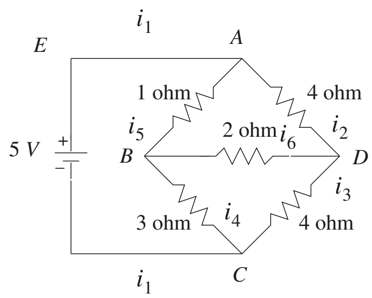

FIGURE 1.22 Exercise 1.61.

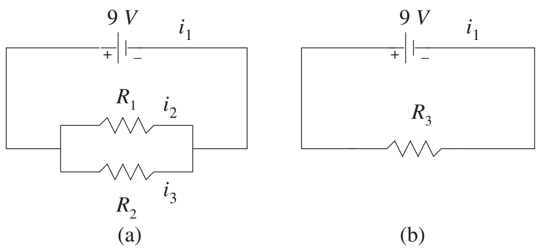

FIGURE 1.23 Exercise 1.62.

1.3 GAUSSIAN ELIMINATION

In this section we would like to consider Gaussian elimination more systematically than in the last section. Consider, for example, the following system, which is the same as system (1.17):

$$\begin{aligned} 4x + 5y + 3z + 3w &= 1 \\ x + y + z + w &= 0 \\ 2x + 3y + z + w &= 1 \\ 5x + 7y + 3z + 3w &= 2 \end{aligned} \tag{1.24}$$

On page 37, we solved this system by transforming its augmented matrix A into the matrix R below.

$$A = \begin{bmatrix} 4 & 5 & 3 & 3 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 1 \\ 5 & 7 & 3 & 3 & 2 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.25}$$

At each step we performed an operation of one of the following types on the augmented matrix:[4]

(I) Interchange two rows.
(II) Add a multiple of one row onto another.
(III) Multiply one row by a nonzero constant.

These operations are called **elementary row operations**. Two matrices A and B are said to be **row equivalent** if there is a sequence of elementary row operations that transforms A into B. Thus, the matrices A and R in formula (1.25) are row equivalent. The following theorem says that *elementary row operations do not change the solution sets for the corresponding systems.* In particular, matrix R in formula (1.25) is the augmented matrix for a system that has exactly the same set of solutions as system (1.24).

Theorem 1.3 *Suppose that A and B are row equivalent matrices. Then the system having A as an augmented matrix has the same solution set as the system having B as its augmented matrix.*

Proof. If we apply a single elementary row operation to A, we obtain a matrix A_1 that is the augmented matrix for a new system. We claim that this new system has the same solution set as the original system with augmented matrix A. To see this, note first that every row of A_1 is a linear combination of rows of the original matrix. Hence, every equation in the new system is a linear combination of equations from the original system, showing that every solution of the original system is also a solution of the new one.

Conversely, every solution of the new system is also a solution of the original. This is due to the reversibility of elementary row operations. We can undo the effect of adding a multiple of a given row onto another by subtracting the same multiple of the given row. Division by a nonzero constant undoes the effect of multiplication by a nonzero constant. Hence, we may transform A_1 back into A using elementary row operations. The same argument as before now shows that every solution of the new system is also a solution of the original system, proving our claim.

Now, since B is row equivalent with A, there is a sequence $A_0, A_1, \ldots, A_n$ of matrices, where $A_0 = A$, $A_n = B$, and, for each i, A_{i+1} was produced by applying a single elementary row operation to A_i. We refer to the system with augmented matrix A_i as "system i" so that system 0 has augmented matrix A and system n has augmented matrix B. From the preceding argument, the solution set of system 0 equals the solution set of system 1, which equals the solution set of system 2, …, which equals the solution set of system n, proving our theorem. $\square$

Matrix R in formula (1.25) is in what is called **echelon form**, that is, recognizable by the "step like" arrangement of the zeros in the lower left corner.

[4]We did not actually use any steps of type III in this particular example.

Definition 1.11 *A matrix A is in echelon form if:*

- *The first nonzero entry in any nonzero row occurs to the right of the first such entry in the row directly above it.*
- *All zero rows are grouped together at the bottom.*

In this case, the first nonzero entry in each nonzero row is referred to as a ***pivot entry****.*

Thus, any matrices of the following forms are in echelon form as long as all the entries in the positions marked "#" are nonzero. (The entries in the positions marked "$*$" can be arbitrary.) The "#" entries are the pivot entries:

$$\begin{bmatrix} \# & * & * & * & * \\ 0 & \# & * & * & * \\ 0 & 0 & 0 & \# & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & \# & * & * & * & * & * \\ 0 & 0 & 0 & \# & * & * & * \\ 0 & 0 & 0 & 0 & \# & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.26}$$

EXAMPLE 1.9

Which of the following matrices are in echelon form?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution. The first three matrices are in echelon form; the last two matrices are not. In the fourth matrix, the first nonzero entry of row 2 is directly below the first nonzero entry of row 1. In the last matrix, rows 3 and 4 must be interchanged to get an echelon form.

A matrix that is not already in echelon form may be reduced further. Hence, we expect that if we keep reducing, we will eventually produce a matrix in echelon form. The following theorem proves this.

Theorem 1.4 *Every matrix A is row equivalent to a matrix in echelon form.*

Proof. We may assume that the first column of A is nonzero since null columns do not affect the reduction process. By exchanging rows (if necessary) to make a_{11} nonzero and then dividing the first row by a_{11}, we may assume $a_{11} = 1$. Subtracting multiples of row 1 from the other rows produces a matrix with all the other entries in the first column equal to zero:

$$\begin{matrix} 1 & \% & \% & \% & \ldots & \% \\ 0 & * & * & * & \ldots & * \\ 0 & * & * & * & \ldots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & * & * & \ldots & * \end{matrix}$$

(The %'s and $*$'s represent arbitrary numbers.)

We ignore the first row and column of the original matrix and apply the reduction process to the resulting smaller matrix, the matrix of $*$'s in our figure. Applying the reduction process to successively smaller matrices eventually results in an echelon form matrix, proving the theorem. □

Recall that the first nonzero entry in each nonzero row of an echelon form matrix is referred to as a pivot entry.

Definition 1.12 *Let R be an echelon form of a matrix A, where A is the augmented matrix for a system in the variables $x_1, x_2, \ldots, x_n$. The variable x_i is said to be a* ***pivot variable*** *for the system, if the ith column of R contains a pivot entry of R.*

Thus, from the echelon matrix R in formula (1.25), we see that x and y are the pivot variables for system (1.24). It is somewhat remarkable that all echelon forms of a given matrix A produce precisely the same set of pivot variables. (See Theorem 1.5 below.)

Let A be the augmented matrix for a system in the variables x_1, x_2, …, x_n and let R be an echelon form for A. We refer to the system with augmented matrix R as the "echelon system." The last equation in the echelon system expresses the last pivot variable in terms of the subsequent variables, all which may be set arbitrarily. The second to last equation in the echelon system expresses the second to last pivot variable in terms of the subsequent variables. Since the last pivot variable has already been expressed in terms of nonpivot variables, we see that, in fact, back substitution expresses the second to last pivot variable in terms of the subsequent nonpivot variables, all which may be set arbitrarily. In general, we see that:

1. All of the nonpivot variables may be set arbitrarily.
2. The values of the pivot variables are uniquely determined by the values of the nonpivot variables.

More specifically, if $X = [x_1, x_2, \dots, x_n]^t$ is a solution to a linear system, then, for all $1 \le i \le n$,

$$x_i = c_i + r_{i1}x_{j_1} + \cdots + r_{ik}x_{j_k}$$

where $x_{j_1}, x_{j_2}, \dots, x_{j_k}$ are the nonpivot variables and the c_i and r_{ij} are scalars that do not depend on X. Hence, the general solution may be expressed as

$$X = T + x_{j_1}X_1 + \cdots + x_{j_k}X_k \tag{1.27}$$

where T and X_j are fixed elements of $\mathbb{R}^n$ and the x_{j_i} are the nonpivot variables. This, of course, is the familiar "parametric form" that we have already used extensively.

The nonpivot variables are examples of what are referred to as **free variables**. In a general linear system, a variable x_i is said to be free if for all real numbers t there exists a solution $X = [x_1, x_2, \dots, x_n]^t$ to the system with $x_i = t$. Pivot variables can also be free. For example, in the single equation

$$x - y - 7z = 0$$

x is the pivot variable. It is also free since we may solve for, say, y, in terms of x and z, allowing us to set x and z arbitrarily. We also refer to the nonpivot variables as "the" free variables. Exercises 1.87 and 1.88 investigate this issue further.

It should be noted that we typically refer to "an echelon form" rather than "the echelon form." This is because a given matrix A can be row equivalent with many different echelon form matrices: the specific echelon form produced by applying Gaussian elimination to a given matrix A will typically depend on the steps used to produce it. Suppose, for example, that in solving system (1.24), we do not first interchange the first and second rows. Then, the Gaussian elimination process proceeds as follows:

$$A = \begin{bmatrix} 4 & 5 & 3 & 3 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 1 \\ 5 & 7 & 3 & 3 & 2 \end{bmatrix}$$

We multiply rows 2–4 by constants and add (in this case, negative) multiples of the first row:

$$\begin{bmatrix} 4 & 5 & 3 & 3 & 1 \\ 0 & -1 & 1 & 1 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 3 & -3 & -3 & 3 \end{bmatrix} \qquad \begin{array}{rcl} \\ R_2 & \to & 4R_2 - R_1 \\ R_3 & \to & 2R_3 - R_1 \\ R_4 & \to & 4R_4 - 5R_1 \end{array}$$

Adding multiples of the second row to lower rows yields the echelon form:

$$\begin{bmatrix} 4 & 5 & 3 & 3 & 1 \\ 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \\ \\ R_3 \to R_3 + R_1 \\ R_4 \to R_4 + 3R_2 \end{array} \tag{1.28}$$

Our answer is quite different from the echelon form found previously [R in formula (1.25) on page 47]. Note, however, that just as before, x and y are pivot variables, while z and w are nonpivot variables. This demonstrates a general principle that we prove at the end of this section.

Theorem 1.5 *Suppose A is the augmented form of a consistent system of linear equations in the variables $x_1, x_2, \ldots, x_n$. Then, the set of pivot variables does not depend on the particular echelon form of A used to determine the pivots.*

Although the specific echelon form of a given matrix obtained from Gaussian elimination depends on the steps used in reducing the matrix, remarkably, the final form of the solution does not. For example, the matrix (1.28) is an echelon form for system (1.24) on page 47. This matrix is the augmented matrix for the system

$$\begin{aligned} 4x + 5y + 3z + w &= 1 \\ -y + z + w &= -1 \end{aligned}$$

We solve the second equation by setting $z = r$ and $w = s$, where r and s are arbitrary real numbers. Then back substitution yields $y = 1 + r + s$ and $x = -2r - 2s - 1$. Thus,

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \tag{1.29}$$

which is identical to equation (1.21) on page 37.

The fact that the expression for the solution does not depend on the reduced form used to derive it is a direct consequence of Theorem 1.5. In fact, from formula (1.27), the translation vector T is the unique solution obtained by setting all nonpivot variables equal to 0. Since the set of nonpivot variables is uniquely determined, the translation vector is unique as well. Similarly, from formula (1.27), the spanning vector X_j is $X - T$, where X is the solution obtained by setting $x_{i_j} = 1$ and all the other nonpivot variables equal to 0, implying the uniqueness of X_j.

Some readers might be tempted to say that we get the same description of the solution because we are, after all, solving the same system. This, however, is not the correct explanation. Suppose, for example, that we were to write system (1.24) on page 47 in the equivalent formulation

$$\begin{aligned}
3z + 4x + 5y + 3w &= 1\\
z + x + y + w &= 0\\
z + 2x + 3y + w &= 1\\
3z + 5x + 7y + 3w &= 2
\end{aligned}$$

The augmented matrix is then

$$\begin{bmatrix} 3 & 4 & 5 & 3 & 1\\ 1 & 1 & 1 & 1 & 0\\ 1 & 2 & 3 & 1 & 1\\ 3 & 5 & 7 & 3 & 2 \end{bmatrix}$$

An echelon form is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0\\ 0 & 1 & 2 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which describes the system

$$\begin{aligned}
z + x + y + w &= 0\\
x + 2y \phantom{{}+w} &= 1
\end{aligned}$$

Now w and y are free variables. Setting $y = t$ and $w = s$ yields

$$\begin{bmatrix} x\\ y\\ z\\ w \end{bmatrix} = \begin{bmatrix} 1\\ 0\\ -1\\ 0 \end{bmatrix} + t\begin{bmatrix} -2\\ 1\\ 1\\ 0 \end{bmatrix} + s\begin{bmatrix} 0\\ 0\\ -1\\ 1 \end{bmatrix} \tag{1.30}$$

which is quite different from equation (1.29).

This example also demonstrates that Theorem 1.5, which states that the set of pivot variables does not depend on the echelon form, assumes a specific ordering of the variables. In system (1.24), if we order the variables z, x, y, w, then, as seen above, the pivots become x and z.

It is sometimes useful to carry the elimination process beyond echelon form. In the matrix R in formula (1.25) on page 47, we may make the entries above the pivot entry equal to zero by subtracting the second row from the row *above it*, getting

$$B = \begin{bmatrix} 1 & 0 & 2 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.31}$$

This matrix is in **row reduced echelon form** (or "reduced form" for short).

Definition 1.13 *A matrix A is in* ***row reduced echelon form (RREF)*** *if all the following conditions hold:*

1. *A is in echelon form.*
2. *All the pivot entries of A are 1.*
3. *All the entries above the pivots are 0.*

The first two matrices in Example 1.9 are in reduced form, the third is not: it has nonzero entries above the pivot in the third row. Also, the pivots in the first and third rows are not 1.

It is clear from Theorem 1.4 that any matrix is row equivalent with a row reduced matrix: we can reduce each pivot entry to 1 by dividing each nonzero row by the value of its first nonzero entry. We can then reduce the entries above each pivot entry to zero by subtracting multiples of the row containing the pivot entry from the higher rows. The advantage of reduced form is that the answer may be obtained directly, without back substitution. Thus, the first row of the preceding matrix tells us that $x = -1 - 2z - 2w$ and the second tells us that $y = 1 + z + w$.

Example 1.10 demonstrates most of the "wrinkles" that can occur in reducing a matrix to row reduced echelon form.

EXAMPLE 1.10

Bring the following matrix into echelon and reduced echelon form.

$$A = \begin{bmatrix} 1 & -1 & 1 & 3 & 0 & 3 & 6 \\ 2 & -2 & 2 & 6 & 0 & 1 & 7 \\ -1 & 1 & 1 & -1 & -2 & 0 & 1 \\ 4 & -4 & 1 & 9 & 3 & 0 & 6 \end{bmatrix} \tag{1.32}$$

Solution. We begin by eliminating the first variable from the equations below the first equation:

$$\begin{bmatrix} 1 & -1 & 1 & 3 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 & -5 & -5 \\ 0 & 0 & 2 & 2 & -2 & 3 & 7 \\ 0 & 0 & -3 & -3 & 3 & -12 & -18 \end{bmatrix} \qquad \begin{array}{ccc} R_2 & \to & R_2 - 2R_1 \\ R_3 & \to & R_3 + R_1 \\ R_4 & \to & R_4 - 4R_1 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 1 & 3 & 0 & 3 & 6 \\ 0 & 0 & 1 & 1 & -1 & 4 & 6 \\ 0 & 0 & 2 & 2 & -2 & 3 & 7 \\ 0 & 0 & 0 & 0 & 0 & -5 & -5 \end{bmatrix} \qquad \begin{array}{ccc} R_4 & \to & R_4/(-3) \\ R_2 & \leftrightarrow & R_4 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 1 & 3 & 0 & 3 & 6 \\ 0 & 0 & 1 & 1 & -1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & 0 & 0 & -5 & -5 \end{bmatrix} \qquad R_3 \to R_3 - 2R_2$$

$$\begin{bmatrix} 1 & -1 & 1 & 3 & 0 & 3 & 6 \\ 0 & 0 & 1 & 1 & -1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{ccc} R_4 & \to & R_4 - R_3 \\ R_3 & \to & R_3/(-5) \end{array}$$

This is the echelon form. In producing the reduced echelon form, it is usually most efficient to begin with the last nonzero row and work from the bottom up. Thus, we subtract multiples of row 3 from the rows above it, yielding

$$\begin{bmatrix} 1 & -1 & 1 & 3 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{ccc} R_2 & \to & R_2 - 4R_3 \\ R_1 & \to & R_1 - 3R_3 \end{array}$$

Next, we subtract row 2 from row 1. This is the reduced echelon form.

$$\begin{bmatrix} 1 & -1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_2 \to R_1 - R_2$$

It is, of course, a simple matter to compute the spanning and translation vectors for a consistent linear system from the row reduced echelon form of its augmented matrix. Interestingly, we can go the other way; we can compute the entries of the row reduced form from the translation and spanning vectors. Since the translation and spanning vectors do not depend on the particular reduced form used in computing them, it follows that *the augmented matrix for a consistent linear system of equations is row equivalent to one, and only one, row reduced echelon form matrix.* This argument may be used to prove the following theorem. We do not present a formal proof as this result is not used elsewhere in the text.

Theorem 1.6 *Each $m \times n$ matrix A is row equivalent to one, and only one, row reduced echelon form matrix R.*

Theorem 1.4 has some very important theoretical consequences. One of the most important is the following:

Theorem 1.7 (More Unknowns Theorem). *A system of linear equations with more unknowns than equations will either fail to have any solutions or will have an infinite number of solutions.*

Proof. Consider an echelon form of the augmented matrix. It might look something like the following matrix:

$$\begin{bmatrix} 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

There is at most one pivot variable per equation. If there are fewer equations than variables, then there are nonpivot variables. If the system is consistent, the values of these variables may be set arbitrarily, proving the theorem. □

We know, of course, that not all systems have solutions. In the following pair of matrices, the matrix on the left is the augmented matrix for a system with no solutions, since the last nonzero row describes the equation $0 = 1$. The matrix on the right is the augmented matrix for a system that does have solutions. The last equation says $x_4 = 1$. A system is inconsistent if and only if its augmented matrix has an echelon form with a row describing the equation $0 = a$, where $a \neq 0$.

$$\begin{bmatrix} 1 & 2 & -1 & 4 & 5 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & -1 & 4 & 5 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

EXAMPLE 1.11

Find conditions on a, b, c, and d for the following system to be consistent:

$$\begin{aligned} x + y + 2z + w &= a \\ 3x - 4y + z + w &= b \\ 4x - 3y + 3z + 2w &= c \\ 5x - 2y + 5z + 3w &= d \end{aligned}$$

Solution. We reduce the augmented matrix:

$$\begin{bmatrix} 1 & 1 & 2 & 1 & a \\ 3 & -4 & 1 & 1 & b \\ 4 & -3 & 3 & 2 & c \\ 5 & -2 & 5 & 3 & d \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 & a \\ 0 & -7 & -5 & -2 & b-3a \\ 0 & -7 & -5 & -2 & c-4a \\ 0 & -7 & -5 & -2 & d-5a \end{bmatrix} \qquad \begin{aligned} R_2 &\to R_2 - 3R_1 \\ R_3 &\to R_3 - 4R_1 \\ R_4 &\to R_4 - 5R_1 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 & a \\ 0 & -7 & -5 & -2 & b-3a \\ 0 & 0 & 0 & 0 & c-a-b \\ 0 & 0 & 0 & 0 & d-2a-b \end{bmatrix} \qquad \begin{aligned} R_3 &\to R_3 - R_2 \\ R_4 &\to R_4 - R_2 \end{aligned}$$

To avoid inconsistencies, we require $c - a - b = 0$ and $d - 2a - b = 0$.

Now that we have an efficient way of solving systems, we can deal much more effectively with some of the issues discussed in Section 1.1, as the next example demonstrates.

EXAMPLE 1.12

Decide whether or not the vector B belongs to the span of the vectors X_1, X_2, and X_3 below.

$$B = [13, -16, 1]^t, \quad X_1 = [2, -3, 1]^t, \quad X_2 = [-1, 1, 1]^t, \quad X_3 = [-3, 4, 0]^t$$

Solution. The vector B belongs to the span if there exist constants x, y, and z such that

$$x\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + z\begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 13 \\ -16 \\ 1 \end{bmatrix}$$

We simplify the left side of this equality and equate coefficients, obtaining the system

$$\begin{aligned} 2x - y - 3z &= 13 \\ -3x + y + 4z &= -16 \\ x + y \qquad &= 1 \end{aligned}$$

The augmented matrix for this system is

$$\begin{bmatrix} 2 & -1 & -3 & 13 \\ -3 & 1 & 4 & -16 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

We now reduce this matrix, obtaining

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix represents an inconsistent system, showing that B *is not* in the span of the X_i.

Spanning in Polynomial Spaces

EXAMPLE 1.13

Write the polynomial $q(x) = -3 + 14x^2$ as a linear combination of the polynomials

$$\begin{aligned} p_1(x) &= 2 - x + 3x^2 \\ p_2(x) &= 1 + x + x^2 \\ p_3(x) &= -5 + 4x + x^2 \end{aligned}$$

Solution. We seek constants a, b, and c such that

$$\begin{aligned} a(2 - x + 3x^2) + b(1 + x + x^2) + c(-5 + 4x + x^2) &= -3 + 14x^2 \\ 2a + b - 5c + (-a + b + 4c)x + (3a + b + c)x^2 &= -3 + 14x^2 \end{aligned}$$

Equating the coefficients of like powers of x, we see that this is equivalent to the system of equations

$$\begin{aligned} 2a + b - 5c &= -3 \\ -a + b + 4c &= 0 \\ 3a + b + c &= 14 \end{aligned}$$

that has augmented matrix

$$\begin{bmatrix} 2 & 1 & -5 & -3 \\ -1 & 1 & 4 & 0 \\ 3 & 1 & 1 & 14 \end{bmatrix}$$

We reduce, obtaining

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Hence, $a = 5$, $b = -3$, and $c = 2$. As a check on our work, we compute

$$5(2 - x + 3x^2) - 3(1 + x + x^2) + 2(-5 + 4x + x^2) = -3 + 14x^2$$

as desired.

EXAMPLE 1.14

Show that the span of the polynomials $p_1(x)$, $p_2(x)$, and $p_3(x)$ from Example 1.13 is the space P_2 of all polynomials of degree ≤ 2.

Solution. We must show that every polynomial $q(x) = u + vx + wx^2$ is a linear combination of the $p_i(x)$—that is, there are scalars a, b, and c such that

$$ap_1(x) + bp_2(x) + cp_3(x) = q(x) \tag{1.33}$$

Reasoning as in Example 1.13, we see that this equation is equivalent to the system

$$\begin{aligned} 2a + b - 5c &= u \\ -a + b + 4c &= v \\ 3a + b + c &= w \end{aligned}$$

which has augmented matrix

$$\begin{bmatrix} 2 & 1 & -5 & u \\ -1 & 1 & 4 & v \\ 3 & 1 & 1 & w \end{bmatrix}$$

We reduce, obtaining

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{9}u & -\frac{2}{3}v & +\frac{1}{3}w \\ 0 & 1 & 0 & \frac{17}{27}v & +\frac{13}{27}u & -\frac{1}{9}w \\ 0 & 0 & 1 & \frac{1}{9}w & -\frac{4}{27}u & +\frac{1}{27}v \end{bmatrix}$$

Hence, equation (1.33) holds with

$$\begin{aligned} a &= -\tfrac{1}{9}u - \tfrac{2}{3}v + \tfrac{1}{3}w \\ b &= \tfrac{17}{27}v + \tfrac{13}{27}u - \tfrac{1}{9}w \\ c &= \tfrac{1}{9}w - \tfrac{4}{27}u + \tfrac{1}{27}v \end{aligned}$$

We end this section with the promised proof of Theorem 1.5 on page 52, which asserts that all echelon forms of a given $m \times n$ matrix A have the same set of pivot variables.

Proof. Let R be an echelon form of A. We refer to the system with augmented matrix A as the "original" system and the system with augmented matrix R as the "echelon system."

While not essential for the proof, it helps our understanding to picture a specific R along with the associated variables:

$$R = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \end{matrix} \\ \begin{bmatrix} \# & * & * & * & * & * & * \\ 0 & 0 & 0 & \# & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \# & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{1.34}$$

Let x_m be the last nonpivot variable in the echelon system [x_5 for matrix (1.34)]. The value of each pivot variable x_i is determined by the values of those nonpivot variables x_j with $j > i$. Since x_m is the last nonpivot variable, it follows that the values of the x_i for $i > m$ are all uniquely determined. Hence, x_m *is also describable as the last free variable for the original system.* Thus, we have succeeded in describing the last nonpivot variable in the echelon system without referring to R. It follows that all reduced forms for A have the same last nonpivot variable.

Now, suppose that we have succeeded in describing the last k nonpivot variables for the original system without referring to R. Call these variables $x_{m_1}, x_{m_2}, \ldots, x_{m_k}$. Consider the new system obtained from the original system by setting all the $x_{m_j} = 0$ for $1 \leq j \leq k$. The augmented matrix for this new system is the matrix A' obtained by deleting the corresponding columns of A. The matrix R' obtained by deleting the same columns from R is an echelon form of A'. [For the system corresponding to matrix (1.34), we would set $x_5 = 0$ and R' would be as indicated below.]

$$\begin{array}{cc} & \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_6 & \end{array} \\ R' = & \begin{bmatrix} \# & * & * & * & * & * \\ 0 & 0 & 0 & \# & * & * \\ 0 & 0 & 0 & 0 & \# & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

From the preceding reasoning, we see that the last nonpivot variable in the system corresponding to R' (x_3 in our example) is the last free variable for the system corresponding to A'. Furthermore, the last nonpivot variable for the system corresponding to R' is the $(k+1)$st to last nonpivot variable in the echelon system. (In our example x_3 is the second to last nonpivot variable for R.)

Hence, each nonpivot variable in the echelon system is describable as the last free variable for the system obtained from the original system by setting all the subsequent nonpivot variables equal to zero. Since this description does not make use of a specific reduced form, it follows that all reduced forms of A have the same set of nonpivots and, hence, the same set of pivots, proving the theorem. □

Computational Issues: Pivoting

Division by very small numbers tends to create inaccuracies in numerical algorithms. Suppose, for example, that we wish to solve the system in the variables x and y with augmented matrix

$$\begin{bmatrix} 0.0001 & 1.0 & 1.0 \\ 1.0 & 1.0 & 2.0 \end{bmatrix}$$

We reduce

$$\begin{bmatrix} 1.0 & 10000.0 & 10,000 \\ 1.0 & 1.0 & 2.0 \end{bmatrix}$$

$$\begin{bmatrix} 1.0 & 10000.0 & 10000.0 \\ 0.0 & -9999.0 & -9998.0 \end{bmatrix}$$

yielding

$$y = 9998/9999 = 0.9998999900$$
$$x = 10000 - 10000y = 1.000100010001$$

Suppose that our computer only carries three places after the decimal. To three places, the answer is $y = 1.000, x = 1.000$. This, however, is not what our computer tells us! Our computer first rounds y to 1.000 and then computes $x = 10000 - 10000y = 0$, which is not even close.

The problem was caused by the division by 0.0001 in our first step. This can be avoided by interchanging the rows before doing the reduction:

$$\begin{bmatrix} 1.0 & 1.0 & 2.0 \\ 0.0001 & 1.0 & 1.0 \end{bmatrix}$$

$$\begin{bmatrix} 1.0 & 1.0 & 2.0 \\ 0.0 & 0.9999 & 0.9998 \end{bmatrix}$$

yielding $y = 1.000$ and $x = 2 - 1 \cdot 1 = 1.000$.

To avoid such problems, many computational algorithms rearrange the rows before each reduction step so that the entry in the pivot position is the one of largest absolute value. This process is called **partial pivoting**. (There is also a process called **full pivoting**, which involves rearranging both rows and columns so as to obtain the largest pivot entry.) Unfortunately, even full pivoting will not eliminate all round-off difficulties. There are certain systems, called **ill-conditioned**, that are inherently sensitive to small inaccuracies in the values of the coefficients on the right sides of the equations in the system. The **condition number**, which is discussed in the computer exercises for Section 3.3 and in Section 8.1, measures this sensitivity.

True-False Questions: Justify your answers.

In these questions, assume that R is the reduced echelon form of the augmented matrix for a system of equations.

1.20 If the system has three unknowns and R has three nonzero rows, then the system has at least one solution.

1.21 If the system has three unknowns and R has three nonzero rows, then the system can have an infinite number of solutions.

1.22 The system below has an infinite number of solutions:

$$\begin{aligned} 2x + 3y + 5z + 6w - 7u - 8v &= 0 \\ 3x - 4y + 7z + 6w + 8u + 5v &= 0 \\ -7x + 9y - 2z - 4w - 5u + 2v &= 0 \\ -5x - 5y + 9z + 3w + 2u + 7v &= 0 \\ -9x + 3y - 9z + 5w - 3u - 4v &= 0 \end{aligned}$$

1.23 The following matrix may be reduced to reduced echelon form *using only one elementary row operation*:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

1.24 The matrix in question 1.23 is the coefficient matrix for a consistent system of equations.

EXERCISES

In these exercises, if you are asked for a general solution, the answer should be expressed in "parametric form" as in the text. Indicate the spanning and translation vectors.

1.63 ✓✓Which of the following matrices are in echelon form? Which are in reduced echelon form?

(a) $\begin{bmatrix} 1 & 0 & 2 & 4 & 0 & 1 \\ 1 & 1 & 0 & 4 & 3 & 2 \\ 0 & 1 & 0 & 4 & 3 & 2 \\ 0 & 0 & 0 & 2 & 1 & 3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 & 1 & 2 & 2 & 4 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 2 & 4 & 1 \\ 3 & 1 & 2 & 6 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 & 1 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 0 & 2 & 4 & 0 & 1 \\ 0 & 1 & 0 & 4 & 3 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

1.64 ✓Bring each of the matrices in Exercise 1.63 that are not already in echelon form to echelon form. Interpret each matrix as the augmented matrix for a system of equations. Give the system and general solution for each system.

1.65 Find the reduced echelon form of each of the following matrices:

(a) ✓✓ $\begin{bmatrix} 2 & 7 & -5 & -3 & 13 \\ 1 & 0 & 1 & 4 & 3 \\ 1 & 3 & -2 & -2 & 6 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 1 \\ 1 & 0 & 2 & 3 & 2 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$

(c) ✓✓ $\begin{bmatrix} 3 & 9 & 13 \\ 2 & 7 & 9 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 1 & 3 & 4 & 0 & -1 \\ -2 & -1 & -3 & -4 & 5 & 6 \\ 4 & 2 & 7 & 6 & 1 & -1 \end{bmatrix}$

(e) ✓✓ $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $ad - bc \neq 0$

(g) ✓✓ $\begin{bmatrix} 2 & 4 & 3 & 0 & 6 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix}$

(h) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 9 & 10 \\ 0 & 0 & 0 & 13 \end{bmatrix}$

(i) $\begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix}$
where a, e, h, and j are all nonzero.

(j) ✓ $\begin{bmatrix} 2 & 5 & 11 & 6 \\ 1 & 4 & 9 & 5 \\ -1 & 2 & 5 & 3 \\ 2 & -1 & -3 & -2 \end{bmatrix}$

1.66 Suppose that the matrices in Exercise 1.65 are the augmented matrices for a system of equations. In each case, write the system down and find all solutions (if any) to the system. ✓[(a), (c), (e), (g)]

1.67 In each of the following systems, find conditions on a, b, and c for which the system has solutions:

(a) ✓✓
$$\begin{aligned} 3x + 2y - z &= a \\ x + y + 2z &= b \\ 5x + 4y + 3z &= c \end{aligned}$$

(b)
$$\begin{aligned} -3x + 2y + 4z &= a \\ -x - 2y + 3z &= b \\ -x - 6y + 23z &= c \end{aligned}$$

(c) ✓✓
$$\begin{aligned} 4x - 2y + 3z &= a \\ 2x - 3y - 2z &= b \\ 4x - 2y + 3z &= c \end{aligned}$$

1.68 ✓The coefficient matrix (the augmented matrix without its last column) for the system in Exercise 1.67.a is the matrix A below. Note that $A_3 = A_1 + 2A_2$, where A_i are the rows of A. On the other hand, the consistency condition for this system is equivalent to $c = a + 2b$, which is essentially the same equation. Explain this "coincidence." [*Hint:* Compute the sum of the first and twice the second equations in the system in Exercise 1.67.a.] Check that there is a similar relationship between the rows of the coefficient matrices and the consistency conditions for the systems in Exercise 1.67.c.

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 2 \\ 5 & 4 & 3 \end{bmatrix}$$

1.69 Find conditions on a, b, c, and d for which the following system has solutions:

$$\begin{aligned} 2x + 4y + z + 3w &= a \\ -3x + y + 2z - 2w &= b \\ 13x + 5y - 4z + 12w &= c \\ 12x + 10y - z + 13w &= d \end{aligned}$$

1.70 Show that the consistency conditions found in Exercise 1.69 are explainable in terms of linear combinations of the rows of the corresponding coefficient matrix.

1.71 Prove that the solution to system (1.24) on page 47 found in equation (1.29) on page 52 is equivalent with the solution given in equation (1.30) on page 53.

[*Hint:* In deriving equation (1.30) on page 53, we set $t = y$ and $s = w$. According to equation (1.21) on page 37, $y = 1 + r + s$ and $w = s$. In equation (1.30), replace t by $1 + r + s$ and simplify to obtain equation (1.21). What you have shown is that every vector X expressible in the form given in equation (1.30) is also expressible in the form given in equation (1.21). You must also prove that every vector X expressible in the form given in equation (1.21) is also expressible in the form given in equation (1.30).]

1.72 ✓We proved that which variables are pivot variables and which are not do not depend on how we reduce the system. This is true only if we keep the variables in the same order.

(a) Find the free variables and the general solution for the following system:

$$\begin{aligned} 2x + 2y + 2z + 3w &= 4 \\ x + y + z + w &= 1 \\ 2x + 3y + 4z + 5w &= 2 \\ x + 3y + 5z + 11w &= 9 \end{aligned} \tag{1.35}$$

(b) Solve the equivalent system obtained by commuting the terms involving z and y in each equation. Express the general solution in the form $[x, y, z, w]^t = \ldots$.

(c) Prove the consistency of the answers to parts (a) and (b). [*Hint:* See the hint for Exercise 1.71.]

1.73 Show that commuting the terms involving x and y in each equation of system (1.35) does not change the free variables. How can we know, without any further work, that the expression for the solution to system (1.35) obtained by solving this new system will be identical with that obtained in Exercise 1.72.a.

1.74 ✓Create an example of each of the following. Construct your examples so that none of the coefficients in your equations are zero and explain why your examples work.

(a) A system of five equations in five unknowns that has a line as its solution set

(b) A system of five equations in five unknowns that has a plane as its solution set

(c) A system of five equations in three unknowns that has a line as its solution set

(d) A system of five equations in three unknowns that has a plane as its solution set

1.75 ✓The following vectors are the translation vector T and spanning vectors X_1 and X_2, obtained by using Gaussian elimination to solve a linear system of three equations in the variables x, y, z, and w.

$$T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad X_1 = \begin{bmatrix} -3 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix},$$

(a) Which are the free variables? How can you tell?

(b) ✓✓Find a row reduced echelon matrix that is the augmented matrix for a system having the stated translation and spanning vectors. Explain why there is only one such matrix.

1.76 One of your engineers wrote a program to solve systems of equations using Gaussian elimination. In a "test" case, the program produced the following answers for the spanning vector T and the translation vectors X_1 and X_2. You immediately knew that the program had an error in it. How? [*Hint:* Which are the free variables?]

$$T = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \quad X_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

1.77 A linear system is homogeneous if the numbers b_i in system (1.8) on page 29 are all 0. What can you conclude from the more unknowns theorem about a homogeneous system that has fewer equations than unknowns?

1.78 ✓✓You are given a vector B and vectors X_i. In each part, decide whether B is in the span of the X_i by attempting to solve the equation $B = x_1X_1 + x_2X_2 + x_3X_3$.

(a) $B = [3, 2, 1]^t, X_1 = [1, 0, -1]^t, X_2 = [1, 2, 1]^t, X_3 = [1, 1, 1]^t$

(b) $B = [a, b, c]^t, X_1 = [1, 0, -1]^t, X_2 = [1, 2, 1]^t, X_3 = [1, 1, 1]^t$

(c) $B = [3, 2, 1]^t, X_1 = [1, 0, -1]^t, X_2 = [1, 2, 1]^t, X_3 = [3, 4, 1]^t$

1.79 In a vector space $\mathcal{V}$, a set of vectors "spans $\mathcal{V}$" if every vector in $\mathcal{V}$ is a linear combination of the given vectors. Show that the vectors $X = [1, 2]^t$ and $Y = [1, -2]^t$ span $\mathbb{R}^2$. Specifically, show that the equation $[a, b]^t = xX + yY$ is solvable regardless of a and b.

1.80 ✓✓Show that the vectors $X_1 = [3, 1, 5]^t, X_2 = [2, 1, 4]^t$, and $X_3 = [-1, 2, 3]^t$ *do not* span $\mathbb{R}^3$ by finding a vector that cannot be expressed as a linear combination of them.

1.81 Prove that a consistent system of five equations in five unknowns will have an infinite number of solutions if and only if the row reduced echelon form of the augmented matrix has at least one row of zeros. [*Hint:* Think about the number of nonpivot variables in the echelon form.]

1.82 ✓Prove that a consistent system of n equations in n unknowns will have an infinite number of solutions if and only if the row reduced echelon form of the augmented matrix has at least one row of zeros. [*Hint:* Think about the number of free variables in the echelon form.]

1.83 ✓✓One possible reduced echelon form of a nonzero 2×2 matrix is shown below. What are the other possibilities?

$$\begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}$$

1.84 What general feature makes it clear that in any matrix in row reduced echelon form the set of nonzero rows is linearly independent?

1.85 ✓✓In any linear system, the last column of the augmented matrix is called the "vector of constants," while the matrix obtained from the augmented matrix by deleting the last column is called the "coefficient" matrix. For example, in Exercise 1.67.a, the vector of constants is $[a, b, c]^t$, which is the vector formed from the constants on the right sides of the equations in this system and the coefficient matrix is

$$\begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 2 \\ 5 & 4 & 3 \end{bmatrix}$$

In applications, it often happens that the coefficient matrix remains fixed while the vector of constants changes periodically. We shall say that the coefficient matrix is "nonsingular" if there is one and only one solution to the system, regardless of the value of the vector of constants.[5]

(a) Is the coefficient matrix for the system in Exercise 1.67.a nonsingular? Explain.

(b) Suppose a given system has three equations in three unknowns with a nonsingular coefficient matrix. Describe the row reduced form of the augmented matrix as explicitly as possible.

(c) Can a system with two equations and three unknowns have a nonsingular coefficient matrix? Explain in terms of the row reduced form of the system.

1.86 Suppose that $ad - bc = 0$. Show that the rows of the matrix A are linearly dependent.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

1.87 ✓✓In the following system, show that for any choice of x and y there is one (and only one) choice of z and w that solves the system. Write the general solution in parametric form using x and y as free variables. Specifically, set $x = s$ and $y = t$, solve for z and w in terms of s and t, and express the general solution as a linear combination of two elements in $\mathbb{R}^4$.

$$\begin{aligned} x \quad + \quad z + w &= 0 \\ y + 2z + w &= 0 \end{aligned}$$

1.88 Can x and y be set arbitrarily in this system? (If you are not sure, try a few values.)

$$\begin{aligned} x \quad + 2z + 2w &= 0 \\ y + \quad z + \quad w &= 0 \end{aligned}$$

Computational Issues: Counting Flops

A rough measure of the amount of time a computer will take to do a certain task is the total number of algebraic operations ($+$, $-$, $\times$, and $/$) required. Each such operation is called a **flop** (floating point operation). The next exercise shows that it requires at most $2n^3/3 + 3n^2/2 - 7n/6$ flops to solve a system of n equations in n unknowns. Thus, for example, with 20 unknowns, the solution could require 5910 flops, while

[5]This exercise is intended as a "preview" of the concept of invertibility that is discussed in depth in Chapter 3.

with 100 unknowns, 681,550 might be required. Such large systems are not at all uncommon in "real-world" applications.

1.89 Let A be an $n \times (n+1)$ matrix.

(a) Explain why it takes at most n flops to reduce A to a matrix of the following form. Note that row exchanges do not count as flops.

$$\begin{bmatrix} 1 & * & \dots & * & * \\ * & * & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \dots & * & * \end{bmatrix}$$

(b) Explain why it takes at most $2n$ additional flops to create a 0 in the $(2, 1)$ position of the matrix in part (a) using row reduction.

(c) Use parts (a) and (b) to explain why it takes at most $n + 2n(n-1) = 2n^2 - n$ flops to reduce A to a matrix with a 1 in the $(1, 1)$ position and 0's in the rest of column 1.

(d) Use part (c) to explain why it takes a total of at most

$$2(n^2 + (n-1)^2 + \cdots + 1) - (n + (n-1) + \cdots + 1) \qquad (1.36)$$

flops to reduce A to an echelon matrix.

(e) Explain why it only takes at most

$$2(n-1) + 2(n-2) + \cdots + 2$$

additional flops to bring the matrix from part (b) into row reduced echelon form.

(f) Use parts (d) and (e) and the following formulas to prove that it requires at most $2n^3/3 + 3n^2/2 - 7n/6$ flops to reduce A to row reduced echelon form.

$$n + (n-1) + \cdots + 1 = \frac{n(n+1)}{2}$$
$$n^2 + (n-1)^2 + \cdots + 1 = \frac{n(n+1)(2n+1)}{6}$$

1.3.1 Computer Projects

MATLAB can row reduce many matrices almost instantaneously. One begins by entering the matrix into MATLAB. We mentioned in the last section that rows can be separated by semicolons. One can also separate rows by putting them on different lines. Thus, we can enter a matrix A by stating that

```
>> A = [  1   -1   1   3   0   6
          2   -2   2   6   0   7
         -1    1   1  -1  -2   1
          4   -4   1   9   3   6  ]
```

To row reduce A, we simply enter

```
>> rref(A)
```

We obtain the row reduced form almost instantaneously:

```
ans  =
      1  -1   0  2   1  0
      0   0   1  1  -1  0
      0   0   0  0   0  1
      0   0   0  0   0  0
```

EXERCISES

1. Enter format long and then compute $(1/99) * 99 - 1$ in MATLAB. You should get zero. Next compute $(1/999) * 999 - 1$ and $(1/9999) * 9999 - 1$. Continue increasing the number of 9's until MATLAB does not produce 0. How many 9's does it require? Why do you not get 0?

   This calculation demonstrates a very important point: MATLAB can interpret some answers that should be zero as nonzero numbers. If this happens with a pivot in a matrix, MATLAB may produce totally incorrect reduced forms. MATLAB guards against such errors by automatically setting all sufficiently small numbers to zero. How small is "sufficiently small" is determined by a number called eps, which, by default, is set to around 10^{-17}. MATLAB will also issue a warning if it suspects that its answer is unreliable.

   In some calculations (especially calculations using measured data), the default value of eps will be too small. For this reason, MATLAB allows you to include a tolerance in rref. If you wanted to set all numbers less than 0.0001 to 0 in the reduction process, you would enter rref(A,.0001).

2. Use rref to find all solutions to the system in Exercise 1.55.i on page 39.

3. We said that the rank of a system is the number of equations left after linearly dependent equations have been eliminated. We commented (but certainly did not prove) that this number does not depend on which equations were kept and which were eliminated. It seems natural to suppose that the rows in the row reduced form of a matrix represent linearly independent equations and hence, that the rank should also be computable as the number of nonzero rows in the row reduced form of the matrix. We can check this conjecture experimentally

using the MATLAB command rank(A), which computes the rank of a given matrix.

Use MATLAB to create four 4×5 matrices A, B, C, and D having respective ranks of 1, 2, 3, and 4. Design your matrices so that none of their entries are zero. Compute their rank by (a) using the rref command and (b) using the rank command.

[*Hint:* Try executing the following sequence of commands and see what happens. This could save you some time!]

```
>> M = [ 1 2 1 5 3
         2 1 1 4 3 ]
>> M(3,:) = 2*M(2,:) + M(1,:)
```

This works because, in MATLAB, M(i,:) represents the *i*th row of M. [Similarly, M(:,j) represents the *j*th column.] The equation M(3,:)=2*M(2,:)+ M(1,:) both creates a third row for M and sets it equal to twice the second plus the first row.

(*Note:* If you choose "nasty" enough coefficients, then you may need to include a tolerance in the rank command in order to get the answer to agree with what you expect. This is done just as with the rref command.)

4. For each of the matrices in Exercise 1.65 on page 63, other than, (f) and (i), use rref to row reduce the *transpose*. The MATLAB symbol for the transpose of A is A′. How does the rank of each matrix compare with that of its transpose?

5. Let

$$X = [1, 2, -5, 4, 3]^t, \quad Y = [6, 1, -8, 2, 10]^t, \quad Z = [-5, 12, -19, 24, 1]^t$$

(a) Determine which of the vectors U and V is in the span of X, Y, and Z by using rref to solve a system of equations.

$$U = [-5, 23, -41, 46, 8]^t, \quad V = [22, 0, -22, 0, 34]^t$$

(b) Imagine as the head of an engineering group that you supervise a computer technician who knows *absolutely nothing* about linear algebra, other than how to enter matrices and commands into MATLAB. You need to get this technician to do problems similar to part (a). Specifically, you will be giving him or her an initial set of three vectors X, Y, and Z from $\mathbb{R}^n$. You will then provide an additional vector U, and you want the technician to determine whether U is in the span of X, Y, and Z.

Write a brief set of instructions that will tell your technician how to do this job. Be as explicit as possible. Remember that the technician cannot do linear algebra! You must provide instructions on constructing the necessary matrices, telling what to do with them and how to interpret the answers.

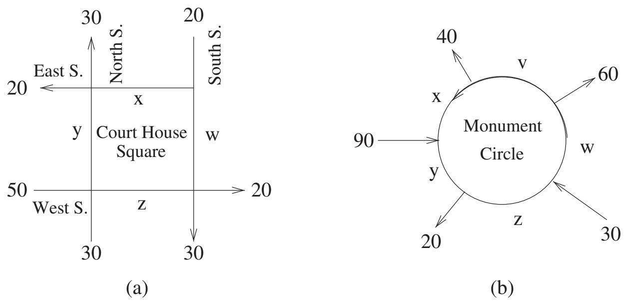

FIGURE 1.24 Two traffic patterns.

The final output to you should be a simple Yes or No. You do not want to see matrices!

(c) One of your assistant engineers comments that it would be easier for the technician in part (b) to use MATLAB's rank command rather than rref. What does your assistant have in mind?

1.3.2 Applications to Traffic Flow

An interesting context in which linearly dependent systems of equations can arise is in the study of traffic flow. Figure 1.24a is a map of the downtown area of a city. Each street is one-way in its respective direction. The numbers represent the average number of cars per minute that enter or leave a given street at 3:30 p.m. The variables also represent average numbers of cars per minute. Of course, barring accidents, the total number of cars entering any intersection must equal the total number leaving. Thus, from the intersection of East and North, we see that $x + y = 50$. Continuing counterclockwise around the square, we get the system

$$\begin{aligned} x + y \qquad\quad &= 50 \\ y + z \qquad &= 80 \\ z + w &= 50 \\ x \qquad\qquad + w &= 20 \end{aligned} \tag{1.37}$$

Notice also that in Figure 1.24, the total number of cars entering the street system per minute is $20 + 30 + 50 = 100$ while the total number leaving is $20 + 30 + 20 + 30 = 100$. All of our examples share the property that the total number of cars per minute that enter the street system per minute equals the total number that leave the street system per minute. *We assume that the total number of cars per minute on the street system at any given time equals the total number of cars entering the street system per minute which equals the total number of cars leaving the street system per minute.* Thus we augment system 1.37 with the additional equation

$$x + y + z + w = 100$$