Notes and Problems from My Research

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1 Autumn 2018

1.1 Problems

Problem 1.1 *Assume that k is a field and let $K = k(t)$ (notice K is a transcendental extension). Prove that* $\mathrm{Hom}_k(K, k) \ncong K$.

Solution:
This is basically just a cardinality argument. I don't think it's particularly worth doing at this juncture. ♠

Problem 1.2 *Let G be a finite group scheme (actually we need only assume that G is a Frobenius algebra so that a module is injective if and only if it is projective). Prove that unless M is projective, its projective dimension is infinite. Conclude that $H^n(G, M) = 0$ for $n > N$ implies that M is projective.*

Solution:
Assume M itself is not projective so that its minimal projective resolution is nontrivial and furthermore that it is finite. That is, let P_i be projective modules such that

$$0 \to P_n \xrightarrow{f_n} P_{n-1} \to \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

is a minimal length projective resolution of M (notice here that $n \geq 1$).

Next consider the short exact sequence

$$0 \to P_n \xrightarrow{f_n} P_{n-1} \to \operatorname{coker} f_n \to 0$$

since P_n is projective (and thus injective!) this sequence splits and therefore $P_{n-1} \cong P_n \oplus \operatorname{coker} f_n$. But then consider the sequence

$$0 \to P_n \xrightarrow{g} P_{n-2} \to \cdots \xrightarrow{f_0} M \to 0$$

where above we are using $P_{n-1} \supseteq P_n \cong f_n(P_n)$ and that $g = f_{n-1}|_{f_n(P_n)}$. This map is injective since $\ker f_{n-1} = \operatorname{coker} f_n$, which is disjoint from $f_n(P_n) \cong P_n$. Exactness everywhere else is evident since the maps are not effectively changed.

But then the existence of this sequence contradicts the minimality of the original sequence, so no finite sequence can exist. ♠

Problem 1.3 *Establish the five-term exact sequence for spectral sequences.*

Solution:
I plan to return to this problem in the future. I have other priorities at the moment, but I will eventually return to cohomology and spectral sequences and this will be a good exercise at that point. ♠

2 Winter 2019

This is my first official quarter as Julia's student!!

2.1 Preparation, Waterhouse, and Görtz & Weddhorn

To give a sense of direction, Julia recommended that I take a look at the following regularity theorem:

2.1.1 Theorem (Smoothness Theorem)
Let G be an algebraic affine group scheme over a field k. Then $k[G] \otimes \bar{k}$ is reduced if and only if $\dim G = \operatorname{rank} \Omega_{k[G]}$.

I have quite a bit of information to process before this, so I will get started!

2.1.2 Definition (Closed Embedding)
If G and H are affine group schemes represented by A and B, respectively and if $\psi : H \to G$ is a homomorphism of affine group schemes (locally a group homomorphism) then if the corresponding algebra map $A \to B$ is surjective, then ψ is called a **closed embedding.**

As its name suggests, this means that ψ is an isomorphism onto a **closed subgroup** H' of G. This is, in fact, a definition of this property. One can also think about it in the following ways: a group scheme H is closed in G if

- H is defined by the relations imposed by G plus some additional ones.
- $H = V(I)$ for some ideal $I \subset k[G]$.

Thinking back to our algebraic geometry, these are not too hard to see as equivalent. For instance, there is a closed embedding of μ_n in G_m (simply adding in $x^n - 1$) and of SL_n in GL_n (adding $\det = 1$).

2.1.1 Hopf Ideals

One problem with the above characterization that one cannot choose I arbitrarily and end up with a group scheme. This is equivalent to arbitrarily adding relations to a group, which is not always guaranteed to work out well (think adding $\det = 2$ to GL_n).

Actually, we can exactly categorize the closed embeddings of subgroups in G by considering certain ideals of the algebra A which represents it.

2.1.3 Definition
Let A be an algebra and $I \lhd A$. Then if

- $\Delta(I)$ goes to zero under the map $A \otimes A \to A/I \otimes A/I$,
- $S(I) \subseteq I$
- $\epsilon(I) = 0$

then I is called a **Hopf Ideal of A.**