Algebraic Geometry

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Abstract

A three-quarter sequence covering the basic theory of affine and projective varieties, rings of functions, the Hilbert Nullstellensatz, localization, and dimension; the theory of algebraic curves, divisors, cohomology, genus, and the Riemann-Roch theorem; and related topics.

1 September 25, 2019

The first thing that one asks is "what is geometry?" One needs to be able to answer this question before they define AG. One idea is that geometry is topology + structure.

1.1 What is Geometry?

Example 1.1
Exotic differentiable structures on a sphere. There are many different smooth structures, all of which are independent of the topology,

$S^1 \times S^1$ has infinitely many complex structures (remember the parallelograms)!

How to you go about defining the geometry of a thing? One idea from manifolds: charts. These describe the local models and the interesting part is how this comes together to a whole space.

There is another idea to capture the "local" model of geometry that underlies modern algebraic geometry: consider the map $\varphi : W \to W' \in \mathbb{CP}^n$ and then say that this map is C^∞ if and only if its coordinate functions are. But the coordinate functions are problematic, so we can replace it with the following idea:

$\phi : W \to W'$ is $C\infty$ if and only if for all C^∞ functions $f : W' \to \mathbb{R}$, the composition

$$\varphi^* f = f \circ \varphi : W \to \mathbb{R}$$

is C^∞.

To capture the manifold structure on M, it is equivalent to know the set of C^∞ fucnctions $U \to \mathbb{R}$ for every open $U \subseteq M$.

1.2 The Big Idea

So then the idea we are talking away here is that *geometry is in the functions* that exist on a particular space!

Fix a field k.

1.2.1 Definition: A **space with functions** is a topological space X along with a collection (a k-algebra!) $\mathscr{O}(U)$ of maps $U \to k$ for each open $U \subseteq X$.

$\mathscr{O}(U)$ are called **regular functions** and must satisfy:

- Given an open cover U_α of U, a function is regular if and only if its restrictions to each element of the cover is regular.
- If $f : U \to k$ is regular, then $D(f) = \{u \in U | f(u) \neq 0\}$ is an open set and $\frac{1}{f} \in \mathscr{O}(D(f))$.

For the next time, try to think of as many examples of this as you can. Next time will be a mind blowing example of a variety.

2 September 27, 2019

Problem 2.1
Do all the exercises in Kempf chapter 1!

For now we assume that k is algebraically closed.

2.1 Examples of spaces with functions

There were lots of suggestions, but here are a couple:

Example 2.1
Let $X = \mathbb{S}^2$ and let $\mathscr{O}_X^{cts}$ be the continuous $\mathbb{C}$-valued functions. Alternatively we could consider a different sheaf $\mathscr{O}_X^{an}$, the holomorphic functions. Or we could consider $\mathscr{O}_X^{\infty}$, the C^∞ functions (under some smooth structure).

2.1.1 Definition: A **morphism** of spaces with functions between $(X,\mathcal{O}_X)$ and $(Y,\mathcal{O}_Y)$ is a continuous map $\varphi : X \to Y$ such that for all $U \subseteq Y$ open and $f \in \mathcal{O}_Y(U)$, the function

$$\phi^* f = f \circ \phi|_{\phi^{-1}(U)} : \phi^{-1}(U) \to k \in \mathcal{O}_X(\phi^{-1}(U))$$

In other words, a morphism of spaces with functions is a map of spaces that *respects the regular functions.*

Example 2.2
Let X,Y be topological spaces and let $\mathcal{O}_X$ and $\mathcal{O}_Y$ be the continuous functions. Then morphisms are just continuous maps.

Example 2.3
When X and Y are manifolds and $\mathcal{O}_\bullet$ are complex-valued functions, then the maorphisms are maps of manifolds.

So now we return to the examples we saw before: $(\mathbb{S}^2,\mathcal{O}^\infty)$, $(\mathbb{S}^2,\mathcal{O}^{cts})$, and $(\mathbb{S}^2,\mathcal{O}^{an})$. A natural question to ask is when we have morphisms between these spaces to see if there exist ones that are the identity on $\mathbb{S}^2$.

Consider the identity map from the continuous to the analytic functions. Then take any map $f \in \mathcal{O}^{an}$ and consider that

$$f = f \circ id_{id^{-1}(U)} : U \to k \in \mathcal{O}^{cts}(U)$$

and there is no map in the other direction.

2.1.2 Remark: Notice that since we are pulling functions back, the maps go in the opposite direction as you may think at first.

We can also talk about **open subspaces**. If $V \subseteq X$ is an open subset, we can let the induced space with functions be $(V,\mathcal{O}_V)$ where if $U \subseteq V$ then $\mathcal{O}_V(U) := \mathcal{O}_X(U)$.

2.2 Varieties

2.2.1 Definition: An **affine k-variety** is a space with functions $(Y, \mathcal{O}_Y)$ such that for every space with functions $(X, \mathcal{O}_X)$, the natural map

$$\mathrm{Hom}((X,\mathcal{O}_X),(Y,\mathcal{O}_Y)) \to \mathrm{Hom}_{\mathbf{Alg}_k}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$$

is a bijection and furthermore $\mathcal{O}_Y(Y) =: k[Y]$ is a finitely generated k-algebra.

2.2.2 Remark: The idea here is that the algebra maps (on the right) are precisely the same as the geometry maps (on the left). Algebraic geometry, baby.

So then this leads to a very simple (loose) definition:

2.2.3 Definition: A **variety** is something that is covered by affine varieties.

Example 2.4
$\mathbb{A}^1 = k$. Give this space the cofinite topology. Then if we have $U = k \setminus \{x_1, \ldots, x_n\} \subset \mathbb{A}^1$,

$$\mathcal{O}_{\mathbb{A}^1}(U) = \{f(t) \in k(t) | \text{poles are in } \{x_i\}\}$$

Problem 2.2
Show that $\mathbb{A}^1$ is an affine variety!

2.2.4 Remark: Notice that this staetment is equivalent to saying that any morphism of spaces with functions gives us a regular map $X \to k$.

3 September 30th, 2019

One question that was asked: if we have fixed the underlying topological space in a space with functions, must there be a morphism between them somehow? Might there instead be a common cover of the two?

Example 3.1
Let k be a field with some topology on it such that every point is closed (you could do the discrete topology). Let $\widetilde{\mathscr{O}}(U)$ be the continuous funcitons $U \to k$. In other words, these functions are locally constant.

Locally constant functions behave nicely under restrictions to opens, of course. The other axioms are also great.

Have we really found an initial object in our category? This would be enough to establish a "tent" (as in localization of categories). Try this out and see what happens!

3.1 The question of affine space

Recall the question about whether $\mathbb{A}^1$ is an affine variety. The idea here is that $\phi : X \to k$ is a morphism of spaces with functions if and only if it is regular (that is, in $\mathscr{O}_{\mathbb{A}^1}$).

One direction is tautological (a morphism to $\mathbb{A}^1$ has a polynomial underlying it), so let ϕ be regular. Then to see that ϕ is continuous can be checked by pulling back all closed sets. The important observation is that $D(f-a) = X \setminus \phi^{-1}(a)$, which is closed (an axiom for spaces with functions).

The last thing to check is where ϕ pulls back regular functions to regular functions. This relies on the facts that $\mathscr{O}_X$ is a k-algebra and that $\phi(x) - b_j$ is regular on U when $b_j \notin U$.

3.2 Algebra maps

Notice that since we have a condition that $\mathscr{O}_X(X)$ must be finitely generated as a k-algebra, this means that

$$\mathrm{Hom}(X,Y) = \mathrm{Hom}_k(\mathscr{O}_Y(Y), \mathscr{O}_X(X)) = \mathrm{Hom}_k(k[x_1,\ldots,x_n]/(f_1,\ldots,f_m), O_X(X))$$

and

$$\mathrm{Hom}(X,Y) = \{(\gamma_1,\ldots,\gamma_n) \in (\mathscr{O}_X(X))^n : f_j(\gamma_i) = 0, \forall j = 1,\ldots,m\}$$

In other words, we are looking at maps that factor through Z:

$$\begin{tikzcd}
(\gamma_1,\ldots,\gamma_n) : X \arrow[r] \arrow[dr] & k^n \\
 & Z = Z(f_i) \arrow[u, hook]
\end{tikzcd}$$

Now what we want to say is that $Y = Z$. That is, *affine varieties are closed subsets of affine spaces.*

Now this is all good, but the problem is that we had to *choose* a presentation of $\mathscr{O}_Y(Y)$ to get this picture. of course we want something more canonical! We will see in this class (and in Kempf) that this can be done.

4 October 2, 2019

4.1 Questions without (complete) answers

4.1.1 Morphisms and stuff

A question to get things started for the day. Let X and Y be spaces with functions and let Y be an affine variety and let $f : Y \to X$ be a map of sets (but with no further assumption on f). This naturally induces amap from $\mathcal{O}_X(X)$ to the functions $\mathrm{Hom}_{\mathbf{Set}}(Y,k)$ (which clearly contains the regular functions on Y).

Further assume that there exists a $\gamma : \mathcal{O}_X(X) \to \mathcal{O}_Y(Y)$. We know that since Y is affine, γ corresponds to a morphism $\varphi : Y \to X$. Then the question is: when does $f = \varphi$? We've already answered this question for $\mathbb{A}^1$, notice.

4.1.2 Algebraic closure

Where do we use algebraic closure of the base field? It has been swept under the rug a bit, but consider the function

$$\frac{1}{x^2+1} : \mathbb{R} \to \mathbb{R}.$$

This certainly seems like it should be a regular function (e.g. it is rational and defined everywhere on $\mathbb{R}$) but this conflicts with the idea that we want to identify $\mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1) = k[t]$, but that is clearly not the case here. Think about this.

4.1.3 Yet another

Consider the set R of all continuous maps $k \to k$ under the cofinite topology. Someone asked if R is a k-algebra. The answer is a bit convoluted, but the short answer is no. Specifically if we are using the product topology on $k \times k$, the addition map isn't continuous! This also points to the question of what topology is the correct one to use on these things.

4.2 Back to affine varieties

Recall that we constructed a (highly-non-canonical) picutre of how any affine variety arises as a closed subset of some affine space k^n.

We want to remove this dependence on presentation, however, and that is what we are working toward.

4.2.1 Affine Space

Now we focus in on $\mathbb{A}^n = k^n$. We really want that the projection functions $x_i : k^n \to k$ should be regular. But since we want this (eventually) to form a k-algebra, we want that each $f \in [x_1, \ldots, x_n]$ should be regular!

The axioms of a space with functions tells us that the **vanishing locus**

$$Z(f) = \{a | f(a) = 0\} \subseteq k^n$$

and furthermore $Z(S)$ should be closed for all $S \subseteq k[x_1, \ldots, x_n]$. This leads us to a definition:

4.2.1 Definition: A subset $Z \subseteq k^n$ is **Zariski-closed** if there exists an $S \subseteq k[x_1, \ldots, x_n]$ such that $Z = Z(S)$.

4.2.2 Lemma
The Zariski closed sets are the closed sets of a topology (called the **Zariski Topology**).

Proof
Just do it. Nike. ✓

4.2.3 Remark: Notice that here the set $\{(a, -a)\} \subseteq k^2$ (the pullback of zero under the addition map) is Zariski closed! This fixes the problem we were running into in the third question (sec. 4.1.3) above.

Now since $Z(S) = Z(I_S)$ where I_S is the ideal generated by S, it is enough to consider vanishing loci of ideals. Furthermore we have the map that extracts the ideal of functions that vanish on a set $Z \subseteq k^n$. There are a ton of great identiies you can prove here. Go to your favorite algebra book (e.g. Dummit & Foote) to see them.

4.2.2 Functions

What about functions on these spaces? If we take $f \in k[x_1, \ldots, x_n]$ these seem like they should be regular functions $k^n \to k$.

4.2.4 Theorem ((Weak) Nullstellensatz)
Say $k = \bar{k}$. Then every maximal ideal $\mathfrak{m} \lhd k[x_1, \ldots, x_n]$ has the form $(x_1 - a_1, \ldots, x_n - a_n)$.

4.2.5 Remark: Equivalently, it is the kernel of a k-algebra morphis $k[x_1, \ldots, x_n] \to k$.

4.2.6 Corollary (Nullstellensatz)
Let J be an ideal of $k[x_1, \ldots, x_n]$. Then $I(Z(J)) = \sqrt{J}$.

Proof
Notice this only works when k is uncountable! Suppose that $\mathfrak{m}$ is a maximal ideal with residue field $L = k[x_i]/\mathfrak{m}$. This gives us a surjection of $k[x_1, \ldots, x_n] \to L$. Thus $\dim_k L$ is countable!

But $\dim_k k(t)$ is uncountable! The proof here is that the $\frac{1}{t-\lambda}$ for $\lambda \in k$ is a linearly-independent collection. So then L/k is algebraic, and since $k = \bar{k}$ $L = k$. ♠

5 October 4, 2019

Today we are going to be talking a bit more about the existence of affine varieties. Max talked a bit about the philosophy of work in this course: he made this extended metaphor concerning butterflies but the take-away is to take learning onto ourselves. :)

5.1 Questions from last time

5.1.1 Maps and elements

In the book we did this silly thing where we defined $\operatorname{Spec} A \stackrel{\text{def}}{=} \operatorname{Hom}_{\mathbf{Alg}_k}(A, k)$ and then idendified A with $k[\operatorname{Spec} A]$ by $a(f) = f(a)$. This seems a bit silly at first, but it may have something to do with the fact that we are looking for a natural way to construct affine varieties without having to choose a presentation. We will hopefully see something about this by the end of the day.

5.2 Back to the Nullstellensatz

Recall that we defined the operators Z and I that "do the work" of the Nullstellensatz. We then wrote (cor. 4.2.6) $I(Z(J)) = \sqrt{J}$. The idea is that this will gives us the function structure on an affine variety.

Proof (Cor. 4.2.6)
One way is not too hard. For the more difficult direction: Let $g \in I(Z(J))$. Then $Z(J) \subseteq Z(g)$. Now notice taht $D(g)$ can be naturally identified with $\operatorname{Spec} k[x_i][1/g]$. Then consider

$$J' = Jk[x_i][1/g]$$

and the key realization is that J' cannot be contained in any maximal ideal. The idea is that you can work by contradiction: this implies that J is contained in an element of $D(g)$, but it isn't!

Thus $J' = (1)$. So we can write $1 = \frac{f}{g^N}$. Thus $g^k(f - g^N) = 0$ in $k[x_i]$ and since g isn't nilpotent, $f = g^N$. ♠

5.2.1 Corollary
There is a lattice anti-isomorphism between the radical ideals in $k[x_i]$ and Zariski-closed subsets $Z \subseteq k^n$ via the maps $J \to Z(J)$ and $Z \mapsto I(Z)$.

5.2.2 Corollary
For any ideal $J \subseteq k[x_i]$,

$$\sqrt{J} = \bigcap_{\text{maximal } \mathfrak{m} \supset J} \mathfrak{m}$$

5.2.3 Remark: "The functions that vanish at the zero locus of J are precisely those that vanish at all the points of J".

5.2.4 Corollary
$D(g) \subseteq k^n$. Then the map
$$k[x_i][1/g] \to \mathrm{Hom}(D(g), k)$$
via the map
$$\frac{f}{g^N} \mapsto \left(x \mapsto \frac{f(x)}{g(x)^n} \right)$$
is injective.

5.3 Affine space

Let's define $\mathbb{A}^n \stackrel{\text{def}}{=} k^n$ with the Zariski topology. Let
$$\mathscr{O}_{\mathbb{A}^n}(U) = \{f \in k(x_1, \ldots, x_n) | \,\mathrm{poles}(f) \subseteq \mathbb{A}^n \setminus U\} \subseteq \mathrm{Hom}(U, k)$$
Then, for example,
$$\mathscr{O}_{\mathbb{A}^n}(D(g)) = k[x_1, \ldots, x_n][1/g].$$

5.3.1 Proposition
$\mathbb{A}^n$ is an affine variety.

Proof
$\phi : X \to \mathbb{A}^n$ gives us maps $\phi_1, \ldots, \phi_n : X \to k$. Then that $\mathbb{A}^n$ is affine relies on the fact: ϕ is a morphis if and only if the ϕ_i are regular. One direction is not too bad since coordinate functions are regular by the axioms of morphisms. The other direction needs to be completed! DO IT! ♠