Notes and Problems from My Research

Nico Courts

1 Autumn 2018

1.1 Problems

Problem 1.1.1. *Assume that k is a field and let $K = k(t)$ (notice K is a transcendental extension). Prove that $\mathrm{Hom}_k(K, k) \not\cong K$.*

Solution. This is basically just a cardinality argument. I don't think it's particularly worth doing at this juncture.

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Problem 1.1.2. *Let G be a finite group scheme (actually we need only assume that G is a Frobenius algebra so that a module is injective if and only if it is projective). Prove that unless M is projective, its projective dimension is infinite. Conclude that $H^n(G, M) = 0$ for $n > N$ implies that M is projective.*

Solution. Assume M itself is not projective so that its minimal projective resolution is nontrivial and furthermore that it is finite. That is, let P_i be projective modules such that

$$0 \to P_n \xrightarrow{f_n} P_{n-1} \to \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

is a minimal length projective resolution of M (notice here that $n \geq 1$).

Next consider the short exact sequence

$$0 \to P_n \xrightarrow{f_n} P_{n-1} \to \operatorname{coker} f_n \to 0$$

since P_n is projective (and thus injective!) this sequence splits and therefore $P_{n-1} \cong P_n \oplus \operatorname{coker} f_n$. But then consider the sequence

$$0 \to P_n \xrightarrow{g} P_{n-2} \to \cdots \xrightarrow{f_0} M \to 0$$

where above we are using $P_{n-1} \supseteq P_n \cong f_n(P_n)$ and that $g = f_{n-1}|_{f_n(P_n)}$. This map is injective since $\ker f_{n-1} = \operatorname{coker} f_n$, which is disjoint from $f_n(P_n) \cong P_n$. Exactness everywhere else is evident since the maps are not effectively changed.

But then the existence of this sequence contradicts the minimality of the original sequence, so no finite sequence can exist.

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Problem 1.1.3. *Establish the five-term exact sequence for spectral sequences.*

Solution. I plan to return to this problem in the future. I have other priorities at the moment, but I will eventually return to cohomology and spectral sequences and this will be a good exercise at that point.

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2 Winter 2019

2.1 Preparation: Waterhouse and Görtz & Weddhorn

Problems I worked from Waterhouse can be found in the appropriate file.