Hopf Algebras

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Winter 2019

Introduction

These are the notes I took in class during the Winter 2019 topics course *Math 582H - Hopf Algebras* at University of Washington, Seattle.

The course description follows:

This course is an introduction to Hopf algebras. In addition to basic material in Hopf algebra, we will present some latest developments in quantum groups and tensor and fusion categories. One of the newer topics is homological properties of Noetherian Hopf algebras of low Gelfand-Kirillov dimension. A good reference for the first two topics in the book *Hopf Algebras and Their Action Rings* by Susan Montgomery. Here is a list of possible topics:

- Classical theorems concerning finite dimensional Hopf algebras.
- Infinite dimensional Hopf algebras and quantum groups.
- Duality and Calabi-Yau property.
- Actions of Hopf algebras and invariant theory.
- Representations of Hopf algebras, tensor and fusion categories.

1 January 7, 2019

If you don't know what a symmetric tensor category is, today is going to be a three star day. Max is 5.

1.1 Overview

We are shooting to understand two conjectures:

Conjecture (Etingof-Ostrik '04): If A is a finite dimentional Hopf algebra, then

$$\bigoplus_{i\geq 0} \mathrm{Ext}^i_A({}_Ak, {}_Ak)$$

is Noetherian.

$$
\begin{tikzcd}
V\otimes V\otimes V \arrow[r, "\mathrm{id}_V\otimes m"] \arrow[d, "m\otimes \mathrm{id}_V"] & V\otimes V \arrow[d, "m"] \\
V\otimes V \arrow[r, "m"] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
k\otimes V \arrow[r, "\sim"] \arrow[dr, "u\otimes \mathrm{id}_V"'] & V & V\otimes k \arrow[l, "\sim"'] \arrow[dl, "\mathrm{id}_V\otimes u"] \\
& V\otimes V \arrow[u, "m"] &
\end{tikzcd}
$$

Figure 1: Diagrams for definition 1.2.1.

CONJECTURE (BROWN-GOODEARL '98): If A is a Noetherian Hopf algebra, then the injective dimension of A_A is finite.

These are both still open! In fact there is a meeting at Oberwolfach this March concerning exactly these conjectures.

1.2 Symmetric Tensor Categories

We are going to be using the following notation throughout:

- k is a field
- Vect_k is the category of k-vector spaces
  - Vect_k is closed under tensor products
  - There is an element $k \in \mathrm{Vect}_k$ such that

    $$k \otimes_k V \cong V \cong V \otimes_k k$$

    where the above isomorphisms are natural.
  - $V \otimes_k W \cong W \otimes_k V$
- An algebra is an object in Vect_k.

1.2.1 Definition
$V \in \mathrm{Vect}_k$ is called an **algebra object** if there are two morphisms

(a) $m : V \otimes V \to V$

(b) $u : k \to V$

such that the diagrams in figure 1.2 commute.

1.2.2 Lemma
$V \in \mathrm{Vect}_k$ is an algebra object iff V is an algebra over k.

1.2.3 Lemma
If C is a symmetric tensor category, so is C^{op}.

Then the natural thing to ask is: what is an algebra object in this opposite category?

1.2.4 Definition
A **coalgebra object** in C is an algebra object in C^{op}. Here we have comultiplication Δ and counit ε.

1.2.5 Remark: Naturally you could go about defining this from first principles and drawing the diagrams in figure 1.2 with the arrows reversed, but we are probably mature enough to do without that (saving my fingers from repetitive strain injury in the process.)

1.2.6 Lemma
Alg_k, defined as the category of algebra objects in Vect_k, is a symmetric tensor category. Furthermore Coalg_k, the category of coalgebra objects in Vect_k, is a symmetric tensor category.

1.2.7 Lemma
The following are equivalent:

(a) V is an algebra object in Coalg_k

(b) V is a coalgebra object in Alg_k

(c) There are morphisms $m, u, \Delta, \varepsilon$ such that

- (V, m, u) is an algebra
- (V, Δ, ε) is a coalgebra
- Equivalently:
  - m and u are coalgebra morphisms
  - Δ and ε are algebra morphisms.

Proof
The nice thing here is that the $(a) \Leftrightarrow (c)$ without the last condition. A similar fact holds for (b) except the second-to-last. The last thing to do is to prove the last two conditions are equivalent. ♠

Problem 1.1
Fill in the details for the proof above.

$$\begin{tikzcd}
V \otimes V \arrow[r, "m"] \arrow[d, "\Delta\otimes\Delta"'] & V \arrow[r, "\Delta"] & V \otimes V \\
V^{\otimes 4} \arrow[rr, "T_{2,3}"] & & V^{\otimes 4} \arrow[u, "m\otimes m"']
\end{tikzcd}
\qquad
\begin{tikzcd}
k \arrow[r, "\Delta"] \arrow[d, "u"] & k \otimes k \arrow[d, "u\otimes u"] \\
V \arrow[r, "\Delta"] & V \otimes V
\end{tikzcd}$$

Figure 2: m and u are coalgebra morphisms.

$$\begin{tikzcd}
V \otimes V \arrow[r, "m"] \arrow[d, "\Delta\otimes\Delta"] & V \arrow[d, "\Delta"] \\
V^{\otimes 4} \arrow[r, "m"] & V \otimes V
\end{tikzcd}
\qquad
\begin{tikzcd}
V \otimes V \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "m"] & k \otimes k \arrow[d, "m"] \\
V \arrow[r, "\varepsilon"] & k
\end{tikzcd}$$

Figure 3: Δ and ε are algebra morphisms.

Solution:
Assume that $(V, m, u, \Delta, \varepsilon)$ is an algebra and coalgebra and further that m and u are coalgebra morphisms. That means in particular that the diagrams in figure 1 commute.

We are looking to prove that Δ and ε are algebra morphisms, or that the diagrams in figure 1 commute.

From here it's actually a bit boring because it's kinda just a definition/notation game. It boils down to the fact that the (co)multiplication on $V \otimes V$ has a twist that exactly lines up so that each square is saying the same thing. ♠

1.2.8 Definition
V is called a **bialgebra object** if V is an algebra object in Coalg_k.

Problem 1.2
(a) Suppose that $\operatorname{char} k \neq 2$. *Classify all bialgebras of* $\dim 2$.

(b) Do the same for $\operatorname{char} k = 2$.

Solution:
Part (a)

Consider $\varepsilon : V \to k$ and consider $\ker \varepsilon \lhd V$. By rank-nullity, $\dim \ker \varepsilon = 1$, so $\ker \varepsilon = kx$ for some $x \in V$. Therefore $x^2 = cx$ for some c, and if $c = 0$, then (as an algebra) $V \cong k[x]/(x^2)$. Otherwise consider $y = \frac{x}{c}$. In this case $y^2 = \frac{x^2}{c^2} = \frac{x}{c} = y$, and $V \cong k[x]/(x^2 - x)$.

Notice that in either case $\varepsilon(x) = 0$, so let

$$\Delta(x) = a(1 \otimes 1) + b(1 \otimes x) + c(x \otimes 1) + d(x \otimes x)$$

and using that $\varepsilon \otimes \mathrm{id} \circ \Delta = \mathrm{id} \otimes \varepsilon \circ \Delta$ and that each should be (essentially) the identity (this is just the diagram we saw before), we get $a = 0$ and $b = c = 1$. Thus the coalgebra structure of any Hopf algebra is given by

$$\varepsilon(x) = 0, \quad \Delta(x) = 1 \otimes x + x \otimes 1 + d(x \otimes x).$$

Consider first the case when $x^2 = 0$. Then since comultiplication will be an algebra morphism,

$$0 = \Delta(x^2) = \Delta(x)^2 = 1 \otimes x^2 + x^2 \otimes 1 + d^2(x^2 \otimes x^2) + 2(x \otimes x) + 2d(x \otimes x^2) + 2d(x^2 \otimes x)$$

and since $x^2 = 0$,

$$0 = 2(x \otimes x).$$

But $x \otimes x$ is a basis element of $V \otimes V$, so V can only have this algebra structure when $\operatorname{char} k = 2$. We will return to this in the next part.

So then $x^2 = x$ and using the computation above,

$$1 \otimes x + x \otimes 1 + d(x \otimes x) = \Delta(x) = \Delta(x^2) = 1 \otimes x + x \otimes 1 + (d^2 + 4d + 2)(x \otimes x)$$

so

$$(d^2 + 3d + 2)(x \otimes x) = 0 \quad \Rightarrow \quad d^2 + 3d + 2 = (d+2)(d+1) = 0$$

and so either $d = -1$ or $d = -2$.

One can verify that Δ is coassociative, so we can conclude that when $\operatorname{char} k \neq 2$, there are precisely two Hopf algebra structures with algebra structure $k[x]/(x^2 - x)$ and comultiplication either

$$\Delta(x) = 1 \otimes x + x \otimes 1 - x \otimes x \quad \text{or} \quad \Delta(x) = 1 \otimes x + x \otimes 1 - 2(x \otimes x)$$

(b)

Now assume that $\operatorname{char} k = 2$ and that $V \cong k[x]/(x^2 - x)$ as an algebra. Then using the analysis above, we see that we can choose comultiplication either

$$\Delta(x) = x \otimes 1 + 1 \otimes x \quad \text{or} \quad \Delta(x) = 1 \otimes x + x \otimes 1 + x \otimes x.$$

If instead $V \cong k[x]/(x^2)$, then *any* value of d will suffice, so there are a full k's worth of Hopf algebra structures that can appear. ♠

2 January 9, 2019

Today we are going to rely heavily on Sweedler notation. :) Notice that if we are looking at actual objects in the diagram for coassociativity, we get

$$\begin{tikzcd}
v \arrow[rr] \arrow[d] & & \sum v_{(1)} \otimes v_{(2)} \arrow[d] \\
\sum v_{(1)} \otimes v_{(2)} \arrow[r] & \sum (v_{(1)})_{(1)} \otimes (v_{(1)})_{(2)} \otimes v_{(2)} = & \sum v_{(1)} \otimes (v_{(2)})_{(1)} \otimes (v_{(2)})_{(2)}
\end{tikzcd}$$

Figure 4: Coassociativity on elements in Sweedler notation

Example 2.1
Let G be a group and kG be the group algebra. The algebra structure arises as normal where $g \cdot h$ comes from the structure on G. Then $\Delta(g) = g \otimes g$ and this extends linearly.

But then if you consider $\Delta(\sum c_g g)$, notice that by the nature of tensors this is not unique! So we will just write

$$\Delta\left(\sum c_g g\right) = \sum_G c_g(g \otimes g) = \sum v_{(1)} \otimes v_{(2)}$$

2.1 Algebra structure on $V \otimes V$

We said earlier on that Alg_k is a symmetric *tensor* category. But how do we define the multiplication on the tensor product?

Well it all comes from the twist! We define

$$m_{V \otimes W} = (m_V \otimes m_W) \circ (\mathrm{id}_V \otimes \tau_{2,3} \otimes \mathrm{id}_W)$$

where $\tau_{2,3}$ is the twist morphism.

More simply, $u_{V \otimes W} : k = k \otimes k \to V \otimes V$ simply defined by $u_V \otimes u_W$.

So then when we say that Δ is an algebra morphism, we are saying that for all $v, w \in V$

$$\Delta(vw) = \sum (vw)_{(1)} \otimes (vw)_{(2)} = (\sum v_{(1)} \otimes v_{(2)})(\sum w_{(1)} \otimes w_{(2)}) = \sum v_{(1)}w_{(1)} \otimes v_{(2)}w_{(2)}$$

2.2 Hopf Algebras

Already to the good stuff!

2.2.1 Definition
$V \in \mathrm{Vect}_k$ is a **Hopf algebra** if V is a bialgebra together with an **antipode** $S : V \to V$ satisfying

$$(S, \mathrm{id}_V) \circ \Delta = \varepsilon = (\mathrm{id}_V, S) \circ \Delta$$

Conjecture: If $V \in \mathrm{Vect}_k$ is a Noetherian Hopf algebra, then S is bijective.

2.3 History and Motivation

Hopf himself was a topologist, so this is the first context in which it arose. In the 1940's, he began studying Hopf algebras over $\mathbb{Z}_2$ graded k vector spaces. For instance, the cohomology ring of topological space X with coefficients in k.

Later, in combinatorics, they ended popping up. Looking at rings of symmetric functions and other places gave some interesting examples.

Then in group theory you can define a functor from groups to Hopf algebras by $F(G) = kG$ with the diagonal map. The antipode is just the inverse.

Then with Lie algebras, you can look at $\mathcal{U}(L)$, the universal enveloping algebra is a Hopf algebra.

Finally with algebraic groups (yay!) we take an algebraic group G and consider the ring of functions on it, which is again a Hopf algebra.

Some "cousins" of Hopf algebras: quasi, weak, multiplier, ribbon, quasi-triangular, etc Hopf algebras. Each has slightly different base category or restrictions.

3 January 11, 2019

The plan for today is to talk about:

- Convolution Algebras
- Antipodes
- Duality
- (Co-)Modules

3.1 Convolution Algebras

Let $\mathcal{T}$ be a symmetric tensor category. We can usually think of $\mathcal{T} = \text{Vect}_k$, but there is a problem since Vect_k is equivalent to the category of Hopf algebras over k, while this is not generally true.

We also need that $\mathcal{T}$ is k-linear (that is, enriched as a category over k). This means that $\text{Hom}_{\mathcal{T}}(A, B) \in \text{Vect}_k$.

3.1.1 Theorem
Let $\mathcal{T}$ be as above. Then $\text{Hom}_{\mathcal{T}}(C, A)$ is an algebra and $\text{Hom}_{\mathcal{T}}(-,-) : (\text{Coalg}_{\mathcal{T}})^{op} \times \text{Alg}_{\mathcal{T}} \to \text{Alg}_k$ is a functor.

Proof
Let A be an algebra object in $\mathcal{T}$ and C be a coalgebra object in $\mathcal{T}$. Then $1_{\text{Hom}} := u_A \circ \varepsilon_C : C \to 1_{\mathcal{T}}$ and define

$$f * g := m_A(f \otimes g)\Delta_C : C \to A.$$

Then using Lemma 3.1.2 and the fact that A and C are (co)algebra objects, we can see that the product $*$ satisfies the axioms required.

Note that actually ♠

3.1.2 Lemma

(a) $1_{\text{Hom}} * f = m_A(u \otimes 1)(1 \otimes f)(\varepsilon \otimes 1)\Delta$.

(b) $f * 1_{\text{Hom}} = m_A(1 \otimes u)(f \otimes 1)(1 \otimes \varepsilon)\Delta$

(c) $(f * g) * h = m_A(m_A \otimes 1)((f \otimes g) \otimes h)(\Delta_C \otimes 1)\Delta_C$

(d) $f * (g * h) = m_A(1 \otimes m_A)(f \otimes (g \otimes h))(1 \otimes \Delta)\Delta$

Proof

(a)

$$\begin{aligned} 1_{\text{Hom}} * f &= m_A(1_{\text{Hom}} \otimes f)\Delta_C \\ &= m_A(u_A \circ \varepsilon \otimes f)\Delta_C \\ &= m_A(u \otimes 1)(\varepsilon \otimes 1)(1 \otimes f)\Delta \end{aligned}$$

(b)

Same as (a), essentially.

(c) and (d)

$$\begin{aligned} (f * g) * h &= m_A((f * g) \otimes h)\Delta \\ &= m_A((m_A(f \otimes g)\Delta) \otimes h)\Delta \\ &= m_A(m_A \otimes 1)((f \otimes g) \otimes h)(\Delta \otimes 1)\Delta \end{aligned}$$

and the other is analogous. ♠

3.1.3 Definition

$V \in \mathcal{T}$ is a **Hopf algebra object** if:

- V is a bialgebra object in $\mathcal{T}$ and
- There is a map $S : V \to V$ that is $(\text{id}_V)^{-1}$ with respect to $*$.

3.1.4 Remark: Notice here that $\text{id}_V \in \text{Hom}_{\mathcal{T}}(V, V)$, the identity map in $\mathcal{T}$. We are *not* taking about $1_{\text{Hom}} = u \circ \varepsilon$.

Also, we call S an **antipode.**

3.2 Duality

Notice that when $C = 1_{\mathcal{T}}$ (that is the tensor identity), $\mathrm{Hom}_{\mathcal{T}}(1_{\mathcal{T}}, -) : \mathrm{Alg}_{\mathcal{T}} \to \mathrm{Alg}_k$ is a functor. Same for the dual from $\mathrm{Coalg}_{\mathcal{T}}$. This second one gives us a chance to talk about duality.

3.2.1 Lemma
Let $\mathcal{T}$ be the category of finite dimensional vector spaces over k. Then $(-)^* : \mathcal{T} \to \mathcal{T}^{op}$ is an equivalence.

This uses $(V \otimes W)^* = W^* \otimes V^*$.

3.2.2 Corollary
V is an algebra over $k \Leftrightarrow V^*$ is a coalgebra over k. And vice versa.

Recall $S = (\mathrm{id}_V)^{-1}$. Thus

$$S * \mathrm{id}_V = 1_{\mathrm{Hom}} = \mathrm{id}_V * S$$

The diagram we have here is

$$\begin{tikzcd}
V \otimes V \arrow[rr, "S\otimes \mathrm{id}_V"] & & V \otimes V \arrow[d, "m_V"] \\
V \arrow[u, "\Delta"] \arrow[r, "\varepsilon_V"] \arrow[d, "\Delta"'] & k \arrow[r, "u_V"] & V \\
V \otimes V \arrow[rr, "\mathrm{id}_V \otimes S"] & & V \otimes V \arrow[u, "m_V"']
\end{tikzcd}$$

Modules/Comodules

3.2.3 Definition
Let A be an algebra object in $\mathcal{T}$. A **left** A **module** is $M \in \mathcal{T}$ with a morphism

$$m_M : A \otimes M \to M$$

such that the diagrams in Figure 3.2 commute.

3.2.4 Remark: Note that we don't necessarily need that M lie in $\mathcal{T}$. We could instead just rely on an algebra homomorphism $\varphi : A \to \mathrm{Hom}_{\mathcal{T}}(M, M)$ and proceed as usual.

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "m_A \otimes 1"] \arrow[d, "1 \otimes m_M"] & A \otimes M \arrow[d, "m_M"] & A \otimes M \arrow[r, "m_M"] & M \\ A \otimes M \arrow[r, "m_M"] & M & 1_{\mathcal{T}} \otimes M \arrow[u, "u_A \otimes 1"] \arrow[ur, "\sim"] & \end{tikzcd}$$

Figure 5: Module diagrams

4 January 14, 2019

Today we are talking about Hopf modules and later in the week we will see the fundamental theorem of Hopf modules as well as a neat result.

Here is the "example of the day."

Example 4.1
Consider $k[x]$ with maps $\Delta(x^n) = \sum_0^n x^i \otimes x^{n-i}$ and $\varepsilon(x^n) = \delta_{1,n}$. $S(x^n) = (-x)^n$. Then notice that $\Delta(x) = 1 \otimes x + x \otimes 1$, so $\delta(x^n) = (\delta x)^n$.

James doesn't want to do the rest of the computations, but they can be done. :)

Problem 4.1
(**) Working with a Hopf algebra V, consider the convolution algebra $Hom(V \otimes V, V)$

(a) $(S \circ m) * m = 1_{Hom(V \otimes V, V)} = m * (m \circ s \otimes s \circ \tau)$ *where s is the antipode in V.*

(b) $(\Delta \circ S) * \Delta = 1_{Hom(V, V \otimes V)} = \Delta * ((s \otimes s) \circ \tau \circ \Delta)$

Solution:

$$(s \circ m) * m$$

Problem 4.2
Classify all Frobenius (to be defined) Hopf algebras of dimension 3.

4.1 Returning to (co)modules

4.1.1 Lemma

Let $\mathcal{T} = \mathrm{Vect}_k$, and V and algebra over k. The following are equivalent:

(a) M is a left V-module (see last lecture).

(b) There is an action of A on M such that $(ab) \cdot m = a \cdot (b \cdot m)$ and $1 \cdot m = m$.

(c) There is an algebra morphism $\varphi : V \to \mathrm{Hom}_k(M, M)$

4.1.2 Definition
Let V be a coalgebra over k. We say M is a left **comodule** if there is a map

$$\rho_M : M \to V \otimes M$$

such that the diagrams in figure 4.1 commute.

$$\begin{tikzcd}
M \arrow[r, "\rho"] \arrow[d, "\rho"] & V \otimes M \arrow[d, "\mathrm{id}_V \otimes \rho"] \\
V \otimes M \arrow[r, "\Delta_V \otimes \mathrm{id}_M"] & V \otimes V \otimes M
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "\rho"] \arrow[d, "\cong"] & V \otimes M \arrow[dl, "\varepsilon \otimes \mathrm{id}_M"] \\
k \otimes M &
\end{tikzcd}$$

Figure 6: Diagrams for the definition of a comodule.

4.1.3 Remark: Notationally speaking, we write ${}_V\mathcal{M}$ and ${}^V\mathcal{M}$ for the categories of left modules and comodules. Similar for right boyes.

4.2 Tensors of V-modules

Say V is a bialgebra (or Hopf if you prefer, but it's not necessary). Then we get the following:

4.2.1 Lemma
Let M and N be two right V-modules. Then

(a) $M \otimes N$ is a right V-module.

(b) If V is cocommutative, then $M \otimes N \cong N \otimes M$ as right V-modules.

(c) $\mathcal{M}_V$ is a tensor (monoidal) category.

Proof
For (a), use the fact that $V \otimes V$ acts on $M \otimes N$ in a natural way. Then you get a map $V \otimes V \to \mathrm{Hom}_k(M \otimes N, M \otimes N)^{op}$ (why opposite?) and by precomposing with Δ to show they are V-modules. This establishes (b).

Finally (c) follows because Vect_k is a tensor category. ♠

4.3 Hopf Modules

4.3.1 Definition
Let V be a bialgebra over k. We say that M is a $\binom{r}{r}$ Hopf V-module if

(a) (M, m_M) is a right V-module

(b) (M, ρ_M) is a right V comodule.

(c) ρ_M is a right V-module map.

- Equivalently, m_M is a right V-comodule map.

4.3.2 Remark: The fact that these two last conditions are equivalent are perhaps not immediately obvious but we have been assured it is true. :)

5 January 16, 2019

5.0.1 Theorem (Larson-Sweedler '69)
Let V be a Hopf algebra over k. Then the following categories are equivalent:

$$ {}_V^V\mathcal{M} \cong \mathrm{Vect}_k \cong \mathcal{M}_V^V $$

5.0.2 Remark: A natural question that one may ask is how to extend this theorem to the enriched setting.

5.0.3 Lemma
S is an anti-homomorphism of an algebra V and an anti-homomorphism of the coalgebra V.

Proof
Use the facts proved in problem 4.1. ♠

Problem 5.1
If $\mathrm{char}\, k = 0$ *any Hopf quotient of* $k[x]$ *is either* k *or* $k[x]$ *itself.*

Problem 5.2
Suppose k *has positive characteristic. Construct two different Hopf algebra quotients of* $k[x]$ *of dimension* $p = \mathrm{char}\, k$.

Example 5.1
$\operatorname{char} k = p > 0$. Let $V = k[x]/(x^p)$ as an algebra and define $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$ and $S(x) = x$. Then notice

$$\Delta(x^p) = \sum \binom{p}{i} x^i \otimes x^{p-i} = x^p \otimes 1 + 1 \otimes x^p.$$

5.1 Hopf Modules Once More

Recall the definitions from section 4.1 and section 4.3.

5.1.1 Remark: An equivalent (more category-theoretical) definition of a Hopf module is a V-comodule object in the category of V-modules. Also can dualize everything.

Example 5.2
Consider $V \in^V_V \mathcal{M}$. THen you can define $\rho = \Delta$ and check it satisfies all the requirements.

5.1.2 Definition
An $\binom{l}{l}$ Hopf V-module M is called **trivial** if $M \cong V \otimes M_0$ for some $M_0 \in \mathrm{Vect}_k$.

5.1.3 Remark: Basically you can think of this being trivial since we can *always* define a module in this way where the entire module structure is inherited from the structure of V (that is, irrespective of M_0).

5.0.1′ Theorem
Suppose that V is a Hopf algebra. Then

(a) Every $\binom{l}{l}$ Hopf V-module is trivial.

(b) Let $M \in^V_V \mathcal{M}$. Then
$$M \xrightarrow{\sim} V \otimes M^{Cov}$$
where
$$M^{Cov} := \{m \in M | \rho(m) = 1 \otimes m\}.$$

(c) ${}^V_V\mathcal{M} \cong \mathrm{Vect}_k$.

5.1.4 Remark: This is actually just a reformulation of theorem 5.0.1 in less compact (but ultimately more usable) notation.

Proof
Some identites that will be useful:

$$\Delta(v) = \sum v_1 \otimes v_2$$
$$\sum S(v_1) \otimes v_2 = \varepsilon(v) = \sum v_1 S(v_2)$$
$$\rho(m) = \sum m_{-1} \otimes m_1$$
$$\sum v_1 \otimes (v_2)_1 \otimes (v_2)_2 = \sum (v_1)_1 \otimes (v_1)_2 \otimes v_2 = \sum v_1 \otimes v_2 \otimes v_3$$
$$\sum (m_{-1})_1 \otimes (m_{-1})_2 \otimes m_0 = \sum m_{-1} \otimes (m_0)_{-1} \otimes (m_0)_0 = \sum m_{-2} \otimes m_{-1} \otimes m_0$$

Then we defin $\phi : M \to M^{Cov}$ by

$$\phi(x) = \sum S(x_{-1})x_0 \in M$$

We claim first that $\phi(x) \in M^{Cov}$ – namely $\rho(\phi(x)) = 1 \otimes \phi(x)$. To see this, compute

$$\begin{aligned}
\rho(\phi(x)) &= \sum (\phi(x))_{-1} \otimes (\phi(x))_0 \\
&= (S(x_{-1})x_0)_{-1} \otimes (S(x_{-1})x_0) \\
&= \sum \Delta(S(x_{-1}))\rho(x_0) \\
&= \sum (S \otimes S) \circ \tau(\Delta(x_{-2})) \cdot [x_{-1} \otimes x_0] \\
&= \sum [S(x_{-2}) \otimes S(x_{-3})][x_{-1} \otimes x_0] \\
&= \text{my fingers are aching...} \\
&= 1 \otimes \phi(x)
\end{aligned}$$

Now define $F : M \to V \otimes M^{Cov}$ by $F = (\mathrm{id} \otimes \phi) \circ \rho$ and $G : V \otimes M^{Cov}$ to be the map taking $v \otimes m$ to vm. The next claim is that GF and FG are the identity. You can see this by similarly pushing around notation.

Finally the last claim will be seen on Friday. ♠

6 January 18, 2019

Recall the Fundamental Theorem of Hopf Modules: ${}^V_V\mathcal{M} \simeq \mathrm{Vect}_k$ or equivalently that every (not necessarily finite dimensional) Hopf module is trivial: $M \cong V \otimes M^{Cov}$.

7 Frobenius Algebras

Today we will discuss the result that

7.0.1 Theorem (Larson-Sweedler, '69)
Every finite dimensional Hopf algebra is Frobenius.

There are some nice categorical equivalences: finite dimensional Frobenius algebras over k is equivalent to 2-D quantum field theories, or equivalently the symmetric functor category from the 2-D cobordism category to Vect_k.

Whoa.

Example 7.1
$V = (kG)^* = \oplus_{g\in G} k\delta g$ where G is a finite group. Define the algebra structure with

$$1_V = \sum_{g\in G} \delta_g$$

and $\delta_g\delta_h = \delta_g$ when $g = h$ and 0 otherwise.

The coalgebra structure is $\varepsilon(\delta_g) = \delta_{g,1_G}$ where on the right it is the Kroneker delta. Furthermore $\Delta(\delta_g) = \sum_{h\in G} \delta_h \otimes \delta_{h^{-1}g}$ and $S(\delta_g) = \delta_{g^{-1}}$.

Problem 7.1
Prove that $kS_3 \cong (kS_3)^*$.

Problem 7.2
Prove that $k\mathbb{Z}_3 \cong (k\mathbb{Z}_3)^*$ *when* $\operatorname{char} k \neq 3$.

7.0.2 Lemma
Let A and B be algebras over k.

(a) If $M \in_A \mathcal{M}_B$, $N \in \mathcal{M}_B$, then $\mathrm{Hom}_{\mathcal{M}_B}({}_AM, N) \in \mathcal{M}_A$.

(b) Take $B = k$, $M \in_A \mathcal{M}$. Then

$$M^* := \mathrm{Hom}_k(M, k) \in \mathcal{M}_A.$$

(c) $(-)^* :_A \mathcal{M}_{f.d.} \to (\mathcal{M}_A)^{op}_{f.d.}$ is an equivalence of categories. This is called the **reflection principle.**

Proof

This is a known fact. (b) follows quickly from (a) and (c) involves extending the dual map to maps of left A-modules. ♠

By the principle of duality, we have:

7.0.3 Lemma

If C and d are finite dimensional coalgebras over k, then

(a) if $M \in^C \mathcal{M}^D_{f.d.}$ and $N \in \mathcal{M}^D_{f.d.}$ then $\mathrm{Hom}_{\mathcal{M}^D}(M, N) \in \mathcal{M}^C_{f.d.}$.

(b) If $M \in^C \mathcal{M}_{f.d.}$ then $\mathrm{Hom}_k(M, k) \in \mathcal{M}^C$

(c) $(-)^* :^C \mathcal{M}_{f.d.} \to (\mathcal{M}^C_{f.d.})^{op}$ is an equivalence of categories.

7.0.4 Definition

A finite dimensional algebra A is called **Frobenius** if one of the following (equivalent) conditions hold:

(a) $(A)^*_A \cong A_A$

(b) ${}_A(A)^* \cong_A A$

7.0.5 Remark: Note that the equivalence of these two things follow due to the equivalence of categories we get from the two lemmas above.

We actually don't need the following lemma, but James is a fan so we're going to write it down.

7.0.6 Lemma

Let A be a finite dimensional algebra over k. Then M is a finite dimensional left A-module if and only if M is a finite dimensional right A^*-comodule.

7.0.7 Remark: The reason we get the module to comodule switch is (partly?) due to the fact that in our monoidal category $(V \otimes W)^* = W^* \otimes V^*$.

7.0.8 Lemma

Let V be a finite dimensional Hopf algebra and M be a finite dimensional $\binom{l}{l}$ Hopf V-module. Then M^* is a finite dimensional $\binom{r}{r}$ Hopf V-module.

Proof

$M \in {}^V_V\mathcal{M}$ if and only if $M \in_V \mathcal{M} \cap {}^V\mathcal{M}$ and the two structures are compatible.

The fact that $M^* \in \mathcal{M}_V \cap \mathcal{M}^V$ is straightforward and checking the compatibility is easy to check. ♠

Now we finally prove that theorem 7.0.1:

PROOF
First of all $V \in {}_V^V\mathcal{M}$ so by the lemma $V^* \in \mathcal{M}_V^V$. By the fundamental theorem of Hopf modules,

$$V^* \cong V \otimes (V^*)^{Cov} \cong V$$

as Hopf modules. Thus $V_V^* \cong V_V$ in particular. ♠

7.0.9 REMARK: The second isomorphism above comes from the fact that $(V^*)^{Cov}$ is one-dimensional, so is the trivial module. :)

7.0.10 Corollary
If A is *not* a Forbenius algebra, then there is *no* Hopf algebra structure on A.

Example 7.2
Some examples of algebras with(out) this property.

(a) $\mathbb{C}[x]/(x^n)$ is finite dimensional and Frobenius, but if $n \geq 2$, there is **no Hopf structure.**

(b) $\mathbb{C}[x,y]/(x^2,y^2,xy) = \mathbb{C}1+\mathbb{C}x+\mathbb{C}y$. This has the property that it is local (unique maximal submodule over itself) but A^* has two maximal submodules, so $A \not\cong A^*$, so not Frobenius.

Finally we conclude with a proof of lemma 7.0.6:

PROOF
Say that $\{a_i\}$ is a basis for A and $\{a_i^*\}$ is a basis for A^* ($a_i^*(a_j) = \delta_{ij}$).

Define $\rho(m) = \sum (a_i \cdot m) \otimes a_i^*$. A lemma (use change of basis matrices) shows that $\sum a_i \otimes a_i^*$ is independent of choice of basis. Another lemma says (scalar) multiplication is associative if and only if ρ is coassociative. This gets us the forward direction.

For the reverse direction, say $\rho(m) = \sum m_0 \otimes m_{-1}$. then define $a \cdot m = \sum m_1(a)m_0$.♠

8 January 23, 2019

Today we are interested in studying the representation category ${}_V\mathcal{M}$ for any Hopf algebra V. It will end up (and this is a bit cryptic for now) that k controls everything here.

9 Represetnations and Modules

Notice that for any Hopf algebra V we have the exact sequence:

$$0 \to \ker\varepsilon \to V \xrightarrow{\varepsilon} k \to 0$$

9.0.1 Definition
The trivial V modules is $V/\ker\varepsilon = k \in_V \mathcal{M}_V$.

Example 9.1
$V = kG$ where G is a group, $\varepsilon(g) = 1$ for all $g \in G$. Then $\ker\varepsilon = \oplus k(g-1)$ and so

$$k = V/\oplus k(g-1).$$

Example 9.2
$V = k[x]$ where $\Delta(x) = 1 \otimes x + x \otimes 1$ and $\varepsilon(x) = 0$. Then $\ker\varepsilon = \oplus_{n\geq 1} kx^n$, whence

$$V/\ker\varepsilon = k[x]/(x).$$

Example 9.3
Let $V = (kG)^* = \oplus_G k\delta_g$ and $\varepsilon(\delta_g) = \delta_{1,1_G}$. THen $\ker\varepsilon = \oplus_{g\neq 1_G} k\delta_g$ and

$$V/\ker\varepsilon = k\delta_1$$

The following uses T_4, the example of the day I missed due to my meeting.

Example 9.4
T_4: $\varepsilon(g) = 1$, $\varepsilon(p) = 0$. Then $\ker\varepsilon = k(g-1) \oplus kp \oplus kpg$ and

$$V/\ker\varepsilon = k.$$

9.0.2 Lemma

- For all $x \in V$ and $a \in k$ (the trivial module), $xa = \varepsilon(x)a$.
- k is the identity object in ${}_V\mathcal{M}$.
- If V is cocommutative, then ${}_V\mathcal{M}$ is a symmetric tensor category.

PROOF

$$x \cdot a = (x - \varepsilon(x))a + \varepsilon(x)a = \varepsilon(x)a$$

since $x - \varepsilon(x) \in \ker \varepsilon$.

For the next part, consider that for every $M \in_V \mathcal{M}$, we have

$$M \otimes k \cong M \cong k \otimes M.$$

♠

But then for all $x \in V$, we have

$$\begin{aligned}\varphi[x\dot(m \otimes 1)] &= \varphi(x_1 \cdot m \otimes x_2 \cdot 1)\\ &= \varphi(x_1 \cdot m \otimes \varepsilon(x_2) \cdot 1)\\ &= \varphi((x_1\varepsilon(x_2))m \otimes 1)\\ &= \varphi(xm \otimes 1) = xm = x\varphi(m \otimes 1).\end{aligned}$$

9.1 Integrals

9.1.1 Definition

An element $x \in V$ is called a **left integral** if $vx = \varepsilon(v)x$ for all $v \in V$. Similar for **right integral.**

9.1.2 Lemma

The following are equivalent:

- $x \in V$ is a left integral.
- $kx \cong$ the trivial module.
- $1 \mapsto x$ defines a left V-module morphism $k \to kx$.

9.1.3 Lemma

Let $\int_V^l$ denote the set of left integrals in V.

- $\int_V^l$ forms a vector subspace of V.
- $\int_V^l$ forms a left V submodule of V. Thus it forms a left ideal of V.
- $\int_V^l$ forms a right ideal of V.
- $\int_V^l \cong \mathrm{Hom}_V(k, V)$ (left V-mods)

Proof

The first three are relatively obvious. The last one comes from the map $\Phi : x \in \int_V^l \mapsto f_x : 1 \to x$. ♠

9.1.4 Remark: Unfortunately it ends up that the collection above ends up usually being zero.

9.1.5 Theorem

Let V be a finite dimensional Hopf algebra. Then $\dim \int_V^l = 1$.

Proof

$$\begin{aligned}\dim \int_V^l &= \dim \mathrm{Hom}_V({}_V k, {}_V V) \\ &= \dim \mathrm{Hom}_V((V^*)_V, k_V^*) \\ &= \dim \mathrm{Hom}_V(V_V, k) = \dim k = 1\end{aligned}$$

Where we used above that V was finite dimensional whence Frobenius to get $V^* \cong V$. ♠

Example 9.5

$V = kG$ for G a finite group. Then $\int_V^l = \sum_G g$ and

$$v \cdot (\sum_G g) = \varepsilon(v)(\sum_G g) = \int_V^r$$

and when $v = h \in G$, we can compute it.

Example 9.6

When $V = k[x]$, which is infinite dimensional, we get no integrals. This is because it is an integral domain, so if

$$x \cdot \int_V^l = \varepsilon(x) \int_V^l = 0 \cdot \int_V^l = 0$$

then this forces $\int_V^l = 0$.

Example 9.7
When $V = (kG)^*$ for G a finite group, then

$$\int_V^l = \int_V^r = \delta_{1_G}$$

Example 9.8
When $V = T_4 = k1 \oplus kg \oplus kp \oplus kgp$,

$$\int_V^l = (g+1)p \neq \int_V^r = p(1+g) = (1-g)p$$

Example 9.9
When $\operatorname{char} k = p$ and $V = k[x]/(x^p)$, then

$$\int_V^l = \int_V^r = x^{p-1}.$$

10 January 28, 2019

Today we will be mostly doing things in homological algebra. :)

Example 10.1
An example of an infinite dimension Hopf algebra is of the following: fix $q \in k^*$. Then

$$T_\infty = k < g, g^{-1}, p > /\langle gg^{-1} = g^{-1}g = 1, gp = qpg\rangle$$

and we can find $\{g^i p^j : i \in \mathbb{Z}, j \in \mathbb{N}\}$ is a k-linear basis.

For the coalgebra structure, define $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$ and $\Delta(p) = p \otimes 1 + g \otimes p$ and $\varepsilon(p) = 0$. Finally $S(g) = g^{-1}$ and $S(p) = q^{-1}pg^{-1}$.

10.1 Some nice results in Homological algebra

Problem 10.1
Prove that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

10.1.1 Theorem (Auslander-Buchsbaum '59)
Every local commutative algebra with finite global dimension is a UFD.

Another result is about algebraic groups:

10.1.2 Theorem
If $\operatorname{char} k = 0$, then every Noetherian commutative Hopf algebra has finite global dimension and is a direct sum of integral domains, each of which is isomorphic.

10.1.3 Definition (Semiprime)
An ideal $I \lhd R$ is called **semiprime** if it is the intersection of (possibly infinitely many) primes.

Conjecture: Every Noetherian Hopf algebra of finite global dimension is semiprime (that is, 0 is semiprime in V).

10.1.4 Definition (Connected Graded)
An algebra A is called **connected graded** if

$$A = k \oplus A_1 \oplus A_2 \oplus \cdots$$

and $1_A = 1_k \in k$, $A_i A_j \subseteq A_{i+j}$.

The new part (connected) refers to the fact that k is a summand.

10.1.5 Theorem
Suppose that $\operatorname{char} k = 0$. Let V be a Noetherian Hopf algebra that is connected graded as an algebra (says nothing about the coalgebra structure). Then the following hold:

- V has finite global dimension.
- V is a domain.
- V is Artin-Schelter regular.
- V is Auslander regular and Cohen-Macaulay.
- V is Calabi-Yau
- V is an iterated Ore extension.

That's a lot of word salad.

10.2 Projective Modules

10.2.1 Definition (Module-Theoretic Definitions)
Let M be a (left) A module.

(a) M is **free** if $M \cong \oplus_{\mathcal{I}} A$

(b) M is **projective** if it is a direct summand of a free module.

10.2.2 Lemma
$p \in {}_A\mathcal{M}$ is projective if and only if the diagram in figure 7 commutes.

$$\begin{tikzcd} & P \arrow[dl, dashed, "F"'] \arrow[d, "f"] & \\ M \arrow[r, "\pi"] & N \arrow[r] & 0 \end{tikzcd}$$

Figure 7: The definition of a projective object.

10.2.3 Remark: This actually is just the definition for any Abelian category. This may generalize even further, but this is enough for us. :)

10.2.4 Definition (Injective Module)
An **injective** V-module is a projective V^{op} module.

11 January 30, 2019

Today we're talking more about homological algebra. In particular, we'll learn about (or see again)

- Complexes
- Projective resolutions and dimension

First the example of the day:

Example 11.1
Consider $GL_2(k)$, a group – in fact, an algebraic group! Then

$$\mathcal{O}(GL_2) = k[x_{11}, x_{12}, x_{21}, x_{22}, det^{-1}]/\langle det^{-1}(x_{11}x_{22} - x_{21}x_{12} - 1\rangle$$

Let $X = (x_{ij}) \in GL_2(k)$.

Define the coalgebra structure via $\Delta(X) = X \otimes X$ and $\Delta(x_{ij}) = \sum_1^2 x_{is} \otimes x_{sj}$, $\varepsilon(X) = I_2$, $\varepsilon(x_{ij}) = \delta_{ij}$. Some computation (this is not trivial!) give us

$$\Delta(det) = det \otimes det \quad \Rightarrow \quad \Delta(det^{-1}) = det^{-1} \otimes det^{-1}$$

Then we can find that $S(X) = X^{-1}$ and $S(det^{\pm}) = det^{\mp}$.

Problem 11.1
Construct $\mathcal{O}(GL_n)$.

Problem 11.2
Construct $\mathcal{O}(G)$.

11.1 Homological stuff

11.1.1 Definition (Complex)
A **complex** of A-modules is a sequence of A-modules connected by homomorphisms

$$\cdots \to M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \to \cdots$$

where $d_i \circ d_{i+1} = 0$ for all i.

Then we went over the standard definitions for:

- Homology
- (Short) Exact sequences
- Projective resolutions
- Projective dimension

11.1.2 Definition (Global (Homological) Dimension)
The (left) **global dimension** is

$$\operatorname{gldim} A = \max\{\operatorname{projdim} M | M \in {}_A\mathcal{M}\}$$

11.1.3 Theorem (Hilbert Syzygy Theorem)

$$\operatorname{gldim} k[x_1, \ldots, x_n] = n$$

11.1.4 Corollary

$$\operatorname{gldim} k = 0$$

11.1.5 Definition
The **finitistic global dimension** of A is defined to be

$$\operatorname{findim} A = \max\{\operatorname{projdim} M | M \in {}_A\mathcal{M}, M \text{ is f.g.}, \operatorname{projdim} M < \infty\}$$

Conjecture (Finitistic Dimension Conjecture): Let A be a finitely generated algebra. Then $\operatorname{findim} A < \infty$.

12 February 1, 2019

A really cool theorem:

12.0.1 Theorem (Lorenz-Lorenz '95)
Let V be a Hopf algebra. Then $\operatorname{gldim} V = \operatorname{projdim}_V k$.

Proof
It is enough to show that $\operatorname{gldim} V \leq \operatorname{projdim}_V k$. Or equivalently that

$$\operatorname{projdim} M \leq \operatorname{projdim} k, \forall M \in {}_V\mathcal{M}$$

but then

$$\operatorname{projdim}(M) = \operatorname{projdim}(k \otimes M) \leq \operatorname{projdim}(k)$$

by lemma 12.1.6 ♠

Example 12.1
Today's example: Quantum group $V = \mathcal{O}_q(GL_2)$. Fix some $q \in k^*$. Then as an algebra,
$$V = k\langle x_{11}, x_{12}, x_{21}, x_{22}, det_q^{-1}\rangle/(\text{relations})$$
where the relations are given by
$$\begin{aligned} x_{ij}x_{kl} = (\delta_{ik}+\delta_{jl})x_{kl}x_{ij} \quad &\text{except when } i=k \text{ and } j=l \\ x_{12}x_{21} &= x_{21}x_{12} \\ x_{22}x_{11} - x_{11}x_{22} &= (q-q^{-1})X_{12}x_{21} \\ det_q^{-1}det_q &= det_q det_q^{-1} = 1 \end{aligned}$$
where $det_q = x_{22}x_{11} - q^{-1}x_{12}x_{21}x_{11}x_{22} - qx_{12}x_{21}$. One can show det_q is central in V.

Now for the coalgebra: $\delta(X) = X \otimes X$, so $\Delta(x_{ij}) = \sum_{s=1}^2 x_{is} \otimes x_{sj}$, $\varepsilon(X) = I_2$ and
$$S(X) = X^{-1} = \begin{pmatrix} x_{22}det_q^{-1} & -qx_{12}det_q^{-1} \\ -q^{-1}x_{21}det_q^{-1} & x_{11}det_q^{-1} \end{pmatrix}$$

Problem 12.1
Show that $\operatorname{gldim}\left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{smallmatrix}\right) = 2$ *while* $\operatorname{gldim}\left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{smallmatrix}\right)^{op} = 1$

12.1 Returning to Homological Algebra

Let A be an algebra, $M \in {}_A\mathcal{M}$. Recall the definitions of projective and global dimension.

12.1.1 Remark: The following are equivalent:

(a) $\operatorname{gldim} A = 0$

(b) $\operatorname{gldim} A^{op} = 0$

(c) A is semisimple Artininan

(d) $A = \oplus_1^b M_{n_i}(D_i)$ by Artin-Wedderburn where D_i is a division ring.

12.1.2 Remark: (a) A PID has global dimension 1.

(b) The free algebra $k\langle x_1, \dots, x_n\rangle$ has global dimension 1.

(c) The path algebra of a finite quiver has global dimension 1.

(d) $\mathcal{O}_q(GL_2)$ has global dimension 4.

From now on, let V always stand for some Hopf algebra.

12.1.3 Lemma

${}_V\mathcal{M}$ is a tensor category. The action on $M \otimes N$ is the following:

$$v \cdot (m \otimes n) = \sum (v_{(1)} \cdot m) \otimes (v_{(2)} \cdot n)$$

12.1.4 Lemma

If $M \in {}_V\mathcal{M}$, then $V \otimes M \in {}_V^V\mathcal{M}$. As a corollary,

$$V \otimes M \cong V \otimes (V \otimes M)^{cov}$$

and as a consequence $V \otimes M$ is a free V-module.

12.1.5 Theorem

Let $P \in {}_V\mathcal{M}$ be projective. Then $P \otimes N \in {}_V\mathcal{M}$ is projective for all $M \in {}_V\mathcal{M}$

Proof

To see this, notice that P is a summand of $V^{\oplus b}$. But then if $P \oplus Q \cong V^b$, then

$$[P \otimes N] \oplus [Q \otimes N] = [P \oplus Q] \otimes N = (V^b) \otimes N = (V \otimes N)^b$$

and by the last lemma, this is free. ♠

12.1.6 Lemma

$$\operatorname{projdim}(M \otimes N) \leq \min\{\operatorname{projdim} M, \operatorname{projdim} N\}$$

Proof

We only show that $\operatorname{projdim} M \otimes N \leq \operatorname{projdim} M$. If $\operatorname{projdim} M = \infty$, we are done. If it is finite, take any minimal projective resolution of M and tensor with N. This is a projective resolution of $M \otimes N$ of length at most $\operatorname{projdim} M$. ♠

12.1.7 Remark: The fact it is still exact holds because we can appeal to the vector space structure.

Some more results (mostly for finite dimensional algebras):

12.1.8 Theorem (Radford '75)

If V is finite dimensional, then the antipode S has finite order:

$$S^d = \mathrm{id}_V$$

12.1.9 Theorem (Larson-Radford '88)

Suppose $\operatorname{char} k \neq 0$ (can be relaxed but gets more ugly). Then the following are equivalent:

(a) $\operatorname{gldim} V = 0$ (V is semisimple Artinian)

(b) $\operatorname{gldim} V^* = 0$

(c) $S^2 = \mathrm{id}_V$

12.1.10 Theorem (Nichols-Zoeller '89)

Let V be finite dimensional. If Q is a Hopf subalgebra of V, then ${}_W V$ and V_W are free.

13 February 6, 2019

First some history: in 1899, Maschke proves the following:

13.0.1 Theorem
Let G be a finite group. Then kG is semisimple (Artinian) if and only if $\operatorname{char} k \nmid |G|$.

Today, we will see an analog of Maschke for Hopf algebras:

13.0.2 Theorem ("Maschke's Theorem" (Larson-Sweedler '69))
Let V be a finite-dimensional Hopf algebra. Then the following are equivalent:

(a) $\operatorname{gldim} V = 0$ (V is semisimple Artinian)

(b) $\varepsilon(\int^l) \neq 0$

(c) $\varepsilon(\int^r) \neq 0$

Now say $V = kG$. Then $\int = \sum_{g\in G} g$ is both a left and right integral of V. This uses that $\varepsilon(g) = 1$.

But then if $\varepsilon(\int) = \sum_G \varepsilon(g) = \sum_G 1 = |G| \neq 0 \Leftrightarrow \operatorname{char} k \nmid |G|$.

13.1 Example of the Day

Example 13.1 (Kac-Paljutkin Algebra)
$k = \mathbb{C}$ or any field with $\operatorname{char} k \neq 2$. Then $V = k\langle x, y, z\rangle$ modulo the relations

$$\begin{aligned} x^2 = y^2 &= 1 \\ xy &= yx \\ xz &= yz \\ xy &= xz \\ x^2 &= \frac{1}{2}(1 + x + y - xy) \end{aligned}$$

Then V has a k-basis $\{1, x, y, z, xy, xz, yz, xyz\}$, so as an algebra $V \cong k^4 \oplus M_2(k)$.

Then $\Delta(x) = x \otimes x$, $\Delta(y) = y \otimes y$, $\Delta(z) = (1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z)$.

And $\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1$ and $S(x) = x$, $S(y) = y$ and $S(z) = z$. Recall that although S looks like the identity map, S is an **antihomomophism**. So actually $S^2 = \mathrm{id}_V$.

Claim:

$$\int = 1 + x + y + z + xy + xz + yz + xyz$$

then $\varepsilon(\int) = 8 \neq 0$, so V is semisimple.

Some nice facts:

(a) $V^* \cong V$

(b) V is the unique eight dimensional noncommutative, noncocommutative semisimple Hopf algebra.

(c) Let Q_8 be the quaternion group. Then kQ_8 is eight dimensional. Therefore $V \cong kQ_8$ as algebras, but **not as coalgebras.** The reason behind this is that kG is *always* cocommutative.

13.2 HW of the day

Problem 13.1
If V is eight dimensional and noncommutative, then $V \cong k^4 \oplus M_2(k)$ as an algebra. Thus if G is nonabelian group of order 8, then $\mathbb{C}G \cong \mathbb{C}^4 \oplus M_2(\mathbb{C})$.

13.3 Proof of "Maschke's theorem"

Now we actually do the proof:

Proof
(a) $\Rightarrow$ (b): Say the global dimension of V is zero. Then V is semisimple by Artin-Wedderburn and we have a decompositon

$$V \cong \oplus_1^s M_{n_i}(D_i)$$

and every minimal nonzero ideal has the form $M_{n_i}(D_i)$.

Consider the ideal $k \int^l = I$ which is an ideal since $v \int^l = \varepsilon(v) \int^l \in I$. But then consider that if $v' \int^l \in I$,

$$v' \int^l v = \varepsilon(v') \int^l v \in k \int^l$$

so in fact it is a two-sided ideal. Thus $k \int^l \cong M_{n_i}(D_i)$, so

$$(k \int^l)^2 = k \int^l \varepsilon(\int^l) \int^l = (\int^l)^2 \qquad = a \int^l$$

for some nonzero a. But then $\varepsilon(\int^l) \neq 0$.

(b) $\Rightarrow$ (a): Now let $e = \frac{1}{\varepsilon(\int^l)} \int^l$. Then $e^2 = e$ and $ke = k \int^l$ is a left V-module. But then $V = Ve \oplus V(1-e)$ so $ke = k \int^l$ is projective, so $\operatorname{projdim}_v k = 0$.

The last part follows by Lorenz-Lorenz – $\operatorname{gldim} V = \operatorname{projdim}_V k$. ♠

13.4 Ext groups

We define these in the usual way through projective resolutions:

Let A be an algebra. Then ${}_A\mathcal{M}$ is an abelian cateogry, so $\mathrm{Hom}_A(M, N) \in \mathrm{Vect}_k$.

13.4.1 Definition
A (covariant) functor $F : {}_A\mathcal{M} \to {}_B\mathcal{M}$ is called **left exact** if, for any short exact sequence

$$0 \to L \to M \to N \to 0$$

we have the exact sequence

$$0 \to F(L) \to F(M) \to F(N).$$

Similar for right exact. If both, it's just exact.

13.4.2 Lemma
For any $W \in {}_A\mathcal{M}$, the functor $\mathrm{Hom}(W, -)$ is left exact.

We also have that $\mathrm{Hom}_A(P, -)$ is exact iff P is projective, so this gives us a nice characterization of projective objects in Abelian categories. Then the category $\mathbf{Ch}({}_A\mathcal{M})$ is the category of chain complexes in A modules with chain maps.

13.4.3 Lemma
$\mathbf{Ch}({}_A\mathcal{M})$ is an Abelian category.

13.4.4 Lemma
A functor $F : {}_A\mathcal{M} \to {}_B\mathcal{M}$ can be extended to a functor between the appropriate chain complex categories.

13.5 Looking Forward

On Friday, we may see the following result:

13.5.1 Theorem
Under some mild hypothesis (including finite dimensionality),

$$\mathrm{projdim}_V\, k = \mathrm{gldim}\, V$$