# Lecture 15

**Exercise 1**

In the two previous exercise sheets, we have considered the following controllable and observable system:

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 7 & -9 \\ 6 & -8 \end{pmatrix} x + \begin{pmatrix} 4 \\ 3 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 1 \end{pmatrix} x \end{aligned} \tag{1}$$

The state feedback

$$u = Fx = \begin{pmatrix} -2 & 2 \end{pmatrix} x$$

was shown to achieve a pole placement corresponding to the char. polynomial $s^2+3s+2 = (s+1)(s+2)$ , and the observer gain

$$L = \begin{pmatrix} -5 \\ -3 \end{pmatrix}$$

was shown to assign the observer poles to $\{-4, -5\}$. In this exercise, we shall try to introduce a reference signal to the observer based controller described by these two gains.

1. The closed loop system for the observer based controller with reference added to the control signal is described by:

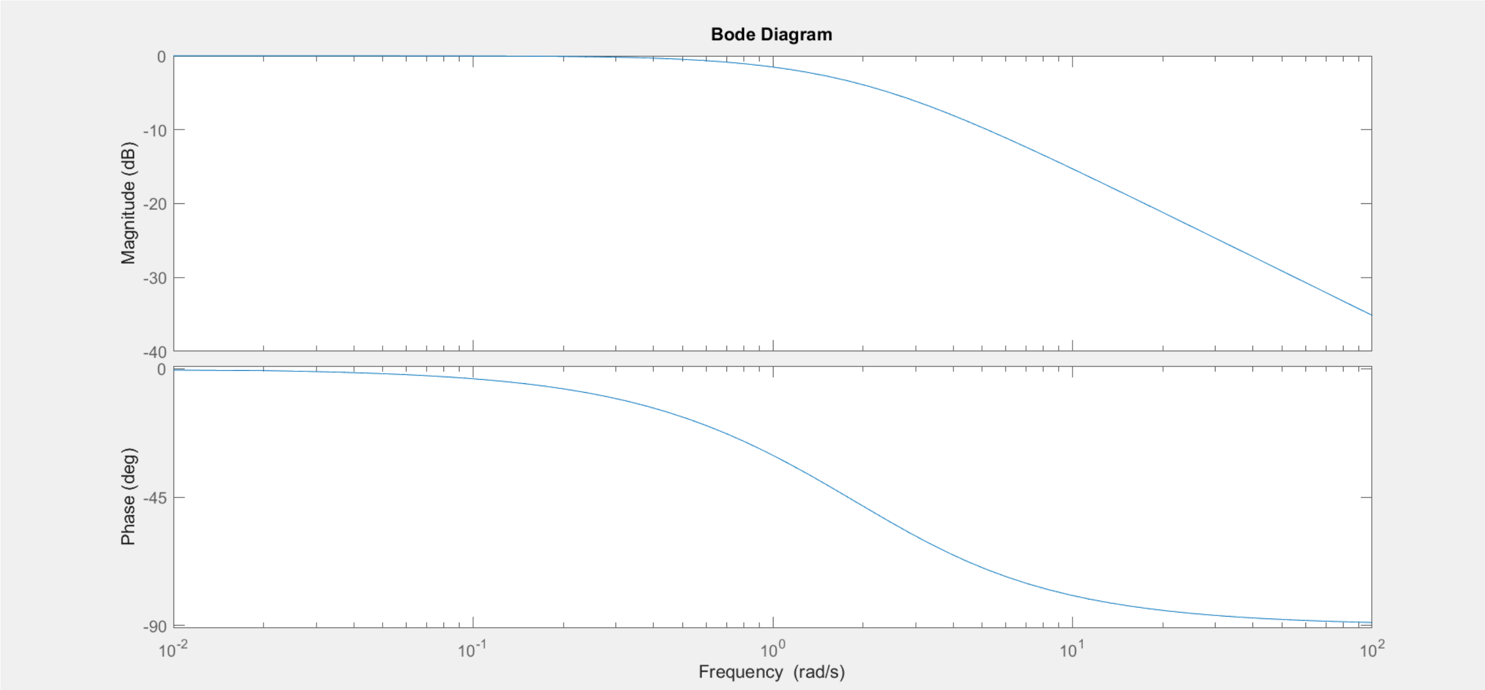
$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} &= \begin{pmatrix} A & BF \\ -LC & A+BF+LC \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} + \begin{pmatrix} B \\ B \end{pmatrix} r \\ y &= \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} \end{aligned}$$

   Generate the Bode plot for this system. Where does it appear appropriate to assign zeros?

```
A = [7 -9; 6 -8];
B = [4;3];
C = [1 1];
F = [-2 2];
L = [-5; -3];

A1 = [2 -3; 4 -5];
B1 = [2;3];
C1 = [-3 2];
F1 = [22 -16];
L1 = [-122;-192];
sys = ss([A B*F; -L*C A+(B*F)+(L*C)],[B;B], [C 0 0; 0 0 0 0; 0 0 0 0; 0 0 0 0], 0);
% bodeplot(sys(1)/4);
```

### Bodeplot:




2. Determine a gain $\tilde{M}$ such that the eigenvalues of $A + BF + LC - \tilde{M}F$ are satisfactory zeros from reference to output. Hint: the MATLAB™ command

```
place((A+B*F+L*C)',F',[pole1,pole2])'
```

can be used. As zeros, you might try $-1.4$ and $-4$, which gives a reasonable result.

```
Mtilde = place((A+(B*F)+(L*C))',F',[-1.4 -4])'
```

```
Mtilde = 2×1
                    7.6
                    4.8
```

3. Compute $N$ as:

$$N = -\left(C_{\text{cl}} A_{\text{cl}}^{-1} \tilde{B}_{\text{cl}}\right)^{-1}$$

where

$$A_{\text{cl}} = \begin{pmatrix} A & BF \\ -LC & A + BF + LC \end{pmatrix}, \quad \tilde{B}_{\text{cl}} = \begin{pmatrix} B \\ \tilde{M} \end{pmatrix}$$
$$C_{\text{cl}} = \begin{pmatrix} C & 0 \end{pmatrix}$$

```
Acl = [A B*F; -L*C A+(B*F)+(L*C)];
Bcl = [B;Mtilde];
Ccl = [C 0 0];

N = -inv((Ccl*inv(Acl)*Bcl))
```

```
N =
       0.892857142857139
```

4. Compute $M = \tilde{M}N$.

```
M = Mtilde*N
```

```
M = 2×1
        6.78571428571426
        4.28571428571427
```

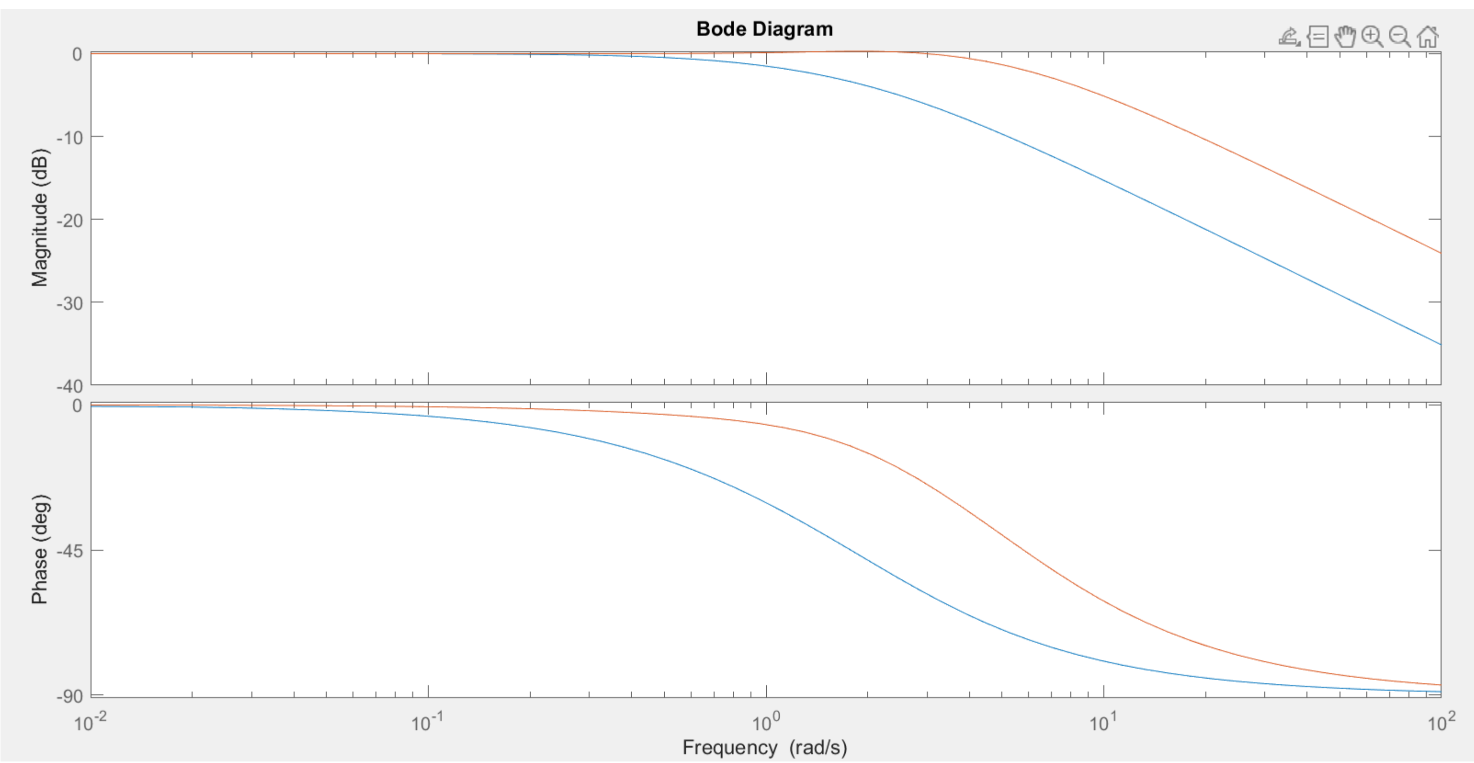
5. Generate a Bode plot for the obtained system:

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} A & BF \\ -LC & A+BF+LC \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} + \begin{pmatrix} BN \\ M \end{pmatrix} r$$
$$y = \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}$$

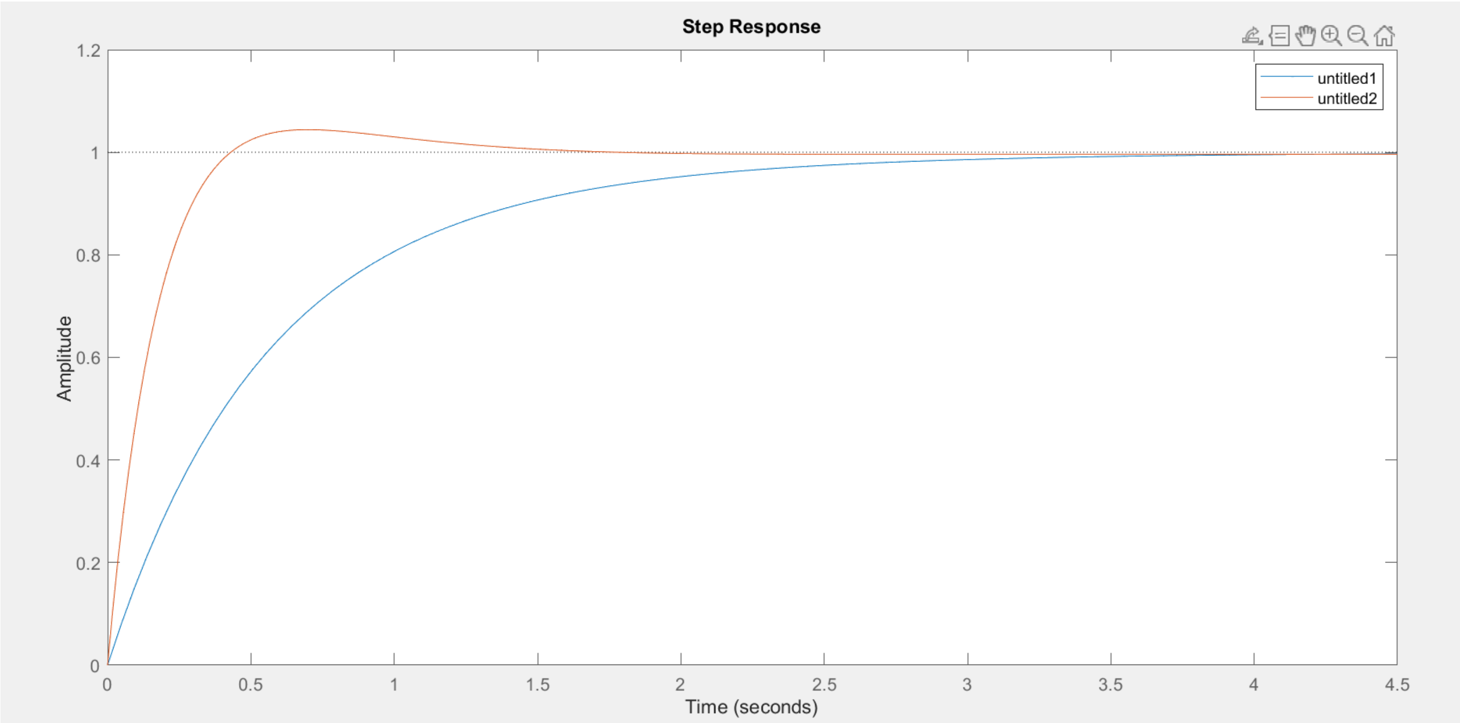
and compare with the original. Compare also the step responses!

```
B1 = [B*N; M];
sys1 = ss([A B*F; -L*C A+(B*F)+(L*C)],B1, [C 0 0], 0)
% bodeplot(sys1(1))
% step(sys(1)/4);
% hold on
% step(sys1(1))
% legend
```

**Bodeplot:**




**Step:**




## Exercise 2

We consider (a final time) the system:

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 7 & -9 \\ 6 & -8 \end{pmatrix} x + \begin{pmatrix} 4 \\ 3 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 1 \end{pmatrix} x \end{aligned}$$

1. Compute an optimal state feedback $F$ for this cost function

$$\mathcal{J} = \int_0^\infty \rho\, y^T y + u^T u \, dt$$

for $\rho = 100$. Try other values (10,1000, etc.), and compute the eigenvalues of $A + BF$ for each value - what happens with the eigenvalue with the largest absolute value as $\rho$ varies? Explain!

```
close all
clear
A = [7 -9; 6 -8];
B = [4;3];
C = [1 1];
F = [-2 2];
L = [-5; -3];
Fopt = -lqr(A,B,1000*C'*C,1);
eig(A + B*Fopt) %Større rho giver poler der rykke mod venstre deraf giver større
styresignal for et hurtigere system og modsat.
```

```
ans = 2×1
         -221.367779711457
         -1.14284978018868
```

2. The smallest eigenvalue seemed to converge to a fixed value as $\rho \to \infty$. Try to compute the zero(s) of the system...

```
sys1 = ss(A,B,C,0);
zero(sys1)
```

```
ans =
         -1.14285714285714
```

3. Compute the step responses of the transfer function:

$$C\left(sI-(A+BF)\right)^{-1}B$$

for the same values of $\rho$. What is the (qualitative) relationship between $\rho$, the gains of $F$ and the rise time for the step response?

```
s = tf('s');
sys2 = C*inv(s*eye(2) - (A + (B*Fopt)))*B;
step(sys2);
```

### Step response: