Algebraic Groups

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Abstract

The topic of algebraic groups is a rich subject combining both group-theoretic and algebro-geometric-theoretic techniques. Examples include the general linear group GL_n, the special orthogonal group SO_n or the symplectic group Sp_n. Algebraic groups play an important role in algebraic geometry, representation theory and number theory.

In this course, we will take the functorial approach to the study of linear algebraic groups (more generally, affine group schemes) equivalent to the study of Hopf algebras. The classical view of an algebraic group as a variety will come up as a special case of a smooth algebraic group scheme. Our algebraic approach will be independent (even complementary) to the analytic approach taken in the course on Lie groups.

1 September 25, 2019

1.1 Group objects

Let $\mathscr{C}$ be a category with a final object and finite products.

1.1.1 Definition: A **group object** G **in** $\mathscr{C}$ is an object in $\mathscr{C}$ along with multiplication, identity, and inverse morphisms satisfying the usual axioms.

One thing is that we are using that there is a final object $*$ along with our identity morphism $e : * \to G$. Here Jarrod explictly used the fact that there is a unique map to $*$.

Example 1.1
If $\mathscr{C}$ is Set, then G is a group. If $\mathscr{C} = \text{Top}$, then G is a topological group, smooth manifolds give Lie groups, and finally (interesting to us):

1.1.2 Definition: Let S be a scheme and let $\mathscr{C}$ be the category of schemes over S. Then a group object G in $\mathscr{C}$ is a **group scheme over S.**

WHen k is a field and $\mathscr{C}$ is schemes of finite type over k, we get a group scheme of finite type over k. There is not a great consensus on what makes an **algebraic group**, but this is what we will use.

When we instead restrict to *affine schemes* we get an affine groupe scheme of finite tipe over k, or a **linear algebraic group.**

1.2 Examples

$\mathbb{G}_m = \operatorname{Spec} k[t]_t$ is one.

If we consider the map $f : \mathbb{G}_m \to \mathbb{G}_m$ which on the level of elements sends $t \mapsto t^p$, the kernel is

$$\mu_p = \ker(f) = \operatorname{Spec} k[t]/(t^p - 1)$$

and that's great, but when $\operatorname{char} k = p$, this causes the group scheme to be **unreduced**. This is (apparently) a case when you need to use schemes.

1.3 The Functorial Approach

Let $\mathscr{C}$ be a category with object X. Define the functor $h_X : \mathscr{C}^{op} \to \mathbf{Set}$ where

$$h_X(Y) = \operatorname{Hom}_{\mathscr{C}}(Y, X).$$

Then we have

1.3.1 Lemma (Yoneda)
Let $G : \mathscr{C}^{op} \to \mathbf{Set}$ be a functor. There is a natural bijection

$$G(X) \simeq \operatorname{Nat}(h_X, G).$$

1.3.2 Proposition
A group object G in $\mathscr{C}$ is the same as an aobject $X \in \mathscr{C}$ together with a choice of factorization of $h_X : \mathscr{C} \to \mathbf{Set}$ through $\mathbf{Grp}$.

1.4 Exercises

(a) Spell out all the details of the proof of the above propositon.

(b) Given a group object G, define in two ways what it means for it to act on another object. (In coordinates and functorially).

1.5 Some Interesting Facts

If we had to write down five results that we'd like to get out of this class:

1.5.1 Proposition
Every affine group scheme of finite type over a field embeds into GL_n as a closed subgroup.

1.5.2 Theorem (Chevalley's Theorem)
Let G be a finite type group scheme over a field. Then it factors as

$$1 \to H \to G \to A \to 1$$

where A is abelian and H is affine (linear algebraic).

1.5.3 Proposition
If G is an affine group scheme of finite type over k, then we have af actorization

$$1 \to U \to G \to R \to 1$$

where U is unipotent and R is reductive.

1.5.4 Proposition
$H \subseteq G$ a subgroup scheme. Then G/H is a projective scheme.

Finally we want to talk about Tanakka duality and how the representations of G define G itself.

2 September 27th, 2019

Last time we defined a group scheme (a group object in the category of schemes over a base scheme). We also mentioned that You could define it as a map $h_G : \mathbf{Sch}/S \to \mathbf{Set}$ along with a factorization through **Grp**.

We defined an **algebraic group** over k as a group scheme over $\operatorname{Spec} k$ of finite type and a **linear algebraic group** to be an *affine* group scheme over k of finite type.

2.1 Hopf Algebras

Let $G = \operatorname{Spec} A$ be a linear algebraic group over k. I have seen most of these before (see Waterhouse or my Hopf algebra notes)

2.1.1 Remark: One think I haven't seen explicitly before: Notice that the augmentation ideal $\ker \varepsilon$, where ε is the counit, is the (maximal!) ideal corresponding in the algebro-geometric sense to the identity element in G.

2.1.2 Definition: A **Hopf algebra** is ...

2.1.3 Definition: Let G be an algebraic group over k. Then if h_G factors through **Ab**, G is called **commutative.**

2.2 Some Examples

2.2.1 Remark: Note that to define a functor from schemes over k, is suffices to define it on affine schemes, thereby defining the (Zariski) local behavior of any such map. Thus we really only need to consider maps in $\mathbf{Alg}_k$.

- $\mathbb{G}_a$. Here we can define it as a functor that sends $S \mapsto \Gamma(S, \mathcal{O}_S)$. Geometrically, $\mathbb{G}_a = \mathbb{A}^1$ where the multiplication is addition, inverses send $x \mapsto -x$ and the unit is the zero map. The Hopf algebraic picture is the usual dual thing.
- $\mathbb{G}_m$ as a scheme isthe map $S \mapsto \Gamma(S, \mathcal{O}_S)^*$. In the geometric picture, $\mathbb{A}^1 \setminus \{0\}$ and the algebra structure comes from multiplciation. Hopf is pretty easy.
- GL_n is a scheme that sends

  $$S \mapsto \left\{A = (a_{ij}) : a_{ij} \in \Gamma(S, \mathcal{O}_S), \det(A) \in \Gamma(S, \mathcal{O}_S)^*\right\}$$

  the algebra is $\mathbb{A}^{n\times n} \setminus \{\det = 0\}$ with the usual multiplication. The coalgebra structure can be seen in the book.

This one requires some more explaination so I am setting it apart.

Example 2.1
Let V be a finite dimensionatl vector space over k. Then we can define the algebraic group V_a which sends

$$S \mapsto \Gamma(S, \mathcal{O}_S) \otimes_k V.$$

Geometrically we are looking at $\mathbb{A}(V) = \operatorname{Spec} \operatorname{Sym}^* V^\vee \simeq \operatorname{Spec} k[x_1, \ldots, x_n]$ where $n = \dim V$.

What about finite groups? As a scheme, we want $G = \sqcup_{g\in G} \operatorname{Spec} k$. The functor sends $S \mapsto \mathrm{Mor}_{\mathrm{Set}}(\pi_0(S), G)$, or maps from the connected components into G.

Example 2.2
Now consider the n^{th} roots of unity: as a scheme, $\mu_n = \operatorname{Spec} k[t]/(t^n - 1) \subseteq \mathbb{G}_m$. If both $k = \bar{k}$ and $\operatorname{char} k \nmid n$, then $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$.

But if (e.g.) $k = \mathbb{Q}$, then μ_3 is $\mathbb{Q}[t]/(t^3 - 1) = \operatorname{Spec}\mathbb{Q} \sqcup \operatorname{Spec}\mathbb{Q}(\xi)$ where ξ is a primitive third root of unity.

If, on the other hand, $k = \bar{\mathbb{F}}_3$ and consider μ_3, we get a single point with residye field $\bar{\mathbb{F}}_3$.

Example 2.3
If we are in the case of positive characteristic, then we get an algebraic group α_p. Here the scheme is $\operatorname{Spec} k[x]/x^p$ and functorially it is the map $S \mapsto \{F \in \Gamma(S, \mathscr{O}_S) | f^p = 0\}$.

2.3 Matrix Groups

We already defined GL_n, but we can also define

$$\mathrm{SL}_n : S \mapsto \{A = (a_{ij}) | \det A = 1\}$$

with scheme $\operatorname{Spec} k[x_{ij}]/(\det - 1)$.

We also have the (upper) triangular matrices T_n and unitary group U_n and diagonal group D_n

2.3.1 Definition: Let G be a linear algebraic group. Then

- G is a **vector group** if $G \cong V_a$ for some finite dimensional V.
- G is a **split torus** if $G \cong \mathbb{G}_m^n$.
- G is a **torus** if there is a field extention $k \to k'$ such that

$$G \times_{\operatorname{Spec} k} \operatorname{Spec} k' \cong \mathbb{G}_{m,k'}^n$$

3 September 30th, 2019

Another example to consider:

Example 3.1
Let $G = \mathrm{PGL}_n$, the projective linear group. Recall we want to define this as GL_n/k^* (from group theory). To do this for algebraic groups, we define

$$\mathrm{PGL}_n = \mathrm{Proj}\, k[x_{ij}]_{det} := \mathrm{Spec}(k[x_{ij}]_{det})_0$$

The geometric picture is difficult since we haven't yet defined quotients, but as a functor we say PGL_n is $\mathrm{Aut}(\mathbb{P}^n)$, the functor that sends $S \mapsto \mathrm{Aut}(\mathbb{P}^n_S)$ where $\mathbb{P}^n_S = \mathbb{P}^n_k \times_{\mathrm{Spec}\, k} S$.

3.1 Non-affine group schemes

Example 3.2
Let $\lambda \neq 0, 1$ be an element in k. Then we can define the elliptic curve

$$E_\lambda = V(y^2 z - x(x-z)(x-\lambda z)) \subset \mathbb{P}^2$$

Which gives us a double cover over $(0,1)$ and (λ, ∞) with singleton fiber (ramified) over $0, 1$, and λ.

Then for any $\lambda \neq 0, 1$, E_λ is a **projective** group scheme.

3.1.1 Remark: If you look at the $\mathbb{C}$-points, you get $E_\lambda(\mathbb{C}) = \Lambda_\lambda$, giving you a torus. Recall (from e.g. complex analysis) that the moduli here is $\mathrm{SL}_2(\mathbb{Z})$ of all elliptic curves.

3.2 Abelian Varieties

3.2.1 Definition: An **abelian variety over** k is asmooth, geometrically connected ($A \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \bar{k}$ is connected), proper group scheme A over k.

Example 3.3
Over $\mathbb{C}$, $\mathbb{C}^g/\Lambda$ where $\Lambda \cong \mathbb{Z}^{2g} \subseteq \mathbb{C}^g$ gives us a genus g example.

3.2.2 Theorem
Any abelian variety over k is commutative and projective.

3.2.3 Theorem (Chevalley)
If G is any group scheme, then the sequence

$$1 \to H \to G \to A \to 1$$

is exact, where H is a linear algebraic group (affine!) and A is an abelian variety.

Example 3.4
Let $X \to \operatorname{Spec} k$ be a geometrically integral projective scheme (proper may suffice). The idea here is that over $\mathbb{C}$ the rings over every open set are integral domains.

Now consider the **Picard functor** $\operatorname{Pic}_X : \operatorname{Pic} : \mathbf{Sch/k} \to \mathbf{Grp}$ sending

$$S \mapsto \operatorname{Pic}(X_S = X \times_k S)/p^k \operatorname{Pic(S)}$$

3.2.4 Theorem
Pic_X is represented by a scheme locally of finite type, thus Pic_X^0, the connected component of the identity in $[\mathscr{O}_X] \in \operatorname{Pic}_X$ is an abelian variety.

3.3 Relative Group Schemes

Example 3.5
Consider $\mathbb{G}_{m,\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[t]_t$. Then $G_{m,S} = \mathbb{G}_{m,\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} S$. In the case that $S = \operatorname{Spec} R$, $\mathbb{G}_{m,S} = \operatorname{Spec} R[t]_t$.

Example 3.6
Let $\mathbb{A}^1 = \operatorname{Spec} k[x]$ and define $G = \operatorname{Spec} k[x,y]_{xy+1} \subseteq \mathbb{A}^2$. Notice this is the plane minus a hyperbola.

Define $\cdot : G \times_{\mathbb{A}^1} G \to G$ to be given by

$$(x,y)\cdot(x,y') = (x, xyy' + y + y')$$

Then the thing here is the fiber (think vertical line in the plane!) over 0 is $\mathbb{G}_a$ and is isomorphic to $\mathbb{G}_m$ otherwise.

Example 3.7
Let $\mathscr{E}_\lambda = V(y^2z - x(x-z)(x-\lambda z))$ over $\operatorname{Spec} k[\lambda]$. Then when $\lambda = 0$, we get the nodal cubic given by $y^2z - x^2(x-z)$ (node at the origin).

Now if you look at the connected component around 0 of $\operatorname{Aut}(\mathscr{E}_\lambda)/\mathbb{A}_\lambda$, you actually find (when $\lambda = 0$) that $\mathbb{G}_m \cong \operatorname{Aut}(\mathscr{E}_0)^0$.

3.4 Some definitions

3.4.1 Definition: A **homomorphism** $\phi : G \to G$ of group schemes over S is a map $\phi : H \to G$ of schemes such that

$$\begin{array}{ccc} H \times_S H & \xrightarrow{m_H} & H \\ \downarrow{\scriptstyle \phi\times\phi} & & \downarrow{\scriptstyle \phi} \\ G \times_S G & \xrightarrow{m_G} & G \end{array}$$

Problem 3.1
Show that this automoatically imples that the identity and inversion maps are respected as well (automatically).

3.4.2 Definition: A **subgroup of** $G \to S$ is a subscheme $H \subseteq G$ such that $H(T) \leq G(T)$ for all T over S.

Problem 3.2
Show that $\ker(\phi) \subseteq H$ *is a subgroup.*

3.4.3 Remark: This gives you a nice way to construct new group schemes. For example, the following are exact:

$$1 \to \mathrm{SL}_n \to GL_n \xrightarrow{\det} \mathbb{G}_m \to 1$$

and

$$1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$$

3.4.4 Proposition
Let $G \to S$ be a group scheme. Then $G \to S$ is separated if andy only if $e : S \to G$ is a closed immersion.

Proof
The idea here is that $S \to G$ is a closed immersion. Then we consider the map $m \circ (\mathrm{id}, S)$: $G \times_S G \to G$ and consider this along with the diagonal map $\Delta : G \to G \times_S G$ and this is a pullback square! ♠

3.4.5 Corollary
Any group scheme over k is separated.

The idea is going to be that if X is any scheme over k, then any point $X \in X(k)$ is closed.