Hopf Algebras

A course by: Prof. James Zhang
Notes by: Nico Courts

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Introduction

These are the notes I took in class during the Winter 2019 topics course *Math 582H - Hopf Algebras* at University of Washington, Seattle.

The course description follows:

This course is an introduction to Hopf algebras. In addition to basic material in Hopf algebra, we will present some latest developments in quantum groups and tensor and fusion categories. One of the newer topics is homological properties of Noetherian Hopf algebras of low Gelfand-Kirillov dimension. A good reference for the first two topics in the book *Hopf Algebras and Their Action Rings* by Susan Montgomery. Here is a list of possible topics:

- Classical theorems concerning finite dimensional Hopf algebras.
- Infinite dimensional Hopf algebras and quantum groups.
- Duality and Calabi-Yau property.
- Actions of Hopf algebras and invariant theory.
- Representations of Hopf algebras, tensor and fusion categories.

1 January 7, 2019

If you don't know what a symmetric tensor category is, today is going to be a three star day. Max is 5.

1.1 Overview

We are shooting to understand two conjectures:

CONJECTURE (ETINGOF-OSTRIK '04): If A is a finite dimentional Hopf algebra, then

$$\bigoplus_{i\geq 0} \operatorname{Ext}^i_A({}_Ak, {}_Ak)$$

is Noetherian.

$$\begin{tikzcd}
V \otimes V \otimes V \arrow[r, "\mathrm{id}_V \otimes m"] \arrow[d, "m \otimes \mathrm{id}_V"] & V \otimes V \arrow[d, "m"] \\
V \otimes V \arrow[r, "m"] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
k \otimes V \arrow[r, "\sim"] \arrow[dr, "u \otimes \mathrm{id}_V"'] & V & V \otimes k \arrow[l, "\sim"'] \arrow[dl, "\mathrm{id}_V \otimes u"] \\
& V \otimes V \arrow[u, "m"] &
\end{tikzcd}$$

Figure 1: Diagrams for definition 1.2.1.

Conjecture (Brown-Goodearl '98): If A is a Noetherian Hopf algebra, then the injective dimension of A_A is finite.

These are both still open! In fact there is a meeting at Oberwolfach this March concerning exactly these conjectures.

1.2 Symmetric Tensor Categories

We are going to be using the following notation throughout:

- k is a field
- Vect_k is the category of k-vector spaces
  - Vect_k is closed under tensor products
  - There is an element $k \in \mathrm{Vect}_k$ such that
    $$k \otimes_k V \cong V \cong V \otimes_k k$$
    where the above isomorphisms are natural.
  - $V \otimes_k W \cong W \otimes_k V$
- An algebra is an object in Vect_k.

1.2.1 Definition
$V \in \mathrm{Vect}_k$ is called an **algebra object** if there are two morphisms

(a) $m : V \otimes V \to V$

(b) $u : k \to V$

such that the diagrams in figure 1.2 commute.

1.2.2 Lemma
$V \in \mathrm{Vect}_k$ is an algebra object iff V is an algebra over k.

1.2.3 Lemma
If C is a symmetric tensor category, so is C^{op}.

Then the natural thing to ask is: what is an algebra object in this opposite category?

1.2.4 Definition
A **coalgebra object** in C is an algebra object in C^{op}. Here we have comultiplication Δ and counit ε.

1.2.5 REMARK: Naturally you could go about defining this from first principles and drawing the diagrams in figure 1.2 with the arrows reversed, but we are probably mature enough to do without that (saving my fingers from repetitive strain injury in the process.)

1.2.6 Lemma
Alg_k, defined as the category of algebra objects in Vect_k, is a symmetric tensor category. Furthermore Coalg_k, the category of coalgebra objects in Vect_k, is a symmetric tensor category.

1.2.7 Lemma
The following are equivalent:

(a) V is an algebra object in Coalg_k

(b) V is a coalgebra object in Alg_k

(c) There are morphisms $m, u, \Delta, \varepsilon$ such that

- (V, m, u) is an algebra
- (V, Δ, ε) is a coalgebra
- Equivalently:
    - m and u are coalgebra morphisms
    - Δ and ε are algebra morphisms.

PROOF
The nice thing here is that the $(a) \Leftrightarrow (c)$ without the last condition. A similar fact holds for (b) except the second-to-last. The last thing to do is to prove the last two conditions are equivalent. ♠

Problem 1.1
Fill in the details for the proof above.

$$\begin{tikzcd} V\otimes V \arrow[r, "m"] \arrow[d, "\Delta\otimes\Delta"'] & V \arrow[r, "\Delta"] & V\otimes V \\ V^{\otimes 4} \arrow[rr, "T_{2,3}"] & & V^{\otimes 4} \arrow[u, "m\otimes m"'] \end{tikzcd} \qquad \begin{tikzcd} k \arrow[r, "\Delta"] \arrow[d, "u"] & k\otimes k \arrow[d, "u\otimes u"] \\ V \arrow[r, "\Delta"] & V\otimes V \end{tikzcd}$$

Figure 2: m and u are coalgebra morphisms.

$$\begin{tikzcd} V\otimes V \arrow[r, "m"] \arrow[d, "\Delta\otimes\Delta"] & V \arrow[d, "\Delta"] \\ V^{\otimes 4} \arrow[r, "m"] & V\otimes V \end{tikzcd} \qquad \begin{tikzcd} V\otimes V \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "m"] & k\otimes k \arrow[d, "m"] \\ V \arrow[r, "\varepsilon"] & k \end{tikzcd}$$

Figure 3: Δ and ε are algebra morphisms.

Solution:
Assume that $(V, m, u, \Delta, \varepsilon)$ is an algebra and coalgebra and further that m and u are coalgebra morphisms. That means in particular that the diagrams in figure 1 commute.

We are looking to prove that Δ and ε are algebra morphisms, or that the diagrams in figure 1 commute.

From here it's actually a bit boring because it's kinda just a definition/notation game. It boils down to the fact that the (co)multiplication on $V \otimes V$ has a twist that exactly lines up so that each square is saying the same thing. ♠

1.2.8 Definition
V is called a **bialgebra object** if V is an algebra object in Coalg_k.

Problem 1.2

(a) *Suppose that* $\operatorname{char} k \neq 2$. *Classify all bialgebras of* $\dim 2$.

(b) *Do the same for* $\operatorname{char} k = 2$.

Solution:
Part (a)

Consider $\varepsilon : V \to k$ and consider $\ker \varepsilon \lhd V$. By rank-nullity, $\dim \ker \varepsilon = 1$, so $\ker \varepsilon = kx$ for some $x \in V$. Therefore $x^2 = cx$ for some c, and if $c = 0$, then (as an algebra) $V \cong k[x]/(x^2)$. Otherwise consider $y = \frac{x}{c}$. In this case $y^2 = \frac{x^2}{c^2} = \frac{x}{c} = y$, and $V \cong k[x]/(x^2 - x)$.

Notice that in either case $\varepsilon(x) = 0$, so let

$$\Delta(x) = a(1 \otimes 1) + b(1 \otimes x) + c(x \otimes 1) + d(x \otimes x)$$

and using that $\varepsilon \otimes \mathrm{id} \circ \Delta = \mathrm{id} \otimes \varepsilon \circ \Delta$ and that each should be (essentially) the identity (this is just the diagram we saw before), we get $a = 0$ and $b = c = 1$. Thus the coalgebra structure of any Hopf algebra is given by

$$\varepsilon(x) = 0, \quad \Delta(x) = 1 \otimes x + x \otimes 1 + d(x \otimes x).$$

Consider first the case when $x^2 = 0$. Then since comultiplication will be an algebra morphism,

$$0 = \Delta(x^2) = \Delta(x)^2 = 1 \otimes x^2 + x^2 \otimes 1 + d^2(x^2 \otimes x^2) + 2(x \otimes x) + 2d(x \otimes x^2) + 2d(x^2 \otimes x)$$

and since $x^2 = 0$,

$$0 = 2(x \otimes x).$$

But $x \otimes x$ is a basis element of $V \otimes V$, so V can only have this algebra structure when $\operatorname{char} k = 2$. We will return to this in the next part.

So then $x^2 = x$ and using the computation above,

$$1 \otimes x + x \otimes 1 + d(x \otimes x) = \Delta(x) = \Delta(x^2) = 1 \otimes x + x \otimes 1 + (d^2 + 4d + 2)(x \otimes x)$$

so

$$(d^2 + 3d + 2)(x \otimes x) = 0 \quad \Rightarrow \quad d^2 + 3d + 2 = (d + 2)(d + 1) = 0$$

and so either $d = -1$ or $d = -2$.

One can verify that Δ is coassociative, so we can conclude that when $\operatorname{char} k \neq 2$, there are precisely two Hopf algebra structures with algebra structure $k[x]/(x^2 - x)$ and comultiplication either

$$\Delta(x) = 1 \otimes x + x \otimes 1 - x \otimes x \quad \text{or} \quad \Delta(x) = 1 \otimes x + x \otimes 1 - 2(x \otimes x)$$

(b)

Now assume that $\operatorname{char} k = 2$ and that $V \cong k[x]/(x^2 - x)$ as an algebra. Then using the analysis above, we see that we can choose comultiplication either

$$\Delta(x) = x \otimes 1 + 1 \otimes x \quad \text{or} \quad \Delta(x) = 1 \otimes x + x \otimes 1 + x \otimes x.$$

If instead $V \cong k[x]/(x^2)$, then *any* value of d will suffice, so there are a full k's worth of Hopf algebra structures that can appear. ♠

2 January 9, 2019

Today we are going to rely heavily on Sweedler notation. :) Notice that if we are looking at actual objects in the diagram for coassociativity, we get

$$\begin{array}{ccc} v & \longrightarrow & \sum v_{(1)} \otimes v_{(2)} \\ \downarrow & & \downarrow \\ \sum v_{(1)} \otimes v_{(2)} & \longrightarrow \sum (v_{(1)})_{(1)} \otimes (v_{(1)})_{(2)} \otimes v_{(2)} = & \sum v_{(1)} \otimes (v_{(2)})_{(1)} \otimes (v_{(2)})_{(2)} \end{array}$$

Figure 4: Coassociativity on elements in Sweedler notation

Example 2.1
Let G be a group and kG be the group algebra. The algebra structure arises as normal where $g \cdot h$ comes from the structure on G. Then $\Delta(g) = g \otimes g$ and this extends linearly.

But then if you consider $\Delta(\sum c_g g)$, notice that by the nature of tensors this is not unique! So we will just write

$$\Delta\left(\sum c_g g\right) = \sum_G c_g(g \otimes g) = \sum v_{(1)} \otimes v_{(2)}$$

2.1 Algebra structure on $V \otimes V$

We said earlier on that Alg_k is a symmetric *tensor* category. But how do we define the multiplication on the tensor product?

Well it all comes from the twist! We define

$$m_{V\otimes W} = (m_V \otimes m_W) \circ (\mathrm{id}_V \otimes \tau_{2,3} \otimes \mathrm{id}_W)$$

where $\tau_{2,3}$ is the twist morphism.

More simply, $u_{V\otimes W} : k = k \otimes k \to V \otimes V$ simply defined by $u_V \otimes u_W$.

So then when we say that Δ is an algebra morphism, we are saying that for all $v, w \in V$

$$\Delta(vw) = \sum (vw)_{(1)} \otimes (vw)_{(2)} = (\sum v_{(1)} \otimes v_{(2)})(\sum w_{(1)} \otimes w_{(2)}) = \sum v_{(1)}w_{(1)} \otimes v_{(2)}w_{(2)}$$

2.2 Hopf Algebras

Already to the good stuff!

2.2.1 Definition
$V \in \mathrm{Vect}_k$ is a **Hopf algebra** if V is a bialgebra together with an **antipode** $S : V \to V$ satisfying

$$(S, \mathrm{id}_V) \circ \Delta = \varepsilon = (\mathrm{id}_V, S) \circ \Delta$$

Conjecture: If $V \in \mathrm{Vect}_k$ is a Noetherian Hopf algebra, then S is bijective.

2.3 History and Motivation

Hopf himself was a topologist, so this is the first context in which it arose. In the 1940's, he began studying Hopf algebras over $\mathbb{Z}_2$ graded k vector spaces. For instance, the cohomology ring of topological space X with coefficients in k.

Later, in combinatorics, they ended popping up. Looking at rings of symmetric functions and other places gave some interesting examples.

Then in group theory you can define a functor from groups to Hopf algebras by $F(G) = kG$ with the diagonal map. The antipode is just the inverse.

Then with Lie algebras, you can look at $\mathcal{U}(L)$, the universal enveloping algebra is a Hopf algebra.

Finally with algebraic groups (yay!) we take an algebraic group G and consider the ring of functions on it, which is again a Hopf algebra.

Some "cousins" of Hopf algebras: quasi, weak, multiplier, ribbon, quasi-triangular, etc Hopf algebras. Each has slightly different base category or restrictions.

3 January 11, 2019

The plan for today is to talk about:

- Convolution Algebras
- Antipodes
- Duality
- (Co-)Modules

3.1 Convolution Algebras

Let $\mathcal{T}$ be a symmetric tensor category. We can usually think of $\mathcal{T} = \mathrm{Vect}_k$, but there is a problem since Vect_k is equivalent to the category of Hopf algebras over k, while this is not generally true.

We also need that $\mathcal{T}$ is k-linear (that is, enriched as a category over k). This means that $\mathrm{Hom}_{\mathcal{T}}(A, B) \in \mathrm{Vect}_k$.

3.1.1 Theorem

Let $\mathcal{T}$ be as above. Then $\mathrm{Hom}_{\mathcal{T}}(C, A)$ is an algebra and $\mathrm{Hom}_{\mathcal{T}}(-,-) : (\mathrm{Coalg}_{\mathcal{T}})^{op} \times \mathrm{Alg}_{\mathcal{T}} \to \mathrm{Alg}_k$ is a functor.

Proof

Let A be an algebra object in $\mathcal{T}$ and C be a coalgebra object in $\mathcal{T}$. Then $1_{\mathrm{Hom}} := u_A \circ \varepsilon_C : C \to 1_{\mathcal{T}}$ and define

$$f * g := m_A(f \otimes g)\Delta_C : C \to A.$$

Then using Lemma 3.1.2 and the fact that A and C are (co)algebra objects, we can see that the product $*$ satisfies the axioms required. ♠

3.1.2 Lemma

(a) $1_{\text{Hom}} * f = m_A(u \otimes 1)(1 \otimes f)(\varepsilon \otimes 1)\Delta$.

(b) $f * 1_{\text{Hom}} = m_A(1 \otimes u)(f \otimes 1)(1 \otimes \varepsilon)$

(c) $(f * g) * h = m_A(m_A \otimes 1)((f \otimes g) \otimes h)(\Delta_C \otimes 1)\Delta_C$

(d) $f * (g * h) = m_A(1 \otimes m_A)(f \otimes (g \otimes h))(1 \otimes \Delta)\Delta$

Proof

(a)

$$\begin{aligned} 1_{\text{Hom}} * f &= m_A(1_{\text{Hom}} \otimes f)\Delta_C \\ &= m_A(u_A \circ \varepsilon \otimes f)\Delta_C \\ &= m_A(u \otimes 1)(\varepsilon \otimes 1)(1 \otimes f)\Delta \end{aligned}$$

(b)

Same as (a), essentially.

(c) and (d)

$$\begin{aligned} (f * g) * h &= m_A((f * g) \otimes h)\Delta \\ &= m_A((m_A(f \otimes g)\Delta) \otimes h)\Delta \\ &= m_A(m_A \otimes 1)((f \otimes g) \otimes h)(\Delta \otimes 1)\Delta \end{aligned}$$

and the other is analogous. ♠

3.1.3 Definition

$V \in \mathcal{T}$ is a **Hopf algebra object** if:

- V is a bialgebra object in $\mathcal{T}$ and
- There is a map $S : V \to V$ that is $(\text{id}_V)^{-1}$ with respect to $*$.

3.1.4 Remark: Notice here that $\text{id}_V \in \text{Hom}_{\mathcal{T}}(V, V)$, the identity map in $\mathcal{T}$. We are *not* taking about $1_{\text{Hom}} = u \circ \varepsilon$.

Also, we call S an **antipode.**

3.2 Duality

Notice that when $C = 1_{\mathcal{T}}$ (that is the tensor identity), $\mathrm{Hom}_{\mathcal{T}}(1_{\mathcal{T}}, -) : \mathrm{Alg}_{\mathcal{T}} \to \mathrm{Alg}_k$ is a functor. Same for the dual from $\mathrm{Coalg}_{\mathcal{T}}$. This second one gives us a chance to talk about duality.

3.2.1 Lemma
Let $\mathcal{T}$ be the category of finite dimensional vector spaces over k. Then $(-)^* : \mathcal{T} \to \mathcal{T}^{op}$ is an equivalence.

This uses $(V \otimes W)^* = W^* \otimes V^*$.

3.2.2 Corollary
V is an algebra over $k \Leftrightarrow V^*$ is a coalgebra over k. And vice versa.

Recall $S = (\mathrm{id}_V)^{-1}$. Thus

$$S * \mathrm{id}_V = 1_{\mathrm{Hom}} = \mathrm{id}_V * S$$

The diagram we have here is

$$\begin{tikzcd}
V \otimes V \arrow[rr, "S \otimes \mathrm{id}_V"] & & V \otimes V \arrow[d, "m_V"] \\
V \arrow[u, "\Delta"] \arrow[r, "\varepsilon_V"] \arrow[d, "\Delta"] & k \arrow[r, "u_V"] & V \\
V \otimes V \arrow[rr, "\mathrm{id}_V \otimes S"] & & V \otimes V \arrow[u, "m_V"']
\end{tikzcd}$$

Modules/Comodules

3.2.3 Definition
Let A be an algebra object in $\mathcal{T}$. A **left** A **module** is $M \in \mathcal{T}$ with a morphism

$$m_M : A \otimes M \to M$$

such that the diagrams in Figure 3.2 commute.

3.2.4 Remark: Note that we don't necessarily need that M lie in $\mathcal{T}$. We could instead just rely on an algebra homomorphism $\varphi : A \to \mathrm{Hom}_{\mathcal{T}}(M, M)$ and proceed as usual.

$$
\begin{tikzcd}
A \otimes A \otimes M \arrow[r, "m_A \otimes 1"] \arrow[d, "1 \otimes m_M"] & A \otimes M \arrow[d, "m_M"] \\
A \otimes M \arrow[r, "m_M"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes M \arrow[r, "m_M"] & M \\
1_{\mathfrak{J}} \otimes M \arrow[u, "u_A \otimes 1"] \arrow[ur, "\sim"] &
\end{tikzcd}
$$

Figure 5: Module diagrams