Hopf Algebras

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Introduction

These are the notes I took in class during the Winter 2019 topics course *Math 582H - Hopf Algebras* at University of Washington, Seattle.

The course description follows:

This course is an introduction to Hopf algebras. In addition to basic material in Hopf algebra, we will present some latest developments in quantum groups and tensor and fusion categories. One of the newer topics is homological properties of Noetherian Hopf algebras of low Gelfand-Kirillov dimension. A good reference for the first two topics in the book *Hopf Algebras and Their Action Rings* by Susan Montgomery. Here is a list of possible topics:

- Classical theorems concerning finite dimensional Hopf algebras.
- Infinite dimensional Hopf algebras and quantum groups.
- Duality and Calabi-Yau property.
- Actions of Hopf algebras and invariant theory.
- Representations of Hopf algebras, tensor and fusion categories.

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If you don't know what a symmetric tensor category is, today is going to be a three star day. Max is 5.

1.1 Overview

We are shooting to understand two conjectures:

Conjecture (Etingof-Ostrik '04): If A is a finite dimentional Hopf algebra, then

$$\bigoplus_{i\geq 0} \operatorname{Ext}^i_A(k,k)$$

is Noetherian.

$$
\begin{tikzcd}
V \otimes V \otimes V \arrow[r, "\mathrm{id}_V \otimes m"] \arrow[d, "m \otimes \mathrm{id}_V"] & V \otimes V \arrow[d, "m"] \\
V \otimes V \arrow[r, "m"] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
k \otimes V \arrow[r, "\sim"] \arrow[dr, "u \otimes \mathrm{id}_V"'] & V & V \otimes k \arrow[l, "\sim"'] \arrow[dl, "\mathrm{id}_V \otimes u"] \\
& V \otimes V \arrow[u, "m"] &
\end{tikzcd}
$$

Figure 1: Diagrams for definition 1.2.1.

Conjecture (Brown-Goodearl '98): If A is a Noetherian Hopf algebra, then the injective dimension of A_A is finite.

These are both still open! In fact there is a meeting at Oberwolfach this March concerning exactly these conjectures.

1.2 Symmetric Tensor Categories

We are going to be using the following notation throughout:

- k is a field
- Vect_k is the category of k-vector spaces
  - Vect_k is closed under tensor products
  - There is an element $k \in \mathrm{Vect}_k$ such that

    $$k \otimes_k V \cong V \cong V \otimes_k k$$

    where the above isomorphisms are natural.
  - $V \otimes_k W \cong W \otimes_k V$
- An algebra is an object in Vect_k.

1.2.1 Definition
$V \in \mathrm{Vect}_k$ is called an **algebra object** if there are two morphisms

(a) $m : V \otimes V \to V$

(b) $u : k \to V$

such that the diagrams in figure 1.2 commute.

1.2.2 Lemma
$V \in \mathrm{Vect}_k$ is an algebra object iff V is an algebra over k.

1.2.3 Lemma
If C is a symmetric tensor category, so is C^{op}.

Then the natural thing to ask is: what is an algebra object in this opposite category?

1.2.4 Definition
A **coalgebra object** in C is an algebra object in C^{op}. Here we have comultiplication Δ and counit ε.

1.2.5 Remark: Naturally you could go about defining this from first principles and drawing the diagrams in figure 1.2 with the arrows reversed, but we are probably mature enough to do without that (saving my fingers from repetitive strain injury in the process.)

1.2.6 Lemma
Alg_k, defined as the category of algebra objects in Vect_k, is a symmetric tensor category. Furthermore Coalg_k, the category of coalgebra objects in Vect_k, is a symmetric tensor category.

1.2.7 Lemma
The following are equivalent:

(a) V is an algebra object in Coalg_k

(b) V is a coalgebra object in Alg_k

(c) There are morphisms $m, u, \Delta, \varepsilon$ such that

- (V, m, u) is an algebra
- (V, Δ, ε) is a coalgebra
- Equivalently:
  - m and u are coalgebra morphisms
  - Δ and ε are algebra morphisms.

Proof
The nice thing here is that the $(a) \Leftrightarrow (c)$ without the last condition. A similar fact holds for (b) except the second-to-last. The last thing to do is to prove the last two conditions are equivalent. ♠

Problem 1.1 *Fill in the details for the proof above.*

1.2.8 Definition
V is called a **bialgebra object** if V is an algebra object in Coalg_k.

Problem 1.2 *(a) Suppose that* $\operatorname{char} k \neq 2$*. Classify all bialgebras of* $\dim 2$.

(b) Do the same for $\operatorname{char} k = 2$.

Solution:
Part (a)

Consider $\varepsilon : V \to k$ and consider $\ker \varepsilon \lhd V$. Since ε cannot be the zero map, $\ker \varepsilon = kx$ for some $x \in V \setminus 0$. ♠