practice exercise of Lecture _ 2

1. $f(x, y, z) = 4x^2 + 2y^2 + 3z^2$

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$= 8x\vec{i} + 4y\vec{j} + 6z\vec{k} = [8x, 4y, 6z]$$

$$\vec{a} = [0, 1, -1], \quad \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{2}}[0, 1, -1] = [0, \tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}]$$

$$D_{\vec{a}}f = \nabla f \cdot \frac{\vec{a}}{|\vec{a}|} = [8x, 4y, 6z] \cdot [0, \tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}] \Big|_{P(1,3,2)}$$

$$= 2\sqrt{2}y - 3\sqrt{2}z \Big|_{P(1,3,2)}$$

$$= 2\sqrt{2} \times 3 - 3\sqrt{2} \times 2$$

$$= 0$$

2.

$\vec{V} = [x^2, 4y^2, 9z^2]$, $P: (-1, 0, 0.5)$

~~div V = ∇·V~~

$$\operatorname{div}\vec{V} = \nabla \cdot \vec{V} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$v_1 = x^2, \quad v_2 = 4y^2, \quad v_3 = 9z^2$$

$$\frac{\partial v_1}{\partial x} = 2x, \quad \frac{\partial v_2}{\partial y} = 8y, \quad \frac{\partial v_3}{\partial z} = 18z$$

$$\operatorname{div}\vec{V} = \nabla \cdot \vec{V} = 2x + 8y + 18z \Big|_{P(-1, 0, 0.5)}$$

$$= 2 \times (-1) + 8 \times 0 + 18 \times 0.5$$

$$= -2 + 0 + 9$$

$$= 7$$

$\vec{v} = [0, \cos(xyz), \sin(xyz)],\ P(2, \frac{1}{2}\pi, 0)$

$\operatorname{div} \vec{v} = \nabla \cdot \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$

$v_1 = 0,\quad v_2 = \cos(xyz),\quad v_3 = \sin(xyz)$

$\frac{\partial v_1}{\partial x} = 0,\quad \frac{\partial v_2}{\partial y} = -xz\sin(xyz),\quad \frac{\partial v_3}{\partial z} = xy\cos(xyz)$

$$\operatorname{div} \vec{v} = \nabla \cdot \vec{v} = 0 - xz\sin(xyz) + xy\cos(xyz) \Big|_{P(2, \frac{1}{2}\pi, 0)}$$

$$= 0 - 2 \times 0 \times \sin(0) + 2 \times \frac{1}{2}\pi \times \cos 0$$

$$= \pi$$

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3. ① $f(x, y, z) = e^{xyz}$

$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

$\frac{\partial f}{\partial x} = yze^{xyz},\quad \frac{\partial f}{\partial y} = xze^{xyz},\quad \frac{\partial f}{\partial z} = xye^{xyz}$

$\frac{\partial^2 f}{\partial x^2} = y^2z^2e^{xyz},\quad \frac{\partial^2 f}{\partial y^2} = x^2z^2e^{xyz},\quad \frac{\partial^2 f}{\partial z^2} = x^2y^2e^{xyz}$

$\nabla^2 f = (y^2z^2 + x^2z^2 + x^2y^2)e^{xyz}$

② $f(\rho, \phi, z) = 2\rho^2 + 3\rho\cos\phi + \rho z^3$

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$\frac{\partial f}{\partial \rho} = 4\rho + 3\cos\phi + z^3,\quad \frac{\partial^2 f}{\partial \rho^2} = 4$

$$\frac{\partial f}{\partial \phi} = -3\rho \sin\phi \ , \ \frac{\partial^2 f}{\partial \phi^2} = -3\rho\cos\phi$$

$$\frac{\partial f}{\partial z} = 3\rho z^2, \ \frac{\partial^2 f}{\partial z^2} = 6\rho z$$

$$\nabla^2 f = 4 + \frac{1}{\rho}(4\rho + 3\cos\phi + z^3) + \frac{1}{\rho^2}(-3\rho\cos\phi) + 6\rho z$$

$$= 4 + 4 + \frac{3\cos\phi}{\rho} + \frac{z^3}{\rho} - \frac{3\cos\phi}{\rho} + 6\rho z$$

$$= 8 + \frac{z^3}{\rho} + 6\rho z$$

③ $f(r, \phi, \theta) = 2r^2\sin\theta + 3r\cos\phi + 3\sin\theta$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

$$= 2\frac{\partial f}{\partial r} + r\frac{\partial^2 f}{\partial r^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial f}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

$$= r\frac{\partial^2 f}{\partial r^2} + 2\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial f}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

$$\frac{\partial f}{\partial r} = 4r\sin\theta + 3\cos\phi \ , \ \frac{\partial^2 f}{\partial r^2} = 4\sin\theta$$

$$\frac{\partial f}{\partial\theta} = 2r^2\cos\theta + 3\cos\theta, \ \frac{\partial^2 f}{\partial\theta^2} = -2r^2\sin\theta - 3\sin\theta$$

$$\frac{\partial f}{\partial\phi} = -3r\sin\phi. \ \frac{\partial^2 f}{\partial\phi^2} = -3r\cos\phi$$

$$\nabla^2 f = r\cdot 4\sin\theta + 2(4r\sin\theta + 3\cos\phi) + \frac{1}{r^2}(-2r^2\sin\theta - 3\sin\theta) + \frac{\cos\theta}{r^2\sin\theta}(2r^2\cos\theta + 3\cos\theta)$$

$$+ \frac{1}{r^2\sin^2\theta}\cdot(-3r\cos\phi)$$

$$= 12r\sin\theta + 6\cos\phi - 2\sin\theta - \frac{3\sin\theta}{r^2} + \frac{2\cos^2\theta}{\sin\theta} + \frac{3\cos^2\theta}{r^2\sin\theta} - \frac{3\cos\phi}{r\sin^2\theta}$$

4. prove $\operatorname{div}(\operatorname{curl}\vec{v}) = 0$

$$\vec{v} = [v_1, v_2, v_3] = v_1\vec{i} + v_2\vec{j} + v_3\vec{j}$$

$$\operatorname{curl}\vec{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\vec{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\vec{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\vec{k}$$

$$\operatorname{div}(\operatorname{curl}\vec{v}) = \nabla \cdot (\operatorname{curl}\vec{v}) = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \operatorname{curl}\vec{v}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$= \frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_2}{\partial x \partial z} + \frac{\partial^2 v_1}{\partial y \partial z} - \frac{\partial^2 v_3}{\partial y \partial x} + \frac{\partial^2 v_2}{\partial z \partial x} - \frac{\partial^2 v_1}{\partial z \partial y}$$

$$= 0$$