Cohomology Theories, Triangulated Categories, and Applications

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Introduction

These are the notes I took in class during the Winter 2019 topics course *Math 583H - Cohomology Theories, Triangulated Categories, and Applications* at University of Washington, Seattle.

The course description follows:

This covers some (co)homology theories and structures of triangulated categories recently used in noncommutative algebra and noncommutative algebraic geometry. In addition, we will discuss some latest developments and applications in representation theory and noncommutative algebraic geometry. Some basics (see first two topics) can be found in the book *An Introduction to Homological Algebra* by Charles Weibel. Here is a list of possible topics:

- Hochschild cohomology and group cohomology.
- Homotopy categories and derived categories.
- Dualizing complexes over noncommutative algebras and applications.
- Artin-Schelter regular algebras and noncommutative projective spaces.
- Semiorthogonal decomposition of derived categories.

1 April 1, 2019

The topic for this week includes dg-algebras and -categories. There is no great textbook for these subjects (for the next topics you can use Weibel), but there is a paper by B. Keller "Deriving dg-categories" which is pretty abstract. We might need a little time to understand it, however.

1.1 What we are working with

Let k be a field throughout the course. As a matter of notation, we let $\mathbf{dgVect}_k$ denote the category of differential graded vector spaces over k. Today we will discuss two ways to view this, as a *category* and as a *dg-category.*

1.2 As a category

Recall that $\mathbf{Vect}_\mathrm{k}$ is the category with k-vector spaces with k-linear maps. Now $\mathbf{dgVect}_k$ consists of objects that are (cochain) complexes of k-modules $(M^i)_{i\in\mathbb{Z}}$ with differential $\partial_M = (\partial_M^i)_{i\in\mathbb{Z}}$ where each

$$\partial_M^i : M^i \to M^{i+1}$$

is a differential in the usual sense in homological algebra.

To describe the morphisms in this category: Let M, N be two objects in $\mathbf{dgVect}_k$. Then a (degree zero) morphism $f : M \to N$ are chain morphisms (the components commute with the differentials).

Example 1.1
$M^i = kx^i \oplus ky^i$ and $\partial_M^i(x^i) = 0$, $\partial_M^i(y^i) = x^{i+1}$. This clearly works with everything above, so $M \in \mathbf{dgVect}_k$.

Now let $N^i = kx^i$ and $\partial_N^i(x^i) = 0$. Then there is a morphism $f : M \to N$ sending $y^i \mapsto x^i$ and $x^i \mapsto 0$.

Problem 1.1
Check that $\mathbf{dgVect}_k$ *is an abelian category.*

1.2.1 Definition
If $M, N \in \mathbf{dgVect}_k$ and $f : M \to N$ such that $f^i : M^i \to N^{i+d}$ for all i, we say $\deg f = |f| = d$.

1.3 As a symmetric monoidal category

The first thing to do is to define the tensor product. Let $M, N \in \mathbf{dgVect}_k$. Then we define $M \otimes N$ to be

$$(M \otimes N)^i = \oplus_{s+t=i} M^s \otimes N^t$$

with differential (if $m \in M^s$ and $n \in N^t$)

$$\partial_{M\otimes N}^i(m \otimes n) = \partial_M^s(m) \otimes n + (-1)^s m \otimes \partial_N^t(n)$$

where we will often use the more universal notation $(-1)^s = (-1)^{|m|}$.

1.3.1 Remark: Notice that $|\partial| = 1$, so the above sign can be more suggestively written as $(-1)^{|m||\partial|}$ which is a nod to the Koszul sign rule below.

To be clear, the Koszul sign rule states, if f and g are graded (that is, chain) maps, then

$$(f \otimes g)(m \otimes n) = (-1)^{|m||g|} f(m) \otimes g(n)$$

and so the map we are considering above is $\mathrm{id}_M \otimes \partial_N + \partial_M \otimes \mathrm{id}_N$

Problem 1.2
Check that $\partial_{M\otimes N}$ is a differential.

We want this monoidal category to be *symmetric*, however. So to see this, define the twist functor

$$\tau : M \otimes N \to N \otimes M$$

such that $\tau(m \otimes n) = (-1)^{|m||n|} n \otimes m$ (Koszul sign rule).

1.3.2 Lemma

(a) $\tau \in \mathrm{Hom}_{\mathbf{dgVect}_k}(M \otimes N, N \otimes M)$.

(b) $\tau \circ \tau|_{M\otimes N} = \mathrm{id}_{M\otimes N}$.

Then since we have the twist functor τ satisfying the above properties, we get that $\mathbf{dgVect}_k$ is symmetric monoidal. Notice that the unit object for the monoidal aspect is the complex

$$\cdots \to 0 \to k \to 0 \to \cdots$$

1.4 Differential Graded Algebras

1.4.1 Definition
A differential graded algebra A is an algebra object in $\mathbf{dgVect}_k$.

1.4.2 Definition
A differential graded algebra A is a $\mathbb{Z}$-graded algebra $A = \oplus_{i\in\mathbb{Z}} A_i$ with differential $\partial_A^i : A^i \to A^{i+1}$ satisfying:

(a) $1 \in A_0$

(b) $A_i A_j \in A_{i+j}$

(c) $\partial_A(1_A) = 0$

(d) $\partial_A(ab) = \partial_A(a)b + a\partial_A(b)$

2 April 3, 2019

At the end of the last lecture, we wondered why we want to do homological algebra (and thus why are we interested in dg-structures)?

2.1 Some History

Suppose we have a class $\mathcal{C}$ of objects. Then around 1900, Poincaré noted we can construct a class of complexes consisting of these objects. Then one can compute (co)homologies and their dimensions (Betti numbers). Then the alternating sum of the Betty numbers gives us the Euler characteristic.

In topology, around 1925, people did this with topological spaces and singular (co)homology. Later, in the 1930's, de Rham introduced de Rham complexes for manifolds and continued similarly. Also around this time, Čech developed Čech complexes for schemes for Čech cohomology. In 1945, Hochschild developed Hochschild complexes. In 1948, Chevalley-Eilenberg did Lie algebras and Harrison did commutative algebras.

2.2 Commutative DGAs

Last time we saw the definition of a differential graded algebra.

2.2.1 Definition
A DGA A is called **commutative** if

$$ab = (-1)^{|a||b|}ba$$

for homogeneous elements $a, b \in A$.

2.3 Returning to dgVect$_k$

Recall that the objects in this category are complexes $M^\bullet$ with differential $\partial^i : M^i \to M^{i+1}$.

2.3.1 Definition
For $M, N \in \mathbf{dgVect}_k$, define

$$\mathrm{Hom}_{\mathbf{dgVect}_k}(M,N) = (\mathrm{Hom}_{\mathbf{dgVect}_k}(M,N)^i, \partial^i)$$

where

$$\mathrm{Hom}_{\mathbf{dgVect}_k}(M,N)^i := \prod_{s\in\mathbb{Z}} \mathrm{Hom}(M^s, N^{s+i})$$

and for each i, $\partial^i_{\mathrm{Hom}} : \mathrm{Hom}_{\mathbf{dgVect}_k}(M,N)^i \to \mathrm{Hom}_{\mathbf{dgVect}_k}(M,N)^{i+1}$, where

$$\partial^i_{\mathrm{Hom}}((f^s)_{s\in\mathbb{Z}}) = (\partial^{s+i}_N \circ f^s - (-1)^{|f||partial|} f^{s+1} \circ \partial^s_M)_{s\in\mathbb{Z}}$$

Problem 2.1
Show that ∂_{Hom} is a differential. Hint: use that ∂_M and ∂_M are.

2.3.2 Remark: The composition in $\mathbf{dgVect}_k$ is defined as: if $f \in \mathrm{Hom}_{\mathbf{dgVect}_k}(M,N)^i$ and $g \in \mathrm{Hom}_{\mathbf{dgVect}_k}(K,M)^j$, then $f \circ g \in \mathrm{Hom}_{\mathbf{dgVect}_k}(K,N)^{i+j}$ where $(f \circ g) = (f^{s+j} \circ g^s)_{s\in\mathbb{Z}}$.

2.3.3 Lemma

(a) $1_M \in \mathrm{Hom}_{\mathbf{dgVect}_k}(M,M)^0$.

(b) Composition has the following properties:

  (a) $\circ$ is associative.

  (b) $\mathrm{Hom}_{\mathbf{dgVect}_k}(N,P) \otimes \mathrm{Hom}_{\mathbf{dgVect}_k}(M,N) \xrightarrow{\circ} \mathrm{Hom}_{\mathbf{dgVect}_k}(M,P)$ is a morphism of complexes.

2.4 dg Categories

2.4.1 Definition
A **dg Category** is a category $\mathcal{C}$ enriched over $\mathbf{dgVect}_k$.

Okay, that is slightly cheating although it is quite nice.

2.4.2 Definition
Let $\mathcal{C}$ be a k-linear category. We say $\mathcal{C}$ is a **dg category** if:

(a) $1_M \in \mathrm{Hom}_{\mathbf{dgVect}_k}(M, M)^0$

(b) The other stuff we talked about in the above lemma.

2.4.3 Remark: So a natural question to arise is whether a specific category admits an "enhancement", or enrichment over $\mathbf{dgVect}_k$. If so, we can do homological algebra!

Next lecture will be more concrete. :)

3 April 5, 2019

Today we will just focus on examples of dg categories! No definitions today. :)

3.1 Examples of dg categories

Example 3.1
$\mathbf{dgVect}_k$ is a dg category. We also use this notation to refer to the category without the dg structure.

Some notation: when we write

$$\hom(M, N) = \{\text{maps of degree zero}\}$$

we are taking about the regular category . Then we call the same category where the maps have arbitrary degree as the dg enchancement.

Example 3.2
A dg algebra A is a dg category with one object. It has a single object $*$ and $\mathrm{Hom}_A(*, *) = A$.

3.1.1 Remark: This gives us a third definition for a dg category: a dg algebra with several objects. This parallels the fact that an algebra over k can be considered a k-linear category with a single object and a k-algebra with several objects is a k-linear category. This is a bit funny.

In this last example, the differential on A is $\operatorname{Hom}(*,*)$ and composition respects grading and the differential. Furthermore, composition is a homomophism of degree zero ($\deg(a \circ b) = \deg a + \deg b$) and finally it is a morphism of complexes of degree zero.

Example 3.3
Let R be a algebra, then the category of complexes of right R-modules $\mathbf{Ch} = \mathbf{Ch}(\mathbf{mod}\text{-}R)$ is a dg category whose objects are complexes of right R-modules and whose morphisms are themselves complexes.

That is, $\operatorname{Hom}_{\mathbf{Ch}}(M, N)^i = \prod_{s\in\mathbb{Z}} \operatorname{Hom}_R(M^s, N^{s+i})$ with differential

$$d_{\mathrm{Hom}}(f) = d_N \circ f - (-1)^{|f|} f \circ d_M.$$

3.1.2 Remark: Usually when one studies $\mathbf{Ch}$, one only considers the regular (non dg) category structure. Actually this is the case in books like Weibel and Rotman.

When one is looking at representation theory, on considers R-modules and so it is natural to consider $\mathbf{D}(\mathbf{mod}\text{-}R)$, the derived category where objects are complexes. Then $\mathbf{Ch}(R)$ has complexes for both objects and morphisms.

Recovering R from the modules is Morita theory. Getting $\mathbf{mod}\text{-}R$ from its derived category is derived Morita theory.

3.2 He lied about no definitions

3.2.1 Definition
Let $A = (A, m, u, \partial)$ be a dg algebra. M is called a **dg right A-module** if

(a) M is a complex with differential ∂_M.

(b) There is a right action on M given by $m_M : M \otimes A \to M$ such that m_M is a morphism of complexes of degree zero (i.e. $\partial_M(ma) = \partial_M(m)a + (-1)^{|m|} m\partial_A(a)$). Furthermore m_M satisfies the usual diagrams for a right action.

3.2.2 Remark: We write $\mathbf{dgmod}\text{-}A$ to denote the right A-modules. As usual, we can define it as a dg category or regular category.

Let M and N be two dg right A-modules. A morphism of degree zero from M to N maps M^i to N^i for all i and commutes with the differential and is A-linear.

When scaling up to the dg structure, we get $\operatorname{Hom}_{\mathbf{dgmod}\text{-}A}(M, N)$ consists of complexes where each component consists of maps of the appropriate degree that are all still (right) A-linear.

Notice that in the general case we don't get commutativity with differentials.

3.2.3 REMARK: Notice that these are nonempty since $\mathrm{id}_M \in \mathrm{Hom}(M,M)^0$ and if I is any dg-ideal of A (graded ideal closed under ∂), then the quotient map is in $\mathrm{Hom}(A, A/I)^0$.

The differential on $\mathrm{Hom}(M,N)$ is the usual thing with the sign trick.

Problem 3.1
Prove that $\partial_{\mathrm{Hom}} \circ \partial_{\mathrm{Hom}} = 0$ *and that* **dgmod**-A *is a dg category.*

Note that if we let $A = (\cdots \to 0 \to R \to 0 \to \cdots)$ then **dgmod**-A is the same as **Ch**(**mod**-R).

4 April 8, 2019

Today we will mostly be playing with morphisms. We expect it to be quite easy. On Wednesday we will play with the objects.

4.1 Morphisms in a dg category

Notice that if A is a dg algebra (or just an algebra) – recall that this is a (dg) Category with one object. Then we can define $B \subseteq A$ subalgebras, $I \subseteq A$ ideals, and quotients A/I as usual.

It is important to notice that properties can change from $A \to B$ and $A \to A/I$. The set of units may change, for instance.

Today we will let $\mathcal{C}$ be a dg category and then analyze 6 (dg) categories and ideals associated to $\mathcal{C}$. Along the way, it is good to keep **dgAlg** and **dgmod**-A in mind as examples.

Recall we had one definition of a dg category $\mathcal{C}$ as an Algebra with several objects. Then we are really studying $\mathcal{B}, \mathcal{I} \subseteq \mathcal{C}$ and $\mathcal{C}/\mathcal{I}$. Here we are assuming that the objects in all related categories are the same.

Problem 4.1
Write down the definitions of subcategory, ideal, and quotient that work with these definitions.

4.2 Homological Categories?!

Let $\mathcal{C}$ be a dg category. Then we will define analogs of the homological algebra objects we look for usually.

4.2.1 Definition
Define $Z(\mathcal{C})$ to be the category with the same objects as $\mathcal{C}$ with

$$Z\operatorname{Hom}_{\mathcal{C}}(M,N) = \operatorname{Hom}_{Z(\mathcal{C})}(M,N) = \{f \in \operatorname{Hom}_{\mathcal{C}}(M,N) | \partial_{\operatorname{Hom}_{\mathcal{C}}}(f) = 0\}$$

4.2.2 Lemma
$Z(\mathcal{C})$ is a dg subcategory of $\mathcal{C}$ (with zero differentials).

Proof
It is clear $Z(\mathcal{C}) \subseteq \mathcal{C}$. Recall that an axiom of a dg category is that $\partial 1_M = 0$. Thus $1_M \in \operatorname{Hom}_{Z(\mathcal{C})}(M,M)$.

Let $f \in \operatorname{Hom}(M,N)$ and $g \in \operatorname{Hom}(N,P)$. Then

$$\partial(g \circ f) = \partial(g) \circ f + (-1)^{|g|} g \circ \partial(f) = 0$$

and we are done. :) ♠

4.2.3 Definition
$B(\mathcal{C})$ again consists of the same objects as $\mathcal{C}$ but now

$$B\operatorname{Hom}_{\mathcal{C}}(M,N) = \operatorname{Hom}_{B(\mathcal{C})}(M,N) = \{f \in \operatorname{Hom}(M,N) | f = \partial_{\operatorname{Hom}} s f'\}$$

4.2.4 Lemma
$B(\mathcal{C})$ is an ideal of $Z(\mathcal{C})$.

4.2.5 Remark: To show that $B(\mathcal{C})$ is an ideal in $Z(\mathcal{C})$, we need to check that for all $f \in Z(\mathcal{C})$ and $g \in B(\mathcal{C})$ that (if f and g can be composed) that $f \circ g$ and/or $g \circ f$ are back in $B(\mathcal{C})$.

Proof
Let $f \in \operatorname{Hom}_Z(M,N)$ and $g \in \operatorname{Hom}_B(L,K)$. Then by definition $g = \partial g'$. If $f \circ g$ is defined, then

$$\begin{aligned}(-1)^{|f|} f \circ g &= f \circ \partial(g') \\ &= \partial(f) \circ g + (-1)^{|f|} f \circ \partial(g') \\ &= \partial(f \circ g') \in B(\mathcal{C})\end{aligned}$$

reversing the composition, if allowable, can be checked similarly. ♠

4.2.6 Definition
$H(\mathcal{C})$ is defined to have the objects of $\mathcal{C}$ and

$$H\operatorname{Hom}(M,N) = \operatorname{Hom}_{H(\mathcal{C})}(M,N) = \operatorname{Hom}_Z(M,N)/\operatorname{Hom}_B(M,N).$$

James uses the book *Rational Homotopy Theory* when he is talking about dgas and dg categories. He says it's a good reference for this part of the class.

4.2.7 Definition
Suppose $\mathcal{C}$ is a dg category with zero differential. Then define $\mathcal{C}^0$ to be the category with the same objects and

$$\operatorname{Hom}_{\mathcal{C}^0}(M,N) = \operatorname{Hom}_{\mathcal{C}}(M,N)^0$$

4.2.8 Lemma
$\mathcal{C}^0$ is a category.

Proof
"It's kinda obvious, right?" ♠

4.3 Homotopy categories

We can define in the more concrete case the category $\mathbf{K}(\mathbf{mod}\text{-}R) := H^0(\mathbf{Ch}(\mathbf{mod}\text{-}R))$.

5 April 10, 2019

Today is going to be a three-star day. We are going to be doing the next natural thing: "playing with objects *and* morphisms."

5.1 Playing with Objects

Last time we defined the categories we were talking about without changing the underlying objects of the category. We need to define:

- Shift or suspension
- Mapping cone
  - To do this we will need some motivation.

Recall that last time we discussed the process of creating $H^0(\mathbf{dgmod}\text{-}A)$. Here the hom sets are just vector spaces (instead of dg vector spaces) so we lose some structure. Then the shift and the cone operations will help recover (some) of the lost information.

5.1.1 Remark: From last time: let A be a (dg) algebra. Then we can define a subalgebra B of A and ideals $I \lhd A$ and quotients A/I. We did everything in categories on Monday: subcategories, ideals, and quotients.

One important process in algebras is localization. The analog also exists in category theory: we can invert a collection of morphisms. This is more complicated (as expected) than in the case of algebras, but analogous.

5.1.2 Remark: $Z^0(\mathbf{dgmod}\text{-}A)$ is an Abelian category. $H^0(\mathbf{dgmod}\text{-}A)$ is usually not Abelian. So we can't really talk about subobjects and quotients, etc, in the homotopy category although we can talk about them in the cycle category and remember the correspondence.

5.1.3 Remark: Consider $\mathbf{Ch}(\mathbf{mod}\text{-}R)$. We can define three different functors:

- $P : \mathbf{mod}\text{-}R \to H^0(\mathbf{Ch}(\mathbf{mod}\text{-}R))$, which takes the projective resolution of a module.
- $i : \mathbf{mod}\text{-}R \to H^0(\mathbf{Ch}(\mathbf{mod}\text{-}R))$, which embeds M as a complex $\cdots \to 0 \to M \to 0 \to \cdots$.
- $I : \mathbf{mod}\text{-}R \to H^0(\mathbf{Ch}(\mathbf{mod}\text{-}R))$, which assigns to each module an injective resolution.

The first and third functors are called the **projective** and **injective deriving functors,** respectively.

We have a natural transformations $P \to i \to I$ in $H^0(\mathbf{Ch}(\mathbf{mod}\text{-}R))$, but when we pass to $D(\mathbf{mod}\text{-}R)$, these become natural isomorphisms.

5.2 Shift/Suspension

5.2.1 Definition

Let $\mathcal{C} = \mathbf{dgmod}\text{-}A$. Let M be a right dg A module (complex). Then the **shift** of M, denoted by ΣM, is another right dg A-module such that:

- $\Sigma M = M$ as a right A-module.
- $(\Sigma M)^i = M^{i+1}$ for all i.
- $\partial_{\Sigma M} = \partial_M$

5.2.2 Remark: Elements in $(\Sigma M)^i = M^{i+1}$ are denoted by sm, where $m \in M^{i+1}$. Then

$$\partial_{\Sigma M}(sm) = (-1)s(\partial_M(m)) \in (\Sigma M)^{i+1} = M^{i+2}.$$

5.2.3 Remark: Actually, Σ is a functor! Just using that $|\partial| = 1$ got us the sign, but we can generalize completely using $\sigma f = (-1)^| f|f$.

Example 5.1
Suppose $A = (\cdots \to 0 \to R \to 0 \to \cdots)$ with R in the zeroth position. Then ΣM is the same complex with R shifted to the $(-1)^{st}$ position.

5.2.4 Lemma
Σ is invertible and the inverse is denoted Σ^{-1} in **dgmod**-A and all cycle, cocycle, and homology categories.

5.2.5 Remark: It is a bit troubling talking about functors on $B^0(\mathcal{C})$, which is an ideal category, but not a category proper. Basically the idea here is that we still get that it preserves morphisms, but identities may not exist and therefore can't be preserved.

5.2.6 Lemma

$$\mathrm{Hom}_{\mathbf{dgmod}\text{-}A}(M, \Sigma N) = \Sigma|_{\mathbf{dgVect}_k} \mathrm{Hom}_{\mathbf{dgmod}\text{-}A}(M, N)$$

Hint: $\mathrm{Hom}(M, \Sigma N)^i = (\Sigma\,\mathrm{Hom}(M, N))^i$.

5.3 Mapping Cone

As a construction, we first consider it as a map of sets

$$\mathrm{cone} : \mathrm{Hom}_{Z^0(\mathbf{dgmod}\text{-}A)}(M, N) \to \mathbf{dgmod}\text{-}A.$$

5.3.1 Definition
If $f : M \to N$ is a cycle in **dgmod**-A, then we define **the cone of** f, denoted $\mathrm{cone}(f)$, to be a right dg A module such that

- $(\mathrm{cone}(f))^i = (\Sigma M \oplus N)^i = M^{i+1} \oplus N^i$
- $(sm + n) \cdot a = s(m \cdot a) + n \cdot a$
- $\partial_{\mathrm{cone}(f)}(sm + n) = -s\partial_M^{i+1}(m) + [f(m) + \partial_N^i(n)]$

5.3.2 Lemma
$\partial^{i+1} \circ \partial^i = 0$.

Proof

$$\partial(sm + n) = -s\partial_M(m) + [f(m) + \partial_N(n)]$$

so applying the differential again:

$$\partial^2(sm + n) = s\partial_M^2(m) + -s\partial$$

I am missing something here but I believe it is clear. ♠

5.3.3 Theorem
cone : $\mathrm{Mor}(Z^0(\textbf{dgmod-}A)) \to \textbf{dgmod-}A$ descends to a map to $H^0(\textbf{dgmod-}A)$.

Here is some homework or just some facts if you are feeling lazy:

- $0 \to N \to \mathrm{cone}(f) \to \Sigma M \to 0$ is short exact (in **dgmod**-A).
- If $K \in \textbf{dgmod-}A$, then

$$\mathrm{Hom}_{\textbf{dgmod-}A}(K, \mathrm{cone}(f)) \cong \mathrm{cone}_{\textbf{dgVect}_k}(\mathrm{Hom}_{\textbf{dgmod-}A}(K, f))$$

6 April 12, 2019

Next week's topic will be Hochschild cohomology, which you can find in Weibel (chapter 9). Today we are going to finish the lecture we began on Wednesday. We will also review projective resolutions.

6.1 Mapping Cone (again)

Recall that if we have $f : M \to N$ in $Z^0(\textbf{dgmod-}A)$, we can construct $\mathrm{cone}(f) \in \textbf{dgmod-}A$ where

$$\mathrm{cone}(f)^i = (\Sigma M \oplus N)^i$$

and

$$\partial_{\mathrm{cone}(f)} = \partial_{\Sigma M} \oplus (f + \partial_N)$$

and more specifically

$$\partial_{\mathrm{cone}(f)}(sm + n) = -s\partial_M^{i+1}(m) + (f(m) + \partial_N^i(n)).$$

6.1.1 Remark: Where does this formula come from? Consider the diagram

$$\begin{tikzcd}
M^{i-1} \arrow[r, "\partial_M^{i-1}"] \arrow[d, "f^{i-1}"] & M^i \arrow[r, "\partial_M^i"] \arrow[d, "f^i"] \arrow[dl] & M^{i+1} \arrow[d, "f^{i+1}"] \arrow[dl] \\
N^{i-1} \arrow[r, "\partial_N"] & N^i \arrow[r, "\partial N"] & N^{i+1}
\end{tikzcd}$$

where the diagonal arrow indicate "pushing down" a copy of M^i to direct sum with N^{i-1}. But then using the maps you have in the diagrams, you get the maps $\partial_M^{i+1}, f^{i+1}$ and ∂_N^i between $M^{i+1} \oplus N^i$ and $M^{i+1} \oplus N^{i+1}$, and $\partial_{\mathrm{cone}(f)}$ is just their sum!

I need to think more to see whether this fact is actually meaningful or just a mnemonic.

Example 6.1
Consider a projective resolution P_M of M and the map $f : P_0 \to M$. Then considering that M is a complex with zeros everywhere except the zero position and consider the chain map f induced by f_0 (and zeros elsewhere) and then we can compute

$$\mathrm{cone}(f) = \cdots \to P^{-1} \to P^0 \to M \to 0.$$

6.2 Facts about $\mathrm{cone}(f)$

6.2.1 Remark: There exists a short exact sequence in **dgmod**-A

$$0 \to N \to \mathrm{cone}(f) \to \Sigma M \to 0$$

where $n \mapsto 0 + n$ under the first man and $sm + n \mapsto sm$ in the second.

This is called the **mapping cone sequence.**

6.2.2 Remark: Let K be another dg right A module. Then

$$Z^0(\mathbf{dgVect}_k) \ni \mathrm{Hom}_{\mathbf{dgmod}\text{-}A}(K, f) : \mathrm{Hom}_{\mathbf{dgmod}\text{-}A}(K, M) \to \mathrm{Hom}_{\mathbf{dgmod}\text{-}A}\, K, N)$$

and furthermore

$$\mathrm{cone}(\mathrm{Hom}_{\mathbf{dgmod}\text{-}A}(K, f)) \cong \mathrm{Hom}_{\mathbf{dgmod}\text{-}A}(K, \mathrm{cone}(f))$$

and

$$\mathrm{cone}(\mathrm{Hom}_{\mathbf{dgmod}\text{-}A}(\Sigma f, K)) \cong \mathrm{Hom}_{\mathbf{dgmod}\text{-}A}(\mathrm{cone}(f), K)$$

6.2.3 Remark: Let K be a left dg A module. Then the map

$$f \otimes_A \mathrm{id}_K : M \otimes_A K \to N \otimes_A K$$

is such that

$$\mathrm{cone}(f \otimes \mathrm{id}_K) = \mathrm{cone}(f) \otimes K.$$

We can also switch "left" and "right."

6.3 A "problem"

The map
$$\text{cone} : \text{Mor}(Z^0(\mathbf{dgmod}\text{-}A)) \to \mathbf{dgmod}\text{-}A$$
is **not** functorial!

However, suppose that $f, g : M \to N$ and $[f] = [g] \in H^0(\mathbf{dgmod}\text{-}A)$. Whenever this is true, there exists a map $\Phi : \text{cone}(f) \to \text{cone}(g)$ that fits into the diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\cong"] & \text{cone}(f) \arrow[r] \arrow[d, dashed, "\Phi"] & \Sigma M \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & N \arrow[r] & \text{cone}(g) \arrow[r] & \Sigma M \arrow[r] & 0
\end{tikzcd}$$

which is clearly an isomorphism via the five lemma. In fact, such a map exists if and only if these two maps are homotopic!

6.4 Classical Homological Algebra

Recall that we can always construct projective resolutions of a (right) R-module by taking a free module that maps onto M and then iterate. Thus such a resolution always exists.

Let $f : M \to N$ be a morphism of right R-modules. Then there exits F, the lift of f, between any projective resolutions P_M and P_N. Notice that the resolutions *don't* include the module themselves. Furthermore any two lifts of f differ only by a boundary in $\mathbf{Ch}(R)$, and thus are equal in the homotopy category.

From this we can conclude that any M in $H^0(\mathbf{Ch}(R))$, there exists a unique projective resolution, up to (a unique) isomorphism. In this way, we can see that "taking a projective resolution" defines a functor from $\mathbf{mod}\text{-}R$ to $H^0(\mathbf{Ch}(R))$.

For any two R modules M and N,
$$\text{Ext}^i_R(M, N) := H^i(\text{Hom}_{\mathbf{Ch}(R)}(P_M, P_N) \left(= H^i(\text{Hom}_{\mathbf{Ch}(R)}(P_M, N)\right).$$

If $M = N$, then $\text{Hom}_{\mathbf{Ch}(R)}(P_M, P_M)$ is a dg algebra. As a consequence,
$$\text{Ext}^*_R(M, M) = \oplus_i H^i(\text{Hom}(P_M, P_M))$$
is a graded algebra.

6.4.1 Definition

Then we can define **the Hochschild cohomology** as
$$\oplus_{i\in\mathbb{Z}} \text{Ext}^i_{R\otimes R^{op}}(R, R)$$

6.4.2 Remark: A nickname of Hochschild cohomology is "the derived functor of the center."

7 April 15, 2019

Today we will focus on understanding $\mathrm{HH}^0(A)$, the zeroth Hochschild cohomology. Recall that Sarah Witherspoon has a preprint (I have it) of a book she is going to release on the subject.

Define the Hochschild cohomology as we saw last week. Here james spent some extra time defining A^{op} as well as going over why it suffices to define it over A^e.

7.1 Hochschild Cohomology

For clarity:

7.1.1 Definition (Hochschild Cohomology)
For every $i \in \mathbb{Z}_{\geq 0}$, the i^{th} Hochschild cohomology of A is defined to be

$$\mathrm{HH}^i(A) = \mathrm{Ext}^i_{A^e}(A, A) = H^i(\mathrm{Hom}_{\mathbf{Ch}(A^e)}(P_A, P_A))$$

where A is any associative algebra.

7.1.2 Remark: In particular, $\mathrm{HH}^0(A) = \mathrm{Ext}^0_{A^e}(A, A) = \mathrm{Hom}_{A^e}(A, A)$.

7.1.3 Remark: "**Slogan 1**": Hochschild cohomology is the derived functor of $\mathrm{Hom}_{A^e}(A, A)$.

7.1.4 Remark: In fact, we can (and will!) define *Hochschild cohomology of an algebra with coefficents in an* A^e *module* M as $\mathrm{HH}^i(M) = \mathrm{Ext}^i_{A^e}(A, M)$.

7.1.5 Lemma
$\mathrm{Hom}_{A^e}(A, A) = Z(A)$.

Proof
Given any $f \in \mathrm{Hom}(A, A)$, set $c = f(1)$. We claim that $c \in Z(A)$. To see this, notice that for every $a \in A$, $ac = af(1) = f(a \cdot 1) = f(1 \cdot a) = f(1)a = ca$.

Thus we can define a map $f \mapsto f(1)$. To define the inverse, just sent $c \in Z(A)$ to f mapping $a \mapsto ca(= ac)$. Then a standard verification shows what we want. ♠

7.1.6 Remark: "**Slogan 2**": Hochschild cohomology is the derived functor of "the center."

This leads us to a really "wild" conjecture:

Conjecture: $\mathrm{HH}^*(A)$ is a commutative algebra.

7.1.7 Remark: We really have only proved this for the zeroth piece, which is why this is so wild. In fact, it is false.

Instead,

7.1.8 Theorem
$\mathrm{HH}^*(A)$ is *graded* commutative.

7.1.9 Remark: We do not have the machinery we need to prove this yet. For today we will prove a weaker result.

7.1.10 Theorem
$\mathrm{HH}^*(A)$ is a graded algebra.

Proof
$\mathrm{HH}^*(A) = \mathrm{Ext}^{\geq 0}_{A^e}(A, A)$ by definition. This can be computed as

$$H^*(\mathrm{Hom}_{\mathbf{Ch}(A^e)}(P_A, P_A)) = H^*(B)$$

where B is a dg algebra. But then $H * (B)$ is a graded algebra. ♠

7.2 Some more categories

Let $\mathcal{C}$ be an abelian category and let $\mathbf{Rex}(\mathcal{C})$ be the category of functors $\mathcal{C} \to \mathcal{C}$ that are right exact and commute with $\oplus$.

7.2.1 Remark: Notice that whenever $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ can be a replacement, but it isn't always abelian.

7.2.2 Theorem (Watts)
If $\mathcal{C} = R$-**mod**, then $\mathbf{Rex}(\mathcal{C}) = A^e$-**mod**.

7.2.3 Remark: More generally, any right exact functor from A modules to B modules that commutes with direct sums is isomorphic to $M \otimes_A -$ for some (B, A)-bimodule.

7.2.4 Lemma
Let $\mathcal{C}$ be A-**mod**. Then $\mathrm{id}_{\mathcal{C}} = A \otimes_A -$.

7.2.5 Definition
Let $\mathcal{C}$ be an abelian category. For every $i \in \mathbb{Z}$, the i^{th} Hochschild cohomology of $\mathcal{C}$ is defined to be

$$\mathrm{HH}^i(\mathcal{C}) = \mathrm{Ext}^i_{\mathbf{Rex}(\mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}).$$

7.2.6 Remark: If $\mathcal{C}$ is A-**mod**, then this agrees with $\mathrm{HH}^i(A)$. We have a map $\Phi : \mathbf{Rex}(\mathcal{C}) \to A^e$-**mod** sending $\mathrm{id}_{\mathcal{C}}$ to A, and when $i = 0$,

$$\mathrm{HH}^0(\mathcal{C}) = \mathrm{Hom}_{\mathbf{Rex}}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}) = \mathrm{Mor}_{\mathcal{C}}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}})$$

where Mor denotes the natural transformations. This forms a commutative ring where the multiplication is composition and the commutativity follows from the naturality squares.

7.2.7 Definition
With $\mathcal{C}$ as above, $\mathrm{Mor}_{\mathcal{C}}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}})$ is called the **center of $\mathcal{C}$**. This definition arises from the analogy with algebras.

7.2.8 Remark: Hochschild cohomology is the derived functor of $\mathrm{Hom}_{\mathbf{Rex}}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}})$.

Next time we will consider HH^1!

8 April 17, 2019

The lectures this week will cover three apparently distinct topics: the center, derivations, and deformations. The underlying math that connects these is the Hochschild complex, which we won't define until next week.

Some homework (for those who haven't seen it):

Example 8.1
Find the definition of a Lie algebra and work out some examples.

8.1 Derivations and $\mathrm{HH}^1(A)$

Recall that $\mathrm{HH}^i(A) = \mathrm{Ext}^i_{A^e}(A, A)$.

8.1.1 Definition
Let A be an algebra. A k-linear map $f : A \to A$ is called a **derivation** if

$$f(1) = 0$$

and for all $a, b \in A$,

$$f(ab) = f(a)b + af(b).$$

The set of all derivations of A is denoted $\mathrm{Der}(A)$.

8.1.2 Definition
Let $x \in A$. Define $I_x : A \to A$ by $I_x(a) = xa - ax$. One can easily check that $I_x \in \mathrm{Der}(A)$. This derivation is very special: it is called an **inner derivation.** Any derivation which is not inner is **outer.**

Example 8.2
$A = k[x]$ with derivation $\frac{d}{dx}$. Furthermore $\mathrm{Der}(A) = \{f(x)\frac{d}{dx} | f(x) \in A\}$. The only inner derivation is zero since A is commutative.

8.1.3 Remark: Notice that $\mathrm{Der}(A)$ is a Lie algebra with $[f, g] = fg - gf \in \mathrm{Der}(A)$. Furthermore $\mathrm{Inn}(A)$ is a Lie ideal, and $\mathrm{Der}(A)/\mathrm{Inn}(A)$ is a quotient Lie algebra.

And finally we get to another definition of HH^1:

8.1.4 Definition
The first Hochschild cohomology group of A is defined to be

$$\mathrm{HH}^1(A) = \mathrm{Der}(A)/\mathrm{Inn}(A)$$

which we sometimes write as $\mathrm{Out}(A)$, the outer derivations of A.

8.1.5 Remark: (**Slogan 4**): $\bigoplus \mathrm{HH}^{i+1}(A)$ is the derived functor of $\mathrm{Out}(A)$.

8.1.6 Theorem
$\bigoplus \mathrm{HH}^{i+1}(A)$ is a graded Lie algebra.

Proof
Nope. ♠

8.1.7 Remark: Notice that if $\mathrm{HH}^1(A) = 0$, then every derivation is inner. For example, when $A = M_n(k)$, every derivation of A is inner. To see this, notice that $A \otimes A^{op} = M_n(k) \otimes M_n(k) = M_{n^2}(k)$, which has global dimension zero. So $\mathrm{Ext}^1_{A^e}(A, A) = 0$.

8.2 Using Categories

Recall that $\mathrm{HH}^i(A) = \mathrm{Ext}^i_{\mathbf{Rex}(\mathcal{C})}(\mathrm{id}_\mathcal{C}, \mathrm{id}_\mathcal{C})$. It ends up that if we pick two algebras A and B which are Morita equivalent (that is, their module categories are equivalent) and let $\mathcal{C} = A$-**mod** or B-**mod**, this construction yields the same thing! So $\mathrm{HH}^i(A) = \mathrm{HH}^i(B)$. This is not at all obvious on the level of complexes, so it is a vote in favor of abstract nonsense.

9 April 19, 2019

Today we are talking about **infinitesimal deformations of algebras** and how they will relate to $\mathrm{HH}^2(A)$.

It ends up that "*every deformation (moduli) problem is related to some cohomology.*" We will use the letter "i" to stand for "infinitesimal." Recall that we have usually been considering algebras A over a field k, but we also want to consider algebras over a ring $R = k[t]/(t^2) \cong k \oplus kt$

9.1 Infinitesimal Deformations and HH^2

9.1.1 Definition
Let (A, m) be a k algebra. An **infinitesimal deformation** of A is a $k[t]/t^2$ algebra structure $m_t = *_t$ (this is the multiplication) on $A \otimes k[t]/t^2$ (note this is free over R, using the k-basis of A).

The multiplication on the deformation is defined as

$$a *_t b = ab + \mu(a,b)t \in A \oplus At \cong A \otimes R$$

for some $\mu : A \otimes A \to A$. This only defines the multiplication for elements in A, but this extends by linearity to $A \oplus At$:

$$(a + a't) *_t (b + b't) = a *_t b + (a' *_t b)t + (a *_t b')t + 0$$

since in R, $t^2 = 0$.

9.1.2 Remark: Notice that it is actually part of the definition of an infinitesimal deformation is **associative.** I didn't see this on the board, but rather than having this family of algebras freely parameterized by μ, we only take those μ such that the deformation is associative.

Example 9.1
The **trivial infinitesimal deformation of** A is determined by $\mu = 0$.

Example 9.2
Let $A = k[x, y]$ (works for arbitrary variables) where we define, for all $f, g \in A$

$$f *_t g = fg + \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)t.$$

It ends up that this algebra is associative! You can show it. :) This version of Nico doesn't feel like typing it.

9.1.3 Lemma (Condition 2-C)
m_t is associative if and only if for all $a, b \in A$,

$$a\mu(b,c) - \mu(ab,c) + \mu(a,bc) - \mu(a,b)c = 0.$$

9.1.4 Lemma
An i-deformation is completely determined by $\mu : A \otimes A \to A$ satisfying the 2-C condition in lemma 9.1.3.

Proof
Suppose $a, b, c \in A$ and that $(a * b) * c = a * (b * c)$. Then computing using the definitions gets us

$$(ab)c + \mu(ab,c)t + \mu(a,b)ct = a(bc) - \mu(a,bc)t + a\mu(b,c)t.$$

To go in the opposite direction, you basically use this identity to recover the equality for associativity. ♠

9.1.5 Corollary
The collection of infinitesimal deformations is in one-to-one correspondence with the collection of k-linear maps $\mu A \otimes A \to A$ satisfying 2-C.

9.1.6 Corollary
The collection of infinitesimal deformations is a k vector space.

9.1.7 Remark: Just formally define each infinitesimal deformation to be a vector corresponding to the μ which defines it. The fact that the set of such μ is a k vector space means the structure can be "pushed over" to the other collection.

9.1.8 Definition
Let m_t^1 and m_t^2 be two infinitesimal deformations of A. We say they are **equivalent** if there is an $R = k[t]/t^2$-algebra isormorphism

$$f_t : (A \otimes R, m_t^1) \to (A \otimes R, m_t^2)$$

such that $f(a) = a + \zeta(a)t$ for all $a \in A$. In this case we write $m_t^1 \sim m_t^2$ or $\mu^1 \sim \mu^2$.

9.1.9 Definition
m_t is called **quasi-trivial** if m_t is equivalent to the trivial infinitesimal deformation.

9.1.10 Lemma (Condition 2-B)
Let m_t^1 and m_t^2 be two infinitesimal deformations of A. Then $m_t^1 \sim m_t^2$ if and only if there exists $\zeta : A \to A$ such that $\mu^1(a,b) - \mu^2(a,b) = a\zeta(b) - \zeta(ab) + \zeta(a)b$ for all $a, b \in A$. This is called condition 2-B.

9.1.11 Lemma
Every equivalence between m_t^1 and m_t^2 is completely determined by ζ in 2-B.

9.1.12 Definition
$\mu : A \otimes A \to A$ is called a **2-coboundary** if there exists $\zeta : A \to A$ such that

$$\mu(a, b) = a\zeta(b) - \zeta(ab) + \zeta(a)b$$

(that is if μ is quasi-trivial).

9.1.13 Definition
μ is called a **2-cocycle** if it satisfies 2-C.

9.1.14 Remark: This gives us a vector subspace of coboundaries (quasi-trivial infinitesimal deformations) of the space of all infinitesimal deformations, so we can do cohomology!

9.1.15 Definition
The **second Hochschild cohomology of A** is

$$\mathrm{HH}^2(A) = \frac{\text{infinitesimal deformations}}{\text{quasi-trivial i-defs}} = \frac{\text{2-cocycles}}{\text{2-coboundaries}}$$

10 April 22, 2019

Today we will focus on connecting the notions that we have seen in the past several lectures (center/derivations/infinitesimal deformations) via the Hochschild complex. On Wednesday we are going to prove that the cohomology associated with the Hochschild complex are what we have computed so far.

Recall the definitions and constructions of $\mathrm{HH}^i(A)$ for $i = 0, 1, 2$. Furthermore we asserted that $\mathrm{HH}^*(A)$ is a graded commutative ring. Furthermore we showed that $\mathrm{HH}^1(A)$ is a Lie algebra and that $\mathrm{HH}^{*+1}(A)$ is a graded Lie algebra.

A natural question one may ask is if there is a “multiplication” operation on HH^2 that we can apply, analogous to the commutative ring/Lie algebra in the first two examples. In that case, can we extend this structure to $\mathrm{HH}^{*+2}(A)$?

10.0.1 Remark: Weibel has a different way to talk about HH^2. In fact, James seems to imply there are several different ways to think about it.

10.1 The Hochschild Complex

10.1.1 Definition
Let A be an algebra over k. Define the **Hochschild complex** $C(A)$ of A to be a cochain complex such that

$$C(A)^n = \mathrm{Hom}_k(A^{\otimes n}, A)$$

for $n \geq 0$ and 0 for $n < 0$. Note that $A^{\otimes 0} = k$.

The differential is defined to be a map $\partial^n : C(A)^n \to C(A)^{n+1}$ where for every $f \in C(A)^n$,

$$\begin{aligned}\partial^n f(a_1 \otimes \cdots \otimes a_{n+1}) = a_1 f(a_2 \otimes a_3 \otimes \cdots \otimes a_{n+1}) \\ + \sum_1^n (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ + (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1}\end{aligned}$$

10.1.2 Remark: The idea here is the first and last terms actually arise from the fact that $\mathrm{Hom}_k(A^{\otimes n}, A)$ has a left- and right A-module structure while the interior terms arise more from the chain complex.

10.1.3 Lemma
$C(A)$ is a cochain complex.

10.1.4 Remark: We omit the proof for the sake of time. James assures us it is true. :P

10.2 Double-checking our work

Let's compute!

10.2.1 Definition
Define the i-cocycles and i-coboundaries as we do with any chain complex.

Then we can compute $H^0(C(A)) = Z^0(C(A))/B^0(C(A)) = Z^0(C(A))$. But $C(A)^0 = \mathrm{Hom}_k(k, A) \cong A$, so we can determine when $\partial(f) = 0$:

$$\partial(f)(a) = af(1) - f(1)a = 0$$

for all a, so this means that $f(1)$ is in the center of A, so $Z^0(C(A)) = H^0(C(A)) \cong \mathcal{Z}(\mathcal{A})$.

What about HH^1? Now Z^1 is the set of $f \in \mathrm{Hom}(A, A)$, such that

$$\partial(f)(a \otimes b) = af(b) - f(ab) + f(a)b = 0$$

and so rewriting this, we get $f(ab) = af(b) + f(a)b$, so f is a derivation!

To compute the coboundaries, say $f = \partial(g) \in C(A)^1$. Then

$$f(a) = \partial(g)(a) = ag(1) - g(1)a = -I_{g(1)}(a)$$

so f is an inner derivation of A. So we recover HH^1.

Now Z^2 is the set of $f \in C(A)^2$ such that $\partial(f) = 0$ – so

$$\partial(f)(a \otimes b \otimes c) = af(b \otimes c) - f(ab \otimes c) + f(a \otimes bc) - f(a \otimes b)c = 0$$

which, via the universal property of the tensor product gets us a μ satisfying condition 2-C, giving us an infinitesimal derivation.

Similarly unfurling definitions gets us that B^2 are the 2-coboundaries.

11 April 24, 2019

Today we are going to talk about the equivalence (in all degrees) of $H^i(C(A))$ and $\mathrm{Ext}^i_A(A, A) = \mathrm{HH}^i(A)$. On Friday, we well begin working on triangulated and derived categories and will spend about two weeks there. For a reference, you can read Weibel, but the lecture will likely not follow his exposition for the most part.

11.1 The bar resolution

11.1.1 Definition

Let A be an algebra over k. The **bar resolution** of A is a chain complex ($\deg \partial = -1$) which is defined as follows:

$$B(A) = B(A; A) = \left(\cdots A^{\otimes 4} \to A^{\otimes 3} \to A^{\otimes 2} \to A \to 0 \to \cdots\right)$$

where the tensor power in the i^{th} term is $i + 2$, whenever that makes sense.

The differential is zero eventually, and otherwise is defined to be $\partial_N : A^{\otimes n+2} \to A^{\otimes n+1}$ via

$$\partial(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

11.1.2 Remark: We could actually define this as a cochain, but it is pretty standard to use a chain instead.

11.1.3 Remark: The name "bar resolution" comes from the fact that the notation for these tensor was (traditionally)

$$a_0 \otimes \cdots \otimes a_{n+1} \sim [a_0|a_1|\cdots|a_{n+1}].$$

Example 11.1
Some computations: $\partial_0(a_0 \otimes a_1) = a_0 a_1$ and

$$\partial(a_0 \otimes a_1 \otimes a_2) = a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2$$

so the idea is we are "generalizing multiplication" to higher tensor powers.

11.1.4 Lemma
$B(A)$ is a complex.

11.1.5 Lemma
Consider each $A^{\otimes n+2}$ as an (A, A)-bimodule using the natural multiplication on each factor. Then ∂_n is an (A, A)-bimodule morphism.

11.1.6 Corollary
This gives us that $B(A)$ is a complex of (A, A)-bimodules.

Proof
For any $x \in A$, we need to show that

$$\partial_n(x \cdots (a_0 \otimes \cdots \otimes a_{n+1})) = x \cdots \partial_n(a_0 \otimes \cdots \otimes a_{n+1})$$

but this is pretty much immediate once you define it the right way:

Basically, define the left action of A on $A^{\otimes n}$ by multiplication (in A on the leftmost factor) and the right action by multiplication on the rightmost factor. There is your structure and it is easy enough to confirm is compatible with your differential. ♠

11.1.7 Theorem
There exist maps (note: not bimodule maps) $s_n : B(A)_n \to B(A)_{n+1}$ such that for all n,

$$\mathrm{id}_{B(A)_n} = s_{n-1} \circ \partial_n + \partial_{n+1} \circ s_n$$

Proof
It is easy enough to confirm that the maps

$$s_n(a_0 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes \cdots \otimes a_{n+1}$$

work. ♠

11.1.8 Lemma
$B(A)$ is exact.

11.1.9 Remark: We've just shown that $\mathrm{id}_{B(A)} \sim 0$, so we're basically done. For a reminder: say $f \in \ker \partial_n$. But then

$$\begin{aligned} f &= \mathrm{id}_{B(A)_n}(f) \\ &= s_{n-1}\partial_n(f) + \partial_{n+1}s_n(f) \\ &= 0 + \partial_{n+1}(s_n(f)) \in \operatorname{Im} \partial_{n+1}. \end{aligned}$$

11.1.10 Theorem
$B(A)_{n\geq 0}$ is a free/projective resolution of the A^e module A.

Proof
We have already seen that the δ are A^e module maps and that this is an exact sequence. It remains to see that each $A^{\otimes n+2}$ is free. But

$$A^{\otimes n+2} = A \otimes A^{\otimes n} \otimes A \cong A \otimes A \otimes A^{\otimes n} \cong A^e \otimes A^{\otimes n}$$

and so this is free over the k-basis for the $A^{\otimes n}$ term. ♠

11.1.11 Corollary
$\mathrm{Ext}^i_{A^e}(A, A) = H^i(\mathrm{Hom}_{\mathbf{Ch}(A^e\textbf{-mod}}(B(A)_{n\geq 0}, (B(A)_{n\geq 0}))) = H^i(\mathrm{Hom}_{\mathbf{Ch}}(B(A)_{n\geq 0}, A))$

11.1.12 Theorem
$\mathrm{Hom}_{\mathbf{Ch}}(B(A)_{n\geq 0}, A) \cong C(A)$

Proof
The tricky part here is defining $\partial_{\mathrm{Hom}_{\mathbf{Ch}}(B(A),A)} : f \mapsto f \circ \partial_{B(A)}$ and then doing the computation. But recall that $\partial_{B(A)}$ and $\partial_{C(A)}$ are almost the same except the first and last terms. Well, use the (A, A)-bimodule structure to pull out a_1 and a_{n+1} and you get what you want! ♠

11.1.13 Corollary
$H^i(C(A)) \cong \mathrm{Ext}^i_{A^e}(A, A)$

11.2 The more general problem

11.2.1 Lemma
Let M be a (left) A module. Then $B(A)_{n\geq 0} \otimes_A M$ is a free resolution of M.

11.2.2 Remark: We could have just as easily have used right module structure by swapping the tensor.

11.2.3 Remark: This gives us a very nice thing – we can get a free resolution of any module! But unfortunately this is quite a terrible resolution to use if you want to do any computations.

12 April 26, 2019

Finally we are talking about the thing I named these notes after: triangulated categories! I was late, but I don't believe that James has done the axioms yet.

12.0.1 Remark: It is an open question (Julia also mentioned this) about whether the octahedral axiom is really required for a triangulated category.

12.1 Triangulated Categories

Some motivation (from Grothendieck): Let X be a projective scheme of dimension d and dualizing sheaf ω_X. Let M be a coherent sheaf on X. Then Serre duality gets us that

$$H^i_X(M)^* \cong H^{d-i}_X(\omega_x \otimes M)$$

for all i (as long as X is Cohen-Macaulay).

But when X is NOT CM, what can we do? Well there is a complex Ω_X in the bounded derived category of coherent sheaves over X such that (in what is now called Grothendieck duality)

$$H^i_X(M)^* \cong H^{d-i}_X(\Omega_X \otimes M).$$

A triangulated (or derived) category is the place where such an Ω_X lives.

12.1.1 Definition
A **triangle** in $\mathcal{C}$ is a diagram in $\mathcal{C}$:

$$X \to Y \to Z \to \Sigma X.$$

12.1.2 Definition
A **morphism of triangles** is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

12.1.3 Lemma
The collection of all triangles in $\mathcal{C}$ with triangle morphisms form a k-linear additive category denoted $\Delta(\mathcal{C})$

12.1.4 Remark: By forgetting all the information except the $X \to Y$ (resp. just X) we can define a forgetful functor $\Delta(\mathcal{C}) \to \mathbf{Mor}(\mathcal{C})$ (resp. $\Delta(\mathcal{C}) \to \mathcal{C}$). Our goal is to define a map in the reverse direction (an adjoint?).

12.1.5 Definition
The **rotation** of a triangle $X \xrightarrow{u} Y \to Z \to \Sigma X$ is $Y \to Z \to \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$. We may be using $\mathscr{R}(\Delta)$ for the rotation of Δ.

12.1.6 Definition
Let $\mathcal{C}$ be a k-linear additive category with a **suspension** or **shift** functor (auto-equivalence, actually)

$$\Sigma : \mathcal{C} \to \mathcal{C}.$$

Then $(\mathcal{C}, \Sigma, D)$ is **pre-triangulated** category where D is a full, nonempty subcategory $D \subseteq \Delta(\mathcal{C})$ with shift functor Σ of D and we have the following axioms:

- (**TR0**) $X \xrightarrow{\mathrm{id}} X \to 0$ is in D for each $X \in \mathcal{C}$ and furthermore D is closed under both shifts and triangle isomorphisms.
- (**TR1**) [*Mapping Cone Axiom*] For any $f : X \to Y$ in $\mathcal{C}$, there is a triangle

$$X \xrightarrow{f} Y \to Z \to \Sigma X$$

- (**TR2**) [*Rotation Axiom*] If $F \in D$, then $\mathscr{R}(F), \mathscr{R}^{-1}(F) \in D$
- (**TR3**) [*Morphism Axiom*] Given two triangles

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

  with maps f and g, there exists an $h : Z \to Z'$. I am unclear about whether the resulting diagram should commute.

12.1.7 Definition
A triple $(\mathcal{C}, \pm, \mathcal{D})$ is a **triangulated category** if it is pre-triangulated and, in addition, satisfies the axiom **TR4** (Verdier/octahedral axiom):

Suppose we are given three triangles: $X \xrightarrow{u} Y \to Z' \to \Sigma X$, $Y \xrightarrow{v} Z \to X' \to \Sigma Y$ and $X \xrightarrow{v \circ u} Z \to Y' \to \Sigma X$. Then there is a triangle $Z' \to Y' \to X' \to \Sigma Z'$ such that

$$\begin{tikzcd}
X \arrow[rr, bend left, "v \circ u"] \arrow[dr, "u"] & & Z \arrow[rr, bend left] \arrow[dr] & & X' \arrow[rr, bend left, dashed] \arrow[dr] & & \Sigma Z' \\
& Y \arrow[ur, "v"] \arrow[dr] & & Y' \arrow[ur, dashed] \arrow[dr] & & \Sigma Y \arrow[ur] & \\
& & Z' \arrow[ur, dashed] \arrow[rr, bend right] & & \Sigma X \arrow[ur] & &
\end{tikzcd}$$

12.1.8 Remark: Notice that the map constructing a triangle for any morphism is not well-defined but it exists. Furthermore it is far from being functorial. This is one of the qualms people have with the definition of these categories. Grothendieck attempted to fix this in an unpublished paper on "derivators" but it is significantly more complicated.

12.1.9 Remark: Topologists generally drop the k-linearity in $\mathcal{C}$. James is unsure if it actually ever comes up.

12.1.10 Remark: The model category you should be thinking about is $H^0(\mathbf{Ch}(\mathbf{mod}\text{-}R))$.

13 April 29, 2019

Recall (notice?) that, given maps $f : X \to X'$ and $g : Y \to Y'$, using the morphism axiom, we can define a morphism of triangles by just this data.

We have a couple other diagrams equivalent to the octahedral diagram (with slightly different information):

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "\sim"] & Y \arrow[r] \arrow[d, "v"] & Z' \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d, "\sim"] \\
X \arrow[r, "v \circ u"] & Z \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d, "\Sigma u"] \\
& X' \arrow[r, "\sim"] \arrow[d] & X \arrow[r] \arrow[d, dashed] & \Sigma Y \\
& \Sigma Y \arrow[r] & \Sigma Z' &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & X \arrow[dl, "u"'] \arrow[dd, "v \circ u"] & \\
 & Y \arrow[dr, "v"] \arrow[ddl] & & \\
 & & Z \arrow[d] \arrow[dr] & \\
Z' \arrow[rr, dashed] & & Y' \arrow[r, dashed] & X'
\end{tikzcd}$$

Example 13.1
Let $\mathcal{C} = K(\textbf{mod}\text{-}R) = H^0(\textbf{Ch}(\textbf{mod}\text{-}R))$. This is a triangulated category. It is easy to check that $\mathcal{C}$ is k-linear additive.

Σ is given by shifting of complexes and is an auto-equivalence. D is the full subcategory containing all triangles that are isomorphic to cone(f) where

$$X \xrightarrow{f} Y \to \text{cone}(f) \to \Sigma X$$

is an object such that cone$(f)^i = (\Sigma X \oplus Y)^i$ and

$$\partial_{\text{cone}} : (sm) + (n) \mapsto (-s\partial(m)) + (f(m) + \partial(m)).$$

Notice that there is some collision of notation: cone(f) can refer either to the complex or the triangle above.

Then we need to find a chain homotopy to establish that $X \to X \to \text{cone}(\text{id}_X) \to \Sigma X$ is isomorphic to $X \to X \to 0 \to \Sigma X$ to show that D satisfies (TR0).

To prove (TR1), let $u : X \to Y$ be any morphism. Then we have the triangle

$$X \xrightarrow{u} Y \to \text{cone}(u) \to \Sigma(X)$$

which gives us the axiom.

For (TR2): start with a triangle $X \to Y \to Z \to \Sigma X = F$ we want to show that $F \in D$ if and only if $\mathcal{R}(F) \in D$. Since all our triangles are isomorphic to the mapping cone, we can just assume that F is the mapping cone of $X \xrightarrow{u} Y$. To show $\mathcal{R}(F)$ is in D, we want to show that

$$(Y \xrightarrow{i} \text{cone}(u) \to \Sigma X \to \Sigma Y) \cong (Y \xrightarrow{i} \text{cone}(u) \to \text{cone}(i) \to \Sigma Y)$$

The idea here is that you notice that cone$(\text{id}_Y) = \Sigma Y \oplus Y$ is nullhomotopic. Then you get your copy of ΣX since cone$(i) = \Sigma Y \oplus (\Sigma X \oplus Y)$ (with the appropriate differential).

For (TR3), if we have $a : X \to X'$ and $b : Y \to Y'$ and $u : X \to Y$ and $u' : X' \to Y'$, then we can define the map $\Sigma a + b : \text{cone}(u) \to \text{cone}(u')$ by

$$\Sigma a + b(sm + n) = sa(m) + b(n).$$

This map is natural and makes the diagram commute so we're golden.

(TR4) is a bit of a pain. But given the triangles

$$X \xrightarrow{u} Y \to Z' \to \Sigma X, \quad X \xrightarrow{v\circ u} Z \to Y' \to \Sigma X, \quad Y \xrightarrow{v} Z \to X' \to \Sigma Y$$

we can again assume that these are mapping cones. That this works is slightly less trivial because of the dimensionality of the diagrams necessary, but can be done.

So here we assume $Z' = \text{cone}(u)$, $Y' = \text{cone}(v \circ u)$ and $X' = \text{cone}(v)$. This gives us the diagram

$$\begin{tikzcd}
x \arrow[r, "u"] \arrow[d, "\sim"] & Y \arrow[r] \arrow[d, "v"] & \text{cone}(u) \arrow[r] \arrow[d, dashed, "\text{id}_{\Sigma X} \oplus v"] & \Sigma X \arrow[d, "\sim"] \\
X \arrow[r, "v\circ u"] & Z \arrow[r] \arrow[d] & \text{cone}(v \circ u) \arrow[r] \arrow[d, dashed, "\Sigma u + \text{id}_Z"] & \Sigma X \arrow[d, "\Sigma u"] \\
& \text{cone}(v) \arrow[r, "\sim"] \arrow[d] & \text{cone}(v) \arrow[r] \arrow[d, dashed, "\pi_{\Sigma Y}"] & \Sigma Y \\
& \Sigma Y \arrow[r] & \Sigma Z' = \Sigma^2 X \oplus \Sigma Y &
\end{tikzcd}$$

Problem 13.1
Fill in the details of the above proof for (TR2) (which is surprisingly similar to the "fake" proof above.)

13.0.1 Definition
We define $\mathbf{Ch}^b(\mathbf{mod}\text{-}R)$, $\mathbf{Ch}^+(\mathbf{mod}\text{-}R)$ and $\mathbf{Ch}^-(\mathbf{mod}\text{-}R)$ to be the subcategories of $\mathbf{Ch}(\mathbf{mod}\text{-}R)$ consisting of the (totally) bounded, bounded below, and bounded above chains.

13.0.2 Proposition
$H^0(\mathbf{Ch}^x(\mathbf{mod}\text{-}R)) = K^x(\mathbf{mod}\text{-}R)$ are triangulated categories for $x \in \{b, +, 0\}$.

13.0.3 Remark: People refer to these as subcategories. Even though this isn't strictly true, it most works out.

13.0.4 Remark: It ends up that you can just pop different properties on your modules (e.g. finite dimensional, projective injective, flat) are all preserved by your shift functor and other operations, so these are also examples. "It works with any property you can think of." –James

13.1 Properties of Triangulated Categories

13.1.1 Proposition
IF $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \in D$ then $v \circ u = 0$.

Proof
Consider the diagram

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"] & X \arrow[r] \arrow[d, "u"] & 0 \arrow[r] \arrow[d, dashed, "0"] & \Sigma X \arrow[d, "\Sigma(\mathrm{id}_X)"] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r] & \Sigma X
\end{tikzcd}$$

where both the above triangles are in D, so we can apply (TR3) to get the existence of the dashed map above, and then by commutativity $v \circ u = 0$. ♠

13.1.2 Proposition
If $(\mathcal{C}, \pm, \mathcal{D})$ is a triangulated category, then so is $(\mathcal{C}^{\wr}\checkmark, \pm^{-\infty}, \mathcal{D}^{\wr}\checkmark)$.

14 May 4, 2019

Missed this day!

15 May 6, 2019

Today we will discuss 3 kinds of localizations. We will construct the derived category $D(\mathbf{mod}\text{-}R)$ via localization.

15.1 Localization

15.1.1 Definition
Let $\mathcal{C}$ be a category. Let S be a collection of morphisms in $\mathcal{C}$. The localization of $\mathcal{C}$ with respect to S is a category $S^{-1}\mathcal{C}$ together with a quotient map $\pi : \mathcal{C} \to \mathcal{S}^{-\infty}\mathcal{C}$ such that

(a) The objects in $S^{-1}\mathcal{C}$ are the same as those in $\mathcal{C}$.

(b) For any $s \in S$, $\pi(s)$ is invertible (an isomorphism).

(c) For any functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is invertible for all $s \in S$, F factors through $S^{-1}\mathcal{C}$.

Example 15.1
If G is a semigroup and $\mathcal{C}$ is a one-object category whose morphisms are the elements in G, then if we let $S = G$, then $S^{-1}\mathcal{C}$ is the category with one object whose morphisms are $\langle G \rangle$.

Example 15.2
If $\mathcal{C}$ is the one-object category for R, a k-algebra, then we can pick a (multiplicative) subset S of R to get $S^{-1}\mathcal{C}$ is essentially R_S.

Example 15.3
$K(\mathbf{mod}\text{-}R)$ is a localization of $\mathbf{Ch}(\mathbf{mod}\text{-}R)$.

15.1.2 Remark: Ignoring set-theoretic issues, $S^{-1}\mathcal{C}$ always exists. If we want to fix any such problems, we can just work over a (locally) small category. Generally if S is a set we are okay, but we will see soon that we will want to include all our identity maps! So we had better have a small $\mathcal{C}$.

15.2 Ore Localization

15.2.1 Definition
A collection S of morphisms in $\mathcal{C}$ is called a **multiplicative system** if

- S is closed under composition and contains id_X for all $X \in \mathcal{C}$.
- (**Right Ore Condition**) If $t : Z \to Y$ is in S and $g : X \to Y$ in $\mathcal{C}$, then there are maps f and s and a commutative diagram

$$\begin{tikzcd} W \arrow[r, dashed, "f"] \arrow[d, dashed, "s"] & Z \arrow[d, "t"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

  such that $s \in S$.
- (**Right Ore Condition**) $\mathcal{C}^{op}$ has the right Ore condition.

- (**Cancellation**) If $f, g : X \to Y$ are two morphisms, then the following are equivalent:
  - $sf = sg$ for some $s : Y \to Z$ in S
  - $ft = gt$ for some $t : V \to X$ in S

15.2.2 Definition
If S is a multiplicative system, then $S^{-1}\mathcal{C}$ is called an **Ore localization** of $\mathcal{C}$.

15.2.3 Theorem
Let S be a multiplicative system in $\mathcal{C}$. Then the Hom in $S^{-1}\mathcal{C}$ can be described as

$$\mathrm{Hom}_{S^{-1}\mathcal{C}}(X,Y) = \{X \overset{s}{\leftarrow} Z \overset{f}{\rightarrow} Y | s \in S\}/\sim = \{(s,f)|s \in S\}/\sim$$

where the equivalence $\sim$ is defined as follows: $(s_1, f_1) \sim (s_2, f_2)$ if there exists (s_3, f_3) and $a_1, a_2 \in S$ such that the following commutes:

$$\begin{tikzcd} & Z_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\ X & Z_3 \arrow[l, "s_3"] \arrow[u, "a_1"'] \arrow[r, "f_3"] \arrow[d, "a_2"] & Y \\ & Z_2 \arrow[ul, "s_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

In this case, the equivalence class of (s, f) is denoted $[fs^{-1}]$ or $[f/s]$.

15.2.4 Remark: Dually we can characterize Hom using diagrams $X \overset{g}{\rightarrow} Z \overset{t}{\leftarrow} Y$ for $t \in S$. These are equivalent but we have to use the left Ore condition instead of right to do composition.

So then we get a functor $\pi : \mathcal{C} \to S^{-1}\mathcal{C}$ that sends $X \mapsto X$ and $(f : X \to Y) \mapsto (X \xleftarrow{\mathrm{id}_X} X \overset{f}{\rightarrow} Y)$. Composition is just what you'd expect, using the ore conditions. Checking that composition is well-defined (modulo the equivalence relation) seems like a bit of a chore, but maybe worth checking.

15.3 Verdier Localization

Let $(\mathcal{C}, \Sigma, D)$ be a triangulated category. let $\mathcal{T}$ be a full triangulated subcategory of $\mathcal{C}$. Then $\mathcal{C}/\mathcal{T}$ is the category such that $X \simeq 0$ for all $X \in \mathcal{T}$.

Recall

15.3.1 Proposition
If $X \xrightarrow{f} Y \to Z \to \Sigma X$ is a triangle, f is an isomorphism if and only if $Z \simeq 0$.

Set $S_{\mathcal{T}}$ to be $\{f | \operatorname{cone}(f) \in \mathcal{T}\}$. Then

15.3.2 Lemma
$\Sigma S_{\mathcal{T}} = S_{\mathcal{T}}$.

15.3.3 Lemma
$S_{\mathcal{T}}$ is a multiplicative system.

15.3.4 Theorem
$S_{\mathcal{T}}^{-1}\mathcal{C}$ is a triangulated category.

15.3.5 Remark: James may or may not prove this on Wednesday. Some sketches: $\Sigma_{S^{-1}\mathcal{C}}$ sends $X \to \Sigma X$ and $[f/s]$ to $[(\Sigma f)/(\Sigma s)]$.

Then $D_{S^{-1}\mathcal{C}}$ is the set of all triangles in $\Delta(S^{-1}\mathcal{C})$ that are isomorphic to some $X \to Y \to Z \to \Sigma X$ in $D(\mathcal{C})$.

16 May 8, 2019

Today we will discuss examples of Verdier localization. First some motivation for it: in commutative algebra or in algebraic geometry, there is always a concept of localization and this (along with "local-global principles") gives us a nice way to get a handle on things.

In a more general case, we no longer have localization! So we don't have hope of chopping up things into local pieces. So instead we can move to a triangulated category! So if we are trying to study **mod**-R for some algebra R, we can instead consider $D(\textbf{mod}\text{-}R)$ where we can localize by any subcategory we choose!

16.1 Examples of Verdier Localization

As a convention, let $(\mathcal{C}, \Sigma, D)$ be a triangulated category and let $\mathcal{T}$ be a full triangulated subcategory such that if $Y \in \mathcal{T}$ and $X \cong Y$, then $X \in \mathcal{T}$ ($\mathcal{T}$ is closed under isomorphism). Furthermore, $\mathcal{T}$ is triangulated via $(\mathcal{T}, \Sigma|_{\mathcal{T}}, D \cap \Delta(\mathcal{T}))$.

Now let P be some property. First, we will say P is a property on objects.

Example 16.1
Let $\mathcal{C} = K(\textbf{mod}\text{-}R)$. Let $P(C)$ mean that C is a (bounded/bounded above/bounded below) complex of (projective/injective/flat/f.g./some mixture). Then take $\mathcal{T} = \{X | \exists Y : Y \cong X, P(Y)\}$. We need to check that P is stable under suspension and that the set of P-objects is closed under cone constructions.

In this case since the cone is just a coproduct, it suffices to show that this preserved under coproducts.

Next P is a property on morphisms:

Example 16.2
Now let X be a P object if $\mathrm{Hom}_{\mathcal{C}}(\Sigma^n X, -)$ commutes with coproducts for all $n \in \mathbb{Z}$. Alternatively, we could let the P objects be those Y such that $\mathrm{Hom}_{\mathcal{C}}(\Sigma^n X, Y) = 0$ for all $n \in \mathbb{Z}$ for some fixed X.

Then we can take the objects of $\mathcal{T}$ to be the objects in $\mathcal{C}$ isomorphic to a P object. Once one proves that these objects are stable under suspension and cone constructions, $\mathcal{C}/\mathcal{T}$ exists.

16.1.1 Lemma
$\mathrm{Hom}_{\mathbf{Ch}(\mathbf{mod}\text{-}R)}(R, X) = X$.

Proof
Look at the i^{th} component of each:

$$\mathrm{Hom}_{\mathbf{Ch}(\mathbf{mod}\text{-}R)}(R, X)^i = \prod_{s \in \mathbb{Z}} \mathrm{Hom}_R(R^s, X^{s+i}) = \mathrm{Hom}_R(R^0, X^i) = X^i$$

It remains to show the differentials are the same, but it can be shown (HW). ♠

16.1.2 Lemma
If X and Y are acyclic complexes and $X \xrightarrow{f} Y \to Z \to \Sigma X$ is a triangle, then Z is acyclic.

16.1.3 Remark: To prove this, one applies $\mathrm{Hom}_{\mathcal{C}}(\Sigma^n R, -)$ to the triangle and uses that $\mathrm{Hom}_{\mathcal{C}}(\Sigma^n R, \Sigma X) = \mathrm{Hom}_{\mathcal{C}}(\Sigma^{n-1} R, X) = 0$.

Example 16.3
Let $\mathcal{C} = K(\mathbf{mod}\text{-}R)$. Let the P objects to be the X such that $\mathrm{Hom}_{\mathcal{C}}(\Sigma^n R, X) = 0$. Now notice

$$\mathrm{Hom}_{\mathcal{C}}(\Sigma^0 R, X) = \mathrm{Hom}_{H^0(\mathbf{Ch}(\mathbf{mod}\text{-}R))}(R, X) = H^0(\mathrm{Hom}_{\mathbf{Ch}(\mathbf{mod}\text{-}R)}(R, X)) = H^0(X)$$

and (a claim) $\mathrm{Hom}_{\mathcal{C}}(\Sigma^n R, X) = H^n(X) = 0$.

Thus X has P if and only if $H^n(X) = 0$ for all n – that is, if X is **acyclic.**

16.1.4 Definition
The **derived category of complexes of right R modules** is

$$D(\mathbf{mod}\text{-}R) := K(\mathbf{mod}\text{-}R)/K_{\text{acyclic}}(\mathbf{mod}\text{-}R)$$

16.1.5 Remark: There are some useful subcategories $D^b(\mathbf{mod}\text{-}R)$, the bounded derived category, as well as $D^+(\mathbf{mod}\text{-}R)$ and $D^-(\mathbf{mod}\text{-}R)$, where are the bounded (resp.) below and above complexes. Here one applies the property to the homotopy category before taking the quotient. One also considers $D^b_{f.g.}(\mathbf{mod}\text{-}R)$ where one puts the finite generation condition on $\mathbf{mod}\text{-}R$ first, then takes the quotient of the appropriate homotopy (sub) categories.

Example 16.4
Let $\mathscr{D} = D^b_{f.d.}(\mathbf{mod}\text{-}R)$ and let P be the complexes of finitely generated projective modules (i.e. **perfect complexes**). Then $\mathcal{T}_P := D^b_{per}(\mathbf{mod}\text{-}R)$, a sub triangulated category of $\mathscr{D}$. Then

$$D_{sing}(\mathbf{mod}\text{-}R) := D^b_{f.g.}(\mathbf{mod}\text{-}R)/D^b_{per}(\mathbf{mod}\text{-}R)$$

is called the **singular category** and corresponds to the singularities of $\operatorname{spec} R$.

16.1.6 Remark: If $\operatorname{gldim} R < \infty$ and R is Noetherian, then $D_{sing}(R) = 0$.

17 May 10, 2019

Today will be largely random remarks. There will continue to be holes in proofs. :)

Last time, we let R be an algebra and considered the derived category $D(\mathbf{mod}\text{-}R)$, which is a Verdier localization of $K(\mathbf{mod}\text{-}R)$ or (somehow equivalently) a quotient. I think he is playing a bit fast and loose here.

Note that the mapping cones (the triangles) are precisely the quasi-isomorphisms. This can be seen by applying $H^0 = \operatorname{Hom}_K(R, -)$ to $\Sigma^{-1}\operatorname{cone}(f) \to X \xrightarrow{f} Y \to \operatorname{cone}(f)$

17.0.1 Remark: If S is the set of quasi-isomorphisms in $\mathbf{Ch}(\mathbf{mod}\text{-}R)$, then

$$S^{-1}\,\mathbf{Ch}(\mathbf{mod}\text{-}R) \cong D(\mathbf{mod}\text{-}R),$$

but this is just (for now?) as k-linear categories, since $\mathbf{Ch}(\mathbf{mod}\text{-}R)$ is not triangulated (it's abelian!).

17.0.2 Remark: There exists a fully faithful embedding $\mathbf{mod}\text{-}R$ into $D(\mathbf{mod}\text{-}R)$ where we have $\operatorname{Hom}_R(M,N) = \operatorname{Hom}_{D(\mathbf{mod}\text{-}R)}(M,N)$.

17.0.3 Remark: $\operatorname{Ext}^i_R(M,N) = \operatorname{Hom}_{D(\mathbf{mod}\text{-}R)}(M, \Sigma^i N)$. This can be useful for computations, but recall that composition in the Verdier localization is all about them roof constructions.

17.0.4 Remark: Let $P_\bullet \to N$ be a projective resolution. Then $P_\bullet$ is quasi-isomorphic to N, so they become (honestly) isomorphic in $D(\mathbf{mod}\text{-}R)$ even though $P_\bullet \not\cong N$ in $K(\mathbf{mod}\text{-}R)$.

17.0.5 Remark: Suppose X is a bounded-above complex of projectives. Then

$$\mathrm{Hom}_{D(\textbf{mod-}R)}(X,Y) = \{X \to Y\}/\sim = \mathrm{Hom}_{K(\textbf{mod-}R)}(X,Y) = H^0(\mathrm{Hom}_{\textbf{Ch}(\textbf{mod-}R)}(X,Y)).$$

This motivates somewhat why we often want to have projective resolutions of our objects – they become more computationally feasible in this case.

17.0.6 Remark: $K^-(\textbf{projmod-}R) \cong D^-(\textbf{mod-}R)$. In other words, (recall we have a fully faithful embedding of $\textbf{mod-}R$ into $D(\textbf{mod-}R)$) there is a fully faithful embedding of $\textbf{mod-}R$ into $K^-(\textbf{projmod-}R)$ sending each M to a projective resolution of M.

Dually, $D^+(\textbf{mod-}R)$ is equivalent to $K^+(\textbf{projmod-}R)$, so $\textbf{mod-}R$ embeds fully faithfully into it.

17.0.7 Remark: In particular, $D^b(\mod - R)$ embeds into either $K^+(\textbf{injmod-}R)$ or $K^-(\textbf{injmod-}R)$.

17.1 Derived Functors

Consider algebras A and B. Let F be an additive k-linear functor from A modules to B modules. This easily extends to a functor between the chain and homotopy categories of these categories. But **there is not a clear extension of F to the derived category.**

Consider the composition of maps (and embeddings):

$$D^-(\textbf{mod-}A) \hookrightarrow K^-(\textbf{projmod-}A) \xrightarrow{F} K^-(\textbf{mod-}B) \xrightarrow{\nu} D^-(\textbf{mod-}B)$$

where ν is the Verdier localization.

Then the composition of these maps is called the **left derived functor of F**, which is written LF.

Example 17.1
Let $F = - \otimes_R N : \textbf{mod-}R \to \textbf{Vect}_{\mathrm{k}}$. Then $H^i(LF(M)) = \mathrm{Tor}^A_{-i}(M,N)$.

Next week we are going to discuss dualizing complexes. From these you can prove some very nice results that otherwise require some complicated argument.

18 May 13, 2019

18.1 Dualizing Complexes

Today we will start with definitions and an introduction to the idea.

Grothendieck introduced dualizing complexes (over schemes) in the 1960's. Using these complexes, he proved what was called "Grothendieck duality", a generalization of Serre duality. In 1990(ish) Yekutieli introduced them in the noncommutative case.

This is a powerful tool in noncommutative/homological algebra. Even the existence of dualizing complexes has many consequences:

18.1.1 Theorem
Take any simple factor ring of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ ($\mathfrak{g}$ finite dimensional) is Auslander-Gorenstein and Cohen-MacCaulay.

18.1.2 Theorem (Bass Theorem (for Hopf algebras))
Let H be a Noetherian Hopf algebra satisfying a polynomial identity. let M be a finitely generated (left) H-module. Then $\operatorname{injdim}_H M$ is either infinite or equal to $\operatorname{injdim}_H H$.

18.1.3 Theorem (Auslander-Buchsbaum)
Let A be a local Noetherian algebra with balanced dualizing complex. Let M be a finitely generated (left) A-module. Then projdim M is either infinite or is $\operatorname{depth}(A) - \operatorname{depth}(M) \leq \operatorname{depth}(A)$.

18.2 Conventions

We will be working a lot in $\mathbf{Alg}_k$, $\mathbf{mod}$-R, and $\mathbf{Ch}$ over the next couple weeks. We should fix some conventions to work with.

- Let A, B, C be Noetherian k-algebras.
- Usually R will stand for a dualizing complex (Except directly below where my notation varied).
- Let M and N be left (or right) modules over A.
- Let X, Y, Z be complexes.
- The category of (A, B)-bimodules is identified with left $A \otimes B^{op}$ modules.
- Usually we work with left A-modules. To study the right A-modules, we study the left A^{op} modules.
- We want to identify $\mathbf{D}^b(R\text{-}\mathbf{mod})$ with a subcategory of $\mathbf{D}(\mathbf{mod}\text{-}R)$, so notice

$$\mathbf{D}^b(R\text{-}\mathbf{mod}) \cong \{X \in \mathbf{D}(\mathbf{mod}\text{-}R) | X \simeq \text{ some object in } \mathbf{D}^b(R\text{-}\mathbf{mod})\}$$

  where the isomorphism there is on the level of triangulated categories.

18.2.1 Proposition
If $X \simeq Y$ for some bounded complex, then $H^n(X) = 0$ for all $n >> 0$. The converse is also true.

This is a slightly nonstandard definition of projective dimension, but it is the one we will use in this class:

18.2.2 Definition
Let $X \in \mathbf{D}^b(R\text{-}\mathbf{mod})$. Then

$$\operatorname{projdim} X = \max\{i | \operatorname{Ext}^i_R(X, M) \neq 0, \text{ some } M \in R\text{-}\mathbf{mod}\}$$

The upshot here is

18.2.3 Lemma
$\operatorname{projdim} \Sigma X = \operatorname{projdim} X + 1$ and therefore $\operatorname{projdim} \Sigma^n X = \operatorname{projdim} X + n$.

18.2.4 Lemma
Let $d = \operatorname{projdim} X < \infty$. Then

$$X \simeq (\cdots 0 \to P^{-d} \to \cdots \to P^s \to 0)$$

with all P^i projective.

Then define $\operatorname{injdim} X$ dually.
Recall that for bounded complexes X and Y of R-modules,

$$R\operatorname{Hom}_R(X, Y) := \operatorname{Hom}_{\mathbf{Ch}(R\text{-}\mathbf{mod})}(X, I_Y) = \operatorname{Hom}_{\mathbf{Ch}(R\text{-}\mathbf{mod})}(P_X, Y) \in \mathbf{D}(\mathbf{Vect}_{\Bbbk})$$

18.2.5 Lemma
Let M be a left $R = A \otimes B^{op}$ module.

(a) If M is free over R, then it is free over both A and B.

(b) If M is projective over R, then it is projective over both A and B.

(c) If M is flat over R, then it is flat over both A and B.

(d) If M is injective over R, then it is injective over both A and B.

Proof
(a): If $M \cong (A \otimes B^{op})^n$, then $A \otimes B^{op} \cong A^{\dim B^{op}}$, so $M \cong A^{n \dim B^{op}}$.
(b): pretty easy.
(c) and (d): a little more tough. Use adjoints. ♠

18.2.6 Lemma
Let M and N be (A, B)-bimodules. Then $\operatorname{Hom}_A(M, N)$ is a (B, B)-bimodule. Furthermore, $\operatorname{Hom}_{B^{op}}(M, N)$ is an (A, A)-bimodule.

18.2.7 Lemma
Let X, Y be complexes of (A, B)-bimodules. Then $\operatorname{Hom}_{\mathbf{Ch}(A\text{-}\mathbf{mod})} \in \mathbf{Ch}(B^e\text{-}\mathbf{mod})$. Also flip it and reverse it.

18.2.8 Lemma
Let X and Y be complexes of (A, B)-bimodules. Then $R\operatorname{Hom}_A(X, Y) \in \mathbf{D}(B \otimes B^{op}\text{-}\mathbf{mod})$ and flip it.

18.3 Dualizing Complexes

18.3.1 Definition
Let A and B be two k-algebras and let R be a complex in $\mathbf{D}^b(A \otimes B^{op}\textbf{-mod})$. Then R is called a **dualizing complex over** (A, B) if

- $H^i(R) \in A \otimes B^{op}\textbf{-mod}$ is finitely generated on both sides.
- $\operatorname{injdim}_A R, \operatorname{injdim}_B R < \infty$.
- The natural maps $A \xrightarrow{\Phi_A} R\operatorname{Hom}_B^{op}(R, R)$ and $B \xrightarrow{\Phi_B} R\operatorname{Hom}_A(R, R)$ are isomorphisms in $\mathbf{D}(A^e\textbf{-mod})$ and $\mathbf{D}(B^e\textbf{-mod})$, respectively.

18.3.2 Remark: The map Recall that $R\operatorname{Hom}_{B^{op}}(R, R) = \operatorname{Hom}_{\mathbf{Ch}(B^{op}\textbf{-mod})}(P_R, P_R)$. Then Φ_A sends $a \mapsto l_a : P_R \to P_R$.

One can verify this more easily when $A = B = R = k$.