Lecture Notes for Data Acquisition

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Chapter 7

Dynamic sensor models

As has been discussed throughout the preceding chapters, the goal of data acquisition is to capture relevant information from physical systems, which are inherently dynamic. This information is carried by signals, which manifest themselves mathematically as functions of time. Often the systems that generate the signals involve various different flavors of physics, and the system models thus have to combine several different sub-systems, each with their own governing differential equations.

Before the signals can be converted to digital values, however, they have to be converted into electric signals that can be sampled (converted from analog to digital representations). This is the task of *sensors*, which are themselves physical devices that react to physical stimuli (the blood pressure of a patient, the acceleration of an autonomous quadcopter, the fuel flow of an engine, the temperature in a baking oven...) and generate electric signals that somehow represent that physical information. However, being physical systems themselves, all sensors necessarily have internal dynamics themselves, even though this is often ignored when modeling larger-scale systems.

This chapter has two main goals. Firstly, it aims to reformulate the input-output-centered view of dynamic operators taken up until now, into a new formalism, called *state space* representation, which better allows capturing different types of physics. In doing so, it will borrow several key concepts from linear algebra.

Secondly, it will present dynamic models of several types of sensors, partly to illustrate some of the techniques required to derive usable models of physical devices, and partly to explain how some common sensors convert different physical inputs into measurable electric signals.

7.1 Coupled first-order linear differential equations

Back in Chapter 3, more specifically in Definition 7, a dynamical operator was rather ruthlessly defined as a collection of multi-variable differential equations, whereupon it was quickly replaced by a more palatable formulation of linear time invariant differential equations; recall:

Definition 7 (repeated): *A* dynamic operator *is a mapping from one signal space $\mathcal{U}$ to another signal space $\mathcal{Y}$, which maps the value $u(t)$ of an element in $\mathcal{U}$ at time t to a unique value of an element $y(t)$ in $\mathcal{Y}$ via a differential equation:*

$$\begin{array}{rrcl} G: & \dot{x}_i & = & f_i(x_1(t), x_2(t), \ldots, x_n(t), u(t)), \quad i = 1, \ldots, n \\ & x_i(0) & = & x_{i0}, \quad i = 1, \ldots, n \\ & y(t) & = & h(x_1(t), x_2(t), \ldots, x_n(t), u(t)) \end{array}$$

where $x_i : \mathbb{R} \to \mathbb{R}$ are differentiable functions called states, x_{i0} are initial values for the state differential equations, and f_i together comprise a smooth vector field; h is a static operator mapping from states and inputs to outputs known as the measurement map.

At this point, we are better equipped to understand what this definition means. First of all, the meaning of $G : \mathcal{U} \to \mathcal{Y}$ is clear; G is an operator between the signal spaces $\mathcal{U}$ and $\mathcal{Y}$, each of which can be identified with $\mathcal{L}_2$, the linear space of square integrable functions. Secondly, the equations specifying the dynamics of G are apparently of the form

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1(t), x_2(t), \ldots, x_n(t), u(t)), & x_1(0) &= x_{1,0} \\ \dot{x}_2(t) &= f_2(x_1(t), x_2(t), \ldots, x_n(t), u(t)), & x_2(0) &= x_{2,0} \\ &\vdots & &\vdots \\ \dot{x}_{n-1}(t) &= f_{n-1}(x_1(t), x_2(t), \ldots, x_n(t), u(t)), & x_{n-1}(0) &= x_{n-1,0} \\ \dot{x}_n(t) &= f_n(x_1(t), x_2(t), \ldots, x_n(t), u(t)), & x_n(0) &= x_{n,0} \end{aligned} \tag{7.1}$$

where the variables $x_i : \mathbb{R} \to \mathbb{R}$, commonly called *states*, are solutions to first-order differential equations in the variables $x_i(t)$ themselves, as well as the external input $u(t)$; $x_{i,0} \in \mathbb{R}$ specify the initial conditions of each differential equation. The functions $f_i : \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \to \mathbb{R}$ determine the evolution of the states and are for convenience assumed to be smooth unless otherwise mentioned.

Finally, the output $y(t)$ is also a multivariable function of the values of $x_i(t)$, as well as—in rare cases—the input $u(t)$.

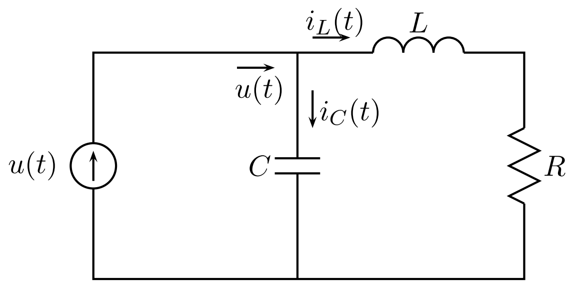

Figure 7.1: Circuit with capacitor and inductor in parallel. The input is the current source $u(t)$, while the measured output $y(t)$ is the voltage drop over the resistor.

To give an example, consider the RLC-circuit depicted in Figure 7.1. Using Kirchoff's current and voltage laws, one finds

$$L\frac{di_L(t)}{dt} = v_C(t) - Ri_L(t)$$

and

$$u(t) = i_C(t) + i_L(t)$$

where i_L is the current flowing through the inductor and the resistor, v_C is the voltage over the capacitor, and R, L and C are the usual component values (parameters). Furthermore,

$$i_C(t) = C\frac{dv_C(t)}{dt}.$$

Now define $x_1(t) = v_C(t)$, $x_2(t) = i_L(t)$ and $y(t) = v_R(t)$; then the circuit equations can be written as a system of two coupled linear first order differential equations and one linear equation mapping the states to the output:

$$\begin{aligned}
\dot{x}_1(t) = f_1(x_1(t), x_2(t), u(t)) &= -\frac{1}{C}x_2(t) + \frac{1}{C}u(t) \\
\dot{x}_2(t) = f_2(x_1(t), x_2(t), u(t)) &= \frac{1}{L}x_1(t) - \frac{R}{L}x_2(t) \\
y(t) = h(x_1(t), x_2(t), u(t)) &= Rx_2(t),
\end{aligned}$$

which clearly fits with the description of the linear dynamic operator above.

The astute reader should notice, how "natural" this description is. It follows immediately from the component equations; for each energy-storing element in the system, we get one first-order differential equation. Once these are written down, it is simply a matter of identifying states and inputs, and renaming variables according to Definition 7.

7.1.1 State space block diagrams

Chapter 3 postulated that a linear, time invariant dynamical operator of the form given in Definition 7 could be replaced by a single n'th order equation, but no justification why this was permissible was given. In the following, it will be made evident that a given n-th order differential equation can indeed be expressed as a set of coupled first-order equations as stated in Definition 7.[1]

To ease the progress towards this goal, we shall make use of the *block diagram* formalism introduced earlier in order to assist in envisioning the behavior of systems described by linear differential equations.

To start, consider the first-order equation

$$\dot{y}(t) = -a_0 y(t) + b_0 u(t) \tag{7.2}$$

with $b_0 \neq 0$. It was explained in Chapter 3 how this equation can be represented by the left block diagram in Figure 7.2. To summarize, start by drawing one integration block and place the time derivative signal in question ($\dot{y}(t)$) as input and its integral ($y(t)$) as output of the block. Then use multiplication blocks and summations to construct the input signal; here, $u(t)$ is multiplied by b_0 and used as one input to the summation point, while $y(t)$ is multiplied by $-a_0$ and used as a feedback input to the summation point.

However, (7.2) may also be constructed as shown in the right block diagram in Figure 7.2. Figuratively speaking, we make use of the linearity of the integrator, multiplication-by-constant, and summation blocks to move the constant factor b_0 "to the output" by introducing the variable $x(t) = \frac{1}{b_0} y(t)$, and (7.2) is rewritten as two coupled equations:

$$\begin{aligned} \dot{x}(t) &= -a_0 x(t) + u(t) \\ y(t) &= b_0 x(t) \end{aligned}$$

where the first equation is a first-order linear differential equation and the second is a linear equation that specifies $y(t)$ as a function of the auxiliary variable $x(t)$.

[1] In a later chapter, we will examine the transformation from state space to higher-order ODE form, in order to verify that the two representations are indeed equivalent.

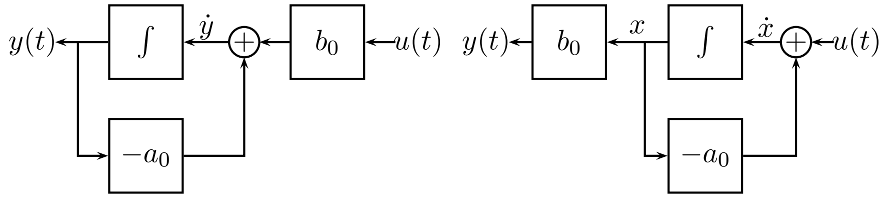

Figure 7.2: Block diagram of first-order system. Left: straightforward implementation of (7.2); right: alternative implementation with auxiliary variable $x(t) = \frac{1}{b_0}y(t)$.

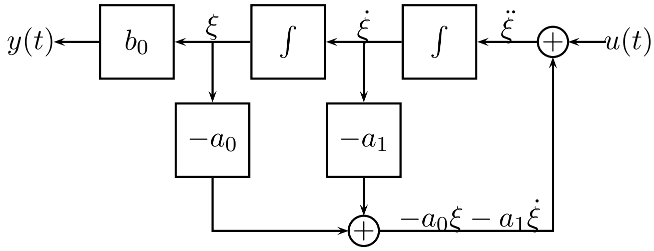

Figure 7.3: Block diagram of second-order system (7.3) with no derivatives of the input present.

For this simple system, the introduction of the auxiliary variable $x(t)$ is a rather trivial complication compared to the original formulation of the first-order equation in y and $\dot{y}$; however, this approach can be extended in a logical and effective way, as we shall see. Consider a second-order system

$$\ddot{y}(t) = -a_1\dot{y}(t) - a_0y(t) + b_0u(t) \tag{7.3}$$

with $b_0 \neq 0$. By introducing the auxiliary variable $\xi(t) = \frac{1}{b_0}y(t)$, we may construct a block diagram for (7.3) as shown in Figure 7.3.

This block diagram is clearly a representation of the two coupled equations:

$$\begin{aligned}\ddot{\xi}(t) &= -a_0\xi(t) - a_1\dot{\xi}(t) + u(t)\\ y(t) &= b_0\xi(t).\end{aligned}$$

However, by introducing yet another set of variables $x_1(t) = \xi(t), x_2(t) = \dot{\xi}(t)$, it may also be interpreted as the following *three* coupled equations; two linear first-order differential equations and one static linear equation:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -a_0x_1(t) - a_1x_2(t) + u(t)\\ y(t) &= b_0x_1(t).\end{aligned}$$

Note that each equation involves at most a first-order derivative wrt. time. This is useful, since it is normally significantly harder to integrate higher-order differential equations than first-order equations.

Next, consider a second-order system with a time-derivative of the input signal being part of the forcing function (ee Equation (3.3)):

$$\ddot{y}(t) = -a_1\dot{y}(t) - a_0y(t) + b_1\dot{u}(t) + b_0u(t). \tag{7.4}$$

For consistence, assume both b_0 and b_1 to be non-zero.[2] Since time-derivatives of the input are present, we have to have the auxiliary variable contain derivatives itself, i.e., be dynamic:

$$\xi(t) = \frac{1}{b_0}(y(t) - b_1\dot{\xi}(t)).$$

Differentiating ξ wrt. time yields

$$\dot{\xi}(t) = \frac{1}{b_0}(\dot{y}(t) - b_1\ddot{\xi}(t))$$

and hence

$$\begin{aligned}y(t) &= b_0\xi(t) + b_1\dot{\xi}(t)\\ \dot{y}(t) &= b_0\dot{\xi}(t) + b_1\ddot{\xi}(t)\\ \ddot{y}(t) &= -a_1(b_0\dot{\xi}(t) + b_1\ddot{\xi}(t)) - a_0(b_0\xi(t) + b_1\dot{\xi}(t)) + b_1\dot{u}(t) + b_0u(t) \qquad (7.5)\end{aligned}$$

which can be reduced to (7.4). The block diagram depicted in Figure 7.4 is thus a representation of (7.4).

7.1.2 Canonical form

The pattern outlined in the previous subsection can be continued as far as needed. Let us therefore consider the general case; a linear time invariant system operator is given by an n'th order differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0y = b_mu^{(m)} + \cdots + b_1\dot{u} + b_0u$$

[2] In the odd case where $b_0 = 0$, simply eliminate the lower connection to the leftmost summation point in Figure 7.4 and let $y(t) = b_1\dot{\xi}$. This situation is virtually never encountered in real-world applications, however.

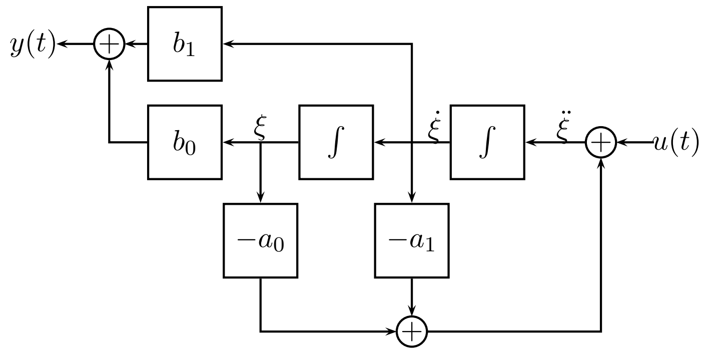

Figure 7.4: Block diagram of second-order system (7.4) with derivative of the input present. Note that the signal leaving the first integral ($\dot{\xi}$) is transmitted to all the arrow target blocks.

where, as usual, $u, y \in \mathcal{L}_2$ and $a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_m$ are real constants. Furthermore, let the initial conditions $y(0), \dot{y}(0), \ldots, y^{(n-1)}(0)$ be zero.

Inspired by the block diagram-building development in the previous subsection, let us introduce a function $\xi(t)$ satisfying

$$\frac{d^n\xi(t)}{dt^n} = -a_{n-1}\frac{d^{n-1}\xi(t)}{dt^{n-1}} - \cdots - a_1\frac{d\xi(t)}{dt} - a_0\xi(t) + u(t). \tag{7.6}$$

Then we postulate that the output can be found as a linear combination of the derivatives of this function:

$$y(t) = b_n\frac{d^n\xi(t)}{dt^n} + b_{n-1}\frac{d^{n-1}\xi(t)}{dt^{n-1}} + \cdots + b_1\frac{d\xi(t)}{dt} + b_0\xi(t) \tag{7.7}$$

where some b-coefficients may be zero. In their most general form, (7.6)–(7.7) may be represented by the block diagram in Figure 7.5.

Letting $x_1 = \xi, x_2 = \dot{x}_1 = \frac{d\xi}{dt}, \ldots, x_n = \dot{x}_{n-1} = \frac{d^{n-1}\xi}{dt^{n-1}}$ and using vector/-

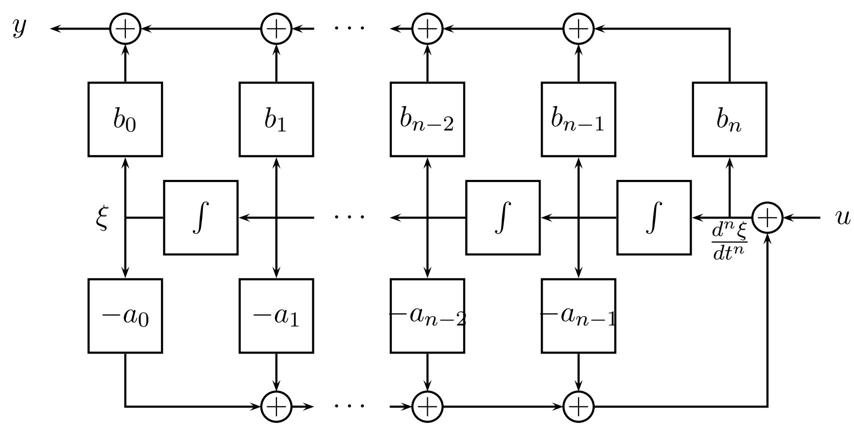

Figure 7.5: Block diagram illustrating how y is generated by sequential integration.

matrix notation then permit the *state space description*

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \tag{7.8}$$

$$y(t) = \begin{bmatrix} b_0 - a_0 b_n & b_1 - a_1 b_n & \cdots & b_{n-1} - a_{n-1} b_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + b_n u(t) \tag{7.9}$$

which is also known as the *canonical form* in literature.[3]

Note that b_n is usually zero for real-world systems (no direct feedthrough),

[3]More precisely, this is the so-called *controller canonical form*. There are other canonical forms, as well as non-canonical forms, but from an operator point of view those are all equivalent to (7.8)–(7.9) and will not be discussed further here.

in which case (7.8)–(7.9) reduce to the simpler form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Example 7.1. Recall the pendulum from Example 3.6. Write the state space representation of its nonlinear model and its linearized version around the stable equilibrium assuming no actuation and no friction.

Solution:

The nonlinear differential equation for the pendulum is

$$\sum_i \tau_i = J\ddot{\theta} = -Mgl\sin\theta$$

where θ is the angular displacement from vertical, J is the pendulum's moment of inertia, M is the pendulum bob mass and l is its length. We may define

$$x_1(t) = \theta(t), \quad x_2(t) = \dot{\theta}(t)$$

which would yield the system of first-order equations

$$\begin{aligned} \dot{x}_1(t) = f_1(x_1(t), x_2(t)) \quad &= x_2(t) \\ \dot{x}_2(t) = f_2(x_1(t), x_2(t)) \quad &= -\frac{Mgl}{J}\sin x_1(t). \end{aligned}$$

The first equation is already linear, since it can be written as $\dot{x}_1(t) = 0x_1(t) + 1x_2(t)$. Linearizing the second equation around $\bar{\theta} = \bar{x}_1 = 0, \bar{\dot{\theta}} =$

$\bar{x}_2 = 0$ yields

$$\begin{aligned}
x_2(t) - \bar{x}_2(t) &\approx f_2(\bar{x}_1, \bar{x}_2) + \left.\frac{\partial f_2}{\partial x_1}\right|_{x_1=\bar{x}_1} (x_1 - \bar{x}_1) \\
&\quad + \left.\frac{\partial f_2}{\partial x_2}\right|_{x_2=\bar{x}_2} (x_2 - \bar{x}_2) + \cdots \\
&= -\frac{Mgl}{J} \sin \bar{x}_1 + \frac{d}{dx_1} \left.\left(-\frac{Mgl}{J} \sin x_1(t) \right)\right|_{x_1=\bar{x}_1} (x_1(t) - \bar{x}_1) \\
&= -\frac{Mgl}{J} \cos \bar{x}_1 x_1(t) \\
&= -\frac{Mgl}{J} x_1(t).
\end{aligned}$$

Thus, the linearized state space description of the pendulum is

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{Mgl}{J} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} \theta_0 \\ \dot{\theta}_0 \end{bmatrix} \tag{7.10}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \tag{7.11}$$

Notice that since there is no actuation, $u \equiv 0$ and therefore left out of the equation.

7.2 Linear state space models

This section formalizes the concept of state space representation of linear time invariant operators. It then moves on to introduce the *matrix exponential function*, which is the vector/matrix extension of the ordinary exponential function. Armed with these tools, we aim for a general solution to linear time invariant higher-order differential equations.

7.2.1 First order LTI vector differential equation

The previous section gave examples of how to define auxiliary or physical variables x_i and first-order differential functions f_i, where each f_i is *linear* in x_i and u. That is,

$$\dot{x}_i(t) = a_{i1}x_1(t) + a_{i2}x_2(t) + \cdots + a_{in}x_n(t) + b_i u(t), \quad i = 1, 2, \ldots, n,$$

where $a_{i1}, \ldots, a_{in}$ and b_i are constants. As the reader may recall, the simulation model for a third-order linear system considered in Chapter 4—more precisely Equation (4.27)—was exactly of this form with a particular choice of x's, namely sequential derivatives of the output $y(t)$.

Using vector/matrix notation, the state equations can be gathered in a *first-order ordinary vector differential equation* where the *state vector* is defined as

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}.$$

As is known from Calculus, differentiating a vector function with respect to its argument corresponds to differentiating every element of the vector with respect to the argument. That is,

$$\dot{\mathbf{x}}(t) = \frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix},$$

which—since the f_i's are assumed to be linear in x_i and u—means that the dynamic operator defined in Definition 7 may be written compactly as a set of vector/matrix equations:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \tag{7.12}$$

$$y(t) = \mathbf{C}\mathbf{x}(t) + Du(t) \tag{7.13}$$

assuming scalar input and output. The coefficient matrices in (7.12)–(7.13)

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}$$

and D are all constants, while $\mathbf{x}(t)$, $y(t)$ and $u(t)$ are functions of time (in particular, u and y are signals). All the *parameters* of the system operator are collected in the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and D (with $D \in \mathbb{R}^{1\times 1}$ considered a 1-by-1 "matrix" for consistency); the matrices are known as the state matrix, the input matrix, the output matrix and the direct feedthrough, respectively. Together, the equations (7.12)–(7.13) are also referred to as a *state space* representation of a linear time invariant (LTI) operator G. Figure 7.6 shows a graphical representation of this configuration.

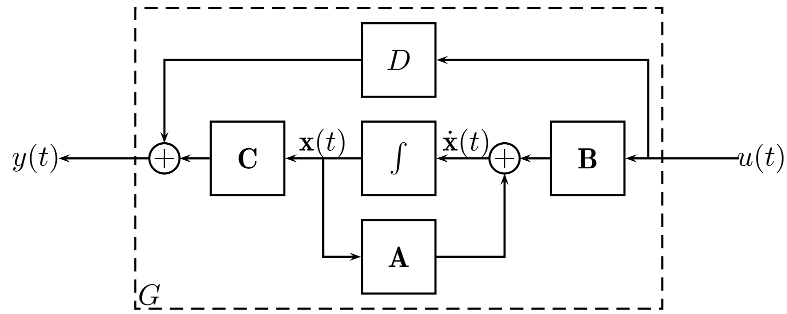

Figure 7.6: State space representation of a dynamic operator $G : \mathcal{L}_2 \to \mathcal{L}_2$. Note that some of the arrows in the diagram—notably, the ones between the **B** and **C** blocks—represent vector signals.

In the RLC-circuit considered at the start of the chapter, for example, we have $n = 2$ and

$$\mathbf{A} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & R \end{bmatrix}, \quad D = 0.$$

Equation (7.12) is a *linear time invariant first order vector differential equation*; its initial condition is also a vector, namely $\mathbf{x}_0 = [x_{1,0}\ x_{2,0}\ \dots\ x_{n,0}]^\top$ ($(\cdot)^\top$ denotes transpose). Matrix multiplication and addition are natural extensions of ordinary (scalar) multiplication and addition, and matrix notation allows us to deal efficiently with *systems* of linear equations (e.g., *solving* such systems by multiplying by an inverse coefficient matrix, or *reducing* them by performing elementary row operations).

7.2.2 Matrix exponential function

Equation (7.12) is a natural extension of the first order linear differential equation discussed in chapters 1 and 3. It is thus not surprising that, just like the ordinary exponential function is the solution to the (homogeneous) scalar first order linear differential equation $\dot{y}(t) = ay(t)$, there exists a *matrix exponential function* which is the solution to the first order linear vector differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$, i.e., the homogeneous version of (7.12).

Definition 16. *Let* $\mathbf{X} \in \mathbb{C}^{n\times n}$*. The matrix exponential function* exp : $\mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ *is defined by the infinite sum*

$$\exp(\mathbf{X}) = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{X}^n \tag{7.14}$$

where $\mathbf{X}^0$ *is defined to be the identity matrix of the same dimension as* $\mathbf{X}$ *and* $n! = n(n-1)(n-2)\cdots 2 \cdot 1$.

The series in (7.14) always converges, so the matrix exponential is well defined for any square matrix. The matrix exponential has several useful properties, a few of which are listed below.

- If $\mathbf{X}$ is real, then $\exp(\mathbf{X}$ is real and $\exp(\mathbf{X}^\top) = \exp(\mathbf{X})^\top$; if $\mathbf{X}$ is complex, $\exp(\mathbf{X}^*) = \exp(\mathbf{X})^*$ ($(\cdot)^*$ denotes transpose and complex conjugate).

- In general, $\exp(\mathbf{X})\exp(\mathbf{Y}) \neq \exp(\mathbf{X}+\mathbf{Y})$; however, for $\alpha, \beta \in \mathbb{R}$, we do have
$$\exp(\alpha\mathbf{X})\exp(\beta\mathbf{X}) = \exp((\alpha+\beta)\mathbf{X}). \tag{7.15}$$

- As a direct corollary of (7.15) and Definition 16,
$$\exp(-\mathbf{X})\exp(\mathbf{X}) = \mathbf{X}^0 = \mathbf{I} \tag{7.16}$$
where $\mathbf{I}$ denotes the $n \times n$ identity matrix.

- If $\mathbf{X}$ is diagonal, the matrix exponential is a diagonal matrix as well, with diagonal entries given by ordinary exponentials:
$$\exp\left(\begin{bmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_n \end{bmatrix}\right) = \begin{bmatrix} e^{x_1} & 0 & \dots & 0 \\ 0 & e^{x_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{x_n} \end{bmatrix} \tag{7.17}$$

Unfortunately, (7.17) is the only case where it is easy to compute the matrix exponential function, i.e., without having to evaluate a lot of matrix products. It can thus sometimes be useful to try *diagonalizing* $\mathbf{X}$ before computing the exponential, as allowed by the following Theorem:

Theorem 7. *Consider any two matrices* $\mathbf{X} \in \mathbb{C}^{n\times n}$ *and* $\mathbf{P} \in \mathbb{C}^{n\times n}$, *and suppose* $\mathbf{P}$ *is invertible. Then*
$$\exp(\mathbf{P}^{-1}\mathbf{X}\mathbf{P}) = \mathbf{P}^{-1}\exp(\mathbf{X})\mathbf{P}. \tag{7.18}$$

Proof. First, note that
$$\begin{aligned}(\mathbf{P}^{-1}\mathbf{X}\mathbf{P})^n &= (\mathbf{P}^{-1}\mathbf{X}\mathbf{P})(\mathbf{P}^{-1}\mathbf{X}\mathbf{P})\cdots(\mathbf{P}^{-1}\mathbf{X}\mathbf{P}) \\ &= \mathbf{P}^{-1}\mathbf{X}^n\mathbf{P}\end{aligned}$$

for any integer $n \geq 0$. Then, using (7.14) we find

$$\begin{aligned}
\exp(\mathbf{P}^{-1}\mathbf{X}\mathbf{P}) &= \mathbf{I} + \mathbf{P}^{-1}\mathbf{X}\mathbf{P} + \frac{1}{2!}(\mathbf{P}^{-1}\mathbf{X}\mathbf{P})^2 + \frac{1}{3!}(\mathbf{P}^{-1}\mathbf{X}\mathbf{P})^3 + \cdots \\
&= \mathbf{I} + \mathbf{P}^{-1}\mathbf{X}\mathbf{P} + \frac{1}{2!}\mathbf{P}^{-1}\mathbf{X}^2\mathbf{P} + \frac{1}{3!}(\mathbf{P}^{-1}\mathbf{X}^3\mathbf{P} + \cdots \\
&= \mathbf{P}^{-1}\left(\mathbf{I} + \mathbf{X} + \frac{1}{2!}\mathbf{X}^2 + \frac{1}{3!}\mathbf{X}^3 + \cdots\right)\mathbf{P} \\
&= \mathbf{P}^{-1}\exp(\mathbf{X})\mathbf{P}. \qquad \square
\end{aligned}$$

It is possible to diagonalize $\mathbf{X}$ if there exists an invertible matrix $\mathbf{P}$ such that $\mathbf{X} = \mathbf{P}^{-1}\mathbf{\Lambda}\mathbf{P}$, where $\mathbf{\Lambda}$ is a diagonal matrix of eigenvalues of $\mathbf{X}$ and $\mathbf{P}$ is a matrix having eigenvectors of $\mathbf{X}$ as its columns. If this is the case, one may write

$$\exp(\mathbf{X}) = \exp(\mathbf{P}^{-1}\mathbf{\Lambda}\mathbf{P}) = \mathbf{P}^{-1}\exp(\mathbf{\Lambda})\mathbf{P} \tag{7.19}$$

according to Theorem 7. Since $\mathbf{\Lambda}$ is diagonal, $\exp(\mathbf{\Lambda})$ may be computed easily as in (7.17).

Example 7.2. Consider the matrix $\mathbf{X} = \begin{bmatrix} 5 & 1 \\ -2 & 2 \end{bmatrix}$. Diagonalize $\mathbf{X}$ and find $\exp(\mathbf{X})$.

Solution:

To find $\mathbf{P}$, it is first necessary to find the eigenvalues of $\mathbf{X}$, which means solving the equation $\det(\mathbf{X} - \lambda\mathbf{I}) = 0$ or

$$(5 - \lambda)(2 - \lambda) - (-2) = 0.$$

The solutions to this equation are $\lambda_1 = 3$ and $\lambda_2 = 4$. The corresponding eigenvectors are found by solving the equation

$$(\mathbf{X} - \lambda_i\mathbf{I})\mathbf{v}_i = \mathbf{0} \qquad i = 1, 2$$

leading to $\mathbf{v}_1 = [1\ -2]^\top$ and $\mathbf{v}_2 = [1\ -1]^\top$, which are linearly independent. Letting $\mathbf{P} = [\mathbf{v}_1\ \mathbf{v}_2]$ then yields

$$\mathbf{X} = \mathbf{P}^{-1}\mathbf{\Lambda}\mathbf{P} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}.$$

Consequently, the matrix exponential of $\mathbf{X}$ is

$$\exp(\mathbf{X}) = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^3 & 0 \\ 0 & e^4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -e^3 + 2e^4 & -e^3 + e^4 \\ 2e^3 - 2e^4 & 2e^3 - e^4 \end{bmatrix}.$$

Unfortunately, not all square matrices are diagonalizable; if any pair of eigenvectors is linearly dependent, the decomposition $\mathbf{X} = \mathbf{P}^{-1}\mathbf{\Lambda}\mathbf{P}$ fails, and Theorem 7 cannot help in computing the matrix exponential. We shall not pursue this issue any further here, as it is much more of a linear algebra-related question than a data acquisition question; at the end of the day, if we cannot perform diagonalization, we may always attempt to evaluate (7.14) for a reasonably large number of terms in order to get a good approximation of the matrix exponential.

7.2.3 Solution to first order vector ODE

Let us now consider the $n \times n$ real[4] matrix function $\mathbf{X}(t) = \mathbf{A}t$, where $t \in \mathbb{R}$ is the independent variable and $\mathbf{A} \in \mathbb{R}^{n \times n}$. Applying the matrix exponential to this expression then yields a new function mapping from $\mathbb{R}$ to $\mathbb{R}^{n \times n}$ defined as

$$\exp(\mathbf{X}(t)) = \exp(\mathbf{A}t) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{A}^n t^n$$

which can be differentiated with respect to its argument to yield

$$\begin{aligned} \frac{d}{dt} \exp(\mathbf{A}t) &= \frac{d}{dt} \left(\sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{A}^n t^n \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \mathbf{A}^n \frac{dt^n}{dt} \\ &= \sum_{n=1}^{\infty} \mathbf{A}^n \frac{nt^{n-1}}{n!} \\ &= \sum_{n=1}^{\infty} \mathbf{A}^n \frac{t^{n-1}}{(n-1)!}. \end{aligned} \tag{7.20}$$

Here, $\mathbf{A}$ may be taken outside the summation to the left, yielding

$$\frac{d}{dt} \exp(\mathbf{A}t) = \mathbf{A} \exp(\mathbf{A}t) \tag{7.21}$$

[4]All of the arguments in this subsection also hold for complex matrices, but since the matrices in (7.12)–(7.13) are real, we limit our attention to real matrices here.

or, equally valid, to the right, yielding

$$\frac{d}{dt}\exp(\mathbf{A}t) = \exp(\mathbf{A}t)\mathbf{A}. \tag{7.22}$$

We can now state the main result of this chapter.

Theorem 8. *Consider the linear time invariant dynamic operator $G : \mathcal{L}_2 \to \mathcal{L}_2$ with state space representation*

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \tag{7.23}$$

$$y(t) = \mathbf{C}\mathbf{x}(t) + Du(t) \tag{7.24}$$

where $x : \mathbb{R} \to \mathbb{R}^n$ is an n-dimensional function of time $t \in \mathbb{R}$ and $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times 1}$, $\mathbf{C} \in \mathbb{R}^{1\times n}$ and $D \in \mathbb{R}$ are constant matrices. Furthermore, let the initial condition be $\mathbf{x}(0) = \mathbf{x}_0$.

Then the output response $y \in \mathcal{L}_2$ to the input $u \in \mathcal{L}_2$ is

$$y(t) = \mathbf{C}\left(\exp(\mathbf{A}t)\mathbf{x}_0 + \int_0^t \exp(\mathbf{A}(t-\tau))\mathbf{B}u(\tau)d\tau\right) + Du(t). \tag{7.25}$$

Proof. Multiplying both sides of (7.23) by $\exp(-\mathbf{A}t)$ yields

$$\exp(-\mathbf{A}t)\dot{\mathbf{x}}(t) = \exp(-\mathbf{A}t)\mathbf{A}\mathbf{x}(t) + \exp(-\mathbf{A}t)\mathbf{B}u(t)$$

or

$$\exp(-\mathbf{A}t)\dot{\mathbf{x}}(t) - \exp(-\mathbf{A}t)\mathbf{A}\mathbf{x}(t) = \exp(-\mathbf{A}t)\mathbf{B}u(t).$$

Now applying the chain rule and (7.22) yields:

$$\begin{aligned}\frac{d}{dt}\left(\exp(-\mathbf{A}t)\mathbf{x}(t)\right) &= \left(\frac{d}{dt}\exp(-\mathbf{A}t)\right)\mathbf{x}(t) + \exp(-\mathbf{A}t)\left(\frac{d}{dt}\mathbf{x}(t)\right)\\ &= \exp(-\mathbf{A}t)(-\mathbf{A})\mathbf{x}(t) + \exp(-\mathbf{A}t)\dot{\mathbf{x}}(t)\end{aligned}$$

which shows that

$$\frac{d}{dt}\left(\exp(-\mathbf{A}t)\mathbf{x}(t)\right) = \exp(-\mathbf{A}t)\mathbf{B}u(t).$$

Multiplying by dt and integrating both sides thus yields

$$\int_0^t d\left(\exp(-\mathbf{A}\tau)\mathbf{x}(\tau)\right) = \int_0^t \exp(-\mathbf{A}\tau)\mathbf{B}u(\tau)d\tau.$$

Here, the left-hand side evaluates to

$$\int_0^t d\left(\exp(-\mathbf{A}\tau)\mathbf{x}(\tau)\right) = \exp(-\mathbf{A}t)\mathbf{x}(t) - \mathbf{x}_0$$

since $\exp(-\mathbf{A}t) = \mathbf{I}$ for $t = 0$ according to Definition 16. Furthermore, since $\exp(\mathbf{A}t)\exp(-\mathbf{A}t) = \mathbf{I}$ for any $\mathbf{A}$ and t according to (7.16),

$$\begin{aligned}\mathbf{x}(t) &= \exp(\mathbf{A}t)\mathbf{x}_0 + \exp(\mathbf{A}t)\int_0^t \exp(-\mathbf{A}\tau)\mathbf{B}u(\tau)d\tau \\ &= \exp(\mathbf{A}t)\mathbf{x}_0 + \int_0^t \exp(\mathbf{A}(t-\tau))\mathbf{B}u(\tau)d\tau. \end{aligned} \tag{7.26}$$

Inserting (7.26) in (7.24) yields (7.25). □

The matrix $\exp(\mathbf{A}t)$ is called the *state transition matrix* and is sometimes denoted by $\Phi(t)$.

7.2.4 Simulation of state space models

Equation (7.25) gives the output response to an input $u(t)$, which can also be expressed as a convolution integral with the system's impulse response:

$$\begin{aligned}y(t) &= h(t) * u(t) \\ &= \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau \\ &= \mathbf{C}\left(\exp(\mathbf{A}t)\mathbf{x}_0 + \int_0^t \exp(\mathbf{A}(t-\tau))\mathbf{B}u(\tau)d\tau\right) + Du(t) \\ &= \mathbf{C}\exp(\mathbf{A}t)\mathbf{x}_0 + \int_0^t \left(\mathbf{C}\exp(\mathbf{A}(t-\tau)\mathbf{B} + D\delta(t-\tau)\right)u(\tau)\, d\tau.\end{aligned}$$

This implies that the impulse response of G must necessarily be given as

$$h(t) = \begin{cases}\mathbf{C}\exp(\mathbf{A}t)\mathbf{B} + D\delta(t), & t \geq 0 \\ 0, & t < 0\end{cases} \tag{7.27}$$

which may be useful when one wants to simulate systems described on state space form. In practice, such systems can be simulated in a few different ways.

Firstly, if one knows $u(t)$ and $\mathbf{x}_0$, one can in principle find an exact expression for $y(t)$ by evaluating (7.25) for any desired t. This approach unfortunately involves evaluating the potentially rather nasty integral $\int_0^t \exp(\mathbf{A}(t-$

$\tau))\mathbf{B}u(\tau)d\tau$, which is generally only possible if $u(t)$ is known for all $t \geq 0$ and only doable by hand in particularly simple cases, as illustrated by the following simple example.

Example 7.3. Consider the system described by the state space model

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \begin{bmatrix} -1 & \frac{1}{2} \\ \frac{3}{2} & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & \frac{1}{2} \end{bmatrix} \mathbf{x}(t).\end{aligned}$$

Find the output response $y(t)$ to the input $u(t) = \cos 2t$ for $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Solution:

The state matrix $\mathbf{A}$ has eigenvalues

$$\lambda_1 = -\frac{5}{2}, \qquad \lambda_2 = -\frac{1}{2}$$

with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

which are linearly independent, so $\mathbf{A}$ is diagonalizable. Setting $\mathbf{P} = [\mathbf{v}_1 \ \mathbf{v}_2]$ yields

$$\mathbf{P} = \begin{bmatrix} -\frac{1}{3} & 1 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{P}^{-1} = \begin{bmatrix} -\frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

implying that

$$\begin{aligned}\exp(\mathbf{A}t) &= \mathbf{P}^{-1} \exp\left(\begin{bmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{bmatrix} \right) \mathbf{P} \\ &= \begin{bmatrix} -\frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} e^{-\frac{5}{2}t} & 0 \\ 0 & e^{-\frac{1}{2}t} \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} e^{-\frac{5}{2}t} + 3e^{-\frac{1}{2}t} & -e^{-\frac{5}{2}t} + e^{-\frac{1}{2}t} \\ -3e^{-\frac{5}{2}t} + 3e^{-\frac{1}{2}t} & 3e^{-\frac{5}{2}t} + e^{-\frac{1}{2}t} \end{bmatrix}.\end{aligned}$$

The impulse response is thus

$$\begin{aligned}
h(t) &= \mathbf{C}\exp(\mathbf{A}t)\mathbf{B} \\
&= \frac{1}{4}\begin{bmatrix}1 & \frac{1}{2}\end{bmatrix}\begin{bmatrix} e^{-\frac{5}{2}t} + 3e^{-\frac{1}{2}t} & -e^{-\frac{5}{2}t} + 3e^{-\frac{1}{2}t} \\ -3e^{-\frac{5}{2}t} + 3e^{-\frac{1}{2}t} & 3e^{-\frac{5}{2}t} + e^{-\frac{1}{2}t} \end{bmatrix}\begin{bmatrix}0 \\ 2\end{bmatrix} \\
&= \frac{-e^{-\frac{5}{2}t} + e^{-\frac{1}{2}t}}{2} + \frac{3e^{-\frac{5}{2}t} + e^{-\frac{1}{2}t}}{4} \\
&= \frac{1}{4}e^{-\frac{5}{2}t} + \frac{3}{4}e^{\frac{1}{2}t}
\end{aligned}$$

for $t \geq 0$.

Finally, the response may be found by evaluating the convolution integral

$$\begin{aligned}
y(t) &= \int_0^t h(t-\tau)u(\tau)d\tau \\
&= \int_0^t \left(\frac{1}{4}e^{-\frac{5}{2}(t-\tau)} + \frac{7}{4}e^{\frac{1}{2}(t-\tau)}\right)\cos 2\tau \; d\tau \\
&= \frac{-e^{-\frac{5}{2}t}(123e^{2t} + 85) + 560\sin 2t + 208\cos 2t}{1394}. \qquad (7.28)
\end{aligned}$$

As indicated by the example, solving these equations by hand quickly becomes overwhelming, and one has to turn to numerical solvers. In order to do so, it is of course possible to write the system equations one by one, as was done in Section 4.4.3. In addition, the `scipy.signal` package provides various efficient classes and functions to deal with integration of linear time invariant systems on state space form, such as `signal.StateSpace` (a class representing a state space model) and `signal.lsim` (a function simulating the response of a linear system to an arbitrary input signal).

Listing 7.1 shows a simple Python implementation that simulates the second-order model and input signal considered in Example 7.3. It generates the plot shown in Figure 7.7. Equation (7.28) is evaluated in a number of points and plotted as well.

```python
"""
signal.lsim Integration
"""
from __future__ import division
import numpy as np
from scipy import signal
import matplotlib.pyplot as plt

def main():
    tstart = 0.0
    tend = 20.0
    N = 1000  # number of time steps
    t = np.linspace(tstart, tend, N)

    # System definition
    A = np.array([[-1., .5], [1.5, -2.]])
    B = np.array([[0.], [2.]])
    C = np.array([[1., .5]])
    D = np.array([[0.]])
    G = signal.StateSpace(A, B, C, D)

    # Perform integration
    u = np.cos(2*t)
    tout, y, x = signal.lsim(G, u, t)

    # Evaluate analytic solution
    T = linspace(tstart, tend, N/5)
    Y = -np.exp(-5*T/2)*(123*np.exp(2*T)+85)
    Y = Y + 560*np.sin(2*T) + 208*cos(2*T)
    Y = Y/1394

    # Plot the results
    plt.plot(t,y,'b',label='y')
    plt.plot(T,Y,'r.',label='Y')
    plt.legend()
    plt.xlabel('t [s]')
    plt.ylabel('y')
    plt.grid()
    plt.show()

if __name__ == "__main__":
    main()
```

Listing 7.1: lsim integration

Having introduced state space representations of linear time invariant operators and found expressions for their responses to arbitrary $\mathcal{L}_2$ inputs, the sequel gives several examples of sensors with dynamics within this framework.