Algebraic Groups

A course by Jarod Alper and Julia Pevtsova
Notes by Nico Courts

Autumn 2019/ Winter 2020

Abstract

The topic of algebraic groups is a rich subject combining both group-theoretic and algebro-geometric-theoretic techniques. Examples include the general linear group GL_n, the special orthogonal group SO_n or the symplectic group Sp_n. Algebraic groups play an important role in algebraic geometry, representation theory and number theory.

In this course, we will take the functorial approach to the study of linear algebraic groups (more generally, affine group schemes) equivalent to the study of Hopf algebras. The classical view of an algebraic group as a variety will come up as a special case of a smooth algebraic group scheme. Our algebraic approach will be independent (even complementary) to the analytic approach taken in the course on Lie groups.

Part I
Quarter 1: Structure Theory

1 September 25, 2019

1.1 Group objects

Let $\mathscr{C}$ be a category with a final object and finite products.

1.1.1 Definition: A **group object** G **in** $\mathscr{C}$ is an object in $\mathscr{C}$ along with multiplication, identity, and inverse morphisms satisfying the usual axioms.

One thing is that we are using that there is a final object $*$ along with our identity morphism $e : * \to G$. Here Jarrod explictly used the fact that there is a unique map to $*$.

Example 1.1
If $\mathscr{C}$ is Set, then G is a group. If $\mathscr{C} = \text{Top}$, then G is a topological group, smooth manifolds give Lie groups, and finally (interesting to us):

1.1.2 Definition: Let S be a scheme and let $\mathscr{C}$ be the category of schemes over S. Then a group object G in $\mathscr{C}$ is a **group scheme over** S**.**

WHen k is a field and $\mathscr{C}$ is schemes of finite type over k, we get a group scheme of finite type over k. There is not a great consensus on what makes an **algebraic group**, but this is what we will use.

When we instead restrict to *affine schemes* we get an affine groupe scheme of finite tipe over k, or a **linear algebraic group.**

1.2 Examples

$\mathbb{G}_m = \operatorname{Spec} k[t]_t$ is one.

If we consider the map $f : \mathbb{G}_m \to \mathbb{G}_m$ which on the level of elements sends $t \mapsto t^p$, the kernel is

$$\mu_p = \ker(f) = \operatorname{Spec} k[t]/(t^p - 1)$$

and that's great, but when $\operatorname{char} k = p$, this causes the group scheme to be **unreduced**. This is (apparently) a case when you need to use schemes.

1.3 The Functorial Approach

Let $\mathscr{C}$ be a category with object X. Define the functor $h_X : \mathscr{C}^{op} \to \mathbf{Set}$ where

$$h_X(Y) = \operatorname{Hom}_{\mathscr{C}}(Y, X).$$

Then we have

1.3.1 Lemma (Yoneda)
Let $G : \mathscr{C}^{op} \to \mathbf{Set}$ be a functor. There is a natural bijection

$$G(X) \simeq \operatorname{Nat}(h_X, G).$$

1.3.2 Proposition
A group object G in $\mathscr{C}$ is the same as an aobject $X \in \mathscr{C}$ together with a choice of factorization of $h_X : \mathscr{C} \to \mathbf{Set}$ through **Grp**.

1.4 Exercises

(a) Spell out all the details of the proof of the above propositon.

(b) Given a group object G, define in two ways what it means for it to act on another object. (In coordinates and functorially).

1.5 Some Interesting Facts

If we had to write down five results that we'd like to get out of this class:

1.5.1 Proposition
Every affine group scheme of finite type over a field embeds into GL_n as a closed subgroup.

1.5.2 Theorem (Chevalley's Theorem)
Let G be a finite type group scheme over a field. Then it factors as

$$1 \to H \to G \to A \to 1$$

where A is abelian and H is affine (linear algebraic).

1.5.3 Proposition
If G is an affine group scheme of finite type over k, then we have af actorization

$$1 \to U \to G \to R \to 1$$

where U is unipotent and R is reductive.

1.5.4 Proposition
$H \subseteq G$ a subgroup scheme. Then G/H is a projective scheme.

Finally we want to talk about Tanakka duality and how the representations of G define G itself.

2 September 27th, 2019

Last time we defined a group scheme (a group object in the category of schemes over a base scheme). We also mentioned that You could define it as a map $h_G : \mathbf{Sch}/S \to \mathbf{Set}$ along with a factorization through **Grp**.

We defined an **algebraic group** over k as a group scheme over $\operatorname{Spec} k$ of finite type and a **linear algebraic group** to be an *affine* group scheme over k of finite type.

2.1 Hopf Algebras

Let $G = \operatorname{Spec} A$ be a linear algebraic group over k. I have seen most of these before (see Waterhouse or my Hopf algebra notes)

2.1.1 Remark: One think I haven't seen explicitly before: Notice that the augmentation ideal $\ker \varepsilon$, where ε is the counit, is the (maximal!) ideal corresponding in the algebro-geometric sense to the identity element in G.

2.1.2 Definition: A **Hopf algebra** is ...

2.1.3 Definition: Let G be an algebraic group over k. Then if h_G factors through **Ab**, G is called **commutative.**

2.2 Some Examples

2.2.1 Remark: Note that to define a functor from schemes over k, is suffices to define it on affine schemes, thereby defining the (Zariski) local behavior of any such map. Thus we really only need to consider maps in **Alg**.

- $\mathbb{G}_a$. Here we can define it as a functor that sends $S \mapsto \Gamma(S, \mathscr{O}_S)$. Geometrically, $\mathbb{G}_a = \mathbb{A}^1$ where the multiplication is addition, inverses send $x \mapsto -x$ and the unit is the zero map. The Hopf algebraic picture is the usual dual thing.
- $\mathbb{G}_m$ as a scheme isthe map $S \mapsto \Gamma(S, \mathscr{O}_S)^*$. In the geometric picture, $\mathbb{A}^1 \setminus \{0\}$ and the algebra structure comes from multiplciation. Hopf is pretty easy.
- GL_n is a scheme that sends

  $$S \mapsto \left\{ A = (a_{ij}) : a_{ij} \in \Gamma(S, \mathscr{O}_S), \det(A) \in \Gamma(S, \mathscr{O}_S)^* \right\}$$

  the algebra is $\mathbb{A}^{n\times n} \setminus \{\det = 0\}$ with the usual multiplication. The coalgebra structure can be seen in the book.

This one requires some more explaination so I am setting it apart.

Example 2.1
Let V be a finite dimensionatl vector space over k. Then we can define the algebraic group V_a which sends

$$S \mapsto \Gamma(S, \mathscr{O}_S) \otimes_k V.$$

Geometrically we are looking at $\mathbb{A}(V) = \operatorname{Spec} \operatorname{Sym}^* V^\vee \simeq \operatorname{Spec} k[x_1, \ldots, x_n]$ where $n = \dim V$.

What about finite groups? As a scheme, we want $G = \sqcup_{g \in G} \operatorname{Spec} k$. The functor sends $S \mapsto \operatorname{Mor}_{\mathsf{Set}}(\pi_0(S), G)$, or maps from the connected components into G.

Example 2.2
Now consider the n^{th} roots of unity: as a scheme, $\mu_n = \operatorname{Spec} k[t]/(t^n - 1) \subseteq \mathbb{G}_m$. If both $k = \bar{k}$ and $\operatorname{char} k \nmid n$, then $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$.

But if (e.g.) $k = \mathbb{Q}$, then μ_3 is $\mathbb{Q}[t]/(t^3 - 1) = \operatorname{Spec} \mathbb{Q} \sqcup \operatorname{Spec} \mathbb{Q}(\xi)$ where ξ is a primitive third root of unity.

If, on the other hand, $k = \bar{\mathbb{F}}_3$ and consider μ_3, we get a single point with residye field $\bar{\mathbb{F}}_3$.

Example 2.3
If we are in the case of positive characteristic, then we get an algebraic group α_p. Here the scheme is $\operatorname{Spec} k[x]/x^p$ and functorially it is the map $S \mapsto \{F \in \Gamma(S, \mathscr{O}_S) | f^p = 0\}$.

2.3 Matrix Groups

We already defined GL_n, but we can also define

$$\mathrm{SL}_n : S \mapsto \{A = (a_{ij}) | \det A = 1\}$$

with scheme $\operatorname{Spec} k[x_{ij}]/(\det - 1)$.

We also have the (upper) triangular matrices T_n and unitary group U_n and diagonal group D_n

2.3.1 Definition: Let G be a linear algebraic group. Then

- G is a **vector group** if $G \cong V_a$ for some finite dimensional V.
- G is a **split torus** if $G \cong \mathbb{G}_m^n$.
- G is a **torus** if there is a field extention $k \to k'$ such that

$$G \times_{\operatorname{Spec} k} \operatorname{Spec} k' \cong \mathbb{G}_{m,k'}^n$$

3 September 30th, 2019

Another example to consider:

Example 3.1
Let $G = \mathrm{PGL}_n$, the projective linear group. Recall we want to define this as GL_n / k^* (from group theory). To do this for algebraic groups, we define

$$\mathrm{PGL}_n = \operatorname{Proj} k[x_{ij}]_{det} := \operatorname{Spec}(k[x_{ij}]_{det})_0$$

The geometric picture is difficult since we haven't yet defined quotients, but as a functor we say PGL_n is $\mathrm{Aut}(\mathbb{P}^n)$, the functor that sends $S \mapsto \mathrm{Aut}(\mathbb{P}^n_S)$ where $\mathbb{P}^n_S = \mathbb{P}^n_k \times_{\operatorname{Spec} k} S$.

3.1 Non-affine group schemes

Example 3.2
Let $\lambda \neq 0, 1$ be an element in k. Then we can define the elliptic curve

$$E_\lambda = V(y^2 z - x(x-z)(x-\lambda z)) \subset \mathbb{P}^2$$

Which gives us a double cover over $(0,1)$ and (λ, ∞) with singleton fiber (ramified) over $0, 1$, and λ.

Then for any $\lambda \neq 0, 1$, E_λ is a **projective** group scheme.

3.1.1 Remark: If you look at the $\mathbb{C}$-points, you get $E_\lambda(\mathbb{C}) = \Lambda_\lambda$, giving you a torus. Recall (from e.g. complex analysis) that the moduli here is $\mathrm{SL}_2(\mathbb{Z})$ of all elliptic curves.

3.2 Abelian Varieties

3.2.1 Definition: An **abelian variety over** k is asmooth, geometrically connected ($A \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$ is connected), proper group scheme A over k.

Example 3.3
Over $\mathbb{C}$, $\mathbb{C}^g/\Lambda$ where $\Lambda \cong \mathbb{Z}^{2g} \subseteq \mathbb{C}^g$ gives us a genus g example.

3.2.2 Theorem
Any abelian variety over k is commutative and projective.

3.2.3 Theorem (Chevalley)
If G is any group scheme, then the sequence

$$1 \to H \to G \to A \to 1$$

is exact, where H is a linear algebraic group (affine!) and A is an abelian variety.

Example 3.4
Let $X \to \operatorname{Spec} k$ be a geometrically integral projective scheme (proper may suffice). The idea here is that over $\mathbb{C}$ the rings over every open set are integral domains.

Now consider the **Picard functor** $\operatorname{Pic}_X : \operatorname{Pic} : \mathbf{Sch/k} \to \mathbf{Grp}$ sending

$$S \mapsto \operatorname{Pic}(X_S = X \times_k S)/p^k \operatorname{Pic(S)}$$

3.2.4 Theorem
Pic_X is represented by a scheme locally of finite type, thus Pic^0_X, the connected component of the identity in $[\mathscr{O}_X] \in \operatorname{Pic}_X$ is an abelian variety.

3.3 Relative Group Schemes

Example 3.5
Consider $\mathbb{G}_{m,\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[t]_t$. Then $G_{m,S} = \mathbb{G}_{m,\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} S$. In the case that $S = \operatorname{Spec} R$, $\mathbb{G}_{m,S} = \operatorname{Spec} R[t]_t$.

Example 3.6
Let $\mathbb{A}^1 = \operatorname{Spec} k[x]$ and define $G = \operatorname{Spec} k[x,y]_{xy+1} \subseteq \mathbb{A}^2$. Notice this is the plane minus a hyperbola.

Define $\cdot : G \times_{\mathbb{A}^1} G \to G$ to be given by

$$(x,y)\cdot(x,y') = (x, xyy' + y + y')$$

Then the thing here is the fiber (think vertical line in the plane!) over 0 is $\mathbb{G}_a$ and is isomorphic to $\mathbb{G}_m$ otherwise.

Example 3.7
Let $\mathscr{E}_\lambda = V(y^2z - x(x-z)(x-\lambda z))$ over $\operatorname{Spec} k[\lambda]$. Then when $\lambda = 0$, we get the nodal cubic given by $y^2z - x^2(x-z)$ (node at the origin).

Now if you look at the connected component around 0 of $\operatorname{Aut}(\mathscr{E}_\lambda)/\mathbb{A}_\lambda$, you actually find (when $\lambda = 0$) that $\mathbb{G}_m \cong \operatorname{Aut}(\mathscr{E}_0)^0$.

3.4 Some definitions

3.4.1 Definition: A **homomorphism** $\phi : G \to G$ of group schemes over S is a map $\phi : H \to G$ of schemes such that

$$\begin{array}{ccc} H \times_S H & \xrightarrow{m_H} & H \\ \downarrow{\scriptstyle \phi\times\phi} & & \downarrow{\scriptstyle \phi} \\ G \times_S G & \xrightarrow{m_G} & G \end{array}$$

Problem 3.1
Show that this automoatically imples that the identity and inversion maps are respected as well (automatically).

3.4.2 Definition: A **subgroup of** $G \to S$ is a subscheme $H \subseteq G$ such that $H(T) \leq G(T)$ for all T over S.

Problem 3.2
Show that $\ker(\phi) \subseteq H$ *is a subgroup.*

3.4.3 Remark: This gives you a nice way to construct new group schemes. For example, the following are exact:

$$1 \to \mathrm{SL}_n \to GL_n \xrightarrow{\det} \mathbb{G}_m \to 1$$

and

$$1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$$

3.4.4 Proposition
Let $G \to S$ be a group scheme. Then $G \to S$ is separated if andy only if $e : S \to G$ is a closed immersion.

Proof
The idea here is that $S \to G$ is a closed immersion. Then we consider the map $m \circ (\mathrm{id}, S) : G \times_S G \to G$ and consider this along with the diagonal map $\Delta : G \to G \times_S G$ and this is a pullback square! ♠

3.4.5 Corollary
Any group scheme over k is separated.

The idea is going to be that if X is any scheme over k, then any point $X \in X(k)$ is closed.

4 October 2, 2019

Notice that a **relative group scheme** (referred to in last lecture) refers to a groups scheme over an arbitrary base scheme S.

4.1 Properties of schemes

Today we are going to be talking about reducedness, connectedness, irreduciblility, regularity, and smoothness.

Recall that a scheme X is **reduced** if and only if $\forall x \in X$, $\mathscr{O}_{X,x}$ is reduced. An example of a non-reduced scheme is $\operatorname{Spec} k[x]/(x^2)$.

4.1.1 Definition: We say a scheme X over k is **geometrically reduced** if for all field extensions k'/k,

$$X_{k'} = X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$$

is reduced.

4.1.2 Remark: It is equivalent that $X_{\bar{k}}$ is reduced if and only if every k'/k is purely inseperable (I think).

4.1.3 Remark: If k is perfect, then X is reduced if and only if X is geometrically reduced.

4.1.4 Definition: A local ring $(A,\mathfrak{m})$ is **regular** if $\dim_{\mathrm{Krull}} A = \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$

4.1.5 Definition: A scheme X is regular if for all $x \in X$, $\mathscr{O}_{X,x}$ is regular.

4.1.6 Remark: If $X \to \operatorname{Spec} k$ and $x \in X(k)$, the tangent space at x is

$$T_{X,x} = (\mathfrak{m}/\mathfrak{m}^2)^\vee = \{f : \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \to X | 0 \mapsto x\}$$

4.1.7 Remark: Notice that if $X \to \operatorname{Spec} k$ is regular and k'/k is a field extension, then $X_{k'}$ is not necessarily regular.

4.1.8 Definition: A Scheme $X \to \operatorname{Spec} k$ of finite type is **smooth** if $X_{\bar{k}}$ is regular.

4.2 Facts about algebraic groups

Then we can return to the proposition we want to prove:

4.2.1 Proposition
Let $G \to \operatorname{Spec} k$ be an algebraic group. Then G is geometrically reduced if and only if G is smooth over $\operatorname{Spec} k$.

Proof
Smoothness over k implies reducedness. Now since we are only interested in the algebraic closure of k, we can say $k = \bar{k}$. Because G is reduced, there exists a nonempty open $U \subseteq G$ that is smooth. Then since $G(k) \subseteq |G|$ is dense in G (as a topological space) and Then $G = \cup_{g \in G(k)} m_g(U)$ for our smooth U, and this gives us a smooth cover of G. ♠

We will see next itme that all linear algebraic groups over k where $\operatorname{char} k = 0$ are all geometrically reduced (and thus smooth).

4.3 Connectedness

Let G be an algebraic group over k. Then we have our maps $e : \operatorname{Spec} k \to G$, so consider it as $e \in G(k)$. Let $G^0 \subseteq G$ be the connected component of e. It is both open and closed.

4.3.1 Remark: If $X \to \operatorname{Spec} k$ is of finite type and $x \in X(k)$, then X being connected implies that X is geometrically connected.
This establishes that G^0 is actually geometrically connected! We actually will see

4.3.2 Proposition
$G^0 \subseteq G$ is an (open and closed) algebraic subgroup.

The idea here is that $G^0 \times G^0$ is connected, so the image of the multipication map on this set lands in a connected component (since it is connected). Since $e \in G^0$, and $m(e,e) = e \in G^0$, this shows that the multiplication map restricts to a well-defined map $G^0 \times G^0 \to G^0$. A similar argument goes through for the inverset map, etc.

The upshot here is that if G is an algebraic group, then there exists a factorization

$$1 \to G^0 \to G \to \pi_0(G) \to 1$$

where $\pi_0(G)$ is given the structure of a discrete group.

4.3.3 Remark: Now we also have that $(G^0)_{k'} = (G_{k'})^0$ for all k'/k. The idea is to get a map of of one into the other and then use clopenness and connectedness to show they are equal.

4.3.4 Proposition
A connected algebraic group over k is irreducible.

Proof
We can assume $k = \bar{k}$. Suppose $G = X \cup Y$, where both are closed, X is irreducible, and $X \cap Y \neq \varnothing$. Thus there exists an element $g \in X \setminus Y$. That is, g lies in a single irreducible component.

But then using the multiplication by h map on G, we get to every other point in G, so every point is in a single irreducible component. But the intersection was nontrivial! Or something.♠

4.3.5 Proposition
If G_{red} is geometrically connected, then $G_{\text{red}} \subseteq G$ is a subgroup. In particular, if k is perfect, then G_{red} is a subgroup of G.

4.3.6 Remark: X is geometrically reduced implies that $X \times X$ is geometrically reduced.

5 October 4, 2019

Some review. Let G be an algebraic group and denote $e : \operatorname{Spec} k \to G$ be the identity. We saw a lot of propositions last time.

Now let k be a nonperfect field and take $t \in k \setminus k^p$. Then define

$$G \stackrel{\text{def}}{=} V(x^{p^2} - t x^p) \subseteq \mathbb{G}_a$$

which Milne claims is not reduced. We can see why it's not geometrically reduced, but we're missing the details here.

5.1 Another special case

5.1.1 Theorem
When $k=\bar{k}$, G is smooth if and only if

$$\dim T_e G_{\text{red.}} = \dim T_e G.$$

5.1.2 Remark: When G is smooth, it is reduced, so the equality is clear. For the other direction, we get that k is pefect, so $G_{\text{red.}}$ which is geoemetrically reduced if and only if G is smooth. But

$$\dim G \le \dim T_e G = \dim T_e G_{\text{red.}} = \dim G_{\text{red.}} = \dim G$$

5.1.3 Theorem
If G is a linear algebraic group over k and $\operatorname{char} k = 0$, G is smooth,

Proof
We can assume $k=\bar{k}$. Then set $G=\operatorname{Spec} A$ where A is a Hopf algebra. THen we get Hopf algebra maps m^* and e^*. Notice that the augmentation ideal $\mathfrak{m}=\ker(e^*)$ is a maximal ideal.

Then we want to prove

(a) $A\cong \mathfrak{m}\oplus k$ as a k-vector space (obvious).

(b) $\forall a\in\mathfrak{m}$, $m^*(a)-a\otimes 1-1\otimes a\in \mathfrak{m}\otimes\mathfrak{m}$.

To see the second, notice that $m^*(a)-a\otimes 1-1\otimes a$ is in the kernel of

$$e^*\otimes \mathrm{id} : A\otimes A\to k\otimes A.$$

This is clear from the commutative diagram

$$\begin{tikzcd} k\otimes A & A\otimes A \arrow[l, "e^*\otimes\mathrm{id}"'] \\ & A \arrow[u, "m^*"] \arrow[ul, "\sim"] \end{tikzcd}$$

Then we conclude $f\in\ker(e^*\otimes\mathrm{id})\cap\ker(\mathrm{id}\otimes e^*) = A\otimes\mathfrak{m}\cap\mathfrak{m}\otimes A$ by a symmetric argument. Finally we notice that $A\otimes\mathfrak{m}\cap\mathfrak{m}\otimes A=\mathfrak{m}\otimes\mathfrak{m}$, and so f lies in this ideal.

Now we want to show that $\dim T_e G = \dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim_k \mathfrak{m}/(\sqrt{0}+\mathfrak{m}^2) = \dim T_e G_{\text{red.}}$. It suffices to show that for all $a\in\sqrt{0}$, $a\in\mathfrak{m}^2$. Suppose the opposite–so let $a\in\sqrt{0}\setminus\mathfrak{m}^2$. Consider the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_{\mathfrak{m}} \arrow[d] \\ A/\mathfrak{m}^2 \arrow[r, "\sim"] & A_{\mathfrak{m}}/(\mathfrak{m}A_{\mathfrak{m}})^2 \end{tikzcd}$$

Now the image of a in $A_\mathfrak{m}$ is nonzero, so there exists $n > 0$ such that $a^n \in A_\mathfrak{m}$ but $a^{n-1} \neq 0$ in $A_\mathfrak{m}$. Thus there exists $f \notin \mathfrak{m}$ such that $a^n f = 0 \in A$. Subsititute af for a, and thus there is an $a \in \sqrt{0}$ such that $a^n = 0$ in A but $a^{n-1} \neq 0$ in $A_\mathfrak{m}$.

Then by fact 2,

$$m^*(a) = 1 \otimes a + a \otimes 1 + r, \quad r \in \mathfrak{m} \otimes \mathfrak{m}$$

and since m^* is a ring homomorphism,

$$0 = m^*(a^n) = (m^*(a))^n = (a \otimes 1 + (1 \otimes a + r))^n = a^n \otimes 1 + n(a^{n-1} \otimes a + (a^{n-1} \otimes 1)r) + X$$

where $X \in A \otimes \mathfrak{m}^2$. But since $a^n = 0$, we get

$$n(a^{n-1} \otimes a + (a^{n-1} \otimes 1)r) \in A \otimes \mathfrak{m}^2$$

Now since $(a^{n-1} \otimes 1)r \in (a^{n-1})\mathfrak{m} \otimes A$, so

$$n(a^{n-1} \otimes a) \in (a^{n-1})\mathfrak{m} \otimes A + A \otimes \mathfrak{m}^2$$

and since $\operatorname{char} k = 0$, we get that n is a unit, so

$$a^{n-1} \otimes a \in (a^{n-1})\mathfrak{m} \otimes A + A \otimes \mathfrak{m}^2$$

Now since this lives in $A \otimes A$, consider the image of the quotient map $A \otimes A \to A \otimes A/\mathfrak{m}^2$. Then

$$a^{n-1} \otimes \bar{a} \in (a^{n-1})\mathfrak{m} \otimes A/\mathfrak{m}^2 \subseteq A \otimes A/\mathfrak{m}^2$$

And note that $a^{n-1} \notin a^{n-1}\mathfrak{m}$ because otherwise $a^{n-1} = a^{n-1}q$ for $q \in \mathfrak{m}$. Then $a^{n-1}(1-q) = 0 \in A_\mathfrak{m}$, which implies that $a^{n-1} = 0 \in A_\mathfrak{m}$ (since $1-q$ is a unit here).

Then somehow we get that $\bar{a} = 0 \in A/\mathfrak{m}^2$, so $a \in \mathfrak{m}^2$. ♠

6 October 7, 2019

Today we will be primarily concerned with

6.1 Group actions

Let G be an algebraic group over k.

6.1.1 Definition: A **group action** of G on a scheme X over k is the data of a morphism

$$\mu : G \times X \to X$$

such that the usual axioms hold. That is,

$$\begin{tikzcd} G\times G\times X \arrow[r, "m\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\mu"] & G\times X \arrow[d, "\mu"] \\ G\times X \arrow[r, "\mu"] & X \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec} k\times x \arrow[r, "e\times\mathrm{id}"] \arrow[dr, "\sim"] & G\times X \arrow[d, "\mu"] \\ & X \end{tikzcd}$$

6.1.2 Remark: Apparently it was an exercise already to show that this is equivalent to an action of h_G on h_X.

6.1.3 Remark: The map $(g,x)\mapsto(g,gx)$ is an automorphism of $G\times X$, so if $p_2 : G\times X\to X$ is projection,

$$\begin{tikzcd} G\times X \arrow[rr, "\sim"] \arrow[dr, "\mu"] & & G\times X \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

commutes.

6.1.4 Definition: Let X and Y be schemes over k with a G action. Then a G**-equivariant morphism** $f : X\to Y$ is one such that for all $g\in G$, the following commutes:

$$\begin{tikzcd} G\times X \arrow[r, "\mathrm{id}\times f"] \arrow[d, "\mu_X"] & G\times Y \arrow[d, "\mu_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

6.1.1 Some examples

- G actions on itself by multiplication or conjugation.
- $\mathbb{G}_m$ acts on $\mathbb{A}^1$. Geometrically, we are just looking at k^* acting on k by scaling. Algebraically, we want a map $\mu\mathbb{G}_m\times\mathbb{A}^1\to\mathbb{A}^1$ given by the map of algebras:

  $$k[x]\xrightarrow{\mu^*}k[t]_t\otimes k[x]\quad\text{via}\quad x\mapsto tx$$

  Functorially, if S is a scheme over k, then $\mathbb{G}_m(S)=\Gamma(S,\mathscr{O}_S)^*$ which acts on $\mathbb{A}^1(S)=\Gamma(S,\mathscr{O}_S)$, again by scaling.
- You can consider GL_n action on $\mathbb{A}^n$ by multiplicaiton or on $\mathbb{A}^{n\times n}$ via multiplication or conjugation.

6.1.2 Orbits and Stabilizers

Let G be an algebraic group over k action on a scheme X over k. Let $x \in X(k)$. Then we have a map

$$\mu_x : G \times \operatorname{Spec} k \xrightarrow{\mathrm{id}\times x} G \times X \xrightarrow{\mu} X$$

where

$$g \mapsto (g,x) \mapsto gx.$$

6.1.5 Definition: The **orbit of x** is $Gx = \mu_x(G) \subseteq |X|$ set-theoretically. The **stabilizer of x in G** is $G_x = \mu_x^{-1}(x) \subseteq G$.

6.1.6 Remark: G_x is always a closed algebraic subgroup of G.

6.1.7 Proposition
$\mu_x(G)$ is open in its closure in $|X|$.

Recall first the following:

6.1.8 Theorem (Chevalley's Theorem (different?))
If $f : X \to Y$ is a map of schemes of finite type over k, then $f(X) \subseteq Y$ is contstructible (i.e. is a disjoint union of finitely many locally closed subsets).
Recall that locally closed means closed in an open subspace.

6.1.9 Corollary
Maybe a definition: The orbit $Gx \subseteq \operatorname{im}(\mu_x) \subseteq X$. If G is reduced, then Gx is reduced.

6.1.3 Applications

Say that $\operatorname{char} k = p$. Then μ_p acts on $\mathbb{G}_m$ by multiplication.

$\mathbb{G}_m$ acts on $\mathbb{A}^1$ with two orbits: $\mathbb{G}_m \cdot 1 = \{x \neq 0\}$ and $\mathbb{G}_m \cdot 0 = \{0\}$. The stabilizers are $G_1 = 1$ and $G_0 \cong \mathbb{G}_m$.

Consider G acting on $\mathbb{A}^2$ via $t(x,y) = (tx, t^{-1}y)$. Then the orbits are hyperbolas! There is also a notion of closed orbits that I didn't quite catch. Also apparently the orbit-stabilizer statement is easy to see in geometry via a fiber bundle G over Gx where the fiber over x is G_x.

6.1.10 Proposition
If $\phi : G \to H$ is a homomorphism of algebraic groups, then $\phi(G) \subseteq |H|$ is closed.

6.1.11 Remark: The proof included reducing first to $k = \bar{k}$. The trick here is to consider the group action induced by ϕ and then consider the map μ_{e_H} of this action. Then $\mu_{e_H}(G) = \phi(G)$ and one can prove that this is closed.

In particular, we have that **subgroups of an algebraic group are always closed.** Note that this stands in stark contrast to Lie theory where you get non-closed subgroups.

7 October 9, 2019

Recall that last time we were considering actions of algebraic groups on schemes of finite type over k. We discussed the orbit and stabilizer of an element $x \in X$ and showed that $G \cdot x$ is open in its closure. We also saw that G_x is a closed subgroup.

We also say that $\phi(H)$ (as a set) is always closed! None of these facts are true for Lie groups or relative group schemes (the base scheme is not $\operatorname{Spec} k$ for k a field).

7.1 Cartier Duality

Let $G \to \operatorname{Spec} k$ be a **finite** group scheme (so $G = \operatorname{Spec} A$ and A is a finite dimensional Hopf algebra). Some examples of finite group schemes are:

- G a finite group. Then $G = \sqcup_{g\in G} \operatorname{Spec} k = \operatorname{Spec}\left(\prod_{g\in G} k\right)$
- $\mu_n = \operatorname{Spec} k[t]/(t^n - 1)$
- $\operatorname{char} k = p$ and $\alpha_p = \operatorname{Spec} k[t]/t^p$

7.1.1 Remark: Recall all the maps and diagrams that A has as a Hopf algebra.
A question one may ask: what if we apply the idea of dualizing $(-)^\vee = \operatorname{Hom}_{\mathbf{Alg}}(-,k)$ to A? Do we get another Hopf algebra?

The short and sweet of it is yes! But notice that we are coming from the commutative world, so we expect A to be commutative. But in general, A is not cocommutative (in fact, it is if and only if G itself was commutative as a group).

Thus $A^\vee$ is indeed a (cocommutative) Hopf algebra, and when G is commutative, $A^\vee$ is as well. So

7.1.2 Proposition
If $G = \operatorname{Spec} A$ is a commutative group scheme, then the Cartier dual $G^D = \operatorname{Spec} A^\vee$ is a commutative group scheme as well.

7.1.3 Remark: The above observations gives us an anti-autoequivalence of the category of commutative affine group schemes. Furthermore $(G^D)^D = G$.

Example 7.1
Consider $\mu_n = \operatorname{Spec} A - \operatorname{Spec} \oplus_0^{n-1} k \cdot t^i$. So then if we let $\{e_i\}$ be the basis for $A^\vee$ dual to $\{t_i\}$. we can compute comultiplication

$$e_i \mapsto \sum_{j=0}^{n-1} e_j \otimes e_{i-j}$$

and multipication

$$e_i \otimes e_j \mapsto \delta_{ij} e_i$$

Then it can be shown that $G^D \cong \mathbb{Z}/n\mathbb{Z}$.

Now given G, an algebraic group over k, define

$$\underline{\mathrm{Hom}}(G, \mathbb{G}_m) : \mathbf{Sch}/k \to \mathbf{Set}$$

which takes

$$T \mapsto \mathrm{Hom}_{\mathbf{AlgGrp}}(G_T, \mathbb{G}_{mT}).$$

7.1.4 Theorem
If G is a commutative finite group scheme over k, then

$$G^D \cong \underline{\mathrm{Hom}}(G, \mathbb{G}_m)$$

Let $H = \operatorname{Spec} B \to \operatorname{Spec} R$ be a group scheme. Then

$$H_{\mathbf{GrpSch}/R}(H, \mathbb{G}_{mR}) \subseteq \mathrm{Mor}_{\mathbf{Sch}/R}(H, \mathbb{G}_{mR})$$

But the left hand side is equivalent to the grouplike elements of B and the right hand side is equivalent to $\mathrm{Hom}_{\mathbf{Alg}_R}(R[t]_t, B)$.

This leads to a proof of thm. 7.1.4:

Proof
Let $G = \operatorname{Spec} A$ and $G^D = \operatorname{Spec} A^\vee$. First look at the k-points:

$$\begin{aligned} G^D(k) &= \mathrm{Mor}_{\mathbf{Sch}/k}(\operatorname{Spec} k, G^D) \\ &= \mathrm{Hom}_{\mathbf{Alg}_k}(A^\vee, k) = \{f \in A \mid m^*(f) = f \otimes f\} \hookrightarrow \mathrm{Hom}_k A^\vee, k) \\ &= \mathrm{Hom}(G, \mathbb{G}_m) \\ &= \underline{\mathrm{Hom}}(G, \mathbb{G}_m)(k) \end{aligned}$$

If we then look at R points for a general R, most things just change over, but we see

$$\{f \in A \otimes R \mid m_R^*(f) = R \otimes R\} = \mathrm{Hom}_{\mathbf{Alg}_k}(A^\vee, R) = \mathrm{Hom}(G_R, \mathbb{G}_m)$$

and the rest follows. ♠

A question one may ask: what is $\mathrm{Hom}_{\mathbf{AlgGrp}}(\mathbb{G}_m, \mathbb{G}_m)$? It ends up it is $\mathbb{Z}$. You can send $t \mapsto t^n$ for all $n \in \mathbb{Z}$. But then $\underline{\mathrm{Hom}}(\mathbb{G}_m, \mathbb{G}_m)$ is $\mathbb{Z}$ as a group scheme over k, which is not quasicompact. There was more but I am le tired.

8 October 11, 2019

Today we're doing problems and stuff. Forgot about that.

8.1 Casey's Presentation

Let G and H be objects in a category $\mathscr{C}$ with finite products. Let $h : H \to G$ be a group homomorphism. That is,

Then we get similar diagrams for the identity and inverse maps (they are respected by h). Then there is a bunch of diagram work. It's too hard to do a diagram without knowing the shape ahead of time.

Oh hey I used Adam's site!

$$\begin{tikzcd}
* \arrow[dr, "{(e,e,e)}"] & & & \\
& H\times H\times H \arrow[r, "(i_g\circ h)\times h\times h"] \arrow[d, "1\times m_H"'] & G\times G\times G \arrow[d, "1\times m_g"] & \\
& H\times H \arrow[r, "(i_G\circ h)\times h"] & G\times G \arrow[dr, "m_G"] & \\
& & & G
\end{tikzcd}$$

$$\begin{tikzcd}
* \arrow[r, "e"] \arrow[dr, "{(e,e,e)}"'] \arrow[ddddr, bend right, "he"'] & H \arrow[dr, "{(h,h)}"] & & \\
& H\times H\times H \arrow[d, "(i_g\circ h)\times h\times h"'] & G\times G \arrow[dl, "{(*,1)\times 1}"] \arrow[dr, "*\times 1"] & \\
& G\times G\times G \arrow[d, "1\times m"'] \arrow[dr, "m\times 1"] & & *\times G \arrow[dl, "e\times 1"'] \arrow[ddll, bend left, "\pi_1"] \\
& G\times G \arrow[d, "m"'] & G\times G \arrow[dl, "m"] & \\
& G & &
\end{tikzcd}$$

$$\begin{tikzcd}
& * \arrow[r, bend left, "*"] \arrow[dl, "{(e,e,e)}"'] \arrow[d, "{(e,e,1)}"'] \arrow[dr, dashed, "he"] \arrow[dd, bend left, "{(e,e)}"] & * \arrow[r, "e"] & G \\
H\times H\times H \arrow[dr, "1\times m"'] & H\times H\times * \arrow[l, "1\times 1\times e"'] \arrow[d, "1\times \pi_2"'] & G \arrow[u, "*"'] \arrow[d, "{(1,1)}"'] \arrow[dr, bend left, "{(i,1)}"] & \\
& H\times H \arrow[r, "h\times h"] \arrow[rr, bend right, "(ih)\times h"'] & G\times G \arrow[r, "i\times 1"] & G\times G \arrow[uu, "m"']
\end{tikzcd}$$

8.1.1 Remark: The idea above is we want to show the first diagram commutes. That is captured in the paths of the second diagram which commutes by the axioms of a group object. The third diagram shows a similar commutativity for the unit e.

9 October 14th, 2019

Let G be a finite group. Recall the definiton of a **representation** (a linear action of G on a vector space V/k). This is the same data as a group homomorphism to $\mathrm{GL}(V)$.

9.1 Representations of Algebraic Groups

Now what if G is an algebraic group over k? Now we have some exra structure of G as a variety.

9.1.1 Definition: A (finite dimensional) **representation** of an algebraic group G/k is a homomorphism $\rho : G \to \mathrm{GL}(V)$ of algebraic groups.

9.1.2 Remark: Notice that when V is infinite-dimensional, $\mathrm{GL}(V)$ is no longer of finite type, so we have to let ρ be a morphism of group schemes.
We have the standard representation of GL_n acting on k^n in the natural way. We also have the regular representation G action on $\Gamma(G, \mathcal{O}_G)$. When $G = \mathbb{G}_m$, we get over $\mathbb{C}$ an action of $\mathbb{G}_m$ on $\mathrm{GL}(V)$ in the usual way (scaling by $\mathbb{C}^*$).

Observe that
$$\rho : G \to \mathrm{GL}(V) = \mathrm{Spec}(\mathrm{Sym}^*(V \otimes V^\vee))_{\det}$$
corresponds to a ring morphism
$$\mathrm{Sym}^*(V \otimes V^\vee)_{\det} \to \Gamma(G, \mathcal{O}_G) \overset{\text{def}}{=} \Gamma(G)$$
which corresponds to a map
$$V \otimes V^\vee \to \Gamma(G)$$
and then tensoring with V, this gives us a map
$$V \xrightarrow{\sigma} \Gamma(G) \otimes V$$

So any group action gives us a **coaction** of $\Gamma(G)$ on V.

9.1.3 Definition: A representation of G is a k-vector space V along with a coaction
$$\sigma : V \to \Gamma(G) \otimes V$$
satisfying the usual dual diagrams to actions.

9.1.4 Remark: As a matter of notation, recall that if $G = \operatorname{Spec} A$, then A is a Hopf algebra. So we call V an A-comodule.

9.2 Reps of diagonalizable group schemes

Let k be a field (or even a ring!) and let A be a finitely-generated abelian group. Define $D(A)$ to be

$$D(A) = \oplus_{a\in A} k \cdot t^a \stackrel{\text{def}}{=} \operatorname{Spec} R$$

Then we get a multiplication

$$R \otimes R \to R \qquad t^a \otimes t^b \mapsto t^{a+b}$$

and comultiplication

$$R \to R \otimes R \qquad t^a \to t^a \otimes t^a$$

and counit ε sending $t^a \to 1$ (all t^a are primitive).

9.2.1 Proposition
R is a Hopf algebra. In particular, $D(A) \to \operatorname{Spec} k$ is a linear algebraic group.

As an example, consider $A = \mathbb{Z}$. Then $R \cong k[t]_t$. Thus $D(A) = \mathbb{G}_m$.
If instead $A = \mathbb{Z}/n$, then $R \cong l[t]/(t^n - 1)$, so $D(A) \cong \mu_n$.
Finally when $A \cong \mathbb{Z}^r \oplus \mathbb{Z}/n_1 \oplus \cdots \oplus \mathbb{Z}/n_k$, then

$$D(A) = \mathbb{G}_m^r \times \mu_{n_1} \times \cdots \times \mu_{n_k}$$

9.2.2 Definition: An algebraic group over k is **diagonalizable** if $G \cong D(A)$ for some A.

Recall the definiton of irreduciblility.

9.2.3 Theorem
Let A be af initely generated abelian group and $G = D(A)$. Then

- Every irreducible representation of G is one-dimensional and isomorphic to I_a, corresponding to $k \to \Gamma(G) \otimes k$ where $1 \mapsto t^a \otimes 1$ for some $a \in A$.
- Every representation decomposes as a direct sum of irreducibles.

Proof
Let $\sigma : V \to \Gamma(G) \otimes V$ be a representation of a diagonalizable group. For $a \in A$, define

$$V_a \stackrel{\text{def}}{=} \{v \in V | \sigma(v) = t^a \otimes v\} \subseteq V$$

Now the claim is that $V_a \cap V_b = 0$ if $a \neq b$ and furthermore $\sum V_a = V$. The first isn't too hard to see.

The second follows by considering $v \in V$ and looking at the image of it under σ. That is,

$$\sigma(v) = \sum_{1}^{N} t^{\alpha_i} \otimes v_i$$

where $\alpha_i \in A$ and $v_i \in V$. Then a very simple argument shows $v = \sum v_i$ (using linearity). Then it remains to show that $v_i \in V_{a_i}$, but this will make things work. (use the other axiom of a coaction). ♠

9.2.4 Remark: When $A = \mathbb{Z}$, $G = D(\mathbb{Z}) = \mathbb{G}_m$, which tells us that representations of $\mathbb{G}_m$ are in bijection with $\mathbb{Z}$-gradings of $V \cong \oplus_{n\in\mathbb{Z}} V_n$!

9.2.5 Definition: A linear algebraic group $G \to \operatorname{Spec} k$ is called **linear reductive** if every representation decomposes as a direct sum of irreducibles.

Problem 9.1
Show the above is equivalent to the statements

- *for each G-representations $W \subseteq V$, there exists $W' \subseteq V$ subrepresentations such that $V \cong W \oplus W'$.*
- *$0 \to W \to V \to W' \to 0$ is exact.*

9.2.6 Remark: Notice that this says that $D(A)$ is linear reducible. In particular, $\mathbb{G}_m$ and $|mu_n$ are in **any characteristic**. This runs counter to Maschke in finite groups.

Consider $\mathbb{Z}/p$ in char p. We get an action $\mathbb{Z}/p$ on k^2 via

$$1 \cdot (x, y) = (x + y, y).$$

But notice that $k \overset{y=0}{\hookrightarrow} k^2$ is a subrepresentation, but has no complement! Thus this group is not linearly reductive!

As another example, consider

$$\mathbb{G}_a \cong \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \right\} \subset \mathrm{GL}_2(k)$$

where $\mathbb{G}_a$ acts on k^2 by $\alpha(x, y) = (a + \alpha y, y)$. Then it can be easily seen not to be a linear representation.

10 October 16th, 2019

Last time we talked about representations! Woot.

Notice that if G is linear (i.e. affine), then the multiplication map induces comultiplication

$$\Gamma(G) \to \Gamma(G) \otimes \Gamma(G)$$

so $\Gamma(G)$ is the **regular** representation with coaction given by multiplication.

We also saw some equivalent contitions similar to Machke for linear reductive groups. Finally we say some examples and diagonalizable groups.

10.1 New Stuff

Given a G-representation V, let V^G be

$$\{v \in V | \sigma(v) = 1 \otimes v\} = \mathrm{Eq}\{V \overset{1\otimes -}{\underset{\sigma}{\rightrightarrows}} \Gamma(G) \otimes V\} = \mathrm{Hom}^G(k, V) \subseteq V.$$

10.1.1 Remark: I need to figure out the TEXfor equalizers/parallel maps.

Example 10.1
Given the representation $\mathbb{G}_m$ action on $V = \oplus V_d$, $V^G = V_0$.

Problem 10.1
If $G(k)$ is dense in G, then $V^G = V^{G(k)}$.

10.1.2 Proposition
A linear algebraic group G over k is linearly reductive if and only if the functor from G-representations to k-vector spaces given by $V \mapsto V^G$ is exact.

Proof
If $V \cong W \oplus W' \twoheadrightarrow W$ is a G representation, then

$$W^G \oplus (W')^G = V^G \twoheadrightarrow W^G$$

is also surjective.

Suppose that we have a short exact sequence

$$0 \to W' \to V \to W \to 0$$

and that this functor is exact. Then we want to show we get a section $\sigma : W \to V$. To do this, consider the functor $\mathrm{Hom}^G(W, -) = \mathrm{Hom}^G(k, W^\vee \otimes -) = (W^\vee \otimes -)^G$, so by the assumption this is exact and we can lift the identity on W to a map in $\mathrm{Hom}^G(W, V)$, giving us our section.♠

10.1.3 Proposition
Let G be a linear algebraic group over k and V a G-representation. Let $W \subseteq V$ be a finite dimensional k-subspace (not necessarily G-invariant). Then there exists $W \subseteq W' \subseteq V$ such that W' is a finite dimensional representation of G.

Proof
We can assume that $W = \langle w \rangle$ for $w \in V$. Apply $\sigma : V \to \Gamma(G) \otimes V$. Then if $\{t_i\}$ is a basis for $\Gamma(G)$, we get

$$w \mapsto \sum t_i \otimes w_i.$$

Then we claim that $w \in \langle w_i \rangle$ and $\langle w_i \rangle \subseteq V$ is a subrepresentation.

For the first, consider the diagrams:

$$\begin{tikzcd} k \otimes V & \Gamma(G) \otimes V \arrow[l, "e^* \otimes \mathrm{id}"'] \\ & V \arrow[u, "\sigma"'] \arrow[ul, "\sim"] \end{tikzcd} \qquad \begin{tikzcd} \sum e^*(t_i) w_i & \sum t_i \otimes w_i \arrow[l] \\ & w \arrow[u] \end{tikzcd}$$

so w is in the span of the w_i.

For the second claim, we need to show that

$$\sigma(w_i) \in \Gamma(G) \otimes \langle w_i \rangle.$$

To see this consider the diagram

$$\begin{tikzcd} \Gamma(G) \otimes \Gamma(G) \otimes V & \Gamma(G) \otimes V \arrow[l, "m^* \otimes \mathrm{id}"'] \\ \Gamma(G) \otimes V \arrow[u, "\mathrm{id} \otimes \sigma"] & V \arrow[l, "\sigma"] \arrow[u, "\sigma"] \end{tikzcd}$$

And tracing through $w \in V$, we get that

$$\sum t_i \otimes \sigma(w_i) = \sum_{i,j,k} \alpha_{i,j,k} t_i \otimes t'_j \otimes w_k$$

and so by looking at coefficients of $t_i \otimes \Gamma(G) \otimes V = \sigma(w_i)$ (look closer here), we see it is

$$\sum_{j,k} \alpha_{i,j,k} t'_j \otimes w - k \in \Gamma(G) \otimes \langle w_i \rangle.$$

♠

10.1.4 Corollary
If V is a G representation, then

$$W = \bigcup_{W \subset V} W$$

where the union is over all finite dimensional subgroups.

10.1.5 Corollary
If G is a linear algebraic group (affine finite type over k), then for some n $G \subseteq \mathrm{GL}_n$ is a closed subgroup. In other workds, there exists a faithful representation V of G.

Now consider the regular representation $\Gamma(G) \to \Gamma(G) \otimes \Gamma(G)$. Notice that $\Gamma(G)$ is a k-algebra of finite type. Choose generators $g_1, \ldots, g_n$ for $\Gamma(G)$. Take a subrepresentation of $\Gamma(G)$ spanned by $\langle h_1, \ldots, h_N \rangle$ containing the span of the g_i.

We have a map $G \to \mathrm{GL}(W)$ where $W = \langle h_i \rangle$ and we aske whether the induced map

$$\mathrm{Sym}^*(W \otimes W^\vee)_{\det}$$

is surjective.

Let's say that $h_i \mapsto \sum_j \gamma_{i,j} \otimes h_j$ under σ. Then using this map and the natural pairing between W and $W^\vee$, we get a map

$$\mathrm{Sym}^*(W \otimes W^\vee) \supset W \otimes W^\vee \to \Gamma(G) \otimes W \otimes W^\vee \to \Gamma(G)$$

where we send

$$h_i \otimes h_j^* \mapsto \gamma_{i,j}$$

So using the counit identiy we can write

$$h_i = \sum_j e^*(\gamma_{i,j}) h_j$$

but we really want to write h_i as a linear combination of the $\gamma_{i,j}$ (since we have shown they all lie in the image of this map). We don't know how to finish up.

11 October 18th, 2019

Last time we say that any linear algebraic group G embeds into GL_n. The argument was basically that you look at the global functions $\Gamma(G)$ and doing cool stuff. Right at the end Taffy and Tuomas figured out that we just needed to use the other "side" of the counit diagram.

11.1 An example

Consider

$$\mathrm{PGL}_2 = (\mathrm{Proj}\, k[a,b,c,d])_{ad-bc]} = \mathrm{Spec}(k[a,b,c,d]_{\det})_0$$

Then consider the representation spanned by

$$\left\langle \frac{a^2}{\det}, \frac{ab}{\det}, \ldots, \frac{d^2}{\det} \right\rangle$$

which has dimension 10 in $\Gamma(\mathrm{PGL}_2)$.

Thus we have a representation $\mathrm{PGL}_2 \to \mathrm{GL}_{10}$. We can compute the matrix representing a matrix (whose determinant can be assumed to be 1 since we are modding out by scalars).

Problem 11.1
Do this! In Sage or something.

11.2 Special Linear Groups

Lets discuss SL_2.

11.2.1 Theorem
If $\operatorname{char} k = 0$, then

- SL_2 is linearly reductive.
- Every irreducible representation of SL_2 is isomorphic to $\mathrm{Sym}^d k^2$ for some d.

where k^2 is the standard representation of SL_2.

Proof
Sketch: Recall that in the proof of Maschke one takes a surjection $V \twoheadrightarrow W$ of G representations and we want to show it has a section. We pick a section s in terms of vector spaces and then "average" it:

$$\widetilde{s} : W \to V \qquad w \mapsto \frac{1}{|G|}\sum_{g\in G} g\cdot s(g^{-1}w)$$

which is our section.

Then using the Harr measure on the group, we get from the inclusion of (compact) SU_2 in SL_2 with quotient $\mathbb{C}$ and we can construct the section via

$$w \mapsto \int_G g\cdot s(g^{-1}w)\,\mathrm{d}g$$

and we get $T_e\,\mathrm{SL}_2 = (T_e\,\mathrm{SU}_2 \otimes_{\mathbb{R}} \mathbb{C})$ and then there is a bit more Lie theory needed to show this makes full sense over $\mathbb{C}$. ♠

Now consider $\mathrm{Sym}^d(k^2)^\vee$, the degree d polynomials on two variables. Then we get an action of SL_2 via

$$g\cdot f(x,y) = f(g^{-1}\begin{pmatrix} x \\ y\end{pmatrix}) - f(dx - by, -cx + ay)$$

Then, as many arguments in linear algebraic groups, we can reduce to a so-called *maximal torus* of matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right) \cong \mathbb{G}_m$. Then we can use techniques on the lie algebra $T_e\,\mathrm{SL}_2 = \mathfrak{sl}_2$.

Whoa coool. The short exact sequence

$$1 \to \mathrm{SL}_2 \to \mathrm{GL}_n \xrightarrow{\det} \mathbb{G}_m \to 1$$

gives us that any representation of SL_2 lifts via $-\otimes \det^i$ to a representation of GL_2. Sean asked whether this actually gets all the representations or just the polynomial ones. I feel like I should know the answer to this. Jarod seems to think this is all of them. I think this stack post says something about that.

I got caught up in thinking and googling and missed an example.

Let $\operatorname{char} k = p$ and let $\alpha \in k^\times$. Then define $\alpha \cdot f = \alpha^p f$. This gives us a map

$$\operatorname{Sym}^d k^n \to \operatorname{Sym}^{dp} k^n$$

that is additive taking p^{th} powers.

So then $\operatorname{Sym}^N k^n$ is **not simple** if $p|N$.

12 October 21st, 2019

Recall that we had that any linear algebraic group over a field k injects into GL_n as a closed subgroup for some n.

An open question is as follows: if $G \to \operatorname{Spec} k[\varepsilon]/\varepsilon^2$ is flat affine group scheme of finite type Then is $G \subseteq \mathrm{GL}_{n,k[\varepsilon]}$ for some n? This question was asked (as far as Jarod knows) by Brian Conrad on Stack Overflow and is still open.

Our goal is to answer the following: if $\phi : H \to H$ then $K = \ker \varphi = H \times_G \operatorname{Spec} k \subseteq H$ is a closed subgroup. What about its image H/K?

12.1 Torsors

Today we are going to be talking about G-torsors.

12.1.1 Definition: If G is a group, a **torsor under** G is a set P with a free and transitive group action.

12.1.2 Remark: So then by fixing a point $p \in P$, we get $G \xrightarrow{\sim} P$ by sending $g \mapsto pg$. In this way, it is like thinking about a group without the identity.
An example is by taking a Galois extension $K(\alpha)/K$ with minimal polynomial f of α. Then $G = \operatorname{Gal}(K(\alpha)/K)$ acts on $\{x|f(x) = 0\}$, which is a G-torsor.

12.1.3 Definition: A **G-torsor over a set** S is a set P with a free right G-action such that $P \to S$ is G-invariant and $S \cong P/G$.

12.1.4 Remark: Notice that a torsor under G is a specialization of this definition by requiring that $S = \{*\}$, the singleton set.

Example 12.1
Let $H \subseteq G$. THen H acting on $H \to H \setminus G$ (left cosets gH of H) is an H-torsor.

12.2 Flatness

12.2.1 Definition: A map of rings $A \to B$ is **flat** if $- \otimes_A B$ is exact.

12.2.2 Remark: Equivalently: for all $p \in \operatorname{Spec} A$, $A_p \to B_p$ is flat. Also: for all $q \in \operatorname{Spec} B$. $A_{\phi^{-1}(q)} \to B_q$ is flat.

12.2.3 Definition: $A \to B$ is **faithfully flat** if and only if $- \otimes_A B$ is **faithfully exact** (exactness and its converse).

12.2.4 Remark: Other equivalence to faithful flatness: $A \to B$ is flat and $\operatorname{Spec} B \twoheadrightarrow \operatorname{Spec} A$; or $A \to B$ is flat and for any A-module M, $M = 0 \longleftrightarrow m \otimes_A B = 0$.

12.2.5 Remark: If $\operatorname{Spec} B \twoheadrightarrow \operatorname{Spec} A$ is faithfully flat and finite presented, then $\operatorname{Spec} A$ has the quotient topology.

12.2.6 Proposition
Let $S = \operatorname{Spec} A$ be Noetherian. Let $G \to S$ be an affine group scheme of flat and finite type over S. Let $P \to S$ be a scheme over S with a right G-action $P \times_S G \xrightarrow{\sigma} P$. Then the following are equivalent:

(a) $P \to S$ is affine, (faithfully?)[1] flat, finite type and $(\sigma, \pi_P) : (p, g) \mapsto (pg, p)$ is an isomorphism.

(b) There exists a faithfully flat S' such that

$$\begin{tikzcd} P_{S'} \arrow[r] \arrow[d] & P \arrow[d] \\ S' \arrow[r, "\sigma"] & S \end{tikzcd}$$

And $P_{S'} \cong G_{S'}$ as $G_{S'}$-modules.

12.2.7 Remark: Note that the above says exactly that P is a G-torsor. Another name that has been mentioned and Jarod seems to like is **principal** G **bundle.**

[1] We tried to prove this in class and it seemed not to be true if we don't say this

12.3 Descent

Along the way of proving the above propositon, we use the idea of descent.

12.3.1 Lemma
Consider $X' = \operatorname{Spec} B' \overset{\text{f.flat, f.type}}{\twoheadrightarrow} X = \operatorname{Spec} B \to Y = \operatorname{Spec} A$ where X', X, and Y are Noetherian. Then

(a) $X' \to Y$ is flat implies that $X \to Y$ is flat.

(b) $X' \to Y$ is faithfully flat implies that $X \to Y$ is faithfully flat.

(c) $X' \to Y$ is finite type implies that $X \to Y$ is.

12.3.2 Remark: The idea for the first two is just looking at the functors using that $B \to B'$ is faithfully flat. For the third, if $B = \cup_\lambda B_\lambda$, then $A \to B_\lambda$ is finitely generated. Then tensoring over B with B' gets us $B' = \cup_\lambda B_\lambda \otimes_B B'$.

But since A is finitely generated over $B_\lambda <$ eventually $B_\lambda \otimes_B B' = B'$. Then consider

$$0 \to B_\lambda \to B \to B/B_\lambda.$$

After tensoring with (faithfully flat!) B' over B, since for some λ

$$0 \to B_\lambda \otimes_B B' \xrightarrow{\sim} B' \to B/B_\lambda \otimes_B B' \to 0$$

is exact, forcing the rightmost term to be zero. But by faithfulness this implies $B/B_\lambda = 0$ and we are done.

12.3.3 Proposition
Consider

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ S' = \operatorname{Spec} A' \arrow[r, two heads, "\text{f.type, f.flat}"'] & S = \operatorname{Spec} A \end{tikzcd}$$

which is a Cartesian square. Then

(a) $X' \to S'$ is an isomorphism iff $X \to S$ is.

(b) $X' \to S'$ is affine iff $X \to S$ is.

13 October 23rd, 2019

Recall the following definition/proposition:

13.0.1 Definition: Let $S = \operatorname{Spec} R$ be Noetherian. Let $G \to S$ be an affine group scheme that is flat and of finite type. Let G be a scheme over S with a right G-action.

Then the following are equiavlent:

- $P \to S$ is a G-**torsor**
- $P \to S$ is faithfully flat and of finite type and $P \times_S G \xrightarrow{(\sigma,\pi_1} P \times_S P$ is an isomorphism.
- There exists $S' \twoheadrightarrow S$ faithfully flat such that

$$\begin{tikzcd} G \times_S S' \cong P \times_S S' \arrow[r] \arrow[d] & P \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

commutes (where the isomorphism shown is $G \times_S S'$-equivariant.)

13.1 Some Examples

We have the trival torsor $P = G \to S$. It is a propositoon that $P \to S$ is trivial iff there exists a section $s : S \to P$.

Let L/K be a finite Galois extension. Then we get $\mathrm{Gal}(L/K)$ acting on $P = \operatorname{Spec} L \to \operatorname{Spec} K$ is a $G = \mathrm{Gal}(L/K)$-torsor. Then in the diagram in the definition above, $\operatorname{Spec} L$ plays the part of S'. Then we get that $L \otimes L \cong L[x]/f \cong \prod_{g\in G} L$ and

$$G \times_{\operatorname{Spec} K} \operatorname{Spec} L \cong \operatorname{Spec}(L \otimes L) \cong \sqcup_{g\in G} \operatorname{Spec} L.$$

Now let X be a scheme and let $\mathbb{G}_m$ act on a line bundle $L \to X$ with section $o : X \to L$. Then $(L \setminus o(X)) \to X$ is a $\mathbb{G}_m$ torsor. It is a result, although we don't have the machinery yet, that

13.1.1 Proposition
There is a bijection between line bundles on X and $\mathbb{G}_m$-torsors.

13.1.2 Remark: We can do womething similar with any vector bundle over X: if $V \to X$ is one, then V_x over $x \in X$ is a vector space. We just send V to $\mathrm{Frame}(V)$, which over any $x \in X$ we have the set of orderd bases of V_x. This gives us a GL_n-torsor.

If we have a subgroup (say of an algebraic group) $H \subseteq G$, recall that we wanted to show the existence of an H-torsor $G \to G/H$.

We begin by talking about *abstract groups*. Assume we have an exact sequence

$$1 \to K \to G \xrightarrow{\pi} Q.$$

Then $G \times K \xrightarrow{\sim} G \times_Q G$ via the map $(g,k) \mapsto (g,gk)$. The proof isn't too bad.

So now consider a geometric group. If we have the same exact sequence of *algebraic groups over k*, we get $K = G \times_Q \operatorname{Spec} k$(??) and then evaluating at any scheme S, we get

$$1 \to K(S) \to G(S) \to Q(S)$$

and consider the map $G(S) \times K(S) \to G(S) \times_{Q(S)} G(S)$ as above. Then by Yoneda we get an isomorphism $G \times K \cong G \times_Q G$ of schemes.

13.1.3 Corollary
If $\pi : G \to Q$ is a faithfully flat map of linear algebraic groups over k, then $G \to Q$ is a torsor under $K = \ker \pi$

Let's do even better!

13.1.4 Corollary
Let $\pi : G \to Q$ be dominant (i.e. the image is dense in Q) and furthermore that Q is reduced. Then $G \to Q$ is faithfully flat and in particular $G \to Q$ is a K-torsor.

Proof
(Of the second corollary): We use the idea of "generic flatness". That is there exists a $U \subseteq Q$ such that $\pi^{-1}(U) \to U$ is flat. Then we can translate this by the G-action (after passing to the algebraic closure of k so that the points are dense) and then flat descent gives us the result we want. :) ♠

13.1.5 Theorem
Let G be alinearly algebraic group over k. Let X be a scheme over k of finite type. Then

$$\{G\text{-bundles on } X\} \cong H^1_{fl}(X,G)$$

where H_{fl} is flat cohomology.

The idea here is clear when $G = \mathbb{G}_m$: you get connections between line bundles on X (i.e. the Picard group of X) and $\mathbb{G}_m$-torsors and similarly between the line bundles and $H^1_{Zar}(X, \mathscr{O}_X^\times)$, using the (usual) Zariski sheaf cohomology.

14 October 25th, 2019

We are going to deviate slightly from the result promised last time, because we want to talk first a bit about

14.1 Descent

Recall that we had two results before about descent involving faithfully flat morphisms of finite type (where the schemes are Noetherian).

Example 14.1
Let $X = \operatorname{Spec} B$ and let U_i be a Zariski-open cover. Then the map

$$\sqcup U_i \to X$$

is faithfully flat and of finite type.

So then one idea is that if we have a nice Zariski cover of X, flatness, faithful flatness, and finite type are all "local on the source" in that you just have to check the property locally on X.

14.1.1 Proposition
Let $A \to B$ be a faithfully flat ring homomorphism. Then

$$A \longrightarrow B \overset{p_1}{\underset{p_2}{\rightrightarrows}} B \otimes_A B$$

where $p_1(b) = b \otimes 1$ and $p_2(b) = 1 \otimes b$ is an exact sequence. More generally,

$$M \longrightarrow M \otimes_A B \overset{\mathrm{id} \otimes p_1}{\underset{\mathrm{id} \otimes p_2}{\rightrightarrows}} M \otimes_A B \otimes_A B$$

is exact.

There is some geometry that lost me a bit. Sorry. :(

14.1.2 Proposition
If $A \to B$ is faithfully flat, then the map $\textbf{mod}\text{-}A \to \{(N, \alpha) | N \in B\text{-}\textbf{mod}, \alpha : p_1^* N \cong p_2^* N, P(\alpha)\}$ sending

$$M \mapsto (M_B = M \otimes_A B, \alpha_{can})$$

is an equivalence of categories where $P(\alpha)$ means that α satisfies a cocyle condition.

14.1.3 Remark: The canonical isomorphism above is

$$\alpha_{can} : M_B \otimes_{B,p_1} (B \otimes_A B) \to M_B \otimes_{B,p_2} (B \otimes_A B)$$

15 November 4th, 2019

Today we are going to do a bit of review as well as do a little more work with G-torsors.

15.1 Review

Recall the following setup: $S = \operatorname{Spec} A$ is Noetherian. Then let $G \to S$ be an affine group scheme of finite type over S. Let $P \to S$ be a scheme with a right G-action:

$$P \times_S G \xrightarrow{\sigma} P.$$

15.1.1 Definition: Then $P \to S$ is a **G-torsor** if either of the following hold:

- $P \to S$ is affine, surjective, of finite type, and furthermore transitive (that is $P \times_S G \xrightarrow{(\sigma,\pi_1)} P \times_S P$ is an isomoporphism).
- There exists a faithfully flat $S' \to S$ with $P \times_S S' \cong G \times_S S'$ withere this isomorphism is $G \times_S S'$-equivariant.

15.1.2 Remark: If G is a linear algebraic gorup over k and S is defined over k, then we say $P \to S$ is a G-torsor if it is a torsor under $G \times_k S$.

Now a **trivial G-torsor** over S is $S \times G \to S$ if and only if $P \to S$ has a section.

Some examples:

- The $\mathbb{G}_m$ torsors over S are in bijection with the line bundles over S.
- The GL_n torsors over S are in bijection with the vector bundles over S.

Since line and vector bundles are tivialized in the Zariski topology, we see that the second condition in definition 15.1.1 can be replaced with: There exists an open cover $\{S_i\}$ such that $P|_{S_i} \cong S_i \times G$.

We proved some stuff with descent. Look at it.

15.1.3 Corollary
There exists an equivalence between $\mathbf{Alg}_A$ and the category of B algebras C along with isomorphisms $\alpha : p_1^* C \to p_2^* C$ along with the cocyle condition.

15.1.4 Remark: To prove this, we descend a (B,α) to an A-module and show that the multiplication on B descends nicely to A.

15.1.5 Corollary
We can also descend G-torsors. That is, we have an equivalence between G-torsors $P \to S$ and G-torsors $P' \to S'$ with isomorphisms and the cocycle condition.

This last result takes quite a bit of doing although we have all the machinery we need. This is the one we're going to be using.

15.2 New Stuff

Let X be a scheme. Then $\mathrm{Pic}(X)$ is the group of line bundles, or equivalently $\mathbb{G}_m$-torsors on X.

15.2.1 Theorem
$\mathrm{Pic}(X) = H^1(X, \mathscr{O}_X^\times) = H^1(X, \mathbb{G}_m)$.

Notice that $\mathscr{O}_X^\times$ is a sheaf assigning to each U $\mathscr{O}_X(U)^\times$. On Zariski opens, this is the same as $\mathbb{G}_m(U)$.

To prove the above result, we need to discuss

15.3 Čech Cohomology

Assume X is separated. Let $\mathscr{U} = \{U_i\}$ be an affine covering where $U_i \cap U_j$ is also affine. Then considering the complex

$$\sqcup U_i \cap U_j \cap U_k \to \sqcup U_i \cap U_j \to \sqcup U_i \to X$$

where we have several a parallel maps for each $n-1$-tuple of intersections in the previous term. Then we can applie $\mathscr{O}_X^\times$:

$$\prod \mathscr{O}_X(U_i)^\times \to \prod \mathscr{O}_X(U_i \cap U_j)^\times \to \cdots$$

where, for instance, the first map above sends a product of maps (s_i) to $(s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j})$.

15.3.1 Definition: $\hat{H}^i_{\mathscr{U}}(X, \mathscr{O}_X^\times)$ is the i^{th} cohomoogy of the above complex.

Then it can be shown that $\hat{H}^i_{\mathscr{U}}(X, \mathscr{O}_X^\times)$ is independent of cover if the U_i are affine.

So what is $H^1(X, \mathscr{O}_X^\times)$? It is the $s_{i,j} in \mathscr{O}_X^\times(U_i \cap U_j)$ modulo the cocycle condition.

15.4 A new result

15.4.1 Theorem
$H^1(X, G)$ is identified with G-torsors.

What is $H^1(X, G)$? We are going to try to recover Čech cohomology. Let X be quasi-compact and let $\cup_1^n U_i \to X$ be faithfully flat. The seperatedness gets us the intersections are affine.

Then we can play the same game, but this time applying G: (notice that $U_i \cap U_j = U_i \times_X U_j$)

$$\prod G(U_i) \to \prod G(U_i \times_X U_j) \to \cdots$$

where (notice now G may be nonabelian!) we map

$$(s_i) \mapsto (s_i|_{U_i \times_X U_j} \cdot s_j|^{-1}_{U_i \times_X U_j})_{i,j}$$

and

$$(s_{i,j}) \mapsto (s_{i,j} \cdot s_{j,k} \cdot s_{i,k}^{-1})$$

where each s is restricted to $U_i \times_X U_j \times_X U_k$.

Then we can define $\hat{H}^1_{\mathscr{U}}(X,G)$ to be the first cohomology of the above chain. Then

15.4.2 Definition: The **flat cohomology** is

$$H^1_{\text{flat}}(X,G) = \operatorname{colim}_{\mathscr{U}} H^1_{\mathscr{U}}(X,G).$$

The overall result here is

15.4.3 Theorem
The G-torsors on X in bijection with the elements of $H^1_{\text{flat}}(X,G)$.

16 November 6th, 2019

Today we are going to prove some more things about representation theory. Down the pipe somewhere we will hope to talk about Tanakka duality, which gives us that Morita equivalence (is this still what it's called?) of two algebraic groups gives us the groups are isomorphic.

Given an algebraic group G/k, we could consider representations, which were either a group scheme morphism $G \to \mathrm{GL}(V)$ or else a coaction $V \xrightarrow{\sigma} \Gamma(G) \otimes_k V$. Then we said that G was linear reductive if and only if every representation is completely reducible,

Example 16.1
If $G = \mathbb{G}_m$, then every irreducible representation is one-dimensional: $\mathbb{G}_m \to \mathbb{G}_m$ sends $t \mapsto t^\alpha$.

A natural question one may ask: what can we say in general?

16.1 The regular representation

The regular representation is $\Gamma(G)$ with coaction given by comultiplicaton.

16.1.1 Proposition
And finite dimensional representation V of G embeds $V \subseteq \Gamma(G)^{\oplus n}$.

Proof
The coaction gives us a map

$$V \xrightarrow{\sigma} \Gamma(G) \otimes V = \Gamma(G) \otimes_k V_{\mathrm{vs}} \cong \Gamma(G)^{\oplus \dim V}$$

where V_{vs} is the underlying vector space of V. Then the claim is that the overall map is injective and that it is a map of G-reps.

The latter property is almost tautological by the property of a coaction. Injectivity follows from the fact that

$$(e^* \otimes \mathrm{id}) \circ \sigma$$

is an isomorphism $V \to k \otimes V$ (again by one of the coaction axioms), so σ is injective. ♠

16.1.2 Proposition
If V is a finite dimensional faithful representation, then every other finite-dimensional representation can be obtained from V by direct sums, tensors, duals, subrepresentations, and quotients.

16.1.3 Remark: More precisely, $W \subseteq (V \oplus V^\vee)^{\otimes n}$.

Proof
$G \subseteq \mathrm{GL}(V)$ is a closed subgroup. Notice that since we have an injective map into $\mathrm{GL}(V)$, G is already a linear algebraic group. ♠

16.1.4 Theorem (Peter-Weyl)
If G is linear reductive, then

$$\Gamma(G) = \oplus_{\mathrm{irr}\ V}(V \otimes V^\vee)$$

16.1.5 Remark: Notice that this is as two-sided representations, but if we only want to look at (say) left representations, we give $V^\vee$ the trivial representation structure and we get the more familiar

$$\Gamma(G) = \oplus_{\mathrm{irr}\ V} V^{\oplus \dim V}$$

and then the formula from finite groups:

$$|G| = \sum_{\mathrm{irr}\ V} (\dim V)^2$$

As an example, consider $\mathbb{G}_m = \operatorname{Spec} k[t]_t$ and $k[t]_t = \oplus_{n \in \mathbb{Z}} k\langle t^n \rangle$. As an exercise, thing about what happens for SL_2.

16.2 Stabilizers of subspaces

If G is an algebraic group over k and V is a G representation, let $W \subseteq V$ be a subspace. We want a subgroup G_W which plays a similar role as the stabilizer in group theory.

Consider G as a functor from $\mathbf{Alg}_k \subseteq \mathbf{Sch}/k \to \mathbf{Grp}$ taking $T \mapsto \mathrm{Hom}(T, G)$. Then for a k-algebra R, $G(R)$ acts on $V_R = V \otimes R$.

16.2.1 Proposition
The functor $\mathbf{Alg}_k \to \mathbf{Grp}$ sending

$$R \mapsto \{g \in G(R) | g W_R = W_R\}$$

is representable by a subgroup of G, which we will denote G_W.

16.2.2 Remark: The idea here is to fix a basis e_i of W and complete them to a basis of V by appending f_i. Then take the coaction $V \to \Gamma(G) \otimes V$ and consider the image

$$f_k \mapsto \sum_{i \in I} a_{ki} \otimes e_i + \sum_{j \in J} a_{kj} \otimes f_j$$

suppose we ahve $g \in G(R) = \mathrm{Hom}(\operatorname{Spec} R, G)$. This gives us a map in $\mathrm{Hom}(\Gamma(G), R)$. Then using the action on $V \otimes R$:

$$g \cdot (f_k \otimes 1) = \sum g(a_{ki}) \otimes e_i + \sum g(a_{kj}) \otimes f_j \in V_R$$

and we see that this is actually in W_R exactly when $g(a_{kj}) = 0$ for all j.

But then this means that g lands in $V(a_{kj})$, the vanishing of the functions $a_{kj} \otimes 1 \in \Gamma(G)_R$, so we get that the closed subscheme $V(a_{kj})$ represents the functor G_W.

For the converse, let G be an algebraic group over k and let $H \subseteq G$ be a subgroup. Then there exists a finite dimensional representation V of G and a $L \subseteq V$ a one-dimensional subspace such that $H = G_L$.

As an example, look at $G = \mathrm{SL}_2$ acting on k^2. Let $L = \langle 1, 0 \rangle$. We can see easily that L is preserved by $g \in G$ if and only if the lower-left coordinate of g is zero. Now if we take $\mathbb{G}_m = \mathrm{diag}(t, t^{-1}) \subseteq \mathrm{SL}_2$, consider the representation $\langle xy \rangle \subseteq \mathrm{Sym}^2 k^2$. This isn't quite it but maybe you can work it out!

17 November 8th, 2019

We started by talking elections. Go Andrew Lewis. You can do it.

Recall that last time we had a result that said that we had an analog of the stabilizer $G_W \subseteq G$ representing the functor

$$R \mapsto \{g \in G(R) | g W_R = W_R\}.$$

The idea was to take a basis for W and complete it to one of V and then to find a group that fixes W. We found that $G_W = V(\{a_{kj}\})$.

17.0.1 Theorem (Chevalley)
If $H \subseteq G$ is a subgroup of an algebraic group over k, then there exists a G representation V and a line $L \subseteq V$ such that $H = G_L$.

Proof

Let $\pi : \Gamma(G) \twoheadrightarrow \Gamma(H)$ be the map induced by $H \hookrightarrow G$. Let $q_1, \ldots, q_n$ be generators of $\ker \pi = I \subseteq \Gamma(G)$, the regular representation. Take a finite dimensional representation V containing I in $\Gamma(G)$ and pick a basis $e_1, \ldots, e_s$ of $W = V \cap I$. Extend this basis to one of V with the additional vectors denoted $f_1, \ldots, f_t$.

Now the image of the coaction gives us

$$e_i \mapsto \sum_k a_{ik} \otimes e_k + \sum_k b_{jk} \otimes f_k$$

and let $I' = (b_{jk})$. We claim that $I = I'$. If this is true, then $H = G_W$ and $W \subseteq V$, so we set

$$L = \bigwedge^{\dim W} W \subseteq \bigwedge^{\dim W} V$$

and now $H = G_L$.

To see the claim, consider that since $e_i \in I$, we get $\sum_k a_{ik} \otimes e_i \in \Gamma(G) \otimes I$. But then the comultiplication on $\Gamma(G)$ gives us that

$$m^*(I) \subseteq I \otimes \Gamma(G) + \Gamma(G) \otimes I \subseteq \Gamma(G) \otimes \Gamma(G)$$

so we get that

$$\sum b_{ik} \otimes f_k \in I \otimes \Gamma(G) + \Gamma(G) \otimes I$$

but since $f_{ij} \notin I$, this forces $b_{ik} \in I$. Thus $I' \subseteq I$.

For the other containment I got behind. ♠

17.1 Quotients

Let $H \subseteq G$ be normal. Now the goal is to construct G/H as an algebraic group. As a preview, we are going to use the last theorem to get a representation V of G such that $H = G_L$ such that $G \to \mathrm{GL}(V)$ descends to a representation $G \to \mathrm{GL}(V^H)$. Then G/H will be the image of this map in $\mathrm{GL}(V^H)$.

17.1.1 Definition: A subgroup $H \subseteq G$ of an algebraic group over k is **normal** if for all k-schemes S, $H(S) \leq G(S)$ is anormal subgroup.

17.1.2 Remark: As an exercise one can show that if $G(k) \subseteq G$, then $H \subseteq G$ is normal if and only if for all $g \in G(k)$, $gHg^{-1} \subseteq H$.

17.1.3 Lemma

If $\pi : H \to G$ is a homomorphism of algebraic groups, then $\ker \pi \subseteq H$ is normal.

To see the above, consider the kernel is the pullback:

$$\begin{tikzcd}
S \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \ker\pi \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d, "e_G"] \\
& H \arrow[r, "\pi"] & G
\end{tikzcd}$$

And then the kernel $\ker \pi(S) = K(S)$ is normal in H. Think about this one some more.

17.1.4 Definition: A homomorphism of algebraic groups $G \to Q$ is a **quotient map** if $G \to Q$ is faithfully flat.

17.1.5 Proposition
A quotient map of linear algebraic groups $\pi : G \to Q$ satisfies the universal property: for all $f : G \to H$,

$$\begin{tikzcd}
K = \ker\pi \arrow[r] \arrow[dr, "0"] & G \arrow[r, "\pi"] \arrow[d, "f"] & Q \arrow[dl, dashed, "!"] \\
& H &
\end{tikzcd}$$

18 November 13, 2019

The big theorem here:

18.0.1 Theorem
Let G be an algebraic group over k. Let $H \subseteq G$ be an algebraic subgroup (over k). Then the quotient G/H exists as a quasi-projective scheme over k.

Moreover if $H \subseteq G$ is normal, then G/H is also an algebraic group.

Today we are going to prove/see the case when G is a linear algebraic group and smooth (reduced) and that $H \subseteq G$ is normal and $k = \bar{k}$. The closedness of k and the smoothness can be dropped at the expense of a couple extra lectures, probably.

18.0.2 Definition: A map $G \xrightarrow{\pi} Q$ of algebraic groups over k is a **quotient** if π is faithfully flat.

Using descent, we showed

18.0.3 Proposition
If $G \xrightarrow{\pi} Q$ is a quotient, then if we have another map $G \xrightarrow{\phi} P$ and furthermore that $\ker \pi \subseteq \ker \phi$, then we get a (unique) factorization $Q \to P$.

We also have

18.0.4 Proposition
Any map of linear algebrai groups $G \xrightarrow{\phi} H$ factors as $G \to Q \hookrightarrow H$ where $G \to Q$ is the quotient map.

Proof
In the case that G is reduced, we get a surjection onto $\operatorname{Im} \phi^*$ of $\Gamma(H)$, and $Q = \operatorname{Spec} \operatorname{Im}(\phi^*)$. Then it just takes showing that the surjection $\Gamma(H) \twoheadrightarrow \operatorname{Im}(\phi^*)$ induces a closed immersion of Q in H.

We also need a lemma that I missed but it can be found both in Milne and Waterhouse. This is where the heavy lifting is done. ♠

19 November 15th, 2019

Today we are going to be talking about properties of algebraic groups as well as properties of lements of $g \in G$. Today, we will be focusing on $g \in \mathrm{GL}(V)$, so essentially talking linear algebra.

19.0.1 Definition: Let V be a finite dimensional vector space over any field k. Let $g \in \mathrm{GL}(V)(k)$. Then g is

- **diagonalizable** if there exists a basis of eigenvectors.
- **semisimple** if there exsits an extension k'/k such that $g \otimes \mathrm{id} \in \mathrm{GL}(V \otimes_k k')$ is diagonalizable.
- **unipotent** if $g - \mathrm{id}$ is nilpotent.
- **triagonalizable** if there exists a basis so that g is upper triangular.

Let $P_g \in k[T]$ be the characteristic polynomial for g. We say the *eigenvalues of g are in k* if P_g splits over k. Thus if g is diagonalizable, the eigenvalues are in k and this is exactly equivalent to g being triagonalizable. Furthermore g is unipotent if and only if all of its eigenvalues are 1 or 0 (in k).

19.0.2 Proposition
If the eigenvalues λ_i are in k, then

$$V \cong \bigoplus \ker(g - \lambda_i \,\mathrm{id})^{a_i})$$

] which gives us our Jordan decomposition in terms of generalized eigenspaces.

19.1 Jordan Decomposition

19.1.1 Theorem

Let k be perfect and let $g \in \mathrm{GL}(V)$ with V finite dimensional over k. Then there exist unique g_s and g_u in $\mathrm{GL}(V)$ such that

- $g = g_s g_u = g_u g_s$
- g_s is semisimple and g_u is unipotent.

Moreover, g_s and g_u are polynomials in g.

Proof

Assume first that $P_g(T)$ splits over k. Thus we can decompose

$$V \cong V_{\lambda_i}$$

where $V_{\lambda_i} = \ker((g - \lambda_i \,\mathrm{id})^{a_i})$. This gives us a Jordan decomposition of g as a matrix with eigenvalues on the diagonal and is upper triangular. Take g_s to be $\mathrm{diag}(\lambda_1, \cdots, \lambda_1, \ldots, \lambda_s)$, the diagonal of the matrix in this form. This is invertible (since $g \in \mathrm{GL}(V)$) so we set $g_u = g_s^{-1} g$. It is clear from this that these matrices commute and that they are semisimple and unipotent, respectively.

To get that these are polynomials in g, you go through an argument I missed. The uniqueness comes from looking at

$$g = g_s g_u = h_s h_u$$

and taking

$$h_s^{-1} g_s = h_u g_u^{-1}$$

and since everything commutes (check this), you get that these two elements are both semisimple and unipotent. But then they are diagonalizable with all eigenvalues 1, so they are the identity and we are done.

For the more general case, choose a finite extension k'/k such that P_g splits. Since k is perfect, this is seperable. So we can assume it is Galois and set $G = \mathrm{Gal}(k'/k)$. So $g \otimes \mathrm{id} \in \mathrm{GL}(V \otimes_k k')$ factors uniquely as $g \otimes \mathrm{id} = g_u g_s$ with $g_u, g_s \in \mathrm{GL}(V)(k')$.

But then using the action of G on $\mathrm{GL}(V)(k')$, by uniqueness of this decomposition $\sigma g_s = g_s$ and $\sigma g_u = g_u$ for all $\sigma \in G$ so the elements are in fact in the field fixed by G (i.e. k). Similarly G acts on $Q(T) \in k'[T]$ but must fix the polynomials describing the factors as polynomials in g. ♠

As an example, consider a field k of characteristic 2 and pick $\alpha \in k/k^2$ (obviously k is not perfect). Then the matrix $g = \left(\begin{smallmatrix} 0 & 1 \\ \alpha, 0 & \end{smallmatrix}\right)$ has polynomial $T^2 - \alpha$, so with a single repeated eigenvalue $\sqrt{a} \notin k$.

19.2 A broader context

So then if $\iota : G \hookrightarrow \mathrm{GL}(V)$ is inclusion (faithful representation) we can decompose $\iota(g) = g_u g_s$ using the above results. Some properties of this phenomenon:

19.2.1 Proposition
If $L : V \to W$ is a linear transformation and $g \in \mathrm{GL}(V)$ and $h \in \mathrm{GL}(W)$ such that $L \circ g = h \circ L$, then $L \circ g_* = h_* \circ L$ for $* = u, s$.

19.2.2 Proposition

$$(g \oplus h)_* = g_* \oplus h_*$$
$$(g \otimes h)_* = g_* \otimes h_*$$

for $* = s, u$.

19.2.3 Proposition
If $\rho : \mathrm{GL}(V) \to \mathrm{GL}(W)$ is a representation, then $\rho(g)_* = \rho(g_*)$.

19.2.4 Definition: For $g \in G(k)$, g is **semisimple (resp. unipotent)** if for all representations $\rho : G \to \mathrm{GL}(V)$, $\rho(g)$ is semisimple (resp. unipotent).

20 November 18th, 2019

20.1 Generalizing the Jordan decomposition

The further generalization of the Jordan decomposition we saw last time is

20.1.1 Theorem
If k is perfect and G is a linear algebraic group over k, then for any $g \in G(k)$, there exist unique $g_s, g_u \in G(k)$ such that

- $g = g_s g_u = g_u g_s$
- g_s is semisimple and g_u is unipotent.
- g_s and g_u are polynomials in g.

Last time we proved this for $G = \mathrm{GL}(V)$. Then we saw a collection of propositions that gave us that semisimplicity and unipotence act nicely with respect to direct sum, tensor, and representations. This allowed us to define what semisimple and unipotent objects are in $G(k)$. This gave us a result which I think I botched last time:

20.1.2 Proposition
If G is alinear algebraic group and V is a faithful finite dimentional representation ($G \hookrightarrow \mathrm{GL}(V)$ is a closed embedding), then eery representation of G is obtained from V using sums, tensors, subrepresentations, and quotients.

20.1.3 Remark: An aside: what does it mean for a subspace $W \subset V$ to be fixed by $g \in G$ ($V \in G$-**mod**)? If V is finite-dimensional, then one can just take the image of G in $\mathrm{GL}(V)$ and ask whether $gW \subseteq W$.

More generally since g is a k-point, there is a maximal ideal $\mathfrak{m}_g \lhd \Gamma(G)$ such that

$$\Gamma(G)/\mathfrak{m}_g \cong k$$

with quotient map q. Then this gives us a map $\Gamma(G)\otimes V \xrightarrow{q\otimes\mathrm{id}} k\otimes V \cong V$ which, when composed with the coaction $V \to \Gamma(G)\otimes V$, gives us a map $m_g : V \to V$. Then we can say that W is fixed by g if $m_g(W) \subseteq W$.

Proof (in general)
Since G is linear algebraic, we can embed $G \hookrightarrow \mathrm{GL}_n$ for some n. This gives us a way to decompose $g = g_s g_u$, but the question is whether g_s and g_u actually lie in (the image of) G. Thus for every representation $\rho : G \to \mathrm{GL}(W)$, we get a pair of elements $\rho(g)_s$ and $\rho(g)_u$. Due to the properties we saw before, these are nice.

The idea here is that one representation we have at our disposal is the regular representation. In general this is large, but it recovers our group G in some way and this will be key to our discussion. Let $V = \Gamma(\mathrm{GL}_n)$, the regular representation of GL_n. Then we get maps

$$G \hookrightarrow \mathrm{GL}_n \to \mathrm{GL}(V)$$

and we have an ideal $I \subseteq \Gamma(GL_n)$ where I is the ideal defining G. Then the claim is that $g \in G(k)$ stabilizes I. This is in Waterhouse chapter 9. ♠

20.2 Group properties

This enables us to make more classifications of groups:

20.2.1 Proposition
We say that G is **diagonalizable** if there is a faithful representation $G \hookrightarrow \mathrm{GL}_n$ that factors through the diagonal matrices in GL_n.

20.2.2 Remark: This is equivalent to the earlier definition of there being a closed embedding of $G \hookrightarrow \mathbb{G}_m^n$.

20.2.3 Definition: We say that G is of **multiplicative type** if $G \times_k \bar{k}$ is diagonalizable.

20.2.4 Proposition
If G is a commutative linear algebraic group over K and all elements $g \in G(k')$ for $k \to k'$ a field extension are semisimple, then G is of multiplicative type.

Proof
We can assume that $k = \bar{k}$ and we know that all $g \in G(k)$ are diagonalizable. Then we use the fact that commuting diagonalizable matrices are simultaneously diagonalizable. The result follows. ♠

20.2.5 Remark: The proof of the linear algebra fact above comes from taking your commuting diagonalizable elements g and h and showing that each λ eigenspace of g is fixed by h and vice versa, giving us the same number and dimension of eigenspaces.

21 November 22, 2019

From last time (which I missed): we had

21.0.1 Proposition
Let G be a linear algebraic group over k. Then the following are equivalent:

- G is diagonalizable (i.e. $G = D(A)$ for some finitely generated abelian group A)
- There exists a faithful representation $G \to \mathrm{GL}(V) \cong \mathrm{GL}_n$ which maps into the diagonal matrices.
- $\Gamma(G)$ is spanned by grouplike elements.
- (Discussed below) All G representations are direct sums of one-dimensional ones.

Earlier this quarter we saw:

21.0.2 Proposition
If $G = D(A)$, then any G representation V can be decomposed as

$$V = \oplus_{a \in AV_a}$$

where

$$V_a = \{v \in V | \sigma^*(v) = t^a \otimes v\}$$

21.0.3 Remark: Here we used the fact (shown yesterday?) that the characters of an abelian group are in bijection with elements.
Note that if k_a is the one-dimensional representation $k_a \to \Gamma(G) \otimes k_a$ via $1 \mapsto t^1 \otimes 1$. One can quickly show that $V_a \cong k_a$, so we get that every representation decomposes as a direct sum of one-dimensional subgroups. In particular, $G = D(A)$ are *linearly reductive*.

21.1 Other properties of diagonalizable groups

First of all, these groups are commutative. That is, $Hom_{\textbf{AlgGrp}}(G,\mathbb{G}_m)\cong A$.

Next, we claim that $\mathrm{Hom}_{\textbf{AlgGrp}}(G,\mathbb{G}_a)=0$. To see this, assume that $f:G\to\mathbb{G}_a$ is such a map and consider the induced map $f^*:k[x]\to k[A]$. Then following through the commutative diagram for comultiplication, we get

$$f^*(x)\otimes f^*(x)=f^*(x)\otimes 1+1\otimes f^*(x)$$

by the diagonalizability of G and the fact that $\mathbb{G}_a$ is "infinitesimal" (I forget the actual term from Hopf algebras.)

21.1.1 Definition: A linear algebraic group G over k is **of multiplicative type** if there exists a finite extension k'/k such that $G_k=G\times_{\operatorname{Spec}k}\operatorname{Spec}k'$ is diagonaliable.

21.1.2 Remark: This is equivalent to $G_{\bar{k}}$ being diagonalizable. One direction is obvious. For the other direction, Supposed we ahve a faithful representation $\bar{V}$ of $\bar{G}=G_{\bar{k}}$ by using some descent ideas.

21.1.3 Theorem

Let G be an algebraic group over k. Then the following are equivalent:

- G is of multiplicative type.
- G is commutative and $\mathrm{Hom}(G,\mathbb{G}_a)=0$.
- G is commutative and $\Gamma(G)$ is coétale.
- G_{k^s} is diagonalizable. That is, there exists k'/k that is finite and seperable such that $G_{k'}$ is seperable.

What does that mean?

21.1.4 Definition: Let (C,Δ,ε) be a coalgebra. Let $A=C^\vee$ be the k-linear dual. This gives us (using that $C^\vee\otimes C^\vee\hookrightarrow(C\otimes C)^\vee$) an algebra. Then

- If C is finite dimensional, we say that C is **coétale** if A is commutative and if $A=C^\vee$ is étale. That is $A=\prod_{i=1}^n k_i'$ where k_i'/k are all finite and seperable extensions.
- If C is infinite dimensional, then C is **coétale** if it is a union of finite dimensional coétales.

Example 21.1
$G = \mathbb{G}_m$, $\Gamma(G) = k[t]_t$. Then let $C = k\langle t^d \rangle \subseteq \Gamma(G)$. Then $A = C^\vee$ is just k, so it is coétale. Furthermore $k\langle t^{-d}, \dots, t, \dots, t^d \rangle$ has that $C^\vee \cong \prod k$, so is as well.

Finally this gives us that $\Gamma(\mathbb{G}_m)$ is coétale since it is a union of these things.

As an aside, if $V \to \Gamma(G) \otimes V$ is a representation, then $\mathrm{Im}(V \otimes V^\vee \to \Gamma(G))$ is a coalgebra.

22 November 25th, 2019

Today we finish our discussion of group schemes of multiplicative type. There was one implication we skipped in Friday's discussion: that if G is commutative then $\mathrm{Hom}(G, \mathbb{G}_a) = 0$ implies that $\Gamma(G)$ is coétale.

22.1 Continuing from last time

Proof (cont.)
Assume that $k = \bar{k}$ and let $C \subseteq \Gamma(G)$ be a subcoalgebra of finite dimension. Let $A = C^\vee$ and $\langle \cdot, \cdot \rangle$ be the natural pairing $C \times A \to k$. If A is not étale, then since A is finite dimensional,

$$A = A_1 \times A_n$$

where the A_i are local Artinian k-algebras.

So there exists a projection map $\pi : A \to k[\varepsilon]/\varepsilon^2 \cong k \oplus k\varepsilon$ that sends

$$x \mapsto \langle x, c \rangle + \varepsilon \langle x, d \rangle$$

for some $c, d \in C$. Then a subclaim is to prove that

- $\Delta(c) = c \otimes c$
- $\Delta(d) = c \otimes d + d \otimes c$
- $\varepsilon(c) = 1$

Then by also showing that c is invertible, one can compute

$$\Delta(dc^{-1}) = 1 \otimes c^{-1}d + c^{-1}d \otimes 1$$

which then gives us a (nontrivial!) map $G \to \mathbb{G}_a$, which contradicts the assumption. ♠

Example 22.1 (Non coétale example)
Consider $\mathbb{G}_a = \operatorname{Spec} k[x]$ with $\Delta(x) = x \otimes 1 + 1 \otimes x$. Then consider th subcoalgebra spanned by $\langle 1, \ldots, x^{n-1} \rangle$ and the dual algebra $A = C^\vee$ spanned by e_i. One can compute that

$$e_i \cdot e_j = \binom{i+j}{i} e_{i+j}$$

if $i + j \leq n-1$ and it is zero otherwise. But then in particular $e_{n-1}^2 = 0$, so A isn't even reduced!

One could ask whether this still holds for any choice of subcoalgebra, but we aren't doing that.

22.1.1 Corollary
Let G be a linear algebraic group over k. Assume that G is commutative (and maybe smooth). Then G is of multiplicative type if and only if $G(\bar{k})$ consists purely of semisimple elements.

22.1.2 Corollary
Let

$$1 \to G' \to G \to G'' \to 1$$

be a short exact sequence. If G is commutative, then G is of multiplciative type if and only if both G' and G'' are.

22.1.3 Remark: That subgroups and quotients of multiplicative type group schemes are multiplicative type is not hard to see. To see this, apply $\operatorname{Hom}(-, \mathbb{G}_a)$ to the sequence and notice that quotients and subgroups of commutative groups are commutative.

The nontrivial direction is that the collection of multiplicative type groups is closed under extension.

23 December 2nd, 2019

We are going to continue our discussion on Jordan decomposition by visiting the idea of

23.1 Unipotent group schemes

Recall that if G is a linear algebraic group over k and $g \in G(k)$, then

- g is semisimple if $\rho(g)$ is for all representations $\rho : G \to \mathrm{GL}(V)$ ($\rho(g) \otimes \bar{k}$ is diagonalizable)
- g is unipotent if $\rho(g)$ is for all representations ρ ($\rho(g) - 1$ is nilpotent).

We also said that, if k is perfect, $g = g_s g_u = g_u g_s$ and this is a unique decomposition (the Jordan decomposition).

When we are talking about an arbitrary group scheme, we no longer use "semisimple" but rather "multiplicative type": If $G \subseteq \mathbb{G}_m^n \subseteq \mathrm{GL}_n$, then we say that G is **diagonaliable**. If $G_{\bar{k}}$ is diagonalizable, then we say G is **of multiplicative type.**

Recall that when G is smooth, we saw that G was of multiplicative type if and only if G is commutative and $G(\bar{k})$ consists entirely of semisimple objects.

Some questions we want to answer: Is there a characterization of groups containing unipotent elements? Perhaps more interestingly: Do we have structure results for factoring algebraic groups analogous to the Jordan decomposition?

23.1.1 Theorem (Kolchin fixed point theorem)

Let V be a finite dimensional vector space over k and let $G \subseteq \mathrm{GL}(V)$ be an abstract subgroup containing unipotent elements. Then there exists a nonzero $v \in V$ fixed by all $g \in G$.

PROOF
Begin by setting

$$V^G = \{v \in V | \forall g \in G, gv = v\}.$$

It is easy to see that if k'/k is a field extension, then

$$(V \otimes_k k')^G = V^G \otimes_k k'$$

thus we can assume that $k = \bar{k}$ (since we want to show that V^G is nonempty).

We can also assume that V is a simple G-module, since it suffices to find a fixed point in a submodule. This reduces the proof to showing that $V^G = V$.

There are two different proofs of this: first, we want to show that for all $g \in G$, that $g - I_V = 0$. Suppose there were an element g' not equal to the identity. We know that for all $g \in G$, (since g is unipotent):

$$\mathrm{tr}(g) = \dim V \quad \Rightarrow \quad \mathrm{tr}(gg') = \dim V \quad \Rightarrow \quad \mathrm{tr}(g(g' - I_V)) = 0$$

Let $E = \{f : V \to V | \mathrm{tr}(gf) = 0, \forall g \in G\}$, which contains $g' - I_V \neq 0$. E is G-invariant under composition. Then let $g' - I_V \in X \subseteq E$ where X is a simple G-module containing this element. THen for any $v \in V$, define

$$\varphi_v : X \to V \quad \text{via} \quad f \mapsto f(v)$$

because $g' \neq I_V$, there exists a $v \in V$ such that $g'v \neq v$. Thus for this specific v, φ_v is an isomorphism!

In particular, this means there exists an $f \in X$ such that $f(v) = v$. THen take

$$A = \{L : V \to V | L \text{ linear, commuting with } G\}$$

which is a division ring over $k = \bar{k}$, which forces $A = k$.

But then for all $w \in V$, we get the composition

$$\begin{tikzcd} V \arrow[r, "\phi_v^{-1}"] \arrow[rr, bend right] & X \arrow[r, "\phi_w"] & V \end{tikzcd}$$

and then we are supposed to show that our choice of f and v above send all other $w \in V$ to the span of v, so f has trace one. Something is fishy in this proof, but it's in Waterhouse. ♠

Proof (Less tricky, more fancy)
By Wedderburn, if A is a ring and M is a faithful left A-module, and if $B = \mathrm{End}_A(M)$, then if M is simple and finitely generated over B, then

$$\mathrm{End}_B(M) = A.$$

The more traditional setting here is when $A = k$, $M = k^n$, and $B = M_n(k)$. Then $\mathrm{End}_{M_n(k)}(k^n) = k$.

Now let $A \subseteq \mathrm{End}_k(V)$ generated by G. Let $M = V$ (which is assumed to be simple as above). Then by letting $B = \mathrm{End}_A(V) = k$, so by the above result, $\mathrm{End}_k(V) = A$ so A is simple! Then let $I = \langle g - I_V : g \in G \rangle$, a two-sided ideal of A, so is either zero (in which case we are done) or is all of A. ♠

24 December 4th, 2019

Recall the Kolchin FPT from last time.

24.1 More unipotent stuff

24.1.1 Corollary
If V is a finite dimensional vector space over k and $G \to \mathrm{GL}(V)$ is an abstract group homomorphism whose image consists of unipotent elements, then there exists a basis of V such that $G \subseteq U_n \subseteq \mathrm{GL}_n$ where U_n is the group of upper-triangular matrices with ones on the diagonal.

24.1.2 Corollary
If G is asmooth linear algebraic group over $k = \bar{k}$ such that $G(k)$ consists of unipotents, then $G \subseteq U_n$ as a closed subgroup.

Examples include $\mathbb{G}_a$ (included as $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$) which is unipotent. In characteristic p, $\mathbb{Z}/p$ is unipotent. There is also some weird α_p stuff.

24.1.3 Definition: A linear algebraic group G over k is unipotent if for all nonzero G-representations, $V^G \neq 0$.

24.1.4 Remark: This is equivalent if we require V to be finite dimensional over k.

24.1.5 Proposition
G is unipotent if and only if for all finite dimensional representations $\rho : G \to \mathrm{GL}(V)$, there exists a basis such that $\rho(G) \subseteq U_n$.

Proof
Choose a filtration of V:

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

with all quotients $W_i \stackrel{\text{def}}{=} V_i/V_{i-1}$ simple. But then since $W_i^G \neq 0$ by the definition of unipotence, $W_i^G = W_i$. So then W_i is the trivial representation and apparently that finishes the forward direction.

The reverse direction is easy (pick a basis and v_1 is fixed by $\rho(G)$). ♠

24.1.6 Theorem
Let G be a linear algebraic group over k. Then G is unipotent if and only if G is closed in U_n for some n.

24.1.7 Lemma
If G is unipotent, so is any subgroup, quotient, or extension.

24.2 Induced representation

Let $H \to G$ be a linear algebraic group homomorphism. Then we have a functor

$$\mathrm{Ind}_H^G : \mathbf{Rep}(H) \to \mathbf{Rep}(G)$$

that is defined via

$$\mathrm{Ind}_H^G(V) = \mathrm{Mor}^H_{\mathbf{Sch}/k}(G, \mathbb{A}(V)) = \{G \to \mathbb{A}(V) | H \text{ invariant}\} = (\Gamma(G) \otimes_k V)^H$$

Notice that the inclusion functor $f : \mathbf{B}H \to \mathbf{B}G$ gives us two maps, f_* (induction) and f^* (forgetful).

Part II
Quarter 2: Representation Theory

25 January 17th, 2020

25.1 Recap

Let $T \subseteq B \subseteq G$ be a maximal torus, Borel, and semisimple algebraic group. Then we talked about:

- The infinitesimal theory by studying $\operatorname{Lie} G$
- Borel subgroups, parabolic subgroups, and the flag variety G/B.
- We also saw the categorization of split connected semisimple algebraic groups according to their root data: This is either given by $(X, R, X^\vee, R^\vee)$ of a group X and a root system R and their duals or equivalently a weight lattice Λ and root latice $\Lambda_r = \mathbb{Z}R$. Then we can define
$$Z/\mathbb{Z}R \cong \Lambda/\Lambda_r$$
to be the **fundamental group.**
- Representation theory: **Rep** G is what is called a **highest weight category.** Let $\Lambda = X$. Then we have the **dominant roots**
$$\Lambda_+ = \{\lambda \in \Lambda | \langle \lambda, \alpha^\vee \rangle \geq 0, \alpha \in R^+\}$$
and for every dominant root λ, we have the **(co)standard** modules $\Delta(\lambda)$ and $\nabla(\lambda)$.

25.1.1 Definition: Let $\lambda \in \Lambda = \operatorname{Hom}(T, \mathbb{G}_m)$. Define the T-module k_λ by
$$t \cdot 1 \stackrel{\text{def}}{=} \lambda(t)1.$$
Then $T(k) = (k^\times)^\Gamma$ gives us $\lambda(t) \in k^\times$.

Example 25.1
Let $G = \mathrm{GL}_n$ and B the upper triangular matrices and T the diagonal ones. Then the unipotent matrices $\mathscr{U}$ of upper triangulars with ones on the diagonal include into B and we have a section $T \to B$ giving us $B = T \rtimes U$.

Then we consider k_λ as a representation of B by having $\mathscr{U}$ act trivially. Then we define

$$\nabla(\lambda) = \operatorname{Ind}_B^G k_\lambda$$

where Ind_B^G is the right adjoint to the restriction functor Res_B^G.

25.1.2 Remark: The induction functor always exists, but the *left* adjoint to Res doesn't. When it does, it is called **coinduction.**

25.2 The Associated Sheaf

Given $H \subseteq G$, there is a functor

$$\mathscr{F} : \mathbf{Rep}\, H \to \mathbf{Sch}\, G/H.$$

this leads to a fact:
25.2.1 Proposition

$$\operatorname{Ind}_B^G k_\lambda = \nabla(\lambda) \cong H^0(G/B, \mathscr{F}_{G/B}(k_\lambda))$$

The standard module $\Delta(\lambda)$ is essentially the dual (after applying an element of the Weyl group of longest length).

25.2.2 Theorem
Let $\lambda \in \Lambda_+$. Then $\nabla(\lambda)$ has a simple socle, denoted $L(\lambda)$. Furthermore, $\{L(\lambda) | \lambda \in \Lambda_+\}$ is a complete set of G-modules.

25.3 Other things to focus on:

Look at Humphreys' *Linear Algebraic Groups*. There is an appendix on root systems that is

25.4 Examples

Let $k = \bar{k}$ and $G = G(\bar{k})$. For type A_{n+1}, we have SL_n as well as PGL_n. In the former case, $\Lambda/\Lambda_r = \mathbb{Z}/n\mathbb{Z}$ and in the latter this quotient is trivial.

For type C_n, we get Sp_{2n}. Let Σ be the matrix given by -1s and 1s on the antidiagonal and zeros elsewhere (negative in the "third quadrant.") Then the matrices in this are those preserving the associated form: that is

$$A^T \Sigma A = \Sigma$$

For type B_n, we look at the $2n+1$ square matrix Σ that is block diagonal with blocks (1) and the previous matrix. Then $\mathrm{SO}(2n+1, k)$ are the matrices preserving Σ. The type D_n matrices are the ones preserving the matrix given by (positive) ones on the antidiagonal.

25.5 More Generally...

Let A be a separable associative unital algebra over k. (That is, $A_{\bar{k}}$ is a product of simple algebras). Fix an involution $\sigma : A \to A$ ($\sigma^2 = \mathrm{id}$). There is always an associated bilinear form for these involutions, which can be symmetric or skew-symmetric. It either preserves the base field or else it descends to an involution of this field.

25.5.1 Definition: $\mathrm{GL}_{1,A}$ is a group scheme whose R-points (R a ring) are given by

$$\mathrm{GL}_{1,A}(R) = \left((A \otimes_k R)^\times\right)^k$$

which contains a subgroup scheme

$$\mathrm{Iso}(A,\sigma)(R) \stackrel{\text{def}}{=} \{a \in (A \otimes_k R)^\times | a \cdot \sigma_R(a) = 1\}$$

We also have $\mathrm{Aut}(A,\sigma) \subset \mathrm{Aut}(A)$ defined as

$$\mathrm{Aut}(A,\sigma)(R) \stackrel{\text{def}}{=} \{\alpha \in \mathrm{Aut}_R(A_R) | \alpha \circ \sigma_R = \sigma_R \circ \alpha\}$$

And finally,

$$\mathrm{Sim}(A,\sigma) \stackrel{\text{def}}{=} \{a \in A_R^\times | a \cdot \sigma_R(a) \in k_R^\times\}$$

Now if A is a central simple algebra and σ is a symplectic form, then

- $\mathrm{Sp}(A,\sigma) = \mathrm{Iso}(A,\sigma)$
- $\mathrm{GSp}(A,\sigma) = \mathrm{Sim}(A,\sigma)$
- $\mathrm{PGSp}(A,\sigma) = \mathrm{Aut}(A,\sigma)$

Want to know more? Look at *The Book of Involutions.*

26 January 22nd, 2020

Now we begin the first part of this class: Infinitesimal theory, derivationts, differentails, and Lie algebras.

26.1 Tangent Spaces

If we have a variety V, we can define the dual space of V at x by considering

$$(m_{\mathcal{O}_x}/m_{\mathcal{O}_x}^2)^*$$

then we can talk about the cotangent space m/m^2.

More formally, set k to be a field (note in Jantzen he uses a commutative ring). Let A be a finitely generated **commutative** k-algebra, so there exists a surjection $k[x_1,\dots,x_n]\to A$.

26.1.1 Definition: Let $M\in A$-**mod**. Then a map $D:A\to M$ is a **derivation** if

$$D(ab)=aD(b)=bD(a).$$

We say that it is a k-**derivation** or k-linear if $D(k)=0$.

As an example, let $B=k[x_1,\dots,x_n]$ and let Ω_B be the free B module of rank n, generated by $dx_1,\dots,dx_n$. Then the map $d_B:B\to\Omega_B$ sending $x_i\mapsto dx_i$ and $1\mapsto 0$ is a derivation.

26.1.2 Lemma

Let M be a B module. Then

$$\mathrm{Der}_k(B,M)\xrightarrow{\sim}\mathrm{Hom}_B(\Omega_B,M).$$

Proof

Given a derivation $D:B\to M$, we can define a map $\varphi:\Omega_B\to M$ via

$$\varphi(dx_i)=D(x_i)$$

and extending B-linearly. One has to prove that this defines an algebra morphism. ♠

26.1.3 Remark: Another way to phrase this is Ω_B is a universal object in B-**mod** for derivations.

26.1.4 Theorem

Let A be a finitely generated commutative k-algebra. Then there exists a (universal) A-module Ω_A and a (universal) derivation $d_A:A\to\Omega_A$ such that for any A-module M,

$$\mathrm{Der}_k(A,M)\xrightarrow{\sim}\mathrm{Hom}_A(\Omega_A,M).$$

26.1.5 Remark: The universality of the pair (Ω_A,d_A) is saying that it is unique up to (unique) isomorphism.

Proof

To see this, we use the last result as well as the fact that

$$\Omega_A=\frac{\Omega_B}{I\Omega_B+B(d_B(I))}.$$

If $I=(f_1,\dots,f_m)$ then Ω_A is an A-module on generators, then Ω_A is an A-module on generators $dx_1,\dots,dx_n$ subject to

$$\sum_{i=1}^n\frac{\partial f_j}{\partial x_i}dx_i=0$$

for all $1 \leq i \leq m$.

The next part of the proof consists of showing that given a derivation $\widetilde{D} : B \twoheadrightarrow A \xrightarrow{D} M$ and the associated map $\widetilde{\varphi} : \Omega_B \to M$, the latter factors through Ω_A. SHow it vanishes on both part of the sum above. ♠

26.1.6 Remark: Note: we just proved that the functor $\mathrm{Der}_k : A\text{-}\mathbf{mod} \to \mathbf{Vect}_k$ is representable! The uniqueness of the representing object is then a consequence of Yoneda lemma!

Example 26.1
Let $A = k[x,y]/(x^2+y^2-1)$. Then Ω_A is generated by d_x, d_y with $2xd_x + 2yd_y = 0$. When $\operatorname{char} k = 2$, we get that Ω_A is free of rank 2.

This defines a group scheme (think matrices in $\mathrm{SL}_2 \subseteq \mathrm{GL}_2$) and this gives us a surjection of $k[\mathrm{SL}_2] \to A$.

When $\operatorname{char} k \neq 2$, we get that Ω_A is a free A-module of rank 1, generated by $dt = xdy - ydx$. I have no idea what this was, but she wrote $dy = -xdt$.

26.1.7 Remark: Notice that the reason we lost smoothness in characteristic 2 was that we lost reducedness: now $(x+y-1)^2 = 0$. There is a general theorem (see deep in Waterhouse) that an affine group scheme is smooth only if its Hopf algebra is reduced.

26.2 Derivations on Hopf algebras.

Recall that if A is a Hopf algebra, then $I = \ker \varepsilon$ is the **augmentation ideal.**

26.2.1 Theorem

Let A be a finitely generated commutative Hopf algebra over k. Then Ω_A is a free A-module.

In fact,

$$\Omega_A \cong A \otimes_k I/I^2$$

. The map $d_A : A \to A \otimes I/I^2$ is given by

$$a \mapsto a' \otimes \pi(a'')$$

where the primes indicate Sweedler notation and $\pi : A \to I/I^2$ is a linear map (not a homomorphism!) that takes a (non-unique) splitting $A = I \oplus k \cdot 1$ and then project on the first coordinate and take a quotient by I^2. It remains to prove that, once we pass to the quotient, the choice of direct sum complement is irrelevant.

26.2.2 Lemma
We say that $D : A \to M$ is an ε-derivation (for $\varepsilon : A \to k$) if

$$D(ab) = \varepsilon(a)D(b) + \varepsilon(b)D(a).$$

The universal module of ε-derivations is

$$\Omega_A \otimes_\varepsilon k = \Omega_A / I\Omega_A = I/I^2$$

27 January 24th, 2020

Let A be a finitely generated commutative Hopf algebra over k. Often we can drop finite generation because any Hopf algebra is a limit of finitely generated ones (because of finite global dimension), so these arguments still hold if they can pass to the limit.

Let $\varepsilon : A \to k$ be the counit and $I = \ker \varepsilon$ be the augmentation ideal.

27.0.1 Remark: We have this duality between algebra and geometry, so the counit can be seen as an element of $G_A(k) = \mathrm{Hom}(A,k)$, specifically the unit element of the group. :)

27.0.2 Lemma
There is a series of isomorphisms

$$\mathrm{Der}_\varepsilon(A,M) \simeq \mathrm{Hom}_k(I/I^2, M) \simeq \mathrm{Hom}_A(A \otimes_k I/I^2, M)$$

This can be proved relatively simply. It is a good exercise.

27.0.3 Theorem

- $\Omega_A \simeq A \otimes_k I/I^2$
- $d_A : A \to \Omega_A = A \otimes I/I^2$
  where the map d_A is the one sending $a \mapsto \sum a' \otimes \pi(a'')$

Proof
Recall that $\pi : A \to I/I^2$ is the map that sends $A \cong k \cdot 1 \oplus I$ to its projection to I modulo I^2.

Let M be an A module. Give $A \oplus M$ an A-algebra structure via

$$(a,m)\cdot(a',m') = (aa', am' + a'm).$$

Then let $G_A(-) = \mathrm{Hom}(A, 0)$ and notice

$$G_A(A \oplus M) = \mathrm{Hom}_{\mathbf{Alg}}(A, A \oplus M)$$

Then we claim that

$$\mathrm{Hom}(A, A\oplus M) = \{(\varphi, D) | \varphi \in \mathrm{Hom}(A,A), D \in \mathrm{Der}_\varphi(A,M)\}$$

then notice that

$$\begin{aligned}(\varphi,D)(ab) &= (\varphi(a)\varphi(b), \varphi(a)D(b) + D(a)\varphi(b))\\ &= (\varphi(a), D(a))\cdot(\varphi(b), D(b)).\end{aligned}$$

so this is indeed an algebra morphism. The other direction is clear, so the bijetion holds.

Now consider the map $p : A \oplus M \to A$ and the induced map

$$\ker p_* \hookrightarrow G_A(A\oplus M) \xrightarrow{G_A(A)}$$

but the projection map admits a section, so $G_A(A \oplus M)$ is a semidirect product. What is $\ker p_*$, though?

$$\ker p_* = \{(\varphi, D) | p_*(\varphi, D) = \varphi = \mathrm{id} \in G_A(A)\}$$

so in particular $\varphi = \varepsilon$. Thus $\ker p_* \cong \mathrm{Der}_\varepsilon(A, M)$, hence $G_A(A \oplus M) \simeq \mathrm{Der}_\varepsilon(A, M) \rtimes G_A(A)$.

For all $D' \in \mathrm{Der}_\varepsilon(A, M)$, the element $(D', \mathrm{id}_A) \in \mathrm{Der}_\varepsilon(A, M) \rtimes G_A(A) = G_A(A \oplus M)$ and this gives us (why?) a "hidden" correspondence between the ε derivations and k-linear derivations.

To find this, take $D' \in \mathrm{Der}_\varepsilon(A, M)$ and consider the element $(\varepsilon, D') \in \ker p_*$. Multiplying, we get (recalling that multiplication of functions is given by

$$(\varphi \cdot \varphi')(x) = m \circ (\varphi \otimes \varphi') \circ \Delta(x)$$

in a Hopf algebra)

$$(\varepsilon, D') \cdot (\mathrm{id}_A, 0) = (\varepsilon \cdot \mathrm{id}_A, a \mapsto \sum a' D'(a''))$$

Now use the map $d_A^\varepsilon : A \to A \otimes I/I^2$ via $a \mapsto 1 \otimes \pi(a)$ and compute $\mathrm{id}_A \cdot d_A^\varepsilon$, which gives us our map $a \mapsto \sum a' \otimes \pi(a'')$ which proves things apparently but some logic is missing. ♠

27.1 Our next goal

We want to show that, in characteristic zero, any algebraic group scheme is reduced (and thus smooth).

27.1.1 Lemma

Let $X - 1, \ldots, x_r$ be a basis for I/I^2. Then $\{x_i^{m_i} \cdots x_r^{m_r} | \sum m_i = n\}$ is a basis for I^n/I^{n+1}.

27.1.2 Remark: Notice that this definitely doesn't work if our algebra is not a Hopf algebra!

Proof

Essentially this boils down to the existsnce of derivations with special properties: there is a $D_i : A \to A$ for all $1 \le i \le r$ such that

$$D_i(x_j) = \delta_{ij} \pmod I,$$

essentially giving us differential operators that distinguish variables. To see why the result follows, notice that if this claim is true then the Leibniz rule and induction gets us

$$D_r^{m_r} D_{r-1}^{m_{r-1}} \cdots D_1^1(x_1^{m_1} \cdots x_r^{m_r}) = m_1! \cdots m_r! \pmod I$$

which implies the statement.

To generate these operators, we will use the last result. Recall that $\mathrm{Der}_\varepsilon(A, A) = \mathrm{Hom}_k(I/I^2, A)$. Then consider

where $d_i(x_j) = \delta_{ij}$. But notice that with such a d_i we get a k-derivation $D_i : A \to A$ defined by

$$D_i(a) = \sum a' d_i(\pi(a''))$$

and

$$\varepsilon D_i(a) = \sum \varepsilon(a') d_i(\pi(a'')) = d_i(\pi(\sum \varepsilon(a') a'')) = d_i(\pi(a)) = d_i(\pi(x_j)) = \delta_{ij}$$ ♠

$$\begin{tikzcd} A \arrow[d, "\pi"] \arrow[drr] & & \\ I/I^2 \arrow[r, "d_i"] & k \arrow[r, hook] & A \end{tikzcd}$$

28 January 27. 2020

We'll pick back up with proving that a finitely generated Hopf algebra in characterisitc zero is reduced.

28.1 The result

28.1.1 Theorem

Let A be a finitely generated Hopf algebra over k with $\operatorname{char} k = 0$. THen A is reduced.

Proof

Let $Y \in I$ be a nilpotent with $y^m = 0$. We want to show that $y = 0$. It sufficies to show this for $m = 2$. We claim that $y \in \cap_{n \geq 1} I^n$. To see this, suppose that $y \in I^n \setminus I^{n+1}$. then we can write

$$y = \sum_{\sum m_i = n} a_{\underline{m}} x^{\underline{m}} + I^{n+1}$$

by the previous lemma. Then consider

$$y^2 = p(x_1, \ldots, x_\Gamma) + I^{2n+1}$$

where p is some monomial of degree $2n$ with some nonzero coefficient. Hence y^2 is nonzero mod I^{2n+1}, a contradiction! Hence $y \in \cap I^n$.

We actually need that $k = \bar{k}$. Let $g : A \to k$ be a point in G_A. Then define the translation map

$$T_g : A \xrightarrow{\Delta} A \otimes A \xrightarrow{g \otimes \mathrm{id}} k \otimes A \simeq A$$

which is a map of algebras and is, in fact, invertible.

Define $\mathfrak{m}_g = \ker g$ (think this is the maximal corresponding to the point g in the spectrum). Then we claim that $T_g(I) = \mathfrak{m}_g$. If $y \in I$ then $y \in \mathfrak{m}_G$ (since it's nilpotent it lies in every maximal). Thus by the previous claim $y \in \cap_{n \geq 1} \mathfrak{m}_g^n$ and by the Krull intersection theorem, $y = 0$ in $A_{\mathfrak{m}_g}$ for all $\mathfrak{m}_g$, whence $y = 0$. ♠

28.2 Properties of Ω_A

- $\Omega_{A \times B} \cong \Omega_A \times \Omega_B$
- Let S be a multiplicative system in A. Then

$$\Omega_{S^{-1}A} \cong \Omega_A \otimes_A S^{-1}A$$

and in particular if A is an integral domain, let $K = \mathrm{Frac}(A)$. Then

$$\Omega_{K/k} \cong \Omega_A \otimes_A K.$$

- Let L/k be a finitely generated seperable field extension. That is we have

  $$L \supset E = k(x_1, \ldots, x_\Gamma) \supset k$$

  where L/E is finite. Then $\dim_L \Omega_{L/k} = \mathrm{trdeg}_k L$ and $dx_1, \ldots, dx_\Gamma$ forma a basis for $\Omega_{L/k}$.

- Let G be an algebraic group scheme. Maybe assume $k = \bar{k}$. Assume that $k[G]$ is an integral domain. Let $K = \mathrm{Frac}(k[G])$. Then

  $$\mathrm{rank}\, \Omega_{k[G]} = \mathrm{trdeg}_k K$$

The last item merits some discussion:

Proof
By the third item above, we get

$$\Omega_{K/k} = \Omega_{k[G]} \otimes_{k[G]} K$$

so

$$\mathrm{trdeg}_k K = \dim_K \Omega_{K/k} = \mathrm{rank}\, \Omega_{k[G]}$$

♠

28.2.1 Definition: Let G be an algebraic group scheme over k. Then G is smooth if

$$\dim G = \Omega_{k[G]}$$

where $\dim G = \dim_{\mathrm{Krull}} k[G]$.

28.2.2 Theorem

Let G be an algebraic group scheme over k. Then if

$$\bar{k}[G] = k[G] \otimes_k \bar{k}$$

is reduced, G is smooth.

28.2.3 Corollary

In characteristic zero, all algebraic group schemes are smooth.

28.2.4 Remark: Smoothness is independent of extending scalars (as the rank and dimension in the definition are), but you do need to extend scalars enough so that the nilpotents aren't "hiding". Thus it suffices to only extend to the seperable closure of k or any perfect field L/k.

The following discussion and proof can be found in section 14 (and partially 6.6) of Waterhouse.

Proof (of thm. 28.2.2)
First extend scalars up to $\bar{k}$. Second recall (from section 6.6) that for $A = k[G]$, the following are equivalent:

- $\pi_0(A)$ is trivial
- $\operatorname{Spec} A$ is connected
- $\operatorname{Spec} A$ is irreducible
- $A/\operatorname{Nil}(A)$ is an integral domain.

The second two are actually a statement about algebraic geometry (recall the geometry doesn't see the nilradical).

Assume that $k[G]$ is reduced. THen we can assume that $G = G^0$ is connected since

$$k[G] \simeq k[G^0] \times \cdots \times k[G^0]$$

where we have $|\pi_0(k[G])|$ copies in the product. Then we use the first property in the last section.

For G^0 connected, the general properties in the last section imply that $k[G^0]$ is irreducible implies that $k[G^0]/\operatorname{Nil}(k[G^0])$ is an integral gdomain. By assumption, the nilradical is trivial, so $k[G^0]$ itself is an integral domain.

Let $K = \operatorname{Frac}(k[G^0])$. Then

$$\dim G^0 = \dim k[G^0] \overset{\text{AG}}{=} \operatorname{trdeg}_k K = \operatorname{rank} \Omega_{k[G^0]}$$

where the *AG* referenced above is essentially the Noether normalization lemma. ♠

29 January 29th, 2020

As a reminder: we are not having classes February 10-14. The following Monday is president's day. So keep that in mind. :)

29.1 Lie algebras for G

29.1.1 Definition: If G is an algebraic group scheme over k, then $\operatorname{Lie} G$ is the space of left-invariant derivations $D : k[G] \to k[G]$.

What is a left-invariant derivation? Let $A = k[G] = k^G$. The Yoneda lemma allows us to identify this with $\mathrm{Hom}(G, \mathbb{A}^1)$. But given $f : G \to k$, we get an action

$$g \cdot f(-) = (T_g f)(-) = f(g-)$$

Note that this actually defines a *right action*, but the left and right invariant derivations are canonically isomorphic and this will help us with the Lie algebra case.

Now let $x \in G$ and by abuse of notation let $x = \mathrm{ev}_x : A \to k$ be the evaluation map. This gives us a diagram:

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "T_g"] & A \otimes A \arrow[r, "(g;\mathrm{id})"] & A \arrow[d, "x"] \\ A \arrow[rr, "x"] & & k \end{tikzcd}$$

Let $T : A \to A$ be a k map. THen T is left invariant if $T_g \circ T = T \circ T_g$ for all G. I wrote some notes in my digital pad. We did some computation. The result is that it is equivalent to having $\Delta \circ T = (\mathrm{id} \otimes T \circ \Delta)$.

Thus we can restate the definition of $\mathrm{Lie}\, G$ as

$$\mathrm{Lie}\, G = \{D \in \mathrm{Der}_k(A, A) | \Delta \circ D = (\mathrm{id} \otimes D) \circ \Delta\}.$$

29.1.2 Lemma

$\mathrm{Lie}\, G$ has the structure of a Lie algebra with bracket

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$$

To prove that this we need to show the three things for the Lie bracket hold! The k bilinearity and $[D, D] = 0$ (careful! We want to consider characteristic 2!) are easy to show but the Jacobi identity requires some work.

Notice that it is equivalent to saying that the operator $\mathrm{ad}\, D$ is a derivation:

$$\mathrm{ad}\, D([D_1, D_2]) = [\mathrm{ad}\, D(D_1), D_2] + [D_1, \mathrm{ad}\, D(D_2)]$$

and one can check this. It has nothing to do with left invariance!

29.2 Another way to think of this algebra

29.2.1 Lemma

There are natural bijections between

(a) $\mathrm{Lie}\, G$ (the left invariant derivations on $k[G]$)

(b) $\mathrm{Der}_\varepsilon(k[G], k)$

(c) $\ker(G(k[\tau]/(\tau^2)) \to G(k))$ where the map is induced from the projection onto the constants in $k[\tau]/(\tau^2)$.

Proof

We can use that $G(k[\tau]/(\tau^2)) = \mathrm{Hom}(A, k \oplus k\tau$ and then study the kernel. ♠

30 January 31, 2020

Recall that we had $\operatorname{Lie} G$ as well as bijections between a couple different things. Today we are going to talk more about that. We scribbled out some easy computations for why (b) and (c) are equivalent. The idea here is that any map $A \to k \oplus k\tau$ can be written as (φ, d) where d is a φ derivation.

To show the equivalence of (a) and (b), you use the maps $D \mapsto \varepsilon D$ and the inverse $d \mapsto (\mathrm{id} \otimes d) \circ \Delta$. One needs to check that you get derivations and that the other thing is left-invariant. To show that they are inverses, consider

$$\varepsilon(\mathrm{id}\bar{\otimes} d) \circ \Delta(x) = \sum \varepsilon(a')d(a'') = d(\sum \varepsilon(a')a'') = d(a).$$

Leave the other direction as an exercise.

Let's check left invariance: we need to check (by the discussion the other day) that $\Delta \circ D = (d \otimes D) \circ \Delta$. Checking this si doing a little diagram chase. Part of the diagram is associativity.

30.1 Lie structures

A natural question to ask: if we have this isomorphism in this last lemma, what are the brackets on the other two objects?

For the first, let $d_1, d_2 \in \mathrm{Der}_\varepsilon(A, k)$. Let $D_i = (\mathrm{id}\bar{\otimes} d_i) \circ \Delta$. Do some work and chase the diagrams for $[D_1, D_2]$! Then you find that

$$[d_1, d_2] = (d_1 \bar{\otimes} d_2 - d_2 \bar{\otimes} d_1) \circ \Delta$$

which really underlines that we need the group/Hopf structure on these things to really be able to do this! This is what makes these equivalences work!

For the kernel $\ker p_*$, take two elements $g_1 = (\varepsilon, d_1) \in k[\tau_1]$ and $g_2 = (\varepsilon, d_2) \in k[\tau_2]$ where we are using different nilpotents so that we have enough degrees of freedom. Consider $R = k[\tau_1, \tau_2]$ and consider the inclusion of g_1 and g_2 in $G(R) = \mathrm{Hom}(A, R)$.

So we think of $g_1 = \varepsilon + d_1\tau_1$ and $g_2 = \varepsilon + d_2\tau_2$ and we compute the multiplication as:

$$g_1 g_2 = \varepsilon + d_1\tau_1 + d_2\tau_2 + ((d_1 \bar{\otimes} d_2) \circ \Delta)\tau_1\tau_2$$

and coompute $g_2 g_1$ as well. Then we really don't have additive inverses in G, but we can compute

$$g_1 g_2 = (\varepsilon + [d_1, d_2]\tau_1\tau_2) g_2 g_1.$$

How do we use this in practice? Consider $\mathrm{GL}_n(S)$. The Lie algebra consists of the matrices $M \in \mathrm{GL}_n(k[\tau])$. Each such matrix can be written as $M_1 + M_2\tau$. But we want the condition that the projection onto $\mathrm{GL}_n(k)$ is the identity, so really we have

$$\operatorname{Lie} \mathrm{GL}_n = \{I + M\tau\} \subset \mathrm{GL}_n(k[\tau])$$

but any such matrix is invertible if M is, so it is no further condition. This gets you your isomorphism with all of $\mathfrak{gl}_n$.

31 February 3rd, 2020

Working from last time, we have some corollaries:

31.0.1 Corollary
Let $f : H \to G$ be a map. Then

- f induces a map $\operatorname{Lie} H \to \operatorname{Lie} G$ (Lie is functorial).
- If f is inclusion, then $\operatorname{Lie} H \hookrightarrow \operatorname{Lie} G$

31.0.2 Remark: This gives us the full equivalences

$$\operatorname{Lie} G \simeq \operatorname{Der}_\varepsilon(k[G], k) \simeq (I/I^2)^* \simeq T_e G.$$

and from this we get

31.0.3 Corollary
G is smooth (or reduced) if and only if $\dim G = \dim_k \operatorname{Lie} G$.

where the above is true since $\dim_k \operatorname{Lie} G = \operatorname{rank} \Omega_{k[G]}$.

31.1 Examples

We talked about GL_n last time. Also if we have $I + \tau_1 M_1, I + \tau_2 M_2 \in \operatorname{Lie} \mathrm{GL}_n = \mathfrak{gl}_n$, the bracked can quickly be seen to be

$$I + \tau_1 \tau_2 [M_1, M_2]$$

as one would hope.

Now consider SL_n. Then $\operatorname{Lie} \mathrm{SL}_n \subset \mathfrak{gl}_n$. To analyze this, consider $\mathrm{SL}_n(k[\tau])$ and we are looking for the ones with determinant one. But

$$\det(I + \tau M) = p(\tau)$$

where p is the characteristic polynomial for $I + \tau M$ over k. But the terms above the linear term die and the linear term is the trace (of M):

$$\det(I + \tau M) = 1 + (\operatorname{tr} M)\tau = 1$$

so the condition is $\operatorname{tr} M = 0$.

31.2 Positive characteristic

Recall that $\mathbb{G}_a$ is the group scheme represented by $k[t]$ where t is primitive. $\mathbb{G}_m$ is represented by $k[t, t^{-1}]$ where t is grouplike.

These are both one-dimensional smooth group schmemes, so the Lie algebras $\mathfrak{g}_a = \operatorname{Lie} \mathbb{G}_a$ and $\mathfrak{g}_m = \operatorname{Lie} \mathbb{G}_m$ are both one-dimensional. If $\operatorname{char} k = 0$, then, there is only one Lie structure! But what if not? Assume for now $\operatorname{char} k \neq 2$.

Do the following problem! It's not hard.

Problem 31.1
Let $\operatorname{char} k = p$. *Take a derivation* $D \in \operatorname{Der}_k(A,A)$ *and show that the* p^{th} *power of* D *is again a derivation.*

This give rise to a "p^{th} power map" which gives rise to a **restricted Lie algebra**. See Weibel for a good introduction (he had to really learn it well to write about it so it is very clear).

Go back to the case of $\mathbb{G}_a$ and $\mathbb{G}_m$. $\mathfrak{g}_a = \operatorname{Der}_\varepsilon(k[t],k) = \langle d_t = \frac{d}{dt}\rangle$ such that $d_t t = 1$.

Then $D_t = (\mathrm{id}\bar{\otimes} d_t)\circ\Delta$ as the corresponding derivation on $k[G]$, and we can compute

$$\begin{aligned} D_t(t) &= (\mathrm{id}\bar{\otimes} d_t)\circ\Delta(t) \\ &= (\mathrm{id}\bar{\otimes} d_t)(1\otimes t + t\otimes 1) = 1 \end{aligned}$$

and so $D_t^2(t) = 0$. Thus $D_t^p(t) = 0$, so $\mathfrak{g}_a$ is a trivial p Lie algebra: $\mathfrak{g}_a = \langle d_t\rangle$ where $d_t^p = 0$.

In the case of $\mathbb{G}_m$, we can compute

$$D_t(t) = t$$

so then $D_t^p = D_t$, so we get $\mathfrak{g}_m = \langle d_t\rangle$ with $d_t^p = d_t$. The punchline here is that the two infinitessimal theories, at first glance, appear to coincide although they actually vary in the right light. $\mathfrak{g}_a \cong \mathfrak{g}_m$ as Lie algebras, but not as *restricted* Lie algebras.

31.3 Frobenius Kernels

For reference, this can be found in Jantzen part I section 7. Some of the proof will be omitted, but can be found there.

Let $\operatorname{char} k = p > 0$ and let G be an algebraic group scheme over k. We assume here that $k = \bar{k}$, but it suffices that k is perfect. This fixes some problems that arise. Recall the Frobenius map $F : \mathbb{A}^n \to \mathbb{A}^n$ that sends $(a_1,\dots,a_n) \mapsto (a_1^p,\dots,a_n^p)$. The **Frobenius twist** of G is the image $G^{(1)}$ of G under the Frobenius twist.

Then we can define the **Frobenius kernel**

$$G_{(1)} \ker F \hookrightarrow G \xrightarrow{F} G^{(1)}$$

which one can show is a connected ($k[G_{(1)}]$ is local) and finite ($k[G_{(1)}]$ is finite dimensional) and a group scheme.

Is the Frobenius kernel smooth? Usually no! The finite dimensionality of $k[G_{(1)}]$ means the dimension of G is zero, so $\operatorname{Lie} G$ is trivial. But this happens only if G is trivial.

An example: $\mathbb{G}_a$. Consider the Frobenius map $F : \mathbb{G}_a \to \mathbb{G}_a$ via $r \mapsto r^p$. Then the kernel is the collection of p-nilpotents,

$$\mathbb{G}_{a(1)}(R) = \{r \in R \mid r^p = 0\}$$

and one can compute that

$$k[\mathbb{G}_{a(1)}] \cong k[t]/t^p$$

with t being a primitive element. We call $\mathbb{G}_{a(1)} = \alpha_p$.

When we switch the picture to $\mathbb{G}_m$, we are now looking at p^{th} roots of unity! The algebra is $k[t,t^{-1}]/(t^p-1) = k[t]/(t^p-1) \simeq k[t-1]/(t-1)^p \simeq k[u]/u^p$. So the coordinate algebra is isomorphic to that in the last case, but with a different coproduct.

32 February 5th, 2020

Today's plan: we're going to go back and talk a bit more about the Frobenius twist to understand it better. Then we'll talk about restricted enveloping algebras and then do some synthesis.

32.1 Frobenius twist

Let's saw we have a commutative ring with R modules A and B. Then we can take the pushout to get $A \otimes_R B$. Instead, let's take a k-algebra and an automorphism $\varphi : k \to k$ we take the pushout which is the twist of A by the automorphism φ. Here the scalars are controlled by φ since

$$\begin{tikzcd} k \arrow[r] \arrow[d, "\varphi"] & A \arrow[d] \\ k \arrow[r] & A \otimes_\varphi k \end{tikzcd}$$

$$\begin{aligned} \lambda(a \otimes \mu) &= a \otimes \lambda\mu \\ &= \varphi^{-1}(\lambda)a \otimes \mu \end{aligned}$$

whenever this all makes sense.

Now letting $\varphi(x) = x^p$ we get the Frobenius twist $A \otimes_\varphi k \stackrel{\text{def}}{=} A^{(1)}$. Whenever k is perfect, this means that if A is an abelian group, so is $A^{(1)}$.

We always have the map $a^{(1)} \stackrel{\text{def}}{=} a \otimes 1 \to a^p$ end extending linearly to the map

$$a \otimes \lambda = \lambda^{1/p} a \otimes 1 \mapsto \lambda a^p.$$

32.1.1 Remark: While this gives us a map $A^{(1)} \to A$, we don't ge a map in the other direction or $M^{(1)} \to M$ for a vector space M. But we do get a map $M^{(1)} \to S^p(M)$.

32.1.2 Remark: Let X/k be a scheme. Then we can define $X^{(1)}/k$. If $X = G$ is a group scheme, then if $A = k[G]$, we can define $G^{(1)}$ to be the group scheme such that

$$k[G^{(1)}] = k[G]^{(1)}$$

You can read more about these in Jantzen, but the map $A^{(1)} \to A$ gives us a map $G \to G^{(1)}$ and the kernel of this map is the Frobenius kernel.

32.2 Representation theory

32.2.1 Remark: If $M \in \mathbf{Rep}\, G$, then this induces a representation of $\mathrm{Lie}\, G$. Why is this true? Well we can use the functoriality of Lie! In particular, a representation M of G is equipped with a morphism $G \to \mathrm{GL}(M)$, yielding a lie algebra morphism

$$\mathrm{Lie}\, G \to \mathrm{Lie}\, GL(M) = \mathfrak{gl}(M)$$

The Frobenius twist also gives us maps on the level of representations. That is, we get a functor

$$(-)^{(1)} : \mathbf{Rep}\, G \to \mathbf{Rep}\, G$$

More clearly, if $M \in \mathbf{Rep}\, G$ we have this vector space as well as a coaction

$$k[G] \to M \otimes k[G].$$

This gives us a map

$$M^{(1)} \to M^{(1)} \otimes k[G]^{(1)} \to M^{(1)} \otimes k[G]$$

so $M \in \mathrm{comod}\, k[G] \simeq \mathbf{Rep}\, G$.

If we have two representations $M, N \in \mathbf{Rep}\, G$, then the map

$$\mathrm{Hom}(M, N) \to \mathrm{Hom}(M^{(1)}, N^{(1)})$$

Considering $\mathrm{End}(M) \to \mathrm{End}(M^{(1)})$, this gives us a map $\mathrm{GL}(M) \to \mathrm{GL}(M^{(1)})$.

32.2.2 Remark: The functor $(-)^{(1)}$ is what is called a "strict polynomial functor". :)

Example 32.1
$\mathbb{G}_{a(1)} \simeq \alpha_p$ is the group represented by $k[t]/t^p$ with t primitive. $\mathbb{G}_{m(1)}$ is the same algebra with t grouplike.

Example 32.2
We have already seen the coordinate algebra of GL_n. What is the Frobenius twist? It is taking a matrix where we take all entries to the p^{th} power (not the whole matrix!). So then the kernel is

$$\mathrm{GL}_{n(1)} = \{(a_{ij}) | a_{ij}^p = \delta_{ij}\}$$

If you try to evaluate at a field, you won't get much: only one point. Thus this is a highly-singluar connected, finite group scheme with coordinate algebra

$$k[x_{ij}]/(x_{ij}^p - \delta_{ij})$$

This gives you the "infinitesimal neighborhood of 1".

32.3 Restricted Lie algebra

32.3.1 Remark: We never mentioned what happens for $\mathfrak{gl}_n$! Here we can honestly multiply matrices, so we can define $A^{[p]} = A^p$. This tells you that for matrix groups we can realize their power operation in this way, but of course this is undesirable as it is dependent on embedding.

Recall that the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ is

$$\mathscr{U}(\mathfrak{g}) = T(\mathfrak{g})/\langle x \otimes y - y \otimes x - [x,y] | x,y \in \mathfrak{g})$$

32.3.2 Definition: If $\mathfrak{g}$ is a restricted Lie algebra (say $\operatorname{Lie} G$), then the **restricted Lie algebra** is

$$u(\mathfrak{g}) = \mathscr{U}(\mathfrak{g})/\langle x^p - x^{[p]} | x \in \mathfrak{g})$$

We have PBW bases for both $\mathscr{U}(\mathfrak{g})$ and $u(\mathfrak{g})$. The latter is better because if $x_1, \ldots, x_n$ are a basis for $\mathfrak{g}$, the basis is

$$\{x_1^{a_1} \cdots x_n^{a_n} | a_i \leq p-1\}.$$

One can show that the ideal generated by $x^p - x^{[p]}$ is a Hopf ideal, so one gets that $u(\mathfrak{g})$ is again a cocommutative Hopf algebra, but this time it is finite dimensional! Then one can consider just the p-restricted representations of $\mathfrak{g}$ and one finds that this category is equivalent to $u(\mathfrak{g})$-**mod**.

Example 32.3
If $\mathfrak{g}_m = \operatorname{Lie} \mathbb{G}_m$, we can compute

$$u(\mathfrak{g}_m) = k[t]/(t^p - 1)$$

but now t is primitive! In fact, $u(\mathfrak{g}_m) = k[\mu_p]^* = k[\mathbb{Z}/p]$.

The above example highlights an important result: if $\mathfrak{g} = \operatorname{Lie} G$,

$$u(\mathfrak{g}) \xrightarrow{\sim} k[G_{(1)}]^*$$

and so we get $\mathbf{Rep}\,\mathfrak{g} \simeq \mathbf{Rep}\, G_{(1)}$. Recall in manifolds that the theories are almost of Lie groups and algebras are very tightly connected (for connected compact manifolds). Here we don't get that! However we get a shadow of it:

Note that we actually get a nested sequence of Frobenius kernels:

$$G \supset G_{(n)} \supset \cdots \supset G_{(1)}$$

meaning that in this way we get better and better approximations of G.

33 February 7th, 2020

Today we are going to talk about induction and restriction.

33.1 The fixed points functor

Recall the *fixed point functor*: let G be an algebraic group scheme and $M \in \mathbf{Rep}\, G$ (which we identify with G-**mod**). Given an action of G on M, we get

$$M^G = \{m \in M | \forall R, g \in G(R), g \cdot (m \otimes 1_R) = m \otimes 1_R\}$$

This definition becomes difficult because it requires an extension of scalars and when you have a lot of nilpotents it messes with things.

An alternative method: define

$$M^G = \{m \in M | \Delta_M(m) = m \otimes 1\}$$

where $\Delta_M : M \to M \otimes k[G]$ is the coaction map. We can also write

$$M^G = \ker(\Delta_M - \mathrm{id}_M \otimes 1).$$

This does indeed create a functor $(-)^G : \mathbf{Rep}\, G \to \mathbf{Vect}_k$.

33.1.1 Proposition
$(-)^G$ is left exact.

33.1.2 Remark: $\mathbf{Rep}\, G$ has an internal Hom. That is, given $M, M' \in \mathbf{Rep}\, G$, we have an action of G on $\mathrm{Hom}_k(M, M')$ (note: note G-equivariant maps!) via the diagonal (in the usual sense of Hopf algebras). All of this, together with tensor-(internal) hom adjunction

$$\mathrm{Hom}_A(M \otimes_k N, M') \cong \mathrm{Hom}_G(M, , \mathrm{Hom}_k(N, M')),$$

gives the category of Hopf modules the structure of a rigid monoidal category.

But now (as vector spaces), we have the isomorphism

$$\mathrm{Hom}_k(M, M')^A \cong \mathrm{Hom}_A(M, M').$$

Let's take a moment to consider $k[G]$ as a G-module. The **right regular representation** on $k[G]$ is as follows: given $f \in k[G]$, we have

$$\rho_r(g)(f)(x) = (g \cdot f)(x) = f(x \cdot g)$$

and

$$\rho_l(g)(f)(x) = (g * f)(x) = f(g^{-1} \cdot x)$$

and by composing the two, we get $\rho_r \circ \rho_l = \rho_l \circ \rho_g$ to get the adjoint representation on $k[G]$.

Now as comodules, we have the action

$$\Delta : k[G] \to k[G] \otimes k[G]$$

which is right regular representation. **Why?** We can define the left regular representation by

$$\tau \circ (S \otimes \mathrm{id}) \circ \Delta.$$

As an exercise, determine whether this is the same element as $(\mathrm{id} \otimes S) \circ \Delta$.

33.2 Induction and restriction

Given $H \hookrightarrow G$, this gives us an obvious functor

$$\operatorname{Res}_H^G : \mathbf{Rep}\, G \to \mathbf{Rep}\, H$$

Then we claim (but don't prove here) that Res has a right adjoint. In some cases, there is a left adjoint as well although we don't have that in general.

In this case, with groups, we can define

$$\operatorname{Ind}_H^G M \overset{\text{def}}{=} (M \otimes_k k[G])^H$$

where each are H-modules ($k[g]$ with the right regular action) and the action on the tensor is via the left regular action.

Another point of view: remember we can define the affine group scheme M_a where $M_a(X) = M \otimes X$. Then

$$M \otimes k[G] = M_a(k[G])$$

and then we can look at the morphisms $G \to M_a$ and consider $(M \otimes k[G])^H$ to be the *H-invariant functions* $f : G \to M$ where H actions on $\operatorname{Mor}(G, M_a)$ via

$$(h \cdot f)(g) = hf(gh)$$

and so

$$\operatorname{Mor}(G, M_a)^H = \{f : G \to M | f(gh) = h^{-1} f(g), \forall R, g \in G(R)\}.$$

33.2.1 Remark: Notice that

$$\operatorname{Ind}_H^G = (-)^H \circ (- \otimes k[G])$$

and so (since tensor is exact and restriction is left exact), Ind_H^G is left exact.

33.3 Frobenius reciprocity

Define the evaluation map $\varepsilon_M : \operatorname{Ind}_H^G M \to M$ via

$$m \otimes f \mapsto \varepsilon(f) m$$

Using the other form, if we have $f \in \operatorname{Mor}(G, M)^H$, we get $f \mapsto f(1) \in M$.

33.3.1 Proposition

(a) $\varepsilon_M : \operatorname{Ind}_H^G M \to M$ is an H-module map.

(b) (*Frobenius reciprocity*) ε_M induces an isomorphism

$$\operatorname{Hom}_G(N, \operatorname{Ind}_H^G M) \xrightarrow{\sim} \operatorname{Hom}_H(\operatorname{Res}_H^G N, M).$$

33.3.2 Corollary

(a) Ind is transitive: $\operatorname{Ind}_H^G = \operatorname{Ind}_{H'}^G \circ \operatorname{Ind}_H^{H'}$.

(b) Ind commutes with field extensions.

(c) Ind preserves injectives.

33.3.3 Remark: Recall the proof that R-**mod** has enough projectives: the idea was to construct projective in **Ab** and then use a kind of induction to bring them up to the module category. In general, property (c) above empowers us to show that categories have enough injectives!
We have a week off! So here's some homework:

Problem 33.1
Prove proposition 33.3.1. Notice if we have

$$N \xrightarrow{f} \mathrm{Ind}_H^G \xrightarrow{\varepsilon_M} M$$

and define $\widetilde{\Psi}^H : N \to \mathrm{Ind}_H^G M$ *via the* H*-map* $\psi : N \to M$ *where*

$$\widetilde{\Psi} : N \to \mathrm{Mor}(G, M)$$

via $n \mapsto \widetilde{\Psi}_n : G \to M$ *where* $\widetilde{\Psi}_n(g) = \Psi(g^{-1}n)$*. Then show* $\widetilde{\Psi}_n$ *is* H*-invariant, a* G*-map, and prove there are mutual inverses.*

34 February 19th, 2020

Welcome back! Today we're taking about the proof for that last problem (Frob. Reciprocity)

Proof
Recall we have the map $\varepsilon_M : \mathrm{Ind}_H^G M \to M$, where we think of

$$\mathrm{Ind}_H^G M = (M \otimes k[G])^H = \{f : G \to M | h^{-1} f(-) = f(- \cdot h), g f(-) = f(g^{-1} \cdot -)\}$$

and $\varepsilon_M(f) = f(1) \in M$. Then the map giving us our Frobenius reciprocity is just composing with ε_M.

We will prove some of this. Let $\widetilde{\psi} : N \to \mathrm{Ind}_H^G M$. Thus it corresponds, for each $n \in N$, to a map

$$\widetilde{\psi}_n : G \to M$$

and we need to define this map, check it is H-invariant, check that the resulting $\widetilde{\psi}$ is G-invariant, and then check that this and $(\varepsilon_M \circ -)$ and this map are mutual inverses.

Then we busted out paper and wrote stuff down. I checked G invariance. ♠

34.0.1 Corollary
(a) Ind is transitive.

(b) Ind takes injectives to injectives.

The way to prove this is to notice that it is a right adjoint to an exact functor.

34.1 Projection formula

34.1.1 Theorem (Tensor identity/Projection formula)
If $M \in H$-**mod** and $N \in G$-**mod**,

$$\mathrm{Ind}_H^G(M \otimes N) \simeq (\mathrm{Ind}_H^G M) \otimes N$$

34.1.2 Remark: If k is a ring instead of a field, we need to make an additional assumption: that N is flat over k.

Proof
We'll write down the maps at least. Since

$$L \stackrel{\text{def}}{=} \mathrm{Ind}_H^G(M \otimes N) = (M \otimes N \otimes k[G])^H \simeq \{f : G \to M \otimes N | f(gh) = (h^{-1} \otimes h^{-1})f(g)\}$$

on the right, we have

$$R \stackrel{\text{def}}{=} \mathrm{Ind}_H^G M \otimes N = (M \otimes k[G])^H \otimes N \simeq \{f : G \to M \otimes N | f(xh) = (h^{-1} \otimes 1)f(x)\}$$

Now we can write down two maps

$$\alpha, \beta : M \otimes N \otimes k[G] \to M \otimes N \otimes k[G]$$

(thinking of these as maps $G \to M \otimes N$) in the following way:

$$\alpha(f)(g) = (1 \otimes g)f(g) \quad \text{and} \quad \beta(f)(g)(1 \otimes g^{-1}f(g))$$

and we claim:

(a) α and β are mutual inverses;

(b) $\alpha(L) = R$ and $\beta(R) = L$;

(c) α, β are G-invariant. ♠

34.1.3 Remark: This is a bit tedious, but this whole induction business is spelled out in Jantzen.

34.2 What do we gain from all this?

Well first, notice that $\operatorname{Ind}_e^G k = k[G]$, which implies taht $k[G]$ is injective. Then we have

34.2.1 Corollary
Let $M \in k$-**mod** and notice

$$\operatorname{Ind}_e^G M \simeq \operatorname{Ind}_e^G (k \otimes kM) \simeq k[G] \otimes M_{\text{triv}}$$

which is just a sum of copies of $k[G]$, which is again projective.

34.2.2 Corollary
Let $M \in \mathbf{Rep}\, G$. Then

$$M \otimes k[G] \simeq M_{\text{triv}} \otimes k[G]$$

Proof
$M_{\text{triv}} \times k[G] \simeq \operatorname{Ind}_e^G M = \operatorname{Ind}_e^G (k \otimes M) \simeq \operatorname{Ind}_e^G k \otimes M \simeq k[G] \otimes M.$ ♠

34.2.3 Remark: The map between these things is nontrivial (we used some twisting in the tensor identity, for example). The map is

$$m \otimes f \mapsto (1 \otimes f)(1 \otimes S)\Delta_M(m)$$

where S is the coinverse.

34.2.4 Theorem ("Rep G has enough injectives")

(a) $k[G]$ is injective.

(b) $\forall M \in \mathbf{Rep}\, G$, there is an injective I with $M \hookrightarrow I$.

(c) Any injetive I is a direct summand of $\bigoplus k[G]$.

Corollary 2 gets us our injectives containing M. This sets us up to do homological algebra, which we will do next time.

35 February 21st, 2020

We did a lot of work last time to show that $k[G]$ was injective and furthermore that every module injects into an injective module. In general, finding projectives is much harder.

35.1 Projectives

35.1.1 Lemma
Let E be a simple G-module. Then there exists an injective hull Q_E such that

$$E \hookrightarrow Q_E$$

with $\operatorname{soc} Q_E = E$ and Q_E is indecomposable.

35.1.2 Remark: Any injective in **Rep** G is a direct sum of indecomposables.

35.1.3 Lemma
Let P be a projective G-module. Then P is injective.

Proof
Let P be a projective module. We claim that for any G-module M, $P \otimes_k M$ is again projective. This follows from tensor-hom adjointness. So $\mathrm{Hom}(-, P)$ is exact. Let

$$0 \to M_1 \to M \to M_2 \to 0$$

be a short exact sequence of finite dimensional modules. Dualizing is exact, as is $- \otimes P$, so

$$0 \leftarrow M_1^* \otimes P \leftarrow M^* \otimes P \leftarrow M_2 \otimes P \leftarrow 0$$

is exact. Thus $M \otimes P$ splits as $(M_1 \otimes P) \oplus M'$.

But then notice that

$$\mathrm{Hom}_G(M_1, P) \simeq (\mathrm{Hom}_k(M, P))^G \simeq (M_1^* \otimes P)^G$$

but then $(M^* \otimes P)^G$ splits as

$$(M_1^* \otimes P)^G \oplus M'^G$$

and this proves surjectivity. ♠

35.1.4 Remark: Notice that this alst proof could possibly be better. The finite dimensionality part could be able to be weakened/dropped by avoiding the tensor product representation.

35.2 Frobenius categories

The upshot to all of this is that

35.2.1 Proposition
If there exists a projective module P in **Rep** G, then the collections of injectives and projectives are the same.

Proof
Let P be projective. Let $P_1 \subseteq P$ be a finite-dimensional submodule. Then

$$M \hookrightarrow M \otimes \mathrm{End}_k(P_1) \simeq M \otimes P_1^* \otimes P_1 \hookrightarrow M \otimes P_1^* \otimes P$$

which we have seen is both injective and projective.

So take a simple M with injective hull Q_M. Then we get maps $Q_M \to M \otimes P_1^* \otimes P$ and $M \otimes P_1^* \otimes P \to Q_M$ by leveraging injectivity and projectivity. Thus Q_M is a direct summand of $M \otimes P_1^* \otimes P$, which is projective. A little clean up gets us the result. ♠

35.2.2 Remark: When P is injective iff projective, the category **Rep** G is a Frobenius category.

35.2.3 REMARK: FOr smooth algebraic groups $\mathbb{G}_m$, $\mathbb{G}_a$, and GL_n, there are no projective modules. If G is finite (finite groups, Frob kernels $\mathbb{G}_{a(r)}$, μ_p, $\mathrm{GL}_{n(r)}$) we have that the collections of projective and injectives are the same.

35.2.4 REMARK: Domkin put together a complete list of affine group schemes with projective modules.

35.3 Cohomology

We know now that $\mathbf{Rep}\, G$ has enough injectives, so if $\mathscr{F} : \mathbf{Rep}\, G \to \mathbf{Vect}_\mathrm{k}$ is left exact we can take an injective resolution $M \to I^\bullet$ and then

$$(\mathscr{R}^n \mathscr{F})(M) \stackrel{\text{def}}{=} H^n(\mathscr{F}(I^\bullet)).$$

Recall the Grothendieck spectral sequence: Let $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ and $\mathscr{F} : \mathscr{D} \to \mathscr{E}$. Then say we want to compute the derived functors of $\mathscr{F}' \circ \mathscr{F}$. Assume that $\mathscr{F}'$ is left exact and that $\mathscr{F}$ takes injectives to $\mathscr{F}'$ acyclic objects. Then we have a spectral sequence whose second page is

$$E_2^{m,n} = \mathscr{R}^m \mathscr{F}' \circ \mathscr{R}^n \mathscr{F} \Rightarrow \mathscr{F}^{m+n}(\mathscr{F}' \circ \mathscr{F})$$

35.3.1 REMARK: Notice that in the Grothendieck spectral sequence if $\mathscr{F}'$ is exact (so $\mathscr{R}^n \mathscr{F} = 0$ when $n > 0$), you get

$$\mathscr{F}' \circ \mathscr{R}^n \mathscr{F} \xrightarrow{\sim} \mathscr{R}^n(\mathscr{F}' \circ \mathscr{F})$$

You can do the same thing with exactness of $\mathscr{F}$.

35.3.1 Notation

We will use the following notation:

$$\begin{aligned} \mathrm{Ext}_G^n(M,-) &= \mathscr{R}^n(\mathrm{Hom}_G(M,-)) \\ H^n(G,M) &= \mathrm{Ext}_G^n(k,M) \\ H^n(G,k) &= \mathrm{Ext}_G^n(k,k) \\ H^*(G,k) &= \oplus_0^\infty \mathrm{Ext}_G^n(k,k) \end{aligned}$$

and this last object is called the cohomology ring of G.

35.3.2 Applications

35.3.2 Proposition
We have the derived adjointness

$$\mathrm{Ext}_G^n(M \otimes V, N) \simeq \mathrm{Ext}_G^n(M, \mathrm{Hom}(V,N))$$

36 February 24th, 2020

Today we are going to talk more about the Grothendieck spectral sequence.

36.1 Examples

Let's begin by proving 35.5.2

Proof
Let $\mathscr{F}(-) = M \otimes -$ and $\mathscr{F}'(-) = \operatorname{Hom}_G(-, N)$. Then

$$\mathscr{F}' \circ \mathscr{F}(V) = \operatorname{Hom}(M \otimes V, N)$$

and so $\mathscr{R}^n(\mathscr{F}') \circ \mathscr{F}(V) = \operatorname{Ext}^n(M \otimes V, N)$ but we know that this is isomorphic to $\mathscr{R}^n(\mathscr{F}' \circ \mathscr{F})(V)$,

I need to work this out once to understand the details. It involved using the regular tensor-hom identity and the convergence of the Grothendieck spectral sequence again. ♠

36.1.1 Proposition

$$\operatorname{Ext}^n_G(M, \mathscr{R}^m \operatorname{Ind}_H^G N) \Rightarrow \operatorname{Ext}^{n+m}_H(m, N)$$

Proof
Leverage Frobenius reciprocity and use the fact $\mathscr{F} = \operatorname{Ind}_H^G$ is left exact and takes injectives to injectives. Then if we let $\mathscr{F}' = \operatorname{Hom}_G(M, -)$, we get $\mathscr{F}' \circ \mathscr{F}(N) = \operatorname{Hom}_H(M, N)$ and it pops out. ♠

36.1.2 Definition: We say a subgroup scheme $H \leq G$ is **exact in** G if Ind_H^G is exact. Equivalently if $\mathscr{R}^{n>0} \operatorname{Ind}_H^G = 0$.

36.1.3 Corollary
If H is exact, we have an isomorphism

$$\operatorname{Ext}^n_G(M, \operatorname{Ind}_H^G N) \xrightarrow{\sim} \operatorname{Ext}^n_H(M, N)$$

This is, of course, a much easier thing to work with. But an interesting question is when H is exact.

36.1.4 Proposition
Let k be a field. Then H is exact if and only if $k[G]$ is an injective H module.

36.1.5 Remark: The direction "$\Leftarrow$" works when k is a commutative ring.

36.1.6 Remark: $k[G] = \operatorname{Ind}_1^G k = \operatorname{Ind}_H^G(\operatorname{Ind}_1^H k) = \operatorname{Ind}_H^G(k[H])$

Proof
Take a short exact sequence $0 \to M_1 \to M \to M_2 \to 0$ of H-modules. Then we do the old argument: take a splitting of $M \otimes k[G]$ and we get that $M_1 \otimes k[G]$ is a summand of $M \otimes k[G]$. Then applying invariants preserves this decomposition.

The rest of this (and all of these properties) live in Jantzen chapter 1 section 4. ♠

36.1.7 Remark: If H is a finite groups scheme, then H is exact (of everything)! The motivation for this: say we have $H \leq G$ any two affine group schemes.

In analogy, there is a statement about Hopf algebras (see my notes for that). If $A \hookrightarrow B$ is a map of two Hopf algebras (perhaps if B is cosemisimple?) This idea ports over here somehow. It can be proved via sheaf cohomology.

Next we're going to try to write down the "generalized version" of the tensor identity using the Grothendieck spectral sequence.

$$(\mathscr{R}^n \operatorname{Ind}_H^G) \circ (- \otimes \operatorname{Res}_H^G N) \cong (\mathscr{R}^n \operatorname{Ind}_H^G(-)) \otimes N$$

37 February 26th, 2020

This is all from Jantzen chapter I section 5. Apparently it is a bit hand-wavy.

37.1 Associated sheaves

Our goal is to describe $\mathscr{R}^n \operatorname{Ind}_H^G(M)$ as $H^n(G/H, \mathscr{L}(M))$ where $\mathscr{L}(M)$ is a sheaf on G/H. We are going to think of $\mathscr{L} : \mathbf{Rep}\, H \to \mathbf{Sh}\, G/H$ as a functor.

37.1.1 Quotients

The history of these quotients go back to Demazure and Gabriel. The idea is that we don't genreally have quotients in **Sch**, so we extend to the **sheaves in the fppf topology**. This is a Grothendieck topology that stands for " finitely presented faithfully flat". It's French, don't question it.

When we have an algebraic group G acting on a scheme X, we want to know when the quotient X/G in this broader context is actually a scheme.

Let us assume that G acts freely (pointwise). We assume that k is a field. We can drop this, but then we have to carry around the assumption of flatness everywhere. Let X be a scheme of finitel type (algebraic) over k and G an affine group scheme (also of finite type). Then we have

37.1.1 Proposition
If G acts freely and X/G is a scheme, then

$$X(R)/G(R) \hookrightarrow (X/G)(R)$$

37.1.2 Proposition
(G always acting freely) If X/G is a scheme, there exists an isomorphism

$$X \times G \xrightarrow{\sim} X \times_{X/G} X \quad \text{via} \quad (x,g) \mapsto (x, xg)$$

where on the level of points, this is $\{(x,x')|x = x'g\}$ (we are using right actions obviously).

From here on out, we are going to assume that X/G is a scheme unless we say something to the contrary.

Example 37.1
$\mathbb{P}^n(R)$ is, as a set, the set of free rank one direct summands of R^{n+1}. Work out how (and if!) this is true. Then we can define

$$\mathbb{A}^{n+1}_*(R) = \{(a_0, \dots, a_n) \in R^{n+1} | \sum Ra_i = R\}$$

which is a subscheme of $\mathbb{A}^{n+1}$. Now we claim that

$$\mathbb{A}^{n+1}_*/\mathbb{G}_m \xrightarrow{\sim} \mathbb{P}^n$$

And we had some trouble making this precise. This is in Jantzen, however.

You can do similar things with the Grassmann schemes $\mathrm{Gr}(k,n)$.

37.1.3 Theorem
Let G be a finite group scheme and X an affine scheme with a free G action. Then X/G is an affine scheme with coordinate ring $k[X]^G$.

37.1.4 Theorem
Let $H \leq G$ be algebraic group schemes (sometimes requires that H is closed, but this is automatic when we look at it from the functorial perspective and require that H is an algebraic subgroup scheme). Then (with H acting on the right of G) G/H (or in the other case $H \setminus G$) is a scheme. In particular, $G \times H \simeq G \times_{G/H} G$.

38 February 28th, 2020

38.1 Associated Sheaves

The set up is the following: we have a scheme X/k, algebraic group scheme G/k acting freely on X such that X/G a scheme. We are constructing a functor

$$\mathscr{L} : \mathbf{Rep}\, G \to \mathbf{Sh} X/G$$

and actually to $\mathscr{O}_{X/G}$ modules. Let $\pi : X \to X/G$ be affine, flat and algebraic.

38.1.1 Theorem

(a) If $Y \subset X/G$ is affine (open), then $\pi^{-1}Y \subset X$ is affine (open). T

(b) If $Y \subset Y/G$ is affine open, then we get the map $\pi : \pi^{-1}Y \to Y$ induces a $k[Y]$ structure on $k[\pi^{-1}Y]$ such that it is a faithfully flat, finitely presented $k[Y]$ module.

Now we construct things in the following way. Let M be a G-modules. Then for any open $U \subset Y$, define

$$\mathscr{L}(M)(U) = \left(M \otimes_k k[\pi^{-1}U]\right)^G$$

where we put the diagonal action on the tensor product. Notice $\pi^{-1}(U) \subset X$.

Why is this the right thing to do? Notice that

$$M \otimes k[\pi^{-1}U] = \mathrm{Mor}(\pi^{-1}U, M_a) = \{f : \pi^{-1}U \to M_a | f(xg) = g^{-1}f(x)\}.$$

We claim that $\mathscr{L}(M)$ is a sheaf, and specifically a sheaf of $\mathcal{O}_{X/G}$-modules. That is, the map $\pi : X \to X/G$ gives us a $\mathcal{O}_{X/G}$ module structure on $\mathcal{O}_X$ via $\pi^{ast} : \pi^{-1}\mathcal{O}_{X/G} \to \mathcal{O}_X$. On any U we have, for $f_1 \in \mathcal{O}_{X/G}(U)$ and $f \in \mathscr{L}(M)(U)$, an element

$$f_1 \cdot f \in \mathscr{L}(M)(U).$$

38.1.2 Proposition

(a) $\mathscr{L}$ is a sheaf of $\mathcal{O}_{X/G}$ modules.

(b) $\mathscr{L}$ is exact.

(c) $\mathscr{L}(M)$ is quasi-coherent.

(d) If M is a finite dimensional G-module, $\mathscr{L}(M)$ is coherent.

We may talk about a proof of this next time.

Example 38.1
Let $M = k$. Then

$$\mathscr{L}(k)(U) = k[\pi^{-1}U]^G = \{f : \pi^{-1}U \to k_a = \mathbb{A}^1 | f(xg) = f(x)\}$$

so we are just looking for invariant functions. The claim is that this is actually

$$\mathscr{L}(k)(U) = \{f : \pi^{-1}U/G \to \mathbb{A}^1\} = \{f : U \to A1\} = k[U] = \mathcal{O}_{X/G}(U)$$

We want to know what happens to injective modules. All of our injectives are going to look like $M \otimes k[G]$ or direct summands of this or will be otherwise comparable to this. So if we have $\pi : X \to X/G$,

$$\mathscr{L}_{X/G}(M \otimes k[G]) \simeq \pi_* \mathscr{L}_{X/e}(M)$$

This is requires some thought and computation.

38.1.3 Proposition

(a) The functor sending
$$M \mapsto H^0(X/G, \mathscr{L}(M)) = \Gamma(\mathscr{L}(M))$$
is left exact (but not right exact!)

(b) If X is affine, then the derived functors are
$$M \mapsto H^n(X/G, \mathscr{L}(M))$$

Proof

The second part looks almost obvious from the Grothendieck spectral sequence. We need, however, that $\mathscr{L}$ takes injective G-modules to acyclic sheaves. Why is that true? Take $M \otimes k[G]$ an injective G-module, then we saw that
$$\mathscr{L}(M \otimes k[G]) \simeq \pi_* \mathscr{I}_{X/e}(M).$$
We need to check that
$$H^n(X/G, \pi_* \mathscr{L}_{X/e}(M))$$
vanishes for all $n \geq 1$. THere is a problem in Hartshorne that gives this to us for an affine morphism between separated (something else) schemes. ♠

38.1.4 Proposition

Let $H \leq G$. Then

(a) $\mathscr{R}^n \operatorname{Ind}_H^G(M) \simeq H^n(G/H, \mathscr{L}_{G/H}(M))$.

(b) If G/H is Noetherian, then $\mathscr{R}^n \operatorname{Ind}_H^G(M) = 0$ for $n > \dim G/H$.

(c) If G/H is projective, then $\mathscr{R}^n \operatorname{Ind}_H^G(M)$ is finitely generated whenever M is finite dimensional.

(d) If G/H is affine, Ind_H^G is exact.

Proof

For the first statement, We are just taking higher derived functors of
$$H^0(GH, \mathscr{L}(M)) = \mathscr{L}(M)(G/H) = (M \otimes k[G])^H = \operatorname{Ind}_H^G(M)$$
where the isomorphisms here are natural. So the derived functors are the same! ♠

38.1.5 Corollary

If H is a finite group scheme, then Ind_H^G is exact.

Proof

When know that G/H is affine and in fact is $\operatorname{Spec} k[G]^H$. ♠