

# Algebraic Groups

A course by Jarod Alper and Julia Pevtsova  
Notes by Nico Courts

Autumn 2019/ Winter and Spring 2020

## Abstract

The topic of algebraic groups is a rich subject combining both group-theoretic and algebro-geometric-theoretic techniques. Examples include the general linear group  $GL_n$ , the special orthogonal group  $SO_n$  or the symplectic group  $Sp_n$ . Algebraic groups play an important role in algebraic geometry, representation theory and number theory.

In this course, we will take the functorial approach to the study of linear algebraic groups (more generally, affine group schemes) equivalent to the study of Hopf algebras. The classical view of an algebraic group as a variety will come up as a special case of a smooth algebraic group scheme. Our algebraic approach will be independent (even complementary) to the analytic approach taken in the course on Lie groups.

## 1 September 25, 2019

### 1.1 Group objects

Let  $\mathcal{C}$  be a category with a final object and finite products.

**1.1.1 Definition:** A **group object**  $G$  in  $\mathcal{C}$  is an object in  $\mathcal{C}$  along with multiplication, identity, and inverse morphisms satisfying the usual axioms.

One thing is that we are using that there is a final object  $*$  along with our identity morphism  $e : * \rightarrow G$ . Here Jarrod explicitly used the fact that there is a unique map to  $*$ .

#### Example 1.1

If  $\mathcal{C}$  is  $\mathbf{Set}$ , then  $G$  is a group. If  $\mathcal{C} = \mathbf{Top}$ , then  $G$  is a topological group, smooth manifolds give Lie groups, and finally (interesting to us):

**1.1.2 Definition:** Let  $S$  be a scheme and let  $\mathcal{C}$  be the category of schemes over  $S$ . Then a group object  $G$  in  $\mathcal{C}$  is a **group scheme over  $S$** .

When  $k$  is a field and  $\mathcal{C}$  is schemes of finite type over  $k$ , we get a group scheme of finite type over  $k$ . There is not a great consensus on what makes an **algebraic group**, but this is what we will use.

When we instead restrict to *affine schemes* we get an affine group scheme of finite type over  $k$ , or a **linear algebraic group**.

## 1.2 Examples

$\mathbb{G}_m = \text{Spec } k[t]_t$  is one.

If we consider the map  $f : \mathbb{G}_m \rightarrow \mathbb{G}_m$  which on the level of elements sends  $t \mapsto t^p$ , the kernel is

$$\mu_p = \ker(f) = \text{Spec } k[t]/(t^p - 1)$$

and that's great, but when  $\text{char } k = p$ , this causes the group scheme to be **unreduced**. This is (apparently) a case when you need to use schemes.

## 1.3 The Functorial Approach

Let  $\mathcal{C}$  be a category with object  $X$ . Define the functor  $h_X : \mathcal{C}^{op} \rightarrow \mathbf{Set}$  where

$$h_X(Y) = \text{Hom}_{\mathcal{C}}(Y, X).$$

Then we have

### 1.3.1 Lemma (Yoneda)

Let  $G : \mathcal{C}^{op} \rightarrow \mathbf{Set}$  be a functor. There is a natural bijection

$$G(X) \simeq \text{Nat}(h_X, G).$$

### 1.3.2 Proposition

A group object  $G$  in  $\mathcal{C}$  is the same as an object  $X \in \mathcal{C}$  together with a choice of factorization of  $h_X : \mathcal{C} \rightarrow \mathbf{Set}$  through **Grp**.

## 1.4 Exercises

- Spell out all the details of the proof of the above proposition.
- Given a group object  $G$ , define in two ways what it means for it to act on another object. (In coordinates and functorially).

## 1.5 Some Interesting Facts

If we had to write down five results that we'd like to get out of this class:

### 1.5.1 Proposition

Every affine group scheme of finite type over a field embeds into  $GL_n$  as a closed subgroup.

### 1.5.2 Theorem (Chevalley's Theorem)

Let  $G$  be a finite type group scheme over a field. Then it factors as

$$1 \rightarrow H \rightarrow G \rightarrow A \rightarrow 1$$

where  $A$  is abelian and  $H$  is affine (linear algebraic).

### 1.5.3 Proposition

If  $G$  is an affine group scheme of finite type over  $k$ , then we have factorization

$$1 \rightarrow U \rightarrow G \rightarrow R \rightarrow 1$$

where  $U$  is unipotent and  $R$  is reductive.

### 1.5.4 Proposition

$H \subseteq G$  a subgroup scheme. Then  $G/H$  is a projective scheme.

Finally we want to talk about Tanakka duality and how the representations of  $G$  define  $G$  itself.

## 2 September 27th, 2019

Last time we defined a group scheme (a group object in the category of schemes over a base scheme). We also mentioned that You could define it as a map  $h_G : \mathbf{Sch}/S \rightarrow \mathbf{Set}$  along with a factorization through  $\mathbf{Grp}$ .

We defined an **algebraic group** over  $k$  as a group scheme over  $\mathrm{Spec} k$  of finite type and a **linear algebraic group** to be an *affine* group scheme over  $k$  of finite type.

### 2.1 Hopf Algebras

Let  $G = \mathrm{Spec} A$  be a linear algebraic group over  $k$ . I have seen most of these before (see Waterhouse or my Hopf algebra notes)

2.1.1 REMARK: One think I haven't seen explicitly before: Notice that the augmentation ideal  $\ker \varepsilon$ , where  $\varepsilon$  is the counit, is the (maximal!) ideal corresponding in the algebro-geometric sense to the identity element in  $G$ .

2.1.2 Definition: A Hopf algebra is ...

**2.1.3 Definition:** Let  $G$  be an algebraic group over  $k$ . Then if  $h_G$  factors through  $\mathbf{Ab}$ ,  $G$  is called **commutative**.

## 2.2 Some Examples

**2.2.1 REMARK:** Note that to define a functor from schemes over  $k$ , it suffices to define it on affine schemes, thereby defining the (Zariski) local behavior of any such map. Thus we really only need to consider maps in  $\mathbf{Alg}$ .

- $\mathbb{G}_a$ . Here we can define it as a functor that sends  $S \mapsto \Gamma(S, \mathcal{O}_S)$ . Geometrically,  $\mathbb{G}_a = \mathbb{A}^1$  where the multiplication is addition, inverses send  $x \mapsto -x$  and the unit is the zero map. The Hopf algebraic picture is the usual dual thing.
- $\mathbb{G}_m$  as a scheme is the map  $S \mapsto \Gamma(S, \mathcal{O}_S)^*$ . In the geometric picture,  $\mathbb{A}^1 \setminus \{0\}$  and the algebra structure comes from multiplication. Hopf is pretty easy.
- $\mathrm{GL}_n$  is a scheme that sends

$$S \mapsto \{A = (a_{ij}) : a_{ij} \in \Gamma(S, \mathcal{O}_S), \det(A) \in \Gamma(S, \mathcal{O}_S)^*\}$$

the algebra is  $\mathbb{A}^{n \times n} \setminus \{\det = 0\}$  with the usual multiplication. The coalgebra structure can be seen in the book.

This one requires some more explanation so I am setting it apart.

### Example 2.1

Let  $V$  be a finite dimensional vector space over  $k$ . Then we can define the algebraic group  $V_a$  which sends

$$S \mapsto \Gamma(S, \mathcal{O}_S) \otimes_k V.$$

Geometrically we are looking at  $\mathbb{A}(V) = \mathrm{Spec} \mathrm{Sym}^* V^\vee \simeq \mathrm{Spec} k[x_1, \dots, x_n]$  where  $n = \dim V$ .

What about finite groups? As a scheme, we want  $G = \bigsqcup_{g \in G} \mathrm{Spec} k$ . The functor sends  $S \mapsto \mathrm{Mor}_{\mathrm{Set}}(\pi_0(S), G)$ , or maps from the connected components into  $G$ .

**Example 2.2**

Now consider the  $n^{th}$  roots of unity: as a scheme,  $\mu_n = \text{Spec } k[t]/(t^n - 1) \subseteq \mathbb{G}_m$ . If both  $k = \bar{k}$  and  $\text{char } k \nmid n$ , then  $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$ .

But if (e.g.)  $k = \mathbb{Q}$ , then  $\mu_3$  is  $\mathbb{Q}[t]/(t^3 - 1) = \text{Spec } \mathbb{Q} \sqcup \text{Spec } \mathbb{Q}(\xi)$  where  $\xi$  is a primitive third root of unity.

If, on the other hand,  $k = \bar{\mathbb{F}}_3$  and consider  $\mu_3$ , we get a single point with residue field  $\bar{\mathbb{F}}_3$ .

**Example 2.3**

If we are in the case of positive characteristic, then we get an algebraic group  $\alpha_p$ . Here the scheme is  $\text{Spec } k[x]/x^p$  and functorially it is the map  $S \mapsto \{F \in \Gamma(S, \mathcal{O}_S) \mid f^p = 0\}$ .

## 2.3 Matrix Groups

We already defined  $\text{GL}_n$ , but we can also define

$$\text{SL}_n : S \mapsto \{A = (a_{ij}) \mid \det A = 1\}$$

with scheme  $\text{Spec } k[x_{ij}]/(\det - 1)$ .

We also have the (upper) triangular matrices  $T_n$  and unitary group  $U_n$  and diagonal group  $D_n$

**2.3.1 Definition:** Let  $G$  be a linear algebraic group. Then

- $G$  is a **vector group** if  $G \cong V_a$  for some finite dimensional  $V$ .
- $G$  is a **split torus** if  $G \cong \mathbb{G}_m^n$ .
- $G$  is a **torus** if there is a field extension  $k \rightarrow k'$  such that

$$G \times_{\text{Spec } k} \text{Spec } k' \cong \mathbb{G}_{m,k'}^n$$

## 3 September 30th, 2019

Another example to consider:

**Example 3.1**

Let  $G = \mathrm{PGL}_n$ , the projective linear group. Recall we want to define this as  $\mathrm{GL}_n/k^*$  (from group theory). To do this for algebraic groups, we define

$$\mathrm{PGL}_n = \mathrm{Proj} k[x_{ij}]_{det} := \mathrm{Spec}(k[x_{ij}]_{det})_0$$

The geometric picture is difficult since we haven't yet defined quotients, but as a functor we say  $\mathrm{PGL}_n$  is  $\mathrm{Aut}(\mathbb{P}^n)$ , the functor that sends  $S \mapsto \mathrm{Aut}(\mathbb{P}_S^n)$  where  $\mathbb{P}_S^n = \mathbb{P}_k^n \times_{\mathrm{Spec} k} S$ .

**3.1 Non-affine group schemes****Example 3.2**

Let  $\lambda \neq 0, 1$  be an element in  $k$ . Then we can define the elliptic curve

$$E_\lambda = V(y^2z - x(x-z)(x-\lambda z)) \subset \mathbb{P}^2$$

Which gives us a double cover over  $(0, 1)$  and  $(\lambda, \infty)$  with singleton fiber (ramified) over  $0, 1$ , and  $\lambda$ .

Then for any  $\lambda \neq 0, 1$ ,  $E_\lambda$  is a **projective** group scheme.

**3.1.1 REMARK:** If you look at the  $\mathbb{C}$ -points, you get  $E_\lambda(\mathbb{C}) = \Lambda_\lambda$ , giving you a torus. Recall (from e.g. complex analysis) that the moduli here is  $\mathrm{SL}_2(\mathbb{Z})$  of all elliptic curves.

**3.2 Abelian Varieties**

**3.2.1 Definition:** An **abelian variety over  $k$**  is a smooth, geometrically connected ( $A \times_{\mathrm{Spec} k} \mathrm{Spec} \bar{k}$  is connected), proper group scheme  $A$  over  $k$ .

**Example 3.3**

Over  $\mathbb{C}$ ,  $\mathbb{C}^g/\Lambda$  where  $\Lambda \cong \mathbb{Z}^{2g} \subseteq \mathbb{C}^g$  gives us a genus  $g$  example.

**3.2.2 Theorem**

Any abelian variety over  $k$  is commutative and projective.

### 3.2.3 Theorem (Chevalley)

If  $G$  is any group scheme, then the sequence

$$1 \rightarrow H \rightarrow G \rightarrow A \rightarrow 1$$

is exact, where  $H$  is a linear algebraic group (affine!) and  $A$  is an abelian variety.

#### Example 3.4

Let  $X \rightarrow \operatorname{Spec} k$  be a geometrically integral projective scheme (proper may suffice). The idea here is that over  $\mathbb{C}$  the rings over every open set are integral domains.

Now consider the **Picard functor**  $\operatorname{Pic}_X : \operatorname{Sch}/k \rightarrow \mathbf{Grp}$  sending

$$S \mapsto \operatorname{Pic}(X_S = X \times_k S) / p^k \operatorname{Pic}(S)$$

### 3.2.4 Theorem

$\operatorname{Pic}_X$  is represented by a scheme locally of finite type, thus  $\operatorname{Pic}_X^0$ , the connected component of the identity in  $[\mathcal{O}_X] \in \operatorname{Pic}_X$  is an abelian variety.

## 3.3 Relative Group Schemes

#### Example 3.5

Consider  $\mathbb{G}_{m,\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[t]_t$ . Then  $G_{m,S} = \mathbb{G}_{m,\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} S$ . In the case that  $S = \operatorname{Spec} R$ ,  $\mathbb{G}_{m,S} = \operatorname{Spec} R[t]_t$ .

#### Example 3.6

Let  $\mathbb{A}^1 = \operatorname{Spec} k[x]$  and define  $G = \operatorname{Spec} k[x, y]_{xy+1} \subseteq \mathbb{A}^2$ . Notice this is the plane minus a hyperbola.

Define  $\cdot : G \times_{\mathbb{A}^1} G \rightarrow G$  to be given by

$$(x, y) \cdot (x, y') = (x, xy y' + y + y')$$

Then the thing here is the fiber (think vertical line in the plane!) over 0 is  $\mathbb{G}_a$  and is isomorphic to  $\mathbb{G}_m$  otherwise.

**Example 3.7**

Let  $\mathcal{E}_\lambda = V(y^2z - x(x-z)(x-\lambda z))$  over  $\text{Spec } k[\lambda]$ . Then when  $\lambda = 0$ , we get the nodal cubic given by  $y^2z - x^2(x-z)$  (node at the origin).

Now if you look at the connected component around 0 of  $\text{Aut}(\mathcal{E}_\lambda)/\mathbb{A}_\lambda$ , you actually find (when  $\lambda = 0$ ) that  $\mathbb{G}_m \cong \text{Aut}(\mathcal{E}_0)^0$ .

**3.4 Some definitions**

**3.4.1 Definition:** A homomorphism  $\phi : G \rightarrow G$  of group schemes over  $S$  is a map  $\phi : H \rightarrow G$  of schemes such that

$$\begin{array}{ccc} H \times_S H & \xrightarrow{m_H} & H \\ \downarrow \phi \times \phi & & \downarrow \phi \\ G \times_S G & \xrightarrow{m_G} & G \end{array}$$

**Problem 3.1**

*Show that this automatically implies that the identity and inversion maps are respected as well (automatically).*

**3.4.2 Definition:** A subgroup of  $G \rightarrow S$  is a subscheme  $H \subseteq G$  such that  $H(T) \leq G(T)$  for all  $T$  over  $S$ .

**Problem 3.2**

*Show that  $\ker(\phi) \subseteq H$  is a subgroup.*

**3.4.3 Remark:** This gives you a nice way to construct new group schemes. For example, the following are exact:

$$1 \rightarrow \text{SL}_n \rightarrow \text{GL}_n \xrightarrow{\det} \mathbb{G}_m \rightarrow 1$$

and

$$1 \rightarrow \mathbb{G}_m \rightarrow \text{GL}_n \rightarrow \text{PGL}_n \rightarrow 1$$



### 3.4.4 Proposition

Let  $G \rightarrow S$  be a group scheme. Then  $G \rightarrow S$  is separated if and only if  $e : S \rightarrow G$  is a closed immersion.

PROOF

The idea here is that  $S \rightarrow G$  is a closed immersion. Then we consider the map  $m \circ (\text{id}, S) : G \times_S G \rightarrow G$  and consider this along with the diagonal map  $\Delta : G \rightarrow G \times_S G$  and this is a pullback square! ♠

### 3.4.5 Corollary

Any group scheme over  $k$  is separated.

The idea is going to be that if  $X$  is any scheme over  $k$ , then any point  $X \in X(k)$  is closed.

## 4 October 2, 2019

Notice that a **relative group scheme** (referred to in last lecture) refers to a groups scheme over an arbitrary base scheme  $S$ .

### 4.1 Properties of schemes

Today we are going to be talking about reducedness, connectedness, irreducibility, regularity, and smoothness.

Recall that a scheme  $X$  is **reduced** if and only if  $\forall x \in X$ ,  $\mathcal{O}_{X,x}$  is reduced. An example of a non-reduced scheme is  $\text{Spec } k[x]/(x^2)$ .

**4.1.1 Definition:** We say a scheme  $X$  over  $k$  is **geometrically reduced** if for all field extensions  $k'/k$ ,

$$X_{k'} = X \times_{\text{Spec } k} \text{Spec } k'$$

is reduced.

**4.1.2 Remark:** It is equivalent that  $X_{\bar{k}}$  is reduced if and only if every  $k'/k$  is purely inseparable (I think).

**4.1.3 Remark:** If  $k$  is perfect, then  $X$  is reduced if and only if  $X$  is geometrically reduced.

**4.1.4 Definition:** A local ring  $(A, \mathfrak{m})$  is **regular** if  $\dim_{\text{Krull}} A = \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$

**4.1.5 Definition:** A scheme  $X$  is regular if for all  $x \in X$ ,  $\mathcal{O}_{X,x}$  is regular.

4.1.6 REMARK: If  $X \rightarrow \operatorname{Spec} k$  and  $x \in X(k)$ , the tangent space at  $x$  is

$$T_{X,x} = (\mathfrak{m}/\mathfrak{m}^2)^\vee = \{f : \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \rightarrow X \mid 0 \mapsto x\}$$

4.1.7 REMARK: Notice that if  $X \rightarrow \operatorname{Spec} k$  is regular and  $k'/k$  is a field extension, then  $X_{k'}$  is not necessarily regular.

**4.1.8 Definition:** A Scheme  $X \rightarrow \operatorname{Spec} k$  of finite type is **smooth** if  $X_{\bar{k}}$  is regular.

## 4.2 Facts about algebraic groups

Then we can return to the proposition we want to prove:

### 4.2.1 Proposition

Let  $G \rightarrow \operatorname{Spec} k$  be an algebraic group. Then  $G$  is geometrically reduced if and only if  $G$  is smooth over  $\operatorname{Spec} k$ .

PROOF

Smoothness over  $k$  implies reducedness. Now since we are only interested in the algebraic closure of  $k$ , we can say  $k = \bar{k}$ . Because  $G$  is reduced, there exists a nonempty open  $U \subseteq G$  that is smooth. Then since  $G(k) \subseteq |G|$  is dense in  $G$  (as a topological space) and Then  $G = \bigcup_{g \in G(k)} m_g(U)$  for our smooth  $U$ , and this gives us a smooth cover of  $G$ . ♠

We will see next time that all linear algebraic groups over  $k$  where  $\operatorname{char} k = 0$  are all geometrically reduced (and thus smooth).

## 4.3 Connectedness

Let  $G$  be an algebraic group over  $k$ . Then we have our maps  $e : \operatorname{Spec} k \rightarrow G$ , so consider it as  $e \in G(k)$ . Let  $G^0 \subseteq G$  be the connected component of  $e$ . It is both open and closed.

4.3.1 REMARK: If  $X \rightarrow \operatorname{Spec} k$  is of finite type and  $x \in X(k)$ , then  $X$  being connected implies that  $X$  is geometrically connected.

This establishes that  $G^0$  is actually geometrically connected! We actually will see

### 4.3.2 Proposition

$G^0 \subseteq G$  is an (open and closed) algebraic subgroup.

The idea here is that  $G^0 \times G^0$  is connected, so the image of the multiplication map on this set lands in a connected component (since it is connected). Since  $e \in G^0$ , and  $m(e, e) = e \in G^0$ , this shows that the multiplication map restricts to a well-defined map  $G^0 \times G^0 \rightarrow G^0$ . A similar argument goes through for the inverset map, etc.

The upshot here is that if  $G$  is an algebraic group, then there exists a factorization

$$1 \rightarrow G^0 \rightarrow G \rightarrow \pi_0(G) \rightarrow 1$$

where  $\pi_0(G)$  is given the structure of a discrete group.

**4.3.3 REMARK:** Now we also have that  $(G^0)_{k'} = (G_{k'})^0$  for all  $k'/k$ . The idea is to get a map of one into the other and then use clopenness and connectedness to show they are equal.

#### 4.3.4 Proposition

A connected algebraic group over  $k$  is irreducible.

PROOF

We can assume  $k = \bar{k}$ . Suppose  $G = X \cup Y$ , where both are closed,  $X$  is irreducible, and  $X \cap Y \neq \emptyset$ . Thus there exists an element  $g \in X \setminus Y$ . That is,  $g$  lies in a single irreducible component.

But then using the multiplication by  $h$  map on  $G$ , we get to every other point in  $G$ , so every point is in a single irreducible component. But the intersection was nontrivial! Or something. ♠

#### 4.3.5 Proposition

If  $G_{\text{red}}$  is geometrically connected, then  $G_{\text{red}} \subseteq G$  is a subgroup. In particular, if  $k$  is perfect, then  $G_{\text{red}}$  is a subgroup of  $G$ .

**4.3.6 REMARK:**  $X$  is geometrically reduced implies that  $X \times X$  is geometrically reduced.

## 5 October 4, 2019

Some review. Let  $G$  be an algebraic group and denote  $e : \text{Spec } k \rightarrow G$  be the identity. We saw a lot of propositions last time.

Now let  $k$  be a nonperfect field and take  $t \in k \setminus k^p$ . Then define

$$G \stackrel{\text{def}}{=} V(x^{p^2} - tx^p) \subseteq \mathbb{G}_a$$

which Milne claims is not reduced. We can see why it's not geometrically reduced, but we're missing the details here.

### 5.1 Another special case

#### 5.1.1 Theorem

When  $k = \bar{k}$ ,  $G$  is smooth if and only if

$$\dim T_e G_{\text{red.}} = \dim T_e G.$$

5.1.2 REMARK: When  $G$  is smooth, it is reduced, so the equality is clear. For the other direction, we get that  $k$  is perfect, so  $G_{\text{red}}$  which is geometrically reduced if and only if  $G$  is smooth. But

$$\dim G \leq \dim T_e G = \dim T_e G_{\text{red}} = \dim G_{\text{red}} = \dim G$$

### 5.1.3 Theorem

If  $G$  is a linear algebraic group over  $k$  and  $\text{char } k = 0$ ,  $G$  is smooth,

PROOF

We can assume  $k = \bar{k}$ . Then set  $G = \text{Spec } A$  where  $A$  is a Hopf algebra. Then we get Hopf algebra maps  $m^*$  and  $e^*$ . Notice that the augmentation ideal  $\mathfrak{m} = \ker(e^*)$  is a maximal ideal.

Then we want to prove

(a)  $A \cong \mathfrak{m} \oplus k$  as a  $k$ -vector space (obvious).

(b)  $\forall a \in \mathfrak{m}, m^*(a) - a \otimes 1 - 1 \otimes a \in \mathfrak{m} \otimes \mathfrak{m}$ .

To see the second, notice that  $m^*(a) - a \otimes 1 - 1 \otimes a$  is in the kernel of

$$e^* \otimes \text{id} : A \otimes A \rightarrow k \otimes A.$$

This is clear from the commutative diagram

$$\begin{array}{ccc} k \otimes A & \xleftarrow{e^* \otimes \text{id}} & A \otimes A \\ & \nwarrow \sim & \uparrow m^* \\ & & A \end{array}$$

Then we conclude  $f \in \ker(e^* \otimes \text{id}) \cap \ker(\text{id} \otimes e^*) = A \otimes \mathfrak{m} \cap \mathfrak{m} \otimes A$  by a symmetric argument. Finally we notice that  $A \otimes \mathfrak{m} \cap \mathfrak{m} \otimes A = \mathfrak{m} \otimes \mathfrak{m}$ , and so  $f$  lies in this ideal.

Now we want to show that  $\dim T_e G = \dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim_k \mathfrak{m}/(\sqrt{0} + \mathfrak{m}^2) = \dim T_e G_{\text{red}}$ . It suffices to show that for all  $a \in \sqrt{0}, a \in \mathfrak{m}^2$ . Suppose the opposite—so let  $a \in \sqrt{0} \setminus \mathfrak{m}^2$ . Consider the diagram

$$\begin{array}{ccc} A & \longrightarrow & A_{\mathfrak{m}} \\ \downarrow & & \downarrow \\ A/\mathfrak{m}^2 & \xrightarrow{\sim} & A_{\mathfrak{m}}/(\mathfrak{m}A_{\mathfrak{m}})^2 \end{array}$$

Now the image of  $a$  in  $A_{\mathfrak{m}}$  is nonzero, so there exists  $n > 0$  such that  $a^n \in A_{\mathfrak{m}}$  but  $a^{n-1} \notin 0$  in  $A_{\mathfrak{m}}$ . Thus there exists  $f \notin \mathfrak{m}$  such that  $a^n f = 0 \in A$ . Substitute  $af$  for  $a$ , and thus there is an  $a \in \sqrt{0}$  such that  $a^n = 0$  in  $A$  but  $a^{n-1} \neq 0$  in  $A_{\mathfrak{m}}$ .

Then by fact 2,

$$m^*(a) = 1 \otimes a + a \otimes 1 + r, \quad r \in \mathfrak{m} \otimes \mathfrak{m}$$

and since  $m^*$  is a ring homomorphism,

$$0 = m^*(a^n) = (m^*(a))^n = (a \otimes 1 + (1 \otimes a + r))^n = a^n \otimes 1 + n(a^{n-1} \otimes a + (a^{n-1} \otimes 1)r) + X$$

where  $X \in A \otimes \mathfrak{m}^2$ . But since  $a^n = 0$ , we get

$$n(a^{n-1} \otimes a + (a^{n-1} \otimes 1)r) \in A \otimes \mathfrak{m}^2$$

Now since  $(a^{n-1} \otimes 1)r \in (a^{n-1})\mathfrak{m} \otimes A$ , so

$$n(a^{n-1} \otimes a) \in (a^{n-1})\mathfrak{m} \otimes A + A \otimes \mathfrak{m}^2$$

and since  $\text{char } k = 0$ , we get that  $n$  is a unit, so

$$a^{n-1} \otimes a \in (a^{n-1})\mathfrak{m} \otimes A + A \otimes \mathfrak{m}^2$$

Now since this lives in  $A \otimes A$ , consider the image of the quotient map  $A \otimes A \rightarrow A \otimes A / \mathfrak{m}^2$ . Then

$$a^{n-1} \otimes \bar{a} \in (a^{n-1})\mathfrak{m} \otimes A / \mathfrak{m}^2 \subseteq A \otimes A / \mathfrak{m}^2$$

And note that  $a^{n-1} \notin a^{n-1}\mathfrak{m}$  because otherwise  $a^{n-1} = a^{n-1}q$  for  $q \in \mathfrak{m}$ . Then  $a^{n-1}(1-q) = 0 \in A_{\mathfrak{m}}$ , which implies that  $a^{n-1} = 0 \in A_{\mathfrak{m}}$  (since  $1-q$  is a unit here).

Then somehow we get that  $\bar{a} = 0 \in A / \mathfrak{m}^2$ , so  $a \in \mathfrak{m}^2$ . ♠

## 6 October 7, 2019

Today we will be primarily concerned with

### 6.1 Group actions

Let  $G$  be an algebraic group over  $k$ .

**6.1.1 Definition:** A **group action** of  $G$  on a scheme  $X$  over  $k$  is the data of a morphism

$$\mu : G \times X \rightarrow X$$

such that the usual axioms hold. That is,

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{m \times \text{id}} & G \times X \\ \downarrow \text{id} \times \mu & & \downarrow \mu \\ G \times X & \xrightarrow{\mu} & X \end{array} \quad \begin{array}{ccc} \text{Spec } k \times x & \xrightarrow{e \times \text{id}} & G \times X \\ & \searrow \sim & \downarrow \mu \\ & & X \end{array}$$

6.1.2 REMARK: Apparently it was an exercise already to show that this is equivalent to an action of  $h_G$  on  $h_X$ .

6.1.3 REMARK: The map  $(g, x) \mapsto (g, gx)$  is an automorphism of  $G \times X$ , so if  $p_2 : G \times X \rightarrow X$  is projection,

$$\begin{array}{ccc} G \times X & \xrightarrow{\sim} & G \times X \\ & \searrow \mu & \swarrow p_2 \\ & X & \end{array}$$

commutes.

**6.1.4 Definition:** Let  $X$  and  $Y$  be schemes over  $k$  with a  $G$  action. Then a  **$G$ -equivariant morphism**  $f : X \rightarrow Y$  is one such that for all  $g \in G$ , the following commutes:

$$\begin{array}{ccc} G \times X & \xrightarrow{\text{id} \times f} & G \times Y \\ \downarrow \mu_X & & \downarrow \mu_Y \\ X & \xrightarrow{f} & Y \end{array}$$

### 6.1.1 Some examples

- $G$  actions on itself by multiplication or conjugation.
- $\mathbb{G}_m$  acts on  $\mathbb{A}^1$ . Geometrically, we are just looking at  $k^*$  acting on  $k$  by scaling. Algebraically, we want a map  $\mu_{\mathbb{G}_m} \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$  given by the map of algebras:

$$k[x] \xrightarrow{\mu^*} k[t]_t \otimes k[x] \quad \text{via} \quad x \mapsto tx$$

Functorially, if  $S$  is a scheme over  $k$ , then  $\mathbb{G}_m(S) = \Gamma(S, \mathcal{O}_S)^*$  which acts on  $\mathbb{A}^1(S) = \Gamma(S, \mathcal{O}_S)$ , again by scaling.

- You can consider  $\text{GL}_n$  action on  $\mathbb{A}^n$  by multiplication or on  $\mathbb{A}^{n \times n}$  via multiplication or conjugation.

### 6.1.2 Orbits and Stabilizers

Let  $G$  be an algebraic group over  $k$  action on a scheme  $X$  over  $k$ . Let  $x \in X(k)$ . Then we have a map

$$\mu_x : G \times \text{Spec } k \xrightarrow{\text{id} \times x} G \times X \xrightarrow{\mu} X$$

where

$$g \mapsto (g, x) \mapsto gx.$$

**6.1.5 Definition:** The **orbit** of  $x$  is  $Gx = \mu_x(G) \subseteq |X|$  set-theoretically. The **stabilizer** of  $x$  in  $G$  is  $G_x = \mu_x^{-1}(x) \subseteq G$ .

6.1.6 REMARK:  $G_x$  is always a closed algebraic subgroup of  $G$ .

### 6.1.7 Proposition

$\mu_x(G)$  is open in its closure in  $|X|$ .

Recall first the following:

### 6.1.8 Theorem (Chevalley's Theorem (different?))

If  $f : X \rightarrow Y$  is a map of schemes of finite type over  $k$ , then  $f(X) \subseteq Y$  is constructible (i.e. is a disjoint union of finitely many locally closed subsets).

*Recall that locally closed means closed in an open subspace.*

### 6.1.9 Corollary

*Maybe a definition:* The orbit  $Gx \subseteq \text{im}(\mu_x) \subseteq X$ . If  $G$  is reduced, then  $Gx$  is reduced.

### 6.1.3 Applications

Say that  $\text{char } k = p$ . Then  $\mu_p$  acts on  $\mathbb{G}_m$  by multiplication.

$\mathbb{G}_m$  acts on  $\mathbb{A}^1$  with two orbits:  $\mathbb{G}_m \cdot 1 = \{x \neq 0\}$  and  $\mathbb{G}_m \cdot 0 = \{0\}$ . The stabilizers are  $G_1 = 1$  and  $G_0 \cong \mathbb{G}_m$ .

Consider  $G$  acting on  $\mathbb{A}^2$  via  $t(x, y) = (tx, t^{-1}y)$ . Then the orbits are hyperbolas! There is also a notion of closed orbits that I didn't quite catch. Also apparently the orbit-stabilizer statement is easy to see in geometry via a fiber bundle  $G$  over  $Gx$  where the fiber over  $x$  is  $G_x$ .

### 6.1.10 Proposition

If  $\phi : G \rightarrow H$  is a homomorphism of algebraic groups, then  $\phi(G) \subseteq |H|$  is closed.

6.1.11 REMARK: The proof included reducing first to  $k = \bar{k}$ . The trick here is to consider the group action induced by  $\phi$  and then consider the map  $\mu_{e_H}$  of this action. Then  $\mu_{e_H}(G) = \phi(G)$  and one can prove that this is closed.

In particular, we have that **subgroups of an algebraic group are always closed**. Note that this stands in stark contrast to Lie theory where you get non-closed subgroups.

## 7 October 9, 2019

Recall that last time we were considering actions of algebraic groups on schemes of finite type over  $k$ . We discussed the orbit and stabilizer of an element  $x \in X$  and showed that  $G \cdot x$  is open in its closure. We also saw that  $G_x$  is a closed subgroup.

We also say that  $\phi(H)$  (as a set) is always closed! None of these facts are true for Lie groups or relative group schemes (the base scheme is not  $\text{Spec } k$  for  $k$  a field).

### 7.1 Cartier Duality

Let  $G \rightarrow \text{Spec } k$  be a **finite** group scheme (so  $G = \text{Spec } A$  and  $A$  is a finite dimensional Hopf algebra). Some examples of finite group schemes are:

- $G$  a finite group. Then  $G = \sqcup_{g \in G} \text{Spec } k = \text{Spec} \left( \prod_{g \in G} k \right)$
- $\mu_n = \text{Spec } k[t]/(t^n - 1)$
- $\text{char } k = p$  and  $\alpha_p = \text{Spec } k[t]/t^p$

7.1.1 REMARK: Recall all the maps and diagrams that  $A$  has as a Hopf algebra.

A question one may ask: what if we apply the idea of dualizing  $(-)^{\vee} = \text{Hom}_{\text{Alg}}(-, k)$  to  $A$ ? Do we get another Hopf algebra?

The short and sweet of it is yes! But notice that we are coming from the commutative world, so we expect  $A$  to be commutative. But in general,  $A$  is not cocommutative (in fact, it is if and only if  $G$  itself was commutative as a group).

Thus  $A^{\vee}$  is indeed a (cocommutative) Hopf algebra, and when  $G$  is commutative,  $A^{\vee}$  is as well. So

#### 7.1.2 Proposition

If  $G = \text{Spec } A$  is a commutative group scheme, then the Cartier dual  $G^D = \text{Spec } A^{\vee}$  is a commutative group scheme as well.

7.1.3 REMARK: The above observations gives us an anti-autoequivalence of the category of commutative affine group schemes. Furthermore  $(G^D)^D = G$ .

#### Example 7.1

Consider  $\mu_n = \text{Spec } A = \text{Spec } \bigoplus_{i=0}^{n-1} k \cdot t^i$ . So then if we let  $\{e_i\}$  be the basis for  $A^{\vee}$  dual to  $\{t_i\}$ , we can compute comultiplication

$$e_i \mapsto \sum_{j=0}^{n-1} e_j \otimes e_{i-j}$$



and multiplication

$$e_i \otimes e_j \mapsto \delta_{ij} e_i$$

Then it can be shown that  $G^D \cong \mathbb{Z}/n\mathbb{Z}$ .

Now given  $G$ , an algebraic group over  $k$ , define

$$\underline{\mathrm{Hom}}(G, \mathbb{G}_m) : \mathrm{Sch}/k \rightarrow \mathrm{Set}$$

which takes

$$T \mapsto \mathrm{Hom}_{\mathrm{AlgGrp}}(G_T, \mathbb{G}_{mT}).$$

#### 7.1.4 Theorem

If  $G$  is a commutative finite group scheme over  $k$ , then

$$G^D \cong \underline{\mathrm{Hom}}(G, \mathbb{G}_m)$$

Let  $H = \mathrm{Spec} B \rightarrow \mathrm{Spec} R$  be a group scheme. Then

$$H_{\mathrm{GrpSch}/R}(H, \mathbb{G}_{mR}) \subseteq \mathrm{Mor}_{\mathrm{Sch}/R}(H, \mathbb{G}_{mR})$$

But the left hand side is equivalent to the grouplike elements of  $B$  and the right hand side is equivalent to  $\mathrm{Hom}_{\mathrm{Alg}_R}(R[t]_t, B)$ .

This leads to a proof of thm. 7.1.4:

PROOF

Let  $G = \mathrm{Spec} A$  and  $G^D = \mathrm{Spec} A^\vee$ . First look at the  $k$ -points:

$$\begin{aligned} G^D(k) &= \mathrm{Mor}_{\mathrm{Sch}/k}(\mathrm{Spec} k, G^D) \\ &= \mathrm{Hom}_{\mathrm{Alg}_k}(A^\vee, k) = \{f \in A \mid m^*(f) = f \otimes f\} \hookrightarrow \mathrm{Hom}_k(A^\vee, k) \\ &= \mathrm{Hom}(G, \mathbb{G}_m) \\ &= \underline{\mathrm{Hom}}(G, \mathbb{G}_m)(k) \end{aligned}$$

If we then look at  $R$  points for a general  $R$ , most things just change over, but we see

$$\{f \in A \otimes R \mid m_R^*(f) = R \otimes R\} = \mathrm{Hom}_{\mathrm{Alg}_k}(A^\vee, R) = \mathrm{Hom}(G_R, \mathbb{G}_m)$$

and the rest follows. ♠

A question one may ask: what is  $\mathrm{Hom}_{\mathrm{AlgGrp}}(\mathbb{G}_m, \mathbb{G}_m)$ ? It ends up it is  $\mathbb{Z}$ . You can send  $t \mapsto t^n$  for all  $n \in \mathbb{Z}$ . But then  $\underline{\mathrm{Hom}}(\mathbb{G}_m, \mathbb{G}_m)$  is  $\mathbb{Z}$  as a group scheme over  $k$ , which is not quasicompact. There was more but I am le tired.

Today we're doing problems and stuff. Forgot about that.

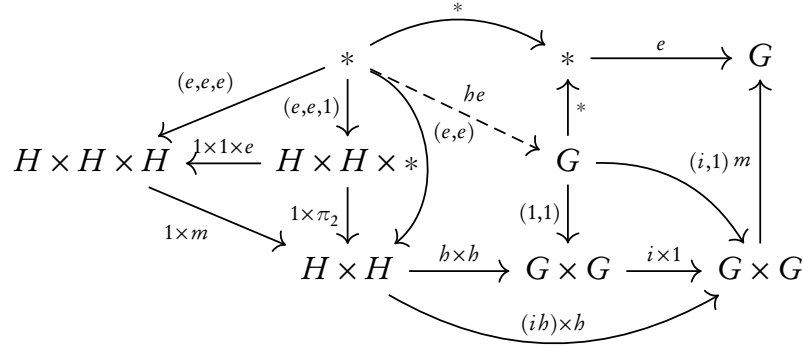
Let  $G$  and  $H$  be objects in a category  $\mathcal{C}$  with finite products. Let  $h : H \rightarrow G$  be a group homomorphism. That is,

Then we get similar diagrams for the identity and inverse maps (they are respected by  $h$ ). Then there is a bunch of diagram work. It's too hard to do a diagram without knowing the shape ahead of time.

Oh hey I used Adam's site!

$$\begin{array}{ccc}
 & & * \\
 & \searrow^{(e,e,e)} & \\
 H \times H \times H & \xrightarrow{(i_g \circ h) \times h \times h} & G \times G \times G \\
 \downarrow 1 \times m_H & & \downarrow 1 \times m_g \\
 H \times H & \xrightarrow{(i_G \circ h) \times h} & G \times G \\
 & & \searrow m_G \\
 & & G
 \end{array}$$

$$\begin{array}{ccccc}
 * & \xrightarrow{e} & H & & \\
 & \searrow (e,e,e) & & \searrow (h,h) & \\
 & & H \times H \times H & & G \times G \\
 & & \downarrow (i_g \circ h) \times h \times h & \swarrow (*,1) \times 1 & \searrow * \times 1 \\
 & & G \times G \times G & & * \times G \\
 & & \downarrow 1 \times m & \searrow m \times 1 & \swarrow e \times 1 \\
 & & G \times G & & G \times G \\
 & & \downarrow m & \swarrow m & \\
 & & G & & 
 \end{array}$$



8.1.1 REMARK: The idea above is we want to show the first diagram commutes. That is captured in the paths of the second diagram which commutes by the axioms of a group object. The third diagram shows a similar commutativity for the unit  $e$ .

## 9 October 14th, 2019

Let  $G$  be a finite group. Recall the definition of a **representation** (a linear action of  $G$  on a vector space  $V/k$ ). This is the same data as a group homomorphism to  $\mathrm{GL}(V)$ .

### 9.1 Representations of Algebraic Groups

Now what if  $G$  is an algebraic group over  $k$ ? Now we have some extra structure of  $G$  as a variety.

**9.1.1 Definition:** A (finite dimensional) **representation** of an algebraic group  $G/k$  is a homomorphism  $\rho : G \rightarrow \mathrm{GL}(V)$  of algebraic groups.

9.1.2 REMARK: Notice that when  $V$  is infinite-dimensional,  $\mathrm{GL}(V)$  is no longer of finite type, so we have to let  $\rho$  be a morphism of group schemes.

We have the standard representation of  $\mathrm{GL}_n$  acting on  $k^n$  in the natural way. We also have the regular representation  $G$  action on  $\Gamma(G, \mathcal{O}_G)$ . When  $G = \mathbb{G}_m$ , we get over  $\mathbb{C}$  an action of  $\mathbb{G}_m$  on  $\mathrm{GL}(V)$  in the usual way (scaling by  $\mathbb{C}^*$ ).

Observe that

$$\rho : G \rightarrow \mathrm{GL}(V) = \mathrm{Spec}(\mathrm{Sym}^*(V \otimes V^\vee))_{\det}$$

corresponds to a ring morphism

$$\mathrm{Sym}^*(V \otimes V^\vee)_{\det} \rightarrow \Gamma(G, \mathcal{O}_G) \stackrel{\mathrm{def}}{=} \Gamma(G)$$

which corresponds to a map

$$V \otimes V^\vee \rightarrow \Gamma(G)$$

and then tensoring with  $V$ , this gives us a map

$$V \xrightarrow{\sigma} \Gamma(G) \otimes V$$

So any group action gives us a **coaction** of  $\Gamma(G)$  on  $V$ .

**9.1.3 Definition:** A representation of  $G$  is a  $k$ -vector space  $V$  along with a coaction

$$\sigma : V \rightarrow \Gamma(G) \otimes V$$

satisfying the usual dual diagrams to actions.

9.1.4 REMARK: As a matter of notation, recall that if  $G = \operatorname{Spec} A$ , then  $A$  is a Hopf algebra. So we call  $V$  an  $A$ -comodule.

## 9.2 Reps of diagonalizable group schemes

Let  $k$  be a field (or even a ring!) and let  $A$  be a finitely-generated abelian group. Define  $D(A)$  to be

$$D(A) = \bigoplus_{a \in A} k \cdot t^a \stackrel{\text{def}}{=} \operatorname{Spec} R$$

Then we get a multiplication

$$R \otimes R \rightarrow R \quad t^a \otimes t^b \mapsto t^{a+b}$$

and comultiplication

$$R \rightarrow R \otimes R \quad t^a \mapsto t^a \otimes t^a$$

and counit  $\varepsilon$  sending  $t^a \rightarrow 1$  (all  $t^a$  are primitive).

### 9.2.1 Proposition

$R$  is a Hopf algebra. In particular,  $D(A) \rightarrow \operatorname{Spec} k$  is a linear algebraic group.

As an example, consider  $A = \mathbb{Z}$ . Then  $R \cong k[t]_t$ . Thus  $D(A) = \mathbb{G}_m$ .

If instead  $A = \mathbb{Z}/n$ , then  $R \cong l[t]/(t^n - 1)$ , so  $D(A) \cong \mu_n$ .

Finally when  $A \cong \mathbb{Z}^r \oplus \mathbb{Z}/n_1 \oplus \cdots \oplus \mathbb{Z}/n_k$ , then

$$D(A) = \mathbb{G}_m^r \times \mu_{n_1} \times \cdots \times \mu_{n_k}$$

**9.2.2 Definition:** An algebraic group over  $k$  is **diagonalizable** if  $G \cong D(A)$  for some  $A$ .

Recall the definition of irreducibility.

### 9.2.3 Theorem

Let  $A$  be a finitely generated abelian group and  $G = D(A)$ . Then

- Every irreducible representation of  $G$  is one-dimensional and isomorphic to  $I_a$ , corresponding to  $k \rightarrow \Gamma(G) \otimes k$  where  $1 \mapsto t^a \otimes 1$  for some  $a \in A$ .
- Every representation decomposes as a direct sum of irreducibles.

PROOF

Let  $\sigma : V \rightarrow \Gamma(G) \otimes V$  be a representation of a diagonalizable group. For  $a \in A$ , define

$$V_a \stackrel{\text{def}}{=} \{v \in V \mid \sigma(v) = t^a \otimes v\} \subseteq V$$

Now the claim is that  $V_a \cap V_b = 0$  if  $a \neq b$  and furthermore  $\sum V_a = V$ . The first isn't too hard to see.

The second follows by considering  $v \in V$  and looking at the image of it under  $\sigma$ . That is,

$$\sigma(v) = \sum_1^N t^{\alpha_i} \otimes v_i$$

where  $\alpha_i \in A$  and  $v_i \in V$ . Then a very simple argument shows  $v = \sum v_i$  (using linearity). Then it remains to show that  $v_i \in V_{\alpha_i}$ , but this will make things work. (use the other axiom of a coaction). ♠

9.2.4 REMARK: When  $A = \mathbb{Z}$ ,  $G = D(\mathbb{Z}) = \mathbb{G}_m$ , which tells us that representations of  $\mathbb{G}_m$  are in bijection with  $\mathbb{Z}$ -gradings of  $V \cong \bigoplus_{n \in \mathbb{Z}} V_n$ !

**9.2.5 Definition:** A linear algebraic group  $G \rightarrow \text{Spec } k$  is called **linear reductive** if every representation decomposes as a direct sum of irreducibles.

#### Problem 9.1

Show the above is equivalent to the statements

- for each  $G$ -representation  $W \subseteq V$ , there exists  $W' \subseteq V$  subrepresentations such that  $V \cong W \oplus W'$ .
- $0 \rightarrow W \rightarrow V \rightarrow W' \rightarrow 0$  is exact.

9.2.6 REMARK: Notice that this says that  $D(A)$  is linear reductive. In particular,  $\mathbb{G}_m$  and  $mu_n$  are in **any characteristic**. This runs counter to Maschke in finite groups.

Consider  $\mathbb{Z}/p$  in char  $p$ . We get an action  $\mathbb{Z}/p$  on  $k^2$  via

$$1 \cdot (x, y) = (x + y, y).$$

But notice that  $k \xrightarrow{\hookrightarrow} k^2$  is a subrepresentation, but has no complement! Thus this group is not linearly reductive!

---

As another example, consider

$$\mathbb{G}_a \cong \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \right\} \subset \mathrm{GL}_2(k)$$

where  $\mathbb{G}_a$  acts on  $k^2$  by  $\alpha(x, y) = (a + \alpha y, y)$ . Then it can be easily seen not to be a linear representation.