Problems from William Waterhouse's *Introduction to Affine Group Schemes*

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1 Affine Group Schemes

Problem 1.1

(a) If R and S are two k algebras and F is a representable functor, show $F(R \times S) \cong F(R) \times F(S)$.

(b) Show there is no representable functor R such that every $F(R)$ has exactly two elements.

(c) Let F be the functor represented by $k \times k$. Show that $F(R)$ has two elements exactly when R has no idempotents besides 0 and 1.

Solution.

(a)

Let A be the k-algebra representing F. Thus $F(R)$ is naturally isomorphic to $\mathrm{Hom}_k(A, R)$ and $F(S) \simeq \mathrm{Hom}(A, S)$. Then define the map $\Phi : \mathrm{Hom}(A, R \times S) \to \mathrm{Hom}(A, R) \times \mathrm{Hom}(A, S)$ via

$$\Phi(\varphi) = (\pi_R \circ \varphi, \pi_S \circ \varphi)$$

where π_X is the canonical projection onto X.

This is surjective since (by the universal property of products) any pair of maps $\varphi_R : A \to R$ and $\varphi_S : A \to S$ factors through the product $R \times S$ and furthermore it does so *uniquely*, giving us injectivity. Thus this map (which is clearly a homomorphism since π_X is) is a bijection.

(b)

By the last problem this is impossible since there are more than 2 k-algebras for any k.

(c)

Let F be such a functor. Consider any $\varphi \in \mathrm{Hom}(k \times k, R) \simeq F(R)$. Assume first that $F(R) \cong \mathbb{Z}/2$ and let r be an idempotent in R.

$\square$

Problem 1.2 Let E be a functor represented by A and let F be any functor. Show that the natural maps $\eta : E \to F$ correspond to elements in $F(A)$.

Solution. Consider the map Φ from natural maps $E \to F$ to elements in $F(A)$ defined by (again leveraging the representability of E)

$$\eta \mapsto \eta(\mathrm{id}_A) \in F(A).$$

Conversely, consider the map Ψ from $F(A)$ to the natural maps $E \to F$ via

$$x \mapsto \xi_x$$

where ξ_x where for any Y and $y \in E(Y) \cong \mathrm{Hom}(A, Y)$ we define the Y^{th} component of ξ_x as

$$\xi_x(y) = F(y)(x) \in F(Y)$$

where (for clarity while I get a grasp here) $F(y) : F(A) \to F(Y)$.

Since we are only looking for a bijection, we only need that these maps are inverses. Consider that for all Y and $y \in E(Y)$,

$$\begin{aligned}\Psi \circ \Phi(\eta)(y) &= \Psi\left(\eta(\mathrm{id}_A)\right)(y)\\ &= \xi_{\eta(\mathrm{id}_A)}(y)\\ &= F(y) \circ \eta(\mathrm{id}_A)\\ &= \eta \circ E(y)(\mathrm{id}_A)\\ &= \eta(y \circ \mathrm{id}_A) = \eta(y)\end{aligned}$$

where above we used the naturality of η along with the fact that $E(y)$ is just precomposition with y. Thus $\Psi \circ \Phi(\eta) = \eta$.

But then for any $x \in F(A)$,

$$\begin{aligned}\Phi \circ \Psi(x) &= \Phi \circ \xi_x\\ &= \xi_x(\mathrm{id}_A)\\ &= F(\mathrm{id}_A)(x)\\ &= \mathrm{id}_{F(A)}(x) = x\end{aligned}$$

completing the proof.

$\square$

Problem 1.3 Let E be a functor represented by A, and let F be any functor. Let $\Psi : F \to E$ be a natural map with surjective component maps. Show there is a natural map $\Phi : E \to F$ with $\Psi \circ \Phi = \mathrm{id}_E$.

Solution. Since in particular Ψ_A is surjective, there is an $x \in F(A)$ such that $\Psi(x) = \mathrm{id}_A$. Then using the map from the last problem, let $\Phi = \xi_x$. Then we can compute for any R and $g \in E(R)$

$$\begin{aligned}\Psi \circ \Phi(g) &= \Psi \circ F(g)(x)\\ &= E(g) \circ \Psi(x)\\ &= E(g)(\mathrm{id}_A)\\ &= g \circ \mathrm{id}_A = g\end{aligned}$$

since $g : A \to R$, so $E(g) : E(A) \to E(R)$, which is just composition with g.

□

Problem 1.5 Write out Δ, ε, and S for the Hopf algebras representing SL_2, μ_n, and α_p.

Solution.

SL_2:

Notice SL_2 is represented by $A = k[X_1, X_2, X_3, X_4]/(X_1X_4 - X_3X_2 - 1)$ so take two elements $f, g \in \mathrm{Hom}(A, R)$ where $f(X_i) = a_i \in R$ and $g(X_i) = b_i \in R$ and notice that we want

$$(f, g)\Delta = h$$

where since

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$$

we want to have that $h(X_i) = c_i$.

So then if $\Delta : A \to A \otimes A$ is defined as follows:

$$X_1 \mapsto X_1 \otimes X_1 + X_2 \otimes X_3$$

□

Problem 1.6 In $A = k[X_{11}, \ldots, X_{nn}, 1/\det]$ representing GL_n, show that $\Delta(X_{ij}) = \sum X_{ik} \otimes X_{kj}$. What is $\varepsilon(X_{ij})$?

Solution. Due to the uniqueness of Δ, ε, and S, we need only find maps satisfying the diagrams. I claim that $\varepsilon(X_{ij}) = \delta_{ij}$. In this case, notice

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta(X_{ij}) = \varepsilon \otimes \mathrm{id}\left(\sum X_{ik} \otimes X_{kj}\right) = \sum \delta_{ik} \otimes X_{kj} = 1 \otimes X_{ij}$$

exactly as we want.

For associativity, notice

$$(\Delta \otimes \mathrm{id}) \circ \Delta(X_{ij}) = \Delta \otimes \mathrm{id}\left(\sum_k X_{ik} \otimes X_{kj}\right) = \sum_k \left(\sum_l X_{il} \otimes X_{lk}\right) \otimes X_{kj}$$

and then the associativity of Δ follows simply from the associativity of the tensor product.

For the last axiom, we compute S such that $(S, \mathrm{id}) \circ \Delta = \iota \circ \varepsilon$ where $\iota : K \to A$ is the map sending $k \mapsto k \cdot 1_A$. That is, we define $S : A \to A$ so that

$$\sum_k S(X_{ik})X_{kj} = \delta_{ij}.$$

We want to leverage the fact that for a fixed i and j, the determinant is

$$\begin{aligned}
\det &= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_l X_{\sigma(l)l} \\
&= \sum_\sigma \mathrm{sgn}(\sigma) X_{\sigma jj} \prod_{l \neq j} X_{\sigma(l)l} \\
&= \sum_i X_{ij} \left(\sum_{\sigma(j)=i} \mathrm{sgn}(\sigma) \prod_{l \neq j} X_{\sigma(l)l}\right)
\end{aligned}$$

and so we want that

$$S(X_{ik}) = \frac{1}{\det} \sum_{\sigma(i)=k} \mathrm{sgn}(\sigma) \prod_{l \neq k} X_{\sigma(l)l}$$

so that when $i = j$,

$$\sum_k S(X_{ik})X_{kj} = \frac{1}{\det} \sum_k X_{kj} \sum_{\sigma(j)=k} \mathrm{sgn}(\sigma) \prod_{l \neq k} X_{\sigma(l)l} = 1 = \delta_{ij}$$

whenever $i \neq j$, however, this equation is the determinant of the matrix where we have replaced the j^{th} column with a copy of the i^{th} column. This is linearly dependent, so

$$\frac{1}{\det} \sum_k S(X_{ik})S_{kj} = 0 = \delta_{ij}.$$

Thus these are precisely the maps we desire.

$\square$

Problem 1.

Solution. a

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