

# Homological Algebra

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## Introduction

These are the notes I took in class during the Winter 2019 topics course *Math 509* - *Homological Algebra* at University of Washington, Seattle.

The course description follows:

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This course is an introductory course on homological algebra. We will be following the book *An Introduction to Homological Algebra* by Charles Weibel. We will be covering the following topics:

- Chain complexes, homotopies, homology and long exact sequence in homology
- Resolutions, derived functors, Ext and Tor. Koszul complexes
- Group (co)homology
- Triangulated and derived categories
- Spectral sequences or open topic depending on the class interests

## 1 April 1, 2019

The idea here is that we are going to spend a significant amount of time doing traditional lectures. After a while we are going to be doing some presentations. We have two weeks to pick groups (and topics, if desired) before we are assigned to them randomly.

Again, we are using Weibel's book. At the beginning we are going to be following the text quite closely, so that is a perfect reference.

### 1.1 Chain Complexes, Maps, Homotopies, and Homology

Let  $R$  be a ring. In general we will say it is unital and associative, but not much more. We will be generally working in the category  $\mathbf{Mod}_R$  of (left)  $R$ -modules.

$$\begin{array}{ccc}
C_n & \xrightarrow{d_n} & C_{n-1} \\
\downarrow f_n & & \downarrow f_{n-1} \\
D_n & \xrightarrow{d'_n} & D_{n-1}
\end{array}$$

Figure 1: Commuting Figure for Chain Maps

**1.1.1 Definition**

A chain complex  $C_\bullet$  is a sequence of  $R$ -modules and  $R$ -module homomorphisms

$$d_n : C_n \rightarrow C_{n-1}$$

where the  $C_n$  are  $R$ -modules and  $f_n$  are  $R$ -module homomorphisms, subject to the relation

$$d_n \circ d_{n-1} = 0.$$

The first HW problem:

**Problem 1.1**

Show that  $\mathbf{Ch}$ , the category of chain complexes forms an abelian category.

**1.1.2 Definition**

Given two chain complexes  $C_\bullet$  and  $D_\bullet$ , then  $f_\bullet : C_\bullet \rightarrow D_\bullet$  is a **chain map** if

$$f_n : C_n \rightarrow D_n$$

is a map of  $R$ -modules and the obvious diagram commutes. (figure 1.1)

**1.1.3 Definition**

Let  $C_\bullet$  be a chain complex. Then  $Z_n = Z_n(C_\bullet)$  is defined to be

$$Z_n = \ker d_n$$

called the  $n$ -**cycles** of  $C_\bullet$ .

Similarly, define

$$B_n = \operatorname{Im} d_{n+1}$$

the  $n^{\text{th}}$  **boundary** of  $C_{\bullet}$ .

Then  $H_n = Z_n/B_n$ , the  $n^{\text{th}}$  **homology group**.

## 1.2 Some Notation

The category  $\mathbf{Ch}_{\bullet}$  (double underline!!!) is the category of (unbounded) complexes, but we are also sometimes interested in  $\mathbf{Ch}_{\bullet}^b$ , the category of **bounded** complexes. Furthermore, we may consider  $\mathbf{Ch}_{\bullet}^{\geq 0}$  or  $\mathbf{Ch}_{\bullet}^{\leq 0}$ .

Furthermore, there are cochain complexes, where we switch the direction and the notation is changed as we do for any dual notion.

## 1.3 Types of Complexes

### 1.3.1 Definition

A complex  $C_{\bullet}$  is called **exact** if it is an exact sequence. Equivalently the homology groups are trivial.

1.3.2 REMARK: A complex with trivial homology is also called **acyclic**. This is obviously equivalent to being exact.

### 1.3.3 Lemma

Let  $f : C_{\bullet} \rightarrow D_{\bullet}$  be a map of chain complexes. Then it induces a well-defined map on cycles, boundaries, and homology.

PROOF

Homework. ♠

### 1.3.4 Definition

Let  $f : C_{\bullet} \rightarrow D_{\bullet}$  be a chain map. It is called a **quasi-isomorphism** if it induces isomorphisms on homology groups.

1.3.5 REMARK: An acyclic complex  $C_{\bullet}$ , then  $C_{\bullet} \rightarrow 0$  is a quasi-isomorphism.

## 1.4 Snake Lemma

### 1.4.1 Lemma (Snake)

Given a commutative diagram with exact rows as in figure 1.4, we have a long exact sequence

$$\ker f \rightarrow \ker g \rightarrow \ker h \xrightarrow{\delta} \operatorname{coker} f \rightarrow \operatorname{coker} g \rightarrow \operatorname{coker} h \rightarrow 0$$

$$\begin{array}{ccccccc}
A' & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0 \\
\downarrow f & & \downarrow g & & \downarrow h & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C
\end{array}$$

Figure 2: Snake Lemma

## 1.5 An Example

We talked about singular homology. Cool!

## 2 April 3, 2019

### 2.1 Long Exact Sequence of Homology

#### 2.1.1 Definition

A short exact sequence  $0 \rightarrow A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \rightarrow 0$  of chain complexes is a width four double complex with exact rows.

#### 2.1.2 Theorem

Let  $0 \rightarrow A_\bullet \rightarrow B_\bullet \rightarrow C_\bullet \rightarrow 0$  be a short exact sequence of complexes. Then there exists a (natural) long exact sequence in homology:

$$\cdots \rightarrow H_n(A) \rightarrow H_n(B) \rightarrow H_n(C) \rightarrow H_{n-1}(A) \rightarrow \cdots$$

**2.1.3 REMARK:** The statement of naturality just means that the maps above are induced from those in the SES under application of  $H_n$  functor.

PROOF

Snaaaaaaaaake.

(Sketch) Use first the fact that  $d : A_n \rightarrow A_{n-1}$  factors through both  $A_n/B_n(A_\bullet)$  and  $Z(A_\bullet)$ , so using the induced maps between these quotients to get a nice little snake-like diagram:

$$\begin{array}{ccccccc}
A_n/B(A_n) & \longrightarrow & B_n/B(B_n) & \longrightarrow & C_n/B(C_n) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_{n-1}(A) & \longrightarrow & Z_{n-1}(B) & \longrightarrow & Z_{n-1}(C) \longrightarrow 0
\end{array}$$

where the exactness of the rows above follows from applications of snake lemma to the original SES of chains.

A final application of snake gets us the SES we desire.



## 2.2 Some Topology

### 2.2.1 Definition

Let  $f : C_\bullet \rightarrow D_\bullet$  be a chain map. We say  $f$  is **nullhomotopic** if there are  $s_n : C_n \rightarrow D_{n+1}$  for all  $n$  such that

$$f_n = d_{n+1} \circ s_n + s_{n-1} \circ d_n.$$

Sometimes we just write  $f = ds + sd$  and let the indices figure themselves out.

### 2.2.2 Definition

If  $f, g : C_\bullet \rightarrow D_\bullet$  are chain maps, we say that they are **chain homotopic** if  $f - g$  is nullhomotopic.

Here we write  $f \sim g$ , so  $f \sim 0$  means  $f$  is nullhomotopic.

### 2.2.3 Definition

We say  $C_\bullet$  and  $D_\bullet$  are **chain homotopy equivalent** if there exist maps  $f : C \rightarrow D$  and  $g : D \rightarrow C$  such that  $f \circ g \sim \text{id}_D$  and  $g \circ f \sim \text{id}_C$ .

### 2.2.4 Definition

We say a complex  $C_\bullet$  is **contractible** if  $\text{id}_C \sim 0$ .

**2.2.5 REMARK:** Some motivation: say we have  $X, Y$  topological spaces and  $f, g : X \rightarrow Y$  continuous maps. Then  $f \sim g$  if there is a continuous map  $H : X \times [0, 1] \rightarrow Y$  such that  $H(x, 0) = f(x)$  and  $H(x, 1) = g(x)$ .

Now if  $f \sim g$ , then the natural (induced) maps  $S_\bullet(f)$  and  $S_\bullet(g)$  on singular chain complexes are chain homotopic.

### 2.2.6 Theorem

If  $f, g : C_\bullet \rightarrow D_\bullet$  are chain homotopic, then they induce the same map on  $H_\bullet$ .

PROOF

We will be using that homology is an additive functor:  $H_n(f) - H_n(g) = H_n(f - g)$ . Then it suffices to show that if  $f \sim 0$  then  $H_n(f) = 0$  for all  $n$ .

Assume  $f \sim 0$  with homotopy  $s_\bullet : C_\bullet \rightarrow D_\bullet[1]$ . Then we have

$$\begin{array}{ccccc} C_{n+1} & \xrightarrow{d} & C_n & \xrightarrow{d} & C_{n-1} \\ & \swarrow s & \downarrow f & \swarrow s & \\ D_{n+1} & \xrightarrow{d} & D_n & \xrightarrow{d} & D_{n-1} \end{array}$$

and so if  $a \in Z_n(C)$ , then say  $H_n(f)(a) = f(a)$ . But

$$f(a) = (ds + sd)(a) = d_{n+1}(s(a)) + s(d_n(a)) = d_{n+1}(s(a)) \in B_n(D). \spadesuit$$

**2.2.7 REMARK:** We can define the **homotopy category**  $K_\bullet(\mathbf{Mod}_R)$  or  $K_\bullet(\mathbf{Ch}_\bullet)$  where the objects are chain complexes and the morphisms are chain homotopy equivalence classes of maps.

This category is additive and in our homework we will investigate whether it is abelian. So then our homology functors  $H_n : \mathbf{Ch}(R) \rightarrow \mathbf{Ab}$  factor through  $K_\bullet(R)$ .

## 2.3 Split exact complexes

### 2.3.1 Definition

$C_\bullet$  is a **split exact sequence** if it is an exact sequence that splits. That is,

$$C_n = B_{n+1}(C_\bullet) \oplus B_n(C_\bullet)$$

or since the sequence is exact, the sum of cycles.

The boundary maps simply map isomorphic summands to one another.

### Example 2.1

$R = k$ . Then  $\mathbf{Mod}_R$  is actually  $\mathbf{Vect}_k$ . All exact sequences here split.

## 3 April 5, 2019

### Example 3.1

Continuing from our example on Wednesday, let  $R = k$  and consider  $\mathbf{Ch}(k)$ . Everything in  $\mathbf{Vect}_k$  splits, so we can write  $C_n = Z_n \oplus C_n/Z_n \cong Z_n \oplus B_{n-1}$ .

But then  $C_n \cong B_n \oplus H_n \oplus B_{n-1}$ , so  $C$  is exact iff  $H_n = 0$  (it is acyclic) iff  $C$  is split exact.

**3.0.1 REMARK:** In general, split exact implies acyclic, but the converse is not true.

**Example 3.2**

Consider  $\mathbf{Ch}(\mathbb{Z})$  and consider the chain of  $\cdots \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/4 \rightarrow \cdots$  with differential  $*2$ .

Notice that  $H_n(C) = 0$ , but it is NOT nullhomotopic/contractible! This is because  $\text{id} = sd + ds$  maps only to even numbers!

**3.1 Operations with complexes**

We can do some operations on complexes  $C \in \mathbf{Ch}$ :

- $(C[d])_i = C_{i+d}$
- Total complexes

**3.1.1 Definition**

A **double (chain) complex** is a collection of  $R$ -modules and maps  $C_{pq}$

$$\begin{array}{ccc} C_{pq} & \xrightarrow{d^h} & C_{p-1q} \\ \downarrow d^v & & \downarrow d^v \\ C_{pq-1} & \xrightarrow{d^h} & C_{p-1q-1} \end{array}$$

such that  $(d^h)^2 = (d^v)^2 = d^h \circ d^v + d^v \circ d^h = 0$ .

**Example 3.3**

Let  $f : C \rightarrow C'$  be a chain map. Then

$$\begin{array}{ccc} C_{n+1} & \xrightarrow{-f} & C'_{n+1} \\ \downarrow -d & & \downarrow d' \\ C_n & \xrightarrow{-f} & C'_n \\ \downarrow -d & & \downarrow d' \\ C_{n-1} & \xrightarrow{-f} & C'_{n-1} \end{array}$$

Is a double complex. Notice we also could use  $f$  instead of  $-f$  but for some reason this is usually what we want.

**3.1.2 Definition**

Let  $(C_{pq}, d)$  be a double complex. Then

$$\mathrm{Tot}_n(C_{\bullet\bullet}) = \bigoplus_{p+q=n} C_{pq}$$

with differential  $d = d^v + d^h$ . This is a chain complex.

**3.1.3 REMARK:** Notice that  $(d^h + d^v)^2 = d^h \circ d^h + d^h \circ d^v + d^v \circ d^h + d^v \circ d^v = 0$ . So this is indeed a chain complex.

**3.1.4 REMARK:** Notice that we have technically used the fact that our category contains coproducts to define  $\mathrm{Tot} = \mathrm{Tot}^\oplus$  as we just did. These should exist in an abelian category.

**Example 3.4**

Let  $f : B \rightarrow C$  be a chain map. Then  $\mathrm{Cone}(f)$  is the total complex of the associated double complex of  $f$ .

Here  $(\mathrm{Cone}(f))_n = B_{n-1} \oplus C_n$  (careful, the indices are a bit wonky here.) Then the differential is given by  $\begin{pmatrix} -d_B & 0 \\ -f & d_C \end{pmatrix}$ .

**3.1.5 REMARK:** Notice that

$$B_\bullet \xrightarrow{f} C_\bullet \rightarrow (\mathrm{Cone}(f))_\bullet \rightarrow B_\bullet[-1]$$

is a short(!) exact sequence of complexes.

**3.1.6 REMARK:** This idea comes from an idea in topology: given a continuous map  $f : X \rightarrow Y$

**April 8, 2019**

Some quick comments about the topological analogy we discussed last time (I trailed off because there was a question about what the map between topological spaces induces this map on homology):

The idea is that although the image of  $f$  is a deformed copy of  $X$  in  $\mathrm{Cone}(f)$ , the suspension  $\Sigma X$  is somehow recovered in the cone over  $\mathrm{Im} f$ .

This is a bit hand-wavy, but we then recover via the long exact sequence in homology the sequence we were considering last time.



## Derived Functors, Ext, and Tor

Recall that we are working over  $R\text{-mod}$ .

### 3.1.7 Definition

An  $R\text{-mod}$   $P$  is projective if one of the following equivalent conditions hold:

- This diagram commutes:

$$\begin{array}{ccc} & P & \\ \tilde{f} \swarrow & \downarrow f & \\ M & \xrightarrow{\varphi} & N \end{array}$$

- $\text{Hom}_R(P, -) : R\text{-mod} \rightarrow \mathbf{Ab}$  is an exact functor.
- $P$  is a direct summand of a free module. *Notice that this only works in  $R\text{-mod}$*

3.1.8 REMARK: Notice that we are generally assuming (as in the definition of an exact functor) that our functors are additive – that is, it preserves sums.

A question one may ask: what are the projective objects in  $\mathbf{Ch}(R)$ ? This will be the topic of a homework question.

### 3.1.9 Definition

A **projective resolution**  $P_\bullet$  of an  $R$  module  $M$  is a complex

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0$$

of projective  $R$  modules such that if we tack on  $\varepsilon : P_0 \rightarrow M$ , it becomes exact.

Equivalently, the chain map induced by  $(\cdots, 0, \varepsilon) : P_\bullet \rightarrow (\cdots \rightarrow 0 \rightarrow M \rightarrow 0)$  is a quasi-isomorphism.

### 3.1.10 Definition

A category  $\mathcal{A}$  has **enough projectives** if, for all  $A \in \mathcal{A}$ , there exists a projective  $P \in \mathcal{A}$  and an epimorphism  $P \rightarrow A$ .

### 3.1.11 Lemma

$R\text{-mod}$  has enough projectives.

PROOF

Let  $P_0$  be any free module over  $M$  (say the free module on all elements of  $M$ ) and say  $\varepsilon : P_0 \rightarrow M$  is the covering map. and let  $\Omega M = \ker \varepsilon$ , the zeroth syzygy of  $M$ . Then let  $P_1$  be a projective cover of  $\Omega_0 M$  and continue in the fashion. ♠

### 3.1.12 Corollary

Any  $R$ -module has a projective resolution.

## Fundamental Theorem of Homological Algebra

### 3.1.13 Theorem (Fundamental Theorem)

If  $P_\bullet$  is a projective complex with map  $\varepsilon_M : P_0 \rightarrow M$  (not necessarily projective) and  $Q_\bullet \rightarrow N$  is a projective resolution of  $N$  and  $f : M \rightarrow N$  is an  $R$ -linear map, then there exists an  $F : P_\bullet \rightarrow Q_\bullet$  extending  $f$ . That is it makes following diagram commute:

$$\begin{array}{ccc} P_\bullet & \xrightarrow{\varepsilon_M} & M \\ \downarrow F & & \downarrow f \\ Q_\bullet & \xrightarrow{\varepsilon_N} & N \end{array}$$

Furthermore  $F$  is unique up to chain homotopy.

3.1.14 REMARK: Notice that while any two extensions are chain homotopic, this doesn't mean there exists a *unique* homotopy between them and this is the cause of many problems in homological algebra.

PROOF

Begin by lifting  $f \circ \varepsilon_M$  to a map  $f_0 : P_0 \rightarrow Q_0$  using the definition of projectivity. Then construct  $f_n$  by induction. Consider

$$\begin{array}{ccccc} P_n & \xrightarrow{d^P} & P_{n-1} & \longrightarrow & P_{n-2} \\ & \searrow & \downarrow f_{n-1} & & \downarrow f_{n-2} \\ Q_n & \longrightarrow & Q_{n-1} & \xrightarrow{d^Q} & Q_{n-2} \end{array}$$

and by the induction hypothesis,  $d^Q \circ f_{n-1} \circ d^P = 0$  whence  $\varphi : P_n \rightarrow Q_{n-1}$  maps to  $\ker d_{n-1}^Q = \text{Im } d_n^Q$ . But this gives us a map  $P_n \rightarrow \text{Im } d^Q$  onto which  $Q_n$  surjects, so again we leverage the projectivity of  $P_n$  to lift to  $f_n$ .

To see the uniqueness, assume  $F$  and  $F'$  are two liftings of  $f$ . We want to show  $H = F - F' \sim 0$ . It will be sufficient once we see that any lifting of  $0 : M \rightarrow N$  is nullhomotopic. We will see this next time. ♠

## 4 April 10, 2019

### 4.0.1 Theorem

... and it is unique up to chain homotopy.

**PROOF**

Say that  $F$  and  $F'$  are two chain maps extending  $f : M \rightarrow N$  in the above scenario. Notice that since we want to show  $F \sim F'$ , it suffices to show that  $0 : M \rightarrow N$  induces a nullhomotopic map  $H$ .

Let  $H$  be a lifting of  $0$  and define  $s_i$  such that

$$\begin{array}{ccccc} P_1 & \longrightarrow & P_0 & \longrightarrow & M \\ \downarrow h_1 & \swarrow s_0 & \downarrow h_0 & \swarrow s_{-1} & \downarrow 0 \\ Q_1 & \longrightarrow & Q_0 & \xrightarrow{\varepsilon_N} & N \end{array}$$

where we need to have that  $h_0 = s_0 \circ d_Q$  for some  $s_0$ . we know that  $h_0(P_0) \subseteq \ker \varepsilon_N = d_Q(Q_1)$  so lifting  $h_0$  through  $Q_1 \twoheadrightarrow d(Q_1)$ , we get a map  $s_0 : P_0 \rightarrow Q_1$  (since  $P_0$  is projective).

Continuing my induction, consider

$$\begin{array}{ccccc} P_{n+1} & \longrightarrow & P_n & \longrightarrow & P_{n-1} \\ \downarrow h_{n+1} & \swarrow s_n & \downarrow h_n & \swarrow s_{n-1} & \downarrow h_{n-1} \\ Q_1 & \longrightarrow & Q_0 & \xrightarrow{\varepsilon_N} & Q_{n-1} \end{array}$$

and consider the map  $s_n = h_n - (s_{n-1} \circ d_P)$ , which maps  $P_n$  into  $d_Q(Q_{n+1})$ . Now  $d_Q(Q_{n+1}) = \ker d_Q$  (easy lemma) This establishes that  $s_n$  exists and satisfies all the requisite properties. ♠

**4.0.2 Corollary**

Any two projective resolutions of  $M$  induce the same homology.

**4.0.3 REMARK:** Lift the identity map  $M \rightarrow M$  in both directions to get chain maps  $(P \rightarrow M) \rightarrow (Q \rightarrow M)$  and backwards. These compose to the identity map! Then apply the theorem!

**4.0.4 Lemma (Horseshoe)**

(Weibel 2.2.8) Say that we have a short exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  and projective resolutions  $P_\bullet^A$  and  $P_\bullet^C$  of  $A$  and  $C$ , respectively. Then  $P_\bullet^B = (P_i^A \oplus P_i^C)_{i \in \mathbb{Z}}$  is a resolution of  $B$  and furthermore we have maps for the following diagram (which commutes everywhere):

$$\begin{array}{ccccccc} P_2^A & \longrightarrow & P_1^A & \longrightarrow & P_0^A & \longrightarrow & A \\ & & \downarrow & & \downarrow & & \downarrow \\ & & P_1^A \oplus P_1^C & \longrightarrow & P_0^A \oplus P_0^C & \longrightarrow & B \\ & & \downarrow & & \downarrow & & \downarrow \\ P_2^C & \longrightarrow & P_1^C & \longrightarrow & P_0^C & \longrightarrow & C \end{array}$$

## 4.1 Left Derived Functors

### 4.1.1 Definition

Let  $\mathcal{A}, \mathcal{B}$  be Abelian categories and let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be an additive functor. We say that  $F$  is **right exact** if for every exact sequence  $A \rightarrow B \rightarrow C \rightarrow 0$  in  $\mathcal{A}$ ,

$$F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow 0$$

is exact in  $\mathcal{B}$ .

### 4.1.2 Definition

Assume  $\mathcal{A}$  is an abelian category with enough projective and that  $\mathcal{B}$  is abelian. Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a right exact functor. Then  $L_i F$  for  $i \geq 0$  are called **the left derived functors of  $F$**  which are defined as follows:

For all  $A \in \mathcal{A}$ , let  $P_\bullet \rightarrow A$  be a projective resolution. Then we have an exact sequence

$$\cdots \rightarrow F(P_2) \rightarrow F(P_1) \rightarrow F(P_0)$$

and we define  $(L_i F)(A) := H_i(F(P_\bullet))$ .

4.1.3 REMARK: (a)  $L_i F$  are functors because of the fundamental theorem.

(b) The functors are well-defined because of the lemma.

The derived functors have some nice properties:

- $L_i F = 0$  for  $i < 0$ .
- Say  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is short exact and  $F$  is right exact. Then we get a long exact sequence

$$\cdots \rightarrow L_1 F(A) \rightarrow L_1 F(B) \rightarrow L_1 F(C) \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow 0$$

## 5 April 17, 2019

Last time we wrote down the definition for (left) derived functors, but never wrote out why Tor is an example.

## 5.1 Examples of $L_i$

Let  $R$  be a ring and  $M$  and  $N$  be right- and left- $R$ -modules, respectively. Then  $M \otimes_R N$  is the set of  $m \otimes n$  for  $m \in M$  and  $n \in N$  where  $m \cdot a \otimes n = m \otimes a \cdot n$  for all  $a \in R$ .

This can also be defined formally as a bifunctor  $- \otimes_R - : R\text{-}\mathbf{mod} \times \mathbf{mod}\text{-}R \rightarrow \mathbf{Ab}$ , or as a regular functor by currying. So then

### 5.1.1 Definition

$$\mathrm{Tor}_i^R(M, N) := L_i(M \otimes_R -)(N)$$

and

$$\mathrm{Tor}_i^R(M, N) = L_i(- \otimes_R M)(N)$$

**5.1.2 REMARK:** Notice that the two definitions are completely equivalent and compatible. That is,  $\mathrm{Tor}_i^R(M, N)$  is what we call **balanced**.

To see this you can do a few computations including computing the homology of the total complex to show that you always get the same thing.

## 5.2 Injective Modules

### 5.2.1 Definition

Let  $\mathcal{A}$  be an abelian category. Then  $E \in \mathcal{A}$  is injective if we have the diagram

$$\begin{array}{ccc} M & \hookrightarrow & N \\ \downarrow f & \nearrow \tilde{f} & \\ E & & \end{array}$$

### 5.2.2 Lemma

If  $\mathcal{A} = R\text{-}\mathbf{mod}$ , then in the above definition it suffices to check that maps  $f : I \rightarrow E$  from every *ideal* lift to maps  $\tilde{f} : R \rightarrow E$  from  $R$ .

### Example 5.1

Injective  $\mathbb{Z}$ -modules are precisely the divisible abelian groups. For instance,  $\mathbb{Q}$ ,  $\mathbb{Q}/\mathbb{Z}$ ,  $\mathbb{Z}_{p^\infty} = \mathbb{Z}_{(p)}/\mathbb{Z}$ .

In fact:

### 5.2.3 Lemma

The indecomposable injective  $\mathbb{Z}$  modules are  $\mathbb{Q}$  and  $\mathbb{Z}_{p^\infty}$ .

### 5.2.4 Theorem

For any ring  $R$ ,  $\mathbf{mod}\text{-}R$  has enough injectives.

### 5.2.5 Corollary

Every  $M \in \mathbf{mod}\text{-}R$  has an injective resolution.

PROOF

It suffices to show that for all  $M \in \mathbf{mod}\text{-}R$ , there is an injective  $R$ -module  $E$  such that  $M \hookrightarrow E$ .

To begin, we will see that  $\mathbb{Z}\text{-}\mathbf{mod} = \mathbf{Ab}$  has enough injectives. Let  $A \in \mathbf{Ab}$  and define the notation

$$A^\vee = \text{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z}).$$

Then  $\mathbb{Z}^\vee = \mathbb{Q}/\mathbb{Z}$ , since any map is determined uniquely by its image. So then if  $F = \mathbb{Z}^n$ , we use linearity of  $\text{Hom}$  to prove  $F^\vee = (\mathbb{Q}/\mathbb{Z})^n$ , so all free groups are divisible (whence injective).

Finally, for all  $A$ , there is a natural map  $A \rightarrow A^{\vee\vee}$ . Now for any  $A \in \mathbf{Ab}$ , there exists a surjective map  $F \rightarrow A^\vee$  from a free module onto the dual of  $A$ . Thus we get an embedding  $A^{\vee\vee} \hookrightarrow F^\vee$  (note that  $\text{Hom}(-, \mathbb{Q}/\mathbb{Z})$  is exact). Since  $A \hookrightarrow A^{\vee\vee}$ , this gets us an embedding of  $A$  into  $F^\vee$ .

Now let  $R$  be any ring. Let  $A \in \mathbf{Ab}$ , then the claim is that  $\text{Hom}_{\mathbb{Z}}(R, A)$  is a (right)  $R$ -module via the action  $(f \cdot a)(b) = f(ba)$  (**check this**).

Next, we claim that there exists a natural isomorphism for all  $M \in \mathbf{mod}\text{-}R$  and  $A \in \mathbf{Ab}$  such that

$$\text{Hom}_{\mathbb{Z}}(M, A) \cong \text{Hom}_R(M, \text{Hom}_{\mathbb{Z}}(R, A)).$$

We claim that one can “follow their nose” here.

Using these facts, we will finish the proof of the main theorem: Let  $A$  be a divisible abelian group. Then  $\text{Hom}_{\mathbb{Z}}(R, A)$  is injective. This is because

$$\text{Hom}_R(-, \text{Hom}_{\mathbb{Z}}(R, A))$$

is exact since  $A$  is injective (by the last claim).

Let  $M$  be any right  $R$ -module. Then in particular it is an abelian group, so  $M \hookrightarrow A \in \mathbf{Ab}$ , where  $A$  is divisible. Thus (again by the last fact) we have a map  $\varphi : M \rightarrow \text{Hom}_{\mathbb{Z}}(R, A)$  is a map of  $R$ -modules. Then it suffices to show that  $\varphi$  is injective.

This follows from the explicit construction of the natural isomorphism between the functors. 

5.2.6 REMARK: The map referred to above takes  $f \in \text{Hom}_{\mathbb{Z}}(M, A)$  to a map  $\varphi_f \in \text{Hom}(M, \text{Hom}_{\mathbb{Z}}(R, A))$  where

$$\varphi_f(a)(r) = (f \cdot a)(r).$$

To show it is injective we just construct the inverse map: let  $\psi_g$  for any  $g : M \rightarrow \text{Hom}_{\mathbb{Z}}(R, A)$  to be

$$\psi_g(a) = (g(a))(1).$$

One can easily check that this composes to the identity in each direction.

## 6 April 19, 2019

I came late to the discussion, but it looks like we defined left exact contravariant functors – those are functors that are left exact as functors  $F : C^{op} \rightarrow D$ .

6.0.1 REMARK:  $\text{Hom}_R(-, W)$  is left exact.

From here, we can continue the sequence to a long exact sequence of homology. The point of this section was to show that we can compute  $\text{Ext}_R^i(V, W)$  *either* from a projective resolution of  $V$  or an injective resolution of  $W$ .

### 6.0.2 Lemma

$P$  is a projective  $R$  module if and only if  $\text{Ext}_R^i(P, -) = 0$  for all  $i > 0$ .

### 6.0.3 Lemma

$I$  is an injective  $R$  module if and only if  $\text{Ext}_R^i(-, I) = 0$  for all  $i > 0$

## 6.1 Adjoint Functors

### 6.1.1 Definition

We say the functors  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$  and  $\mathcal{G} : \mathcal{B} \rightarrow \mathcal{A}$  are **adjoint functors** if there exists a natural isomorphisms

$$\eta_{AB} : \text{Hom}_{\mathcal{B}}(\mathcal{F}(A), B) \rightarrow \text{Hom}_{\mathcal{A}}(A, \mathcal{G}(B))$$

and in this case we say  $\mathcal{F}$  is left adjoint to  $\mathcal{G}$ .

### Example 6.1

Recall we had the functors  $\mathcal{F}$  and  $\mathcal{G}$  from  $\mathbb{Z}$ -modules to  $R$ -modules and back, respectively, where  $\mathcal{F} = \text{Hom}_{\mathbb{Z}}(\mathcal{R}, -)$ , assigning to each  $\mathbb{Z}$  module an  $R$ -module with the action we discussed last time.

Then  $\mathcal{G} = \text{Res}_{\mathbb{Z}}^{\mathcal{R}}$ , the forgetful functor, is its left adjoint.

**Example 6.2**

In rep theory we have Frobenius Reciprocity! Woo.

**7 April 22, 2019**

Recall the tensor-hom adjunction:  $- \otimes_S B$  and  $\text{Hom}_R(B, -)$  are adjunctions between **mod**- $R$  and **mod**- $S$ . There is some shadiness here (mirroring what we saw before) but the fact that

$$(f.s)(b) = f(sb)$$

defines a right  $S$  action on  $\text{Hom}_R(B, C)$  is not immediately evident to me.

Again, many of the common adjunctions we see are actually just special cases of tensor-hom.

**7.1 Why Adjointness Matters****7.1.1 Lemma**

If  $\mathcal{F}$  and  $\mathcal{G}$  are adjoints ( $\mathcal{F}$  is left adjoint to  $\mathcal{G}$ ), then  $\mathcal{F}$  is right exact and  $\mathcal{G}$  is left exact.

PROOF

Let  $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$  be a short exact sequence. Then consider the diagram

$$\begin{array}{ccccccc} \text{Hom}_{\mathcal{A}}(A', \mathcal{G}(\mathcal{B})) & \longleftarrow & \text{Hom}_{\mathcal{A}}(A, \mathcal{G}(\mathcal{B})) & \longleftarrow & \text{Hom}_{\mathcal{A}}(A'', \mathcal{G}(\mathcal{B})) & \longleftarrow & 0 \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ \text{Hom}_{\mathcal{B}}(\mathcal{F}(\mathcal{A}'), \mathcal{B}) & \longleftarrow & \text{Hom}_{\mathcal{B}}(\mathcal{F}(\mathcal{A}), \mathcal{B}) & \longleftarrow & \text{Hom}_{\mathcal{B}}(\mathcal{F}(\mathcal{A}''), \mathcal{B}) & \longleftarrow & 0 \end{array}$$

and since the top sequence is exact (by left exactness of  $\text{Hom}$ ), and by the naturality of the isomorphism between the functors, we get exactness below. ♠

**7.1.2 REMARK:** There is a fact that  $C' \rightarrow C \rightarrow C'' \rightarrow 0$  is exact **if and only if**  $\text{Hom}_{\mathcal{B}}(C_{\bullet}, B)$  is exact *for all*  $B$ . One direction is just left exactness, the other basically uses Yoneda lemma.

**7.1.3 Theorem**

If  $\mathcal{F}$  and  $\mathcal{G}$  are adjoint (left and right, resp.), then

- (a)  $\mathcal{F}$  commutes with colimits (e.g. coproducts,  $\oplus$ , cokernels).
- (b)  $\mathcal{G}$  commutes with limits (e.g. products and kernels).

**7.1.4 Corollary**

Let  $\mathcal{A}$  be an abelian category with enough projectives and let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$  be right exact. Then

$$(L_i \mathcal{F}) \left( \bigoplus_{\mid} \mathcal{A}_{\mid} \right) \cong \bigoplus_{\mid} \mathcal{L}_i \mathcal{F}(\mathcal{A}_{\mid}).$$



**Example 7.1**

Let  $R$  be a PID. Consider torsion modules (free ones are uninteresting when computing Tor). Let's compute  $\mathrm{Tor}_*^R(R/(a), M)$  for any  $M$ . Take the resolution

$$0 \rightarrow R \xrightarrow{a} R \rightarrow R/(a)$$

which is free whence projective. Then compute the homology of

$$0 \rightarrow R \otimes_R M \xrightarrow{a} R \otimes_R M \rightarrow 0$$

and since  $R \otimes_R M \cong M$ , we get  $\mathrm{Tor}_{\geq 2}^R(R/(a), M) = 0$ . Then  $\mathrm{Tor}_0^R(R/(a), M) = R/(a) \otimes_R M \cong M/aM$ .

Finally  $\mathrm{Tor}_1^R(R/(a), M) = {}_aM$ , the  $a$ -torison in  $M$ .

**7.1.5 REMARK:** Because of the fact that derived functors of right exact functors preserve colimits, we have actually effectively computed  $\mathrm{Tor}_i^R(N, M)$  for all  $N$ .

**Example 7.2**

In the case when  $R = \mathbb{Z}$  and  $N = \mathbb{Q}/\mathbb{Z}$ , we can use the fact that  $\mathbb{Q}/\mathbb{Z}$  is the colimit (directed limit) of its finite subgroups. Then you can pull this out and say that  $\mathrm{Tor}_*^R(\mathbb{Q}/\mathbb{Z}, M)$  is the “total torsion module” of  $M$ , the directed limit of all finite torsion subgroups (is this last sentence correct?).

## 8 April 24, 2019

We're going to start by doing some more computations.

### 8.1 Computing Ext

Begin by letting  $A = \mathbb{Z}$ . Then  $\mathrm{Ext}_{\mathbb{Z}}^{i>0}(\mathbb{Z}, B) = 0$  since  $\mathbb{Z}$  is free whence projective. When  $A = \mathbb{Z}/n$ , we can compute Ext via the free resolution

$$0 \rightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \rightarrow \mathbb{Z}/n$$

and then applying  $\mathrm{Hom}_{\mathbb{Z}}(-, B)$ , we get the complex

$$0 \rightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, B) \xrightarrow{n} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, B) \rightarrow 0$$

so  $\mathrm{Ext}_{\mathbb{Z}}^{i \geq 2}(\mathbb{Z}/n, B) = 0$  and  $\mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/n, B) = B/nB$  and  $\mathrm{Ext}_{\mathbb{Z}}^0(\mathbb{Z}/n, B) = {}_nB/$ .

**8.1.1 Theorem**

$$\mathrm{Ext}_{\mathbb{Z}}^{i \geq 2}(A, B) = 0.$$

PROOF

Take an injective resolution of  $B$ :

$$B \rightarrow I^0 \rightarrow I^1 \cong \Omega^{-1}B \rightarrow 0$$

and every resolution can end there! This is because the cokernel of  $B \rightarrow I^0$  is the quotient of a divisible group, whence divisible. ♠

8.1.2 REMARK: Notice that  $\mathrm{gldim}(\mathbb{Z}) = 1$ .Notice that when  $B = \mathbb{Z}$ , we can consider the injective resolution

$$\mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

and then compute from the complex

$$\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}) \rightarrow \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$$

where the first is zero and the latter is  $A^{\vee}$ , the **Pontragin dual** of  $A$ . Here we get that  $\mathrm{Ext}_{\mathbb{Z}}^1(A, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) = A^{\vee}$ .

Now let  $A = \mathbb{Z}/p^{\infty} = \varinjlim \mathbb{Z}/p^n$ . Then

$$\mathrm{Ext}_{\mathbb{Z}}^1(\varinjlim \mathbb{Z}/p^n, \mathbb{Z}) = \varprojlim (\mathbb{Z}/p^n, \mathbb{Z}) = \varprojlim \mathbb{Z}/p^n = \widehat{\mathbb{Z}}_p$$

the  $p$ -adic numbers.

Recall:

**8.1.3 Lemma**

$P$  is projective iff  $\mathrm{Ext}^1(P, -)$  is exact if and only if  $\mathrm{Ext}^i(P, -)$  is exact for all  $i \geq 1$ .

**8.1.4 Definition**

$M$  is a flat module if  $- \otimes_R M$  is exact.

**8.1.5 Theorem**

The following are equivalent

- $M$  is flat
- $- \otimes_R M$  is exact
- if  $N' \hookrightarrow N$  then  $N' \otimes_R M \hookrightarrow N \otimes_R M$ .
- $\mathrm{Tor}_i^R(-, M) = 0$  for  $i \geq 1$
- $\mathrm{Tor}_1^R(-, M) = 0$

8.1.6 REMARK: Any projective module is flat.

## 8.2 Local Properties of Modules

Here let  $R$  be a commutative ring,  $\mathfrak{p} \triangleleft R$  is a prime, and  $R_{\mathfrak{p}}$  is localization at  $\mathfrak{p}$  and  $(-)_{\mathfrak{p}}$  is a functor from  $R\text{-mod}$  to  $R_{\mathfrak{p}}\text{-mod}$ .

### 8.2.1 Proposition

$(-)_{\mathfrak{p}}$  is exact.

### 8.2.2 Definition (Informal)

We say that a property  $P$  is local if  $M$  satisfies  $P$  if and only if  $M_{\mathfrak{p}}$  satisfies  $P$  for all  $\mathfrak{p} \triangleleft R$ .

### 8.2.3 Proposition

$M \cong 0$  if and only if

- $M_{\mathfrak{p}} = 0$  for all  $\mathfrak{p}$
- $M_{\mathfrak{m}} = 0$  for all maximal  $\mathfrak{m}$

8.2.4 REMARK: Other examples of local properties are flatness of a module and exactness of a short exact sequence.

### 8.2.5 Proposition

If  $R$  is commutative, and if  $S$  is flat as an  $R$ -algebra (flatness is just referring to the module structure), then

$$S \otimes_R \text{Tor}_i^R(A, B) \cong \text{Tor}_i^S(S \otimes_R A, S \otimes_R B).$$

8.2.6 REMARK: The idea above is “flat base changes commute with Tor.”

### 8.2.7 Corollary

The following are equivalent:

- $\text{Tor}_i^R(M, N) = 0$
- For all  $\mathfrak{p}$ ,  $\text{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = 0$
- For all maximal  $\mathfrak{m}$ ,  $\text{Tor}_i^{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, N_{\mathfrak{m}}) = 0$

## 9 April 26, 2019

### 9.1 A bit more about flat modules

#### 9.1.1 Proposition

Let  $R$  be a local ring and let  $M$  be a finitely generated  $R$ -module. Then the following are equivalent:

- $M$  is flat.
- $M$  is free.
- $M$  is projective.

9.1.2 REMARK: In other words, every flat module is locally free.

9.1.3 REMARK: In general we have the inclusions: free, projective, flat, torsion free.

## 9.2 Ext as Extensions

Recall the five lemma

### 9.2.1 Lemma (Weak 5-lemma)

If we have the commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow \sim & & \downarrow f & & \downarrow \sim \\
 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \longrightarrow 0
 \end{array}
 \quad \text{then } f \text{ is an isomorphism.}$$

### 9.2.2 Definition

Define  $\mathcal{E}xt_R^1(M, N)$  to be the set of short exact sequences  $0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$  modulo the relation that if there is a map (necessarily an isomorphism)  $\varphi : E \rightarrow E'$  extending the identity maps on  $M$  and  $N$ , then the two sequences are equivalent.

We call  $0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$  a **extension of  $M$  by  $N$  of length 1**.

### 9.2.3 Proposition

$\mathcal{E}xt_R^1(N, M)$  has the structure of an abelian group.

9.2.4 REMARK: Let  $\alpha : M' \rightarrow M$  and let  $E \times^M M'$  be the pullback over

$$E' \xrightarrow{g} M \xleftarrow{\alpha} M'$$

where  $0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$  is an extension.

Then setting  $E' = E \times^M M'$ , we can pull back along  $\alpha$  to a short exact sequence

$$0 \rightarrow N \rightarrow E' \rightarrow M' \rightarrow 0.$$

Here we usually refer to this map (of short exact sequences) as  $\alpha^* : \mathcal{E}xt^1(M, N) \rightarrow \mathcal{E}xt^1(M', N)$ .

Dually, we can use a  $\beta : N \rightarrow N'$  and the pushforward  $E'' = N' \times_N E$  under

$$N' \xleftarrow{\beta} N \xrightarrow{f} E$$

and this gives us a map to the sequence  $0 \rightarrow N' \rightarrow E'' \rightarrow M \rightarrow 0$ . This gives us the map  $\beta_* : \mathcal{E}xt^1(M, N) \rightarrow \mathcal{E}xt^1(M, N')$ .

**9.2.5 REMARK:** A natural thing to want to do is sum SES's by using direct sums of the elements in the sequence. Call this  $\xi \oplus \xi'$ . That is great, but it doesn't quite work. To do so, you use the (co)diagonal maps  $\Delta : M \rightarrow M \oplus M$  sending  $m \mapsto (m, m)$  and  $\nabla : N \oplus N \rightarrow N$  via  $(n, n') \mapsto n + n'$ .

Then we can define a sum of extensions:

$$\xi + \xi' := \nabla_* \Delta^*(\xi \oplus \xi') = \Delta^* \nabla_*(\xi \oplus \xi')$$

The last equality isn't completely trivial.

From here we talked about how one proves that this is indeed a group. The zero element is the split exact sequence.

### 9.2.6 Proposition

There exists a natural group isomorphism

$$\mathcal{E}xt_R^1(M, N) \cong \text{Ext}_R^1(M, N).$$

**9.2.7 REMARK:** The proof isn't too hard. Check Weibel.

In general, for  $i \geq 1$ , we use an analogous definition but now we define two extensions to be equivalent if there exist a collection of (i) maps  $f_k : E_k \rightarrow E'_k$  making the diagram commute.

The general group law via pushbacks and -forwards like before. Then we define the addition the same way.

### 9.2.8 Proposition

There exists a natural isomorphism  $\mathcal{E}xt_R^i(M, N) \rightarrow \text{Ext}_R^i(M, N)$ .

PROOF

(Sketch) Take a projective resolution  $P_\bullet \rightarrow M \rightarrow 0$  and for any extension

$$0 \rightarrow N \rightarrow E_1 \rightarrow \cdots \rightarrow E_i \rightarrow M \rightarrow 0$$

lift the identity map on  $M$  to a map of resolutions (not both projective) of  $M$ .

Say that  $f : P_i \rightarrow N$  is the map you get in this chain map. We claim that  $f$  is a cycle in  $\text{Hom}(P_\bullet, N)$ . If  $d_i^*$  is the differential on this complex, then since  $d_i^*(f) = f \circ d_i$ , and using the extension since the map  $f \circ d_i : P_{i+1} \rightarrow P_i \rightarrow N$  is the same as  $0 : P_{i+1} \rightarrow 0 \rightarrow N$ , we have it.

But then  $\bar{f}$  is in  $\text{Ext}_R^i(M, N)$ . ♠

## 9.3 Yoneda Product

There is a relatively natural pairing

$$\text{Ext}_R^i(L, M) \times \text{Ext}_R^j(M, N) \rightarrow \text{Ext}_R^{i+j}(L, N)$$

where we just concatenate the two exact sequences (eliminating  $M$  along the way). It is easy to check that it is still exact.

## 10 April 29, 2019

Today we will finish up the last bit on  $\text{Ext}$ . Let  $M = N$  and consider  $\text{Ext}_R^*(M, M)$ . Then the Yoneda product gives the structure of a graded ring.

Usually when you are working in another context (e.g. group cohomology) where you have another product (e.g. the cup product) this ends up being the same as the Yoneda product. There is something to prove here.

### Example 10.1

Recall  $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{Z}/p$ . Then consider the SES

$$0 \rightarrow \mathbb{Z}/p \hookrightarrow \mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p$$

where the embedding on the first map takes (written multiplicatively)  $a \mapsto ap$ . Then there are  $p-1$  different choices we have for the surjection where  $a \mapsto ia$  for  $1 \leq a \leq p-1$ .

The one remaining extension comes from the split exact sequence.

10.0.1 REMARK: These are the same embeddings we say in  $\mathbb{Z}/p^\infty$ !

## 10.1 The Künneth formula and the Universal Coefficient Theorem

The motivation here is that, given a space  $X$ , it is most natural to consider  $H_*(X, \mathbb{Z})$ , but if there is some kind of an action of  $M$  on these cycles, one may want to compute  $H_*(X, M)$ .

### 10.1.1 Definition

Let  $P_\bullet$  and  $Q_\bullet$  be complexes. Then  $P_\bullet \otimes Q_\bullet$  be defined by

$$(P_\bullet \otimes_R Q_\bullet)_n = \bigoplus_{i+j=n} P_i \otimes_R Q_j$$

and the differential

$$d_n = \sum_{i+j=n} d_{ij} : (P \otimes Q)_n \rightarrow (P \otimes Q)_{n-1}$$

is such that if  $p$  is in degree  $i$  and  $q$  is in degree  $j$ , then

$$d_n(p \otimes q) = d_i(p) \otimes q + (-1)^i p \otimes d_j(q).$$

A natural question to ask: what is the homology of this complex?

10.1.2 REMARK: We will be using the fact that, to compute Tor, it suffices to take flat (rather than projective) resolutions.

There is a more general principle here that one can use modules that are “acyclic” (higher derived functors are zero) for a functor and these resolutions give us homology.

### 10.1.3 Theorem

Let  $P$  and  $Q$  be complexes of (right- and left-)  $R$  modules such that the  $P_n$  and

$$B_n^P = d_{n+1}(P_{n+1})$$

are flat.

Then there exists a short exact sequence

$$0 \rightarrow \bigoplus_{i+j=n} H_i(P) \otimes H_j(Q) \rightarrow H_n(P \otimes Q) \rightarrow \bigoplus_{i+j=n} \text{Tor}_1^R(H_i(P), H_j(Q)) \rightarrow 0$$

PROOF

We will not use boundaries and cycles for  $Q$ , so throughout  $Z$  and  $B$  refer to those for  $P$ . Now since we have the short exact sequence

$$- \rightarrow Z_i \rightarrow P - i \rightarrow B_{i-1} \rightarrow 0$$

and the long exact sequence of Tor gives us that  $Z_i$  are flat as well. Furthermore (again by the long exact sequence) we get

$$0 \rightarrow Z_i \otimes Q_j \rightarrow P_i \otimes Q_j \rightarrow B_{i-1} \otimes Q_j \rightarrow 0$$

is short exact.

Now consider

$$0 \rightarrow \bigoplus_{i+j=n} Z_i \otimes Q_j \rightarrow \bigoplus_{i+j=n} P_i \otimes Q_j \rightarrow \bigoplus_{i+j=n-1} B_i \otimes Q_j \rightarrow 0$$

which gives us a SES of complexes

$$- \rightarrow Z_\bullet \otimes Q_\bullet \rightarrow P_\bullet \otimes Q_\bullet \rightarrow B_\bullet \otimes Q_\bullet[-1] \rightarrow 0$$

and by taking the long exact sequence in homology (note the index shift because of the shift on  $Q$ ):

$$\cdots \rightarrow H_n(B \otimes Q) \rightarrow H_n(Z \otimes Q) \rightarrow H_n(P \otimes Q) \rightarrow H_{n-1}(B \otimes Q) \rightarrow H_{n-1}(Z \otimes Q) \rightarrow \cdots$$

Let  $\partial : H_n(B \otimes Q) \rightarrow H_n(Z \otimes Q)$  be the map above. Then we get the SES

$$0 \rightarrow \text{coker } \partial \rightarrow H_n(P \otimes Q) \rightarrow \ker \delta \rightarrow 0$$

where  $\text{coker } d_{ij}$  is  $H_i(P) \otimes H_j(Q)$  (this needs some work – see the book!)

Finally  $\ker \partial$  can be computed via using the (flat!) resolution  $0 \rightarrow B_i \rightarrow Z_i \rightarrow H_i(P)$  and computing Tor. ♠

10.1.4 REMARK: Notice that it is *not true* that if  $A$  and  $B$  are flat that  $A/B$  are flat!

## 11 May 1, 2019

### 11.0.1 Corollary (Universal Coefficient Theorem – Corollary of Künneth Formula)

Let  $P_\bullet$  be a chain complex of flat modules and that  $d_n(P)$  are flat, and that  $M$  is an  $R$  module. Then we have a short exact sequence

$$0 \rightarrow H_n(P_\bullet) \otimes M \rightarrow H_n(P_\bullet \otimes M) \rightarrow \operatorname{Tor}_1^R(H_{n-1}(P_\bullet), M) \rightarrow 0$$

and in particular if  $M$  is flat, then  $H_n(P_\bullet) \otimes M \cong H_n(P_\bullet \otimes M)$

### 11.0.2 Proposition (UCT for $\mathbb{Z}$ )

Let  $P_\bullet$  be a complex of free abelian groups. Then there exists a (non-canonical) isomorphism

$$H_n(P \otimes M) \cong H_n(P_\bullet) \otimes M \bigoplus \operatorname{Tor}_1^R(H_{n-1}(P_\bullet), M)$$

where the statement that this isomorphism is non-canonical amounts to saying that this splitting in the sequence from the UCT doesn't behave well with respect to maps.

11.0.3 REMARK: What does this come from? Well in topology, if  $X$  is a topological space, then “homology with coefficients” is just  $H_n(X, M) = H_n(S(X) \otimes M)$ . Then the universal coefficient theorem gives us a way to compute this.

### 11.0.4 Proposition (UCT for $H^n$ )

Let  $P_\bullet$  be a chain complex of projective  $R$ -modules where the  $d_n(P_\bullet)$  are projective and  $M$  is any  $R$ -module. Then we can apply  $\operatorname{Hom}(-, M)$  to  $P_\bullet$  to compute cohomology, yielding the short exact sequence

$$0 \rightarrow \operatorname{Ext}_R^1(H_{n-1}(P_\bullet), M) \rightarrow H^n(\operatorname{Hom}(P_\bullet, M)) \rightarrow \operatorname{Hom}(H_n(P_\bullet), M) \rightarrow 0$$

### 11.0.5 Proposition (Künneth for $H^n$ )

Let  $P_\bullet$  be a chain complex of projectives with  $d_n(P_\bullet)$  projective. Let  $Q_\bullet$  be a cochain complex. Then we have the short exact sequence

$$0 \rightarrow \prod_{i+j=n-1} \operatorname{Ext}_R^1(H_i(P), H^j(Q)) \rightarrow H^n(\operatorname{Hom}(P, Q)) \rightarrow \prod_{i+j=n} \operatorname{Hom}(H_i(P), H^j(Q))$$

## 11.1 A reminder on dimension theory

### 11.1.1 Definition

Let  $M$  be a (left)  $R$ -module. Then

$$\operatorname{projdim}_R M := \inf_n \{0 \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow M\}$$

Notice that  $\operatorname{projdim}$  can be infinite.

11.1.2 REMARK: A module has zero projective dimension if and only if it is projective.



11.1.3 REMARK: One can define (analogously) injective and flat dimension.

#### 11.1.4 Lemma

Let  $M$  be an  $R$ -module. The following are equivalent:

- (a)  $\text{projdim}_R M \leq d$
- (b)  $\text{Ext}_R^i(M, N) = 0$  for all  $i \geq d + 1$ , for all  $N$ .
- (c)  $\text{Ext}_R^{d+1}(M, N) = 0$  for all  $N$ .
- (d) If  $0 \rightarrow \Omega^d M \rightarrow P_{d-1} \rightarrow P_{d-2} \rightarrow \cdots \rightarrow P_0 \rightarrow M$  is a resolution, then  $\Omega^d(M)$  is projective.

11.1.5 REMARK: Everything can be dualized (e.g. use injective resolutions and injective dimension and  $\Omega^{-d}$ ) and get a similar set of equivalences.

11.1.6 REMARK: Notice here that the syzygies are not uniquely defined, but they are uniquely defined *up to projective summands*. In fancier parlance, they are uniquely defined in the stable module category.

11.1.7 REMARK: Notice that  $\Omega(\Omega^{d-1}(M)) = \Omega^d(M)$ . Using this, we can compute

$$\text{Ext}_R^{d+1}(M, N) \cong \text{Ext}_R^1(\Omega^d M, N) \cong \underline{\text{Hom}}_R(\Omega^{d+1}, N)$$

where  $\underline{\text{Hom}}$  is Hom in the stable module category.

11.1.8 REMARK: Sometimes the **syzygy** is called **Heller shift** in representation theory or just **shift** when using a categorical viewpoint.

#### 11.1.9 Theorem

The following numbers are the same:

- (a)  $\sup_{M \in R\text{-mod}} \text{projdim}_R M$
- (b)  $\sup_{M \in R\text{-mod}} \text{injdim}_R M$
- (c)  $\sup_{I \text{ a left ideal}} \text{projdim}_R R/I$
- (d)  $\sup_{M, N \in R\text{-mod}} \{d \mid \text{Ext}_R^d(M, N) \neq 0\}$

#### 11.1.10 Definition

The above number is called the **left global dimension of  $R$** .

11.1.11 REMARK: This is where commutative algebra diverges: we have notions of *right* global dimension (just change to **mod- $R$** ) as well as global dimension proper.

We also have a notion of *weak global dimension* or Tor global dimension:

**11.1.12 Definition**

The Tor dimension of  $R$  is any of the following numbers:

- (a)  $\sup_{M \in R\text{-mod}} \text{flatdim}_R M$
- (b)  $\sup_{M \in \text{mod-}R} \text{flatdim}_R M$
- (c)  $\sup_{I \text{ left ideal}} \text{flatdim}_R R/I$
- (d)  $\sup_{J \text{ right ideal}} \text{flatdim}_R R/J$
- (e)  $\sup_{M \in \text{mod-}R, N \in R\text{-mod}} \{d \mid \text{Tor}_d^R(M, N) \neq 0\}$

11.1.13 REMARK: If  $R$  is (left) Noetherian, then  $\text{flatdim}_R(M) = \text{projdim}_R(M)$  for all finitely generated left  $R$ -modules. Furthermore the **left**  $\text{gldim } R = \text{Tordim } R$ .

11.1.14 REMARK: If  $R$  is (right) Noetherian, you get all the same stuff but flipped.

11.1.15 REMARK: Rings of global dimension zero are exactly the semisimple ones!

**12 May 3, 2019**

Recall that semisimple rings are precisely those of global dimension zero.

**12.1 Frobenius Algebras****12.1.1 Definition**

Let  $R$  be a finite dimensional algebra over  $k$ . Then  $R$  is **Frobenius** if there exists a non-degenerate, associative, bilinear form  $\sigma : R \times R \rightarrow k$ .

**12.1.2 Definition**

If  $R$  is Frobenius and furthermore  $\sigma$  is a symmetric form, then  $R$  is **symmetric**.

**12.1.3 Proposition**

If  $R$  is Frobenius, then

$$R^* = \text{Hom}_k(R, k) \cong R$$

as (left)  $R$ -modules.

**PROOF**

Consider the map  $R \rightarrow R^*$  where we send  $a \mapsto \varphi_a = \sigma(-, a)$ . One can check that this actually gives you an isomorphism. ♠

**12.1.4 Corollary**

$R$  is injective as an  $R$ -module.

**12.1.5 REMARK:** The idea here is that  $R^*$  is projective, and dualizing takes projectives to injectives.

**12.1.6 Corollary**

When  $R$  is Frobenius, all projectives are injective (and vice versa).

**12.1.7 REMARK:**  $R$  is Frobenius implies that  $R$  is “self-injective”, or that  $\text{injdim}_R R = 0$ . Note that this then implies that  $R$  is *Gorenstein*!

It often happens in representation theory that one proves something for finite dimensional algebras, then Frobenius algebras, then sees that it generalizes to Gorenstein ones.

**12.1.8 Definition**

$R$  is called **quasi-Frobenius** (according to Wikipedia) if the collection of injectives and projectives are the same.

**Example 12.1**

(This is homework) If  $G$  is a finite group then  $kG$  is symmetric. Note that it is uninteresting when we have Artin-Wedderburn, so the fun stuff happens in the modular case.

**Example 12.2**

Consider the Lie algebra  $\mathfrak{g}$  over  $k$ , a field of characteristic  $p$ . Then  $[p] : \mathfrak{g} \rightarrow \mathfrak{g}$  is a  $p^{\text{th}}$  power map or **restriction operation** if

- (a)  $\text{ad } x^{[p]} = (\text{ad}(x))^p$
- (b)  $(\lambda x)^{[p]} = \lambda^{[p]} x^{[p]}$
- (c)  $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_1^{p-1} s_i(x, y)$

where  $s_i(x, y)$  is the coefficient of  $\lambda^{i-1}$  in

$$(\text{ad}(\lambda x + y))^{p-1}(x) = [\lambda x + y, [\lambda x + y, [\cdots, [\lambda x + y, x] \cdots, x], x], x]$$

12.1.9 REMARK: If  $\mathfrak{g}$  is linear (sits in  $\mathfrak{gl}_n$ ), then we have the  $p^{th}$  power map that takes every matrix to its  $p^{th}$  power. This is what we are trying to capture without having to rely on what is going to be an inherently coordinate-dependent definition.

### 12.1.10 Definition

The universal enveloping algebra  $\mathfrak{U}(\mathfrak{g})$  is the universal object representing the smallest associative algebra capturing the relations of  $\mathfrak{g}$ . Look to my Lie algebra notes for more details.

### 12.1.11 Definition

The **restricted enveloping algebra**  $\mathfrak{u}(\mathfrak{g})$  is

$$\mathfrak{u}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g}) / \langle x^p - x^{[p]} \rangle$$

### 12.1.12 Proposition

$\mathfrak{u}(\mathfrak{g})$  has a “PBW” style basis. The monomials take a basis for  $\mathfrak{g}$  and then the monomials look like

$$x_1^{i_1} \cdots x_n^{i_n}$$

for  $0 \leq i_1, \dots, i_n \leq p-1$ .

In general, the category of representations of  $\mathfrak{g}$  is equivalent to  $\mathfrak{U}(\mathfrak{g})\text{-mod}$  and the restricted representations of  $\mathfrak{g}$  are equivalent to  $\mathfrak{u}(\mathfrak{g})$  modules.

### 12.1.13 Proposition

$\mathfrak{u}(\mathfrak{g})$  is Frobenius.

## 12.2 Hereditary Rings

### 12.2.1 Definition

A ring  $R$  is **(left) hereditary** if every submodule of a free module is projective.

12.2.2 REMARK: Equivalently, the (left)  $\text{gldim } R \leq 1$ .

### Example 12.3

PIDs are hereditary, of course. Furthermore, a commutative integral domain  $R$  is hereditary if and only if  $R$  is a Dedekind domain.

**12.2.3 Proposition**

If  $f : R \rightarrow S$  is a ring homomorphism and  $M$  is an  $S$ -module, then

$$\mathrm{projdim}_R(M) \leq \mathrm{projdim}_R(S) + \mathrm{projdim}_S(M).$$

PROOF

The proof of this statement (which we may have seen with Sándor?) uses Cartan-Eilenberg resolutions. Let  $\mathcal{A}$  be an abelian category with enough projectives and let  $C$  be a chain complex in  $\mathcal{A}$ . Then  $P = P_{\bullet,\bullet}$  (a double complex) is a **Cartan-Eilenberg Resolution of  $C$**  if we have a commutative diagram

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \\ & & P_{n+1,1} & & P_{n,1} & & \\ & & \downarrow & & \downarrow & & \\ & & P_{n+1,0} & & P_{n,0} & & \\ & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & \cdots \end{array}$$

where we have the properties:

- $P_{n\bullet} \rightarrow C_n$  is a projective resolution.
- $P_{nj} = 0$  for  $j < 0$
- $B(P_{n\bullet}) \rightarrow B(C_\bullet)$  and  $H(P_n) \rightarrow H_n(C_\bullet)$  are projective resolutions. ♠

**13 May 8, 2019****13.0.1 Lemma**

Cartan-Eilenberg resolutions exist for any complex  $C$ .

**13.0.2 Proposition**

Let  $f : R \rightarrow S$  be a ring homomorphism and let  $M$  be an  $S$ -module and let  $f^*(M)$  be the  $R$ -module taken by acting on  $M$  via  $f$ . Then

$$\mathrm{projdim}_S(f^*(M)) = \mathrm{projdim}_S(M) + \mathrm{projdim}_R(S)$$

PROOF

*Sketch.* Let  $n = \mathrm{projdim}_S(M)$  and let  $P_\bullet \rightarrow M$  be a projective resolution of  $M$  (as an  $S$ -module) of length  $n$ . Now we use the fact (exercise) that  $\mathrm{projdim}_R(P_i) \leq \mathrm{projdim}_R(S)$ .

Let  $Q_{\bullet\bullet} \rightarrow P_\bullet$  be a Cartan-Eilenberg resolution of  $P_\bullet$  as  $R$ -modules. Now we can “chop off” higher terms of the CE complex, replacing the  $m^{\mathrm{th}}$  row with  $\Omega^m(Q_{n\bullet})$  and

the higher rows with zeros. One can show without too much work that this is still a CE resolution of  $P_\bullet$ .

Now we have already that  $P_\bullet \sim M$  is a quasi isomorphism and the total complex  $\text{Tot}(Q_{\bullet\bullet})$  is quasi isomorphic to  $P_\bullet$ , so we have, in effect, created a projective resolution of  $M$  using  $R$ -modules of length  $n + m$ . ♠

### 13.1 Koszul Complex/Resolution

As motivation, let  $R$  be a regular local ring of dimension  $n$ . Then the Koszul resolution is a resolution of the residue field  $k = R/\mathfrak{m}$  of length  $n$ .

#### Example 13.1

Let  $x$  be a central element in  $R$ . Consider the map  $R \xrightarrow{x} R$ . This is a complex! We can compute  $H_1 = {}_x R$  and  $H_0 = R/xR$ . In particular, if  $x$  is not a zero divisor, then  $K_\bullet(x) = R \rightarrow R$  is a projective resolution of  $R/xR$ .

This gives us the Koszul complex on a single element.

#### 13.1.1 Definition (Koszul Complex)

Let  $\underline{x} = (x_1, \dots, x_n)$  be a sequence (not yet regular) of central elements. Then

$$K_\bullet(\underline{x}) = K_\bullet(x_1) \otimes \cdots \otimes K_\bullet(x_n)$$

is the Koszul complex of  $\underline{x}$ .

#### Example 13.2

Let's consider what happens when  $n = 2$ . Then the length of this complex is going to be 2. This comes from the diagonals of the double complex:

$$\begin{array}{ccc} R & \xrightarrow{x_1} & R \\ -x_2 \downarrow & & \downarrow x_2 \\ R & \xrightarrow{x_1} & R \end{array}$$

giving us the complex

$$R \xrightarrow{\begin{pmatrix} -x_2 & x_1 \end{pmatrix}} R \oplus R \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} R$$

Now in general if  $K_p(\underline{x})$  is the  $p^{\text{th}}$  component of the Koszul complex, we get  $\dim_R(K_p(\underline{x})) = \binom{n}{p}$ . We denote the generators for  $K_p(\underline{x})$  as  $e_{i_1} \wedge \cdots \wedge e_{i_p}$  for  $1 \leq i_1 \cdots i_p \leq n$ .

Then we claim that the differential on this complex is

$$d_p(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum (-1)^{k+1} x_k e_{i_1} \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p}$$

### 13.1.2 Proposition

$K_*(\underline{x})$  is a differential graded algebra with product formed by concatenation modulo the relation  $e_i \wedge e_j = -e_j \wedge e_i$  and  $e_i \wedge e_j = 0$ . If  $V = Rx_1 \oplus \cdots \oplus Rx_n$ , then  $K_*(\underline{x}) \cong \Lambda^*(V)$  and  $K_p(\underline{x}) \cong \Lambda^p(V) \cong \Lambda^p(R^n)$ .

13.1.3 REMARK: Note that in the proposition above,  $K_*(\underline{x}) \cong \Lambda^*(V)$  is meant to imply isomorphism *on the level of graded algebras*.

### 13.1.4 Definition

If  $M$  is any (left)  $M$  module then  $K_\bullet(\underline{x}) \otimes M$  is a Koszul complex of  $M$ .

13.1.5 REMARK: Then, as a matter of notation,  $H_i(\underline{x}, M) = H_i(K_\bullet(\underline{x}) \otimes M)$ .

## 14 May 10. 2019

### 14.1 Koszul Complexes Continued

Let  $R$  be a commutative local ring  $(R, \mathfrak{m}, k)$ .

#### 14.1.1 Theorem

Let  $(x_1, \dots, x_n) \in \mathfrak{m}$  is an  $M$ -regular sequence for  $M$ , then

- $H_0(\underline{x}, M) = M/((x_1, \dots, x_n)M)$ .
- $H_i(\underline{x}, M) = 0$  when  $i > 0$ .

14.1.2 REMARK: Equivalently,  $K_\bullet(\underline{x}) \otimes M$  is a projective resolution of  $M/((x_1, \dots, x_n)M)$ .

In particular, when  $M = R$ , and when  $\dim R = n$ , then the (length  $n$ ) sequence is just a regular sequence. Then  $K_\bullet(\underline{x}) \rightarrow k$  is a free resolution of  $k = R/\mathfrak{m}$ .

#### 14.1.3 Corollary

If  $R$  is a regular ring,  $\text{projdim}_R k = n = \dim R$

#### 14.1.4 Lemma

Let  $x$  be a non-zero-divisor in the center of  $R$  and let  $C \in \mathbf{Ch}(R\text{-mod})$ . Then there exists a natural short exact sequence

$$0 \rightarrow H_i(C)/xH_i(C) \rightarrow H_i(K_\bullet(x) \otimes C) \rightarrow {}_xH_i(C) \rightarrow 0$$

PROOF

Künneth formula! ♠

### 14.1.5 Theorem

Let  $R$  be a commutative local ring,  $M$  an  $R$ -module, and a sequence  $(x_1, \dots, x_n) \in \mathfrak{m}$ . Then

- $(x_1, \dots, x_n)$  is  $M$ -regular.
- $H_i(\underline{x}, M) = 0$  for  $i > 0$ .
- $H_1(\underline{x}, M) = 0$

### 14.1.6 Corollary

If  $(x_1, \dots, x_n)$  is a regular sequence in a local ring, then any permutation of it is also regular!

14.1.7 REMARK: Definitely not true when  $R$  is non-local. We did an example with Sándor.

## 14.2 Group (co)homology

Let  $G$  be a group. Then we can consider  $\mathbb{Z}G$  modules and compute

$$\mathrm{Ext}_G^i(M, N) = R^i \mathrm{Hom}_G(M, N).$$

Recall that we can consider representations of  $G$  over  $R$  (a commutative ring). Then if  $M$  and  $N$  are two representations of  $G$  ( $RG$ -modules), then  $M \otimes_R N$  can be defined to have the diagonal action of  $G$ . Similarly  $\mathrm{Hom}_R(M, N)$  becomes a representation via  $(g \cdot f)(m) = g \cdot (f(g^{-1}m))$ .

14.2.1 REMARK: Notice here that  $\mathrm{Hom}_R(-, -)$  is *internal hom* since it becomes again a  $G$ -module.

### 14.2.2 Proposition

$\mathrm{Hom}_R(M, N)^G \cong \mathrm{Hom}_{RG}(M, N)$  where the later is the external (categorical)  $\mathrm{Hom}$ .

The upshot of all of this is that we can compute the group cohomology:

$$H^i(G, M) := \mathrm{Ext}_{RG}^i(R, M) = R^i \mathrm{Hom}_{RG}(R, M) = R^i \mathrm{Hom}_R(Z, M)^G = R^i M^G$$

so we can define the group cohomology functor

$$H^i(G, -) = R^i(-)^G.$$

Now  $H_i(G, M) = \mathrm{Tor}_i^{RG}(R, M)$ . But  $R \otimes_{RG} M$  is going to be isomorphic to  $M_G$ , the **coinvariance** of  $M$ , which is  $M / \langle m - gm \mid m \in M, g \in G \rangle$ . So then  $H_i(G, -) = L_i(-)_G$ .

14.2.3 REMARK: How do we compute this in general? This is not actually known. For instance, consider  $GL_n(\mathbb{F}_p)$  is a relatively simple group (finite!) and we don't know the minimal  $i > 0$  such that  $H^i(GL_n(\mathbb{F}_p), \mathbb{F}_p) \neq 0$ .

Naïvely, we can use that  $H^i(G, M) = \mathrm{Ext}_G^i(\mathbb{Z}, M)$  (here setting  $R = \mathbb{Z}$ ) and then compute this via a resolution either of  $P_\bullet \rightarrow \mathbb{Z}$  and compute  $H^i(\mathrm{Hom}(P_\bullet, M))$  or  $H^i((I^\bullet)^G)$ .



## 15 May 13, 2019

### 15.1 Sample Computations

#### Example 15.1

Let  $G = \mathbb{Z}/p = C_p$  (not necessarily a prime yet). For instance, let  $p = 0$  first and  $G = \mathbb{Z}$ . Then to compute  $H^i(C_p, A) = \text{Ext}_{C_p}^i(\mathbb{Z}, A)$  is computed either with a resolution of  $\mathbb{Z}$  or  $A$ . Since the latter is allowed to be general, we need to do the former. Consider

$$\cdots \rightarrow \mathbb{Z}C_p \xrightarrow{N} \mathbb{Z}C_p \xrightarrow{\sigma-1} \mathbb{Z}C_p \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0$$

where  $N = (\sigma^p - 1)/(\sigma - 1)$ .

Let's begin by noticing that  $\text{Hom}_{\mathbb{Z}C_p}(\mathbb{Z}C_p, A) \cong A$ , so we get a resolution

$$\cdots \leftarrow A \xleftarrow{\sigma-1} A \xleftarrow{N} A \xleftarrow{\sigma-1} A$$

Doing our computations, we get  $H^0(C_p, A) = \ker\{\sigma - 1 : A \rightarrow A\} = A^{C_p}$ . As we expected!

Then  $H^1(C_p, A) = \ker(N)/(\sigma-1)A$  and  $H^2(C_p, A) = A^{C_p}/NA$ , and this just cycles.

15.1.1 REMARK: The above is an example of a **periodic resolution of  $\mathbb{Z}$** . We will not talk much about this, but there is a measure of *complexity of a module* which refers to the length of the period in such a resolution.

#### Example 15.2

In a specific case, consider  $A = \mathbb{Z}$  (the trivial  $\mathbb{Z}C_p$  module). Then  $H^0$  is  $\mathbb{Z}$ ,  $H^1 = 0$  and  $H^2 = \mathbb{Z}/p$ .

#### Example 15.3

Now let's let  $k$  be a field of characteristic  $p$  (now  $p$  is a prime, obviously) and let  $A$  be a  $k$ -module. Then the same thing goes through in computing  $H^i(C_p, A) = \text{Ext}_{kC_p}^i(k, A)$ , but we just replace  $\mathbb{Z}$  with  $k$ .

If instead we compute  $H^i(C_p, k)$ , we get  $k$  every time (since  $\sigma - 1$  is the zero map as well as  $1 + \sigma + \cdots + \sigma^{p-1}$ ).

**Example 15.4**

Let  $\langle t \rangle = C$  be a cyclic group and consider the group algebra  $\mathbb{Z}C \cong \mathbb{Z}[t, t^{-1}]$ . But then a resolution of  $\mathbb{Z}$  is

$$0 \rightarrow \mathbb{Z}\langle t \rangle \xrightarrow{t-1} \mathbb{Z}\langle t \rangle \xrightarrow{\varepsilon} \mathbb{Z}$$

And so you get a result about Poincaré duality that I missed but don't have time to figure out.

**Example 15.5**

Now if  $A = \mathbb{Z}\langle t \rangle$  and  $C = \langle t \rangle$ . Then  $H^0(C, \mathbb{Z}C) = 0$  and  $H^1(C, \mathbb{Z}C) = \mathbb{Z}$ . Replacing  $\mathbb{Z}$  with  $k$ , we get  $H^0(C, \mathbb{Z}C) = 0$  and  $H^1(C, \mathbb{Z}C) = k$ .

15.1.2 REMARK: This gives us one of the examples of the differences between finite and infinite groups: if  $C$  were finite,  $kC$  would be Frobenius, whence self-injective, whence  $H^1(C, kC) = 0$ .

## 15.2 Products in Cohomology

We will begin by discussing the idea of a “coss-product.” Let  $G$  and  $H$  be groups. Then we want to define a map

$$H^p(G, \mathbb{Z}) \otimes H^q(H, \mathbb{Z}) \xrightarrow{\times} H^{p+q}(G \times H, \mathbb{Z}).$$

Eventually we can let  $G = H$  and using the coproduct (diagonal) map  $\Delta(g) = (g, g)$ , whose image under  $H^{p+q}$  is  $\Delta^* : H^{p+q}(G \times G, \mathbb{Z}) = H^{p+q}(G, \mathbb{Z})$  to get our cup product.

**15.2.1 Proposition**

Let  $P_\bullet \rightarrow \mathbb{Z}$  be a projective resolution as  $\mathbb{Z}G$  modules and  $Q_\bullet \rightarrow \mathbb{Z}$  a resolution as  $\mathbb{Z}H$  modules. Then  $P_\bullet \otimes Q_\bullet \rightarrow \mathbb{Z}$  is a projective resolution as  $\mathbb{Z}G \otimes \mathbb{Z}H = \mathbb{Z}(G \times H)$  modules.

Let's begin by defining a map

$$f : \text{Hom}_G(P_\bullet, \mathbb{Z}) \otimes \text{Hom}_H(Q_\bullet, \mathbb{Z}) \rightarrow \text{Hom}_{G \times H}(P \otimes Q, \mathbb{Z})$$

via

$$f(\mu \otimes \nu)(x \otimes y) = \mu(x)\nu(y)$$

for all  $x \in P$  and  $y \in Q$ . Notice here that if  $|\mu| \neq |x|$ , then we set  $\mu(x) = 0$ .

Now using that  $H^i(G, \mathbb{Z}) = H^i(\text{Hom}_G(P_\bullet, \mathbb{Z}))$ , and using the first map from the Künneth formula,

$$H^i(\text{Hom}_G(P_\bullet)) \otimes H^j(\text{Hom}_H(Q_\bullet, \mathbb{Z})) \rightarrow H^{i+j}(\text{Hom}_{G \times H}(P \otimes Q, \mathbb{Z}))$$

and then it can be checked that this map behaves well with respect to cycles and boundaries, so we get a map

$$H^{i+j}(\mathrm{Hom}_G(P_\bullet, \mathbb{Z}) \otimes \mathrm{Hom}_H(Q_\bullet, \mathbb{Z})) \rightarrow H^{i+j}(\mathrm{Hom}_{G \times H}(P \otimes Q, \mathbb{Z})) \cong H^{i+j}(G \times H, \mathbb{Z}).$$

## 16 HW1 – Due April 17

Throughout the assignment,  $R$  is an associative ring with 1.

### Problem 16.1

Show that  $\mathbf{Ch}(R)$  is an abelian category.

#### Solution:

We begin by proving a lemma:

##### 16.0.1 Lemma

$\mathbf{Ch} = \mathbf{Ch}(R)$  is additive.

##### PROOF

Let  $C, D \in \mathbf{Ch}$  and  $f \in \text{Hom}_{\mathbf{Ch}}(C, D)$ . Then  $f = (f_i)_{i \in \mathbb{Z}}$  is a collection of maps of  $R$ -modules. Then, leveraging that  $R\text{-mod}$  is Abelian (*the* Abelian category in some ways) and the fact that addition of chain maps is performed component-wise, this gives  $\text{Hom}_{\mathbf{Ch}}(C, D)$  the structure of an Abelian group as the direct product of Abelian groups. That the component-wise sum is actually a chain map is evident since

$$d((f+g)(x)) = d(f(x)+g(x)) = d(f(x)) + d(g(x)) = f(d(x)) + g(d(x)) = (f+g)(d(x)).$$

The chain  $0_{\bullet} = \cdots 0 \rightarrow 0 \rightarrow 0 \rightarrow \cdots$  is the zero object in  $\mathbf{Ch}$  since there is a unique map  $0 = (0)_{i \in \mathbb{Z}}$  both from and to  $0_{\bullet}$  from any chain (again, since maps are component-wise  $R$ -module maps, so each component must be 0). This map is indeed a chain map since  $0 = 0 \circ d = d' \circ 0$ . Therefore  $\mathbf{Ch}$  is additive. ♠

Now for the final three properties (that (co)kernels exist, that every monic is the kernel of its cokernel, and that every epi is the cokernel of its kernel) follow from the fact that  $R\text{-mod}$  is abelian (and thus satisfy these properties) and that (co)kernels in  $\mathbf{Ch}$  are “what we would hope they would be” component-wise. Therefore the properties pass from  $R\text{-mod}$  up to  $\mathbf{Ch}$ .

##### 16.0.2 Lemma

Let  $C, D \in \mathbf{Ch}$  and  $f \in \text{Hom}_{\mathbf{Ch}}(C, D)$ . Then there is a complex  $K$  and a map  $i : K \rightarrow C$  that is the kernel of  $f$  and furthermore corresponds to the ( $R\text{-mod}$ ) kernel

$$i_n : K_n = \ker f_n \rightarrow C_n$$

on the components.

##### PROOF

16.0.3 REMARK: I have just noticed that I am basically re-proving part of snake lemma below for the existence of the maps between kernels. Oh well, good practice.

Consider the following diagram:

$$\begin{array}{ccc}
\ker f_n & \xrightarrow{\partial_n} & \ker f_{n+1} \\
\downarrow i_n & & \downarrow i_{n+1} \\
C_n & \xrightarrow{d_n} & C_{n+1} \\
\downarrow f_n & & \downarrow f_{n+1} \\
D_n & \xrightarrow{d'_n} & D_{n+1}
\end{array}$$

Let  $f \in \text{Hom}_{\mathbf{Ch}}(C, D)$  as above. Define  $K = (\ker f_i, \partial) \in \mathbf{Ch}$  where  $\partial^i : \ker f_i \rightarrow \ker f_{i+1}$  is defined as follows: let  $x \in \ker f_n$ . Then since  $f$  is a chain map,

$$0 = d'_n(0) = d'_n(f_n(x)) = f_{n+1}(d_n(x))$$

so in particular  $d_n(x) \in \ker f_{n+1}$ . So  $d_n(\ker f_n) \subseteq \ker f_{n+1}$ , so we can lift  $d_n \circ i_n$  through  $i_{n+1}$  to a map  $\partial_n : \ker f_n \rightarrow \ker f_{n+1}$ . Then  $K$  is a chain and  $i$  is a chain map by construction.

That this is the kernel of  $f$  in  $\mathbf{Ch}$  follows since if  $g \in \text{Hom}(A, C)$  and  $f \circ g = 0$ , each component map  $g_n$  must factor through  $\ker f_n$ . It remains to see that this map is a chain map. Consider the diagram

$$\begin{array}{ccccc}
& & A_n & \xrightarrow{d_A} & A_{n+1} \\
& \swarrow \tilde{g}_n & & & \swarrow \tilde{g}_{n+1} \\
\ker f_n & \xrightarrow{d_K} & \ker f_{n+1} & & \\
\downarrow i_n & \searrow g_n & \downarrow i_{n+1} & \searrow g_{n+1} & \\
C_n & \xrightarrow{d_C} & C_{n+1} & & 
\end{array}$$

where  $\tilde{g}_n$  is the canonical map through the kernel of  $f_n$ . Pick any  $a \in A_n$  and notice

$$\begin{aligned}
(i_{n+1} \circ d_K) \circ \tilde{g}_n(a) &= (d_C \circ i_n) \circ \tilde{g}_n(a) \\
&= d_C \circ g_n(a) \\
&= g_{n+1} \circ d_A(a) \\
&= i_{n+1} \circ \tilde{g}_{n+1} \circ d_A(a)
\end{aligned}$$

and so since  $i_{n+1}$  is mono, this establishes

$$d_K \circ \tilde{g}_n = \tilde{g}_{n+1} \circ d_A$$

so  $\tilde{g}$  is a chain map and therefore  $i : K \rightarrow C$  is the kernel of  $f$ , which lies in  $\mathbf{Ch}$ . ♠

**16.0.4 Lemma**

Let  $C, D \in \mathbf{Ch}$  and  $f \in \text{Hom}_{\mathbf{Ch}}(C, D)$ . Then there is a complex  $E$  and a map  $q : D \rightarrow E$  that is the cokernel of  $f$  and furthermore corresponds to the  $(R\text{-}\mathbf{mod})$  cokernel

$$q_n : D_n \rightarrow E_n = \text{coker } f_n$$

on the components.

PROOF

Consider

$$\begin{array}{ccc} C_n & \xrightarrow{d_n} & C_{n+1} \\ \downarrow f_n & & \downarrow f_{n+1} \\ D_n & \xrightarrow{d'_n} & D_{n+1} \\ \downarrow q_n & & \downarrow q_{n+1} \\ \text{coker } f_n & \xrightarrow{\partial_n} & \text{coker } f_{n+1} \end{array}$$

16.0.5 REMARK: This also follows from snake lemma and is less “elementy”.

Given  $\hat{x} \in \text{coker } f_n$ , let  $x$  be any preimage of  $\hat{x}$  under  $q_n$  and define

$$\partial_n(\hat{x}) = q_{n+1} \circ d'_n(x).$$

To show that this map is well-defined, let  $y$  be any other preimage of  $x$  under  $q_n$ . Then since  $q_n(x - y) = 0$ , we can pull  $x - y$  back through  $f_n$  to some  $c \in C_n$ . Then

$$\begin{aligned} (q_{n+1} \circ d'_n)(x - y) &= q_{n+1} \circ (d'_n \circ f_n)(c) \\ &= q_{n+1} \circ (f_{n+1} \circ d_n)(c) \\ &= (q_{n+1} \circ f_{n+1}) \circ d_n(c) \\ &= 0 \circ d_n(c) = 0 \end{aligned}$$

so  $q_{n+1} \circ d'_n(x) = q_{n+1} \circ d'_n(y)$ , so  $\partial_n(\hat{x})$  is well-defined. Then  $E = (\text{coker } f_n)_{n \in \mathbb{Z}}$  is a chain complex with differential  $\partial$ . To show that  $q : D \rightarrow E$  is the cokernel of  $f$ , let  $A \in \mathbf{Ch}$  and  $g \in \text{Hom}_{\mathbf{Ch}}(D, A)$  with  $g \circ f = 0$  then from the universal property in  $R\text{-}\mathbf{mod}$ , the component maps must factor through the  $E_n$ :

$$\begin{array}{ccccc} & & A_n & \xrightarrow{d_A} & A_{n+1} \\ & \nearrow \tilde{g}_n & & & \nearrow \tilde{g}_{n+1} \\ \text{coker } f_n & \xrightarrow{d_E} & \text{coker } f_{n+1} & & \\ \uparrow q_n & \nearrow g_n & \uparrow q_{n+1} & \nearrow g_{n+1} & \\ D_n & \xrightarrow{d_D} & D_{n+1} & & \end{array}$$

Then for any  $d \in D_n$ ,

$$\begin{aligned}\tilde{g}_{n+1} \circ d_E \circ q_n(d) &= \tilde{g}_{n+1} \circ q_{n+1} \circ d_D(d) \\ &= g_{n+1} \circ d_D(d) \\ &= d_A \circ g_n(d) \\ &= d_A \circ \tilde{g}_n \circ q_n(d)\end{aligned}$$

and since  $q_n$  is epi, we conclude that

$$\tilde{g} \circ d_E = d_A \circ \tilde{g}$$

so since  $E$  is the (unique) object satisfying this property, it is the cokernel of  $f$  in **Ch**.♠

### Problem 16.2

Show that a chain map  $f : C_\bullet \rightarrow D_\bullet$  induces a well-defined map  $H_n(f)$  on homology.

#### Solution:

Since  $f$  is a chain map, it is easy to see it induces a map on cycles: let  $c \in Z_n(C)$ . Then

$$d_D(f(c)) = f(d_C(c)) = f(0) = 0$$

so  $f(Z_n(C)) \subseteq Z_n(D)$ . Thus it suffices to show that if  $x$  and  $y$  differ by a boundary,  $f(x)$  and  $f(y)$  do as well.

To see this, assume that  $x - y \in B_n(C)$ , so there is a  $z \in C_{n+1}$  such that  $d_C(z) = x - y$ . But then

$$f(x) - f(y) = f(x - y) = f(d_C(z)) = d_D(f(z)) \in B_n(D)$$

so the map  $H_n(f)$  is well-defined. ♠

### Problem 16.3

(Weibel 1.4.5)

- (a) Show that the chain homotopy equivalence is an equivalence relation on the set of all chain maps from  $C$  to  $D$ . Let  $\text{Hom}_{\mathbf{K}}(C, D)$  denote the equivalence classes of such maps. Show that  $\text{Hom}_{\mathbf{K}}(C, D)$  is an Abelian group.
- (b) Let  $f \sim g$  where  $f, g : C \rightarrow D$ . If  $u : B \rightarrow C$  and  $v : D \rightarrow E$  are chain maps, show  $vfu$  and  $vgu$  are chain homotopic. Deduce that there is a category  $\mathbf{K}$  whose objects are chain complexes and whose morphisms are given in (a).
- (c) Let  $f_0, f_1, g_0$ , and  $g_1$  be chain maps  $C \rightarrow D$  such that  $f_i$  is chain homotopic to  $g_i$ . Show that  $f_0 + f_1 \sim g_0 + g_1$ . Deduce that  $\mathbf{K}$  is an additive category and that  $\mathbf{Ch} \rightarrow \mathbf{K}$  is an additive functor.
- (d) (Optional) Is  $\mathbf{K}$  an abelian category? Explain.

**Solution:**

(a): Let  $f, g, h \in \text{Hom}_{\mathbf{Ch}}(C, D)$ . It is clear that  $f \sim f$  since  $f - f = 0$  is nullhomotopic via  $s = 0$ . Thus  $\sim$  is reflexive. If  $f \sim g$ , then there is a chain homotopy  $s$  such that  $f - g = ds + sd$ . Setting  $s' = -s$ , we get

$$ds' + s'd = -(ds + sd) = -(f - g) = g - f$$

so  $g \sim f$ , whence  $\sim$  is symmetric.

Finally, assume  $f \sim g$  via  $s$  and  $g \sim h$  via  $t$ . But then notice that

$$\begin{aligned} f - h &= f - g + g - h \\ &= ds + sd + dt + td \\ &= ds + dt + sd + td \\ &= d(s + t) + (s + t)d \end{aligned}$$

so  $f - h \sim 0$  via the chain homotopy  $s + t$ . Therefore  $\sim$  is an equivalence relation.

The last statement follows since

$$\text{Hom}_{\mathbf{K}}(C, D) = \text{Hom}_{\mathbf{Ch}}(C, D) / \sim$$

and every quotient of an Abelian group is Abelian.

(b): Say  $s$  is the homotopy from  $f - g$  to 0. Then notice since everything is linear:

$$\begin{aligned} vfu - vgu &= v(f - g)u \\ &= v(sd_C + d_Ds)u \\ &= vs(d_Cu) + (vd_D)su \\ &= vsud_B + d_Evsu \end{aligned}$$

so  $vfu \sim vgu$  via the chain homotopy  $vsu$ . This means that if  $[f]$  is the chain homotopy equivalence class of  $f$ , that the composition of chain maps is well-defined. To see this, we can let either  $B = C$  and  $u = \text{id}_C$  or else  $D = E$  and  $v = \text{id}_D$ .

The identity map in  $\text{Hom}_{\mathbf{K}}(C, C)$  is  $[\text{id}_C]$ , containing all nullhomotopic self-maps of  $C$ . Since  $\mathbf{K}$  is a collection of objects and morphisms with an identity map for each object and that is closed under composition, we get that  $\mathbf{K}$  is a category.

(c): Let  $f_0, f_1, g_0$  and  $g_1$  be defined as provided. Then

$$\begin{aligned} f_0 + f_1 - g_0 - g_1 &= (f_0 - g_0) + (f_1 - g_1) \\ &= s_0d_C + d_Ds_0 + s_1d_C + d_Ds_1 \\ &= (s_0 + s_1)d_C + d_D(s_0 + s_1) \end{aligned}$$

so  $f_0 + f_1 \sim g_0 + g_1$ . Then  $[f] + [g] = [f + g]$  for all  $f, g \in \text{Hom}_{\mathbf{Ch}}(C, D)$ . Now  $0_{\bullet}$  is the zero object in  $\mathbf{K}$  since there is a unique map in  $\text{Hom}_{\mathbf{Ch}}(0, C)$  and  $\text{Hom}_{\mathbf{Ch}}(C, 0)$ , so the same is true in  $\mathbf{K}$ . Therefore  $\mathbf{K}$  is additive.



The functor  $Q : \mathbf{Ch} \rightarrow \mathbf{K}$  sending complexes to themselves and chain maps  $Q(f) = [f]$  (which is a functor since composition is respected by (b) and identity maps are respected by definition) is additive since for any  $A, B \in \mathbf{Ch}$ ,

$$\mathrm{Hom}_{\mathbf{Ch}}(A, B) \rightarrow \mathrm{Hom}_{\mathbf{K}}(A, B)$$

is a group homomorphism as shown above.

(d): I suspect the answer to this is no. ♠

**Problem 16.4**

*Show that a chain complex  $C$  is split exact if and only if it is nullhomotopic.*

**Solution:**

Begin by assuming that  $C$  is split exact. Then consider the diagram

$$\begin{array}{ccccccc} C_{n-1}^0 \oplus C_{n-1}^1 & \xrightarrow{d} & C_n^0 \oplus C_n^1 & \xrightarrow{d} & C_{n+1}^0 \oplus C_{n+1}^1 \\ \downarrow \mathrm{id} & \swarrow s & \downarrow \mathrm{id} & \swarrow s & \downarrow \mathrm{id} \\ C_{n-1}^0 \oplus C_{n-1}^1 & \xrightarrow{d} & C_n^0 \oplus C_n^1 & \xrightarrow{d} & C_{n+1}^0 \oplus C_{n+1}^1 \end{array}$$

where  $C_n = C_n^0 \oplus C_n^1$  and  $d_n|_{C_n^0}$  is the zero map and  $d_n|_{C_n^1}$  is an isomorphism onto  $C_{n+1}^0$ .

Define  $s$  to be

$$(0, \pi_0)_i : C_i^0 \oplus C_i^1 \rightarrow C_{i-1}^0 \oplus C_{i-1}^1$$

where

$$\pi_0 : C_i^0 \oplus C_i^1 \rightarrow C_i^0 \cong C_{i-1}^1$$

is projection onto the first coordinate. Then clearly

$$f((a, b)) = (a, b) = (a, 0) + (0, b) = d((0, a)) + s((b, 0)) = d(s((a, b))) + s(d((a, b)))$$

whence  $\mathrm{id} \sim 0$  so  $C$  is nullhomotopic.

Now assume there exists a map  $\mathrm{id}_C \sim 0$  via chain homotopy  $s : C \rightarrow C[-1]$ . The exactness of  $C$  is immediate since homotopic maps induce isomorphisms on homology. Consider the SES

$$0 \longrightarrow \ker d \xrightarrow{i} C_n \xrightarrow[d]{s'} d(C_n) \longrightarrow 0$$

where

$$s' = s_{n+1}|_{d(C_n)}$$

But then consider the restriction of  $\text{id} = \text{id}_{C_{n+1}}$ , to  $d(C_n) \subseteq C_{n+1}$ :

$$\text{id}|_{d(C_n)} = (s_{n+2} \circ d + d \circ s_{n+1})|_{d(C_n)} = (s_{n+2} \circ d)|_{d(C_n)} + (d \circ s_{n+1})|_{d(C_n)} = (d \circ s_{n+1})|_{d(C_n)}$$

so  $d \circ s'$  is the identity on  $d(C_n)$  whence  $s$  is a splitting of the above short exact sequence.

Since this holds for an arbitrary  $n$ , this establishes that  $C_n \cong \ker d \oplus d(C_n)$ , and thus by exactness  $C$  is split exact. ♠

### Problem 16.5

Show that a chain complex  $P_\bullet$  is a projective object in the category of chain complexes over a ring  $R$  if and only if  $P_\bullet$  is a split exact complex of projective modules.

### Solution:

Assume first that  $P$  is a projective object in **Ch**. Following a hint in Weibel, we consider the surjection  $\text{cone}(P) \xrightarrow{\delta} P[-1] \rightarrow 0$ , which admits a splitting map  $\sigma : P[-1] \rightarrow \text{cone}(P)$  via the standard argument – lift  $\text{id}_P : P \rightarrow P$  to a map  $\sigma : P \rightarrow \text{cone}(P)$  such that  $\delta \circ \sigma = \text{id}_P$ .

From Weibel, we know  $\delta(a, b) = -a$ , so if  $\sigma_i = (s, t)_i : P_{i-1} \rightarrow P_{i-1} \oplus P_i$ , we get that

$$\delta \circ (s, t)(p) = -s(p) = p = \text{id}_P(p)$$

by the property of the lift  $\sigma$  and therefore  $s = -\text{id}_P$ .

Then using the fact that  $\sigma$  is a chain map, we can compute (if  $C = \text{cone}(P)$ ):

$$(s \circ d_P, t \circ d_{P[-1]}) = (s, t) \circ d_P = d_C \circ (s, t) = (d_P \circ s, d_P \circ t - \text{id}_P \circ s)$$

and therefore by focusing on the second component:

$$t \circ d_{P[-1]} - d_P \circ s = -t \circ d_P - d_P \circ s = \text{id}_P$$

whence  $-\text{id}_P \sim \text{id}_P \sim 0$ , so by problem three above  $P$  is split exact.

To see every element is injective, we can “localize” by considering a lift of  $P \rightarrow M_\bullet = (\cdots \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow \cdots)$  through the surjection  $(\cdots \rightarrow 0 \rightarrow N \rightarrow 0 \rightarrow \cdots) \twoheadrightarrow M_\bullet$ . Since a lift always exists for  $M$  and  $N$  in any component, this shows all the terms in the complex  $P$  are projective.

Now assume that  $P$  is a split exact. Then since the  $P_i$  are projective, for any chain map  $f : P \rightarrow M$  and surjection  $g : N \rightarrow M$ , we get maps  $\tilde{f}_i : P_i \rightarrow N_i$ . Recall that since  $P$  is split exact, there exists a splitting map  $\sigma_n : P_n \rightarrow P_{n+1}$  for every  $n$  such that  $d_P \circ \sigma_n = \text{id}_{P_n}$ .

Now define  $F_n : P_n \rightarrow N_n$  by

$$F_n = d_N \circ \tilde{f}_{n+1} \circ \sigma_n + \tilde{f}_n \circ \sigma_{n-1} \circ d_P$$

and so we can compute

$$\begin{aligned}
 d_N \circ F_n &= d_N^2 \circ \tilde{f}_{n+1} \circ \sigma_n + d_N \circ \tilde{f}_n \circ \sigma_{n-1} \circ d_P \\
 &= d_N \circ \tilde{f}_n \circ \sigma_{n-1} \circ d_P \\
 &= d_N \circ \tilde{f}_n \circ \sigma_{n-1} \circ d_P + \tilde{f}_{n-1} \circ \sigma_{n-2} \circ d_P^2 \\
 &= F_{n-1} \circ d_P
 \end{aligned}$$

so  $F$  is a chain map, and therefore  $F$  is a lift of  $f$ , so since  $P$  satisfies the usual diagram, it is a projective object. ♠

## 17 HW2 – Due XXXXX

### Problem 17.1

Let  $R$  be a commutative ring, and  $N$  be an  $R$ -module. Show that the following are equivalent:

- (a)  $N$  is flat
- (b)  $\mathrm{Tor}_i^R(M, N) = 0$  for any  $i > 0$  and any  $R$ -module  $M$
- (c)  $\mathrm{Tor}_1^R(M, N) = 0$  for any  $R$ -module  $M$
- (d)  $\mathrm{Tor}_1^R(R/I, N) = 0$  for any ideal  $I \subset R$ .

### Solution:

(a) $\Rightarrow$ (b) $\Rightarrow$ (c) is clear since  $\mathrm{Tor}_i^R(-, N)$  is a left derived functor of  $- \otimes N$ . But when  $N$  is flat,  $- \otimes N$  is exact.

That (c) $\Rightarrow$ (a) can be seen by using the long exact sequence of Tor arising from any short exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ :

$$\cdots \rightarrow \mathrm{Tor}_2^R(C, N) \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow A \otimes N \rightarrow B \otimes N \rightarrow C \otimes N \rightarrow 0 \quad \spadesuit$$

but then in particular  $0 \rightarrow A \otimes N \rightarrow B \otimes N \rightarrow C \otimes N \rightarrow 0$  is exact, so  $- \otimes N$  is exact whence  $N$  is flat.

Clearly (c) $\Rightarrow$ (d), so it remains to see that (d) implies any of (a), (b), or (c).

### Problem 17.2

Let  $(R, \mathfrak{m}, k)$  be a commutative local ring and  $M$  be a finitely generated  $R$ -module. Prove that the following are equivalent:

- (a)  $M$  is free
- (b)  $M$  is flat
- (c) The map  $\mathfrak{m} \otimes_R M \rightarrow R \otimes_R M = M$  induced by the embedding  $\mathfrak{m} \subset R$  is injective
- (d)  $\mathrm{Tor}_1(k, M) = 0$ .

### Solution:

(a) $\Leftrightarrow$ (b): First let  $M$  be free (and finitely generated). Then  $M \cong R^k$  for some finite  $k$ . Then take an injection  $\spadesuit$

**Problem 17.3**

Let  $R$  be a Frobenius algebra over a field  $k$ . Show that the global dimension of  $R$  is either zero or infinity.

**Solution:**

Assume that  $\text{gldim}(R) = n < \infty$ . Then for any  $R$ -module  $M$  both  $\text{projdim}_R(M)$  and  $\text{injdim}_R(M)$  are less than or equal to  $n$ . Take minimal projective and injective resolutions  $P_\bullet \xrightarrow{\varepsilon} M$  and  $M \xrightarrow{\eta} I_\bullet$  where in particular we have  $I_k = P_k = 0$  for all  $k > n$ . But then consider the diagram

$$\begin{array}{ccccccc}
 P_n & \longrightarrow & \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \xrightarrow{\eta \circ \varepsilon} & I_0 & \longrightarrow & \cdots & \longrightarrow & I_n & \longrightarrow & 0 \\
 & & & & & & \searrow \varepsilon & & \nearrow \eta & & & & & & \\
 & & & & & & & M & & & & & & & 
 \end{array}$$

where, since  $\varepsilon$  is epi and  $\eta$  is mono, the top row is still exact. Therefore since  $R$  is Frobenius, the  $I_k$  are all projective as well, so this is a projective resolution of  $I_n$ . Then  $\text{Ext}_R^i(I_n, N)$  (for any  $N$ ) is zero everywhere except (possibly) the zeroth position.

Here we consider two cases. Notice that  $n = 0$  if and only if  $M$  is projective and injective. In this case we are done since  $\text{projdim}_R(M) = \text{injdim}_R(M) = 0$ . If this is not the case,  $P_1$  and  $I_1$  are both not trivial. Then we can continue by noticing that the homology at  $\text{Hom}_R(P_1, N)$  is unaffected by this splicing of sequences. Observe

$$\text{Ext}_R^{n+1}(I_n, N) = \text{Ext}_R^1(M, N) = 0$$

which implies that  $M$  is projective (since  $N$  was arbitrary) and injective. But then all  $R$ -modules are projective and injective, so  $\text{gldim}(R) = 0$ . ♠

**Problem 17.4**

Let  $k$  be a field of positive characteristic  $p$ .

- (a) Show that the group algebra  $kG$  for a finite group  $G$  is Frobenius (in fact, symmetric).
- (b) Show that the restricted enveloping algebra  $\mathbf{u}(\mathfrak{gl}_n)$  is Frobenius.

**Solution:**

**Problem 17.5**

Let  $\underline{x} = (x_1, x_2, \dots, x_n)$  be a sequence of elements in a commutative ring  $R$ , and let  $K(\underline{x}) = K(x_1) \otimes \dots \otimes K(x_n)$  be the corresponding Koszul complex.

- (a) Prove the explicit formula for the differential in  $K(\underline{x})$ .
- (b) Show that  $K(\underline{x})$  is a graded commutative DGA and, moreover, that  $K(\underline{x}) \simeq \Lambda^*(V)$  where  $V = \bigoplus R x_i$ .

**Solution:**