DISCRETE TIME SYSTEMS AND Z-TRANSFORM

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What we have learned in the previous lecture

- Inverse z-transform

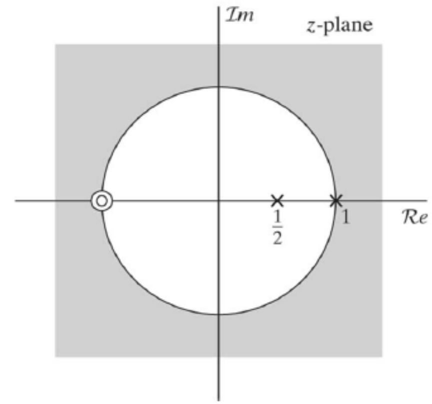
  $X(z) \Rightarrow x[n]$

- How to calculate it?
  - Inspection method
  - Partial fraction expansion
  - Power series expansion

What we have learned in Module 4

- Example

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{(1 + z^{-1})^2}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}, \qquad |z| > 1.$$



$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}.$$

$$\begin{array}{r|l} & \quad 2 \\ \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & z^{-2} + 2z^{-1} + 1 \\ & \underline{z^{-2} - 3z^{-1} + 2} \\ & \quad 5z^{-1} - 1 \end{array}$$

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}.$$

$$A_1 = \left[\left(2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}\right)\left(1 - \frac{1}{2}z^{-1}\right)\right]_{z=1/2} = -9,$$

$$A_2 = \left[\left(2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}\right)\left(1 - z^{-1}\right)\right]_{z=1} = 8.$$

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}.$$

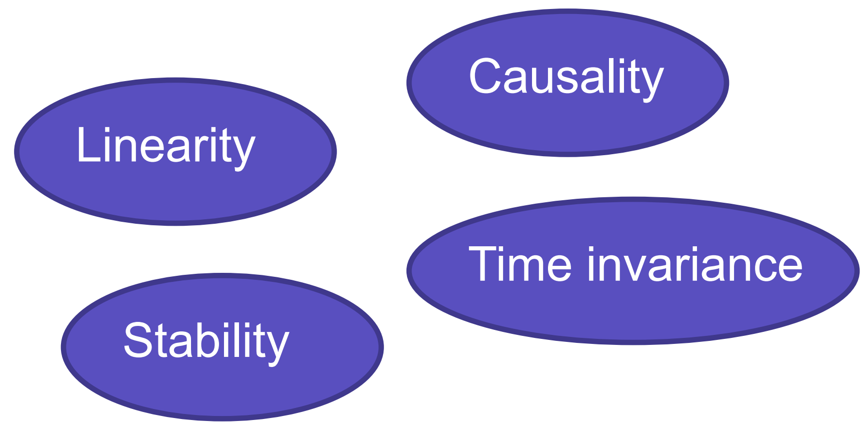
$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n].$$

What we have learned in module 1

- Linear discrete time system

$$\begin{aligned} T\{x_1[n] + x_2[n]\} &= T\{x_1[n]\} + T\{x_2[n]\} &&= y_1[n] + y_2[n] \\ T\{a \cdot x[n]\} &= a \cdot T\{x[n]\} &&= a \cdot y[n] \end{aligned}$$

- Other discrete system properties
  - Causality
  - Time invariance
  - Stability



Today’s agenda

- Discrete time signals
  - Basic sequences and operations
  - Linear systems
  - Stability, causality, time invariance
- Linear time invariant (LTI) systems
  - Inpulse response and convolution
  - Parallel and cascade system combination
- Fourier transform of LTI systems
  - Definition and conditions for existence
- Z-transform
  - Definition and region of convergence (ROC)
  - Right, left-sided and finite duration sequences
  - ROC analysis
- Inverse z-transform
  - Definition and inspection method
  - Partial fraction expansion
  - Power series expansion
- **Transform analysis of LTI systems**
  - **Stability and causality**
  - **Linear constant coefficient difference equations**
  - **Inverse systems**

Stability and causality

- System is causal → h[n] must be a right handed sequence, and therefore the region of convergence of H(z) must be outside the outermost pole

- System is stable → impulse response absolutely summable

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty.$$

→ ROC of H[z] include the unit circle

Linear constant coefficient difference equations

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- An important class of LTI systems takes the following form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m].$$

or $$y[n] = \frac{1}{a_0}\left(\sum_{m=0}^{M} b_m \cdot x[n-m] - \sum_{k=1}^{N} a_k \cdot y[n-k]\right)$$

- Applying z-tranform and linearity and time shift property, we obtain

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z),$$

$$\Downarrow$$

$$\left(\sum_{k=0}^{N} a_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z).$$

Linear constant coefficient difference equations

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$$\left(\sum_{k=0}^{N} a_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z). \quad \Rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}.$$

- It is convenient to express the former in factored form:

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}.$$

- Important: a linear coefficient difference equation does not provide an <u>unique</u> specification of the output for a given input → for a given H(z), each choice of ROC leads to a different impulse response, but they will all correspond to the same difference equation.

Linear constant coefficient difference equations

Example

Consider the LTI system with input and output related through the difference equation

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n]. \tag{5.25}$$

From the previous discussions, $H(z)$ is given by

$$H(z) = \frac{1}{1 - \frac{5}{2}z^{-1} + z^{-2}} = \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - 2z^{-1}\right)}. \tag{5.26}$$

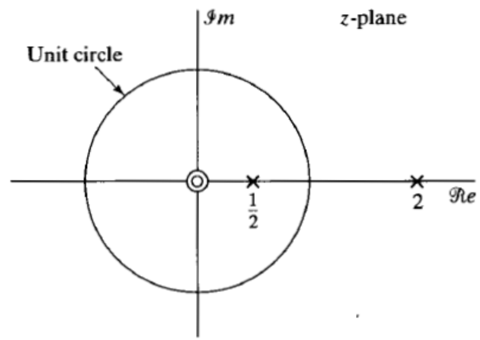

Figure 5.4 Pole–zero plot for Example 5.3.

The pole–zero plot for $H(z)$ is indicated in Figure 5.4. There are three possible choices for the ROC. If the system is assumed to be causal, then the ROC is outside the outermost pole, i.e., $|z| > 2$. In this case the system will not be stable, since the ROC does not include the unit circle. If we assume that the system is stable, then the ROC will be $\frac{1}{2} < |z| < 2$. For the third possible choice of ROC, $|z| < \frac{1}{2}$, the system will be neither stable nor causal.

Design of LTI systems

Linear constant coefficient difference equation

Impulse response
h(n)

Transfer function
H(z)

Implementation

Requirements & Analysis

Inverse systems

inverse

$$G(z) = H(z)H_i(z) = 1. \quad \Rightarrow \quad H_i(z) = \frac{1}{H(z)}.$$

- Equivalent time domain condition $$g[n] = h[n] * h_i[n] = \delta[n].$$

- Frequency response of an inverse system $$H_i(e^{j\omega}) = \frac{1}{H(e^{j\omega})};$$

Inverse systems

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}, \quad \Rightarrow \quad H_i(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{N}(1 - d_k z^{-1})}{\prod_{k=1}^{M}(1 - c_k z^{-1})};$$

- The poles of Hi(z) are the zeros of H(z) and viceversa.
- ROCs of Hi(z) and H(z) must overlap.
- If H(z) is causal, its region of convergence is $|z| > \max_k |d_k|$.
- Any appropriate region of convergence that overlaps with the region specified above is a valid region of convergence for Hi(z).

Inverse systems

- Example

Let $H(z)$ be

$$H(z) = \frac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}}$$

with ROC $|z| > 0.9$. Then $H_i(z)$ is

$$H_i(z) = \frac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}}.$$

Since $H_i(z)$ has only one pole, there are only two possibilities for its ROC, and the only choice for the ROC of $H_i(z)$ that overlaps with $|z| > 0.9$ is $|z| > 0.5$. Therefore, the impulse response of the inverse system is

$$h_i[n] = (0.5)^n u[n] - 0.9(0.5)^{n-1} u[n-1].$$

In this case, the inverse system is both causal and stable.