Hopf Algebras

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Introduction

These are the notes I took in class during the Winter 2019 topics course *Math 582H - Hopf Algebras* at University of Washington, Seattle.

The course description follows:

This course is an introductory course on homological algebra. We will be following the book *An Introduction to Homological Algebra* by Charles Weibel. We will be covering the following topics:

- Chain complexes, homotopies, homology and long exact sequence in homology
- Resolutions, derived functors, Ext and Tor. Koszul complexes
- Group (co)homology
- Triangulated and derived categories
- Spectral sequences or open topic depending on the class interests

1 April 1, 2019

If you don't know what a symmetric tensor category is, today is going to be a three star day. Max is 5.

1.1 Overview

We are shooting to understand two conjectures:

Conjecture (Etingof-Ostrik '04): If A is a finite dimentional Hopf algebra, then

$$\bigoplus_{i\geq 0} \operatorname{Ext}^i_A({}_Ak, {}_Ak)$$

is Noetherian.

Conjecture (Brown-Goodearl '98): If A is a Noetherian Hopf algebra, then the injective dimension of A_A is finite.

These are both still open! In fact there is a meeting at Oberwolfach this March concerning exactly these conjectures.

$$\begin{tikzcd}
V \otimes V \otimes V \arrow[r, "\mathrm{id}_V \otimes m"] \arrow[d, "m \otimes \mathrm{id}_V"] & V \otimes V \arrow[d, "m"] \\
V \otimes V \arrow[r, "m"] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
k \otimes V \arrow[r, "\sim"] \arrow[dr, "u \otimes \mathrm{id}_V"'] & V & V \otimes k \arrow[l, "\sim"'] \arrow[dl, "\mathrm{id}_V \otimes u"] \\
& V \otimes V \arrow[u, "m"] &
\end{tikzcd}$$

Figure 1: Diagrams for definition 1.2.1.

1.2 Symmetric Tensor Categories

We are going to be using the following notation throughout:

- k is a field
- $\mathbf{Vect}_{\mathrm{k}}$ is the category of k-vector spaces
  - $\mathbf{Vect}_{\mathrm{k}}$ is closed under tensor products
  - There is an element $k \in \mathbf{Vect}_{\mathrm{k}}$ such that

    $$k \otimes_k V \cong V \cong V \otimes_k k$$

    where the above isomorphisms are natural.
  - $V \otimes_k W \cong W \otimes_k V$
- An algebra is an object in $\mathbf{Vect}_{\mathrm{k}}$.

1.2.1 Definition: $V \in \mathbf{Vect}_{\mathrm{k}}$ is called an **algebra object** if there are two morphisms

(a) $m : V \otimes V \to V$

(b) $u : k \to V$

such that the diagrams in figure 1.2 commute.

1.2.2 Lemma
$V \in \mathbf{Vect}_{\mathrm{k}}$ is an algebra object iff V is an algebra over k.

1.2.3 Lemma
If C is a symmetric tensor category, so is C^{op}.

Then the natural thing to ask is: what is an algebra object in this opposite category?

1.2.4 Definition: A **coalgebra object** in C is an algebra object in C^{op}. Here we have comultiplication Δ and counit ε.

1.2.5 Remark: Naturally you could go about defining this from first principles and drawing the diagrams in figure 1.2 with the arrows reversed, but we are probably mature enough to do without that (saving my fingers from repetitive strain injury in the process.)

1.2.6 Lemma
$\mathbf{Alg}_k$, defined as the category of algebra objects in $\mathbf{Vect}_{\mathrm{k}}$, is a symmetric tensor category. Furthermore $\mathbf{Coalg}_k$, the category of coalgebra objects in $\mathbf{Vect}_{\mathrm{k}}$, is a symmetric tensor category.

1.2.7 Lemma
The following are equivalent:

(a) V is an algebra object in $\mathbf{Coalg}_k$

(b) V is a coalgebra object in $\mathbf{Alg}_k$

(c) There are morphisms $m, u, \Delta, \varepsilon$ such that

- (V, m, u) is an algebra
- (V, Δ, ε) is a coalgebra
- Equivalently:
  - m and u are coalgebra morphisms
  - Δ and ε are algebra morphisms.

Proof
The nice thing here is that the $(a) \Leftrightarrow (c)$ without the last condition. A similar fact holds for (b) except the second-to-last. The last thing to do is to prove the last two conditions are equivalent. ♠

Problem 1.1
Fill in the details for the proof above.

Solution:
Assume that $(V, m, u, \Delta, \varepsilon)$ is an algebra and coalgebra and further that m and u are coalgebra morphisms. That means in particular that the diagrams in figure 1 commute.

$$\begin{tikzcd}
V \otimes V \arrow[r, "m"] \arrow[d, "\Delta\otimes\Delta"'] & V \arrow[r, "\Delta"] & V \otimes V \\
V^{\otimes 4} \arrow[rr, "T_{2,3}"] & & V^{\otimes 4} \arrow[u, "m\otimes m"']
\end{tikzcd}
\qquad
\begin{tikzcd}
k \arrow[r, "\Delta"] \arrow[d, "u"] & k \otimes k \arrow[d, "u\otimes u"] \\
V \arrow[r, "\Delta"] & V \otimes V
\end{tikzcd}$$

Figure 2: m and u are coalgebra morphisms.

$$\begin{tikzcd} V\otimes V \arrow[r, "m"] \arrow[d, "\Delta\otimes\Delta"] & V \arrow[d, "\Delta"] \\ V^{\otimes 4} \arrow[r, "m"] & V\otimes V \end{tikzcd} \qquad \begin{tikzcd} V\otimes V \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "m"] & k\otimes k \arrow[d, "m"] \\ V \arrow[r, "\varepsilon"] & k \end{tikzcd}$$

Figure 3: Δ and ε are algebra morphisms.

We are looking to prove that Δ and ε are algebra morphisms, or that the diagrams in figure 1 commute.

From here it's actually a bit boring because it's kinda just a definition/notation game. It boils down to the fact that the (co)multiplication on $V \otimes V$ has a twist that exactly lines up so that each square is saying the same thing. ♠

1.2.8 Definition: V is called a **bialgebra object** if V is an algebra object in $\mathbf{Coalg}_k$.

Problem 1.2

(a) Suppose that $\operatorname{char} k \neq 2$. *Classify all bialgebras of* $\dim 2$.

(b) Do the same for $\operatorname{char} k = 2$.

Solution:
Part (a)

Consider $\varepsilon : V \to k$ and consider $\ker\varepsilon \lhd V$. By rank-nullity, $\dim\ker\varepsilon = 1$, so $\ker\varepsilon = kx$ for some $x \in V$. Therefore $x^2 = cx$ for some c, and if $c = 0$, then (as an algebra) $V \cong k[x]/(x^2)$. Otherwise consider $y = \frac{x}{c}$. In this case $y^2 = \frac{x^2}{c^2} = \frac{x}{c} = y$, and $V \cong k[x]/(x^2 - x)$.

Notice that in either case $\varepsilon(x) = 0$, so let

$$\Delta(x) = a(1\otimes 1) + b(1\otimes x) + c(x\otimes 1) + d(x\otimes x)$$

and using that $\varepsilon\otimes\mathrm{id}\circ\Delta = \mathrm{id}\otimes\varepsilon\circ\Delta$ and that each should be (essentially) the identity (this is just the diagram we saw before), we get $a = 0$ and $b = c = 1$. Thus the coalgebra structure of any Hopf algebra is given by

$$\varepsilon(x) = 0, \quad \Delta(x) = 1\otimes x + x\otimes 1 + d(x\otimes x).$$

Consider first the case when $x^2 = 0$. Then since comultiplication will be an algebra morphism,

$$0 = \Delta(x^2) = \Delta(x)^2 = 1\otimes x^2 + x^2\otimes 1 + d^2(x^2\otimes x^2) + 2(x\otimes x) + 2d(x\otimes x^2) + 2d(x^2\otimes x)$$

and since $x^2 = 0$,

$$0 = 2(x \otimes x).$$

But $x \otimes x$ is a basis element of $V \otimes V$, so V can only have this algebra structure when $\operatorname{char} k = 2$. We will return to this in the next part.

So then $x^2 = x$ and using the computation above,

$$1 \otimes x + x \otimes 1 + d(x \otimes x) = \Delta(x) = \Delta(x^2) = 1 \otimes x + x \otimes 1 + (d^2 + 4d + 2)(x \otimes x)$$

so

$$(d^2 + 3d + 2)(x \otimes x) = 0 \quad \Rightarrow \quad d^2 + 3d + 2 = (d + 2)(d + 1) = 0$$

and so either $d = -1$ or $d = -2$.

One can verify that Δ is coassociative, so we can conclude that when $\operatorname{char} k \neq 2$, there are precisely two Hopf algebra structures with algebra structure $k[x]/(x^2 - x)$ and comultiplication either

$$\Delta(x) = 1 \otimes x + x \otimes 1 - x \otimes x \quad \text{or} \quad \Delta(x) = 1 \otimes x + x \otimes 1 - 2(x \otimes x)$$

(b)

Now assume that $\operatorname{char} k = 2$ and that $V \cong k[x]/(x^2 - x)$ as an algebra. Then using the analysis above, we see that we can choose comultiplication either

$$\Delta(x) = x \otimes 1 + 1 \otimes x \quad \text{or} \quad \Delta(x) = 1 \otimes x + x \otimes 1 + x \otimes x.$$

If instead $V \cong k[x]/(x^2)$, then *any* value of d will suffice, so there are a full k's worth of Hopf algebra structures that can appear. ♠

2 January 9, 2019

Today we are going to rely heavily on Sweedler notation. :) Notice that if we are looking at actual objects in the diagram for coassociativity, we get

$$\begin{tikzcd}
v \arrow[rr] \arrow[d] & & \sum v_{(1)} \otimes v_{(2)} \arrow[d] \\
\sum v_{(1)} \otimes v_{(2)} \arrow[r] & \sum (v_{(1)})_{(1)} \otimes (v_{(1)})_{(2)} \otimes v_{(2)} = & \sum v_{(1)} \otimes (v_{(2)})_{(1)} \otimes (v_{(2)})_{(2)}
\end{tikzcd}$$

Figure 4: Coassociativity on elements in Sweedler notation

Example 2.1
Let G be a group and kG be the group algebra. The algebra structure arises as normal where $g \cdot h$ comes from the structure on G. Then $\Delta(g) = g \otimes g$ and this extends linearly.

But then if you consider $\Delta(\sum c_g g)$, notice that by the nature of tensors this is not unique! So we will just write

$$\Delta\left(\sum c_g g\right) = \sum_G c_g (g \otimes g) = \sum v_{(1)} \otimes v_{(2)}$$

2.1 Algebra structure on $V \otimes V$

We said earlier on that $\mathbf{Alg}_k$ is a symmetric *tensor* category. But how do we define the multiplication on the tensor product?

Well it all comes from the twist! We define

$$m_{V\otimes W} = (m_V \otimes m_W) \circ (\mathrm{id}_V \otimes \tau_{2,3} \otimes \mathrm{id}_W)$$

where $\tau_{2,3}$ is the twist morphism.

More simply, $u_{V\otimes W} : k = k \otimes k \to V \otimes V$ simply defined by $u_V \otimes u_W$.

So then when we say that Δ is an algebra morphism, we are saying that for all $v, w \in V$

$$\Delta(vw) = \sum (vw)_{(1)} \otimes (vw)_{(2)} = (\sum v_{(1)} \otimes v_{(2)})(\sum w_{(1)} \otimes w_{(2)}) = \sum v_{(1)}w_{(1)} \otimes v_{(2)}w_{(2)}$$

2.2 Hopf Algebras

Already to the good stuff!

2.2.1 Definition: $V \in \mathbf{Vect}_\mathrm{k}$ is a **Hopf algebra** if V is a bialgebra together with an **antipode** $S : V \to V$ satisfying

$$(S, \mathrm{id}_V) \circ \Delta = \varepsilon = (\mathrm{id}_V, S) \circ \Delta$$

Conjecture: If $V \in \mathbf{Vect}_\mathrm{k}$ is a Noetherian Hopf algebra, then S is bijective.

2.3 History and Motivation

Hopf himself was a topologist, so this is the first context in which it arose. In the 1940's, he began studying Hopf algebras over $\mathbb{Z}_2$ graded k vector spaces. For instance, the cohomology ring of topological space X with coefficients in k.

Later, in combinatorics, they ended popping up. Looking at rings of symmetric functions and other places gave some interesting examples.

Then in group theory you can define a functor from groups to Hopf algebras by $F(G) = kG$ with the diagonal map. The antipode is just the inverse.

Then with Lie algebras, you can look at $\mathcal{U}(L)$, the universal enveloping algebra is a Hopf algebra.

Finally with algebraic groups (yay!) we take an algebraic group G and consider the ring of functions on it, which is again a Hopf algebra.

Some "cousins" of Hopf algebras: quasi, weak, multiplier, ribbon, quasi-triangular, etc Hopf algebras. Each has slightly different base category or restrictions.

3 January 11, 2019

The plan for today is to talk about:

- Convolution Algebras
- Antipodes
- Duality
- (Co-)Modules

3.1 Convolution Algebras

Let $\mathcal{T}$ be a symmetric tensor category. We can usually think of $\mathcal{T} = \mathbf{Vect}_{\mathrm{k}}$, but there is a problem since $\mathbf{Vect}_{\mathrm{k}}$ is equivalent to the category of Hopf algebras over k, while this is not generally true.

We also need that $\mathcal{T}$ is k-linear (that is, enriched as a category over k). This means that $\mathrm{Hom}_{\mathcal{T}}(A, B) \in \mathbf{Vect}_{\mathrm{k}}$.

3.1.1 Theorem

Let $\mathcal{T}$ be as above. Then $\mathrm{Hom}_{\mathcal{T}}(C, A)$ is an algebra and $\mathrm{Hom}_{\mathcal{T}}(-,-) : (\mathbf{Coalg}_{\mathcal{T}})^{op} \times \mathbf{Alg}_{\mathcal{T}} \to \mathbf{Alg}_k$ is a functor.

PROOF

Let A be an algebra object in $\mathcal{T}$ and C be a coalgebra object in $\mathcal{T}$. Then $1_{\mathrm{Hom}} := u_A \circ \varepsilon_C : C \to 1_{\mathcal{T}}$ and define

$$f * g := m_A(f \otimes g)\Delta_C : C \to A.$$

Then using Lemma 3.1.2 and the fact that A and C are (co)algebra objects, we can see that the product $*$ satisfies the axioms required.

Note that actually ♠

3.1.2 Lemma

(a) $1_{\text{Hom}} * f = m_A(u \otimes 1)(1 \otimes f)(\varepsilon \otimes 1)\Delta$.

(b) $f * 1_{\text{Hom}} = m_A(1 \otimes u)(f \otimes 1)(1 \otimes \varepsilon)\Delta$

(c) $(f * g) * h = m_A(m_A \otimes 1)((f \otimes g) \otimes h)(\Delta_C \otimes 1)\Delta_C$

(d) $f * (g * h) = m_A(1 \otimes m_A)(f \otimes (g \otimes h))(1 \otimes \Delta)\Delta$

Proof

(a)

$$\begin{aligned} 1_{\text{Hom}} * f &= m_A(1_{\text{Hom}} \otimes f)\Delta_C \\ &= m_A(u_A \circ \varepsilon \otimes f)\Delta_C \\ &= m_A(u \otimes 1)(\varepsilon \otimes 1)(1 \otimes f)\Delta \end{aligned}$$

(b)

Same as (a), essentially.

(c) and (d)

$$\begin{aligned} (f * g) * h &= m_A((f * g) \otimes h)\Delta \\ &= m_A((m_A(f \otimes g)\Delta) \otimes h)\Delta \\ &= m_A(m_A \otimes 1)((f \otimes g) \otimes h)(\Delta \otimes 1)\Delta \end{aligned}$$

and the other is analogous. ♠

3.1.3 Definition: $V \in \mathcal{T}$ is a **Hopf algebra object** if:

- V is a bialgebra object in $\mathcal{T}$ and
- There is a map $S : V \to V$ that is $(\text{id}_V)^{-1}$ with respect to $*$.

3.1.4 Remark: Notice here that $\text{id}_V \in \text{Hom}_{\mathcal{T}}(V, V)$, the identity map in $\mathcal{T}$. We are *not* taking about $1_{\text{Hom}} = u \circ \varepsilon$.

Also, we call S an **antipode.**

3.2 Duality

Notice that when $C = 1_{\mathcal{T}}$ (that is the tensor identity), $\mathrm{Hom}_{\mathcal{T}}(1_{\mathcal{T}}, -) : \mathbf{Alg}_{\mathcal{T}} \to \mathbf{Alg}_k$ is a functor. Same for the dual from $\mathbf{Coalg}_{\mathcal{T}}$. This second one gives us a chance to talk about duality.

3.2.1 Lemma
Let $\mathcal{T}$ be the category of finite dimensional vector spaces over k. Then $(-)^* : \mathcal{T} \to \mathcal{T}^{op}$ is an equivalence.

This uses $(V \otimes W)^* = W^* \otimes V^*$.

3.2.2 Corollary
V is an algebra over $k \Leftrightarrow V^*$ is a coalgebra over k. And vice versa.

Recall $S = (\mathrm{id}_V)^{-1}$. Thus

$$S * \mathrm{id}_V = 1_{\mathrm{Hom}} = \mathrm{id}_V * S$$

The diagram we have here is

$$\begin{tikzcd}
V \otimes V \arrow[rr, "S\otimes \mathrm{id}_V"] & & V \otimes V \arrow[d, "m_V"] \\
V \arrow[u, "\Delta"] \arrow[r, "\varepsilon_V"] \arrow[d, "\Delta"'] & k \arrow[r, "u_V"] & V \\
V \otimes V \arrow[rr, "\mathrm{id}_V \otimes S"] & & V \otimes V \arrow[u, "m_V"']
\end{tikzcd}$$

Modules/Comodules

3.2.3 Definition: Let A be an algebra object in $\mathcal{T}$. A **left** A **module** is $M \in \mathcal{T}$ with a morphism

$$m_M : A \otimes M \to M$$

such that the diagrams in Figure 3.2 commute.

$$\begin{tikzcd}
A \otimes A \otimes M \arrow[r, "m_A\otimes 1"] \arrow[d, "1\otimes m_M"] & A \otimes M \arrow[d, "m_M"] \\
A \otimes M \arrow[r, "m_M"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes M \arrow[r, "m_M"] & M \\
1_{\mathcal{T}} \otimes M \arrow[u, "u_A\otimes 1"] \arrow[ur, "\sim"] &
\end{tikzcd}$$

Figure 5: Module diagrams

3.2.4 Remark: Note that we don't necessarily need that M lie in $\mathcal{T}$. We could instead just rely on an algebra homomorphism $\varphi : A \to \mathrm{Hom}_{\mathcal{T}}(M, M)$ and proceed as usual.

4 January 14, 2019

Today we are talking about Hopf modules and later in the week we will see the fundamental theorem of Hopf modules as well as a neat result.

Here is the "example of the day."

Example 4.1
Consider $k[x]$ with maps $\Delta(x^n) = \sum_0^n x^i \otimes x^{n-i}$ and $\varepsilon(x^n) = \delta_{1,n}$. $S(x^n) = (-x)^n$. Then notice that $\Delta(x) = 1 \otimes x + x \otimes 1$, so $\delta(x^n) = (\delta x)^n$.

James doesn't want to do the rest of the computations, but they can be done. :)

Problem 4.1
(**) Working with a Hopf algebra V, consider the convolution algebra $\mathrm{Hom}(V \otimes V, V)$

(a) $(S \circ m) * m = 1_{\mathrm{Hom}(V\otimes V,V)} = m * (m \circ s \otimes s \circ \tau)$ *where s is the antipode in V.*

(b) $(\Delta \circ S) * \Delta = 1_{\mathrm{Hom}(V,V\otimes V)} = \Delta * ((s \otimes s) \circ \tau \circ \Delta)$

Solution:

$$(s \circ m) * m$$

Problem 4.2
Classify all Frobenius (to be defined) Hopf algebras of dimension 3.

4.1 Returning to (co)modules

4.1.1 Lemma

Let $\mathcal{T} = \mathbf{Vect}_{\mathrm{k}}$, and V and algebra over k. The following are equivalent:

(a) M is a left V-module (see last lecture).

(b) There is an action of A on M such that $(ab) \cdot m = a \cdot (b \cdot m)$ and $1 \cdot m = m$.

(c) There is an algebra morphism $\varphi : V \to \mathrm{Hom}_k(M, M)$

4.1.2 Definition: Let V be a coalgebra over k. We say M is a left **comodule** if there is a map

$$\rho_M : M \to V \otimes M$$

such that the diagrams in figure 4.1 commute.

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\rho"] & V \otimes M \arrow[d, "\mathrm{id}_V \otimes \rho"] \\ V \otimes M \arrow[r, "\Delta_V \otimes \mathrm{id}_M"] & V \otimes V \otimes M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\cong"] & V \otimes M \arrow[dl, "\varepsilon \otimes \mathrm{id}_M"] \\ k \otimes M & \end{tikzcd}$$

Figure 6: Diagrams for the definition of a comodule.

4.1.3 Remark: Notationally speaking, we write ${}_V\mathcal{M}$ and ${}^V\mathcal{M}$ for the categories of left modules and comodules. Similar for right boyes.

4.2 Tensors of V-modules

Say V is a bialgebra (or Hopf if you prefer, but it's not necessary). Then we get the following:

4.2.1 Lemma
Let M and N be two right V-modules. Then

(a) $M \otimes N$ is a right V-module.

(b) If V is cocommutative, then $M \otimes N \cong N \otimes M$ as right V-modules.

(c) $\mathcal{M}_V$ is a tensor (monoidal) category.

Proof
For (a), use the fact that $V \otimes V$ acts on $M \otimes N$ in a natural way. Then you get a map $V \otimes V \to \mathrm{Hom}_k(M \otimes N, M \otimes N)^{op}$ (why opposite?) and by precomposing with Δ to show they are V-modules. This establishes (b).

Finally (c) follows because $\mathbf{Vect}_\mathrm{k}$ is a tensor category. ♠

4.3 Hopf Modules

4.3.1 Definition: Let V be a bialgebra over k. We say that M is a $\binom{r}{r}$ Hopf V-module if

(a) (M, m_M) is a right V-module

(b) (M, ρ_M) is a right V comodule.

(c) ρ_M is a right V-module map.

- Equivalently, m_M is a right V-comodule map.

4.3.2 Remark: The fact that these two last conditions are equivalent are perhaps not immediately obvious but we have been assured it is true. :)

5 January 16, 2019

5.0.1 Theorem (Larson-Sweedler '69)

Let V be a Hopf algebra over k. Then the following categories are equivalent:

$$ {}^V_V\mathcal{M} \cong \mathbf{Vect}_\mathrm{k} \cong \mathcal{M}^V_V $$

5.0.2 Remark: A natural question that one may ask is how to extend this theorem to the enriched setting.

5.0.3 Lemma

S is an anti-homomorphism of an algebra V and an anti-homomorphism of the coalgebra V.

Proof

Use the facts proved in problem 4.1. ♠

Problem 5.1

If $\operatorname{char} k = 0$ *any Hopf quotient of* $k[x]$ *is either* k *or* $k[x]$ *itself.*

Problem 5.2

Suppose k *has positive characteristic. Construct two different Hopf algebra quotients of* $k[x]$ *of dimension* $p = \operatorname{char} k$.

Example 5.1
$\operatorname{char} k = p > 0$. Let $V = k[x]/(x^p)$ as an algebra and define $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$ and $S(x) = x$. Then notice

$$\Delta(x^p) = \sum \binom{p}{i} x^i \otimes x^{p-i} = x^p \otimes 1 + 1 \otimes x^p.$$

5.1 Hopf Modules Once More

Recall the definitions from section 4.1 and section 4.3.

5.1.1 Remark: An equivalent (more category-theoretical) definition of a Hopf module is a V-comodule object in the category of V-modules. Also can dualize everything.

Example 5.2
Consider $V \in^V_V \mathcal{M}$. THen you can define $\rho = \Delta$ and check it satisfies all the requirements.

5.1.2 Definition: An $\binom{l}{l}$ Hopf V-module M is called **trivial** if $M \cong V \otimes M_0$ for some $M_0 \in \mathbf{Vect}_\mathrm{k}$.

5.1.3 Remark: Basically you can think of this being trivial since we can *always* define a module in this way where the entire module structure is inherited from the structure of V (that is, irrespective of M_0).

5.0.1′ Theorem
Suppose that V is a Hopf algebra. Then

(a) Every $\binom{l}{l}$ Hopf V-module is trivial.

(b) Let $M \in^V_V \mathcal{M}$. Then
$$M \xrightarrow{\sim} V \otimes M^{Cov}$$
where
$$M^{Cov} := \{m \in M | \rho(m) = 1 \otimes m\}.$$

(c) ${}^V_V\mathcal{M} \cong \mathbf{Vect}_\mathrm{k}$.

5.1.4 Remark: This is actually just a reformulation of theorem 5.0.1 in less compact (but ultimately more usable) notation.

Proof
Some identites that will be useful:

$$\begin{aligned}\Delta(v) &= \sum v_1 \otimes v_2 \\ \sum S(v_1) \otimes v_2 = \varepsilon(v) &= \sum v_1 S(v_2) \\ \rho(m) &= \sum m_{-1} \otimes m_1 \\ \sum v_1 \otimes (v_2)_1 \otimes (v_2)_2 = \sum (v_1)_1 \otimes (v_1)_2 \otimes v_2 &= \sum v_1 \otimes v_2 \otimes v_3 \\ \sum (m_{-1})_1 \otimes (m_{-1})_2 \otimes m_0 = \sum m_{-1} \otimes (m_0)_{-1} \otimes (m_0)_0 &= \sum m_{-2} \otimes m_{-1} \otimes m_0\end{aligned}$$

Then we defin $\phi : M \to M^{Cov}$ by

$$\phi(x) = \sum S(x_{-1})x_0 \in M$$

We claim first that $\phi(x) \in M^{Cov}$ – namely $\rho(\phi(x)) = 1 \otimes \phi(x)$. To see this, compute

$$\begin{aligned}\rho(\phi(x)) &= \sum (\phi(x))_{-1} \otimes (\phi(x))_0 \\ &= (S(x_{-1})x_0)_{-1} \otimes (S(x_{-1})x_0) \\ &= \sum \Delta(S(x_{-1}))\rho(x_0) \\ &= \sum (S \otimes S) \circ \tau(\Delta(x_{-2})) \cdot [x_{-1} \otimes x_0] \\ &= \sum [S(x_{-2}) \otimes S(x_{-3})][x_{-1} \otimes x_0] \\ &= \text{my fingers are aching...} \\ &= 1 \otimes \phi(x)\end{aligned}$$

Now define $F : M \to V \otimes M^{Cov}$ by $F = (\mathrm{id} \otimes \phi) \circ \rho$ and $G : V \otimes M^{Cov}$ to be the map taking $v \otimes m$ to vm. The next claim is that GF and FG are the identity. You can see this by similarly pushing around notation.

Finally the last claim will be seen on Friday. ♠

6 January 18, 2019

Recall the Fundamental Theorem of Hopf Modules: ${}^V_V\mathcal{M} \simeq \mathbf{Vect}_\mathrm{k}$ or equivalently that every (not necessarily finite dimensional) Hopf module is trivial: $M \cong V \otimes M^{Cov}$.

7 Frobenius Algebras

Today we will discuss the result that

7.0.1 Theorem (Larson-Sweedler, '69)
Every finite dimensional Hopf algebra is Frobenius.

There are some nice categorical equivalences: finite dimensional Frobenius algebras over k is equivalent to 2-D quantum field theories, or equivalently the symmetric functor category from the 2-D cobordism category to $\mathbf{Vect}_{\mathrm{k}}$.

Whoa.

Example 7.1
$V = (kG)^* = \oplus_{g\in G} k\delta g$ where G is a finite group. Define the algebra structure with

$$1_V = \sum_{g\in G} \delta_g$$

and $\delta_g\delta_h = \delta_g$ when $g = h$ and 0 otherwise.

The coalgebra structure is $\varepsilon(\delta_g) = \delta_{g,1_G}$ where on the right it is the Kroneker delta. Furthermore $\Delta(\delta_g) = \sum_{h\in G} \delta_h \otimes \delta_{h^{-1}g}$ and $S(\delta_g) = \delta_{g^{-1}}$.

Problem 7.1
Prove that $kS_3 \cong (kS_3)^*$.

Problem 7.2
Prove that $k\mathbb{Z}_3 \cong (k\mathbb{Z}_3)^*$ *when* $\operatorname{char} k \neq 3$.

7.0.2 Lemma
Let A and B be algebras over k.

(a) If $M \in_A \mathcal{M}_B$, $N \in \mathcal{M}_B$, then $\mathrm{Hom}_{\mathcal{M}_B}({}_AM, N) \in \mathcal{M}_A$.

(b) Take $B = k$, $M \in_A \mathcal{M}$. Then

$$M^* := \mathrm{Hom}_k(M, k) \in \mathcal{M}_A.$$

(c) $(-)^* :_A \mathcal{M}_{f.d.} \to (\mathcal{M}_A)^{op}_{f.d.}$ is an equivalence of categories. This is called the **reflection principle.**

Proof

This is a known fact. (b) follows quickly from (a) and (c) involves extending the dual map to maps of left A-modules. ♠

By the principle of duality, we have:

7.0.3 Lemma

If C and d are finite dimensional coalgebras over k, then

(a) if $M \in^C \mathcal{M}^D_{f.d.}$ and $N \in \mathcal{M}^D_{f.d.}$ then $\mathrm{Hom}_{\mathcal{M}^D}(M,N) \in \mathcal{M}^C_{f.d.}$.

(b) If $M \in^C \mathcal{M}_{f.d.}$ then $\mathrm{Hom}_k(M,k) \in \mathcal{M}^C$

(c) $(-)^* :^C \mathcal{M}_{f.d.} \to (\mathcal{M}^C_{f.d.})^{op}$ is an equivalence of categories.

7.0.4 Definition: A finite dimensional algebra A is called **Frobenius** if one of the following (equivalent) conditions hold:

(a) $(A)^*_A \cong A_A$

(b) ${}_A(A)^* \cong_A A$

7.0.5 Remark: Note that the equivalence of these two things follow due to the equivalence of categories we get from the two lemmas above.

We actually don't need the following lemma, but James is a fan so we're going to write it down.

7.0.6 Lemma

Let A be a finite dimensional algebra over k. Then M is a finite dimensional left A-module if and only if M is a finite dimensional right A^*-comodule.

7.0.7 Remark: The reason we get the module to comodule switch is (partly?) due to the fact that in our monoidal category $(V \otimes W)^* = W^* \otimes V^*$.

7.0.8 Lemma

Let V be a finite dimensional Hopf algebra and M be a finite dimensional $\binom{l}{l}$ Hopf V-module. Then M^* is a finite dimensional $\binom{r}{r}$ Hopf V-module.

Proof

$M \in {}^V_V\mathcal{M}$ if and only if $M \in_V \mathcal{M} \cap {}^V\mathcal{M}$ and the two structures are compatible.

The fact that $M^* \in \mathcal{M}_V \cap \mathcal{M}^V$ is straightforward and checking the compatibility is easy to check. ♠

Now we finally prove that theorem 7.0.1:

Proof
First of all $V \in {}^V_V\mathcal{M}$ so by the lemma $V^* \in \mathcal{M}^V_V$. By the fundamental theorem of Hopf modules,

$$V^* \cong V \otimes (V^*)^{Cov} \cong V$$

as Hopf modules. Thus $V^*_V \cong V_V$ in particular. ♠

7.0.9 Remark: The second isomorphism above comes from the fact that $(V^*)^{Cov}$ is one-dimensional, so is the trivial module. :)

7.0.10 Corollary
If A is *not* a Forbenius algebra, then there is *no* Hopf algebra structure on A.

Example 7.2
Some examples of algebras with(out) this property.

(a) $\mathbb{C}[x]/(x^n)$ is finite dimensional and Frobenius, but if $n \geq 2$, there is **no Hopf structure.**

(b) $\mathbb{C}[x, y]/(x^2, y^2, xy) = \mathbb{C}1 + \mathbb{C}x + \mathbb{C}y$. This has the property that it is local (unique maximal submodule over itself) but A^* has two maximal submodules, so $A \ncong A^*$, so not Frobenius.

Finally we conclude with a proof of lemma 7.0.6:

Proof
Say that $\{a_i\}$ is a basis for A and $\{a_i^*\}$ is a basis for A^* $(a_i^*(a_j) = \delta_{ij})$.

Define $\rho(m) = \sum(a_i \cdot m) \otimes a_i^*$. A lemma (use change of basis matrices) shows that $\sum a_i \otimes a_i^*$ is independent of choice of basis. Another lemma says (scalar) multiplication is associative if and only if ρ is coassociative. This gets us the forward direction.

For the reverse direction, say $\rho(m) = \sum m_0 \otimes m_{-1}$. then define $a \cdot m = \sum m_1(a) m_0$. ♠

8 January 23, 2019

Today we are interested in studying the representation category ${}_V\mathcal{M}$ for any Hopf algebra V. It will end up (and this is a bit cryptic for now) that k controls everything here.

9 Represetnations and Modules

Notice that for any Hopf algebra V we have the exact sequence:

$$0 \to \ker \varepsilon \to V \xrightarrow{\varepsilon} k \to 0$$

9.0.1 Definition: The trivial V modules is $V/\ker\varepsilon = k \in_V \mathcal{M}_V$.

Example 9.1
$V = kG$ where G is a group, $\varepsilon(g) = 1$ for all $g \in G$. Then $\ker\varepsilon = \oplus k(g-1)$ and so

$$k = V/\oplus k(g-1).$$

Example 9.2
$V = k[x]$ where $\Delta(x) = 1\otimes x + x\otimes 1$ and $\varepsilon(x) = 0$. Then $\ker\varepsilon = \oplus_{n\geq 1} kx^n$, whence

$$V/\ker\varepsilon = k[x]/(x).$$

Example 9.3
Let $V = (kG)^* = \oplus_G k\delta_g$ and $\varepsilon(\delta_g) = \delta_{1,1_G}$. THen $\ker\varepsilon = \oplus_{g\neq 1_G} k\delta_g$ and

$$V/\ker\varepsilon = k\delta_1$$

The following uses T_4, the example of the day I missed due to my meeting.

Example 9.4
T_4: $\varepsilon(g) = 1$, $\varepsilon(p) = 0$. Then $\ker\varepsilon = k(g-1)\oplus kp\oplus kpg$ and

$$V/\ker\varepsilon = k.$$

9.0.2 Lemma

- For all $x \in V$ and $a \in k$ (the trivial module), $xa = \varepsilon(x)a$.
- k is the identity object in ${}_V\mathcal{M}$.
- If V is cocommutative, then ${}_V\mathcal{M}$ is a symmetric tensor category.

Proof

$$x \cdot a = (x - \varepsilon(x))a + \varepsilon(x)a = \varepsilon(x)a$$

since $x - \varepsilon(x) \in \ker \varepsilon$.

For the next part, consider that for every $M \in_V \mathcal{M}$, we have

$$M \otimes k \cong M \cong k \otimes M.$$

♠

But then for all $x \in V$, we have

$$\begin{aligned}\varphi[x\dot{(}m \otimes 1)] &= \varphi(x_1 \cdot m \otimes x_2 \cdot 1)\\ &= \varphi(x_1 \cdot m \otimes \varepsilon(x_2) \cdot 1)\\ &= \varphi((x_1\varepsilon(x_2))m \otimes 1)\\ &= \varphi(xm \otimes 1) = xm = x\varphi(m \otimes 1).\end{aligned}$$

9.1 Integrals

9.1.1 Definition: An element $x \in V$ is called a **left integral** if $vx = \varepsilon(v)x$ for all $v \in V$. Similar for **right integral.**

9.1.2 Lemma

The following are equivalent:

- $x \in V$ is a left integral.
- $kx \cong$ the trivial module.
- $1 \mapsto x$ defines a left V-module morphism $k \to kx$.

9.1.3 Lemma

Let $\int_V^l$ denote the set of left integrals in V.

- $\int_V^l$ forms a vector subspace of V.
- $\int_V^l$ forms a left V submodule of V. Thus it forms a left ideal of V.
- $\int_V^l$ forms a right ideal of V.
- $\int_V^l \cong \mathrm{Hom}_V(k, V)$ (left V-mods)

PROOF

The first three are relatively obvious. The last one comes from the map $\Phi : x \in \int_V^l \mapsto f_x : 1 \to x$. ♠

9.1.4 REMARK: Unfortunately it ends up that the collection above ends up usually being zero.

9.1.5 Theorem

Let V be a finite dimensional Hopf algebra. Then $\dim \int_V^l = 1$.

PROOF

$$\begin{aligned}\dim \int_V^l &= \dim \mathrm{Hom}_V({}_V k, {}_V V)\\ &= \dim \mathrm{Hom}_V((V^*)_V, k_V^*)\\ &= \dim \mathrm{Hom}_V(V_V, k) = \dim k = 1\end{aligned}$$

Where we used above that V was finite dimensional whence Frobenius to get $V^* \cong V$. ♠

Example 9.5

$V = kG$ for G a finite group. Then $\int_V^l = \sum_G g$ and

$$v \cdot (\sum_G g) = \varepsilon(v)(\sum_G g) = \int_V^r$$

and when $v = h \in G$, we can compute it.

Example 9.6

When $V = k[x]$, which is infinite dimensional, we get no integrals. This is because it is an integral domain, so if

$$x \cdot \int_V^l = \varepsilon(x) \int_V^l = 0 \cdot \int_V^l = 0$$

then this forces $\int_V^l = 0$.

Example 9.7
When $V = (kG)^*$ for G a finite group, then

$$\int_V^l = \int_V^r = \delta_{1_G}$$

Example 9.8
When $V = T_4 = k1 \oplus kg \oplus kp \oplus kgp$,

$$\int_V^l = (g+1)p \neq \int_V^r = p(1+g) = (1-g)p$$

Example 9.9
When $\operatorname{char} k = p$ and $V = k[x]/(x^p)$, then

$$\int_V^l = \int_V^r = x^{p-1}.$$

10 January 28, 2019

Today we will be mostly doing things in homological algebra. :)

Example 10.1
An example of an infinite dimension Hopf algebra is of the following: fix $q \in k^*$. Then

$$T_\infty = k < g, g^{-1}, p > / \langle gg^{-1} = g^{-1}g = 1, gp = qpg \rangle$$

and we can find $\{g^i p^j : i \in \mathbb{Z}, j \in \mathbb{N}\}$ is a k-linear basis.

For the coalgebra structure, define $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$ and $\Delta(p) = p \otimes 1 + g \otimes p$ and $\varepsilon(p) = 0$. Finally $S(g) = g^{-1}$ and $S(p) = q^{-1}pg^{-1}$.

10.1 Some nice results in Homological algebra

Problem 10.1
Prove that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

10.1.1 Theorem (Auslander-Buchsbaum '59)
Every local commutative algebra with finite global dimension is a UFD.

Another result is about algebraic groups:

10.1.2 Theorem
If $\operatorname{char} k = 0$, then every Noetherian commutative Hopf algebra has finite global dimension and is a direct sum of integral domains, each of which is isomorphic.

10.1.3 Definition (Semiprime): An ideal $I \lhd R$ is called **semiprime** if it is the intersection of (possibly infinitely many) primes.

Conjecture: Every Noetherian Hopf algebra of finite global dimension is semiprime (that is, 0 is semiprime in V).

10.1.4 Definition (Connected Graded): An algebra A is called **connected graded** if

$$A = k \oplus A_1 \oplus A_2 \oplus \cdots$$

and $1_A = 1_k \in k$, $A_i A_j \subseteq A_{i+j}$.

The new part (connected) refers to the fact that k is a summand.

10.1.5 Theorem
Suppose that $\operatorname{char} k = 0$. Let V be a Noetherian Hopf algebra that is connected graded as an algebra (says nothing about the coalgebra structure). Then the following hold:

- V has finite global dimension.
- V is a domain.
- V is Artin-Schelter regular.
- V is Auslander regular and Cohen-Macaulay.
- V is Calabi-Yau
- V is an iterated Ore extension.

That's a lot of word salad.

10.2 Projective Modules

10.2.1 Definition (Module-Theoretic Definitions): Let M be a (left) A module.

(a) M is **free** if $M \cong \oplus_{\mathcal{I}} A$

(b) M is **projective** if it is a direct summand of a free module.

10.2.2 Lemma
$p \in {}_A\mathcal{M}$ is projective if and only if the diagram in figure 7 commutes.

$$\begin{tikzcd} & P \arrow[dl, dashed, "F"'] \arrow[d, "f"] & \\ M \arrow[r, "\pi"] & N \arrow[r] & 0 \end{tikzcd}$$

Figure 7: The definition of a projective object.

10.2.3 Remark: This actually is just the definition for any Abelian category. This may generalize even further, but this is enough for us. :)

10.2.4 Definition (Injective Module): An **injective** V-module is a projective V^{op} module.

11 January 30, 2019

Today we're talking more about homological algebra. In particular, we'll learn about (or see again)

- Complexes
- Projective resolutions and dimension

First the example of the day:

Example 11.1
Consider $GL_2(k)$, a group – in fact, an algebraic group! Then

$$\mathcal{O}(GL_2) = k[x_{11}, x_{12}, x_{21}, x_{22}, det^{-1}]/\langle det^{-1}(x_{11}x_{22} - x_{21}x_{12} - 1\rangle$$

Let $X = (x_{ij}) \in GL_2(k)$.

Define the coalgebra structure via $\Delta(X) = X \otimes X$ and $\Delta(x_{ij}) = \sum_1^2 x_{is} \otimes x_{sj}$, $\varepsilon(X) = I_2$, $\varepsilon(x_{ij}) = \delta_{ij}$. Some computation (this is not trivial!) give us

$$\Delta(det) = det \otimes det \quad \Rightarrow \quad \Delta(det^{-1}) = det^{-1} \otimes det^{-1}$$

Then we can find that $S(X) = X^{-1}$ and $S(det^{\pm}) = det^{\mp}$.

Problem 11.1
Construct $\mathcal{O}(GL_n)$.

Problem 11.2
Construct $\mathcal{O}(G)$.

11.1 Homological stuff

11.1.1 Definition (Complex): A **complex** of A-modules is a sequence of A-modules connected by homomorphisms

$$\cdots \to M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \to \cdots$$

where $d_i \circ d_{i+1} = 0$ for all i.

Then we went over the standard definitions for:

- Homology
- (Short) Exact sequences
- Projective resolutions
- Projective dimension

11.1.2 Definition (Global (Homological) Dimension): The (left) **global dimension** is

$$\operatorname{gldim} A = \max\{\operatorname{projdim} M | M \in {}_A\mathcal{M}\}$$

11.1.3 Theorem (Hilbert Syzygy Theorem)

$$\operatorname{gldim} k[x_1, \dots, x_n] = n$$

11.1.4 Corollary

$$\operatorname{gldim} k = 0$$

11.1.5 Definition: The **finitistic global dimension** of A is defined to be

$$\operatorname{findim} A = \max\{\operatorname{projdim} M | M \in {}_A\mathcal{M}, M \text{ is f.g.}, \operatorname{projdim} M < \infty\}$$

Conjecture (Finitistic Dimension Conjecture): Let A be a finitely generated algebra. Then $\operatorname{findim} A < \infty$.

12 February 1, 2019

A really cool theorem:

12.0.1 Theorem (Lorenz-Lorenz '95)
Let V be a Hopf algebra. Then $\operatorname{gldim} V = \operatorname{projdim}_V k$.

Proof
It is enough to show that $\operatorname{gldim} V \leq \operatorname{projdim}_V k$. Or equivalently that

$$\operatorname{projdim} M \leq \operatorname{projdim} k, \forall M \in {}_V\mathcal{M}$$

but then

$$\operatorname{projdim}(M) = \operatorname{projdim}(k \otimes M) \leq \operatorname{projdim}(k)$$

by lemma 12.1.6 ♠

Example 12.1
Today's example: Quantum group $V = \mathcal{O}_q(GL_2)$. Fix some $q \in k^*$. Then as an algebra,
$$V = k\langle x_{11}, x_{12}, x_{21}, x_{22}, det_q^{-1}\rangle/(\text{relations})$$
where the relations are given by
$$\begin{aligned} x_{ij}x_{kl} = (\delta_{ik} + \delta_{jl})x_{kl}x_{ij} \quad \text{except when } i = k \text{ and } j = l \\ x_{12}x_{21} = x_{21}x_{12} \\ x_{22}x_{11} - x_{11}x_{22} = (q - q^{-1})X_{12}x_{21} \\ det_q^{-1}det_q = det_q det_q^{-1} = 1 \end{aligned}$$
where $det_q = x_{22}x_{11} - q^{-1}x_{12}x_{21}x_{11}x_{22} - qx_{12}x_{21}$. One can show det_q is central in V.

Now for the coalgebra: $\delta(X) = X \otimes X$, so $\Delta(x_{ij}) = \sum_{s=1}^{2} x_{is} \otimes x_{sj}$, $\varepsilon(X) = I_2$ and
$$S(X) = X^{-1} = \begin{pmatrix} x_{22}det_q^{-1} & -qx_{12}det_q^{-1} \\ -q^{-1}x_{21}det_q^{-1} & x_{11}det_q^{-1} \end{pmatrix}$$

Problem 12.1
Show that $\operatorname{gldim}\left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{smallmatrix}\right) = 2$ *while* $\operatorname{gldim}\left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{smallmatrix}\right)^{op} = 1$

12.1 Returning to Homological Algebra

Let A be an algebra, $M \in {}_A\mathcal{M}$. Recall the definitions of projective and global dimension.

12.1.1 Remark: The following are equivalent:

(a) $\operatorname{gldim} A = 0$

(b) $\operatorname{gldim} A^{op} = 0$

(c) A is semisimple Artininan

(d) $A = \oplus_1^b M_{n_i}(D_i)$ by Artin-Wedderburn where D_i is a division ring.

12.1.2 Remark: (a) A PID has global dimension 1.

(b) The free algebra $k\langle x_1, \ldots, x_n\rangle$ has global dimension 1.

(c) The path algebra of a finite quiver has global dimension 1.

(d) $\mathcal{O}_q(GL_2)$ has global dimension 4.

From now on, let V always stand for some Hopf algebra.

12.1.3 Lemma
${}_V\mathcal{M}$ is a tensor category. The action on $M \otimes N$ is the following:

$$v \cdot (m \otimes n) = \sum (v_{(1)} \cdot m) \otimes (v_{(2)} \cdot n)$$

12.1.4 Lemma
If $M \in {}_V\mathcal{M}$, then $V \otimes M \in {}_V^V\mathcal{M}$. As a corollary,

$$V \otimes M \cong V \otimes (V \otimes M)^{cov}$$

and as a consequence $V \otimes M$ is a free V-module.

12.1.5 Theorem
Let $P \in {}_V\mathcal{M}$ be projective. Then $P \otimes N \in {}_V\mathcal{M}$ is projective for all $M \in {}_V\mathcal{M}$

Proof
To see this, notice that P is a summand of $V^{\oplus b}$. But then if $P \oplus Q \cong V^b$, then

$$[P \otimes N] \oplus [Q \otimes N] = [P \oplus Q] \otimes N = (V^b) \otimes N = (V \otimes N)^b$$

and by the last lemma, this is free. ♠

12.1.6 Lemma

$$\operatorname{projdim}(M \otimes N) \leq \min\{\operatorname{projdim} M, \operatorname{projdim} N\}$$

Proof
We only show that $\operatorname{projdim} M \otimes N \leq \operatorname{projdim} M$. If $\operatorname{projdim} M = \infty$, we are done. If it is finite, take any minimal projective resolution of M and tensor with N. This is a projective resolution of $M \otimes N$ of length at most $\operatorname{projdim} M$. ♠

12.1.7 Remark: The fact it is still exact holds because we can appeal to the vector space structure.

Some more results (mostly for finite dimensional algebras):

12.1.8 Theorem (Radford '75)
If V is finite dimensional, then the antipode S has finite order:

$$S^d = \mathrm{id}_V$$

12.1.9 Theorem (Larson-Radford '88)
Suppose $\operatorname{char} k \neq 0$ (can be relaxed but gets more ugly). Then the following are equivalent:

(a) $\operatorname{gldim} V = 0$ (V is semisimple Artinian)

(b) $\operatorname{gldim} V^* = 0$

(c) $S^2 = \mathrm{id}_V$

12.1.10 Theorem (Nichols-Zoeller '89)
Let V be finite dimensional. If Q is a Hopf subalgebra of V, then ${}_W V$ and V_W are free.

13 February 6, 2019

First some history: in 1899, Maschke proves the following:

13.0.1 Theorem
Let G be a finite group. Then kG is semisimple (Artinian) if and only if $\operatorname{char} k \nmid |G|$.

Today, we will see an analog of Maschke for Hopf algebras:

13.0.2 Theorem ("Maschke's Theorem" (Larson-Sweedler '69))
Let V be a finite-dimensional Hopf algebra. Then the following are equivalent:

(a) $\operatorname{gldim} V = 0$ (V is semisimple Artinian)

(b) $\varepsilon(\int^l) \neq 0$

(c) $\varepsilon(\int^r) \neq 0$

Now say $V = kG$. Then $\int = \sum_{g\in G} g$ is both a left and right integral of V. This uses that $\varepsilon(g) = 1$.

But then if $\varepsilon(\int) = \sum_G \varepsilon(g) = \sum_G 1 = |G| \neq 0 \Leftrightarrow \operatorname{char} k \nmid |G|$.

13.1 Example of the Day

Example 13.1 (Kac-Paljutkin Algebra)
$k = \mathbb{C}$ or any field with $\operatorname{char} k \neq 2$. Then $V = k\langle x, y, z\rangle$ modulo the relations

$$\begin{aligned} x^2 = y^2 &= 1 \\ xy &= yx \\ xz &= yz \\ xy &= xz \\ x^2 &= \frac{1}{2}(1 + x + y - xy) \end{aligned}$$

Then V has a k-basis $\{1, x, y, z, xy, xz, yz, xyz\}$, so as an algebra $V \cong k^4 \oplus M_2(k)$.

Then $\Delta(x) = x \otimes x$, $\Delta(y) = y \otimes y$, $\Delta(z) = (1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z)$.

And $\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1$ and $S(x) = x$, $S(y) = y$ and $S(z) = z$. Recall that although S looks like the identity map, S is an **antihomomophism**. So actually $S^2 = \mathrm{id}_V$.

Claim:

$$\int = 1 + x + y + z + xy + xz + yz + xyz$$

then $\varepsilon(\int) = 8 \neq 0$, so V is semisimple.

Some nice facts:

(a) $V^* \cong V$

(b) V is the unique eight dimensional noncommutative, noncocommutative semisimple Hopf algebra.

(c) Let Q_8 be the quaternion group. Then kQ_8 is eight dimensional. Therefore $V \cong kQ_8$ as algebras, but **not as coalgebras.** The reason behind this is that kG is *always* cocommutative.

13.2 HW of the day

Problem 13.1
If V is eight dimensional and noncommutative, then $V \cong k^4 \oplus M_2(k)$ as an algebra. Thus if G is nonabelian group of order 8, then $\mathbb{C}G \cong \mathbb{C}^4 \oplus M_2(\mathbb{C})$.

13.3 Proof of "Maschke's theorem"

Now we actually do the proof:

PROOF
(a) $\Rightarrow$ (b): Say the global dimension of V is zero. Then V is semisimple by Artin-Wedderburn and we have a decompositon

$$V \cong \oplus_1^s M_{n_i}(D_i)$$

and every minimal nonzero ideal has the form $M_{n_i}(D_i)$.

Consider the ideal $k \int^l = I$ which is an ideal since $v \int^l = \varepsilon(v) \int^l \in I$. But then consider that if $v' \int^l \in I$,

$$v' \int^l v = \varepsilon(v') \int^l v \in k \int^l$$

so in fact it is a two-sided ideal. Thus $k\int^l \cong M_{n_i}(D_i)$, so

$$(k\int^l)^2 = k\int^l \varepsilon(\int^l)\int^l = (\int^l)^2 \qquad\qquad = a\int^l$$

for some nonzero a. But then $\varepsilon(\int^l) \neq 0$.

(b) $\Rightarrow$ (a): Now let $e = \frac{1}{\varepsilon(\int^l)}\int^l$. Then $e^2 = e$ and $ke = k\int^l$ is a left V-module. But then $V = Ve \oplus V(1-e)$ so $ke = k\int^l$ is projective, so $\operatorname{projdim}_v k = 0$.

The last part follows by Lorenz-Lorenz – $\operatorname{gldim} V = \operatorname{projdim}_V k$. ♠

13.4 Ext groups

We define these in the usual way through projective resolutions:

Let A be an algebra. Then ${}_A\mathcal{M}$ is an abelian cateogry, so $\operatorname{Hom}_A(M,N) \in \mathbf{Vect}_{\mathrm{k}}$.

13.4.1 Definition: A (covariant) functor $F : {}_A\mathcal{M} \to {}_B\mathcal{M}$ is called **left exact** if, for any short exact sequence

$$0 \to L \to M \to N \to 0$$

we have the exact sequence

$$0 \to F(L) \to F(M) \to F(N).$$

Similar for right exact. If both, it's just exact.

13.4.2 Lemma
For any $W \in {}_A\mathcal{M}$, the functor $\operatorname{Hom}(W,-)$ is left exact.

We also have that $\operatorname{Hom}_A(P,-)$ is exact iff P is projective, so this gives us a nice characterization of projective objects in Abelian categories. Then the category $\mathbf{Ch}({}_A\mathcal{M})$ is the category of chain complexes in A modules with chain maps.

13.4.3 Lemma
$\mathbf{Ch}({}_A\mathcal{M})$ is an Abelian category.

13.4.4 Lemma
A functor $F : {}_A\mathcal{M} \to {}_B\mathcal{M}$ can be extended to a functor between the appropriate chain complex categories.

13.5 Looking Forward

On Friday, we may see the following result:

13.5.1 Theorem
Under some mild hypothesis (including finite dimensionality),

$$\operatorname{projdim}_V k = \operatorname{gldim} V$$

14 February 13, 2019

Here's the theorem for today (with the hypotheses in place):

14.0.1 Theorem
Assume $\operatorname{char} k = 0$ and $S^2 = \mathrm{id}_V$. Then

$$\operatorname{gldim} V = \operatorname{projdim}_V M$$

for all finite dimensional V modules M.

Another result is the theorem 12.1.9:

PROOF
Assuming theorem 14.0.1, (c)⇒(a) is obvious. That's only part of it, obviously. :) ♠

14.1 Example of the Day

Example 14.1
$\mathcal{U}(\mathfrak{g})$, the universal enveloping algebra which (as an algebra) is isomorphic to $k\langle\mathfrak{g}\rangle/(xy-yx-[xy])$. As a coalgebra, we define $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon = 0$ and $S(x) = -x$.

This is a nice example for today, because notice that $S^2(x) = x$ and in fact $S^2 = \mathrm{id}_{\mathcal{U}(\mathfrak{g})}$, so we get that $\operatorname{gldim} \mathcal{U}(\mathfrak{g}) = \operatorname{projdim}_{\mathcal{U}(\mathfrak{g})} M$ for any finite dimensional M (by the theorem).

14.2 Today's Content

14.2.1 Lemma (Reflection Principle)
Let A be an algebra. Then

(a) $(-)^* = \mathrm{Hom})k(-,k) : ({}_A\mathcal{M})_{fd} \to (\mathcal{M}_A)_{fd}$ is a contravariant equivalence.

(b) $(M^*)^* \cong M$ for each $M \in {}_AM_{fd}$. In linear algebra, this says that we have a perfect pairing $M \otimes M^* \to k$.

14.2.2 Lemma (Restriction of Scalars)
Let $f : A \to B$ be a map of algebras. Then

(a) There is a functor $f^* : {}_B\mathcal{M} \to {}_A\mathcal{M}$; also

(b) $f^* : \mathcal{M}_B \to \mathcal{M}_A$.

14.2.3 Lemma (Restriction of Scalars)
Let $f : A \to B$ be an antihomomophism of algebras. Then

(a) There is a functor $f^* : {}_B\mathcal{M} \to \mathcal{M}_A$; also

(b) $f^* : \mathcal{M}_B \to {}_A\mathcal{M}$.

14.2.4 Lemma
Let V be a Hopf algebra with antipode S. Then

$$\widetilde{S} \circ (-)^* : {}_V\mathcal{M} \to \mathcal{M}_V \to {}_V\mathcal{M}$$

is a contravariant functor.

14.2.5 Corollary
If V is finite dimensional, then $\widetilde{S} \circ (-)^*$ is a duality of $({}_V\mathcal{M})_{fd}$.

For the future we will use the notation $M^{*S} = \widetilde{S}(M^*)$. Notice that for all $M \in {}_V\mathcal{M}$, $M^{*S} \in {}_V\mathcal{M}$.

This leads us to our proposition:

14.2.6 Theorem
V is a Hopf algebra.

(a) If $M \in {}_V\mathcal{M}$ then there is a natural morphism

$$ev_M : M^{*S} \otimes M \to {}_V k$$

(b) if $M \in {}_V\mathcal{M}_{fd}$, then there is a natural map

$$coev_M : {}_V k \to M \otimes M^{*S}$$

Proof
$ev_M(f \otimes m) = f(m) \in k$ is the map we're talking about. For all $v \in V$,

$$\begin{aligned} ev_M(v \cdot (f \otimes m)) &= ev_M(v_1 f \otimes v_2 m) \\ &= ev_M(fS(v_1) \otimes v_2 m) \\ fS(v_1)(v_2 m) & \\ &= f(S(v_1)v_2 m) \\ &= f(\varepsilon(v)m) = \varepsilon(v)f(m) = v \cdot ev_M(f \otimes m). \end{aligned}$$

If instead M is finite dimensional, pick a k basis $m_1, \ldots, m_d$ for M and let m_i^* be the dual basis. Set $coev_M(1) = \sum m_i \otimes m_i^* = \Phi$. Then use the fact that (under the identification of $M \otimes M^*$ with $\mathrm{Hom}_k(M, M)$) we get $\Phi = \mathrm{id}_M$. Then prove

$$coev_M(v \cdot 1) = v \cdot coev_M(1). \qquad \spadesuit$$

14.2.7 Definition: The map ev_M above is called the **evaluation map** and $coev_M$ is called the **coevaluation map.**

14.2.8 Lemma

(a) $(M^{*S})^{*S} \cong \widetilde{S}^2(M)$ for all finite dimensional $M \in {}_V\mathcal{M}$.

(b) If $S^2 = \mathrm{id}$ then $(M^{*S})^{*S} \cong M$.

14.2.9 Lemma

Let $M \in {}_V M_{fd}$ and suppose $S^2 = 1$. Then

$$ev_{M^{*S}} \circ coev_M = (\dim M)\,\mathrm{id}_k$$

15 February 15, 2019

Here we use the (slightly extended) lemma:

15.0.1 Lemma

(a) $ev_M \circ coev_{M^{*S}} = (\dim M)\,\mathrm{id}_k$

(b) $\frac{1}{\dim M} ev_M \circ coev_{M^{*S}} = \mathrm{id}$

(c) (since the identity factors through $M^{*S} \otimes M$) $M^{*S} \otimes M \cong {}_V k \oplus N$ as V-modules.

(d) $\operatorname{projdim} k \leq \operatorname{projdim} M^{*S} \otimes M$.

Then using this lemma, we can prove the theorem 14.0.1:

Proof

Basically use this, Lorenz-Lorenz, and the result bounding projdim of a tensor module by its factors. ♠

Notice that if we change things up slightly we get a new theorem:

14.0.1′ Theorem

Let V be a Hopf algebra. Suppose that $S^2 = \mathrm{id}$. Then

$$\operatorname{projdim}_V M = \operatorname{gldim} V$$

for all $M \in {}_V M_{fd}$ such that $\dim M \neq 0$ in k.

Conjecture: Let V be a Noetherian Hopf algebra over a field k of characteristic zero and $S^2 = \mathrm{id}$. Then

$$\operatorname{gldim} V < \infty$$

15.0.2 Corollary

If V is a finite module over $\mathcal{Z}(V)$, its center, then the above conjecture holds.

15.1 Today's example

He did the quantized enveloping algebra of $\mathfrak{sl}_2$! Might be worth looking up.

An interesting part of this example is that S^2 is **not** the identity! In fact, this example is important in constructing Joon (sp?) polynomials in knot theory.

15.2 Today's Homework

Problem 15.1
Prove the following "dimensional shift" lemma: Let

$$0 \to X \to P \to Y \to 0$$

be a SES of A modules, where P is projective. If M is an A module, then for all $n \geq 1$:

$$\operatorname{Ext}_A^{n+1}(Y, M) \cong \operatorname{Ext}_A^n(X, M).$$

Solution:
Take a projective resolution for X and try to attach it to the SES. ♠

15.3 Classifying Algebras

We have the (descending) chain of classes of algebras:

- Algebras
- Noetherian Algebras
- Cohen-Macaulay Algebras
- Gorenstein Algebras
- Complete Intersections
- Hypersurface Rings
- Regular Algebras

The question we have is: where to (Noetherian) Hopf algebras lie in this hierarchy?

If we restrict to finite-dimensional Hopf algebras, Brown-Goodearl proved that Noetherian Hopf algebras are all Gorenstein rings.

15.3.1 Definition: A Noetherian algebra A is called **regular** if $\operatorname{gldim} A < \infty$.

15.3.2 REMARK: Notice that since A is Noetherian, we didn't have to talk about left vs. right global dimension.

Example 15.1
If L is a finite dimensional Lie algebra, then $U(L)$ is regular:

$$\operatorname{gldim} U(L) = \dim L.$$

15.3.3 Definition: An element $x \in A$ is called **normal** if $xA = Ax$ (NOT $xa = ax$)

15.3.4 Definition: A Noetherian algebra A is called a **hypersurface ring** if there exists a Noetherian regular algebra B and a normal non-zero-divisor $x \in B$ such that $A \cong B/(x)$

Example 15.2
$\operatorname{char} k = p > 0$, $V = k[x]/(x^p)$, a Hopf algebra with comultiplication $\Delta(x) = x \otimes 1 + 1 \otimes x$. Then V is a hypersurface ring.

15.3.5 Definition: A Noetherian algebra A is called a **complete intersection** if there is a Noetherian regular algebra B and a sequence of elements $x_1, \dots, x_d$ such that:

- x_1 is a non-zero-divisor in B.
- x_i is a NZD in $B/(x_1, \dots, x_{i-1})$ for all i
- $A \cong B/(x_i)$.

15.3.6 Definition: A Noetherian algebra A is called **Gorenstein** if

$$\operatorname{injdim}{}_A A = \operatorname{injdim} A_A < \infty.$$

15.3.7 Definition: A Noetherian algebra A is called **Cohen-Macaulay** if there is an A-bimodule M such that

- $\operatorname{injdim} {}_AM < \infty$, $\operatorname{injdim} M_A < \infty$
- $\operatorname{End}(M_A) \cong A$, $\operatorname{End}({}_AM) \cong A^{op}$
- ${}_AM$ and M_A are Noetherian.

16 February 20, 2019

Last time we wrote out a big chain of inclusions for algebras. We didn't really discuss why all complete intersections are Gorenstein algebras. Actually, this is one of the harder inclusions to prove. We will at least try today, but will definitely get to it by Friday.

16.1 Example of the day

16.1.1 Definition: $V = k[g^{\pm 1}][y;\delta]$, an Orr extension of the Laurent polynomial ring by the derivation $\delta = (g^n - g)\frac{d}{dg}$. For those (me) wo haven't seen Orr extensions before, we also have

$$V \cong k\langle g, g^{-1}, y\rangle/(yg - gy - (g^n - g), gg^{-1} - 1, g^{-1}g - 1)$$

This is a domain where $\operatorname{gldim} V = 2$ (also the GK dimension, which we haven't defined). As a coalgebra, we get g is grouplike and $\Delta y = y \otimes g^{n-1} + 1 \otimes y$, $\varepsilon(y) = 0$, and $S(y) = -yg^{-(n+1)}$.

Notice that V is regular.

16.2 Problem of the day

Problem 16.1

Let x be a normal non-zero-divisor in A where $xA = Ax$. Then for all $a \in A$ let $\sigma \in \operatorname{Aut} A$ be defined such that $\sigma(a)x = xa$.

Let $M \in {}_A\mathcal{M}$ and define a left A action on ${}^\sigma M$ whose underlying set is M and the action is defined by $a \cdot m = \sigma(a)m$. In terms of the action (morphism), this is equivalent to precomposing with σ :

$$A \xrightarrow{\sigma} A \xrightarrow{\rho} \operatorname{End}(M)$$

Prove the following lemma:

16.2.1 Lemma (Rees Lemma)

Let x be a normal non-zero-divisor in A. If L is a left $A/(x)$-module and M is an x-torsion-free A-module, then

$$\mathrm{Ext}^n_{A/(x)}(L, M/xM) \cong \mathrm{Ext}^{n+1}_A(L, {}^\sigma M).$$

16.3 Today's Results

We begin with some corollaries of Rees Lemma:

16.3.1 Corollary

(a) $\mathrm{Ext}^n_{A/(x)}(L, A/(x)) \cong \mathrm{Ext}^{n+1}_A(L, {}^\sigma A) \cong \mathrm{Ext}^{n+1}_A(L, A)$

(b) $\mathrm{injdim}_{A/(x)}(A/(x)) \leq \mathrm{injdim}_A A - 1$

(c) If A is Gorenstein, so is $A/(x)$.

16.3.2 Remark: Then the idea here is that we just have to prove that regular rings are Gorenstein, since then complete intersections are just (finite) sequences of quotients by regular elements.

16.4 Doing Ext again

Let $P^\bullet$ and $Q^\bullet$ be complexes of A-modules. Then $\mathrm{Hom}_A(P^\bullet, Q^\bullet)$ is a complex. Define

$$\mathrm{Hom}_A(P^\bullet, Q^\bullet) := \prod_{i\in\mathbb{Z}} \mathrm{Hom}_A(P^i, Q^{i+n} = \prod_{-i+j=n} \mathrm{Hom}(P^i, Q^j)$$

where we define the boundary map to be

$$d_{\mathrm{Hom}_A(P^\bullet, Q^\bullet)} : (f_i)_{i\in\mathbb{Z}} \mapsto (d_Q^{n+1} \circ f^i - (-1)^n f^{i+1} \circ d_P^i)_{i\in\mathbb{Z}}$$

Problem 16.2

*The **real** homework for this class (the other one is what we're showing, apparently) is to show that this is a chain complex – i.e. that $d^2 = 0$.*

Define the projective resolution to be the regular projective resolution, but do not include M itself. But then we have

$$\begin{aligned} R\underline{\mathrm{Hom}} = \mathrm{Ext}^n_A(M, N) &:= H^n(\mathrm{Hom}_A(P_M^\bullet, N)) \\ &:= H^n(\mathrm{Hom}_A(P_M^\bullet, P_N^\bullet)) \\ &:= H^n(\mathrm{Hom}_A(I_M^\bullet, I_N^\bullet)) \\ &:= H^n(\mathrm{Hom}_A(M, I_N^\bullet)) \end{aligned}$$

16.5 Proof plz

Today we will try to get $n = 0$ in the inductive argument.

16.5.1 Lemma

(a) There is a short exact sequence of left A modules

$$0 \to {}^\sigma A \xrightarrow{l_x} A \to A/(x) \to 0$$

where l_x sis left multiplication by x.

(b) If M is x-torsion-free, then there is a SES of left A modules

$$0 \to {}^\sigma M \xrightarrow{l_x} M \to M/xM \to 0$$

The second item implies the first and neither is hard to show directly.

16.5.2 Lemma

Let M be an x-torsion-free left A modules and let I_M be an injective resolution of M. THen there is a morphism of complexes $l_x : {}^\sigma I_M \to I_m$.

Again, not too hard to check.

Notice now that $xL = 0$ by the hypotheses. Combining this with the last result, we have

16.5.3 Lemma

(a) For any $L \in {}_A\mathcal{M}$,

$$\operatorname{Ext}^n_A(L, l_x) : \operatorname{Ext}^n_A(L, {}^\sigma M) \to \operatorname{Ext}^n_A(L, M)$$

(b) If $xL = 0$ then the above map is zero for all $n \geq 0$.

Again, follow your nose a bit. $f : L \to {}^\sigma I^i$ is an A homomorphism and pull that x in.

Applying $\operatorname{Hom}_A(L, -)$ to $0 \to {}^\sigma M \xrightarrow{l_x} M \to M/xM \to 0$, we obtain the long exact sequence for Ext. But then notice that $\operatorname{Hom}_A(L, M) = \operatorname{Ext}^1_A(L, M) = 0$ since M is x-torsion-free (and L is killed by x). and since $\operatorname{Ext}^1_A(L, {}^\sigma M) \cong \operatorname{Hom}_A(L, M/xM) \cong \operatorname{Hom}_{A/(x)}(L, M/xM)$, we get the base case proven.

17 February 22, 2019

Today we are going to finish the proof of Rees, then give some remarks, then give the example of the day. :)

17.0.1 Continuation of Proof of Rees

Last time we proved that $\mathrm{Ext}^1_A(L, {}^\sigma M) \cong \mathrm{Ext}^0_{A/(x)}(L, M/xM)$. Now we proceed by induction.

Consider the SES $0 \to K \to F \to L \to 0$ of left $A/(x)$-modules where F is free. Then we get a long exact sequence

$$\begin{aligned}0 \to \mathrm{Hom}_{A/(x)}(L, M/xM) \to \mathrm{Hom}_{A/(x)}(F, M/xM) \to \mathrm{Hom}_{A/(x)}(K, M/xM)\\ \to \mathrm{Ext}^1_{A/(x)}(L, M/xM) \to \mathrm{Ext}^1_{A/(x)}(F, M/xM) \to \cdots\end{aligned}$$

as well as the sequence

$$\begin{aligned}\cdots \to \mathrm{Ext}^1_A(L, {}^\sigma M) \to \mathrm{Ext}^1_A(F, {}^\sigma M) \to \mathrm{Ext}^1_A(K, {}^\sigma M)\\ \to \mathrm{Ext}^2_A(L, {}^\sigma M) \to \mathrm{Ext}^2_A(F, {}^\sigma M \to \cdots)\end{aligned}$$

and we have isomorphisms between all the corresponding parts (except the fourth term). THe first three follow from $n = 0$ and the last one follows since F is a projective $A/(x)$ module and since F has projective dimension (over A) less than or equal to one. Then five lemma proves that this holds for $n = 1$.

Then you can just extend both sequences and iteratively prove this holds for each term. Woot. :)

17.1 Return to Classifications

Recall we have the (descending) category chain

- Algebras
- Noetherian Algebras
- Cohen-Macaulay Algebras
- Gorenstein Algebras
- Complete Intersections
- Hypersurface Rings
- Regular Algebras

Now we can show by applying Rees directly:

17.1.1 Corollary
$\operatorname{injdim} A/(x) \leq \operatorname{injdim} A$

Which then allows us to prove:

17.1.2 Corollary
Every complete intersection ring is Gorenstein.

Proof

Let A be a CI ring. Then there exists a Noetherian regular ring B and a sequence of elements $b_1, \ldots, b_n \in B$ such that each b_i is a normal nonzerodivisor on $B/(b_1, \ldots, b_{i-1})$.

Since B is regular, it is Gorenstein. By the corollary $B/(b_1)$ is Gorenstein. Continuing this way, A itself is Gorenstein and we are done. ♠

17.2 Where do Hopf algebras fit in?

Recall that we proved that (finite dimensional) Hopf algebras are Frobenius algebras. Well it ends up that Frobenius algebras are also Gorenstein! Thus all (f.d) Hopf algebra is Gorenstein.

Recall the Brown-Goodearl conjecture: all Noetherian Hopf algebras are Gorenstein. This is still an open conjecture! But it gets worse!

Conjecture: Every Noetherian Hopf algebra is a complete intersection.

Conjecture: Every finite dimensional Hopf algebra is a complete intersection.
There are no known counterexamples to either! This has been open for decades with no satisfactory result.

17.2.1 Even more classes of algebras!

Inside the category of Gorenstein rings, there are a couple more classes (not related to complete intersections): We get the decreasing chain

- Gorenstein rings
- Artin-Schelter Gorenstein
- A-S Gorenstein with skew Calabi-Yau property

and over at regular algebras we get the decreasing chain

- Regular algebras
- A-S regular algebras
- Skew C-Y algebras
- C-Y algebras

17.2.1 Definition: A Noetherian Gorenstein ring A of injdim d is called $A-S$ Gorenstein if, for all $S \in {}_A\mathcal{M}_{fd}$,

$$\operatorname{Ext}^i_A(S, A) = \begin{cases} 0 & i \neq d, \\ \in {}_A\mathcal{M}_{fd}, & i = d \end{cases}$$

where the above property is called the Artin-Schelter (AS) property.

17.2.2 Definition: A Noetherian A-S Gorenstein algebra A of injective dimension d is called **skew Calabi-Yau** if

$$\mathrm{Ext}^i_{A^e}(A, A^e) = \begin{cases} 0 & i \neq d, \\ {}^\sigma A, & i = d \end{cases}$$

(remember that $A^e \cong A \otimes A^{op}$).

17.2.3 Definition: A is Calabi-Yau if it is skew C-Y and also $\mathrm{Ext}^d_{A^e}(A, A^e) = A$.

17.2.4 Remark: The idea here (and honestly I missed some parts of the discussion) is that C-Y algebras are ones where we can get Poincaré duality.

17.3 Example of the Day

Example 17.1
V is generated by g, g^{-1}, x_2, x_3 subject to

$$gg^{-1} = g^{-1}g = 1, \quad gx_i = \eta^i x_i g, \quad x_2^3 = x_3^2 \quad x_2x_3 = x_3x_2$$

where η is a primitive 6^{th} root of unity.

Then $\mathrm{gldim}\, V = \infty$ and it is a kypersurface ring, since

$$V \cong k[x_1, x_2][g^{\pm 1}; \phi]/(x_2^3 - x_3^2)$$

where $\phi : x_i \to \eta^i x_i$.

As a coalgebra, g is grouplike and x_i are such that $\Delta(x_i) = x_i \otimes 1 + g^i \otimes x_i$ and $\varepsilon(x_i) = 0$. Then we get $S(x_i) = -g^{-i}x_i$.

17.4 HW for the day

Problem 17.1
Let A and B be rings. Let ${}_AM$, ${}_AN_B$, and ${}_BC$. Then

$$\mathrm{Hom}_A(N \otimes_B C, M) \cong \mathrm{Hom})B(C, \mathrm{Hom}_A(N, M))$$

and if instead C_A, ${}_AN_B$ and M_B, then you can do things over A^{op} and B^{op}.

18 February 25, 2019

Topic for today: **Hopf actions on algebras.** One can naturally extend everything we talk about today to (co)actions on (co)algebras. :)

18.1 Example of the Day

Example 18.1
$V = k\langle g, g^{-1}, x\rangle/(gg^{-1} = 1 = g^{-1}g)$. Then g is grouplike and $\Delta(x) = x \otimes 1 + g \otimes x$, $\varepsilon(x) = 0$, and $S(x) = -g^{-1}x$.

We can see that $\operatorname{gldim} V = 1$ (use resolution $0 \to V \oplus V \to V \to {}_V k \to 0$). Note that it is NOT AS-Gorenstein. $\operatorname{Ext}^1_V(k, V)$ is infinite dimensional over k (not easy!).

18.1.1 Remark: We've seen elements like x above several times – these elements are called $(1, g)$-primitive (generalizing the concept of primitive, obviously).

Also, these are examples of **pointed Hopf algebras**(!!) These are V such that (as a coalgebra) the coradical is a sum of one-dimensional modules. This is actually related to the fact that every simple submodule of V is of the form kg^n (in this case $n = 1$).

18.2 HW of the Day

Problem 18.1
This problem considers (one of the) internal Hom*(s) in* ${}_V\mathcal{M}$*. Suppose* V *has bijective antipode. Then for any* M *and* N*,*

$$\operatorname{Hom}^{S^{-1}}(M, N) := \operatorname{Hom}_k(M, N)$$

where a V *action is defined by (for* $v \in V$ *and* $f \in \operatorname{Hom}^{S^{-1}}(M, N)$*)*

$$(v \cdot f)(m) = \sum v_2 f(S^{-1}(v_1)m)$$

where, as usual, $\Delta v = \sum v_1 \otimes v_2$*.*

(a) Show that this makes $\operatorname{Hom}^{S^{-1}}(M, N)$ *into a left* V*-module.*

(b) Show that $\operatorname{Hom}^{S^{-1}}(-, -)$ *is right adjoint to* $- \otimes_k -$*.*

18.3 Automorphisms and Group Actions

Let A be an algebra. $\operatorname{Aut}_{alg}(A)$ denotes the **group of autormorphisms of** A**.**

18.3.1 Definition: Let G be a group. We say G acts on A ($G \curvearrowright A$) fi there is a group homomorphism $G \to \mathrm{Aut}_{alg}(A)$.

18.3.2 Definition: Alternatively, we say $G \curvearrowright A$ if

- A is a G-module.
- $\forall g \in G$, $g(1_A) = 1_A$
- $\forall a, b \in A$, $g(ab) = g(a)g(b)$
- g is bijective.

18.3.3 Definition: Suppse $G \curvearrowright A$. Then the **invariant subring** of this action is

$$A^G := \{a \in A | g(a) = a, \forall g \in G\} \subseteq A$$

This was fairly popular around 1880-1960 or so.

Example 18.2
$A = k[x_1, x_2]$, $G = \langle \sigma \rangle \cong C_2$ where $\sigma \cdot f(x_i) = f(-x_i)$. One can compute that this just switches signs on monomial of odd degree (and fixes the even ones) so

$$A^G = \left\{ \sum c_{ij} x_1^i x_2^j | i + j = 2k \right\} \cong k[a, b, c]/(ac - b^2)$$

Generalizing this, many of the questions of this time revolved around asking what category of algebra $k[x_1, \ldots, x_n]^G$ is for arbitrary G. We may return to this later.

18.4 Derivations

18.4.1 Definition: Let A be an algebra. A map $d \in \mathrm{Hom}_k(A, A)$ is called a **derivation** if it satisfies the Leibniz rule.

18.4.2 Lemma
The collection of all derivations of an algebra from a Lie algebra – denoted $\mathrm{Der}(A)$.

18.4.3 Definition: Let L be a Lie algebra. We say L acts on A if there is a Lie algebra morphism $L \to \mathrm{Der}(A)$. Equivalently, if

(a) A is a left L-module

(b) $\forall l \in L$, $l(1_A) = 0$

(c) $\forall a, b \in A$, $l(ab) = l(a)b + al(b)$.

18.4.4 Remark: The second item above is implied by the first (in fact, James included a similar statement in the definition of a derivation) but apparently there is a reason to include it!

Example 18.3
$A = k[x_1, x_2]$, $d = \frac{\partial}{\partial x_1}$. Then $L = kd$ is a Lie algebra that acts on A.
Then A^L (see below) is $k[x_2]$

18.4.5 Definition: If L is a Lie algebra acting on A, then the invariant subring under the action is

$$A^L = \{a \in A | l(a) = 0, \forall l \in L\}$$

18.5 Putting These Things Together

Let A be an algebra and V be a Hopf algebra.

18.5.1 Definition: We say V actions on A if

- A is a left V-module
- A is an algebra object in $({}_V\mathcal{M}, \otimes)$

Note that the second item above implies the first!
Equivalently, V acts on A if

- $A \in {}_V\mathcal{M}$
- For all $v \in V$, $v \cdot (1_A) = \varepsilon(v)1_A$
- For all $a, b \in A$, $v(ab) = \sum v_1(a)v_2(b)$.

Returning to the example from the beginning of class (18.1), if V acts on A, then

- $g \in \mathrm{Aut}_{alg}(A)$
- $x \in \mathrm{Der}_g(A)$

In particular, notice that $x(1) = 0$ and $x(ab) = x(a)b + g(a)x(b)$.

19 March 11, 2019

We took a week off since James was in Germany for a conference. We're coming back together for this last week of the quarter.

Here is one result that was brought up at the conference. It was proved by a grad student and might have some missing hypotheses:

19.0.1 Theorem

If char $k = 0$ (and some other mild hypotheses) then the Etingof-Ostrik conjecture holds for H if and only if it holds for H^*. That is,

$$\mathrm{kdim} \oplus \mathrm{Ext}^i_H(k, k) = \mathrm{kdim} \oplus \mathrm{Ext}^i_{H^*}(k, k)$$

where kdim means Krull dimension.

19.1 Example of the Day

Example 19.1 (Small Quantum Group of $\mathfrak{sl}_2$)

Let $q \in k$ be a primitive l^{th} root of unity ($l \geq 2$). Then

$$u_q(\mathfrak{sl}_2) = k\langle K, K^{-1}, E, F\rangle / R$$

where R is the ideal for the relations

$$KK^{-1} = K^{-1}K = 1$$
$$KEK^{-1} = q^2 E$$

$$\begin{aligned} KFK^{-1} &= q^{-2}F \\ EF - FE &= \frac{K-K^{-1}}{q-q^{-1}} \\ E^l = F^l &= K^0 \\ K^l &= 1 \end{aligned}$$

Now $\dim_k u_q(\mathfrak{sl}2) = l^3$ and this is (symmetric) Frobenius and noncommutative and noncocommutative. As a coalgebra, K is grouplike and E, F are almost primitive:

$$\Delta E = E \otimes 1 + K \otimes E, \quad \Delta F = F \times K^{-1} + 1 \otimes F$$

and $S(E) = -K^{-1}E$ and $S(F) = -FK$.

Then $\mathfrak{sl}_2 = ke \oplus kf \oplus kh$ and there is a map $e \mapsto E$, $f \mapsto F$ and $K = \exp(h)$. Hmm

19.2 Last Lecture

I wasn't here, so I missed this part. Let A be a Noetherian connected graded AS-regular algebra and let V be a Hopf algebra action on A such that each A_i is a left V-module.

(a) $k := A/A_{\geq 1}$ and $\oplus_0^\infty \operatorname{Ext}_A^i(k,k)$ is an algebra.

(b) (S. Paul Smith's Result) $\oplus \operatorname{Ext}_A^i(k,k)$ is Frobenius and $\operatorname{Ext}_A^d(k,k)$ is one dimensional over k when $d = \operatorname{gldim} A$.

(c) Each $\operatorname{Ext}_A^i(k,k)$ is a left V module.

(d) $(\oplus \operatorname{Ext}_A^i(k,k))^{op}$ is a left V algebra.

Now pick any nonzero element $e \in \operatorname{Ext}_A^d(k,k)$ where d is as above. For all $v \in V$, $v \cdot e \in \operatorname{Ext}_A^d(k,k)$ and we write $v \cdot e = \operatorname{hdet}(v)v$ for $\operatorname{hdet}(v) \in k$.

19.2.1 Definition: A map $\operatorname{hdet} : V \to k$ that fits in as above is called the **homological determinant** of the V-action on A.

19.2.2 Lemma
hdet is an algebra map.

19.2.3 Lemma
If $A = k[x_1, \ldots, x_n]$, $V = kG$, $G \subseteq GL(n)$ then

$$\operatorname{hdet} g = \det g$$

for all $g \in G$.

19.2.4 Definition: We say that hdet is trivial if $\mathrm{hdet}(v) = \varepsilon(v)$ for all $v \in V$.

Recall the theorem

19.2.5 Theorem (Watanabe)
Suppose G is a finite subgroup in GL_n. If $\deg g = 1$ for all $g \in G$, then $k[x_1, \ldots, x_n]^G$ is Gorenstein.

19.2.6 Theorem (Noncommutative Watanabe)
Let A be a Noetherian connected graded AS-regular algebra. Let V be a semisimple Hopf algebra acting on A such that each $A)i$ is a left V-module. If hdet is trivial, then A^V is AS Gorenstein.

19.2.7 Remark: This last result is important because it distinguishes between the commutative and noncommutative cases: in the commutative case the only AS-regular ring is the polynomial ring. This shows us that there is in fact a vast class of them in the noncommutative case. In fact it's even bigger!

Recall that $g \in GL_n$ is a reflection if g is similar to $\mathrm{diag}(1, \ldots, 1, \sqrt{-1})$. Let R be the subgroup of G generated by the reflections in G.

19.2.8 Theorem (Watanabe II)
Suppose G is a finite subgroup of GL_n. Then $k[x_1, \ldots, x_n]^G$ is Gorenstein if and only if $\det \bar{g} = 1$ for all $g \in G/R$.

19.3 Noncommutative Reflections

19.3.1 Definition: Let $g \in \mathrm{Aut}_{gralg}(A)$. Then

$$\mathrm{Tr}(g) = \sum_0^\infty \mathrm{tr}(g|_{A_i})t^i \in k[[t]].$$

Then a result (which?) gets us that $\mathrm{Tr}(g) = \frac{1}{p(t)}$ for a polynomial p. Consider $\widetilde{A} = k_{p_{ij}}[x_1, \ldots, x_n]$ where $x_j x_i = p_{ij} x_i x_j$ for all $i < j$. Then

19.3.2 Definition: $g \in \mathrm{Aut}_{gralg}(\widetilde{A})$ is a reflection if $\mathrm{Tr}(g) = \frac{1}{(1-t)^{n-1}(1-\lambda t)}$.

20 March 13, 2019

Today we are going to discuss why considering algebras over A is not quite sufficient in many cases – why we may be instead interested in studying ${}_A\mathcal{M}$.

Let A be an algebra and V a Hopf algebra. Let V act on A and define

$$A^V = \{a \in A | va = \varepsilon(v)a, \forall v \in V\}$$

We can see that we get $A^v = \mathrm{Hom}_V({}_Vk, A)$ and then use this to compute the cohomology

$$\mathrm{Ext}^i_V(k, A)$$

We forgot the

20.1 (Non)-Examples of the Day

Example 20.1
Let A be an algebra, $n \geq 2$ an integer. Then $M_n(A)$ can't be a Hopf algebra.

The idea here is that there is no counit $\varepsilon : M_n(A) \to k$.

Example 20.2
If V_1 and V_2 are Hopf algebras, then $V_1 \oplus V_2$ is not a Hopf algebra. Notice here that it is both an algebra and a coalgebra!

Here, use that $1_V = 1_{V_1} + 1_{V_2}$ and compute $\Delta(1_V) \neq 1_V \otimes 1_V$.

Example 20.3
If A is finite dimensional and not Frobenius, then A can't be a Hopf algebra.

20.2 Some Category Theory

Let k be af field.

20.2.1 Definition: Let $\mathcal{C}$ be a category. $\mathcal{C}$ is called **k-linear** if

- $\mathrm{Hom}_{\mathcal{C}}(M, N) \in \mathbf{Vect}_k$ for all M and N in $\mathcal{C}$
- Composition is k-linear.

$$\begin{tikzcd}
((W\otimes X)\otimes Y)\otimes Z \arrow[r, "a_{W\otimes X,Y,Z}"] \arrow[d, "a_{W,X,Y}\otimes \mathrm{id}_Z"] & (W\otimes X)\otimes(Y\otimes Z) \arrow[dd, "a_{W,X,Y\otimes Z}"] \\
(W\otimes(X\otimes Y))\otimes Z \arrow[d, "a_{W,X\otimes Y,Z}"] & \\
W\otimes((X\otimes Y)\otimes Z) \arrow[r, "\mathrm{id}_W\otimes a_{X,Y,Z}"] & W\otimes(X\otimes(Y\otimes Z))
\end{tikzcd}$$

Figure 8: The Pentagon Axioms

20.2.2 Definition: A category $\mathcal{C}$ is called **finite** if $\mathcal{C} \cong {}_A\mathcal{M}_{fd}$ for some finite dimensional algebra A.

20.2.3 Definition: A category $\mathcal{C}$ is called **monoidal** (notice that sometimes this books call this a tensor category) if it is equipped with

- A bifunctor $-\otimes - : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$
- An associativity isomorphism: for all $X, Y, X \in \mathcal{C}$:

$$a_{X,Y,Z} : (X\otimes Y)\otimes Z \xrightarrow{\sim} X\otimes(Y\otimes Z)$$

- An identity object $\mathbf{1}\in\mathcal{C}$
- An isomorphism $i : \mathbf{1}\otimes\mathbf{1}\to\mathbf{1}$.

That satisfies the pentagon axiom (Figure 20.2) and the unit axioms

$$L_{\mathbf{1}}X \xrightarrow{\sim} \mathbf{1}\otimes X, \qquad R_{\mathbf{1}} : X \xrightarrow{\sim} X\otimes\mathbf{1}$$

Example 20.4
Let A be an algebra and let $\mathcal{C} = ({}_A\mathcal{M}_A, \otimes_A, \mathbf{1} = A)$. Clearly this works. :)

Recall that if V is a Hopf algebra and $M \in {}_V\mathcal{M}_{fd}$, then $M^* := \mathrm{Hom}_k(M,k) \in \mathcal{M}_V$ then we defined $M^{*S} = \widetilde{S}\circ M^* \in {}_V\mathcal{M}$. This is the right dual.

We can also define the left dual if S is bijective. In this case, we define $M^{*S^{-1}} = \widetilde{S}^{-1}\circ M^* \in {}_V\mathcal{M}$.

Now let $(\mathcal{C}, \otimes, \mathbf{1})$ be a monoidal category.

20.2.4 Definition: Let $X \in \mathcal{C}$. An object X^* in $\mathcal{C}$ is said to be a **left dual of** X if there exist an **evaluation map**

$$\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}$$

and a **coevaluation map**

$$\mathrm{coev}_X : \mathbf{1} \to X \otimes X^*$$

such that

$$X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} (X \otimes X^*) \otimes X \to X \otimes (X^* \otimes X) \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X$$

is an isomorphism as well as the map "on the other side".

20.2.5 Definition: In the context of the above, a **right dual of** $X \in \mathcal{C}$ is a left dual of X in $\mathcal{C}^{op}$. The right dual is denoted (confusingly) by ${}^*X$.

So in the Hopf algebra case, ${}^*M = M^{*S}$ and $M^* = M^{*S^{-1}}$.

20.2.6 Definition: A category $\mathcal{C}$ is called **rigid** if every $X \in \mathcal{C}$ has both left and right duals.

20.2.7 Definition: $\mathcal{C}$ is called a **tensor category** if

- $\mathcal{C}$ is finite (this implies k-linear)
- It is monoidal
- it is rigid
- $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) = k$

20.2.8 Remark: If we remove the last condition and replace it with $\mathrm{End}(\mathbf{1}) = \oplus k$, then we call $\mathcal{C}$ **multi-tensor.**

20.2.9 Definition: $\mathcal{C}$ is called **fusion** if

- It is tensor
- It is semisimple (every object is a direct sum of simple objects).

20.2.10 Remark: If instead $\mathcal{C}$ is multitensor and semisimple, then it is called **multifusion.**

Example 20.5
(a) $\mathbf{Vect}_{\mathrm{k}}$ is an example.

(b) If V is a finite dimensional Hopf algebra, ${}_V\mathcal{M}$ is tensor.

(c) If V is also semisimple, ${}_V\mathcal{M}$ is fusion.

Recall this conjecture for the first day (Etinghof-Ostrik restated):

Conjecture: If $\mathcal{C}$ is a tensor category then $\mathrm{Ext}^*(\mathbf{1},\mathbf{1})$ is Noetherian.

21 March 15, 2019

On Wednesday, we said that if $M \in {}_V\mathcal{M}_{fd}$ that $M^* = M^{*S^{-1}}$ and ${}^*M = M^{*S}$, but in fact these are reversed.

21.1 Example of the Day

Example 21.1
Usually we have been defining Hopf algebras over vector spaces, but this will be an object in $\textsc{grVect}_k$, that is $V = \oplus_{i\in\mathbb{Z}} V_i$ and if $M = \oplus M_i$ and $N = \oplus N_i$, then $M \otimes N = \oplus_i(\oplus_{s+t=i} M_s \otimes N_t)$.

Now putting the "symmetric" back into the monoidal category, we define $\tau : M \otimes N \to N \otimes M$ via

$$m \otimes n \mapsto (-1)^{|m||n|} n \otimes m.$$

Then we have our symmetric monoidal category and we can define a Hopf algebra in it.

$V = \Lambda(x_1, \ldots, x_n)$, the exterior algebra on n elements. Define the coalgebra structure so that each x_i is primitive. Note that here V is a Hopf algebra over graded vector spaces, but NOT over vector spaces (unless $\operatorname{char} k = 2$).

To see this, we can compute

$$0 = \Delta(x_i^2) = \Delta(x_i)^2 = (x_i \otimes 1 + 1 \otimes x)^2 = x_i^2 \otimes 1 + x_i \otimes x_i + \tau(x_i \otimes x_i) + 1 \otimes x_i^2$$

and if τ is the usual vector space twist (and the characteristic isn't 2) we get a contradiction.

21.2 Some History

Before 1941 people studied rings like kG,$\mathcal{U}(\mathfrak{g})$ and $\mathcal{O}(G)$. An open question at that point was "If $\mathcal{U}(\mathfrak{g})$ is Noetherian, then is $\dim \mathfrak{g} < \infty$?"

In 1941, Hopf algebras were born, although they were originally defined (by Hopf) to be over graded vector spaces. Eventually people started considering them over vector spaces.

In 1967, Sweedler wrote the first book about Hopf algebra over vector spaces. Then later, near 1970, Rota began working more on combinatorial Hopf algebras. It ends up in this context that often the Möbius function ends up being the co(unit? multiplication?).

In 1975, Kaplansky posited 10 conjectures. Of those only one more remains:

Conjecture: Kaplansky, '75 (#6) If V is a semisimple Hopf algebra and M is a simple V-module, then

$$\dim_k M | \dim_k V.$$

From 1960-90, the studies of kG,$\mathcal{U}(G)$ and $\mathcal{O}(G)$ and semisimple Hopf algebra continued, led in large part by Radford. In 1987, Drinfeld and Jimbo came out with quantum groups. At the same time Drinfeld introduced quasi- and braided Hopf algebras and weak Hopf algebras.

Of course we all know about Nichols algebras. Also we switched to different tensor categories, including fusion, pivotal, and spherical ones.

In 1993 Lusztig invented the small quantum groups, which are of particular interest to representation theorists. '97 was Brown-Goodearl. Then around 1990 there were lots of classification projects including the one by Anruskiewitsch-Schneider project. Near this time Zhu proved that every Hopf algebra of dimension p is a group algebra $k\mathbb{Z}/p$.

In 2004 was Etingof-Ostrik. The next year, Etingof-Nikshych-Ostrik wrote a big paper "On Fusion Categories" which really shifted the perspective from Hopf algebras to things in fusion categories, where many of the big results were transferred.

In 2015 Etingof, Nikshych, Ostrik (and someone else) wrote "Tensor Categories".

21.3 Extensions

We can look at extensins of Brown-Goodearl. The question "Is every Noetherian X Gorenstein?" can be extended to be asked about any noun in {Hopf Algebras, weak Hopf algebras, braided hopf algebras, Nichols algebgras, quasi-Hopf algebras}.

We can also consider extensions of Etingof-Ostrik: "Let V be a finite dimensional X and A be a Notherian left V-module such that A^V is Noetherian. Is then $\mathrm{Ext}_V^*(k, A)$ Noetherian?" Again we can take X to be anything in the above set.

In 2019, who knows!