Algebraic Geometry

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Abstract

A three-quarter sequence covering the basic theory of affine and projective varieties, rings of functions, the Hilbert Nullstellensatz, localization, and dimension; the theory of algebraic curves, divisors, cohomology, genus, and the Riemann-Roch theorem; and related topics.

1 September 25, 2019

The first thing that one asks is "what is geometry?" One needs to be able to answer this question before they define AG. One idea is that geometry is topology + structure.

1.1 What is Geometry?

Example 1.1
Exotic differentiable structures on a sphere. There are many different smooth structures, all of which are independent of the topology,

$S^1 \times S^1$ has infinitely many complex structures (remember the parallelograms)!

How to you go about defining the geometry of a thing? One idea from manifolds: charts. These describe the local models and the interesting part is how this comes together to a whole space.

There is another idea to capture the "local" model of geometry that underlies modern algebraic geometry: consider the map $\varphi : W \to W' \in \mathbb{CP}^n$ and then say that this map is C^∞ if and only if its coordinate functions are. But the coordinate functions are problematic, so we can replace it with the following idea:

$\phi : W \to W'$ is $C\infty$ if and only if for all C^∞ functions $f : W' \to \mathbb{R}$, the composition

$$\varphi^* f = f \circ \varphi : W \to \mathbb{R}$$

is C^∞.

To capture the manifold structure on M, it is equivalent to know the set of C^∞ fucnctions $U \to \mathbb{R}$ for every open $U \subseteq M$.

1.2 The Big Idea

So then the idea we are talking away here is that *geometry is in the functions* that exist on a particular space!

Fix a field k.

1.2.1 Definition: A **space with functions** is a topological space X along with a collection (a k-algebra!) $\mathcal{O}(U)$ of maps $U \to k$ for each open $U \subseteq X$.

$\mathcal{O}(U)$ are called **regular functions** and must satisfy:

- Given an open cover U_α of U, a function is regular if and only if its restrictions to each element of the cover is regular.
- If $f : U \to k$ is regular, then $D(f) = \{u \in U | f(u) \neq 0\}$ is an open set and $\frac{1}{f} \in \mathcal{O}(D(f))$.

For the next time, try to think of as many examples of this as you can. Next time will be a mind blowing example of a variety.

2 September 27, 2019

Problem 2.1
Do all the exercises in Kempf chapter 1!

For now we assume that k is algebraically closed.

2.1 Examples of spaces with functions

There were lots of suggestions, but here are a couple:

Example 2.1
Let $X = \mathbb{S}^2$ and let $\mathcal{O}_X^{cts}$ be the continuous $\mathbb{C}$-valued functions. Alternatively we could consider a different sheaf $\mathcal{O}_X^{an}$, the holomorphic functions. Or we could consider $\mathcal{O}_X^{\infty}$, the C^∞ functions (under some smooth structure).

2.1.1 Definition: A **morphism** of spaces with functions between $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a continuous map $\varphi : X \to Y$ such that for all $U \subseteq Y$ open and $f \in \mathcal{O}_Y(U)$, the function

$$\phi^* f = f \circ \phi|_{\phi^{-1}(U)} : \phi^{-1}(U) \to k \in \mathcal{O}_X(\phi^{-1}(U))$$

In other words, a morphism of spaces with functions is a map of spaces that *respects the regular functions.*

Example 2.2
Let X, Y be topological spaces and let $\mathcal{O}_X$ and $\mathcal{O}_Y$ be the continuous functions. Then morphisms are just continuous maps.

Example 2.3
When X and Y are manifolds and $\mathcal{O}_\bullet$ are complex-valued functions, then the maorphisms are maps of manifolds.

So now we return to the examples we saw before: $(\mathbb{S}^2, \mathcal{O}^\infty)$, $(\mathbb{S}^2, \mathcal{O}^{cts})$, and $(\mathbb{S}^2, \mathcal{O}^{an})$. A natural question to ask is when we have morphisms between these spaces to see if there exist ones that are the identity on $\mathbb{S}^2$.

Consider the identity map from the continuous to the analytic functions. Then take any map $f \in \mathcal{O}^{an}$ and consider that

$$f = f \circ id_{id^{-1}(U)} : U \to k \in \mathcal{O}^{cts}(U)$$

and there is no map in the other direction.

2.1.2 Remark: Notice that since we are pulling functions back, the maps go in the opposite direction as you may think at first.

We can also talk about **open subspaces**. If $V \subseteq X$ is an open subset, we can let the induced space with functions be $(V, \mathcal{O}_V)$ where if $U \subseteq V$ then $\mathcal{O}_V(U) := \mathcal{O}_X(U)$.

2.2 Varieties

2.2.1 Definition: An **affine k-variety** is a space with functions $(Y, \mathcal{O}_Y)$ such that for every space with functions $(X, \mathcal{O}_X)$, the natural map

$$\mathrm{Hom}((X,\mathcal{O}_X),(Y,\mathcal{O}_Y)) \to \mathrm{Hom}_{\mathbf{Alg}_k}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$$

is a bijection and furthermore $\mathcal{O}_Y(Y) =: k[Y]$ is a finitely generated k-algebra.

2.2.2 Remark: The idea here is that the algebra maps (on the right) are precisely the same as the geometry maps (on the left). Algebraic geometry, baby.

So then this leads to a very simple (loose) definition:

2.2.3 Definition: A **variety** is something that is covered by affine varieties.

Example 2.4
$\mathbb{A}^1 = k$. Give this space the cofinite topology. Then if we have $U = k \setminus \{x_1, \ldots, x_n\} \subset \mathbb{A}^1$,

$$\mathcal{O}_{\mathbb{A}^1}(U) = \{f(t) \in k(t) | \text{poles are in } \{x_i\}\}$$

Problem 2.2
Show that $\mathbb{A}^1$ is an affine variety!

2.2.4 Remark: Notice that this staetment is equivalent to saying that any morphism of spaces with functions gives us a regular map $X \to k$.