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# Limiting shapes of birth-and-death processes on Young diagrams

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A B S T R A C T

We consider a family of birth processes and birth-and-death processes on Young diagrams of integer partitions of $n$. This family incorporates three famous models from very different fields: Rost's totally asymmetric particle model (in discrete time), Simon's urban growth model, and Moran's infinite alleles model. We study stationary distributions and limit shapes as $n$ tends to infinity, and present a number of results and conjectures.


## 1. Introduction

Draw a Young diagram at random under some probability distribution on the partitions of $n$; scale its row lengths by a factor $1/a_n$ and column heights by a factor $a_n/n$ so that the total area of the diagram is 1. Vershik and others have studied how such random Young diagrams may approach a limit shape as $n$ grows [20]. The historically first and most famous example of such results is the limit shape of partitions chosen according to the Plancherel measure [21,12]; see [1] for a discussion of the importance of this result. Here we will be more concerned with another of Vershik's examples:

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Under the uniform distribution and scaling $a_n = \sqrt{n}$ random Young diagrams have the symmetric limit shape

$$e^{-(\pi/\sqrt{6})x} + e^{-(\pi/\sqrt{6})y} = 1. \tag{1}$$

For some probability distributions on partitions there is no limit shape, i.e., no single shape is approached in probability. Vershik calls such cases "non-ergodic" [20]. These non-ergodic cases were recently studied in some detail by Yakubovich [24] who, among other examples, discusses why the Ewens distribution on partitions,

$$\mathrm{Prob}(\lambda) = \frac{n!}{\theta(\theta+1)\cdots(\theta+n-1)} \prod_k \frac{\theta^{r_k(\lambda)}}{r_k(\lambda)!k^{r_k(\lambda)}}, \tag{2}$$

has no limit shape (assuming $\theta$ is constant as $n \to \infty$).

In this paper we will study limit shapes obtained from *stochastic processes on Young diagrams*. More specifically, we will deal with birth-and-death processes where in each step one square is added to or removed from the Young diagram, cf. [2]. We were inspired by three famous models from different applied areas which turn out to be of similar flavors when framed in terms of Young diagrams:

*The Rost model.* Rost [16] studied the limit behavior of a one-dimensional asymmetric particle system. In discrete time, this system can be described as a randomly growing Young diagram where in every step an inner corner is drawn uniformly at random and filled with a new square [17]. From Rost's result it follows that this process tends to the limit shape $\sqrt{x} + \sqrt{y} = 6^{1/4}$ under scaling $a_n = n^{1/2}$.

*The Simon model.* One of the many accomplishments of Nobel prize winning economist Herbert Simon was a mathematical model of urban growth proposed to explain the universal observation that the distribution of city sizes tend to satisfy a power law (sometimes referred to as "Zipf's law for cities") [8,11,18]. This model has been rediscovered many times; it is also known as "Polya's infinite urn model" or a "Yule process" or a "preferential attachment model." Simon's model can be formulated as a randomly growing Young diagram where in each step the new square forms a new row with some probability $\mu$, and otherwise it is placed in the inner corner associated with the length of the row of a square drawn uniformly at random among the already existing squares. Simon's result says that as the number of squares tend to infinity, the expected number $E[r_k]$ of rows of length $k$ will approach a power law. Chung and Lu showed that asymptotically the scaled expected number $E[r_k]/n$ is attained in probability. In Section 4.2 we derive a limit shape from these results.

*The Moran model.* A very important model in mathematical population genetics is the so-called Moran model with infinitely many alleles [7]. This is a birth-and-death process which, formulated in terms of Young diagrams, consists of alternating births and deaths of squares so that at the end of each birth-and-death period the Young diagram has a fixed size $n$. Each birth obeys the same rule as in Simon's model above (with some parameter $\mu$), and the birth is followed by a death occurring in the outer corner associated with the length of the row of a square drawn uniformly at random. Ewens showed that the stationary distribution of this process is the Ewens distribution (2), with $\theta = n\mu/(1-\mu)$. As mentioned above, for fixed $\theta$ (i.e., for $\mu \sim 1/n$) this distribution is known not to yield any limit shape. In Section 6.1 we show that a limit shape exists, under scaling $a_n = 1/\mu$, whenever $n\mu \to \infty$.

As far as we know, the parallels between these three models have never been pointed out before. Considering these models within a common framework, we also obtain a number of other processes worth studying. For instance, in our framework the Moran model is obtained from the Simon model through addition of a death step that is analogous to the birth step. If we do the same thing to the Rost model, we obtain the birth-and-death process where births occur in uniformly drawn inner corners and deaths occur in uniformly drawn outer corners. In Section 6.4 we show that its limit shape satisfies Eq. (1), i.e., the same limit shape as Vershik found for the uniform distribution.

We will present a rather large collection of theorems and conjectures. The theorems are obtained through various methods; indeed, we still lack a general approach to these questions. The conjectures

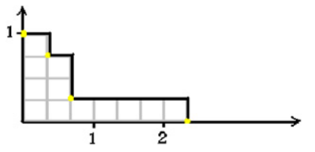


**Fig. 1.** The Young diagram $(7, 2, 2, 1) \vdash 12$ with scaling $a_{12} = 3$ (so row lengths are scaled down by a factor 3 and column lengths are scaled down by a factor $12/3 = 4$). The four inner corners are marked.

are based on simulations of the processes, through which we obtain approximate average shapes for large values of $n$. Through simulations we can investigate which scaling $a_n$ seems to give a limit average shape as $n$ tends to infinity.

The rest of the paper is organized as follows. After first presenting the basic definitions on limit shapes (Section 2) we present our framework for describing a family of birth processes and birth-and-death processes on Young diagrams (Section 3). We then discuss the limit shapes of the birth processes of our framework (Section 4). Moving on to birth-and-death processes with birth steps and death steps of analogous types, we first analyze stationary distributions (Section 5) and then limit shapes (Section 6). One can also combine birth steps of one type with death steps of another type. For these processes, we have no analytical results but a few conjectures (Section 7).

## 2. Preliminaries

Let $\mathcal{P}_n = \{\lambda\colon \lambda \vdash n\}$ be the set of partitions of the integer $n > 0$ into parts $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N > 0$. The number of parts of the partition is denoted by $N = N(\lambda)$. For $i > N$, it will be convenient to define $\lambda_i := 0$.

To describe a partition $\lambda$ we list its parts, like $(\lambda_1, \lambda_2, \ldots)$. We will identify a partition $\lambda$ with its *Young diagram*, consisting of rows of squares such that the $i$th row has length $\lambda_i$. Thus the number of rows is $N(\lambda) = \sum_k r_k(\lambda)$ and the total number of squares is $n = \sum_i \lambda_i = \sum_k k r_k(\lambda)$, where $r_k$ denotes the number of parts of size $k > 0$.

The boundary of a Young diagram is defined by an alternating sequence of *inner corners* and *outer corners*. As illustrated in Fig. 1 there is one inner corner and one outer corner for each positive row length present in the diagram, and an additional inner corner corresponding to rows of length zero. We will let $N_{\text{out}} = N_{\text{out}}(\lambda)$ denote the number of outer corners of $\lambda$. Hence, the number of inner corners is $1 + N_{\text{out}}$.

As $n$ grows we can rescale Young diagrams so that their areas are always 1. Following Vershik [20], we say that the scaling is $a_n$ if row lengths are multiplied by $1/a_n$ and column heights are multiplied by $a_n/n$ (Fig. 1). Thus the border of the rescaled diagram of $\lambda \vdash n$ is the stepwise decreasing function $\tilde{\phi}_\lambda^{(n)} : [0, \infty) \to [0, \infty)$ given by

$$\tilde{\phi}_\lambda^{(n)}(x) := \frac{a_n}{n} \sum_{k \geqslant a_n x} r_k(\lambda). \tag{3}$$

For each positive integer $n$, let $\nu^{(n)}$ be some probability distribution on $\mathcal{P}_n$. As discussed by Vershik and others, cf. [6,20,24], it is often possible to find a sequence $\{a_n\}$ of scalings such that the rescaled diagrams approach a *limit shape* $\phi$ in probability as $n$ grows to infinity. By this one means that

$$\lim_{n\to\infty} \nu^{(n)}\bigl\{\lambda\colon \bigl|\tilde{\phi}_\lambda^{(n)}(x) - \phi(x)\bigr| < \epsilon\bigr\} = 1$$

holds for any $\epsilon > 0$ and any point $x$ in the domain $[0, \infty)$. If it holds for any point $x > 0$ but not necessarily for $x = 0$ we call it an *open limit shape*. If $\lim_{n\to\infty} E[\tilde{\phi}_\lambda^{(n)}(x)] = \phi(x)$ for any $x \geqslant 0$, we say that $\phi(x)$ is a *limit average shape*.

## 3. A family of processes on Young diagrams

We will here give a structured account of a family of random processes on Young diagrams. We are interested in processes where each step entails either the *birth of one square* or the *combined death of one square and birth of another*.

Births and deaths always occur in inner corners and outer corners, respectively. We will consider three main ways of choosing an inner corner, and three corresponding ways of choosing an outer corner. First, we need a tool to associate squares with row lengths, and row lengths with corners.

**Definition 1.** Consider some given Young diagram $\lambda$. For any square $s$ let $\kappa(s)$ denote the length of the row to which $s$ belongs. If $\kappa$ is a row length, let $\mathrm{Out}(\kappa)$ and $\mathrm{Inn}(\kappa)$ denote the unique outer corner and inner corner, respectively, for which the row coordinate is $\kappa$:

$$\mathrm{Out}(\kappa) = \bigl(\kappa, \max\{i \mid \lambda_i = \kappa\}\bigr) \quad \text{and} \quad \mathrm{Inn}(\kappa) = \bigl(\kappa, \max\{i \mid \lambda_i > \kappa\}\bigr).$$

For the inner corners at the ends, this definition is interpreted as $\mathrm{Inn}(0) = (0, N)$ and $\mathrm{Inn}(\lambda_1) = (\lambda_1, 0)$.

### 3.1. Types of death

Assume a current Young diagram $\lambda$. We shall consider the following three ways of killing an outer corner.

DESQUARE Choose a square $s$ uniformly at random, and remove the corresponding outer corner $\mathrm{Out}(\kappa(s))$.

DEROW Choose a non-empty row $i$ uniformly at random, and remove the corresponding outer corner $\mathrm{Out}(\lambda_i)$.

DECORNER Choose an outer corner of $\lambda$ uniformly at random and remove it.

Observe that if one obtains $\lambda'$ from $\lambda$ by removing outer corner $\mathrm{Out}(\kappa)$, then the number of rows of length $k \geqslant 1$ changes as follows:

$$r_k\bigl(\lambda'\bigr) = \begin{cases} r_\kappa(\lambda) - 1 & \text{if } k = \kappa, \\ r_{\kappa-1}(\lambda) + 1 & \text{if } k = \kappa - 1, \\ r_k & \text{otherwise.} \end{cases} \tag{4}$$

The total number of rows changes only if the removed square was in a row by itself:

$$N\bigl(\lambda'\bigr) = \begin{cases} N(\lambda) - 1 & \text{if } \kappa = 1, \\ N(\lambda) & \text{otherwise.} \end{cases} \tag{5}$$

### 3.2. Types of birth

Again assume that the current Young diagram is $\lambda$ with $n$ squares. For each of the three types of death, we will define analogous births. These definitions are somewhat more complicated because births involve an extra degree of freedom: the probability $\mu$ of birth occurring at inner corner $\mathrm{Inn}(0)$, i.e. creating a new row of length 1. For the birth step corresponding to DESQUARE, it is natural to have $\mu$ as a parameter—indeed, this is exactly the step used in Simon's model.

SQUARE($\mu$): With probability $\mu$ create a new row of length 1. Otherwise make a uniformly random choice of a square $s$ and insert a new square at the corresponding inner corner $\mathrm{Inn}(\kappa(s))$.

For the birth step corresponding to DECORNER, one chooses directly among the $1 + N_{\text{out}}(\lambda)$ inner corners. If all inner corners are of equal probability, then we obtain $\mu = 1/(1 + N_{\text{out}}(\lambda))$—indeed, this is exactly the step used in Rost's model. In order to vary $\mu$ we find it natural to give a special weight $w$ to the inner corner Inn(0) that corresponds to creation of a new row, with Rost's model obtained at $w = 1$.

CORNER$_w$: With probability

$$\mu = \frac{w}{w + N_{\text{out}}(\lambda)}$$

create a new row of length 1, i.e., insert a square at inner corner Inn(0). Otherwise, make a uniformly random choice among the other $N_{\text{out}}(\lambda)$ inner corners and insert a new square there. Thus, the probability of any given inner corner other than Inn(0) is $1/(w + N_{\text{out}}(\lambda))$.

The birth step corresponding to DEROW lies conceptually in-between the previous two birth steps. We thus find it natural to consider two versions. The first version is the ROW($\mu$) step, which has a fixed probability $\mu$ of creation of a new row—in analogy with the SQUARE($\mu$) step. The second version is the ROW$_w$ step, where the probability $\mu$ of creating a new row depends on the current number of rows—in analogy with the CORNER$_w$ step (where $\mu$ depends on the current number of inner corners). The precise definitions are as follows.

ROW($\mu$): With probability $\mu$ create a new row of length 1. Otherwise make a uniformly random choice of a *row* $i$ among the $N(\lambda)$ non-empty rows and insert a new square at the corresponding inner corner Inn($\lambda_i$).

ROW$_w$: With probability

$$\mu = \frac{w}{w + N(\lambda)}$$

create a new row of length 1. Otherwise make a uniformly random choice of a *row* $i$ among the $N(\lambda)$ non-empty rows and insert a new square at the corresponding inner corner Inn($\lambda_i$). In other words, for $\kappa > 0$ the probability of inner corner Inn($\kappa$) is

$$\frac{r_\kappa}{w + N(\lambda)}.$$

### 3.3. *Combination of birth steps and death steps*

Every combination of a type of death and a type of birth, with alternating deaths and births, defines a *birth-and-death process* on Young diagrams. In the course of such processes, the size of the diagram will oscillate by one square. In the introduction we discussed the Moran model with infinitely many alleles, which in the above scheme is equivalent to the SQUARE($\mu$)-DESQUARE process. However, here we will follow the convention from Strimling et al. [19] to think of these processes as "death-then birth" and instead consider the DESQUARE-SQUARE($\mu$) process. The limit shape is, of course, the same whether we look at Young diagrams after the death step or after the birth step.

## 4. Limit shapes and average shapes of birth processes

Here we will consider what limit shapes, if any, that we obtain for the birth processes of our framework. After dealing with the Rost and Simon models, we will spend the most effort on the ROW($\mu$) and ROW$_w$ processes, which seem to be heretofore undocumented.

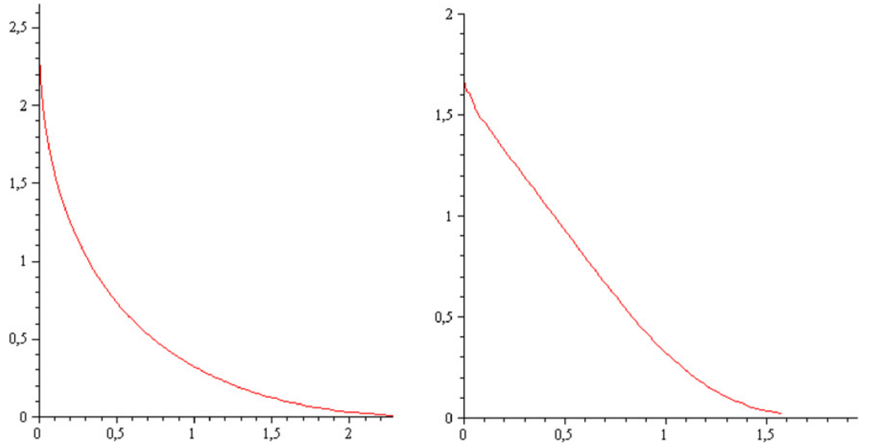


**Fig. 2.** Average shapes sampled through simulation of the CORNER$_w$ process for $n = 10{,}000$ with $w = 1$ (left) and $w = 0.25$ (right) with scaling $a_n = (n/w)^{1/2}$.

### *4.1. Birth process* CORNER$_w$*: generalizing the Rost model*

The CORNER$_w$ process with parameter $w = 1$ (so that all inner corners are equally probable) is equivalent to the discrete time version of the process studied by Rost [16,17]. It follows from Rost's results that for scaling $a_n = n^{1/2}$ the process gives the limit shape $\sqrt{x} + \sqrt{y} = 6^{1/4}$.

Simulations show that for smaller values of $w$ and scaling $a_n = (n/w)^{1/2}$ the shape fast approaches a straight line with slope $-1$, as shown in Fig. 2. The explanation is that for $w \ll 1$, new rows are created at a much slower rate than existing rows grow; every row will typically host an inner corner and grow at a rate $1/w$ times the rate at which new rows are created. Thus all rows grow with the same rate and the typical difference in length between two adjacent rows will be $1/w$, so the slope of the unscaled diagram is $-w$, which means slope $-1$ after scaling by $a_n = (n/w)^{1/2}$. Such argument can be made rigorous with great effort, as we show below in our treatment of the ROW$_w$ process.

It is an open question whether one can mathematically describe the interpolation between these two limit shapes as one gradually changes $w$ from 1 to 0.

### *4.2. Birth process* SQUARE$(\mu)$*: the Simon model*

The SQUARE$(\mu)$ birth process is equivalent to Simon's model of urban growth [18]. The process is also known as Polya's infinite urn model [3], as well as many other names such as a Yule process or a preferential attachment model (see [13] for a review). Although the model has been the subject of very much attention, we have never seen it treated in terms of limit shapes of integer partitions.

Let $\rho := 1/(1-\mu) > 1$. In our terminology, Simon proved an exact formula for $E[r_k^{(n)}]$, the expected number of rows of length $k$ in a diagram of $n$ squares obtained through the SQUARE birth process. For convenience, define

$$e_k^{(n)} := \frac{E[r_k^{(n)}]}{n}.$$

In terms of Eq. (3), this corresponds to using scaling $a_n = 1$. Simon's formula then says that

$$e_k^{(n)} = \frac{\mu(k-1)!}{(2-\mu)(k+\rho)(k+\rho-1)\cdots(2+\rho)} = \frac{\mu\Gamma(2+\rho)\Gamma(k)}{(2-\mu)\Gamma(k+\rho+1)}. \tag{6}$$

Simon also pointed out that Stirling's approximation of the Gamma function can be applied to (6) to yield

$$\frac{\Gamma(k)}{\Gamma(k+\rho+1)} = k^{-(\rho+1)}\bigl(1+O(1/k)\bigr). \tag{7}$$

Chung and Lu [3] used the same model, under the name of the "infinite Polya process," as an application of their work on concentration inequalities. They proved that

$$\left|\frac{r_k^{(n)}}{n} - e_k^{(n)}\right| \leqslant 2\sqrt{\frac{k^3 \ln n}{n}} \text{ holds with probability at least } 1 - \frac{2}{n}\left(1+\frac{1}{n}\right)^{k-1} \tag{8}$$

for all $n > 0$ [3, Theorem 40]. The result of Chung and Lu does not immediately give us a limit shape, because the shape is given by sums of the $r_k$, and potentially there are $n$ such terms in which case the sum of the error terms does not tend to zero. However, we can first deduce that we only need to consider a much smaller number of terms.

**Lemma 1.** *Choose $\lambda \in \mathcal{P}_n$ by running the* SQUARE *birth process. Let $m := \lfloor \ln n \rfloor$. Then*

$$\frac{1}{n}\sum_{k=m+1}^{n} r_k(\lambda) \to 0$$

*in probability as $n \to \infty$.*

**Proof.** By definition we have $\sum_{k=1}^{n} k r_k(\lambda)/n = 1$ and hence also $\sum_{k=1}^{n} k e_k^{(n)} = 1$. By these equalities and Eqs. (6), (7) and (8) we have, with probability at least $1 - \frac{2m}{n}(1+\frac{1}{n})^m$,

$$\begin{aligned}
\sum_{k=m+1}^{n} k r_k(\lambda)/n &= 1 - \sum_{k=1}^{m} k r_k(\lambda)/n \\
&= 1 - \sum_{k=1}^{m}\left[k e_k^{(n)} + O\left(\sqrt{\frac{k^3 \ln n}{n}}\right)\right] \\
&= \sum_{k=m+1}^{n} k e_k^{(n)} + O\left(m\sqrt{\frac{m^3 \ln n}{n}}\right) \\
&= \sum_{k=m+1}^{n} \frac{\mu\Gamma(2+\rho)}{2-\mu} k^{-\rho}\bigl(1+O(1/k)\bigr) + O\left(m\sqrt{\frac{m^3 \ln n}{n}}\right).
\end{aligned}$$

As $n \to \infty$ with $m = \ln n$, the last expression clearly tends to zero, and the equation holds with a probability that tends to one. $\square$

It is now relatively straightforward to derive a limit shape.

**Theorem 1.** *With scaling $a_n = 1$, the* SQUARE *birth process has a discrete open limit shape satisfying*

$$y = \frac{\mu\Gamma(2+\rho)}{(2-\mu)\rho} x^{-\rho} + O\bigl(x^{-(\rho+1)}\bigr), \quad \textit{for } x = 1, 2, 3, \ldots.$$

**Proof.** The scaled Young diagram for a fixed $n$ is given by

$$y_n(x) = \frac{1}{n}\sum_{k=x}^{n} r_k(\lambda)$$

and Lemma 1 says that in probability this sum is asymptotically equal to

$$\frac{1}{n}\sum_{k=x}^{\ln n} r_k(\lambda).$$

Much like the proof of Lemma 1, this sum can be approximated by means of Eqs. (6), (7) and (8) to yield

$$y(x) = \lim_{n\to\infty}\sum_{k=x}^{\ln n}\frac{\mu\Gamma(2+\rho)}{(2-\mu)}k^{-(\rho+1)}\bigl(1+O(1/k)\bigr)$$

in probability. Approximation by an integral yields the formula in the theorem. □

### *4.3. Birth process* SQUARE$(\mu_n)$ *with* $\mu_n$ *varying during the process*

Already Simon [18] observed that it may be interesting to study the SQUARE$(\mu)$ birth process also in cases where $\mu$ is not constant during the process but depends on $n$, the current number of squares. As far as we have seen in the literature only one such case has been thoroughly studied: the case where step $n$ uses $\mu_n = \theta/(n+\theta)$, so that $n\mu_n/(1-\mu_n) = \theta$ is constant. In mathematical population biology this model is known under the name of Hoppe's urn model [4]. Hoppe [9] showed that partitions generated by $n$ steps of this model are sampled from the Ewens distribution (2). As we discussed in the introduction, it is known in the literature on limit shapes that the Ewens distribution does not yield a limit shape for constant $\theta$ [24]. We can thus immediately state the following result.

**Theorem 2.** *Let $\theta$ be any positive constant and let $\mu_n = \theta/(n+\theta)$. Then the* SQUARE$(\mu_n)$ *birth process has no limit shape.*

Nothing is known for other dependencies of $\mu_n$ on $n$.

### *4.4. Birth processes* ROW$_w$ *and* ROW$(\mu)$

The ROW$_w$ and ROW$(\mu)$ processes on Young diagrams seem not to have been explicitly studied before, and are among our main contributions in this paper. We are grateful to Timo Seppäläinen for pointing out to us that the ROW$_w$ process is equivalent to the following continuous time process on particles moving along the integer points of the $x$-axis from the origin and to the right:

*The independent particle process*: Start with an infinite supply of particles at the origin. Particles move from the origin to site 1 according to a Poisson process with rate $w$. Particles that reach site 1 begin their own, independent, random walks to the right, taking each step at rate one.

If we interpret particle positions as row lengths, every event in the independent particle process corresponds to a step of the ROW$_w$ process. We shall use this interpretation to prove the following limit shape result.

**Theorem 3.** *With scaling $a_n = (n/w)^{1/2}$, the* ROW$_w$ *process has the limit shape*

$$y = \max\{0, \sqrt{2} - x\}.$$

Let us now look at the $\textsc{row}(\mu)$ process, where the probability of creating a new row is constant $\mu$ in every step. As the number of rows increases, the probability of any given row being chosen in the birth step decreases. Thus, if we consider the ratio between the probability that a new row is created and the probability that any given row is chosen in the birth step, this ratio increases linearly with the number of rows. Thus, the $\textsc{row}(\mu)$ process is equivalent to the following continuous time particle process.

*The accelerating independent particle process*: Let $N_t$ be the total number of particles at sites to the right of the origin at time $t$. Particles move from the origin to site 1 with the (increasing) rate $N_t\mu$, and then begin their own independent random walks to the right with rate 1. To start the process, the first particle enters site 1 at time 0.

Using this interpretation, we shall prove the following limit shape result.

**Theorem 4.** *For any fixed $n$ and $\mu_n$, choose a random partition $\kappa^{(n)} \in \mathcal{P}_n$ by starting with a diagram consisting of a single square and running the $\textsc{row}(\mu_n)$ process over $n-1$ birth steps. Suppose $\mu_n \log(\mu_n n) \to 0$ and $\mu_n n \to \infty$ as $n \to \infty$. Then $\kappa^{(n)}$ with scaling $a_n = 1/\mu_n$ has the limit shape*

$$y = e^{-x}.$$

The remainder of this section is devoted to the quite technical proofs of Theorems 3 and 4, starting with three Chernoff type inequalities. For these arguments we find it convenient to adopt the following notation.

**Definition 2.** For a real number $x$, we will write $x^+ = \max\{0, x\}$ and $x^- = \min\{0, x\}$.

For $x \neq 0$, the symbol $>^x$ means $>$ if $x > 0$ and $<$ if $x < 0$. Similarly, the symbol $\geqslant^x$ means $\geqslant$ if $x > 0$ and $\leqslant$ if $x < 0$. Finally, $<^x$ and $\leqslant^x$ have the same meaning as $>^{-x}$ and $\geqslant^{-x}$, respectively.

*4.5. Three propositions*

The first proposition is a standard result due to Chernoff.

**Proposition 1** *(Chernoff bounds). Let $X$ be a real-valued random variable. Then, for any real $a$ and $r \neq 0$, we have*

$$P\big((X-a) \geqslant^r r\big) \leqslant \exp \inf_{t \geqslant^r 0} f_r(t),$$

*where we define $f_r(t) := \log E(e^{tX}) - (a+r)t$ for any real $t$ such that $E(e^{tX})$ exists (and the infimum are taken only over those $t$).*

**Proof.** If $t \geqslant^r 0$, Markov's inequality yields that

$$P\big((X-a) \geqslant^r r\big) = P\big(e^{(X-a)t} \geqslant e^{rt}\big) \leqslant e^{-rt} E\big(e^{(X-a)t}\big) = e^{f_r(t)}. \qquad \square \tag{9}$$

The next two propositions are relatively straightforward applications of Chernoff bounds.

**Proposition 2.** *Let $X_1, X_2, \ldots$ be independent random variables, exponentially distributed with mean 1, and define $S_n := X_1 + X_2 + \cdots + X_n$. Then, for any $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$P\big((S_n - n) \geqslant^\zeta \zeta n\big) \leqslant e^{-|\zeta| n\delta}$$

*for all non-negative integers $n$ and all $\zeta \in \mathbb{R}$ such that $|\zeta| > \varepsilon$.*

**Proof.** If $n = 0$ there is nothing to prove, so assume that $n \geqslant 1$. If $\zeta \leqslant -1$ there is nothing to prove since $S_n > 0$. Thus, in the following we assume that $\zeta > -1$.

As in Proposition 1, define

$$f_{\zeta n}(t) := \log E\left(e^{tS_n}\right) - (1+\zeta)nt = -(1+\zeta)nt - \log(1-t).$$

It is easy to verify that $f_{\zeta n}(t)$ is minimized by setting $t = \zeta/(1+\zeta)$, so the following bound is optimal:

$$\inf_{t \geqslant^\zeta 0} f_{\zeta n}(t) \leqslant f_{\zeta n}\big(\zeta/(1+\zeta)\big) = -|\zeta| n \cdot \left(\operatorname{sgn}(\zeta) - |\zeta|^{-1} \log(1+\zeta)\right).$$

It is also easy to verify that the continuous function $g(\zeta) := \operatorname{sgn}(\zeta) - |\zeta|^{-1}\log(1+\zeta)$ is bounded below from zero as $\zeta > -1$ and $|\zeta| > \varepsilon$, i.e. there is a $\delta > 0$ such that $g(\zeta) > \delta$ for all such $\zeta$. The proposition now follows from Proposition 1. $\square$

**Proposition 3.** *Let $X_1, X_2, \ldots$ be independent exponentially distributed random variables with $E(X_i) = 1/i$. For any $1 \leqslant k \leqslant \ell$, put $S_{k,\ell} := X_k + X_{k+1} + \cdots + X_{\ell-1}$. Then, for each $\varepsilon > 0$, there are $\delta, \delta' > 0$ such that,*

$$P\left(\left(S_{k,\ell} - \log\frac{\ell}{k}\right) \geqslant^r \zeta \cdot \left(1 + \log\frac{\ell}{k}\right)\right) \leqslant e^{-|\zeta| \cdot ((1+\log\frac{\ell}{k})k\delta' - \frac{1}{2})},$$
$$P\left(\left(S_{k,\ell} - \log\frac{\ell}{k}\right) \geqslant^r \zeta \cdot \left(1 + \log\frac{\ell}{k}\right)\right) \leqslant e^{-(1+\log\frac{\ell}{k})k\delta},$$

*for any $\zeta \in \mathbb{R}$, and $k, \ell \in \mathbb{Z}^+$ such that $|\zeta| > \varepsilon$ and $k \leqslant \ell$.*

**Proof.** The case $k = \ell$ is trivial, so let us assume that $k < \ell$.

The cumulant-generating function for $X_i$ is given by $\log E(e^{tX_i}) = -\log(1 - \frac{t}{i})$ and since $X_1, \ldots, X_{\ell-1}$ are independent, we obtain

$$\log E\left(e^{tS_{k,\ell}}\right) = -\sum_{i=k}^{\ell-1} \log\left(1 - \frac{t}{i}\right)$$

which is defined for $t < k$.

Put $r := \zeta \cdot (1 + \log\frac{\ell}{k})$. By Proposition 1,

$$P\left(\left(S_{k,\ell} - \log\frac{\ell}{k}\right) \geqslant^r r\right) \leqslant \exp \inf_{t \geqslant^\zeta 0} f_r(t), \tag{10}$$

where we define

$$f_r(t) := -rt - t\log\frac{\ell}{k} - \sum_{i=k}^{\ell-1} \log\left(1 - \frac{t}{i}\right).$$

Since $|\log(1 - \frac{t}{x})|$ is a decreasing function of $x$ for $x \geqslant k > t$, we obtain an upper bound for $f_r(t)$ by integration from $k$ to $\ell$ if we add the first term of the sum if $t > 0$:

$$\begin{aligned} f_r(t) &\leqslant -rt - t\log\frac{\ell}{k} - \log\left(1 - \frac{t^+}{k}\right) - \int_k^\ell \log\left(1 - \frac{t}{x}\right) dx \\ &= -rt - (\ell - t)\log\left(1 - \frac{t}{\ell}\right) + \left(k - t - (\operatorname{sgn} t)^+\right)\log\left(1 - \frac{t}{k}\right). \end{aligned}$$

Observing that $-x\log(1-\frac{t}{x})$ is a decreasing function of $x$ for $x \geqslant k > t$, we get

$$f_r(t) \leqslant -rt + t\log\left(1-\frac{t}{\ell}\right) - \left(t + (\operatorname{sgn} t)^+\right)\log\left(1-\frac{t}{k}\right),$$

and since the middle term is non-positive,

$$f_r(t) \leqslant -rt - \left(t + (\operatorname{sgn} t)^+\right)\log\left(1-\frac{t}{k}\right). \tag{11}$$

Put $u_\zeta := 1 - e^{-\zeta/2}$. Then, by (11),

$$\begin{aligned}\inf_{t\colon t \geqslant^\zeta 0} f_r(t) &\leqslant f_r(u_\zeta k) \leqslant -ru_\zeta k - \left(u_\zeta k + (\operatorname{sgn}\zeta)^+\right)\log(1-u_\zeta) \\ &= -|\zeta| \cdot \left(\left(1 + 2\log\frac{\ell}{k}\right)k\frac{|u_\zeta|}{2} - \frac{1}{2}(\operatorname{sgn}\zeta)^+\right) \\ &\leqslant -|\zeta| \cdot \left(\left(1 + \log\frac{\ell}{k}\right)k\delta' - \frac{1}{2}\right),\end{aligned}$$

if we choose

$$\delta' = \frac{1}{2}\min\bigl\{|u_\varepsilon|, |u_{-\varepsilon}|\bigr\} < \frac{1}{2}|u_\zeta|.$$

The first inequality in the proposition now follows from Proposition 1. The second inequality is a simple consequence of the first one. □

### 4.6. *Proof of Theorem 3*

Let $X_1$ be the time until the first particle leaves the origin, and, for $k = 2, 3, \ldots$, let $X_k$ be the time elapsing from when the $(k-1)$th particle leaves the origin to when the $k$th particle leaves the origin. For $k, i = 1, 2, \ldots$, let $Y_{k,i}$ be the time particle $k$ is waiting at site $i$ before going to site $i+1$. Thus $X_k$ is exponentially distributed with mean $1/w$, and $Y_{k,i}$ is exponentially distributed with mean 1; all $X_k$ and $Y_{k,i}$ are independent.

Let $\nu(t)$ be the particle composition at time $t$, i.e. $\nu_k(t)$ is the position of particle $k$ at time $t$. At the time $t = T$ when $\nu(t)$ gets its $n$th square, we have obtained a composition that we call $\lambda^{(n)} = \nu(T)$ (but the particle process continues for ever). If we sort the parts of $\lambda^{(n)}$ we obtain the sampled partition in $\mathcal{P}_n$.

For any $k \in \mathbb{Z}^+$, $t > 0$ and $\alpha \neq 0$, let $s_k^\alpha(t) := ((1+\alpha)t - w^{-1}k)^+$. Intuitively, after time $t$, the position of the $k$th particle is probably approximately $(t - w^{-1}k)^+ = s_k^0(t)$. To be more precise, we have the following lemma.

**Lemma 2.** *For any $\alpha \neq 0$,*

$$\lim_{t\to\infty} P\bigl(\forall k \in \mathbb{Z}^+\colon\ \nu_k(t) \leqslant^\alpha s_k^\alpha(t)\bigr) = 1.$$

**Proof.** For convenience, let $S_k := X_1 + X_2 + \cdots + X_k$ and $T_k(a) := Y_{k,1} + Y_{k,2} + \cdots + Y_{k,\lceil a\rceil - 1}$ for $a > 0$.

First, we observe that for $t, a > 0$ and $k \in \mathbb{Z}^+$ the following equivalence holds by the definition of the particle process:

$$\nu_k(t) \geqslant a \quad \Longleftrightarrow \quad S_k + T_k(a) \leqslant t. \tag{12}$$

Next, we will give an upper bound for the probability that $S_k + T_k(s_k^\alpha(t)) \leqslant^\alpha t$. If $s_k^\alpha(t) > 0$, we have

$$t - E\big[S_k + T_k\big(s_k^\alpha(t)\big)\big] = t - w^{-1}k - \big\lceil (1+\alpha)t - w^{-1}k\big\rceil + 1 = -\alpha t + R,$$

where $|R| \leqslant 1$. If the sum of $S_k$ and $T_k(s_k^\alpha(t))$ is far from its mean, then at least one of the terms must be far from its mean:

$$\begin{aligned} P\big(S_k + T_k\big(s_k^\alpha(t)\big) \leqslant^\alpha t\big) &= P\big(S_k + T_k\big(s_k^\alpha(t)\big) - E\big[S_k + T_k\big(s_k^\alpha(t)\big)\big] \leqslant^\alpha -\alpha t + R\big) \\ &\leqslant P\big(S_k - E[S_k] \leqslant^\alpha (-\alpha t + R)/2\big) \\ &\quad + P\big(T_k\big(s_k^\alpha(t)\big) - E\big[T_k\big(s_k^\alpha(t)\big)\big] \leqslant^\alpha (-\alpha t + R)/2\big). \end{aligned}$$

By Proposition 2 there is a $\delta > 0$ such that the above is less than

$$e^{-|\alpha t - R|\cdot w\delta/2} + e^{-|\alpha t - R|\cdot\delta/2} = e^{-\Theta(t)},$$

and we conclude that there is a function $f(t) = e^{-\Theta(t)}$ such that

$$P\big(S_k + T_k\big(s_k^\alpha(t)\big) \leqslant^\alpha t\big) \leqslant f(t) \tag{13}$$

whenever $s_k^\alpha(t) > 0$. If $s_k^\alpha(t) > 0$, then by (12) we have

$$\begin{aligned} P\big(\nu_k(t) >^\alpha s_k^\alpha(t)\big) &\begin{cases} \leqslant P(S_k + T_k(s_k^\alpha(t)) \leqslant t) & \text{if } \alpha > 0, \\ = P(S_k + T_k(s_k^\alpha(t)) > t) & \text{if } \alpha < 0 \end{cases} \\ &\leqslant P\big(S_k + T_k\big(s_k^\alpha(t)\big) \leqslant^\alpha t\big). \end{aligned}$$

Combining this with (13) yields

$$P\big(\exists k \in \mathbb{Z}^+ \colon s_k^\alpha(t) > 0 \text{ and } \nu_k(t) >^\alpha s_k^\alpha(t)\big) \leqslant \#\big\{k \in \mathbb{Z}^+ \colon s_k^\alpha(t) > 0\big\} \cdot e^{-\Theta(t)} = \Theta(t) \cdot e^{-\Theta(t)} \to 0$$

as $t \to \infty$.

What happens when $k$ is so large that $s_k^\alpha(t) = 0$? If $\alpha < 0$, obviously

$$P\big(\exists k \in \mathbb{Z}^+ \colon s_k^\alpha(t) = 0 \text{ and } \nu_k(t) >^\alpha s_k^\alpha(t)\big) = 0$$

as $\nu_k(t)$ is always non-negative. Assume instead that $\alpha > 0$. Clearly, $s_k^\alpha(t) = 0$ if and only if $k \geqslant (1+\alpha)wt$, so

$$P\big(\exists k \in \mathbb{Z}^+ \colon s_k^\alpha(t) = 0 \text{ and } \nu_k(t) > 0\big) \leqslant P\big(\nu_{\lfloor (1+\alpha)wt\rfloor}(t) > 0\big) = P(S_{\lfloor (1+\alpha)wt\rfloor} \leqslant t)$$

which is bounded by $e^{-\Theta(t)}$ by Proposition 2. $\square$

Our next lemma gives an approximation of $\lambda_k^{(n)}$.

**Lemma 3.** *For any $\gamma \neq 0$,*

$$\lim_{n\to\infty} P\big(\forall k \in \mathbb{Z}^+ \colon \lambda_k^{(n)} \leqslant^\gamma s_k^\gamma(\sqrt{2n/w})\big) = 1.$$

**Proof.** Recall that

$$T = \inf\left\{t\colon \sum_{k=1}^{\infty} \nu_k(t) = n\right\} = \inf\bigl\{t\colon \forall k \in \mathbb{Z}^+,\ \nu_k(t) = \lambda_k^{(n)}\bigr\}$$

is the time $t$ until $\nu(t)$ has exactly $n$ squares. First, we will use Lemma 2 to show that $T$ is probably approximately $\sqrt{2n/w}$. Let $0 < |\beta| < 1$ and define $t^\beta := (1+\beta)\sqrt{2n/w}$. Then, for any $\alpha \in \mathbb{R}$ such that $0 < |\alpha| < 1$, we have

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{\infty} s_k^\alpha\bigl(t^\beta\bigr) &= \sqrt{w/n}\int_0^\infty s^\alpha_{\lceil y\sqrt{wn}\,\rceil}\bigl(t^\beta\bigr)\,dy \\
&= \sqrt{w/n}\int_0^\infty \bigl((1+\alpha)(1+\beta)\sqrt{2n/w} - w^{-1}\lceil y\sqrt{wn}\,\rceil\bigr)^+\,dy \\
&\xrightarrow[n\to\infty]{} \int_0^\infty \bigl((1+\alpha)(1+\beta)\sqrt{2} - y\bigr)^+\,dy \\
&= (1+\alpha)^2(1+\beta)^2.
\end{aligned}$$

Choose $\alpha \neq 0$ such that $\operatorname{sgn}\alpha = -\operatorname{sgn}\beta$ and $|\alpha|$ is small enough so that $(1+\alpha)^2(1+\beta)^2 >^\beta 1$. Lemma 2 now yields that

$$\lim_{n\to\infty} P\left(\sum_{k=1}^{\infty}\nu_k\bigl(t^\beta\bigr) >^\beta n\right) = 1,$$

and we conclude that

$$\lim_{n\to\infty} P\bigl(\forall k \in \mathbb{Z}^+\colon\ \nu_k\bigl(t^\beta\bigr) \geqslant^\beta \lambda_k^{(n)}\bigr) = 1.$$

If we apply Lemma 2 once more, but this time with $\alpha := \beta$, we obtain

$$1 = \lim_{n\to\infty} P\bigl(\forall k \in \mathbb{Z}^+\colon\ \lambda_k^{(n)} \leqslant^\beta s_k^\beta\bigl(t^\beta\bigr)\bigr) \leqslant P\bigl(\forall k \in \mathbb{Z}^+\colon\ \lambda_k^{(n)} \leqslant^\gamma s_k^\gamma(\sqrt{2n/w}\,)\bigr)$$

if we choose $\beta \neq 0$ such that $\operatorname{sgn}\beta = \operatorname{sgn}\gamma$ and $|\beta|$ is small enough such that $(1+\beta)^2 <^\gamma 1+\gamma$. $\square$

Now we are ready to prove Theorem 3.

**Proof of Theorem 3.** Take any $x \geqslant 0$ and $\varepsilon \neq 0$ and let

$$Q := P\bigl((wn)^{-1/2}\#\bigl\{k \in \mathbb{Z}^+\colon\ \sqrt{w/n}\cdot\lambda_k^{(n)} > x\bigr\} >^\varepsilon \bigl((1+\varepsilon)\sqrt{2} - x\bigr)^+\bigr).$$

We have to prove that

$$\lim_{n\to\infty} Q = 0. \tag{14}$$

For $x \geqslant (1+\varepsilon)\sqrt{2}$, clearly $Q$ is non-increasing in $x$, so we may assume, without loss of generality, that $x < (1+\varepsilon)\sqrt{2}$.

Define $K^{(n)} := ((1+\varepsilon)\sqrt{2} - x)\sqrt{wn} > 0$ and $R := \lceil K^{(n)} \rceil - K^{(n)} \in [0, 1)$. Pick a $\gamma \in \mathbb{R}$ such that $0 < \gamma/\varepsilon < 1$. We have

$$\sqrt{w/n} \cdot s^{\gamma}_{\lceil K^{(n)} \rceil}(\sqrt{2n/w}) = \bigl(x + (\gamma - \varepsilon)\sqrt{2} - (wn)^{-1/2}R\bigr)^{+},$$

which is $\leqslant x$ if $\varepsilon > 0$ and $> x$ for large $n$ if $\varepsilon < 0$.

Since $s^{\gamma}_k(\sqrt{2n/w})$ is non-increasing in $k$, it follows that

$$s^{\gamma}_k(\sqrt{2n/w}) \leqslant \sqrt{n/w} \cdot x \quad \text{for } k \geqslant \lceil K^{(n)} \rceil \text{ if } \gamma > 0, \tag{15}$$

$$s^{\gamma}_k(\sqrt{2n/w}) > \sqrt{n/w} \cdot x \quad \text{for } k \leqslant \lceil K^{(n)} \rceil \text{ if } \gamma < 0, \tag{16}$$

for large $n$.

Finally, we have all the tools we need to prove (14).

$$\begin{aligned}
Q &= P\bigl((wn)^{-1/2}\#\bigl\{k \in \mathbb{Z}^+ \colon \sqrt{w/n} \cdot \lambda^{(n)}_k > x\bigr\} >^{\varepsilon} \bigl((1+\varepsilon)\sqrt{2} - x\bigr)^{+}\bigr) \\
&\leqslant \begin{cases} P(\exists k \geqslant \lceil K^{(n)} \rceil \colon \sqrt{w/n} \cdot \lambda^{(n)}_k > x) & \text{if } \varepsilon > 0, \\ P(\exists k \leqslant \lceil K^{(n)} \rceil \colon \sqrt{w/n} \cdot \lambda^{(n)}_k \leqslant x) & \text{if } \varepsilon < 0 \end{cases} \\
&\leqslant P\bigl(\exists k \in \mathbb{Z}^+ \colon \lambda^{(n)}_k >^{\gamma} s^{\gamma}_k(\sqrt{2n/w})\bigr) \to 0 \quad (\text{as } n \to 0)
\end{aligned}$$

where the last inequality follows from (15) and (16) and the limit follows from Lemma 3. □

### 4.7. Proof of Theorem 4

For $k = 1, 2, \ldots$, let $X^{\mu}_k$ be the time elapsing from when the $k$th particle leaves the origin to when the $(k+1)$st particle leaves the origin. For $k, i = 1, 2, \ldots$, let $Y_{k,i}$ be the time particle $k$ is waiting at site $i$ before going to site $i+1$. Thus $X^{\mu}_k$ is exponentially distributed with mean $(\mu k)^{-1}$, and $Y_{k,i}$ is exponentially distributed with mean 1; all $X^{\mu}_k$ and $Y_{k,i}$ are independent.

For convenience, let $S^{\mu}_{k,\ell} := X^{\mu}_k + X^{\mu}_{k+1} + \cdots + X^{\mu}_{\ell-1}$ and $T_k(a) := Y_{k,1} + Y_{k,2} + \cdots + Y_{k,\lfloor a \rfloor}$ for $a \geqslant 0$.

Let $\nu^{\mu}(t)$ be the particle composition at time $t$, i.e. $\nu^{\mu}_k(t)$ is the position of the $k$th particle at time $t$. At the time $t = T$ when $\nu(t)$ gets its $n$th square, we have obtained a composition that we call $\lambda^{(n)} = \nu(T)$ (but the particle process continues for ever). If we sort the parts of $\lambda^{(n)}$ we obtain the sampled partition in $\kappa^{(n)}\mathcal{P}_n$.

For any integers $1 \leqslant k \leqslant \ell$, let $\nu_k(\ell, \mu) := \nu^{\mu}_k(S_{1,\ell})$ and let $s^{\alpha}_k(\ell, \mu) := ((1+\alpha)\log\frac{(1+\alpha)\ell}{k})^{+}\mu^{-1}$.

The following lemma gives an approximation for $\nu_k(\ell_n, \mu_n)$.

**Lemma 4.** *Given a sequence $0 < \mu_n \to 0$ and a positive integer sequence $\ell_n \to \infty$ such that $\mu_n \log \ell_n \to 0$, for any $\alpha \neq 0$ the following holds*:

$$\lim_{n\to\infty} P\bigl(\forall k \in \mathbb{Z}^+ \colon \nu_k(\ell_n, \mu_n) \leqslant^{\alpha} s^{\alpha}_k(\ell_n, \mu_n)\bigr) = 1.$$

**Proof.** For convenience, let us write $\nu_k$ and $s^{\alpha}_k$ instead of $\nu_k(\ell_n, \mu_n)$ and $s^{\alpha}_k(\ell_n, \mu_n)$.

If $k > \ell_n$ clearly $\nu_k = s^{\alpha}_k = 0$, and if $\alpha < 0$ and $k > (1+\alpha)\ell_n$ then $s^{\alpha}_k < 0$; in either case $\nu_k \leqslant^{\alpha} s^{\alpha}_k$ holds. Thus, it suffices to show that

$$\lim_{n\to\infty} P\bigl(\forall k \leqslant \bigl(1 + \alpha^{-}\bigr)\ell_n \colon \nu_k \leqslant^{\alpha} s^{\alpha}_k\bigr) = 1.$$

In what follows we assume that $k \leqslant (1 + \alpha^{-})\ell_n$.

For $a \geqslant 0$ the following equivalence holds by the definition of the particle process:

$$\nu_k > a \quad \Longleftrightarrow \quad S^{\mu_n}_{k,\ell_n} > T_k(a). \tag{17}$$

Thus,

$$\begin{aligned}
P\big(\nu_k >^{\alpha} s_k^{\alpha}\big) &= P\big(S^{\mu_n}_{k,\ell} - T_k\big(s_k^{\alpha}\big) >^{\alpha} 0\big) && (18)\\
&= P\big(S^{\mu_n}_{k,\ell} - T_k\big(s_k^{\alpha}\big) - s_k^0 + \lfloor s_k^{\alpha}\rfloor >^{\alpha} \lfloor s_k^{\alpha}\rfloor - s_k^0\big) && (19)\\
&\leqslant P\big(S^{\mu_n}_{k,\ell} - s_k^0 >^{\alpha} \big(\lfloor s_k^{\alpha}\rfloor - s_k^0\big)/2\big) + P\big(T_k\big(s_k^{\alpha}\big) - \lfloor s_k^{\alpha}\rfloor >^{-\alpha} \big(s_k^0 - \lfloor s_k^{\alpha}\rfloor\big)/2\big). && (20)
\end{aligned}$$

Assume without loss of generality that $|\alpha| < 1/2$ and hence $(1+\alpha)\log(1+\alpha) >^{\alpha} \alpha/2$. For some $0 \leqslant R < 1$, we have

$$\begin{aligned}
\lfloor s_k^{\alpha}\rfloor &= \mu_n^{-1}(1+\alpha)\log\frac{(1+\alpha)\ell_n}{k} - R,\\
\lfloor s_k^{\alpha}\rfloor - s_k^0 &= \mu_n^{-1}\left((1+\alpha)\log(1+\alpha) + \alpha\log\frac{\ell_n}{k}\right) - R\\
&>^{\alpha} \mu_n^{-1}\frac{\alpha}{2}\left(1+\log\frac{\ell_n}{k}\right) - R.
\end{aligned}$$

If $n$ is large, $(\lfloor s_k^{\alpha}\rfloor - s_k^0)/\lfloor s_k^{\alpha}\rfloor$ is bounded away from zero, so Propositions 2 and 3 yield that there is a $\delta > 0$ such that (20) is bounded by

$$e^{-\delta\cdot k(1+\log\frac{\ell_n}{k})} + e^{-\delta\cdot|\mu_n^{-1}\frac{\alpha}{2}(1+\log\frac{\ell_n}{k})-R|}.$$

We conclude that there is a function $f(n) = e^{-\Theta(\mu_n^{-1})}$ such that

$$P\big(\nu_k >^{\alpha} s_k^{\alpha}\big) < e^{-\delta\cdot k(1+\log\frac{\ell_n}{k})} + f(n)$$

for every $k \leqslant \ell_n$.

By summing over $k$, we obtain

$$P\big(\exists k \leqslant \ell_n\colon\ \nu_k >^{\alpha} s_k^{\alpha}\big) \leqslant \sum_{1\leqslant k\leqslant \ell_n} e^{-\delta\cdot k(1+\log\frac{\ell_n}{k})} + \sum_{1\leqslant k\leqslant \ell_n} f(n).$$

From our assumption that $\mu_n \log \ell_n \to 0$, it follows that

$$\sum_{1\leqslant k\leqslant \ell_n} f(n) = O\big(\ell_n e^{-\Theta(\mu_n^{-1})}\big) \to 0,$$

and a simple calculation reveals that the first sum also tends to zero as $n \to \infty$:

$$\begin{aligned}
\sum_{1\leqslant k\leqslant \ell_n} e^{-\delta\cdot k(1+\log\frac{\ell_n}{k})} &\leqslant \sum_{1\leqslant k\leqslant \log \ell_n} e^{-\delta\cdot\log\frac{\ell_n}{\log \ell_n}} + \sum_{\log \ell_n < k < \infty} e^{-\delta\cdot k}\\
&\sim \ell_n^{-\delta}(\log \ell_n)^{1+\delta} + \frac{\ell_n^{-\delta}}{1-e^{-\delta}} \to 0. \qquad \square
\end{aligned}$$

Our next lemma gives an approximation of $\lambda_k^{(n)}$.

**Lemma 5.** *Given a sequence* $0 < \mu_n \to 0$ *such that* $\mu_n \log(\mu_n n) \to 0$, *for any* $\gamma \neq 0$ *the following holds*:

$$\lim_{n\to\infty} P\big(\forall k \in \mathbb{Z}^+ \colon \lambda_k^{(n)} \leqslant^\gamma s_k^\gamma(\mu_n n, \mu_n)\big) = 1.$$

**Proof.** First, we will use Lemma 2 to show that the number of rows of $\lambda^{(n)}$ is probably approximately $\mu_n n$.

Let $0 < |\beta| < 1$ and define $\ell_n^\beta := \lfloor (1+\beta)\mu_n n \rfloor$. Then, for any $\alpha \in \mathbb{R}$ such that $0 < |\alpha| < 1$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\ell_n^\beta} s_k^\alpha\big(\ell_n^\beta, \mu_n\big) = (1+\alpha)(1+\beta) \int_0^1 \log\frac{1}{u}\,du = (1+\alpha)(1+\beta).$$

Choose $\alpha \neq 0$ such that $\operatorname{sgn}\alpha = -\operatorname{sgn}\beta$ and $|\alpha|$ is small enough so that $(1+\alpha)(1+\beta) >^\beta 1$. Lemma 4 now yields that

$$\lim_{n\to\infty} P\left(\sum_{k=1}^{\infty} \nu_k\big(\ell_n^\beta, \mu_n\big) >^\beta n\right) = 1,$$

and we conclude that

$$\lim_{n\to\infty} P\big(\forall k \in \mathbb{Z}^+ \colon \nu_k\big(\ell_n^\beta, \mu_n\big) \geqslant^\beta \lambda_k^{(n)}\big) = 1.$$

If we apply Lemma 4 once more, but this time with $\alpha := \beta$, we obtain

$$1 = \lim_{n\to\infty} P\big(\forall k \in \mathbb{Z}^+ \colon \lambda_k^{(n)} \leqslant^\beta s_k^\beta\big(\ell_n^\beta, \mu_n\big)\big) \leqslant P\big(\forall k \in \mathbb{Z}^+ \colon \lambda_k^{(n)} \leqslant^\gamma s_k^\gamma(\mu_n n, \mu_n)\big)$$

if we choose $\beta \neq 0$ such that $\operatorname{sgn}\beta = \operatorname{sgn}\gamma$ and $|\beta|$ is small enough such that $(1+\beta)^2 <^\gamma 1+\gamma$. $\square$

Now we are ready to prove Theorem 4.

**Proof of Theorem 4.** Take any $x \geqslant 0$ and $\varepsilon \neq 0$ with $|\varepsilon| < 1$ and let

$$Q := P\big((\mu_n n)^{-1} \cdot \#\big\{k \in \mathbb{Z}^+ \colon \mu_n \lambda_k^{(n)} > x\big\} >^\varepsilon e^{-x}\big).$$

We have to prove that

$$\lim_{n\to\infty} Q = 0. \tag{21}$$

Define $K^{(n)} := \mu_n n(1+\varepsilon)e^{-x/(1+\varepsilon)} > 0$ and $R := \lceil K^{(n)} \rceil - K^{(n)} \in [0,1)$. Pick a $\gamma \in \mathbb{R}$ with $\operatorname{sgn}\gamma = \operatorname{sgn}\varepsilon$ and $|\gamma|$ positive but very small. We have

$$\lim_{n\to\infty} \mu_n s_{\lceil K^{(n)} \rceil}^\gamma(\mu_n n, \mu_n) = (1+\gamma)\left(\frac{x}{1+\varepsilon} + \log\frac{1+\gamma}{1+\varepsilon}\right)^+,$$

which, if $|\gamma|$ is small enough, is $\leqslant x$ if $\varepsilon > 0$ and $> x$ if $\varepsilon < 0$.

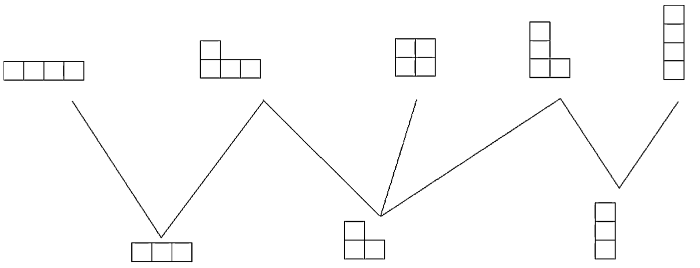

**Fig. 3.** The bipartite graph $G_4$.

Since $s_k^{\gamma}(\mu_n n, \mu_n)$ is non-increasing in $k$, it follows that

$$s_k^{\gamma}(\mu_n n, \mu_n) \leqslant \mu_n^{-1} x \quad \text{for } k \geqslant \lceil K^{(n)} \rceil \text{ if } \gamma > 0, \tag{22}$$

$$s_k^{\gamma}(\mu_n n, \mu_n) > \mu_n^{-1} x \quad \text{for } k \leqslant \lceil K^{(n)} \rceil \text{ if } \gamma < 0, \tag{23}$$

for large $n$.

Finally, we have all the tools we need to prove (21).

$$\begin{aligned} Q &= P\big((\mu_n n)^{-1} \cdot \#\big\{k \in \mathbb{Z}^+\colon\ \mu_n \lambda_k^{(n)} > x\big\} >^{\varepsilon} e^{-x}\big) \\ &\leqslant \begin{cases} P(\exists k \geqslant \lceil K^{(n)} \rceil\colon\ \mu_n \lambda_k^{(n)} > x) & \text{if } \varepsilon > 0, \\ P(\exists k \leqslant \lceil K^{(n)} \rceil\colon\ \mu_n \lambda_k^{(n)} \leqslant x) & \text{if } \varepsilon < 0 \end{cases} \\ &\leqslant P\big(\exists k \in \mathbb{Z}^+\colon\ \lambda_k^{(n)} >^{\gamma} s_k^{\gamma}(\mu_n n, \mu_n)\big) \to 0 \quad (\text{as } n \to 0) \end{aligned}$$

where the last inequality follows from (22) and (23) and the limit follows from Lemma 5. □

## 5. Stationary distributions of birth-and-death processes of consistent types

We shall here consider the stationary distributions of the birth-and-death processes that are of “consistent” types, in the sense that the birth step is analogous to the death step: DESQUARE-SQUARE($\mu$), DEROW-ROW($\mu$), DEROW-ROW$_w$ and DECORNER-CORNER$_w$. For a given $n$, processes with alternating births and deaths can be regarded as random walks on the Hasse diagram for $\mathcal{P}_n \cup \mathcal{P}_{n-1}$ partially ordered by inclusion of Young diagrams, i.e., the bipartite graph $G_n$ on the vertex set $\mathcal{P}_n \cup \mathcal{P}_{n-1}$ where there is an edge between $\lambda \in \mathcal{P}_n$ and $\lambda' \in \mathcal{P}_{n-1}$ if $\lambda'$ is the result of removing some outer corner from $\lambda$ (see Fig. 3).

For each direction of an edge $(\lambda, \lambda')$ of $G_n$ we define a weight by

$$\vec{\pi}\big(\lambda, \lambda'\big) := \text{the probability of the death step taking } \lambda \text{ to } \lambda'$$

and

$$\overleftarrow{\pi}\big(\lambda, \lambda'\big) := \text{the probability of the birth step taking } \lambda' \text{ to } \lambda.$$

Let $i = 0, 1, 2, \ldots$ indicate time. At even time steps ($t = 2i$) the unit probability mass is distributed over $\mathcal{P}_n$; at odd time steps ($t = 2i + 1$) it is instead distributed over $\mathcal{P}_{n-1}$. Between time steps the probability mass travels along the edges according to their weights:

$$p_{2i+1}\big(\lambda'\big) = \sum_{\lambda} \vec{\pi}\big(\lambda, \lambda'\big) p_{2i}(\lambda), \qquad p_{2i+2}(\lambda) = \sum_{\lambda'} \overleftarrow{\pi}\big(\lambda, \lambda'\big) p_{2i+1}\big(\lambda'\big).$$

$p_{2i} = p_{\text{even}}$ is an (even-step) *stationary distribution* on $\mathcal{P}_n$ if it satisfies $p_{2i+2}(\lambda) = p_{2i}(\lambda)$ for all $\lambda \vdash n$. Analogously, $p_{2i+1} = p_{\text{odd}}$ is an *odd-step stationary distribution* on $\mathcal{P}_{n-1}$ if $p_{2i+3}(\lambda') = p_{2i+1}(\lambda')$ for all $\lambda' \vdash n-1$.

A sufficient condition for a pair of distributions $(p_{\text{even}}, p_{\text{odd}})$ to be stationary on $\mathcal{P}_n$ and odd-step stationary on $\mathcal{P}_{n-1}$, respectively, is that along each edge $(\lambda, \lambda')$ the probability mass that travels from $\lambda$ to $\lambda'$ in a death step equals the probability mass that returns from $\lambda'$ to $\lambda$ in the following birth step. This condition can be expressed as

$$p_{\text{even}}(\lambda)\overrightarrow{\pi}\big(\lambda, \lambda'\big) = p_{\text{odd}}\big(\lambda'\big)\overleftarrow{\pi}\big(\lambda, \lambda'\big). \tag{24}$$

Below we will use the sufficient condition (24) to prove exact expressions for the stationary distributions. In particular, we will retrieve the Ewens distribution (2) on $\mathcal{P}_{n-1}$ as the odd-step stationary distribution for the DESQUARE-SQUARE($\mu$) process because of the equivalence between the infinite alleles Moran model and the SQUARE($\mu$)-DESQUARE process.

**Proposition 4.** *The stationary distribution of the* DESQUARE-SQUARE($\mu$) *process on* $\mathcal{P}_n$ *is given by*

$$p_{\text{even}}(\lambda) = \text{const}_1\, n \frac{\theta^{N(\lambda)}}{\prod_k r_k(\lambda)! k^{r_k(\lambda)}}, \tag{25}$$

*where* $\theta = (n-1)\mu/(1-\mu)$. *The corresponding odd-step stationary distribution on* $\mathcal{P}_{n-1}$ *has*

$$p_{\text{odd}}\big(\lambda'\big) = \text{const}_1 \frac{n-1}{1-\mu} \frac{\theta^{N(\lambda')}}{\prod_k r_k(\lambda')! k^{r_k(\lambda')}}. \tag{26}$$

*The stationary distribution of the* DEROW-ROW($\mu$) *process on* $\mathcal{P}_n$, *is given by*

$$p_{\text{even}}(\lambda) = \text{const}_2 \frac{\phi^{N(\lambda)} N(\lambda)!}{\prod_k r_k(\lambda)!}, \tag{27}$$

*where* $\phi = \mu/(1-\mu)$. *The corresponding odd-step stationary distribution on* $\mathcal{P}_{n-1}$ *has*

$$p_{\text{odd}}\big(\lambda'\big) = \text{const}_2 \frac{\phi^{N(\lambda')} N(\lambda')!}{(1-\mu)\prod_k r_k(\lambda')!}. \tag{28}$$

*The stationary distribution of the* DEROW-ROW$_w$ *process on* $\mathcal{P}_n$, *is given by*

$$p_{\text{even}}(\lambda) = \text{const}_3 \frac{N(\lambda) w^{N(\lambda)}}{\prod_k r_k(\lambda)!}. \tag{29}$$

*The corresponding odd-step stationary distribution on* $\mathcal{P}_{n-1}$ *has*

$$p_{\text{odd}}\big(\lambda'\big) = \text{const}_3 \frac{(w + N(\lambda')) w^{N(\lambda')}}{\prod_k r_k(\lambda')!}. \tag{30}$$

*The stationary distribution of the* DECORNER-CORNER$_w$ *process on* $\mathcal{P}_n$, *is given by*

$$p_{\text{even}}(\lambda) = \text{const}_4\, N_{\text{out}}(\lambda) w^{N(\lambda)}, \tag{31}$$

*where $N_{\text{out}}(\lambda)$ denotes the number of outer corners, which is equal to the number of different row lengths present in $\lambda$. The corresponding odd-step stationary distribution on $\mathcal{P}_{n-1}$ has*

$$p_{\text{odd}}(\lambda') = \text{const}_4\big(w + N_{\text{out}}(\lambda')\big)w^{N'(\lambda)}. \tag{32}$$

**Proof.** The DESQUARE-SQUARE($\mu$) process has edge weights

$$\vec{\pi}(\lambda, \lambda') = \frac{\kappa r_\kappa(\lambda)}{n},$$

because there are a total of $\kappa r_\kappa(\lambda)$ choices of squares in rows of length $\kappa$, and

$$\overleftarrow{\pi}(\lambda, \lambda') = \begin{cases} (1-\mu)\frac{(\kappa-1)r_{\kappa-1}(\lambda')}{n-1} & \text{if } \kappa > 1, \\ \mu & \text{if } \kappa = 1, \end{cases}$$

because $\kappa = 1$ means that the new square shall create a new row. It is not difficult to check the sufficient condition for stationarity of the pair (25) and (26) of distributions:

$$\begin{aligned} p_{\text{even}}(\lambda)\vec{\pi}(\lambda, \lambda') &= \text{const}_1\, \kappa r_\kappa(\lambda) \prod_k \frac{\theta^{r_k(\lambda)}}{r_k(\lambda)!k^{r_k(\lambda)}} \\ &= \begin{cases} \text{const}_1 (\kappa-1) r_{\kappa-1}(\lambda') \prod_k \frac{\theta^{r_k(\lambda')}}{r_k(\lambda')!k^{r_k(\lambda')}} & \text{if } \kappa > 1, \\ \text{const}_1\, \theta \prod_k \frac{\theta^{r_k(\lambda')}}{r_k(\lambda')!k^{r_k(\lambda')}} & \text{if } \kappa = 1 \end{cases} \\ &= p_{\text{odd}}(\lambda')\overleftarrow{\pi}(\lambda, \lambda'). \end{aligned}$$

The DEROW-ROW($\mu$) process has edge weights

$$\vec{\pi}(\lambda, \lambda') = \frac{r_\kappa(\lambda)}{N(\lambda)}$$

and

$$\overleftarrow{\pi}(\lambda, \lambda') = \begin{cases} (1-\mu)\frac{r_{\kappa-1}(\lambda')}{N(\lambda')} & \text{if } \kappa > 1, \\ \mu & \text{if } \kappa = 1. \end{cases}$$

As in the previous case, we can check the sufficient condition for stationarity of the pair (27) and (28) of distributions:

$$\begin{aligned} p_{\text{even}}(\lambda)\vec{\pi}(\lambda, \lambda') &= \text{const}_2 \frac{r_\kappa(\lambda)}{N(\lambda)} \frac{\phi^{N(\lambda)} N(\lambda)!}{\prod_k r_k(\lambda)!} \\ &= \begin{cases} \text{const}_2 \frac{r_{\kappa-1}(\lambda')}{N(\lambda')} \frac{\phi^{N(\lambda')} N(\lambda')!}{\prod_k r_k(\lambda')!} & \text{if } \kappa > 1, \\ \text{const}_2 \frac{\phi^{1+N(\lambda')} N(\lambda')!}{\prod_k r_k(\lambda')!} & \text{if } \kappa = 1 \end{cases} \\ &= p_{\text{odd}}(\lambda')\overleftarrow{\pi}(\lambda, \lambda'). \end{aligned}$$

The DEROW-ROW$_w$ process has edge weights as follows:

$$\vec{\pi}(\lambda, \lambda') = r_\kappa(\lambda)/N(\lambda)$$

and

$$\overleftarrow{\pi}\big(\lambda,\lambda'\big)=\begin{cases}\frac{r_{\kappa-1}(\lambda')}{w+N(\lambda')} & \text{if } \kappa>1,\\ \frac{w}{w+N(\lambda')} & \text{if } \kappa=1.\end{cases}$$

Again it is straightforward to check the sufficient condition for stationarity of the pair (29) and (30) of distributions:

$$\begin{aligned}p_{\text{even}}(\lambda)\overrightarrow{\pi}\big(\lambda,\lambda'\big)&=\text{const}_3\,\frac{r_\kappa(\lambda)w^{N(\lambda)}}{\prod_k r_k(\lambda)!}\\&=\begin{cases}\text{const}_3\,\frac{r_{\kappa-1}(\lambda')w^{N(\lambda')}}{\prod_k r_k(\lambda')!} & \text{if } \kappa>1,\\ \text{const}_3\,\frac{w^{1+N(\lambda')}}{\prod_k r_k(\lambda')!} & \text{if } \kappa=1\end{cases}\\&=p_{\text{odd}}\big(\lambda'\big)\overleftarrow{\pi}\big(\lambda,\lambda'\big).\end{aligned}$$

The DECORNER-CORNER$_w$ process has edge weights $\overrightarrow{\pi}(\lambda,\lambda')=1/N_{\text{out}}(\lambda)$ and

$$\overleftarrow{\pi}\big(\lambda,\lambda'\big)=\begin{cases}\frac{1}{w+N_{\text{out}}(\lambda')} & \text{if } \kappa>1,\\ \frac{w}{w+N_{\text{out}}(\lambda')} & \text{if } \kappa=1.\end{cases}$$

The sufficient condition for stationarity of the pair (31) and (32) of distributions is easily checked:

$$\begin{aligned}p_{\text{even}}(\lambda)\overrightarrow{\pi}\big(\lambda,\lambda'\big)&=\text{const}_4\,w^{N(\lambda)}\\&=\begin{cases}\text{const}_4\,w^{N(\lambda')} & \text{if } \kappa>1,\\ \text{const}_4\,w^{1+N(\lambda')} & \text{if } \kappa=1\end{cases}\\&=p_{\text{odd}}\big(\lambda'\big)\overleftarrow{\pi}\big(\lambda,\lambda'\big).\qquad\square\end{aligned}$$

We have not been able to find expressions for the stationary distributions of the processes for inconsistent combinations of births and deaths, such as DEROW-SQUARE($\mu$). Small examples reveal that their stationary distributions do not satisfy condition (24), and hence we cannot use the same approach.

## 6. Limit shapes of birth-and-death processes of consistent types

Here we shall attempt to find the limit shapes of the birth-and-death processes of consistent type. For the DESQUARE-SQUARE($\mu$) process (the Moran model) we prove the limit shape $y=E_1(x)$ under scaling $a_n=1/\mu$ and certain conditions on $\mu$. For the DECORNER-CORNER$_w$ process with $w=1$ we prove that the limit shape is the same as the well-known limit shape under the uniform distribution. We also offer three mutually related conjectures, saying that $y=e^{-x}$ seems to be the limit shape for each of the following three processes: the DEROW-ROW$_w$ process with scaling $a_n=\sqrt{n/w}$; the DEROW-ROW($\mu$) process with $\mu\to 0$ and scaling $a_n=1/\mu$; and the DECORNER-CORNER$_w$ process with $w\to 0$ and scaling $a_n=\sqrt{n/w}$.

### 6.1. *Birth-and-death process* DESQUARE-SQUARE($\mu$)*: the Moran model*

As we have mentioned, the DESQUARE-SQUARE($\mu$) process is equivalent to the Moran model. The stationary distribution is the Ewens distribution, for which it is known that it has no limit shape for fixed $\theta$ [24]. From the point of view of the DESQUARE-SQUARE($\mu$) process, however, the important parameter is $\mu$. A fixed value of $\theta$ corresponds to the special case $n\mu\to\theta$ as $n\to\infty$. If instead we allow $n\mu$ to grow indefinitely, we do obtain a nice limit shape.

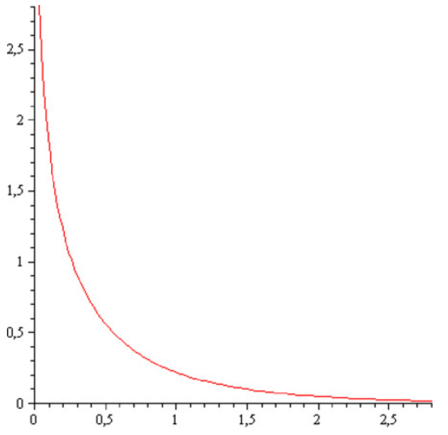


**Fig. 4.** The limit shape $y(x) = E_1(x)$ from Theorem 5.

**Theorem 5.** *Assume that $\mu \to 0$ and $\mu n \to \infty$ when $n$ tends to infinity. Choose the scaling $a_n = 1/\mu$. Then the stationary distribution of the* DESQUARE-SQUARE($\mu$) *process has the open limit shape*

$$y(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt.$$

This shape, illustrated in Fig. 4, is also known as the "exponential integral" $E_1(x)$, or as $\Gamma(0,x)$ where $\Gamma(\beta,x)$ denotes the incomplete Gamma function.

In order to prove Theorem 5, we first need a couple of lemmas. The first one is due to Strimling et al. [19, Eq. (2)] and gives an exact formula for the expected number of rows of any given length.

**Lemma 6** *(Strimling et al.). For $\lambda \in \mathcal{P}_n$ sampled according to the stationary distribution over for the* DESQUARE-SQUARE *process, the expected number of parts of size $k$ is*

$$E\big[r_k(\lambda)\big] = \mu n \frac{(1-\mu)^{k-1}}{k} \prod_{i=1}^{k-1} \frac{n-i}{n-1-(1-\mu)i}. \tag{33}$$

Our next lemma gives an approximation of $E[r_k(\lambda)]$ that is more handy than the exact formula above.

**Lemma 7.** *Assume that $\mu \to 0$ and $\mu n \to \infty$ when $n \to \infty$. Then, for any $\varepsilon > 0$ and $a > 0$, the following holds for all sufficiently large $n$.*

$$\begin{aligned} \forall k\colon \quad & E\big[r_k(\lambda)\big] < (1+\varepsilon)\mu n(1-\mu)^k/k, \\ \forall k \leqslant a/\mu\colon \quad & E\big[r_k(\lambda)\big] > (1-\varepsilon)\mu n(1-\mu)^k/k. \end{aligned}$$

**Proof.** By (33) it suffices to show that the product

$$H := \prod_{i=1}^{k-1} \frac{n-i}{n-1-(1-\mu)i}$$

is eventually smaller than $(1+\varepsilon)$ for all $k$, and that it tends to 1 uniformly for $k \leqslant a/\mu$, when $n \to \infty$. It is easily verified that $\frac{n-i}{n-1-(1-\mu)i}$ is a decreasing function of $i$ and that it is smaller than 1 if $i > 1/\mu$. Thus we obtain upper and lower bounds for $H$ as follows:

$$H \leqslant \left(\frac{n-1}{n-2+\mu}\right)^{\lceil 1/\mu \rceil} = \left(1 + \frac{1-\mu}{n-2+\mu}\right)^{\lceil 1/\mu \rceil} \sim \exp \frac{1}{\mu n}$$

which approaches 1 since $n\mu \to \infty$, and

$$\begin{aligned} H &\geqslant \left(\frac{n-(k-1)}{n-1-(1-\mu)(k-1)}\right)^{k-1} = \left(1 + \frac{1-\mu(k-1)}{n-1-(1-\mu)(k-1)}\right)^{k-1} \\ &\sim \exp\left(\frac{(1-\mu(k-1))(k-1)}{n-1-(1-\mu)(k-1)}\right), \end{aligned}$$

which tends to 1 uniformly for $k \leqslant a/\mu$ since $\mu n \to \infty$. $\square$

Finally, we need to bound the variance of sums of $r_k(\lambda)$ in order to show that $E_1(x)$ is not only a limit average shape but a limit shape.

**Lemma 8.** *For all m,*

$$\operatorname{Var}\left[\sum_{k=m}^{\infty} r_k(\lambda)\right] \leqslant \sum_{k=m}^{\infty} E\big[r_k(\lambda)\big].$$

**Proof.** This is Proposition 2 in [5]. (There, $r_k(\lambda)$ is called $X_k$ and $E[r_k(\lambda)]$ is called $f_k$.) $\square$

At last, we are ready to settle the theorem.

**Proof of Theorem 5.** Fix an $x \geqslant 0$.

First we will show that $E_1(x)$ is the limit average shape, i.e. $E[\tilde{\phi}_\lambda^{(n)}(x)] \to E_1(x)$.

Let $a$ be an arbitrary number greater than $x$, and write

$$E\big[\tilde{\phi}_\lambda^{(n)}(x)\big] = \frac{1}{\mu n} \sum_{x/\mu \leqslant k \leqslant a/\mu} E\big[r_k(\lambda)\big] + \frac{1}{\mu n} \sum_{k > a/\mu} E\big[r_k(\lambda)\big]. \tag{34}$$

Let $\Delta t := \mu$ and define $\tilde{x} := \mu \lceil x/\mu \rceil$ and $\tilde{a} := \mu \lfloor a/\mu \rfloor$. By Lemma 7, the first sum on the right-hand side of (34) is

$$\begin{aligned} \frac{1}{\mu n} \sum_{x/\mu \leqslant k \leqslant a/\mu} E\big[r_k(\lambda)\big] &\to \sum_{x/\mu \leqslant k \leqslant a/\mu} (1-\mu)^k/k \\ &= \sum_{t \in \{\tilde{x}, \tilde{x}+\Delta t, \tilde{x}+2\Delta t, \tilde{a}\}} \frac{(1-\mu)^{t/\mu}}{t} \Delta t \\ &\to \int_x^a \frac{e^{-t}}{t}\, dt, \end{aligned}$$

and, for any $\varepsilon > 0$, the second sum on the right-hand side of (34) is

$$\begin{aligned}
\frac{1}{\mu n}\sum_{k>a/\mu} E\big[r_k(\lambda)\big] &< (1+\varepsilon)\sum_{k>a/\mu}(1-\mu)^k/k \\
&\leqslant (1+\varepsilon)\sum_{k>a/\mu} e^{-\mu k}/k \\
&\leqslant (1+\varepsilon)\sum_{k\geqslant \tilde{a}/\mu} e^{-\mu k}\mu/\tilde{a} \\
&= (1+\varepsilon)\frac{\mu}{\tilde{a}}\frac{e^{-\tilde{a}}}{1-e^{-\mu}} \\
&\sim (1+\varepsilon)e^{-\tilde{a}}/\tilde{a}.
\end{aligned}$$

Since $a$ was chosen arbitrarily, we conclude that

$$E\big[\tilde{\phi}_\lambda^{(n)}(x)\big] \to \int_x^\infty \frac{e^{-t}}{t}\,dt = E_1(x).$$

Now we must show that $E_1(x)$ is a not only a limit average shape, but a limit shape, i.e. for any $\epsilon > 0$,

$$P\big(\big|\tilde{\phi}_\lambda^{(n)}(x) - E_1(x)\big| > \epsilon\big) \to 0. \tag{35}$$

By the triangle inequality,

$$P\big(\big|\tilde{\phi}_\lambda^{(n)}(x) - E_1(x)\big| > \epsilon\big) \leqslant P\big(\big|\tilde{\phi}_\lambda^{(n)}(x) - E\big[\tilde{\phi}^{(n)}(x)\big]\big| > \epsilon - \big|E\big[\tilde{\phi}^{(n)}(x)\big] - E_1\big|\big)$$

which, for large $n$, is no more than $P(|\tilde{\phi}_\lambda^{(n)}(x) - E[\tilde{\phi}^{(n)}(x)]| > \epsilon/2)$ since $E_1(x)$ is the limit average shape. By Chebyshev's inequality,

$$P\big(\big|\tilde{\phi}_\lambda^{(n)}(x) - E\big[\tilde{\phi}^{(n)}(x)\big]\big| > \epsilon/2\big) \leqslant \frac{\mathrm{Var}[\tilde{\phi}_\lambda^{(n)}(x)]}{(\epsilon/2)^2},$$

and by Lemma 8 we have

$$\begin{aligned}
\mathrm{Var}\big[\tilde{\phi}_\lambda^{(n)}(x)\big] &= \frac{1}{(\mu n)^2}\mathrm{Var}\Big[\sum_{k\geqslant x/\mu} r_k(\lambda)\Big] \\
&\leqslant \frac{1}{(\mu n)^2}E\Big[\sum_{k\geqslant x/\mu} r_k(\lambda)\Big] \\
&= \frac{1}{\mu n}E\big[\tilde{\phi}^{(n)}(x)\big] \\
&\sim \frac{1}{\mu n}E_1(x) \to 0,
\end{aligned}$$

where we use the assumption that $\mu n \to \infty$. Finally, combining the inequalities above yields (35). $\square$

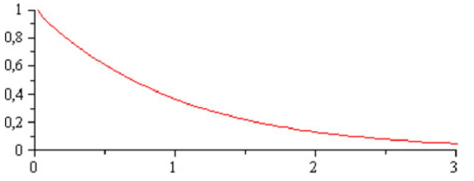


**Fig. 5.** The average of 10,000 shapes sampled through simulation of the DEROW-ROW$_w$ process for $n = 25{,}600$ with $w = 1$ and scaling $a_n = n^{1/2}$.

### 6.2. Birth-and-death process DEROW-ROW$_w$

Proposition 4 states that for $w = 1$ the stationary distribution of the DEROW-ROW$_w$ process is

$$p_{\text{even}}(\lambda) = \text{const}\,\frac{N(\lambda)}{\prod_k r_k(\lambda)!}.$$

From this expression it is easy to see that for $n$ a triangular number $L(L+1)/2$, the Young diagram with the highest probability is the staircase shaped diagram $(L, L-1, L-2, \dots, 1)$. However, simulations show that the limit average shape is exponential, see Fig. 5, and *not* the straight line corresponding to a staircase. This means that the single most probable partition is *not* of the same shape as the limit average shape.

**Conjecture 1.** *Choose the scaling $a_n = (n/w)^{1/2}$. Then the stationary distribution of the* DEROW-ROW$_w$ *process has the limit shape*

$$y(x) = e^{-x}.$$

In addition to evidence from simulations, the conjecture is supported by the following handwaving argument. First construct a new model where we fix an $N$, and in each time step make a random choice between a ROW$_w$ birth step and a DEROW-death step, with $\mathrm{P}(\text{birth}) = \frac{(w+N')/(w+N)}{N'/N+(w+N')/(w+N)}$ where $N'$ signifies the current number of rows. Clearly, if $N' > N$ then $\mathrm{P}(\text{birth}) < 1/2$ and so the tendency will be for $N'$ to decrease. Similarly, if $N' < N$ then $\mathrm{P}(\text{birth}) > 1/2$ and the tendency will be for $N'$ to decrease. Therefore on average we expect $N' = N$ and equally often occurring births and deaths, and hence we expect this model to behave similarly to the original DEROW-ROW$_w$ model.

The new model is equivalent to a particle model with independent particles that leave the origin at rate $w/(w+N)$ and then move to the right at rate $1/(w+N)$ and move to the left at rate $1/N$ until they are absorbed when they hit the origin. For this model it is obvious that the equilibrium distribution must have $E[r_k] = w(N/(w+N))^k$. This gives $\sum_{k=1}^{\infty} E[r_k] = N$ and $n := \sum_{k=1}^{\infty} E[kr_k] = N(N+w)/w$. After scaling $a_n = (n/w)^{1/2}$ we obtain the average limit shape

$$y(x) = \lim_{N\to\infty} \frac{1}{\sqrt{nw}} \sum_{k \geqslant x\sqrt{n/w}} E[r_k] = e^{-x}.$$

Further, using the toolbox in [3] it seems doable to show that this is a proper limit shape for the new model. However, we do not know of any way of making rigorous the similarity between the new and original models.

### 6.3. Birth-and-death process DEROW-ROW$(\mu)$

**Conjecture 2.** *Let $\mu \to 0$ as $n \to \infty$ and choose the scaling $a_n = 1/\mu$. Then the stationary distribution of the* DEROW-ROW$(\mu)$ *process has limit shape*

$$y(x) = e^{-x}.$$

This conjecture is consistent with the previous one. As discussed in Section 3.2, the parameter $\mu$ in a $\textsc{row}(\mu)$ step corresponds to $w/(w+N)$ in a $\textsc{row}_w$ step. According to Conjecture 1, in the $\textsc{derow-row}_w$ process the number of rows of a typical partition is asymptotically $N = (n/a_n)y(0) = (nw)^{1/2}$ with small deviations. Thus a $\textsc{derow-row}(\mu)$ process with $\mu = w/(w+(nw)^{1/2})$ is approximated by a $\textsc{derow-row}_w$ process. This implies a discrete shape approximated by $y(x) = e^{-x}$ for scaling $a_n = (n/w)^{1/2} = (1-\mu)/\mu$. As $\mu \to 0$ a continuous shape is approached, and asymptotically the scaling is $a_n = 1/\mu$.

### 6.4. *Birth-and-death process* $\textsc{decorner-corner}_w$

Proposition 4 states that the stationary distribution of the $\textsc{decorner-corner}_w$ process for $w = 1$ is given by

$$p_{\text{even}}(\lambda) = \text{const}\, N_{\text{out}}(\lambda),$$

where $N_{\text{out}}(\lambda)$ denotes the number of outer corners. As was the case for the $\textsc{derow-row}_w$ process, this probability is maximized by the staircase shape, but again this turns out not to be the limit shape.

**Theorem 6.** *Choose the scaling $a_n = n^{1/2}$. Then, as $n$ tends to infinity, the shape of $\lambda \in \mathcal{P}_n$ sampled according to the stationary distribution of the $\textsc{decorner-corner}_w$ process with $w = 1$ has the limit shape*

$$e^{-(\pi/\sqrt{6})x} + e^{-(\pi/\sqrt{6})y} = 1,$$

*which is the same limit shape as for the uniform distribution on $\mathcal{P}_n$.*

**Proof.** Let $\phi(x) := -\frac{\sqrt{6}}{\pi}\log(1-e^{-\pi x/\sqrt{6}})$ be the limit shape for the uniform distribution. Fix $x > 0$ and $\varepsilon > 0$ and let $\nu$ be a random partition drawn uniformly from $\mathcal{P}_n$. It is well known (see e.g. [15, Lemma 1]) that

$$P\left(\left|\tilde{\phi}_\nu^{(n)}(x) - \phi(x)\right| > \varepsilon\right) = O\left(a^{\sqrt{n}}\right)$$

for some $0 < a < 1$.

Since a partition of size $n$ cannot have more than $n$ outer corners, for any fixed $\kappa \in \mathcal{P}_n$ we have $P(\lambda = \kappa) \leqslant nP(\nu = \kappa)$. Thus,

$$\begin{aligned}
P\left(\left|\tilde{\phi}_\lambda^{(n)}(x) - \phi(x)\right| > \varepsilon\right) &= \sum_{\substack{\kappa \in \mathcal{P}_n \\ |\tilde{\phi}_\kappa^{(n)}(x) - \phi(x)| > \varepsilon}} P(\lambda = \kappa) \\
&\leqslant \sum_{\substack{\kappa \in \mathcal{P}_n \\ |\tilde{\phi}_\kappa^{(n)}(x) - \phi(x)| > \varepsilon}} nP(\nu = \kappa) \\
&= nP\left(\left|\tilde{\phi}_\nu^{(n)}(x) - \phi(x)\right| > \varepsilon\right) \\
&= O\left(na^{\sqrt{n}}\right)
\end{aligned}$$

which tends to zero as $n \to \infty$. $\square$

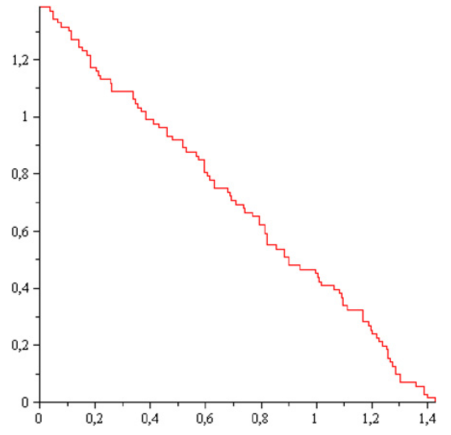


**Fig. 6.** A partition sampled from the DECORNER-ROW$_w$ process for $w = 0.5$ and $n = 10{,}000$ with scaling $a_n = (n/w)^{1/2}$.

We also have a conjecture about the limit shape for small $w$.

**Conjecture 3.** *If $w \to 0$ as $n \to \infty$, the shape of $\lambda \in \mathcal{P}_n$ sampled according to the stationary distribution of the* DECORNER-CORNER$_w$ *process has the limit shape*

$$y = e^{-x}$$

*under scaling $a_n = (n/w)^{1/2}$.*

This conjecture is supported by simulations and follows intuitively from Conjecture 1, because for small $w$ every row will typically host a corner, in which case the model is approximately equivalent to the DEROW-ROW$_w$ model.

## 7. Conjectures on limit shapes of birth-and-death processes of inconsistent types

Simulations of the processes where births are not analogous to deaths suggest a varying collection of phenomena. Here we mention just a few conjectures that we find particularly interesting: a conjectured limit shape and a conjectured non-ergodic case.

Fig. 6 shows the shape of a sample partition obtained after one million steps of the DECORNER-ROW$_w$ process for $w = 0.5$ and $n = 10{,}000$, illustrating the following conjecture.

**Conjecture 4.** *The stationary distribution of the* DECORNER-ROW$_w$ *process with scaling $a_n = (n/w)^{1/2}$ has limit shape*

$$y = \max\{0, \sqrt{2} - x\}.$$

In contrast, for some parameter values the DEROW-SQUARE($\mu$) process exhibits a phenomenon of multiple attracting shapes that suggests the process may not have a limit shape. For instance, we ran two simulations of the DEROW-SQUARE($\mu$) process of one million steps each, for $n = 400$, $\mu = 0.95$, and the same initial partition. The two simulations gave two utterly different average shape: one had a very long longest row (about 370 squares long, thus containing more than 90 percent of the total area), whereas the other average shape had a very short longest row (3.5 squares long). Simulations of the DECORNER-SQUARE($\mu$) process exhibits the same phenomenon.

**Conjecture 5.** *The stationary distributions of the* DEROW-SQUARE($\mu$) *and* DECORNER-SQUARE($\mu$) *processes are non-ergodic cases, i.e., do not yield limit shapes under any scaling.*

## 8. Discussion

In this paper we have studied the limit shapes of a family of birth and birth-and-death processes on Young diagrams that include the Moran model from population genetics, the Simon model of urban growth, and Rost's particle model. We introduced a framework that proved useful to organize this family of processes. However, we have found no general approach to limit shapes that is successful for all processes in the framework. For instance, although for birth-and-death processes with analogous types of birth steps and death steps we were able to find expressions for the stationary distributions, we could not use them to find all limit shapes. Indeed, we have only conjectures for the limit shapes of processes of type DEROW-ROW. For mixed types of births and deaths we do not even have the stationary distributions. Thus our investigation has opened up a number of new research questions.

As pointed out to us by an anonymous reviewer, the limit shape $y(x) = e^{-x}$ in Conjectures 1–3 coincides with the limit shape of uniformly random partitions of $n$ into $k$ parts in the asymptotic regime where both $n$ and $k$ grow to infinity and $k$ grows at a rate asymptotically slower than $\sqrt{n}$ [22,23]. We do not know whether there is a connection.

We have here only considered processes where each step deals with the birth or death of a single square. Multi-square steps are also worth considering, though. For instance, Jockusch, Propp and Shor studied the birth process where every inner corner is filled with a fixed probability, and found the limit shape to be a quarter-ellipse [10,17]. Similarly, the "Bulgarian solitaire" can be interpreted as a birth-and-death process with multi-square steps [14]. We are currently working on an extension of our framework to incorporate such multi-square step processes.

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