Multidimensional Fourier Transform

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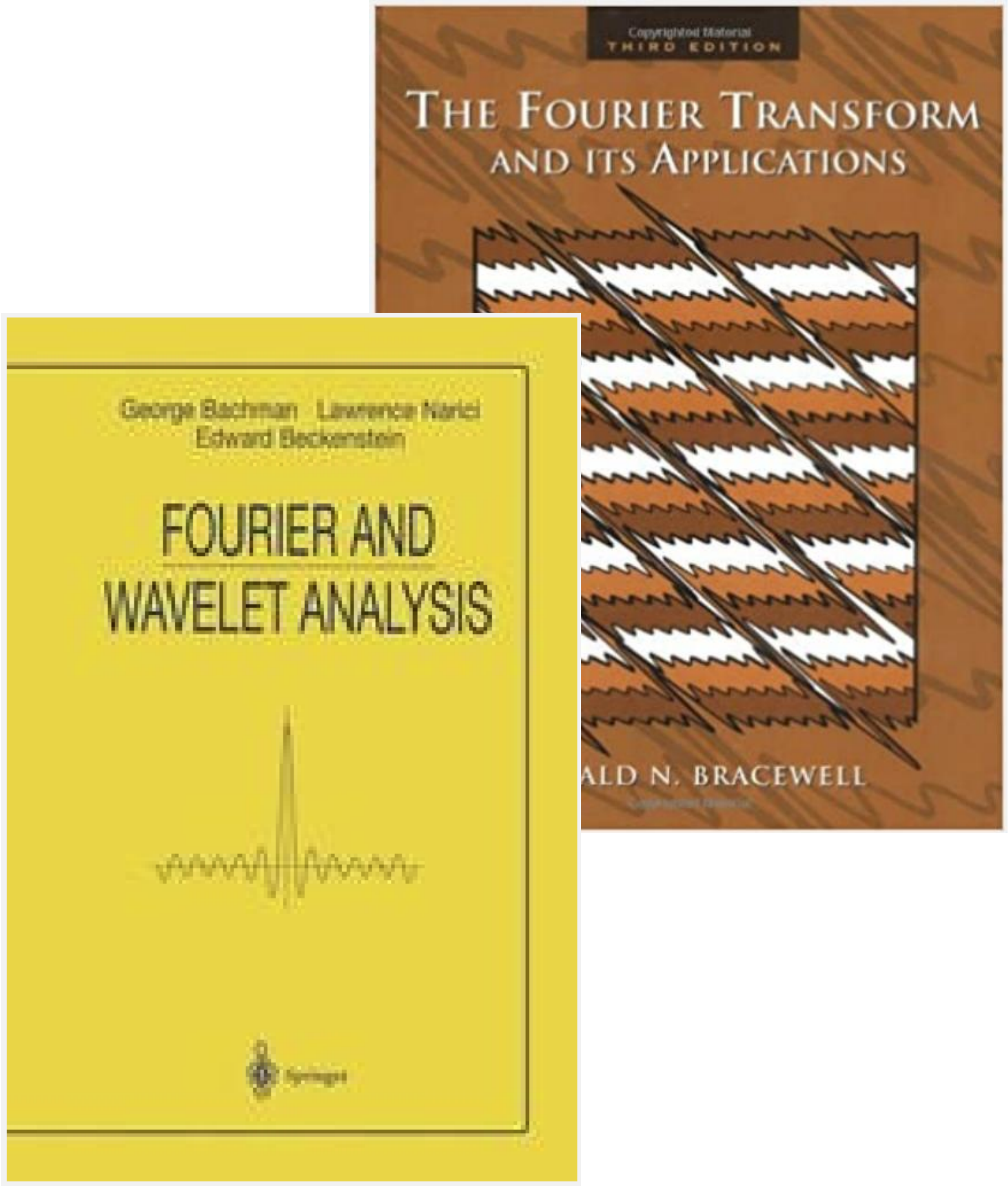
Agenda

- Overview of the general idea behind Fourier series
- Fourier transform and basic properties
- Extension of the FT to multiple dimensions



Signals, systems, and measurements

- We want to study a particular system, by measuring a signal f.
- *How is the signal measured in practice?*
- **Possible solutions:**
  - We have an instrument I, which can perfectly measure the signal in time, $f(t)$.
  - Alternatively (and practically), we may have a number of samples of the signal, e.g., at equidistant sampling intervals, $f[n] = f(nT)$, where T is the sampling period.

The Fourier Series

- Recall that we want to measure a real signal.
- Consider and orthogonal basis of functions, f_n.
- Using this, we may express our signal as a linear combination of these functions in a bounded interval, $t \in [-T/2; T/2]$:

$$f(t) = \sum_{n=-\infty}^{\infty} a_n f_n(t)$$

Coefficient associated with the n'th basis function

The Fourier Series

- Already in the 18th century, Fourier claimed that any **periodic** function can be written as a series of **trigonometric** function.
- Mathematically, this can be expressed as:

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T}$$

- For analysis, we can use the inverse relationship:

$$a_k = \frac{1}{T} \int_T f(t) e^{-j2\pi kt/T} dt$$

Importance of Periodic Signals

- Many physical phenomena exhibit periodic behaviour and can thus be modeled and analyzed using periodic basis functions:
  - Sound waves (musical instruments).
  - AC electrical signals
  - Planetary motion (orbits)
  - Pendulum motion (simple harmonic oscillator)
  - Rotating machinery
  - Heartbeats (ECG)
  - Seasonal climate patterns.
  - Biological rhythms (circadian rhythms)
- Moreover, sinusoidal functions are eigenfunctions of LSI systems.

Beyond Periodic Functions

- While many natural phenomena are periodic, not all are.
- Moreover, even some of the aforementioned phenomena are not perfectly periodic.
- We can extend the Fourier series, by
  - considering an unbounded interval, $T \to \infty$.
  - considering the frequency as being continuous.
- This functional operator is referred to as the *Fourier Transform*.

Fourier Transform

- **Fourier Transform (analysis)**

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\omega t} dt$$

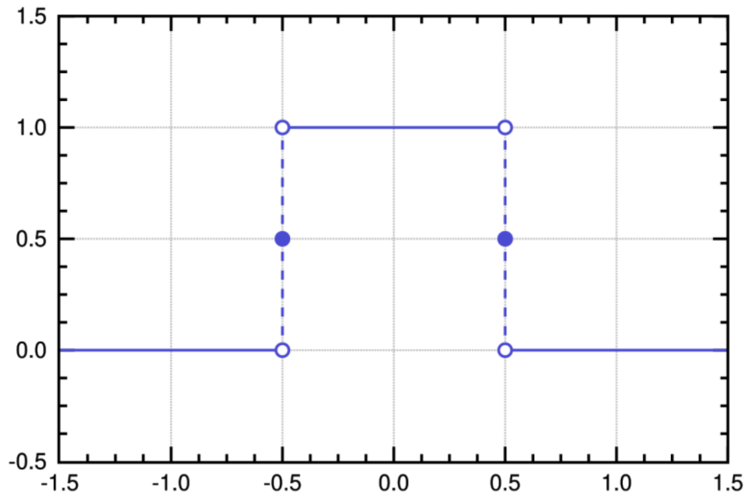
- **Inverse Fourier Transform (synthesis)**

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \int_{-\infty}^{\infty} F(\omega) e^{j2\pi\omega t} d\omega$$

Examples

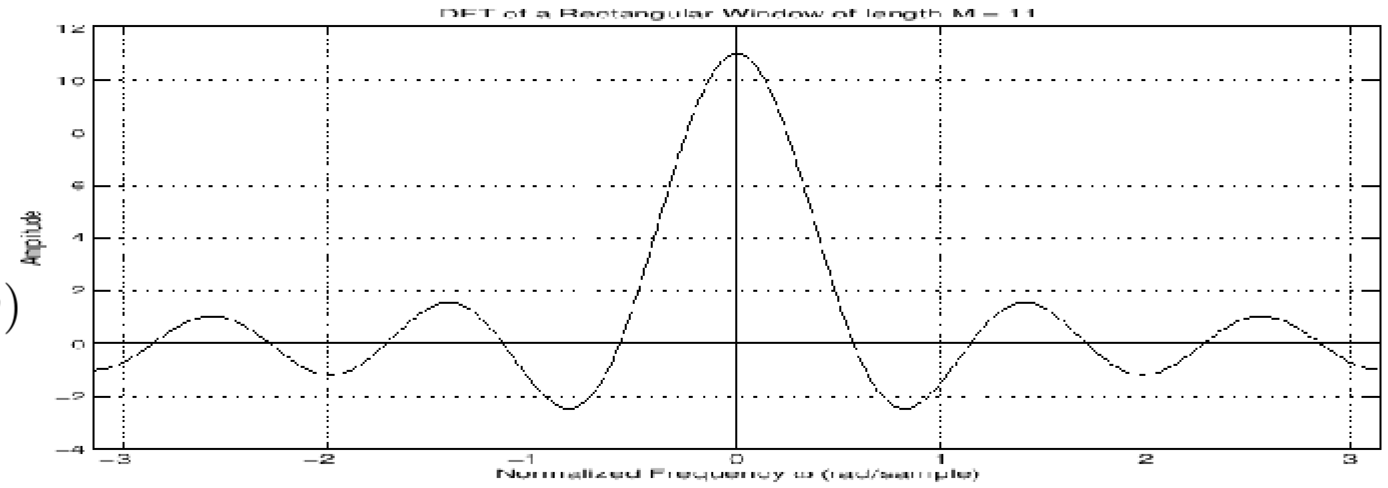
- **Square function**

$$f(t) = \begin{cases} 1 & -\frac{1}{2} \leq 0 \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\omega t} dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi\omega t} dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(2\pi\omega t) dt = \frac{\sin(\pi\omega)}{\pi\omega} = \mathrm{sinc}(\omega)$$



Fourier Transform and Periodic Signals

- While the Fourier Transform is useful for representing/analyzing non-periodic signals, we can generalize it to periodic signals.
- This can be achieved using distributions (i.e., the delta distribution or unit impulse):

$$\delta(x) = 0 \quad \forall x \neq 0$$

- with

$$\int \delta(x) dx = 1$$

Fourier Transform and Periodic Signals

- If we have a unit impulse in the frequency domain:

$$\mathcal{F}\{\delta(\omega-\omega_0)\} = \int_{-\infty}^{\infty} \delta(\omega-\omega_0) e^{j2\pi\omega t} d\omega = e^{j2\pi\omega_0 t}$$

- That is, a unit impulse in the frequency domain corresponds to a conventional (complex) sinusoidal function:

$$\delta(\omega-\omega_0) \xrightarrow{\mathcal{F}} e^{j2\pi\omega_0 t}$$

General Properties

- **Spectrum**
  - Given a signal *f*, we define its spectrum S_f as:

$$S_f(\omega) = |F(\omega)|^2, \quad F(\omega) = \mathcal{F}\{f(t)\}$$

- **Bandwidth**
  - Given a signal *f* we can define the bandwith, *B*, as:

$$F(\omega) = 0 \quad \forall \omega \notin B$$

General Properties

Theorem	$f(x)$	$F(k)$
Similarity	$f(ax)$	$\frac{1}{\lvert a \rvert} F\left(\frac{k}{a}\right)$
Addition	$f(x) + g(x)$	$F(k) + G(k)$
Homogeneity	$af(x)$	$aF(k)$
Shift	$f(x - a)$	$e^{-i2\pi ak} F(k)$

Convolutional Theorem

- The Fourier Transform converts *convolutions into multiplications*:

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\}\mathcal{F}\{g\}$$

- Proof:

$$\begin{aligned}
\mathcal{F}\{f * g\} &= \int (f * g)(x)e^{-j2\pi\omega x}dx = \int \left(\int f(t)g(x-t)dt \right) e^{-j2\pi\omega x}dx \\
&= \int f \left(\int g(x-t)e^{-j2\pi\omega x}dx \right) dt = \int f(t) \left(e^{-j2\pi\omega t} \right) G(\omega)dt \\
&= G(\omega) \int f(t)e^{-j2\pi\omega t}dt = F(\omega)G(\omega)
\end{aligned}$$

Example: Triangular window

- **Triangular window:**

$$\Lambda(t) = \max(1 - |t|, 0) = \begin{cases} 1 - |t| & |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

- Alternatively: $\Lambda(t) = \Pi(t) * \Pi(t)$

Rectangular window

- That is, with the convolution theorem, we get

$$\begin{aligned} \mathcal{F}\{\Lambda(t)\} &= \mathcal{F}\{\Pi(t) * \Pi(t)\} = \mathcal{F}\{\Pi(t)\}\mathcal{F}\{\Pi(t)\} \\ &= \text{sinc}(t)\text{sinc}(t) = \text{sinc}^2(t) \end{aligned}$$

Differentiation

- If a function *f*(*t*) is *differentiable*, and both *f* and *f'* are *integrable*:

$$\mathcal{F}\{f'(t)\}(\omega) = j2\pi\omega F(\omega)$$

- **Proof:**

$$f(t) = \int F(\omega)e^{j2\pi\omega t}dt$$

$$f'(t) = \frac{d}{dt}\left(\int F(\omega)e^{j2\pi\omega t}d\omega\right) = \int \underbrace{j2\pi\omega F(\omega)}_{\mathcal{F}\{f'(t)\}} e^{j2\pi\omega t}d\omega$$

Energy and Power Theorems

- **Rayleigh's Theorem (Energy)**

$$\int |f(t)|^2 dt = \int |F(\omega)|^2 d\omega$$

- **Parseval's Theorem (Power)**

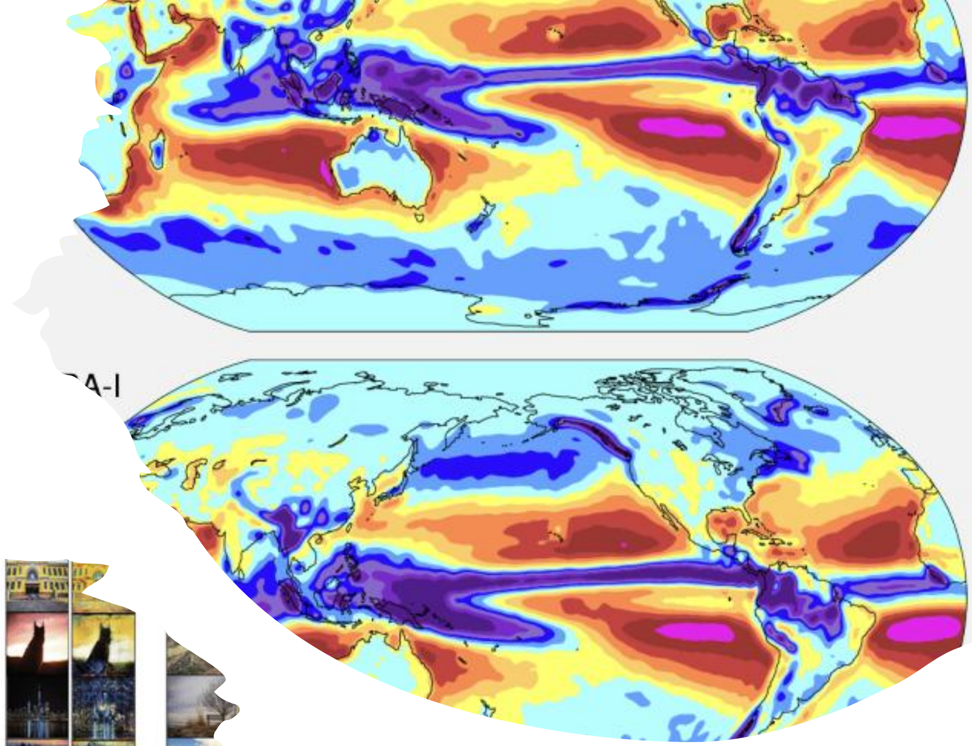
$$\int f(t) g^*(t) dt = \int F(\omega) G^*(\omega) d\omega$$

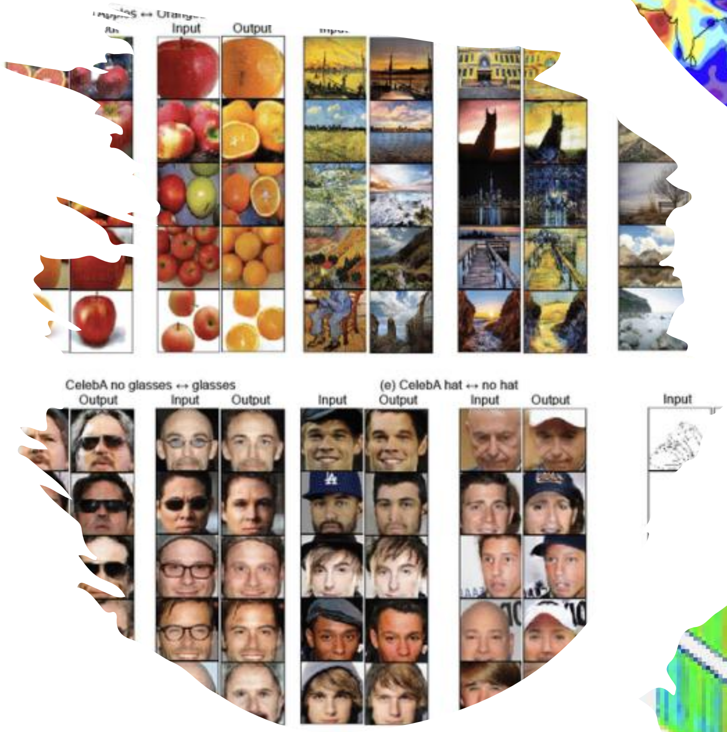
Summary

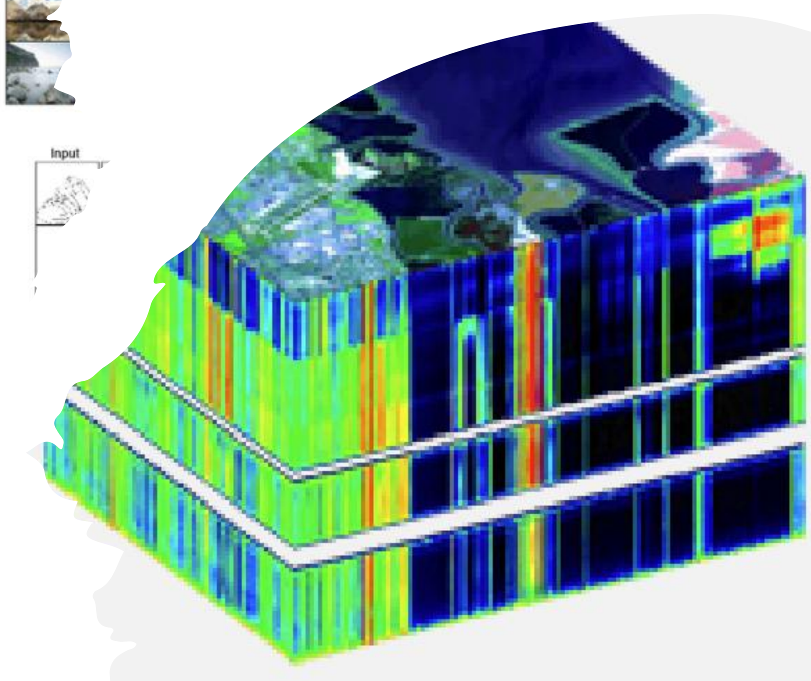
Theorem	$f(x)$	$F(k)$
Similarity	$f(ax)$	$\frac{1}{\lvert a\rvert}F\left(\frac{k}{a}\right)$
Addition	$f(x)+g(x)$	$F(k)+G(k)$
Homogeneity	$af(x)$	$aF(k)$
Shift	$f(x-a)$	$e^{-i2\pi ak}F(k)$
Convolution	$f(x)*g(x)$	$F(k)G(k)$
Derivative	$f'(x)$	$i2\pi kF(k)$
Rayleigh	$\int_{-\infty}^{+\infty}\lvert f(x)\rvert^2dx=\int_{-\infty}^{+\infty}\lvert F(k)\rvert^2dk$	
Parseval	$\int_{-\infty}^{+\infty}f(x)\overline{g}(x)dx=\int_{-\infty}^{+\infty}F(k)\overline{G}(k)dk$	

Multiple Dimensions

- Until now we focused on one-dimensional signals.
- In nature, we can find a wide range of signals, depending on two or more variables, i.e., lieing in a multidimensional space.



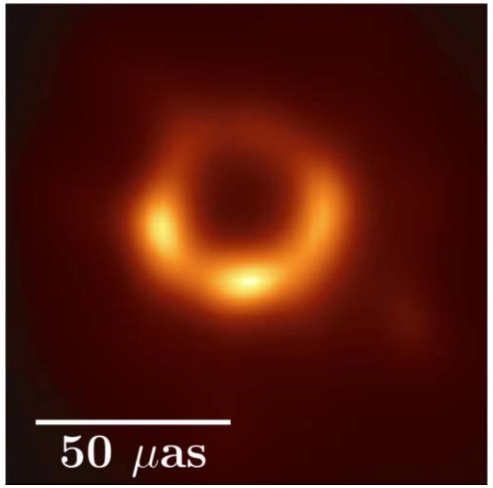




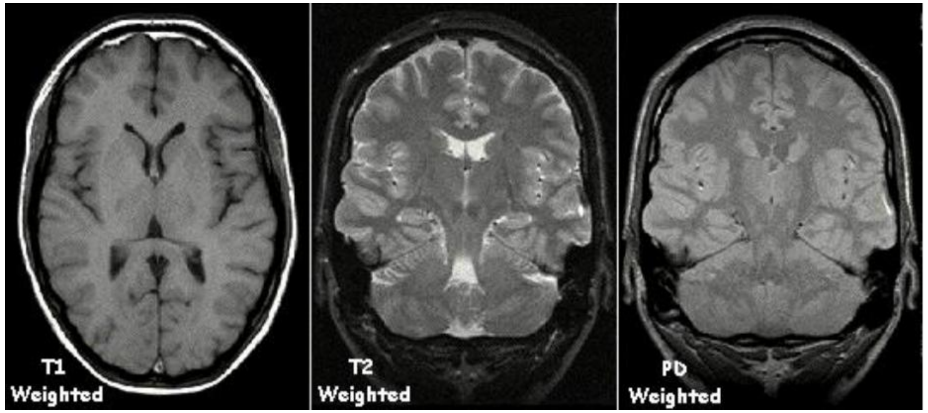
Multidimensional Fourier Domain

- The multidimensional Fourier frequency domain can by used for analysis and imaging.
- **Examples**

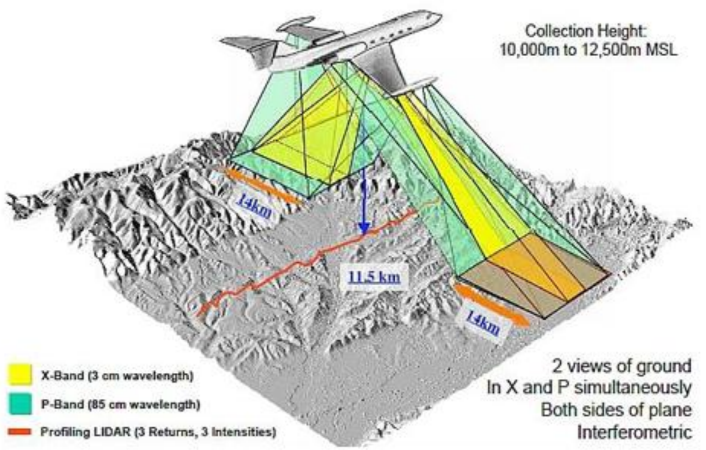
Astronomy



Medical imaging

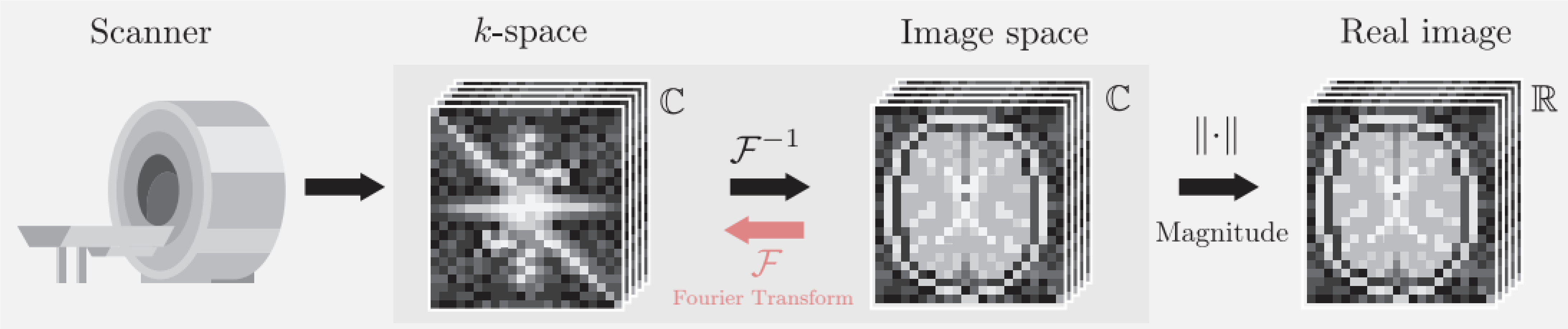


Synthetic Aperture Radar



Multidimensional Fourier Domain

- In contrast to conventional imaging (intensity measured pixel-by-pixel), MRI scanners measures variation in response to different frequencies.
- The result is a complex image in a two-dimensional space of spatial frequencies (*k*-space).



Multidimensional Fourier Transform

- Consider a signal $f(\mathbf{x})$ with $\mathbf{x} = [x_1, x_2, \ldots, x_N]^T$, ($x_n$ is real). Then,
- **Fourier Transform (analysis)**

$$F(\boldsymbol{\omega}) = \mathcal{F}\{f(\mathbf{x})\} = \int \cdots \int f(x_1, \ldots, x_N) e^{-j2\pi(x_1\omega_1 + \cdots + x_N\omega_N)} dx_1 \ldots dx_N$$
$$= \int_{\mathbb{R}^N} f(\mathbf{x}) e^{-j2\pi\mathbf{x}^T\boldsymbol{\omega}} d\mathbf{x}$$

- **Inverse Fourier Transform (synthesis)**

$$f(\mathbf{x}) = \mathcal{F}^{-1}\{F(\boldsymbol{\omega})\} = \int_{\mathbb{R}^N} F(\boldsymbol{\omega}) e^{j2\pi\mathbf{x}^T\boldsymbol{\omega}} d\boldsymbol{\omega}$$

- where

$$\boldsymbol{\omega} = \begin{bmatrix}\omega_1 & \cdots & \omega_N\end{bmatrix}^T$$ (Multidimensional frequency vector)

Vector of Frequencies

- Let examine a two-dimensional case:

$$e^{\pm 2\pi \mathbf{x}^T \boldsymbol{\omega}} = e^{\pm j2\pi(x_1\omega_1 + x_2\omega_2)}$$

- Regardless of the sign, the exponential function equals 1 if the vector product is an integer, *n*

$$x_1\omega_1 + x_2\omega_2 = n$$

- For a fixed frequency vector, the equation corresponds to a *family of parallel lines* in the (x_1, x_2) plane, normal to the frequency vector
- *Distance* between two consecutive lines:

$$\text{distance} = \frac{1}{\sqrt{\omega_1^2 + \omega_2^2}} = \frac{1}{\|\boldsymbol{\omega}\|}$$

Harmonics and Periodicity

- We may now parametrize **x** using the normalized frequency vector ($\mathbf{a} = [a_1, a_2]^T$) is a starting point)

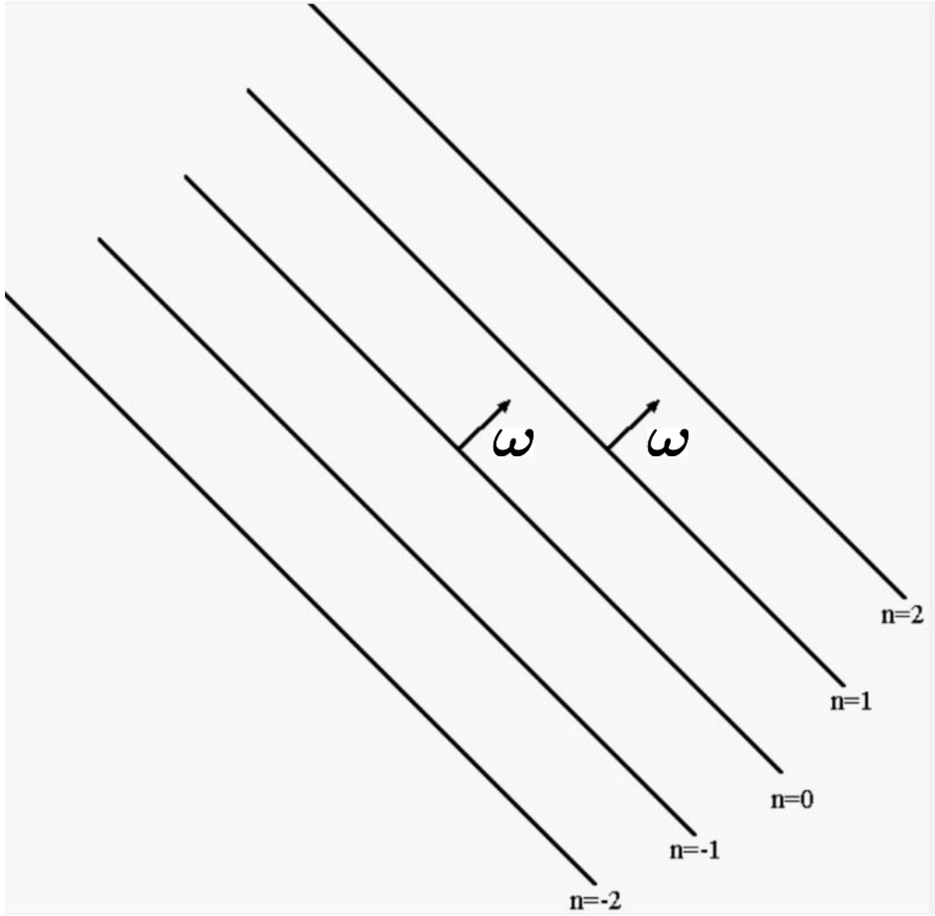
$$\mathbf{x}(t) = \mathbf{a} + t\frac{\boldsymbol{\omega}}{||\boldsymbol{\omega}||}$$

- The complex exponential function then becomes

$$e^{\pm j2\pi\mathbf{x}^T(t)\boldsymbol{\omega}} = e^{\pm j2\pi\mathbf{a}^T\boldsymbol{\omega}}e^{\pm j2\pi t||\boldsymbol{\omega}||}$$

- The first term is *constant*, not depending on t.
- The second term is *periodic* with period $1/||\boldsymbol{\omega}||$.
- Allows us to keep most of the terminology from the 1-D case.

Harmonics and Periodicity



Separable Functions

- In some cases, a function *f*(**x**) of *N* variables may be written as *N* functions of one variable:

$$f(x_1, \ldots, x_N) = f_1(x_1) f_2(x_2) \cdots f_N(x_N)$$

- This significantly simplifies the Fourier analysis, since (in 2-D)

$$\begin{aligned}
\mathcal{F}\{f(x_1, x_2)\} &= \int\int e^{-j2\pi(x_1\omega_1 + x_2\omega_2)} f(x_1, x_2) dx_1 dx_2 \\
&= \int\int e^{-j2\pi x_1\omega_1} e^{-j2\pi x_2\omega_2} f_1(x_1) f_2(x_2) dx_1 dx_2 \\
&= \mathcal{F}\{f_1(x_1)\}\mathcal{F}\{f_2(x_2)\}
\end{aligned}$$

- **In general**

$$\mathcal{F}\{f(x_1, x_2, \ldots, x_N)\} = \mathcal{F}\{f_1(x_1)\}\mathcal{F}\{f_2(x_2)\} \cdots \mathcal{F}\{f_N(x_N)\}$$

General Properties

Theorem	$f(\boldsymbol{x})$	$F(\boldsymbol{\xi})$
Similarity	$f(\boldsymbol{a}\boldsymbol{x})$ with $\boldsymbol{a} \in \mathbb{R}^n$	$\frac{1}{\lvert a_1 a_2 \cdots a_n \rvert} F\left(\frac{\xi_1}{a_1}, \frac{\xi_2}{a_2}, \ldots, \frac{\xi_n}{a_n}\right)$
Addition	$f(\boldsymbol{x}) + g(\boldsymbol{x})$	$F(\boldsymbol{\xi}) + G(\boldsymbol{\xi})$
Shift	$f(\boldsymbol{x} - \boldsymbol{a})$	$e^{-i2\pi \boldsymbol{a}\boldsymbol{\xi}} F(\boldsymbol{\xi})$
Convolution	$f(\boldsymbol{x}) * g(\boldsymbol{x})$	$F(\boldsymbol{\xi}) G(\boldsymbol{\xi})$
Derivative	$\frac{\partial}{\partial x_i} f(\boldsymbol{x})$	$i2\pi\xi_i F(\boldsymbol{\xi})$
Rayleigh	$\int_{\mathbb{R}^n} \lvert f(\boldsymbol{x})\rvert^2 dx = \int_{\mathbb{R}^n} \lvert F(\boldsymbol{\xi})\rvert^2 dk$	
Parseval	$\int_{\mathbb{R}^n} f(\boldsymbol{x})\overline{g}(\boldsymbol{x}) dx = \int_{\mathbb{R}^n} F(\boldsymbol{\xi})\overline{G}(\boldsymbol{\xi}) dk$	

General Stretch Theorem

- For any linear transformation, **A**, of the input data, its corresponding Fourier transform is given by:

$$\mathcal{F}\{f(\mathbf{A}\mathbf{x})\} = \frac{1}{|\det(\mathbf{A})|} F(\mathbf{A}^{-T}\boldsymbol{\omega})$$

- Rotations is a particular kind of linear transformation.
- The rotation matrix, **R**, has the special property that **R** = $\mathbf{R}^{-T}$ and det(**R**) = 1, i.e.,

$$\mathcal{F}\{f(\mathbf{R}\mathbf{x})\} = F(\mathbf{R}\boldsymbol{\omega})$$

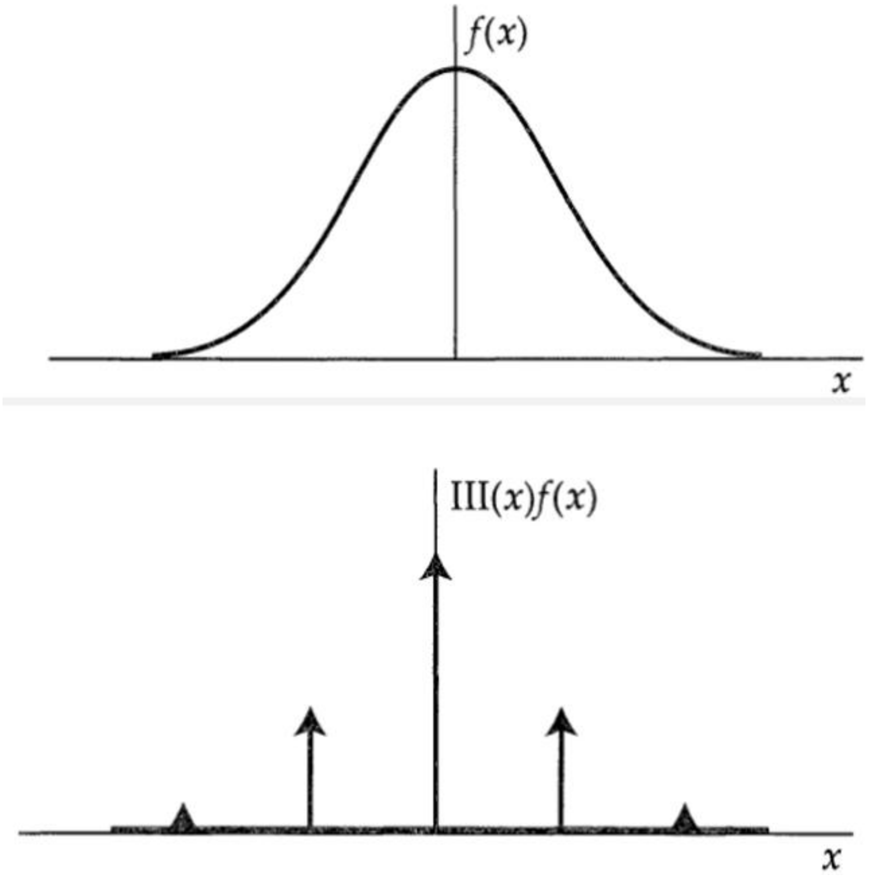
Sampling

- We can define the sampling function as

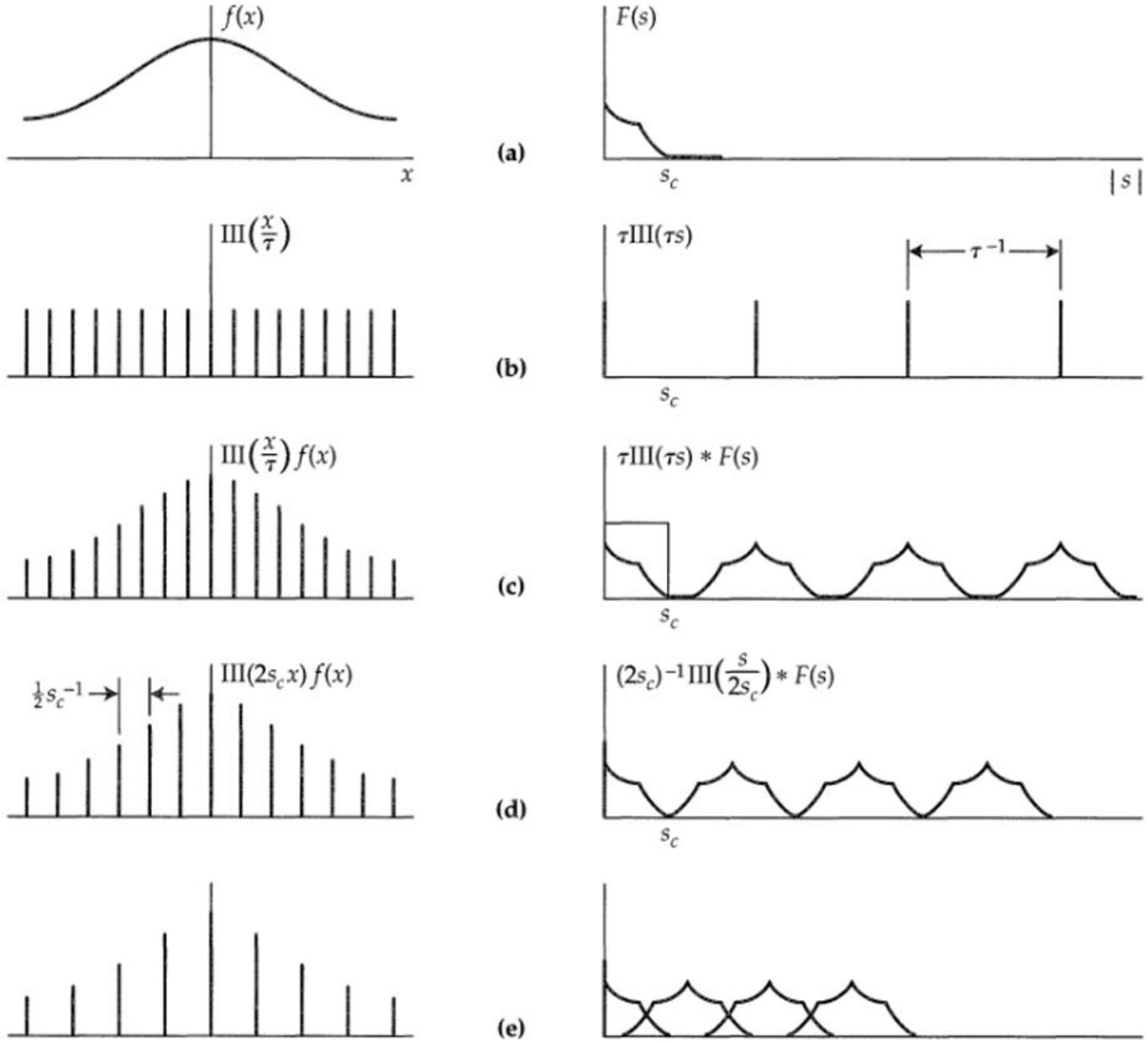
$$\text{Ш}_T(x) = \sum_{n=-\infty}^{\infty} \delta(x - nT)$$

- Using this function, we can sample any other function as

$$\text{Ш}(x) f(x) = \sum_{n=-\infty}^{\infty} f(x - n)$$



Sampling Rate



Nyquist Sampling

- Suppose we are given samples of a function → Can we reconstruct the function *perfectly*?

- **Nyquist Sampling Theorem**
  - Let *f* be a signal with band-limit *B*.
  - Then, we can reconstruct *f* *exactly* as long as the sampling frequency is greater than 2*B*.

Multidimensional Discrete Fourier Transform

- The MD DFT is a sampled version of the discrete-domain FT.
- Obtained by evaluating it at uniformly spaced sample frequencies:

$$F(\mathbf{k}) = \sum_{n_1=0}^{N_1-1} \cdots \sum_{n_m=0}^{N_m-1} f(\mathbf{x}) e^{-j2\pi\left(\frac{n_1 K_1}{N_1} + \cdots + \frac{n_m K_m}{N_m}\right)}$$

- The inverse MD DFT:

$$f(\mathbf{x}) = \frac{1}{N_1 \cdots N_m} \sum_{n_1=0}^{N_1-1} \cdots \sum_{n_m=0}^{N_m-1} F(\mathbf{k}) e^{j2\pi\left(\frac{n_1 K_1}{N_1} + \cdots + \frac{n_m K_m}{N_m}\right)}$$