Power series

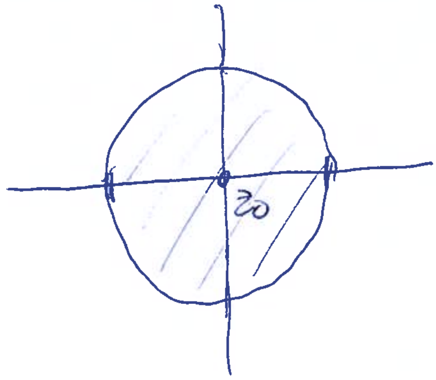
$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$ z is a complex variable

Convergence

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{if } |z|<1$$

$a_n = 0 \quad n=0,1,\dots,\infty$

$z_0 = 0$

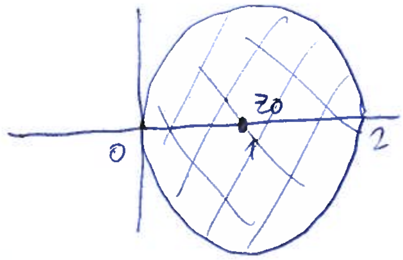


$$\sum_{n=0}^{\infty} (z-1)^n = \frac{1}{1-(z-1)} = \frac{1}{2-z} \quad \text{if } |z-1|<1 \Rightarrow \begin{cases} z-1<1 \Rightarrow z<2 \\ z-1>-1 \Rightarrow z>0 \end{cases}$$

$0<z<2$

$a_n = 0 \quad n=0,1,\dots,\infty$

$z_0 = 1$



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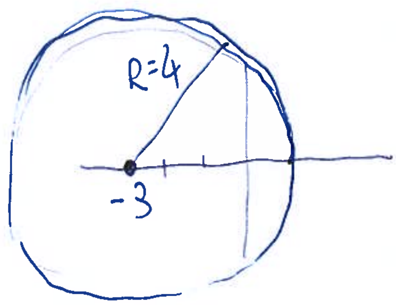
examples of calculation of radius of convergence

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} (z+3)^n \qquad \text{center} = -3$$

$$\lim_{n\to\infty} \left| \frac{(-1)^n n}{4^n} \cdot \frac{4^{n+1}}{(-1)^{n+1}(n+1)} \right| = 4 \Rightarrow R=4$$



center: 2

$$\sum_{n=1}^{\infty} \frac{2^n}{n}(4z-8)^n = \sum_{n=1}^{\infty} \frac{2^n \cdot 4^n}{n}(z-2)^n = \sum_{n=1}^{\infty} \frac{8^n}{n}(z-2)^n$$

$$\lim_{n\to\infty} \frac{8^n}{n} \cdot \frac{n+1}{8^{n+1}} = \frac{1}{8} \Rightarrow R = \frac{1}{8}$$

Differentiation and integration of power series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \longrightarrow R$$

$\downarrow$ termwise derivative

$$\sum_{n=0}^{\infty} n a_n (z-z_0)^{n-1} \longrightarrow R$$

same radius of convergence

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-z_0)^{n+1} \longrightarrow R$$

example

$$\sum_{n=\cancel{0}}^{\infty} \frac{1}{n} z^n \longrightarrow R=1$$

$$\downarrow \frac{\delta}{\delta z}$$

$$\sum_{n=\cancel{0}}^{\infty} \frac{n}{n} z^{n-1} \longrightarrow R=1$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} z^n \to \frac{\delta}{\delta z} \to \sum_{n=1}^{\infty} \frac{n}{n^2} z^{n-1} = \sum_{n=1}^{\infty} \frac{1}{n} z^{n-1} \to \text{same } R$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} z^n \to \int dz \to \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} z^{n+1} = \sum_{n=1}^{\infty} \frac{1}{n^3+n^2} z^{n+1} \to \text{same } R$$

Demonstration Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + a_4(z-z_0)^4 + \dots$$

$$f(z_0) = a_0$$

$$f'(z) = a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + 4a_4(z-z_0)^3 + \dots$$

$$f'(z_0) = a_1$$

$$f''(z) = 2a_2 + 3\cdot 2\, a_3(z-z_0) + 4\cdot 3\, a_4(z-z_0)^2 + \dots$$

$$f''(z_0) = 2a_2 \rightarrow a_2 = \frac{1}{2} f''(z_0)$$

$$f'''(z) = 3\cdot 2\, a_3 + 4\cdot 3\cdot 2\, a_4(z-z_0) + \dots$$

$$f'''(z_0) = 3\cdot 2\cdot a_3 \rightarrow a_3 = \frac{1}{3!} f'''(z_0)$$

$$\vdots$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$

Taylor & Maclaurin series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{with } a_n = \frac{1}{n!} f^{(n)}(z_0)$$

example

$f(z) = \frac{1}{1-z} = (1-z)^{-1}$ find Maclaurin series

$$f^{(0)}(0) = 1$$

$$f'(z) = \frac{1}{(1-z)^2}\bigg|_{z=0} = 1$$

$$f''(z) = \frac{2\cdot 1}{(1-z)^3}\bigg|_{z=0} = 2$$

$$f'''(z) = \frac{3\cdot 2\cdot 1}{(1-z)^4}\bigg|_{z=0} = 6$$

$$\Rightarrow f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}\bigg|_{z=0} = n! \Rightarrow a_n = 1, \quad n = 0, 1, \ldots \infty$$

$$f(z) = \frac{1}{1-z} = 1 + z + z^2 + \cdots + \cancel{z} \cdots$$

$$f(z) = e^z$$

$$f^0(0) = 1$$

$$f'(z) = e^z\big|_{z=0} = 1$$

$$f''(z) = e^z\big|_{z=0} = 1$$

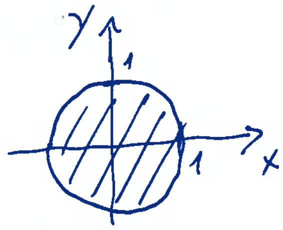
$$\Rightarrow f^{(n)}(z) = 1 \Rightarrow a_n = \frac{1}{n!}$$

$$\vdots$$

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots$$

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

I can represent this function this way only in the unit circle



What about representing the function in other points, e.g. including $z=2$?

I can calculate my Taylor series centered in $z=2$

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} a_n (z-2)^n \qquad a_n = \frac{1}{n!} f^{(n)}(z)$$

$$a_0 = \frac{1}{0!} \cdot f(2) = -1$$

$$a_1 = \frac{1}{1!} f'(2) = 1 \qquad f'(z) = \frac{1}{(1-z)^2}\Big|_{z=2} = 1$$

$$a_2 = \frac{1}{2!} f''(2) = -\frac{1}{2!} \cdot 2 = -1 \qquad f''(z) = \frac{2}{(1-z)^3}\Big|_{z=2} = -1$$

$$a_3 = \frac{1}{3!} f'''(2) = \frac{1}{3 \cdot 2} \cdot 3 \cdot 2 = 1 \qquad f'''(z) = \frac{3 \cdot 2}{(1-z)^4}\Big|_{z=2} = 3 \cdot 2$$

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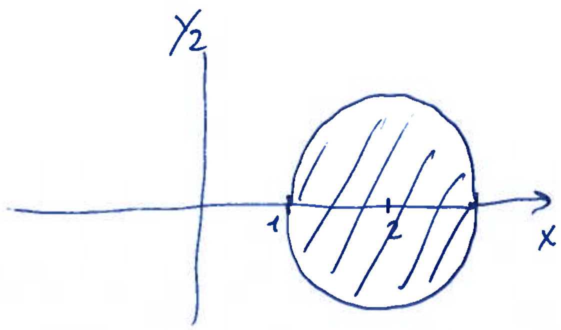
$$f(z) = \frac{1}{1-2} = -1 + (z-2) - (z-2)^2 + (z-2)^3 - \dots$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} (z-2)^n$$



What about complex exponential?

$$f(z) = e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = (i)^n \frac{z^n}{n!}$$

$$(i)^n = \begin{cases} 1 & n = 0, 4, 8 \ldots \\ i & n = 1, 5, 9 \\ -1 & n = 2, 6, 10 \\ -i & n = 3, 7, 11 \ldots \end{cases}$$

$$= \underbrace{\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} + \ldots\right)}_{\Downarrow \ \cos z} + i \underbrace{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots\right)}_{\Downarrow \ \sin z}$$

$$\cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} \qquad \sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

$$f(z) = \ln(1+z)$$

$$f(0) = \ln(1) = 0$$

$$f'(z) = \frac{1}{1+z}\Big|_{z=0} = 1$$

$$f''(z) = \frac{-1}{(1+z)^2}\Big|_{z=0} = -1$$

$$f'''(z) = \frac{2}{(1+z)^3}\Big|_{z=0} = 2$$

$$\cancel{f^{IV}(z) = \frac{1}{(1+z)^4}\Big|_{z=0}}$$

$$f^{IV}(z) = \frac{-6}{(1+z)^4}\Big|_{z=0} = -6$$

$$\Rightarrow f^{(n)}(z) = \frac{(-1)^{n+1}(n-1)!}{(1+z)^n} \ \Big| \Rightarrow a_n = \frac{1}{n!} f^{(n)}(z_0) = \frac{(-1)^{n+1}}{n}$$

$$f(z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n$$

$f(z) = \frac{1}{1+z^2}$

we know that $f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$

Substitution method

we define

$q := -z^2$

$$\Rightarrow f(q) = \frac{1}{1-q} = \sum_{n=0}^{\infty} q^n = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n} = 1 - z^2 + z^4 - z^6 + \dots \qquad |z|<1$$