Lie Algebras and Groups

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Introduction

These notes are my best attempt at following along with our *Math 508 – Lie Algebras* course at UW. This is my first time trying to type my notes on-the-fly in class so we'll see how well this goes. The course reference is Humphreys' *Introduction to Lie Algebras and Representation Theory.*

The course description follows:

This is the second course in the Algebraic Structures sequence. I will classify finite-dimensional complex semisimple Lie algebras, also proving some structural results on general Lie algebras along the way. Although one usually first encounters Lie algebras in a manifolds course, the treatment (following the text) will be entirely algebraic.

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The homework is posted on Monty's website. :)

1.1 Lie algebras

This course will be studying Lie algebras, but as opposed to their treatment in manifolds, we will be studying them from a purely algebraic point of view. The book (Humphreys) actually never defines a Lie group.

1.1.1 Definition

A **Lie Algebra** L or $\mathfrak{g}$ over a field k is a k-vector space (usually f.d.) along with a *bracket operation* $[vw] : L \times L \to L$ such that $[\cdot\cdot]$ is

- anticommutative,
- bilinear,
- $[x[yz]] = [[xy]z] + [y[xz]]$

1.1.2 Remark: The last principle above is actually equivalent to the *Jacobi identity:*

$$[x[yz]] + [y[xz]] + [z[xy]].$$

This follows from bilinearity and anticommutativity of the bracket.

The most natural place for these to arise is as *derivations* on an algebra!

1.1.3 Definition
A **k-derivation** $d : A \to A$ on an algebra A over k is a k-linear map satisfying the Leibniz rule.

1.1.4 Remark: Some key facts about derivations (for us):

- Given a fixed $a \in A$, the map d_a sending $b \mapsto ab - ba$, the **commutator** $[ab]$ is a derivation.
- If d, e are derivations, then so is $[de] = de - ed$, where de is the *composite* of d and e as opposed to the product.

1.2 Examples

A main source of Lie algebras is (associative) algebras! *Any associative k-algebra A* becomes a Lie algebra over k, taking $[ab] = ab - ba$. In particular, one obvious choice for k-algebra is $M_n(k) = \mathfrak{gl}_n(k)$, the (Lie) algebra of $n \times n$ matrices over k.

Lie subalgebras are what you'd expect (including closure under brackets). Notice that if $L' \leq L$, then they **must both be over the same field.**

If L is a k-Lie algebra and $I \lhd L$ is an ideal of L, then the quotient space L/I becomes a Lie algebra with $[x + I, y + I] = [xy] + I$ as the bracket.

A **Lie algebra homomorphism** is a map $\varphi : L \to L'$ such that φ is k-linear and $\varphi([xy]) = [\varphi(x)\varphi(y)]$.

We get the usual first isomorphism theorem $L/\ker\varphi \cong \varphi(L)$.

Associative algebras are not the only source of Lie algebras, however! One example is $\mathfrak{sl}(n, k) = \{n \times n \text{ matrices over } k \text{ with trace zero}\}$

Note that this is **not closed under product** since $\operatorname{tr}(AB) \neq \operatorname{tr} A \operatorname{tr} B$ but $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ so $\operatorname{tr}(AB - BA) = \operatorname{tr}(AB) - \operatorname{tr}(BA) = 0$.

1.2.1 Definition
We call this algebra (or, in fact any subalgebra of $\mathfrak{gl}(n, k)$) **linear**. Think "Linear" means "of matrices."

We say that $\mathfrak{sl}(n,k)$ has **type** A_{n-1}. Eventually we will see seven types $A-G$ of semisimple Lie algebras. The shift in index will emerge later.

$\mathfrak{sl}(n,k)$ is, in fact, a simple Lie algebra: for $k=\mathbb{C}$, $\mathfrak{sl}(n,\mathbb{C})$ has no ideals apart from the trivial ones.

Other non-associative examples include k^n with a bilinear form $(\cdot,\cdot)$ which is either symmetric or skew-symmetric and (in either case) is nondegenerate.

1.2.2 Definition
$(\cdot,\cdot)$ is **nondegenerate** if the map $v \mapsto (v,\cdot)$ is injective. Equivalently there is no $v \in V$ such that $(v,w)=0$ for all $w \in V$.

Given $V=k^n$ and a bilinear form on V, we can look at all $X \in \mathfrak{gl}(n,k)=\mathfrak{gl}(V)$ such that $(Xv,w)=(v,Xw)$. Then X is **adjoint** with respect to the form. There is a similar definition for when X is **skew-adjoint.** One can check that $[XY]$ is skew-adjoint whenever both X and Y are.

1.3 Generating (skew) symmetric forms

It ends up that the dot product (which is a symmetric form) is misleadingly simple – thus we will look elsewhere.

If $M \in \mathfrak{gl}(n,k)$ is symmetric, so that $M^t=M$, then $(v,w)=v^tMw$ is a symmetric. If instead M is skew-symmetric, then the same definition yields a skew-symmetric form. This actually induces a one-to-one correspondence between matrices and forms.

In both cases, if M is invertible, then the form will be nondegenerate. As a consequence, since skew-symmetric matrices are always singular in odd dimensions, we see that nondegenerate skew-symmetric forms (over $\operatorname{char} k \neq 2$ where the two families of forms coincide) exist only in even dimensions.

1.4 A peek at classifications

If we have a nondegenerate symmetric form where $n=2m$ is going to give us an algebra of type D_m. If $n=2m+1$, then it is of type B_m. Both of these cases are called **orthogonal.**

If instead we have a skew-symmetric form and $n=2m$, then this is of type C_m, and we call this algebra **symplectic.**

We will make a particular choice for our matrix M and then study the resulting Lie algebras in much more detail next time. The choices will be:

- For type D_m:
$$\begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$$

- For type C_m:

$$\begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}$$

- For type B_m:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_m & 0 \end{pmatrix}$$

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Today we will be looking deeply into the stucture of linear Lie algebras of types A-D.

2.1 Linear Lie Algebras Revisited

Recall that the **matrix unit** e_{ij} is the matrix with 1 in the (i, j) entry and zero elsewhere. And then $e_{ij}e_{kl} = \delta_{jk}e_{il}$ and furthermore

$$[e_{ij}e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}.$$

This is especially nice when $j = i$, called the **diagonal matrix unit**.

Then we look at type A_{n-1} ($\mathfrak{sl}(n, k)$). Let D be the set of diagonal matrices in this algebra. Notice the dimension is $n - 1$ since then n^{th} term on the diagonal is determined as the negative of the sum of the other $n - 1$ terms. Let $A = \text{diag}(d_1, \ldots, d_n)$. Then consider the eigenvalues associated with e_{ij}:

$$[Ae_{ij}] = (d_i - d_j)e_{ij} = (E_i - E_j)Ae_{ij}$$

where E_i is the linear functional selecting the i^{th} entry in A. Moreover, D is abelian as a Lie algebra, so D acts diagonally on $L = \mathfrak{sl}(n, k)$ by commutation with eigenvalues $E_i - E_j$ and zero for $1 \leq i, j \leq n$ and $i \neq j$.

In the other classical cases B-D, there is always a matrix M which defines the form $(v, w) = v^t M w$ as we saw yesterday. In all three cases, the Lie algebra exists consists of all skew-adjoint matrices X relative to the form. $B_m = \mathfrak{so}(2m+1, k)$ as well as $D_m = \mathfrak{so}(2m, k)$ and $C_m = \mathfrak{sp}(2m, k)$.

This condition translates to the form of matrices in the above Lie groups and the condition is always $Mx = -x^t M$ in all cases.

Type B:

$$\begin{pmatrix} 0 & b_1 & b_2 \\ c_1 & m & n \\ c_2 & p & q \end{pmatrix}$$

where $c_1 = -b_2^t$, $c_2 = -b_1^t$, $q = -m^t$, $n^t = -n$ and $p^t = -p$.

Type C:

$$\begin{pmatrix} m & n \\ p & q \end{pmatrix}$$

where $n^t = n, p^t = p$, and $m^t = -q$.

Type D:

$$\begin{pmatrix} m & n \\ p & q \end{pmatrix}$$

where $n^t = -n, p^t = -p$, and $m^t = -q$.

Looking at the eigenvalues of elements of D associated to vectors $e_{ij} - e_{m+i,m+j}$. Look at photos

Using a similar analysis, we can look at types B and D. We define the functions E_i similarly on the space of diagonalmatrices and gives a rise to the following collection of linear functions: in $B_m : \pm E_i$ and $\pm(E_i \pm E_j)$ and D_m gives us $\pm(E_i \pm E_j)$.

This collection of functions in each case is called the **root system of the Lie algebra**, Then any complex simple finite-dimensional Lie algebra is classified by its root system. The (perhaps surprising) fact is that this already encompasses all but finitely many of these things up to ismorphism: the classical Lie algebras. Eventually we will learn more about the **exceptional Lie groups.**

This section was a little hard to follow and the handling in Humphreys is easier to follow, but delays speaking about root systems and actually deriving the eigenfunctions (is that the right word?) until significantly later. Monty seemed to think it was acceptable to delay the understanding of this a bit.

2.2 Derivations and exp

Look an an arbitrary Lie algebra over a field k where $\operatorname{char} k = 0$ (which we will mostly be assuming from here on) . Let δ be a derivation of L, so that $[\delta x, y] + [x \delta y] = \delta[x, y]$. Assume that δ is nilpotent.

Then the "power series" (polynomial) is

$$\exp \delta = \sum_{i>0} \frac{\delta^i}{i!}$$

Problem 0.1 *This is a good exercise to go through: Check that*

$$[(\exp \delta)x, (\exp \delta)y) = [xy]$$

for each $x, y \in L$

2.2.1 Remark: This actually shows that $\exp \delta$ is an automorphism of L. Furthermore you'd find that

$$(\exp \delta)(\exp(-\delta)) = 1.$$

What if $k = \mathbb{R}$ or $\mathbb{C}$? Then the power series (even when δ is not nilpotent!) always converges and defines an automorphism as before.

2.2.2 Lemma
For all **complex** semisimple Lie algebras L it turns out that the group generated by $\exp \operatorname{ad} x$ ($\operatorname{ad} x(y) = [xy]$) coincides with the group generated by all nilpotent $\operatorname{ad} x$.

2.3 Adjoint group

The last thing for today is to define the adjoint group:

2.3.1 Definition
Let L be a Lie algebra, then

$$\operatorname{Int}(L) = \exp \operatorname{ad} L$$

is the **Adjoint group** of L. It is a subgroup (so we believe) of the Lie Group associated to L.

Some examples of adjoint groups:

- If $L = \mathfrak{sl}(n, \mathbb{C})$, then $\operatorname{Int}(L) = PSL(n, \mathbb{C}) = SL(n, \mathbb{C})/\text{center}$
- If $L = \mathfrak{so}(n, \mathbb{C})$, then $\operatorname{Int}(L) = PSO(n, \mathbb{C})$
- If $L = \mathfrak{sp}(2n, \mathbb{C})$ then $\operatorname{Int}(L) = PSp(2n, \mathbb{C})$.