DISCRETE TIME SYSTEMS AND Z-TRANSFORM

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What we have learned in the previous lecture

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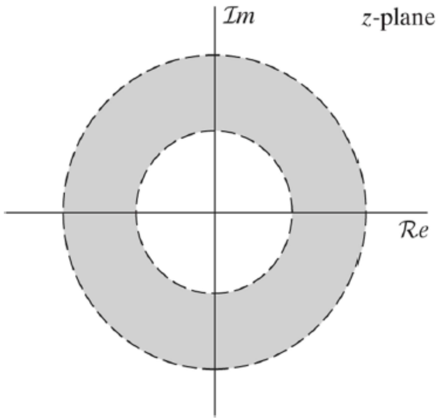
Z- transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}.$$



The set of values of z for which the z-tranform converges is called the region of convergence (ROC).

We have calculated z-trasform and ROC of
- Right sided exponential sequence
- Left Sided exponential sequence
- Sum of exponential sequences

We have studied the properties of the ROC, and common *z*-transform pairs.

The z-transform

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Today’s agenda

- Discrete time signals
  - Basic sequences and operations
  - Linear systems
  - Stability, causality, time invariance
- Linear time invariant (LTI) systems
  - Inpulse response and convolution
  - Parallel and cascade system combination
- Fourier transform of LTI systems
  - Definition and conditions for existence
- Z-transform
  - Definition and region of convergence (ROC)
  - Right, left-sided and finite duration sequences
  - ROC analysis
- **Inverse z-transform**
  - **Definition and inspection method**
  - **Partial fraction expansion**
  - **Power series expansion**
- Transform analysis of LTI systems
  - Linear constant coefficient difference equations
  - Stability and causality
  - Inverse systems
  - FIR and IIR systems

Inverse z-transform

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- Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

$z \in C$
$ROC: \{z|\, X(z) < \infty\}$

- Inverse z-transform:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz,$$

where C is a closed circle around the origin which includes all the poles for X(z).
In case C is the unit circle:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} (X(e^{i\omega}) \cdot e^{i\omega n}) d\omega$$

Inverse z-transform

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Inverse z-transform

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- How to calculate inverse z-transform?
  - Inspection method
  - Partial fraction expansion
  - Power series expansion

Inverse z-transform

- Less formal procedure than the Cauchy-Schwartz integral are preferable.
- **Inspection method:** ”recognizing” certain transform pairs

$$x[n] = \underline{(a)^n \cdot u[n]} \quad -\infty < n < \infty \quad \stackrel{Z}{\longleftrightarrow} \quad X(z)_h = \sum_{n=0}^{\infty} a^n \cdot z^{-n} = \frac{1}{1-a\cdot z^{-1}}; \quad \underline{|a| < |z|}$$

$$x[n] = \underline{-(a)^n \cdot u[-n-1]} \quad -\infty < n < \infty \stackrel{Z}{\longleftrightarrow} X(z)_v = -\sum_{n=1}^{\infty} a^{-n} \cdot z^n = \frac{1}{1-a\cdot z^{-1}}; \quad \underline{|z| < |a|}$$

- Example

$$X(z) = \left(\frac{1}{1-\frac{1}{2}z^{-1}}\right); \quad |z| > \frac{1}{2}$$

$$\Downarrow$$

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$

Inverse z-transform

- **Partial fraction expansion**

Sometimes inverse tranforms cannot be found via simple inspection. However, it might be possible to obtain an alternative expression for X(z) as a sum of simpler terms, each of which is tabulated.

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}.$$

$$X(z) = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}}.$$

M zeros and N poles at non-zero locations in the *z*-plane

M-N poles at z=0 if M>N
or
N-M zeros at z=0 if M<N

X(z) can be factorized as

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})},$$

If M<N and the poles are all first order, then

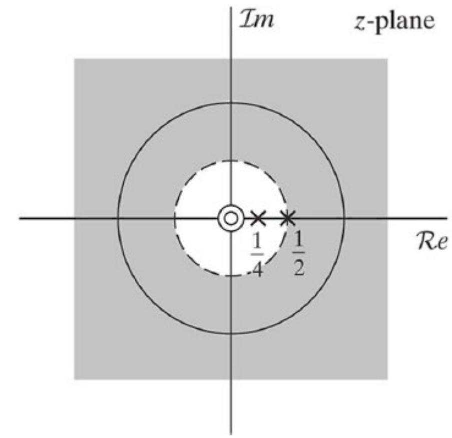
$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}.$$

Coefficients can be found as

$$A_k = (1 - d_k z^{-1}) X(z)\big|_{z=d_k}.$$

Inverse z-transform

- Example



$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}, \qquad |z| > \frac{1}{2}.$$

$$X(z) = \frac{A_1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{A_2}{\left(1 - \frac{1}{2}z^{-1}\right)}.$$

$$A_1 = \left(1 - \tfrac{1}{4}z^{-1}\right) X(z)\Big|_{z=1/4} = -1,$$

$$A_2 = \left(1 - \tfrac{1}{2}z^{-1}\right) X(z)\Big|_{z=1/2} = 2.$$

$$X(z) = \frac{-1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{2}z^{-1}\right)}.$$

$$x[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n].$$

Inverse z-transform

- If M>=N, the partial fraction expansion has the form

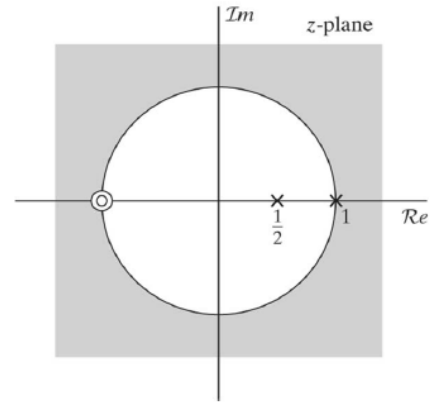
$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}.$$

- The Br coefficients can be found by long division of the numerator by the denumerator, with the division process terminating when the remainder is of lower degree of the denominator.

Inverse z-transform

- Example

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{(1 + z^{-1})^2}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}, \qquad |z| > 1.$$



$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}.$$

$$\begin{array}{r|l} & \quad 2 \\ \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & z^{-2} + 2z^{-1} + 1 \\ & \underline{z^{-2} - 3z^{-1} + 2} \\ & \quad 5z^{-1} - 1 \end{array}$$

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}.$$

$$A_1 = \left[\left(\frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}\right)\left(1 - \frac{1}{2}z^{-1}\right)\right]_{z=1/2} = -9,$$

$$A_2 = \left[\left(\frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}\right)(1 - z^{-1})\right]_{z=1} = 8.$$

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}.$$

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n].$$

Inverse z-transform

- The defining expression for the z-transform is a Laurent series where the sequence values x[n] are the coefficients of z^{-n}.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$
$$= \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots,$$

- Any value of the sequence can be then determined by finding the coefficient of the appropriate power of z^{-1}
- Example .

$$X(z) = z^2 - \tfrac{1}{2}z - 1 + \tfrac{1}{2}z^{-1}. \quad \Rightarrow \quad x[n] = \begin{cases} 1, & n = -2, \\ -\frac{1}{2}, & n = -1, \\ -1, & n = 0, \\ \frac{1}{2}, & n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$x[n] = \delta[n+2] - \tfrac{1}{2}\delta[n+1] - \delta[n] + \tfrac{1}{2}\delta[n-1].$$

Inverse z-transform

- Transform by power series expansion (example 2)

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a|.$$

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots.$$

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}.$$

$$x[n] = \begin{cases} (-1)^{n+1} \dfrac{a^n}{n}, & n \geq 1, \\ 0, & n \leq 0. \end{cases}$$