

# Algebraic Geometry

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## Abstract

A three-quarter sequence covering the basic theory of affine and projective varieties, rings of functions, the Hilbert Nullstellensatz, localization, and dimension; the theory of algebraic curves, divisors, cohomology, genus, and the Riemann-Roch theorem; and related topics.

## 1 September 25, 2019

The first thing that one asks is “what is geometry?” One needs to be able to answer this question before they define AG. One idea is that geometry is topology + structure.

### 1.1 What is Geometry?

#### Example 1.1

Exotic differentiable structures on a sphere. There are many different smooth structures, all of which are independent of the topology,

$S^1 \times S^1$  has infinitely many complex structures (remember the parallelograms)!

How to you go about defining the geometry of a thing? One idea from manifolds: charts. These describe the local models and the interesting part is how this comes together to a whole space.

There is another idea to capture the “local” model of geometry that underlies modern algebraic geometry: consider the map  $\varphi : W \rightarrow W' \in \mathbb{CP}^n$  and then say that this map is  $C^\infty$  if and only if its coordinate functions are. But the coordinate functions are problematic, so we can replace it with the following idea:

$\varphi : W \rightarrow W'$  is  $C^\infty$  if and only if for all  $C^\infty$  functions  $f : W' \rightarrow \mathbb{R}$ , the composition

$$\varphi^* f = f \circ \varphi : W \rightarrow \mathbb{R}$$

is  $C^\infty$ .

To capture the manifold structure on  $M$ , it is equivalent to know the set of  $C^\infty$  functions  $U \rightarrow \mathbb{R}$  for every open  $U \subseteq M$ .

## 1.2 The Big Idea

So then the idea we are talking away here is that *geometry is in the functions* that exist on a particular space!

Fix a field  $k$ .

**1.2.1 Definition:** A **space with functions** is a topological space  $X$  along with a collection (a  $k$ -algebra!)  $\mathcal{O}(U)$  of maps  $U \rightarrow k$  for each open  $U \subseteq X$ .

$\mathcal{O}(U)$  are called **regular functions** and must satisfy:

- Given an open cover  $U_\alpha$  of  $U$ , a function is regular if and only if its restrictions to each element of the cover is regular.
- If  $f : U \rightarrow k$  is regular, then  $D(f) = \{u \in U \mid f(u) \neq 0\}$  is an open set and  $\frac{1}{f} \in \mathcal{O}(D(f))$ .

For the next time, try to think of as many examples of this as you can. Next time will be a mind blowing example of a variety.

## 2 September 27, 2019

### Problem 2.1

*Do all the exercises in Kempf chapter 1!*

For now we assume that  $k$  is algebraically closed.

### 2.1 Examples of spaces with functions

There were lots of suggestions, but here are a couple:

#### Example 2.1

Let  $X = \mathbb{S}^2$  and let  $\mathcal{O}_X^{cts}$  be the continuous  $\mathbb{C}$ -valued functions. Alternatively we could consider a different sheaf  $\mathcal{O}_X^{an}$ , the holomorphic functions. Or we could consider  $\mathcal{O}_X^\infty$ , the  $C^\infty$  functions (under some smooth structure).

**2.1.1 Definition:** A **morphism** of spaces with functions between  $(X, \mathcal{O}_X)$  and  $(Y, \mathcal{O}_Y)$  is a continuous map  $\phi : X \rightarrow Y$  such that for all  $U \subseteq Y$  open and  $f \in \mathcal{O}_Y(U)$ , the function

$$\phi^* f = f \circ \phi|_{\phi^{-1}(U)} : \phi^{-1}(U) \rightarrow k \in \mathcal{O}_X(\phi^{-1}(U))$$

In other words, a morphism of spaces with functions is a map of spaces that *respects the regular functions*.

### Example 2.2

Let  $X, Y$  be topological spaces and let  $\mathcal{O}_X$  and  $\mathcal{O}_Y$  be the continuous functions. Then morphisms are just continuous maps.

### Example 2.3

When  $X$  and  $Y$  are manifolds and  $\mathcal{O}_\bullet$  are complex-valued functions, then the morphisms are maps of manifolds.

So now we return to the examples we saw before:  $(\mathbb{S}^2, \mathcal{O}^\infty)$ ,  $(\mathbb{S}^2, \mathcal{O}^{cts})$ , and  $(\mathbb{S}^2, \mathcal{O}^{an})$ . A natural question to ask is when we have morphisms between these spaces to see if there exist ones that are the identity on  $\mathbb{S}^2$ .

Consider the identity map from the continuous to the analytic functions. Then take any map  $f \in \mathcal{O}^{an}$  and consider that

$$f = f \circ id_{id^{-1}(U)} : U \rightarrow k \in \mathcal{O}^{cts}(U)$$

and there is no map in the other direction.

**2.1.2 REMARK:** Notice that since we are pulling functions back, the maps go in the opposite direction as you may think at first.

We can also talk about **open subspaces**. If  $V \subseteq X$  is an open subset, we can let the induced space with functions be  $(V, \mathcal{O}_V)$  where if  $U \subseteq V$  then  $\mathcal{O}_V(U) := \mathcal{O}_X(U)$ .

## 2.2 Varieties

**2.2.1 Definition:** An **affine  $k$ -variety** is a space with functions  $(Y, \mathcal{O}_Y)$  such that for every space with functions  $(X, \mathcal{O}_X)$ , the natural map

$$\mathrm{Hom}((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)) \rightarrow \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$$

is a bijection and furthermore  $\mathcal{O}_Y(Y) =: k[Y]$  is a finitely generated  $k$ -algebra.

**2.2.2 Remark:** The idea here is that the algebra maps (on the right) are precisely the same as the geometry maps (on the left). Algebraic geometry, baby.

So then this leads to a very simple (loose) definition:

**2.2.3 Definition:** A **variety** is something that is covered by affine varieties.

#### Example 2.4

$\mathbb{A}^1 = k$ . Give this space the cofinite topology. Then if we have  $U = k \setminus \{x_1, \dots, x_n\} \subset \mathbb{A}^1$ ,

$$\mathcal{O}_{\mathbb{A}^1}(U) = \{f(t) \in k(t) \mid \text{poles are in } \{x_i\}\}$$

#### Problem 2.2

*Show that  $\mathbb{A}^1$  is an affine variety!*

**2.2.4 Remark:** Notice that this statement is equivalent to saying that any morphism of spaces with functions gives us a regular map  $X \rightarrow k$ .

## 3 September 30th, 2019

One question that was asked: if we have fixed the underlying topological space in a space with functions, must there be a morphism between them somehow? Might there instead be a common cover of the two?

**Example 3.1**

Let  $k$  be a field with some topology on it such that every point is closed (you could do the discrete topology). Let  $\tilde{\mathcal{O}}(U)$  be the continuous functions  $U \rightarrow k$ . In other words, these functions are locally constant.

Locally constant functions behave nicely under restrictions to opens, of course. The other axioms are also great.

Have we really found an initial object in our category? This would be enough to establish a “tent” (as in localization of categories). Try this out and see what happens!

**3.1 The question of affine space**

Recall the question about whether  $\mathbb{A}^1$  is an affine variety. The idea here is that  $\phi : X \rightarrow k$  is a morphism of spaces with functions if and only if it is regular (that is, in  $\mathcal{O}_{\mathbb{A}^1}$ ).

One direction is tautological (a morphism to  $\mathbb{A}^1$  has a polynomial underlying it), so let  $\phi$  be regular. Then to see that  $\phi$  is continuous can be checked by pulling back all closed sets. The important observation is that  $D(f - a) = X \setminus \phi^{-1}(a)$ , which is closed (an axiom for spaces with functions).

The last thing to check is where  $\phi$  pulls back regular functions to regular functions. This relies on the facts that  $\mathcal{O}_X$  is a  $k$ -algebra and that  $\phi(x) - b_j$  is regular on  $U$  when  $b_j \notin U$ .

**3.2 Algebra maps**

Notice that since we have a condition that  $\mathcal{O}_X(X)$  must be finitely generated as a  $k$ -algebra, this means that

$$\mathrm{Hom}(X, Y) = \mathrm{Hom}_k(\mathcal{O}_Y(Y), \mathcal{O}_X(X)) = \mathrm{Hom}_k(k[x_1, \dots, x_n]/(f_1, \dots, f_m), \mathcal{O}_X(X))$$

and

$$\mathrm{Hom}(X, Y) = \{(\gamma_1, \dots, \gamma_n) \in (\mathcal{O}_X(X))^n : f_j(\gamma_i) = 0, \forall j = 1, \dots, m\}$$

In other words, we are looking at maps that factor through  $Z$ :

$$\begin{array}{ccc} (\gamma_1, \dots, \gamma_n) : X & \longrightarrow & k^n \\ & \searrow & \uparrow \\ & & Z = Z(f_i) \end{array}$$

Now what we want to say is that  $Y = Z$ . That is, *affine varieties are closed subsets of affine spaces*.

Now this is all good, but the problem is that we had to *choose* a presentation of  $\mathcal{O}_Y(Y)$  to get this picture. of course we want something more canonical! We will see in this class (and in Kempf) that this can be done.

## 4 October 2, 2019

### 4.1 Questions without (complete) answers

#### 4.1.1 Morphisms and stuff

A question to get things started for the day. Let  $X$  and  $Y$  be spaces with functions and let  $Y$  be an affine variety and let  $f : Y \rightarrow X$  be a map of sets (but with no further assumption on  $f$ ). This naturally induces a map from  $\mathcal{O}_X(X)$  to the functions  $\text{Hom}_{\text{set}}(Y, k)$  (which clearly contains the regular functions on  $Y$ ).

Further assume that there exists a  $\gamma : \mathcal{O}_X(X) \rightarrow \mathcal{O}_Y(Y)$ . We know that since  $Y$  is affine,  $\gamma$  corresponds to a morphism  $\varphi : Y \rightarrow X$ . Then the question is: when does  $f = \varphi$ ? We've already answered this question for  $\mathbb{A}^1$ , notice.

#### 4.1.2 Algebraic closure

Where do we use algebraic closure of the base field? It has been swept under the rug a bit, but consider the function

$$\frac{1}{x^2 + 1} : \mathbb{R} \rightarrow \mathbb{R}.$$

This certainly seems like it should be a regular function (e.g. it is rational and defined everywhere on  $\mathbb{R}$ ) but this conflicts with the idea that we want to identify  $\mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1) = k[t]$ , but that is clearly not the case here. Think about this.

#### 4.1.3 Yet another

Consider the set  $R$  of all continuous maps  $k \rightarrow k$  under the cofinite topology. Someone asked if  $R$  is a  $k$ -algebra. The answer is a bit convoluted, but the short answer is no. Specifically if we are using the product topology on  $k \times k$ , the addition map isn't continuous! This also points to the question of what topology is the correct one to use on these things.

### 4.2 Back to affine varieties

Recall that we constructed a (highly-non-canonical) picture of how any affine variety arises as a closed subset of some affine space  $k^n$ .

We want to remove this dependence on presentation, however, and that is what we are working toward.

#### 4.2.1 Affine Space

Now we focus in on  $\mathbb{A}^n = k^n$ . We really want that the projection functions  $x_i : k^n \rightarrow k$  should be regular. But since we want this (eventually) to form a  $k$ -algebra, we want that each  $f \in [x_1, \dots, x_n]$  should be regular!

The axioms of a space with functions tells us that the **vanishing locus**

$$Z(f) = \{a \mid f(a) = 0\} \subseteq k^n$$

and furthermore  $Z(S)$  should be closed for all  $S \subseteq k[x_1, \dots, x_n]$ . This leads us to a definition:

**4.2.1 Definition:** A subset  $Z \subseteq k^n$  is **Zariski-closed** if there exists an  $S \subseteq k[x_1, \dots, x_n]$  such that  $Z = Z(S)$ .

#### 4.2.2 Lemma

The Zariski closed sets are the closed sets of a topology (called the **Zariski Topology**).

PROOF

Just do it. Nike. ✓

**4.2.3 REMARK:** Notice that here the set  $\{(a, -a)\} \subseteq k^2$  (the pullback of zero under the addition map) is Zariski closed! This fixes the problem we were running into in the third question (sec. 4.1.3) above.

Now since  $Z(S) = Z(I_S)$  where  $I_S$  is the ideal generated by  $S$ , it is enough to consider vanishing loci of ideals. Furthermore we have the map that extracts the ideal of functions that vanish on a set  $Z \subseteq k^n$ . There are a ton of great identities you can prove here. Go to your favorite algebra book (e.g. Dummit & Foote) to see them.

#### 4.2.2 Functions

What about functions on these spaces? If we take  $f \in k[x_1, \dots, x_n]$  these seem like they should be regular functions  $k^n \rightarrow k$ .

#### 4.2.4 Theorem ((Weak) Nullstellensatz)

Say  $k = \bar{k}$ . Then every maximal ideal  $\mathfrak{m} \triangleleft k[x_1, \dots, x_n]$  has the form  $(x_1 - a_1, \dots, x_n - a_n)$ .

**4.2.5 REMARK:** Equivalently, it is the kernel of a  $k$ -algebra morphism  $k[x_1, \dots, x_n] \rightarrow k$ .

#### 4.2.6 Corollary (Nullstellensatz)

Let  $J$  be an ideal of  $k[x_1, \dots, x_n]$ . Then  $I(Z(J)) = \sqrt{J}$ .

PROOF

*Notice this only works when  $k$  is uncountable!* Suppose that  $\mathfrak{m}$  is a maximal ideal with residue field  $L = k[x_i]/\mathfrak{m}$ . This gives us a surjection of  $k[x_1, \dots, x_n] \rightarrow L$ . Thus  $\dim_k L$  is countable!

But  $\dim_k k(t)$  is uncountable! The proof here is that the  $\frac{1}{t-\lambda}$  for  $\lambda \in k$  is a linearly-independent collection. So then  $L/k$  is algebraic, and since  $k = \bar{k}$   $L = k$ . ♠

## 5 October 4, 2019

Today we are going to be talking a bit more about the existence of affine varieties. Max talked a bit about the philosophy of work in this course: he made this extended metaphor concerning butterflies but the take-away is to take learning onto ourselves. :)

### 5.1 Questions from last time

#### 5.1.1 Maps and elements

In the book we did this silly thing where we defined  $\text{Spec} A \stackrel{\text{def}}{=} \text{Hom}_{\text{Alg}}(A, k)$  and then identified  $A$  with  $k[\text{Spec} A]$  by  $a(f) = f(a)$ . This seems a bit silly at first, but it may have something to do with the fact that we are looking for a natural way to construct affine varieties without having to choose a presentation. We will hopefully see something about this by the end of the day.

### 5.2 Back to the Nullstellensatz

Recall that we defined the operators  $Z$  and  $I$  that “do the work” of the Nullstellensatz. We then wrote (cor. 4.2.6)  $I(Z(J)) = \sqrt{J}$ . The idea is that this will give us the function structure on an affine variety.

PROOF (COR. 4.2.6)

One way is not too hard. For the more difficult direction: Let  $g \in I(Z(J))$ . Then  $Z(J) \subseteq Z(g)$ . Now notice that  $D(g)$  can be naturally identified with  $\text{Spec } k[x_i][1/g]$ . Then consider

$$J' = Jk[x_i][1/g]$$

and the key realization is that  $J'$  cannot be contained in any maximal ideal. The idea is that you can work by contradiction: this implies that  $J$  is contained in an element of  $D(g)$ , but it isn't!

Thus  $J' = (1)$ . So we can write  $1 = \frac{f}{g^N}$ . Thus  $g^N(f - g^N) = 0$  in  $k[x_i]$  and since  $g$  isn't nilpotent,  $f = g^N$ . ♠

#### 5.2.1 Corollary

There is a lattice anti-isomorphism between the radical ideals in  $k[x_i]$  and Zariski-closed subsets  $Z \subseteq k^n$  via the maps  $J \rightarrow Z(J)$  and  $Z \mapsto I(Z)$ .

#### 5.2.2 Corollary

For any ideal  $J \subseteq k[x_i]$ ,

$$\sqrt{J} = \bigcap_{\text{maximal } \mathfrak{m} \supset J} \mathfrak{m}$$

5.2.3 REMARK: “The functions that vanish at the zero locus of  $J$  are precisely those that vanish at all the points of  $J$ ”.



### 5.2.4 Corollary

$D(g) \subseteq k^n$ . Then the map

$$k[x_i][1/g] \rightarrow \text{Hom}(D(g), k)$$

via the map

$$\frac{f}{g^N} \mapsto \left( x \mapsto \frac{f(x)}{g(x)^N} \right)$$

is injective.

## 5.3 Affine space

Let's define  $\mathbb{A}^n \stackrel{\text{def}}{=} k^n$  with the Zariski topology. Let

$$\mathcal{O}_{\mathbb{A}^n}(U) = \{f \in k(x_1, \dots, x_n) \mid \text{poles}(f) \subseteq \mathbb{A}^n \setminus U\} \subseteq \text{Hom}(U, k)$$

Then, for example,

$$\mathcal{O}_{\mathbb{A}^n}(D(g)) = k[x_1, \dots, x_n][1/g].$$

### 5.3.1 Proposition

$\mathbb{A}^n$  is an affine variety.

PROOF

$\phi : X \rightarrow \mathbb{A}^n$  gives us maps  $\phi_1, \dots, \phi_n : X \rightarrow k$ . Then that  $\mathbb{A}^n$  is affine relies on the fact:  $\phi$  is a morphism if and only if the  $\phi_i$  are regular. One direction is not too bad since coordinate functions are regular by the axioms of morphisms. The other direction needs to be completed! DO IT! ♠

## 6 October 7th, 2019

### 6.1 Questions/Discussions

#### 6.1.1 Initial and final objects

We asked before whether the space with functions  $(X, \mathcal{O}^{\text{loc. constant}})$  is an initial or terminal object. Adam asserts that it is a final object in the category of spaces with functions where the underlying space is  $X$  (I believe this is sheaves over  $X$ ).

Notice we can't use all continuous maps where  $k$  has the discrete topology, since  $0 \in k$  is not closed.

#### 6.1.2 Subalgebras and rings of functions

Assume that  $f : B \rightarrow A$ . This gives us a nice map  $\tilde{f} : \text{Spec } A \rightarrow \text{Spec } B$ . One question we may have is "if  $f$  is injective, does this imply that  $\tilde{f}$  is surjective?"

Consider an example: Say  $B = k[t] \hookrightarrow k[s, t] = A$ . Then any function on  $A$  looks like  $(s-a, t-b)$  and the map induced on functions is just projection, so this gives us the map  $(t-b)$ , which is all the maps on  $B$ .

Another example: consider the map  $\mathbb{C}[t] \rightarrow \mathbb{C}[s]$  sending  $t \rightarrow s^2$ . Then this induces a map  $z \rightarrow z^2$  from  $\mathbb{C} \rightarrow \mathbb{C}$  (why?) which is again surjective.

Next consider  $k[x, y]/(xy-1) = k[x][1/x] = A$ , which is a hyperbola over  $\mathbb{R}$ . Then the localization map  $B = k[x] \hookrightarrow A$  induces a map that is basically the identity everywhere *except zero*. So it is **not surjective**.

Some properties to notice: examples one and two are *flat* extensions. The second is a **finite** extension. The third is neither. We will investigate what is going on further later on.

One idea: consider whether the map  $X = \text{Spec } B \rightarrow \mathbb{A}^2 \setminus (\mathbb{A}^1 \setminus \{0\})$  exists. One of the things that he keeps questioning is whether the target space is open or closed as a subset of affine space. (Note that a map is proper if it sends closed to closed).

## 6.2 Back to affine varieties

Continuing our proof/discussion from last time, we were considering  $\varphi : X \rightarrow \mathbb{A}^n$  which we said was a morphism iff each coordinate  $\phi_i : X \rightarrow k$  is regular. (This basically follow since  $\varphi$  is continuous and sends regular functions to regular functions.

For the regularity, consider  $U \subseteq \mathbb{A}^n$ . Then  $U$  admits a cover of  $D(g)$ , so it suffices to check where  $\phi$  pulls back regular maps on  $D(g)$ . Of course this is  $k[x_1, \dots, x_n][1/g]!$  So consider the image of

$$\frac{1}{g^N} \sum a_i x^i$$

and its image in  $\phi^{-1}(D(g)) = D(\phi^*(g))$  is

$$\frac{\sum a_i (\phi^* x)^i}{(\phi^* g)^N}$$

**6.2.1 REMARK:** Big idea: We started with the “dream”: that there is a correspondence between the algebra and the geometry. This is our main guiding principle, so we know we’ve found the “right” topology when we have found one that supports this dream. This is an answer to the question “why is the Zariski topology not just a degenerate case?”

### 6.2.1 Affine varieties in general

Let  $J \subseteq k[x_1, \dots, x_n]$  be a radical ideal. We know this corresponds (uniquely!) to a subset  $Z \subseteq \mathbb{A}^n$ . Here we can consider  $(Z, \mathcal{O}_Z)$  where  $Z$  is a subspace of  $\mathbb{A}^n$ .

Then take any closed  $W \subseteq Z$ , which is the intersection (by definition of the topology)

$$\bigcap_{i \in I} Z(f_i) \quad f_i \in k[x_i]/J$$

and then  $\mathcal{O}_Z(D(g)) = \frac{k[x_i]}{J}[1/g]$  (note we used the nullstellensatz here!).

So then we claim that  $Z$  is an affine variety. To see this, consider a map  $\varphi : X \rightarrow Z$  and the composition

$$X \xrightarrow{\varphi} Z \hookrightarrow \mathbb{A}^n$$

so topologically  $X$  factors through  $Z$  if and only if  $J \subseteq \mathcal{O}^{\mathbb{A}^n}(\mathbb{A}^n)$  maps to zero in  $\mathcal{O}_X(X)$ .

The takeaway here is that a morphism  $X \rightarrow \mathbb{A}^n$  factors through  $Z$  topologically if and only if it factors in the categorification of spaces with functions.

## 7 October 11, 2019

Today we are going to talk some more about varieties. Coming up on the horizon is a discussion of the functor-of-points perspective and we'll talk about Yoneda.

### 7.1 Questions/Discussion

A group of students met up on Wednesday (which we skipped for Yom Kippur) and were talking about how to prove the statement: "points in an affine variety are closed."

The Nullstellensatz gives us that a point  $(x_1, \dots, x_n) \in X$  corresponds to the vanishing locus of  $(x_i - a_i)$ . What if we use the definition in the book, though? If  $X = \text{Spec } A = \text{Hom}_{\text{Alg}_k}(A, k)$  (where  $A$  is a reduced finitely generated  $k$ -algebra). But then the points are the maximal ideals  $\mathfrak{m} \in k[X] = A$ , which are exactly the points we want!

### 7.2 Back to Varieties

Recall the definition of a variety: we are considering  $(X, \mathcal{O}_X)$  is a space of functions. Then the idea we want is that we want to say that this thing is locally affine. But if you consider infinitely many copies of  $\mathbb{A}^1$  intersecting pairwise, there is an obvious cover by (affine)  $\mathbb{A}^1$ 's. This shouldn't be affine. It doesn't embed in an affine space, for example.

So the definition we used (and the one in Kempf) is that we must have a *finite* cover by affine spaces. We wanted to discuss some examples.

- Clearly all affine varieties are varieties.
- $\mathbb{P}^1$  is our first nontrivial example. As a space it is the one-point compactification of  $\mathbb{A}^1$ . The functions  $\mathcal{O}_X(U)$  are the rational functions in  $t$  with poles not in  $U$ . You can also construct it by taking the morphism

$$\mathbb{G}_m \xrightarrow{t \mapsto 1/t} \mathbb{G}_m$$

and gluing along this morphism to get a copy of  $\mathbb{P}^1$ ! We've already shown in Kempf that it is not affine. What happens if we were to pick the identity above? We get the line with two origins. It's a variety! But notice that (under the Euclidean topology) the space isn't Hausdorff!

**Problem 7.1**

*If  $X$  and  $Y$  are affine varieties, is  $X \sqcup Y$  affine?*

**Problem 7.2**

*How can we express the non-Hausdorffness of the line/plane with two origins in the Zariski topology?!*

### 7.3 Varieties Glue

We've been throwing things around, but this is important to write down: Start with  $U_i, i \in I$ , where  $I$  is finite (although we could drop finiteness if we don't care about the thing being a variety). For all  $i, j$ , we have open subsets  $V_{ij} \subseteq U_i$  such that  $V_{ii} = U_i$  and isomorphisms  $\varphi_{ij} : V_{ij} \xrightarrow{\sim} V_{ji}$  of varieties such that

- (a)  $\varphi_{ii} = \text{id}$
- (b)  $\varphi_{ij}(V_{ik} \cap V_{ij}) = V_{ji} \cap V_{jk}$
- (c)  $\forall i, j, k, \varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$  on  $V_{ij} \cap V_{ik}$

then there exists a unique package  $(X, \iota_i : U_i \hookrightarrow X)$  where  $X$  is a variety and each  $\iota_i$  is an open embedding.

One idea is you can rephrase this in categorical language as the colimit of a diagram in a category. Hmmm

## 8 October 14th, 2019

We're going off-book a bit to talk about

### 8.1 Yoneda Lemma

Recall that we said that  $Y$  was affine if

$$\text{Hom}(X, Y) \cong \text{Hom}_k(\mathcal{O}_Y(Y), \mathcal{O}_X(X)).$$

Now we are going to take some time to put this into a broader context. Let  $\mathcal{C}$  be a category. We get naturally two functors

$$h_a : \mathcal{C}^{op} \rightarrow \mathbf{Set} \quad h^a : \mathcal{C} \rightarrow \mathbf{Set}$$

where

$$h_a(b) = \text{Hom}_{\mathcal{C}}(b, a) \quad h^a(b) = \text{Hom}(a, b).$$

Now given a map  $f : a \rightarrow a'$ , we get natural transformations  $f \circ - : h_a \rightarrow h_{a'}$  and  $- \circ f : h^{a'} \rightarrow h^a$ . Now notice that if  $*$  is a one-point variety, then  $h_X(*) = |X|$ , the underlying point set of  $X$ . Notice that in a similar way  $h_X(\mathbb{A}^1)$  is something like the “line space” of  $X$ .

Notice that we then get a functor

$$h_{(-)} : \mathcal{C} \rightarrow \mathbf{Func}(\mathcal{C}^{op}, \mathbf{Set})$$

and we should **think** that a functor  $\mathcal{C}^{op} \rightarrow \mathbf{Set}$  is a **space over**  $\mathcal{C}$ . That is, we can think of  $\mathcal{C}$  as the category of open sets of a topological space. We can say

$$\text{Hom}(U, V) = \begin{cases} \emptyset, & U \not\subseteq V \\ \{\emptyset\}, & \text{otherwise.} \end{cases}$$

Notice the category  $\mathbf{Open}(X)$  gives us a functor  $\phi : \mathbf{Open}(X)^{op} \rightarrow \mathbf{Set}$  where we map an open  $U \subseteq X$  to  $\phi(U)$ . For instance if  $F$  is a vector bundle over  $X$ , we can define

$$\phi_F(U)$$

to be the sections over  $U$  of the covering map. Cool.

So we’re working with the slice category  $\mathbf{Top}/X = \{X \rightarrow Y, \text{cts}\}$ . Here morphisms are maps  $Y \rightarrow X$  satisfying

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & Z \\ & \searrow & \swarrow \\ & X & \end{array}$$

So now fix  $F \rightarrow X$ . Then the map  $\phi_F(Y \rightarrow X)$  which is the collection of diagrams of the form above ( $Z = F$ ).

Then if we look at points  $x \hookrightarrow X$ , we get that  $\phi_F(x \rightarrow X)$  is the fiber of  $F$  over  $x$ . One could hope that somehow you could recover all of  $F$  from these maps, and that is precisely the content of

### 8.1.1 Lemma (Yoneda)

The functor  $h : \mathcal{C} \rightarrow \mathbf{Func}(\mathcal{C}^{op}, \mathbf{Set})$  is fully faithful.

8.1.2 REMARK: That is,

$$\text{Hom}_{\mathcal{C}}(a, a') \rightarrow \text{Hom}_{\mathbf{Func}(\mathcal{C}^{op}, \mathbf{Set})}(h_a, h_{a'})$$

is a bijection.

PROOF

This proof is so tautological it is sometimes confusing to prove. We can look it up in any of our old notes or books but the idea is to look at the image of the identity map. ♠

Next time we will see a bunch of examples and exercises.

## 9 October 16

We're going to finish up with Yoneda! The milestone for the next week or so: we should have seen and digested some of the algebra in the early sections of chapter 2. Max will assume we'll have seen it already starting Friday(ish). We should shoot for having all the problems in chapter 2 done by a week from Friday (or so).

### 9.1 Old Questions

We spoke a bit about lines glued together and the automorphisms of one of the lines that extend to the entire space. This necessitates talking about what the automorphisms of  $\mathbb{A}^1$  are.

We also talked about whether the coordinate axes in  $\mathbb{A}^3$  are isomorphic as varieties to the projection of these axes onto a plane. We discussed that this must induce an isomorphism between the coordinate algebras since they are affine.

### 9.2 Back to Yoneda

Some useful notation for encoding Yoneda: given a functor  $F : \mathcal{C}^{op} \rightarrow \mathbf{Set}$ , we say that an object  $a \in \mathcal{C}$  *represents*  $F$  if  $h_a \simeq F$ . Then another formulation of Yoneda says that a representing object of a representable functor is unique up to isomorphism of representing objects.

$$h_a \xrightarrow{\sim} F \xleftarrow{\sim} h_{a'} \\ \quad \quad \quad \sim$$

$$a \rightarrow a'$$

9.2.1 REMARK:  $h_a$  is called the *functor of points of*  $a$ .

#### Example 9.1

Given  $f_1, \dots, f_n \in k[x_i]$ , consider the  $k$ -algebra  $k[x_i]/(f_i)$ . Then Yoneda gives us that  $A$  is uniquely determined by the functor

$$h^A : \mathbf{Alg}_k \rightarrow \mathbf{Set}$$

via

$$B \mapsto \mathrm{Hom}(A, B) = \{(b_1, \dots, b_n) \in B^n \mid f_i(b_1, \dots, b_n) = 0\}.$$

**Example 9.2**

The functor  $h_{\mathbb{P}^n} : \mathbf{Alg}_k \rightarrow \mathbf{Set}$  is

$$A \mapsto \{A \twoheadrightarrow L \mid L \text{ invertible } A\text{-module}\} / \cong$$

This is apparently very important.

**9.3 Representable or not?**

- $\mathbb{G}_a$  as a functor from  $\mathbf{SpcFun}^{op} \rightarrow \mathbf{Set}$  is represented by  $\mathbb{A}^1$ .
- $\mathbb{G}_m$  as a functor from  $\mathbf{SpcFun}^{op} \rightarrow \mathbf{Set}$  is represented by  $\mathbb{A}^1 \setminus \{0\}$ .
- $\mathrm{GL}_n : \mathbf{SpcFun}^{op} \rightarrow \mathbf{Set}$  which is represented by  $\mathrm{Spec} k[x_{ij}]_{\det}$
- $|\cdot| : \mathbf{SpcFun} \rightarrow \mathbf{Set}$  underlying set represented by  $*$
- $\emptyset : \mathbf{SpcFun}^{op} \rightarrow \mathbf{Set}$  sending  $X \mapsto \emptyset$ . Not representable!
- $\{\emptyset\} : \mathbf{SpcFun}^{op} \rightarrow \mathbf{Set}$  sending  $X \mapsto \{\emptyset\}$ . Representable by  $*$ .
- $\mu_n : \mathbf{SpcFun}^{op} \rightarrow \mathbf{Set}$  sending  $X \mapsto \{f \in \mathcal{O}_X(X) \mid f^n = 1\}$  represented by  $\mathrm{Spec} k[t](t^n - 1)$

**10 October 18th, 2019**

This is something that Max just mentioned: what is  $\mathcal{O}_X(\emptyset)$ ? There is a single map here!

**10.1 Old Questions**

Is  $U = \mathbb{A}^2 \setminus (0,0)$  affine? Well first notice

$$\mathcal{O}_U(U) \subseteq \bigcap_{f \in k[x,y] \text{ irred}} k[x,y][1/f] = k[x,y]$$

**Why is the last equality true?! We're working over a UFD.** But then of course  $U$  can't be affine since  $U \neq \mathbb{A}^2 = \mathrm{Spec} k[U]$ . Another way to think of this is that every isomorphism of coordinate rings yields an isomorphism of spaces!

## 10.2 Back (again) to affine varieties

### 10.2.1 Theorem

- Let  $\mathcal{A} \subseteq \mathbf{Alg}_k$  be the subcategory of finitely-generated, reduced  $k$ -algebras. Then  $\mathrm{Spec}$  defines an equivalence of categories

$$\mathrm{Spec} : \mathcal{A}^{op} \rightarrow \mathbf{Aff}$$

- The functor

$$X \mapsto \mathrm{Spec} \mathcal{O}_X(X) : \mathbf{SpcFun}^* \rightarrow \mathbf{Aff}$$

is left adjoint to the canonical inclusion  $\mathbf{Aff} \subseteq \mathbf{SpcFun}^*$ , where  $\mathbf{SpcFun}^*$  is the subcategory of  $\mathbf{SpcFun}$  such that the global sections are finitely generated.

Thus  $X \rightarrow \mathrm{Spec} \mathcal{O}_X(X)$  is universal for maps to affine varieties.

#### Problem 10.1

*The last thing is saying that every morphism  $X \rightarrow \mathrm{Spec} A$  factors (uniquely) through  $X \rightarrow \mathrm{Spec} \mathcal{O}_X(X)$ . Show this!*

## 11 October 21st, 2019

Today we are moving on from Yoneda to speak a bit about the topological properties of algebraic varieties.

Later we will do a lot more examples with the functor of points perspective, but for now the idea to keep around is that this enables us to study something *in relation to something else*.

### 11.1 Questions

We thought about  $\mu_n : \mathbf{SpcFun}^{op} \rightarrow \mathbf{Set}$ . Notice that (at least when  $\mathrm{char} k \nmid n$ ), we get that  $k[t]/(t^n - 1)$ , the representing algebra, is  $\prod_{\zeta \in \mu_n(k)} k$ . Probably the best way to understand this is that it is  $n$  points, but moreso that it is embedded in a natural way in  $\mathbb{G}_m$  via the SES:

$$\mu_n \hookrightarrow \mathbb{G}_a \rightarrow \mathbb{G}_m$$

### 11.2 Topological properties of varieties

**11.2.1 Definition:** A topological space  $X$  is **quasi-compact** if every open cover has a finite subcover.

**11.2.2 Remark:** In France, compactness requires a space be Hausdorff. Thus in the development of AG we used the French definition and it stuck!



**11.2.3 Lemma**

An affine variety is quasi-compact.

PROOF

The idea was that if  $X = \cup U_i$ , then since  $D(f)$  generate the topology for  $X$ , we can refine the cover so that  $U_i = D(f_i)$ . By the Nullstellensatz,  $X = \cup D(f_i)$ , we know that  $(f_i) \triangleleft \mathcal{O}_X(X)$  is the whole ring!

But then  $1 = \sum_{j=1}^n a_j f_{i_j}$ , so we can restrict to a finite cover! ♠

But note that we didn't need that our ring was Noetherian! So the exact same proof shows that  $\text{Spec } \mathbb{C}[x_1, x_2, \dots]$  is quasi-compact! But  $\text{Spec } \mathbb{C}[x_i]/(x_1, \dots, x_n, \dots)$  is not quasi-compact!!!

**11.2.4 REMARK:** A very cool (“AMAZING” according to Max) fact is the following: if  $U \subset X$  is any open in an affine space, then  $U$  is quasi-compact. Compare a similar idea in Hausdorff spaces, where this is actually false.

Let  $X = \text{Spec } A$  where  $A = k[x_1, \dots, x_n]/(f_1, \dots, f_m)$ , which is Noetherian by Hilbert basis and correspondence theorem. Now we have an anti-isomorphism of lattices between the radical ideals in  $A$  and the closed sets in  $X$ . Thus

**11.2.5 Definition:** A topological space  $X$  is Noetherian if any **descending** chain of closed subspaces stabilizes.

**11.2.6 REMARK:** Assuming the axiom of choice, this is equivalent to the statement “A non-empty set of closed subsets in  $X$  has a minimal element.”

**11.2.7 REMARK:** Note that  $\text{Spec } A$  is Noetherian for any (finitely-generated)  $k$ -algebra.

So let  $X$  be a Noetherian topological space.

**11.2.8 Lemma**

$Z \subseteq X$  is closed implies that  $Z$  is Noetherian and quasi-compact.

**11.2.9 REMARK:** The Noetherian bit is pretty clear! The quasi-compact part needs a bit of arguing. Use the definition that  $X$  is quasi-compact if and only if  $\cap Z_i = \emptyset \Rightarrow \cap_{j=1}^n Z_{i_j} = \emptyset$  for some  $i_j$ .

Then using choice and the contrapositive it falls out!

**11.2.10 Definition:** A topological space  $X$  is **irreducible** if  $X \neq \emptyset$  and if  $X = X_1 \cup X_2$  where the  $X_i$  are both closed implies that either  $X = X_1$  or  $X = X_2$ .

**11.2.11 REMARK:** Equivalently, if  $U_1$  and  $U_2$  are open and nonempty then  $U_1 \cap U_2 \neq \emptyset$ . We also have that any non-empty open set is dense.

## 12 October 23rd, 2019

The plan moving forward: we are going to talk about some more ideas not from the book about topology and other topics. Then we will rejoin the book with chapter three (skipping the discussion of dimension theory).

Here's where we're aiming: last time we talked about what it meant for a Noetherian topological space (or variety) to be irreducible. In particular for any Noetherian topological space.

### 12.0.1 Theorem

$X$  admits a decomposition

$$X = X_1 \cup \dots \cup X_n$$

with each  $X_i$  irreducible and for all  $i \neq j$ ,  $X_i \not\subseteq X_j$ .

Furthermore, given two such decompositions

$$X = X_1 \cup \dots \cup X_n = Y_1 \cup \dots \cup Y_m$$

then there exists a function  $\iota : [n] \rightarrow [m]$  such that  $X_i = Y_{\iota(i)}$ .

Recall that when  $X = \text{Spec } A$ , the irreducible components  $X_i$  of  $X$  correspond precisely to the minimal primes  $\mathfrak{p} \triangleleft A$ .

## 12.1 Noetherian Induction

Let  $X$  be a Noetherian topological space and let  $P$  be a set of closed subsets of  $X$ . Suppose that for all  $Y \subseteq X$  closed, if

$$\text{for all closed } Z \subsetneq Y, Z \in P \text{ implies that } Y \in P$$

Then  $X \in P$

PROOF

By contrapositive: assume that  $X \in P^c$ . But now we can use axiom of choice: there exists a minimal  $W \in P^c$ . But then for any closed  $Z \subsetneq W$ , we have  $Z \in P$ , so  $W \in P$ , a contradiction. But then  $P^c = \emptyset = X$ . ♠

Now we can prove theorem 12.0.1

PROOF

$\emptyset$  has the empty decomposition. Thus  $\emptyset \in P$ . Now if  $Y \subset X$  is closed, then either  $Y$  is irreducible and  $Y \in P$  ("admits a decomposition") or else  $Y = Z \cup W$  and assuming each are in  $P$  we get that  $Y \in P$  by concatenating (and eliminating). Thus by Noetherian induction we get our decomposition.

To get the last bit, take two decompositions. Then consider

$$X_1 = X_1 \cap X = (Y_1 \cap X_1) \cup \dots \cup (Y_m \cap X_1).$$

Then since  $X_1$  is irreducible,  $X_1 \subseteq X_1 \cap Y_j$  for some  $j$  and thus  $X_1 \subseteq Y_j$ . But then similarly  $Y_j \subseteq X_i$  for some  $i$ . But by assumption on the  $X_k$ ,  $X_1 = Y_j = X_i$ , giving us our  $\iota$ . ♠

## 12.2 Important Morphisms

**12.2.1 Definition:** A morphism  $f : X \rightarrow Y$  of varieties is

- **affine** if there exists an affine covering  $U_i \subseteq Y$  such that  $f^{-1}(U_i) \subseteq X$  is affine.
- **finite** if there exists an affine covering  $U_i \subseteq Y$  such that  $f^{-1}(U_i)$  is affine and

$$\mathcal{O}_Y(U_i) \rightarrow \mathcal{O}_X(f^{-1}(U_i))$$

is a finite ring extension (finitely generated as a module over the base).

### 12.2.2 Proposition

$f : X \rightarrow Y$  is affine (resp. finite) if and only if for all  $U \subseteq Y$  affine,  $f^{-1}(U)$  is affine (resp.  $f^{-1}(U)$  is affine and  $\mathcal{O}_X(U) \rightarrow \mathcal{O}_X(f^{-1}(U))$  is finite).

## 13 October 25th, 2019

Another takeaway from this class: *algebraic geometry is relative and not absolute*. One way we are starting to dip our toes into this idea is via the definition of affine and finite morphisms. Notice that the definition of finite is a bit clunky because it requires us to restate the definition of affine withing the definition of a finite morphism. Gross. We should be able to write it as “*affine* +  $\square$ .”

For some examples, let  $Y = \operatorname{Spec} k$ . Then  $X \rightarrow \operatorname{Spec} k$  is affine if and only if  $X$  is affine. So when is  $X \rightarrow \operatorname{Spec} k$  finite? Well since  $X$  is affine,  $X = \operatorname{Spec} A$  where each  $A$  is a finite dimensional (reduced!)  $k$ -algebra. Therefore it is Artinian and we get a decomposition

$$A = A_1 \times \cdots \times A_n$$

where each  $A_i$  are finite dimensional local rings with maximal  $\mathfrak{m}_i \subset A_i$  that are nilpotent (use Nakayama!). So by this theorem (since  $A$  is reduced), we get  $A_i = k$ , so

$$A \cong k \times \cdots \times k.$$

So then  $\operatorname{Spec} A = \sqcup \operatorname{Spec} k$ , just a handful of points!

Let's look at another example we saw on the first day:  $k[x] \rightarrow k[t]$  sending  $x \mapsto t^2$ . This corresponds to a map  $\mathbb{A}^1 \rightarrow \mathbb{A}^1$ , and we can talk about the fibers of this morphism. They are all (well, almost all) of size two. But they are definitely finite! Similarly if we have a finite map  $A \rightarrow B$ , we can ask if it has finite fibers. Here if  $\varphi : A \rightarrow B$  is our morphism and  $f : \operatorname{Spec} B \rightarrow \operatorname{Spec} A$  is the map of rings, then since

$$f^{-1}(\mathfrak{m}) = \{\mathfrak{n} \subseteq B \mid \varphi(\mathfrak{n}) = \mathfrak{m}\}$$

we get a lemma

**13.0.1 Lemma**

Given a map  $f : \operatorname{Spec} B \rightarrow \operatorname{Spec} A$  and  $\mathfrak{m} \in \operatorname{Spec} A$ ,

$$f^{-1}(\mathfrak{m}) = \{\mathfrak{n} \subseteq B/\mathfrak{m}B, \text{ maximal}\}$$

And so (think this through) the answer is **yes, the fibers are finite**.

A natural question involves the following

**13.0.2 Definition:** A morphism  $f : X \rightarrow Y$  is **quasi-finite** if for all  $y \in U$ ,  $f^{-1}(y)$  is finite.

and we ask: is this equivalent to finiteness? The answer is no. Consider the map  $\mathbb{A}^1 \setminus 0 \hookrightarrow \mathbb{A}^1$ . Both of these are affine, so the map is affine. But definitely the  $k[t] \rightarrow k[t, t^{-1}]$  is not finite!

**13.0.3 Proposition**

Let  $X \rightarrow Y$  be finite and  $Z \subseteq X$  closed. Then  $f(Z) \subseteq Y$  closed.

Try it!

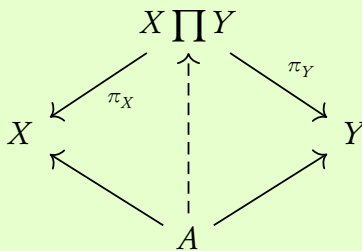
**14 October 28th, 2019**

Max heard some things from the feedback: he is interested in seeing if he can't help us put things together at the end of class. That should help!

**14.1 Products**

Today we are going to be talking about products (and fiber products) in the category of varieties.

**14.1.1 Definition:** Let  $\mathcal{C}$  be a category and  $X, Y \in \mathcal{C}$ . Then the product is the usual thing with the usual universal property.



**14.1.2 REMARK:** Notice that we can also “think in terms of functors.” Then we can define the product of  $h_X$  and  $h_Y$  as  $h_X \times h_Y$  in terms of the (Cartesian) product of sets.

Also recall the definition of a fiber product. Notice that if we translate back to functors, we are again trying to figure out what to make of the set

$$h_X \times_{h_Z} h_Y.$$

If  $f$  and  $g$  are maps from  $h_X$  and  $h_Y$ , respectively, into  $h_Z$ , then the fiber over  $x \in h_Z$  in  $h_X \times_{h_Z} h_Y$  is precisely the product of the fibers (whhaaaaaat *like a fibered product*) in  $f$  and  $g$ .

**Problem 14.1**

If  $\mathcal{C}$  has a final object  $\Omega$ , then  $X \times Y = X \times_{\Omega} Y$ . Think about this.

**14.1.3 Theorem**

Fiber products exist in the category of varieties.

Today we are going to be discussing the affine case: given  $X \rightarrow Z \leftarrow Y$  of affines, constructing  $X \times_Z Y$ . To start, let  $Z = \text{Spec } k$ . We are looking to study the functor  $h_X \times h_Y$  and then show representability.

Let's look at this pointwise: let  $X = \text{Spec } A$  and  $Y = \text{Spec } B$ . Then

$$h_X(S) \times h_Y(S) = \text{Hom}(S, X) \times \text{Hom}(S, Y) = \text{Hom}(A, \mathcal{O}_S) \times \text{Hom}(B, \mathcal{O}_S) = \text{Hom}(A \otimes B, \mathcal{O}_S) = h_{\text{Spec } A \otimes B}(S)$$

**Problem 14.2**

Show that the coproduct in  $\mathbf{Alg}_k$  is the tensor product.

14.1.4 REMARK: For the above problem, notice that  $A \otimes B$  is odd notation because it is not just the underlying vector space, but carries with it an algebra structure.

**Problem 14.3**

Is the tensor product of a finitely-generated algebra finitely generated? Can you do it without coordinates?

But notice for what we wrote above to be “okay” we need that  $A \otimes_k B$  is a reduced finitely generated  $k$ -algebra. The finite generation isn't too hard to see. The reducedness comes from the following (in Kempf):

We have a map  $A \otimes B \rightarrow \{|\text{Spec } A| \times |\text{Spec } B| \rightarrow k\}$  as set of functions. The claim is that this map is injective. The idea is that if  $\sum_i f_i \otimes g_i \mapsto 0$ , we can assume that the  $f_i$  and  $g_i$  are linearly independent.

But then for all  $x$ , we get  $\sum_i f_i(x)g_i = 0 : \text{Spec } B \rightarrow k$ . But then since the  $g_i$  are linearly independent, this means the  $f_i(x)$  are all zero. Thus the nullstellensatz tells us that all  $f_i = 0$ .