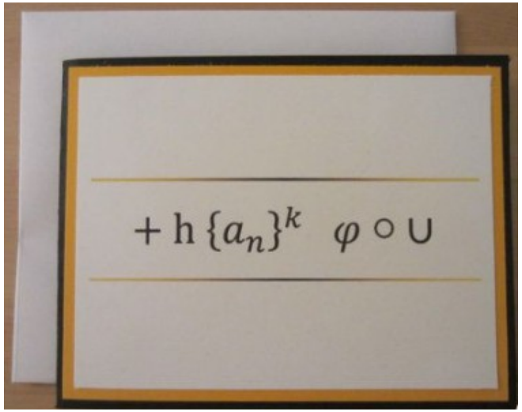
We also have that $\cos\alpha dA = dydz$, $\cos\beta dA = dzdx$, $\cos\gamma dA = dxdy$, then:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dA = \iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy).$$

Practice exercise:

Compute the flux of water through the parabolic cylinder S: $y = x^2$, $0 \leqq x \leqq 2$, $0 \leqq z \leqq 3$ (Fig. 245) if the velocity vector is $\mathbf{v} = \mathbf{F} = [3z^2, \quad 6, \quad 6xz]$, speed being measured in meters/sec. (Generally, $\mathbf{F} = \rho\mathbf{v}$, but water has the density $\rho = 1 \text{ g/cm}^3 = 1 \text{ ton/m}^3$.)




$+ \mathrm{h} \{a_n\}^k \quad \varphi \circ \cup$