Fundamental Vector Calculus

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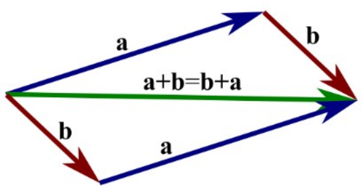
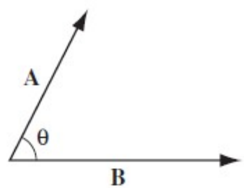
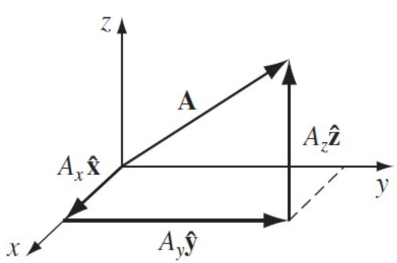
Learning objectives:

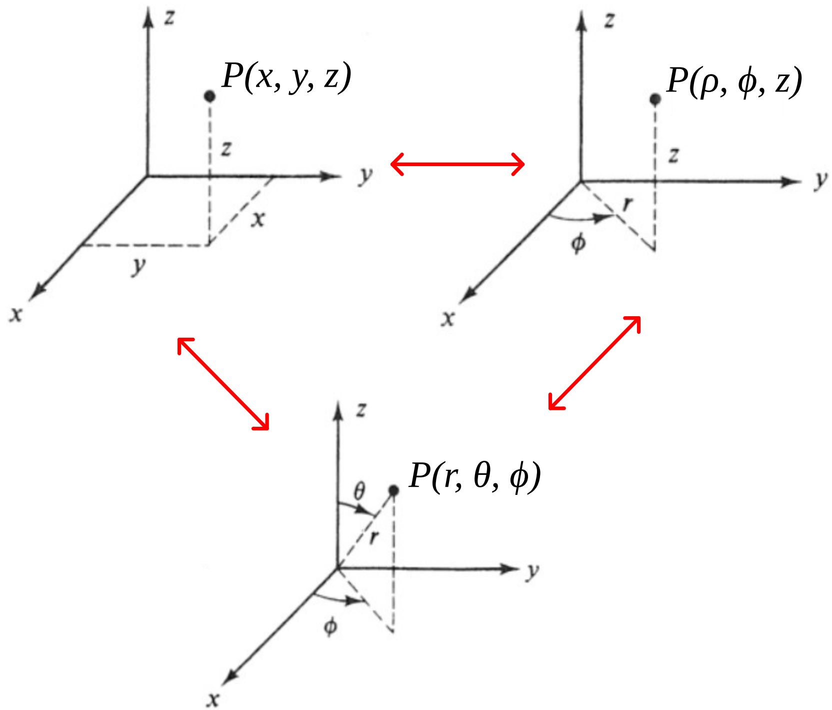
- **Vector algebra;**
  - Vector operations
  - Inner product;
  - Cross product;
  - …



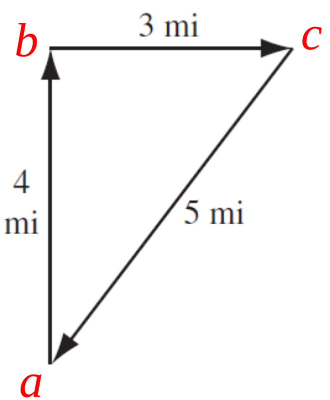




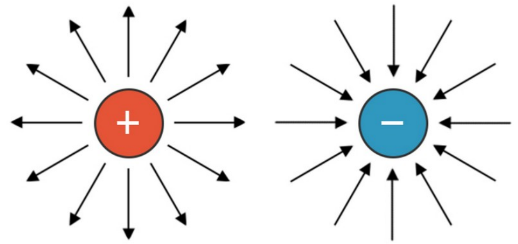
- **Coordinate transformation;**



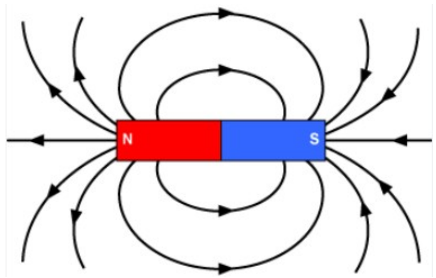
Why do we need to do vector algebra?



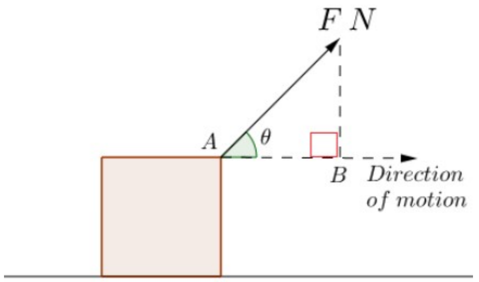
Electric field



Magnetic field



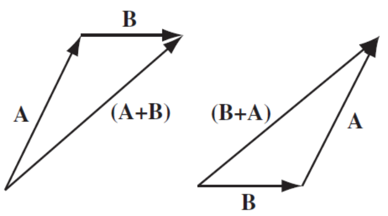
Force



……

Recap of fundamental vector operations:

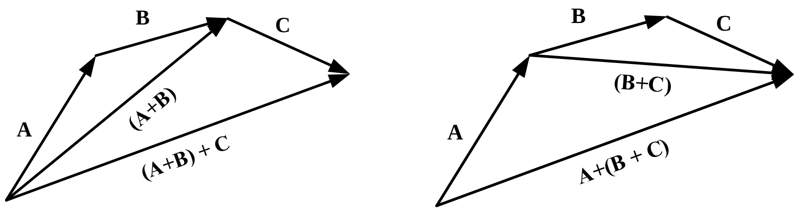
- **Addition of two vectors**:



$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A};$

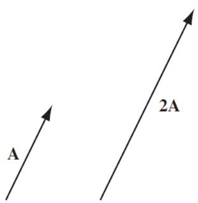
(***commutative***)

$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}).$ (***associative***)



- **Multiplication by a scalar**:

$a(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B}.$ (***distributive***)

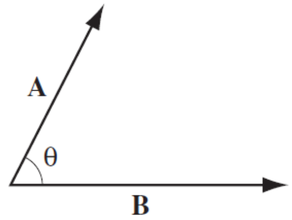


How to prove?

- **Dot product of a two vectors**:

$\mathbf{A} \cdot \mathbf{B} \equiv AB\cos\theta,$

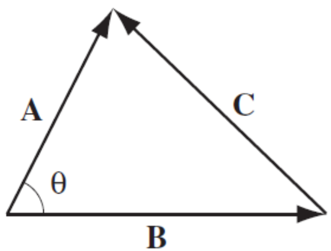
$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A},$



$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}.$ ***How to prove?***

Recap of fundamental vector operations (continued):

Using dot product to establish the relationship between the three lengths of a triangle:

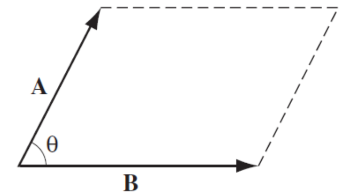


$$\mathbf{C}\cdot\mathbf{C} = (\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) = \mathbf{A}\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{A} + \mathbf{B}\cdot\mathbf{B},$$

(***distributive***)

$$C^2 = A^2 + B^2 - 2AB\cos\theta.$$

- **Cross product of two vectors**:



$$\mathbf{A}\times\mathbf{B} \equiv AB\sin\theta\,\hat{\mathbf{n}},$$ (right-hand rule)

$$(\mathbf{B}\times\mathbf{A}) = -(\mathbf{A}\times\mathbf{B}).$$ (not ***commutative***)

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = (\mathbf{A}\times\mathbf{B}) + (\mathbf{A}\times\mathbf{C}),$$ (***distributive***)

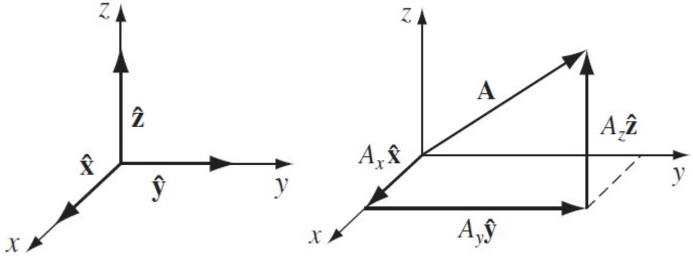
How to prove?

Is the cross product associative?

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} \stackrel{?}{=} \mathbf{A}\times(\mathbf{B}\times\mathbf{C}).$$

Recap of fundamental vector operations (continued):

A vector expressed in component form:



$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}.$$

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1;$$

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{x}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = 0.$$

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \cdot (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) \\ &= A_xB_x + A_yB_y + A_zB_z.\end{aligned}$$

$$\begin{aligned}\hat{\mathbf{x}} \times \hat{\mathbf{x}} &= \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = \mathbf{0}, \\ \hat{\mathbf{x}} \times \hat{\mathbf{y}} &= -\hat{\mathbf{y}} \times \hat{\mathbf{x}} = \hat{\mathbf{z}}, \\ \hat{\mathbf{y}} \times \hat{\mathbf{z}} &= -\hat{\mathbf{z}} \times \hat{\mathbf{y}} = \hat{\mathbf{x}}, \\ \hat{\mathbf{z}} \times \hat{\mathbf{x}} &= -\hat{\mathbf{x}} \times \hat{\mathbf{z}} = \hat{\mathbf{y}}.\end{aligned}$$

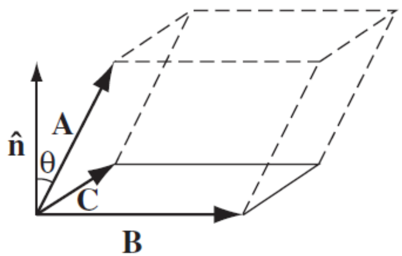
$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \times (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) \\ &= (A_yB_z - A_zB_y)\hat{\mathbf{x}} + (A_zB_x - A_xB_z)\hat{\mathbf{y}} + (A_xB_y - A_yB_x)\hat{\mathbf{z}}.\end{aligned}$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$

Recap of fundamental vector operations (continued):

- **Scalar triple products**:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}),$$



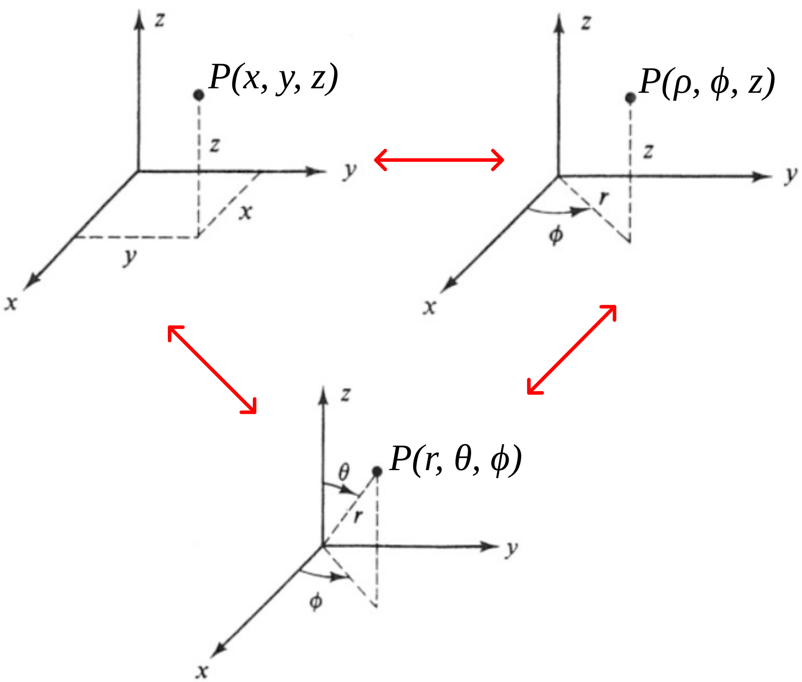
$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}.$$

- **Vector triple products**:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}).$$

How to prove?

Coordinate transformation:



In the Cartesian coordinate system:

$$P(x, y, z) \qquad -\infty < x < +\infty,\quad -\infty < y < +\infty,\quad -\infty < z < +\infty$$

In the Cylindrical coordinate system:

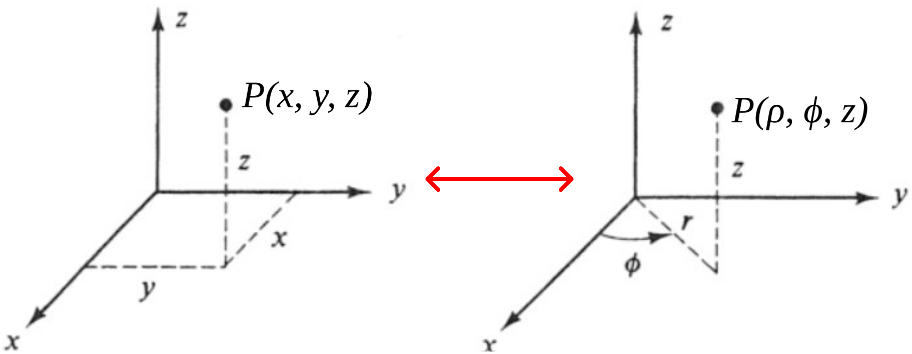
$$P(\rho, \phi, z) \qquad 0 \le \rho < +\infty,\quad 0 \le \phi \le 2\pi,\quad -\infty < z < +\infty$$

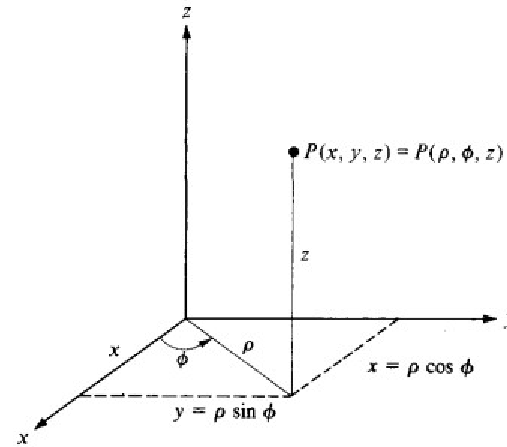
In the Spherical coordinate system:

$$P(r, \theta, \phi) \qquad 0 \le r < +\infty,\quad 0 \le \theta \le \pi,\quad 0 \le \phi \le 2\pi$$

How to perform coordinate transformation?

From the Cartesian to the Cylindrical coordinates:





A vector **A** in the Cylindrical coordinates can be written as:

$$(A_\rho, A_\phi, A_z) \qquad \vec{A} = \vec{a}_\rho A_\rho + \vec{a}_\phi A_\phi + \vec{a}_z A_z$$

$$\mathbf{a}_\rho \cdot \mathbf{a}_\rho = \mathbf{a}_\phi \cdot \mathbf{a}_\phi = \mathbf{a}_z \cdot \mathbf{a}_z = 1$$

$$\mathbf{a}_\rho \cdot \mathbf{a}_\phi = \mathbf{a}_\phi \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_\rho = 0$$

$$\mathbf{a}_\rho \times \mathbf{a}_\phi = \mathbf{a}_z$$

$$\mathbf{a}_\phi \times \mathbf{a}_z = \mathbf{a}_\rho$$

$$\mathbf{a}_z \times \mathbf{a}_\rho = \mathbf{a}_\phi$$

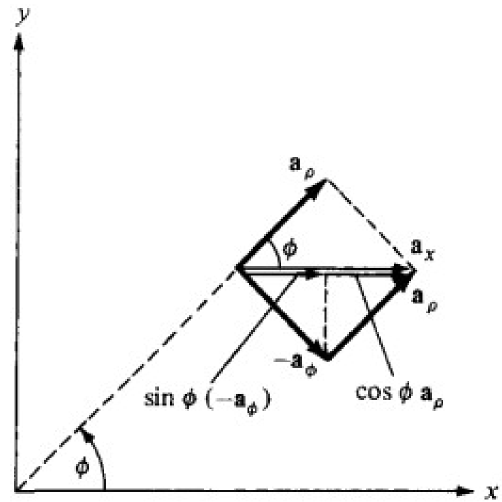
From the Cartesian to the Cylindrical coordinates:

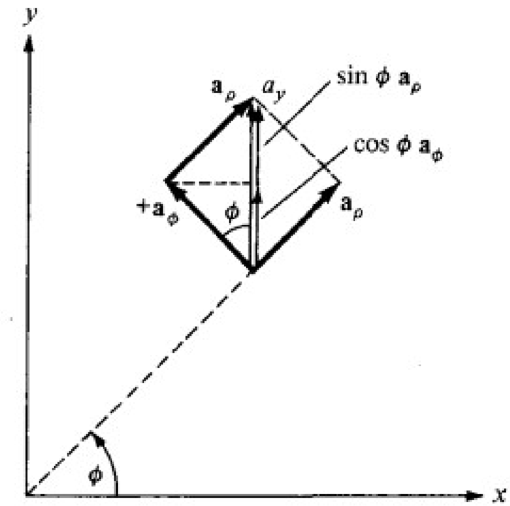
$$\boxed{\rho = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}, \qquad z = z}$$

From the Cylindrical to the Cartesian coordinates:

$$\boxed{x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad z = z}$$

The relationships between the unit vectors in the Cartesian and the Cylindrical coordinates :





$$\mathbf{a}_x = \cos\phi\,\mathbf{a}_\rho - \sin\phi\,\mathbf{a}_\phi$$

$$\mathbf{a}_y = \sin\phi\,\mathbf{a}_\rho + \cos\phi\,\mathbf{a}_\phi$$

$$\mathbf{a}_z = \mathbf{a}_z$$

$$\mathbf{a}_\rho = \cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y$$

$$\mathbf{a}_\phi = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y$$

$$\mathbf{a}_z = \mathbf{a}_z$$

$$\mathbf{a}_x = \cos\phi\,\mathbf{a}_\rho - \sin\phi\,\mathbf{a}_\phi$$

$$\mathbf{a}_y = \sin\phi\,\mathbf{a}_\rho + \cos\phi\,\mathbf{a}_\phi$$

$$\mathbf{a}_z = \mathbf{a}_z$$

$$\vec{A} = \vec{a}_x A_x + \vec{a}_y A_y + \vec{a}_z A_z \Longrightarrow$$

$$\begin{aligned}\vec{A} &= \vec{a}_x A_x + \vec{a}_y A_y + \vec{a}_z A_z \\ &= (\vec{a}_\rho \cos\phi - \vec{a}_\phi \sin\phi)A_x + (\vec{a}_\rho \sin\phi + \vec{a}_\phi \cos\phi)A_y + \vec{a}_z A_z \\ &= \vec{a}_\rho (A_x \cos\phi + A_y \sin\phi) + \vec{a}_\phi(-A_x \sin\phi + A_y \cos\phi) + \vec{a}_z A_z\end{aligned}$$

$$\boxed{\begin{aligned}A_\rho &= A_x \cos\phi + A_y \sin\phi \\ A_\phi &= -A_x \sin\phi + A_y \cos\phi \\ A_z &= A_z\end{aligned}}$$

$$\mathbf{a}_\rho = \cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y$$

$$\mathbf{a}_\phi = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y$$

$$\mathbf{a}_z = \mathbf{a}_z$$

$$\vec{A} = \vec{a}_\rho A_\rho + \vec{a}_\phi A_\phi + \vec{a}_z A_z \Longrightarrow$$

$$\begin{aligned}\vec{A} &= \vec{a}_\rho A_\rho + \vec{a}_\phi A_\phi + \vec{a}_z A_z \\ &= (\vec{a}_x \cos\phi + \vec{a}_y \sin\phi)A_\rho + (-\vec{a}_x \sin\phi + \vec{a}_y \cos\phi)A_\phi + \vec{a}_z A_z \\ &= \vec{a}_x (A_\rho \cos\phi - A_\phi \sin\phi) + \vec{a}_y (A_\rho \sin\phi + A_\phi \cos\phi) + \vec{a}_z A_z\end{aligned}$$

$$\boxed{\begin{aligned}A_x &= A_\rho \cos\phi - A_\phi \sin\phi \\ A_y &= A_\rho \sin\phi + A_\phi \cos\phi \\ A_z &= A_z\end{aligned}}$$

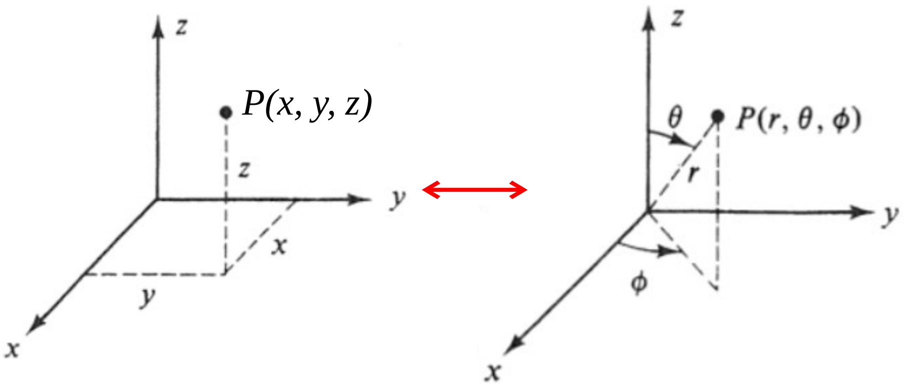
In matrix form, we have the transformation of vector **A** from (A_x, A_y, A_z) to (A_ρ, A_ϕ, A_z) as:

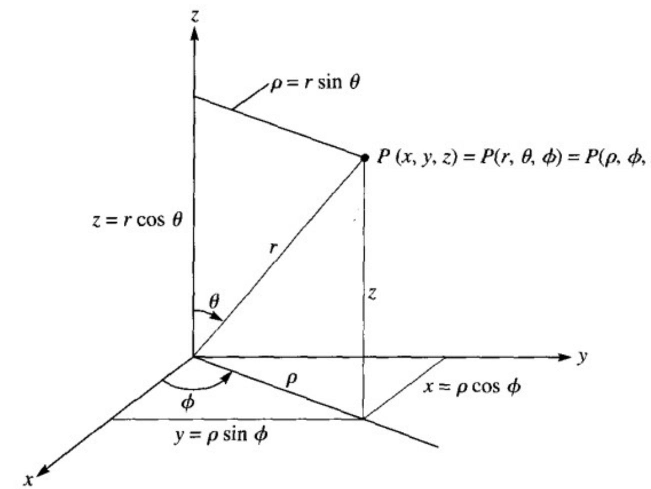
$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

from (A_ρ, A_ϕ, A_z) to (A_x, A_y, A_z) as:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

From the Cartesian to the Spherical coordinates:





A vector **A** in the Spherical coordinate can be written as:

(A_r, A_θ, A_ϕ) $\vec{A} = \vec{a}_r A_r + \vec{a}_\theta A_\theta + \vec{a}_\phi A_\phi$

$$\mathbf{a}_r \cdot \mathbf{a}_r = \mathbf{a}_\theta \cdot \mathbf{a}_\theta = \mathbf{a}_\phi \cdot \mathbf{a}_\phi = 1$$

$$\mathbf{a}_r \cdot \mathbf{a}_\theta = \mathbf{a}_\theta \cdot \mathbf{a}_\phi = \mathbf{a}_\phi \cdot \mathbf{a}_r = 0$$

$$\mathbf{a}_r \times \mathbf{a}_\theta = \mathbf{a}_\phi$$

$$\mathbf{a}_\theta \times \mathbf{a}_\phi = \mathbf{a}_r$$

$$\mathbf{a}_\phi \times \mathbf{a}_r = \mathbf{a}_\theta$$

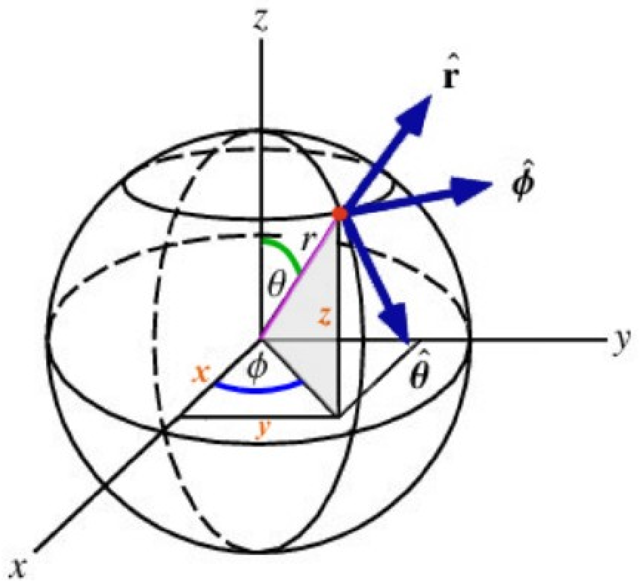
From the Cartesian to the Spherical coordinates:

$$\boxed{r = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}, \qquad \phi = \tan^{-1}\frac{y}{x}}$$

From the Spherical to theCartesian coordinates:

$$\boxed{x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta}$$

The relationships between the unit vectors in the Cartesian and the Spherical coordinates :



$$\mathbf{a}_x = \sin\theta\cos\phi\,\mathbf{a}_r + \cos\theta\cos\phi\,\mathbf{a}_\theta - \sin\phi\,\mathbf{a}_\phi$$

$$\mathbf{a}_y = \sin\theta\sin\phi\,\mathbf{a}_r + \cos\theta\sin\phi\,\mathbf{a}_\theta + \cos\phi\,\mathbf{a}_\phi$$

$$\mathbf{a}_z = \cos\theta\,\mathbf{a}_r - \sin\theta\,\mathbf{a}_\theta$$

$$\mathbf{a}_r = \sin\theta\cos\phi\,\mathbf{a}_x + \sin\theta\sin\phi\,\mathbf{a}_y + \cos\theta\,\mathbf{a}_z$$

$$\mathbf{a}_\theta = \cos\theta\cos\phi\,\mathbf{a}_x + \cos\theta\sin\phi\,\mathbf{a}_y - \sin\theta\,\mathbf{a}_z$$

$$\mathbf{a}_\phi = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y$$

$$\vec{A} = \vec{a}_x A_x + \vec{a}_y A_y + \vec{a}_z A_z$$

$$\mathbf{a}_x = \sin\theta\cos\phi\,\mathbf{a}_r + \cos\theta\cos\phi\,\mathbf{a}_\theta - \sin\phi\,\mathbf{a}_\phi$$
$$\mathbf{a}_y = \sin\theta\sin\phi\,\mathbf{a}_r + \cos\theta\sin\phi\,\mathbf{a}_\theta + \cos\phi\,\mathbf{a}_\phi$$
$$\mathbf{a}_z = \cos\theta\,\mathbf{a}_r - \sin\theta\,\mathbf{a}_\theta$$

$$\begin{aligned}
\vec{A} &= \vec{a}_x A_x + \vec{a}_y A_y + \vec{a}_z A_z \\
&= (\vec{a}_r \sin\theta\cos\phi + \vec{a}_\theta \cos\theta\cos\phi - \vec{a}_\phi \sin\phi)A_x + \\
&\quad (\vec{a}_r \sin\theta\sin\phi + \vec{a}_\theta \cos\theta\sin\phi + \vec{a}_\phi \cos\phi)A_y + (\vec{a}_r \cos\theta - \vec{a}_\theta \sin\phi)A_z \\
&= \vec{a}_r (A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z \cos\theta) + \\
&\quad \vec{a}_\theta (A_x \cos\theta\cos\phi + A_y \cos\theta\sin\phi - A_z \sin\theta) + \\
&\quad \vec{a}_\phi (-A_x \sin\phi + A_y \cos\phi)
\end{aligned}$$

$$\boxed{\begin{aligned}
A_r &= A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z \cos\theta \\
A_\theta &= A_x \cos\theta\cos\phi + A_y \cos\theta\sin\phi - A_z \sin\theta \\
A_\phi &= -A_x \sin\phi + A_y \cos\phi
\end{aligned}}$$

$$\vec{A} = \vec{a}_r A_r + \vec{a}_\theta A_\theta + \vec{a}_\phi A_\phi$$

$$\mathbf{a}_r = \sin\theta\cos\phi\,\mathbf{a}_x + \sin\theta\sin\phi\,\mathbf{a}_y + \cos\theta\,\mathbf{a}_z$$
$$\mathbf{a}_\theta = \cos\theta\cos\phi\,\mathbf{a}_x + \cos\theta\sin\phi\,\mathbf{a}_y - \sin\theta\,\mathbf{a}_z$$
$$\mathbf{a}_\phi = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y$$

$$\begin{aligned}
\vec{A} &= \vec{a}_r A_r + \vec{a}_\theta A_\theta + \vec{a}_\phi A_\phi \\
&= (\vec{a}_x \sin\theta\cos\phi + \vec{a}_y \sin\theta\sin\phi + \vec{a}_z \cos\theta)A_r + \\
&\quad (\vec{a}_x \cos\theta\cos\phi + \vec{a}_y \cos\theta\sin\phi - \vec{a}_z \sin\theta)A_\theta + (-\vec{a}_x \sin\phi + \vec{a}_y \cos\phi)A_\phi \\
&= \vec{a}_x (A_r \sin\theta\cos\phi + A_\theta \cos\theta\cos\phi - A_\phi \sin\phi) + \\
&\quad \vec{a}_y (A_r \sin\theta\sin\phi + A_\theta \cos\theta\sin\phi + A_\phi \cos\phi) + \\
&\quad \vec{a}_z (A_r \cos\theta - A_\theta \sin\theta)
\end{aligned}$$

$$\boxed{\begin{aligned}
A_x &= A_r \sin\theta\cos\phi + A_\theta \cos\theta\cos\phi - A_\phi \sin\phi \\
A_\theta &= A_r \sin\theta\sin\phi + A_\theta \cos\theta\sin\phi + A_\phi \cos\phi \\
A_\phi &= A_r \cos\theta - A_\theta \sin\theta
\end{aligned}}$$

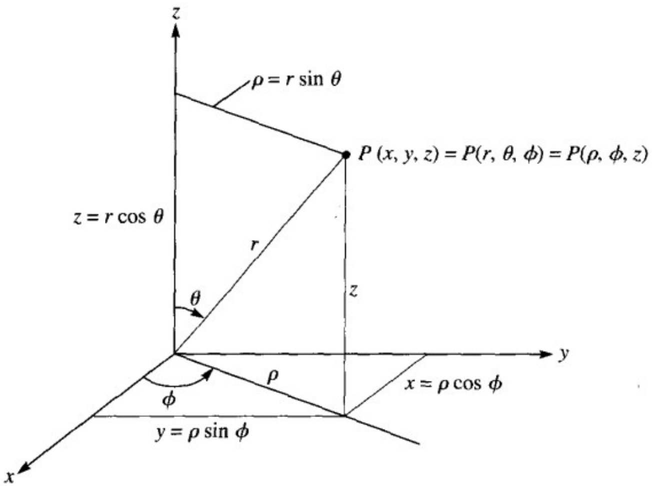
In matrix form, we have the transformation of vector **A** from (A_x, A_y, A_z) to (A_r, A_θ, A_ϕ) as:

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -\cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

from (A_r, A_θ, A_ϕ) to (A_x, A_y, A_z) as:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

From the Cylindrical to the Spherical coordinates:



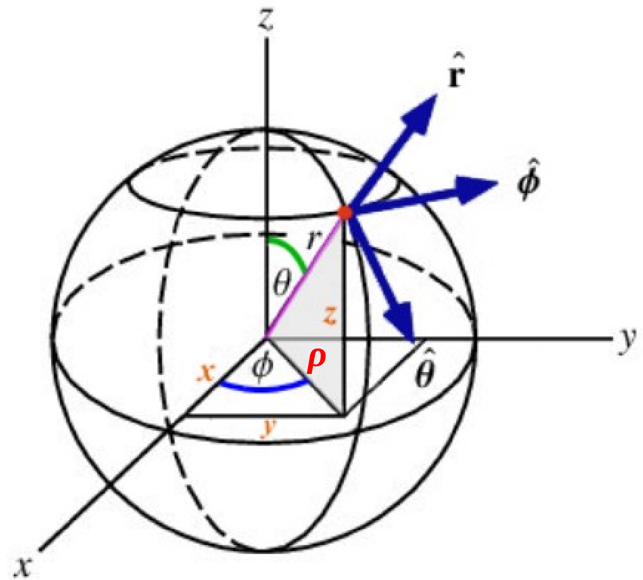
From the Cylindrical to the Spherical coordinates:

$$r = \sqrt{\rho^2 + z^2} \qquad \theta = \tan^{-1}(\rho/z) \qquad \phi = \phi$$

From the Spherical to the Cylindrical coordinates:

$$\rho = r\sin\theta \qquad \phi = \phi \qquad z = r\cos\theta$$

The relationships between the unit vectors in the Cylindrical and the Spherical coordinates :



$$\vec{a}_r = \vec{a}_\rho \sin\theta + \vec{a}_z \cos\theta$$
$$\vec{a}_\theta = \vec{a}_\rho \cos\theta - \vec{a}_z \sin\theta$$
$$\vec{a}_\phi = \vec{a}_\phi$$

$$\vec{a}_\rho = \vec{a}_r \sin\theta + \vec{a}_\theta \cos\theta$$
$$\vec{a}_\phi = \vec{a}_\phi$$
$$\vec{a}_z = \vec{a}_r \cos\theta - \vec{a}_\theta \sin\theta$$

$$\vec{a}_\rho = \vec{a}_r \sin\theta + \vec{a}_\theta \cos\theta$$
$$\vec{a}_\phi = \vec{a}_\phi$$
$$\vec{a}_z = \vec{a}_r \cos\theta - \vec{a}_\theta \sin\theta$$

$$\vec{A} = \vec{a}_\rho A_\rho + \vec{a}_\phi A_\phi + \vec{a}_z A_z \Longrightarrow$$

$$\begin{aligned}\vec{A} &= \vec{a}_\rho A_\rho + \vec{a}_\phi A_\phi + \vec{a}_z A_z \\ &= (\vec{a}_r \sin\theta + \vec{a}_\theta \cos\theta) A_\rho + \vec{a}_\phi A_\phi + (\vec{a}_r \cos\theta - \vec{a}_\theta \sin\theta) A_z \\ &= \vec{a}_r (A_\rho \sin\theta + A_z \cos\theta) + \vec{a}_\theta (A_\rho \cos\theta - A_z \sin\theta) + \vec{a}_\phi A_\phi\end{aligned}$$

$$\boxed{\begin{aligned} A_r &= A_\rho \sin\theta + A_z \cos\theta \\ A_\theta &= A_\rho \cos\theta - A_z \sin\theta \\ A_\phi &= A_\phi \end{aligned}}$$

$$\vec{a}_r = \vec{a}_\rho \sin\theta + \vec{a}_z \cos\theta$$
$$\vec{a}_\theta = \vec{a}_\rho \cos\theta - \vec{a}_z \sin\theta$$
$$\vec{a}_\phi = \vec{a}_\phi$$

$$\vec{A} = \vec{a}_r A_r + \vec{a}_\theta A_\theta + \vec{a}_\phi A_\phi \Longrightarrow$$

$$\begin{aligned}\vec{A} &= \vec{a}_r A_r + \vec{a}_\theta A_\theta + \vec{a}_\phi A_\phi \\ &= (\vec{a}_\rho \sin\theta + \vec{a}_z \cos\theta) A_r + (\vec{a}_\rho \cos\theta - \vec{a}_z \sin\theta) A_\theta + \vec{a}_\phi A_\phi \\ &= \vec{a}_\rho (A_r \sin\theta + A_\theta \cos\theta) + \vec{a}_\phi A_\phi + \vec{a}_z (A_r \cos\theta - A_\theta \sin\theta)\end{aligned}$$

$$\boxed{\begin{aligned} A_\rho &= A_r \sin\theta + A_\theta \cos\theta \\ A_\phi &= A_\phi \\ A_z &= A_r \cos\theta - A_\theta \sin\theta \end{aligned}}$$

Cartesian to Cylindrical	$\rho = \sqrt{x^2 + y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\vec{a}_\rho = \vec{a}_x \cos\phi + \vec{a}_y \sin\phi$ $\vec{a}_\phi = -\vec{a}_x \sin\phi + \vec{a}_y \sin\phi$ $\vec{a}_z = \vec{a}_z$	$A_\rho = A_x \cos\phi + A_y \sin\phi$ $A_\phi = -A_x \sin\phi + A_y \cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = \rho\cos\phi$ $y = \rho\sin\phi$ $z = z$	$\vec{a}_x = \vec{a}_\rho \cos\phi - \vec{a}_\phi \sin\phi$ $\vec{a}_y = \vec{a}_\rho \sin\phi + \vec{a}_\phi \cos\phi$ $\vec{a}_z = \vec{a}_z$	$A_x = A_\rho \cos\phi - A_\phi \sin\phi$ $A_y = A_\rho \sin\phi + A_\phi \cos\phi$ $A_z = A_z$
Cartesian to Spherical	$r = \sqrt{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}(\sqrt{x^2 + y^2}/z)$ $\phi = \tan^{-1}(y/x)$	$\vec{a}_r = \vec{a}_x \sin\theta\cos\phi + \vec{a}_y \sin\theta\sin\phi + \vec{a}_z \cos\theta$ $\vec{a}_\theta = \vec{a}_x \cos\theta\cos\phi + \vec{a}_y \cos\theta\sin\phi - \vec{a}_z \sin\theta$ $\vec{a}_\phi = -\vec{a}_x \sin\phi + \vec{a}_y \cos\phi$	$A_r = A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z \cos\theta$ $A_\theta = A_x \cos\theta\cos\phi + A_y \cos\theta\sin\phi - A_z \sin\theta$ $A_\phi = -A_x \sin\phi + A_y \cos\phi$
Spherical to Cartesian	$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$	$\vec{a}_x = \vec{a}_r \sin\theta\cos\phi + \vec{a}_\theta \cos\theta\cos\phi - \vec{a}_\phi \sin\phi$ $\vec{a}_y = \vec{a}_r \sin\theta\sin\phi + \vec{a}_\theta \cos\theta\sin\phi + \vec{a}_\phi \cos\phi$ $\vec{a}_z = \vec{a}_r \cos\theta - \vec{a}_\theta \sin\theta$	$A_x = A_r \sin\theta\cos\phi + A_\theta \cos\theta\cos\phi - A_\phi \sin\phi$ $A_y = A_r \sin\theta\sin\phi + A_\theta \cos\theta\sin\phi + A_\phi \cos\phi$ $A_z = A_r \cos\theta - A_\theta \sin\theta$
Cylindrical to Spherical	$r = \sqrt{\rho^2 + z^2}$ $\theta = \tan^{-1}(\rho/z)$ $\phi = \phi$	$\vec{a}_r = \vec{a}_\rho \sin\theta + \vec{a}_z \cos\theta$ $\vec{a}_\theta = \vec{a}_\rho \cos\theta - \vec{a}_z \sin\theta$ $\vec{a}_\phi = \vec{a}_\phi$	$A_r = A_\rho \sin\theta + A_z \cos\theta$ $A_\theta = A_\rho \cos\theta - A_z \sin\theta$ $A_\phi = A_\phi$
Spherical to Cylindrical	$\rho = r\sin\theta$ $\phi = \phi$ $z = r\cos\theta$	$\vec{a}_\rho = \vec{a}_r \sin\theta + \vec{a}_\theta \cos\theta$ $\vec{a}_\phi = \vec{a}_\phi$ $\vec{a}_z = \vec{a}_r \cos\theta - \vec{a}_\theta \sin\theta$	$A_\rho = A_r \sin\theta + A_\theta \cos\theta$ $A_\phi = A_\phi$ $A_z = A_r \cos\theta - A_\theta \sin\theta$

Practice exercise:

Express vector

$$\mathbf{B} = \frac{10}{r}\,\mathbf{a}_r + r\cos\theta\,\mathbf{a}_\theta + \mathbf{a}_\phi$$

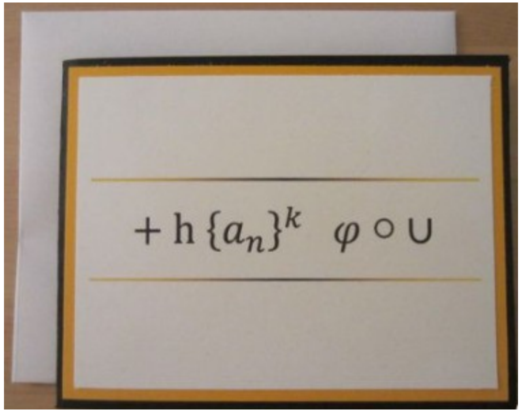
in Cartesian and cylindrical coordinates. Find $\mathbf{B}\,(-3, 4, 0)$ and $\mathbf{B}\,(5, \pi/2, -2)$.

Practice exercise:

Express the following vectors in Cartesian coordinates:

(a). $\vec{A} = \vec{a}_\rho \rho z \sin\phi + \vec{a}_\phi 3\rho \cos\phi + \vec{a}_z \rho \cos\phi \sin\phi$

(b). $\vec{B} = \vec{a}_r r^2 + \vec{a}_\phi \sin\theta$


$+ \mathrm{h} \{a_n\}^k \quad \varphi \circ \cup$