Problems from Peter Webb's
A Course in Finite Group Representation Theory

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1 Representations, Maschke's Theorem, and Semisimplicity

Problem 1.1 In Example 1.1.6, prove that there are no invariant subspaces other than the ones listed.

Solution. Recall that in this example we are considering the representation of $C_2 = \mathbb{Z}_2$ given by $\rho : C_2 \to GL(\mathbb{R}^2)$ via

$$\rho(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \rho(-1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

In the example we noted that $\{0\}$, $\langle \binom{1}{0} \rangle$, $\langle \binom{0}{1} \rangle$ and $\mathbb{R}^2$ are all invariant subspaces.

Assume that V is any $\mathbb{R}$-subspace of $\mathbb{R}^2$ not listed above. Then necessarily it has $\mathbb{R}$-dimension one, so is spanned by an element $\binom{a}{b} \in \mathbb{R}^2$. If V is stable under the C_2 action, it must be that (in particular) the image of $\binom{a}{b}$ under nontrivial matrix in (1) above lands back in V. But then for some $\alpha \in \mathbb{R}$,

$$\alpha \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix}$$

whence either $a \neq 0$ and so $\alpha = 1$ and $b = 0$ or else $a = 0$ and (since V is nontrivial) $b \neq 0$ whence $\alpha = -1$ and $a = 0$. But these two case describe precisely the two one dimensional subspaces given above so this must in fact be a complete list.

$\square$

Problem 1.2 (*The Modular Law*) – Let A be a ring and $U = V \oplus W$ an A-module that is a direct sum of A-modules V and W. Show by example that if X is any submodule of U then it need not be the case that $X = (V \cap X) \oplus (W \cap X)$. Show that if we make the assumption that $V \subseteq X$ then it is true that $X = (V \cap X) \oplus (W \cap X)$.

Solution. Consider $A = \mathbb{R}$ and let $U = \langle e_1, e_2 \rangle$ be $\mathbb{R}^2$ spanned by the standard basis vectors. Now define $V = \langle e_1 + e_2 \rangle$ and $W = \langle e_1 \rangle$. $V \cap W = \{0\}$ and $V + W = U$, so the hypotheses are satisfied. Now let $X = \{e_2\}$ and notice

$$(X \cap A) \oplus (X \cap B) = \{0\} \oplus \{0\} = \{0\} \neq X.$$

Now assume that $V \subseteq X$, so that $V \cap X = V$. Now we want to prove that $X = V \oplus (W \cap X)$. Clearly since V and $W \cap X$ are in X, their sum is as well. Therefore $V \oplus (W \cap X) \subseteq X$.

Assume now that $x \in X$. But then $x \in U$ so $x = v + w$ for $v \in V$ and $w \in W$. Since $V \subseteq X$, $v \in X$ and the same goes for $x - v = w$. But then $x = (v) + (x - v)$ where the former part is in V and the latter is in $X \cap W$. This gives us the reverse inclusion and thus equality.

□

Problem 1.3 Suppose that ρ is a finite-dimensional representation of a finite group G over $\mathbb{C}$. Show that for each $g \in G$, the matrix $\rho(g)$ is diagonalizable.

Solution. Let $|G| = n$ and $\rho : G \to GL(\mathbb{C})$ be a degree $k < \infty$ representation. Since G has finite order, $\rho(g)^n = I_k$ so $\rho(g)$ satisfies the polynomial $x^n - 1$. Since $\mathbb{C}$ is algebraically closed, $\rho(g)$ has k eigenvalues, each of which must be an n^{th} root of unity. Furthermore $x^n - 1$ has no repeated roots (separable) so the minimal polynomial of $\rho(g)$ has distinct roots.

Therefore $\rho(g)$ has k distinct eigenvalues in $\mathbb{C}$ which is sufficient to show that it is diagonalizable.

□

Problem 1.4 Let $\phi : U \to V$ be an A-module homomorphism. Show that $\phi(\operatorname{Soc} U) \subseteq \operatorname{Soc} V$ and that if ϕ is an isomorphism then it restricts to an isomorphism $\operatorname{Soc} U$ to $\operatorname{Soc} V$.

Solution. Recall that the image of a simple module is itself simple. To see this, assume that S is simple and $0 \neq X \subseteq \phi(S)$ is a submodule. Then $\phi^{-1}(X)$ is a submodule of U intersecting S nontrivially and by simplicity $\phi^{-1}(X) \cap S = S$. But then $\phi(S) \subseteq X$, so they are equal. Thus the only nonzero submodule of $\phi(S)$ is itself.

Thus since $\operatorname{Soc} U$ is a sum of simple modules of U, $\varphi(\operatorname{Soc} U)$ is a sum of simple modules of V. So $\phi(\operatorname{Soc} U) \subseteq \operatorname{Soc} V$, the sum of *all* simple modules of V.

Assume now that ϕ is an isomorphism. Then by the same argument as above $\phi^{-1}(\operatorname{Soc} V) \subseteq \operatorname{Soc} U$ and so

$$\phi \circ \phi^{-1}(\operatorname{Soc} V) \subseteq \phi(\operatorname{Soc} U) \quad \Rightarrow \quad \operatorname{Soc} V \subseteq \phi(\operatorname{Soc} U)$$

so they are equal.

Finally that $\operatorname{Soc} V = \phi(\operatorname{Soc} U)$ and $\phi^{-1}(\operatorname{Soc} V) = \operatorname{Soc} U$ and the injectivity of ϕ and its inverse gives us that ϕ restricts to an isomorphism $\operatorname{Soc} U \to \operatorname{Soc} V$.

□

Problem 1.5 Let $U = S_1 \oplus \cdots \oplus S_r$ be an A-module that is the direct sum of finitely many simple S_i. Show that if T is any simple submodule of U then $T \cong S_i$ for some i.

Solution. Since $T \subseteq U$, $T \cap S_i$ is nontrivial for some i. But then $T \cap S_i = S_i$ by the simplicity of S_i and so $S_i \subseteq T$. By problem 1.2, since we have

$$T \subset S_i \oplus \left(S_1 \oplus \cdots \oplus \hat{S}_i \oplus \cdots \oplus S_r\right) = S_i + S$$

we have that

$$T = (T \cap S_i) \oplus (T \cap S) = S_i \oplus (T \cap S)$$

and by the simplicity of T, $T \cap S = 0$ and $T \cong S_i$.

□

Problem 1.6 Let V be an A-module for some ring A and suppose that V is a sum $V = V_1 + \cdots + V_n$ of simple submodules. Assume further that the V_i are pairwise nonisomorphic. Show that the V_i are the only simple submodules of V and that $V = V_1 \oplus \cdots \oplus V_n$ is their direct sum.

Solution. Let W be any simple submodule of V. By Lemma 1.2.3 in the book $V = V_{i_1} \oplus \cdots \oplus V_{i_k}$ for some subset of the V_i. By problem 1.5 above, this gives us $W \cong V_j$ for some j. But the V_i are pairwise nonisomorphic so $W = V_j$ so the V_i are the only simple submodules of V.

That all V_i occur in the direct sum holds by again applying the last problem and noticing that each V_i must be isomorphic to some V_j in the direct sum, whence V_i itself must appear.

□

Problem 1.7 Let $G = \langle x, y | x^2 = y^2 = 1 = [x, y]\rangle$ be the Klein 4-group, $R = \mathbb{F}_2$, and consider the two representations ρ_1 and ρ_2 given by

$$\rho_1(x) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho_1(y) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\rho_2(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho_2(y) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Compute the socles of these representations. Show that neither representation is semisimple.

Solution. Notice that every subrepresentation is automatically a vector subspace of $\mathbb{F}_2^3$, so begin by considering the one dimensional subspaces. If $\langle (a, b, c)^T \rangle$ is invariant under ρ_1, consider the images:

$$\rho_1(x)(a, b, c)^T = (a + b, b, c)^T, \quad \rho_1(y)(a, b, c)^T = (a + c, b, c)^T$$

each of which are either $(a,b,c)^T$ or $\mathbf{0}$. In either case we are forced to have $b = c = 0$, so the only invariant one dimensional subspace is the one spanned by $(1,0,0)^T$.

In the two dimensional case, consider any subspace spanned by $\mathbf{v} = (a,b,c)^T$ and $\mathbf{w} = (d,e,f)^T$. If $b \neq e$ or $c \neq f$, we must have that ρ_1 fixes each of these vectors, so the resulting module must not be simple. Therefore we can assume that $b = e$ and $c = f$. But then since it must be two dimensional, $a = d + 1 \pmod 2$. But then

$$\mathbf{v} + \mathbf{w} = (1,0,0)^T$$

which spans an invariant subspace under ρ_1 so this is also not simple.

So there are no simple degree two subrepresentations of ρ_1 so the socle of ρ_1 is the one dimensional space spanned by $(1,0,0)^T$. Thus ρ_1 is clearly not semisimple.

Let $\mathbf{v} = (a,b,c)^T$ be any vector spanning a degree one subrepresentation of ρ_2. Then

$$\rho_2(x)\mathbf{v} = (a, b+c, c)^T, \qquad \rho_2(y)\mathbf{v} = (a+c, b, c)^T.$$

Neither can be $\mathbf{0}$ since otherwise $\mathbf{v} = \mathbf{0}$. But this implies $c = 0$. Therefore there are three one-dimensional invariant subspaces spanned by $(1,0,0)^T$, $(0,1,0)^T$ and $(1,1,0)^T$.

Now given vectors $\mathbf{v} = (a,b,c)^T$ and $\mathbf{w} = (d,e,f)^T$ spanning a degree 2 subrepresentation, notice that if $c = f$ then $\mathbf{v} + \mathbf{w}$ is either $\mathbf{0}$ (whence $\mathbf{v} = \mathbf{w}$) or else this sum is one of the vectors from the previous paragraph. In either case $\mathbf{v}$ and $\mathbf{w}$ don't span a simple degree two representation space, so we have $c \neq f$.

But then without loss of generality if $c = 0$, $\mathbf{v}$ spans an invariant subspace so we can conclude that there are *no* degree two simple subrepresentations of ρ_2. Thus the socle is computed as the sum of the spaces spanned by the three vectors from the first paragraph:

$$\langle (1,0,0)^T, (0,1,0)^T, (1,1,0)^T \rangle = \{(a,b,0)^T \in \mathbb{F}_2^3\} \subsetneq \mathbb{F}_2^3$$

so this representation is not semisimple either.

$\square$

Problem 1.8 Let $G = C_p = \langle x \rangle$ and $R = \mathbb{F}_p$ for some prime $p \geq 3$. Consider the two representations ρ_1 and ρ_2 specified by

$$\rho_1(x) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho_1(x) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Calculate the socles of these two representations and show that neither representation is semisimple. Show that the second representation is nevertheless the direct sum of two nonzero subrepresentations.

Solution. Consider first ρ_1. If $\mathbf{v} = (a,b,c)$ (lazily eschewing the transpose in this problem for notational simplicity) spans a one dimensional invariant subspace, then for some $\alpha \in \mathbb{Z}_p$ we have

$$\rho_1(x)\mathbf{v} = \alpha \begin{pmatrix} a+b \\ b+c \\ c \end{pmatrix}$$

Assume by contradiction that $c \neq 0$. Then $\alpha = 1$ and $b = b + c \Rightarrow c = 0$, a contradiction. Thus $c = 0$. Similarly if $b \neq 0$, $a + b = a$, and we get another contradiction. Thus $b = c = 0$. Any vector $(a, 0, 0) \in \mathbb{F}_p^3$ spans an invariant subspace with the trivial action by ρ_1 so these are all the degree one subrepresentations.

Notice that if V is the representation space of a degree two subrepresentation of ρ_1 and if $\mathbf{v} \in V$ where the third component of $\mathbf{v}$ is zero, we have that $\mathbf{v} = 0$. This can be seen by considering

$$\big(\rho_1(x^2) - \rho_1(x)\big)\begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix} = \begin{pmatrix} v_1 + 2v_2 \\ v_2 \\ 0 \end{pmatrix} - \begin{pmatrix} v_1 + v_2 \\ v_2 \\ 0 \end{pmatrix} = \begin{pmatrix} v_2 \\ 0 \\ 0 \end{pmatrix}$$

and if $v_2 \neq 0$, V contains one of the one-dimensional subspaces from above and is therefore not simple. Otherwise $v_2 = 0$ and $\mathbf{v}$ spans a degree 1 invariant subspace, so again V is not simple.

Not finished.

$\square$

Problem 1.9 Let k be an infinite field of characteristic 2, and $G = \langle x, y\rangle \cong C_2 \times C_2$ be the noncyclic group of order 4. For each $\lambda \in k$, let $\rho_\lambda(x), \rho_\lambda(y)$ be

$$\rho_\lambda(x) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \rho_\lambda(y) = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix},$$

regarded as linear maps $U_\lambda \to U_\lambda$ where U_λ is a k-vector space of dimension 2 with basis $\{e_1, e_2\}$.

(a) Show that ρ_λ defines a representation of G with representation space U_λ.

(b) Find a basis for $\operatorname{Soc} U_\lambda$.

(c) By considering the effect on $\operatorname{Soc} U_\lambda$, show that any kG-module homomorphism $\alpha : U_\lambda \to U_\mu$ has a triangular matrix $\alpha = \left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right)$ with respect to the given bases.

(d) Show that if $U_\lambda \cong U_\mu$ as kG-modules then $\lambda = \mu$. Deduce that kG has infinitely many nonisomorphic 2-dimensional representations.

Solution. Let k, G, and ρ_λ be defined as above.

(a)

Some routine computations gives us

$$\rho_\lambda(x)\rho_\lambda(y) = \begin{pmatrix} 1 & 0 \\ \lambda + 1 & 1 \end{pmatrix} = \rho_\lambda(y)\rho_\lambda(x)$$

and

$$(\rho_\lambda(x))^2 = (\rho_\lambda(y))^2 = I_2$$

proving ρ_λ is a group homomorphism into $GL_2(k)$, whence is a representation (which acts on U_λ by definition).

(b)

Any one dimensional subspace spanned by (a, b) must have $a = 0$ in order to be fixed by x (referring for simplicity to the element and its action on U_λ rather than its image under the map). But then the only proper subspace is the one spanned by e_2.

(c)

Leveraging the result from problem 1.4, if $\alpha : U_\lambda \to U_\mu$ is a kG-module homomorphism, then

$$\alpha(\operatorname{Soc} U_\lambda) = \alpha(\langle e_2 \rangle) \subseteq \langle e_2 \rangle = \operatorname{Soc} U_\mu.$$

That is, $\alpha(e_1) = ce_1$ for some $c \in k$, giving us that α has the form

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$$

for some $a, b \in k$ in the standard basis.

(d)

Assume that $U_\lambda \cong U_\mu$ via α. By the fact that the G action pulls through α, we get that

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ a+b & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ b+c & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and so $a + b = b + c$ whence $a = c = k$. That this matrix is invertible means $k \neq 0$.

Using a similar computation, we know

$$\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ \lambda a+b & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ b+\mu c & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}$$

so $\lambda a = \mu c$ or $\lambda k = \mu k$ and since $k \neq 0$, $\lambda = \mu$.

Thus since ρ_λ can be defined for any $\lambda \in k$, this gives us the existence of infinitely many non-isomorphic two dimensional representations, as desired.

□

Problem 1.10 Let

$$\rho_1 : G \to GL(V), \qquad \rho_2 : G \to GL(V)$$

be two representations of G on the same R-module V that are injective as homomorphisms. (We say that such a representation is *faithful*.) Consider the three properties that

(1) the RG-modules given by ρ_1 and ρ_2 are isomorphic,

(2) the subgroups $\rho_1(G)$ and $\rho_2(G)$ are conjugate in $GL(V)$,

(3) for some automorphism $\alpha \in \operatorname{Aut}(G)$, the representations ρ_1 and $\rho_2\alpha$ are isomorphic.

Show that (1)⇒(2) and that (2)⇒(3). Show also that if $\alpha \in \mathrm{Aut}(G)$ is an inner automorphism then ρ_1 and $\rho_1\alpha$ are isomorphic.

Solution. Let $\rho_i : G \to GL(V)$ be faithful representations acting on ${}_RV$.

(1)⇒(2)

Let V_1 and V_2 be V with the RG-module structure defined by ρ_1 and ρ_2, respectively and assume that $V_1 \cong V_2$, say via a map α. Let $v \in V$ be arbitrary and notice that by virtue of being a RG-module homomorphism,

$$\alpha(\rho_1(g)(v)) = \rho_2(g)\alpha(v), \quad \forall g \in G.$$

That is, $\alpha \circ \rho_1(g) = \rho_2(g) \circ \alpha$ as maps in $GL(V)$ (since α is invertible and R-linear) and furthermore

$$\rho_1(g) = \alpha^{-1} \circ \rho_2(g) \circ \alpha$$

for each $g \in G$ whence $\rho_1(G)$ is conjugate to $\rho_2(G)$ in $GL(V)$.

(2)⇒(3)

Now assume that $T^{-1}\rho_1(G)T = \rho_2(G)$ for some $T \in GL(V)$. Now since ρ_1 and ρ_2 are faithful, $\exists! h_g \in G$ for each $g \in G$ such that

$$T^{-1}\rho_1(g)T = \rho_2(h).$$

Define $\alpha \in \mathrm{Aut}(G)$ by $\alpha(g) = h_g$. This is a homomorphism since ρ_2 is injective and

$$\rho_2(\alpha(gg')) = T^{-1}\rho_1(gg')T = T^{-1}\rho_1(g)TT^{-1}\rho_1(g')T = \rho_2(\alpha(g))\rho_2(\alpha(g')) = \rho_2(\alpha(g)\alpha(g'))$$

and an automorphism since it is both surjective and injective.

Now define the map $\varphi = T^{-1} \in GL(V)$. This map is bijective and R linear, so we need only show it is G-linear to prove it is a module isomorphism. But consider

$$\rho_2(\alpha^{-1}g)(\varphi(v)) = T^{-1}\rho_1(g)T(T^{-1}v) = T^{-1}(\rho_1(g)(v)) = \varphi(\rho_1(g)(v)).$$

which proves G-linearity and we can conclude that $\rho_1 \cong \rho_2\alpha^{-1}$.

Finally,

Let $\alpha \in \mathrm{Aut}(G)$ be an inner automorphism corresponding to conjugation by $h \in G$. But then

$$\rho_1(\alpha(g)) = \rho_1(h^{-1}gh) = \rho_1(h^{-1})\rho_1(g)\rho_1(h) = H^{-1}\rho_1(g)H$$

for each $g \in G$, so $\rho_1(\alpha(G)) = H^{-1}\rho_1(G)H$, so (2) applies and the result follows from the argument above.

□

Problem 1.11 One version of the Jordan-Zassenhaus Theorem asserts that for each n, $GL(n,\mathbb{Z}) = \mathrm{Aut}(\mathbb{Z}^n)$ has only finitely many conjugacy classes of subgroups of finite order. Assuming this, show that for each finite group G and each integer n, there are only finitely many isomorphism classes of representations of G on $\mathbb{Z}^n$.

Solution. Fix a finite G and $n \in \mathbb{N}$. Assume by contradiction that there were infinitely many pairwise nonisomophic representations $\rho_i : G \to \mathbb{Z}^n$. Now consider the subgroups $H_i = \rho_i(G) \leq Z^n$.

Now if $H_i = k^{-1}H_jk$ then $H_i = H_j$ since $\mathbb{Z}^n$ is abelian. But since G is finite, so are the H_i, so there are only finitely many maps ρ_i that can map onto each finite $H_i \leq \mathbb{Z}^n$. Furthermore by Jordan-Zassenhaus, there exist only finitely many conjugacy classes among the H_i, giving that there must be, in fact, only finitely many H_i. This contradicts that there be infinitely many ρ_i and proves the statement.

□

Problem 1.12 Write out a proof of Maschke's theorem in the case of representations over $\mathbb{C}$ along the following lines.

(a) Given a representation $\rho : G \to GL(V)$ where V is a vector space over $\mathbb{C}$, let $(-,-)$ be a positive definite Hermitian form on V. Define a new form $(-,-)_1$ on V by

$$(v,w)_1 = \frac{1}{|G|}\sum_{g\in G}(gv,gw).$$

Show that $(-,-)_1$ is a positive definite Hermitian form, preserved by the action by G; that is $(v,w)_1 = (gv,gw)_1$ always. If W is a subrepresentation of V then show that $V = W \oplus W^\perp$, as representations.

(b) Show that any finite subgroup of $GL(n,\mathbb{C})$ is conjugate to a subgroup of $U(n,\mathbb{C})$ (the unitary group with matrices satisfying $A\bar{A}^T = I$). Show that any finite subgroup of $GL(n,\mathbb{R})$ is conjugate to a subgroup of $O(n,\mathbb{R})$, the orthogonal group of matrices satisfying $AA^T = I$.

Solution. **(a)** Define $(-,-)_1$ as above and we can see that this is a bilinear form since

$$(a\alpha+b\beta,\gamma)_1 = \frac{1}{|G|}\sum_g(ag\alpha+bg\beta,g\gamma) = \frac{1}{|G|}\sum_g\left(a(g\alpha,g\gamma)+b(g\beta,g\gamma)\right) = a(\alpha,\gamma)_1+b(\beta,\gamma)_1$$

and linearity in the second coordinate is proved via an identical argument. It is positive definite since $(-,-)$ is positive definite and the image of $(-,-)_1$ is simply the average of images of $(-,-)$.

Finally this form is Hermitian since

$$(v,w)_1 = \frac{1}{|G|}\sum(gv,gw) = \frac{1}{|G|}\sum\overline{(gw,gv)} = \overline{\frac{1}{|G|}\sum(gw,gv)} = \overline{(w,v)_1}.$$

That this form is preserved by action by G is evident since multiplying u and v by g is the same as simply permuting the summands in the definition.

Now let ${}_{\mathbb{C}G}W \leq V$. $W^{\perp}$ is a vector space in V, but we must show it is stable under the G-action. Assume that for $g \in G$ and $0 \neq w \in W^{\perp}$, $gw \in W$. But then since W is a subrepresentation, $g^{-1}(gw) = w \in W$, a contradiction. Thus W and $W^{\perp}$ are disjoint (from the definition/linear algebra) subrepresentations of V and it suffices to show $V = W + W^{\perp}$.

To see this,

□

Problem 1.13

(a) Using proposition 1.2.4, show that if A is a ring such that the regular representation ${}_AA$ is semisimple, then every finitely generated A-module is semisimple.

(b) Extend the result of part (a) using Zorn's lemma, to show that if A is a ring for which ${}_AA$ is semisimple then every A-module is semisimple.

Solution. **(a)** Let A be a ring with semisimple regular representation.

□