# Notes and Problems from My Research

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# Part I
# Autumn 2018

## 1 Problems

**Problem 1.1** *Assume that $k$ is a field and let $K = k(t)$ (notice $K$ is a transcendental extension). Prove that $\mathrm{Hom}_k(K, k) \ncong K$.*

**Solution:**
This is basically just a cardinality argument. I don't think it's particularly worth doing at this juncture. ♠

**Problem 1.2** *Let $G$ be a finite group scheme (actually we need only assume that $G$ is a Frobenius algebra so that a module is injective if and only if it is projective). Prove that unless $M$ is projective, its projective dimension is infinite. Conclude that $H^n(G, M) = 0$ for $n > N$ implies that $M$ is projective.*

**Solution:**
Assume $M$ itself is not projective so that its minimal projective resolution is nontrivial and furthermore that it is finite. That is, let $P_i$ be projective modules such that

$$0 \to P_n \xrightarrow{f_n} P_{n-1} \to \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

is a minimal length projective resolution of $M$ (notice here that $n \geq 1$).

Next consider the short exact sequence

$$0 \to P_n \xrightarrow{f_n} P_{n-1} \to \operatorname{coker} f_n \to 0$$

since $P_n$ is projective (and thus injective!) this sequence splits and therefore $P_{n-1} \cong P_n \oplus \operatorname{coker} f_n$. But then consider the sequence

$$0 \to P_n \xrightarrow{g} P_{n-2} \to \cdots \xrightarrow{f_0} M \to 0$$

where above we are using $P_{n-1} \supseteq P_n \cong f_n(P_n)$ and that $g = f_{n-1}|_{f_n(P_n)}$. This map is injective since $\ker f_{n-1} = \operatorname{coker} f_n$, which is disjoint from $f_n(P_n) \cong P_n$. Exactness everywhere else is evident since the maps are not effectively changed.

But then the existence of this sequence contradicts the minimality of the original sequence, so no finite sequence can exist.

The last statement (as discussed with Julia) is actually false. ♠

**Problem 1.3** *Establish the five-term exact sequence for spectral sequences.*

**Solution:**
I plan to return to this problem in the future. I have other priorities at the moment, but I will eventually return to cohomology and spectral sequences and this will be a good exercise at that point. ♠

**Problem 1.4 (Waterhouse 1.1)**

*(a) If $R$ and $S$ are two $k$ algebras and $F$ is a representable functor, show $F(R \times S) \cong F(R) \times F(S)$.*

*(b) Show there is no representable functor $R$ such that every $F(R)$ has exactly two elements.*

*(c) Let $F$ be the functor represented by $k \times k$. Show that $F(R)$ has two elements exactly when $R$ has no idempotents besides 0 and 1.*

**Solution:**
**(a)**

Let $A$ be the $k$-algebra representing $F$. Thus $F(R)$ is naturally isomorphic to $\mathrm{Hom}_k(A, R)$ and $F(S) \simeq \mathrm{Hom}(A, S)$. Then define the map $\Phi : \mathrm{Hom}(A, R \times S) \to \mathrm{Hom}(A, R) \times \mathrm{Hom}(A, S)$ via

$$\Phi(\varphi) = (\pi_R \circ \varphi, \pi_S \circ \varphi)$$

where $\pi_X$ is the canonical projection onto $X$.

This is surjective since (by the universal property of products) any pair of maps $\varphi_R : A \to R$ and $\varphi_S : A \to S$ factors through the product $R \times S$ and furthermore it does so *uniquely*, giving us injectivity. Thus this map (which is clearly a homomorphism since $\pi_X$ is) is a bijection.

**(b)**

By the last problem this is impossible since if $|F(k)| = 2$ then

$$|F(k \times k)| = |F(k) \times F(k)| = 4.$$

**(c)**

Let $F$ be such a functor. Consider any $\varphi \in \mathrm{Hom}(k \times k, R) \simeq F(R)$. Assume first that $F(R) \cong \mathbb{Z}/2$ and let $r$ be an idempotent in $R$. ♠

**Problem 1.5 (Waterhouse 1.2)** *Let $E$ be a functor represented by $A$ and let $F$ be any functor. Show that the natural maps $\eta : E \to F$ correspond to elements in $F(A)$.*

**Solution:**
Consider the map $\Phi$ from natural maps $E \to F$ to elements in $F(A)$ defined by (again leveraging the representability of $E$)

$$\eta \mapsto \eta(\mathrm{id}_A) \in F(A).$$

Conversely, consider the map $\Psi$ from $F(A)$ to the natural maps $E \to F$ via

$$x \mapsto \xi_x$$

where $\xi_x$ where for any $Y$ and $y \in E(Y) \cong \mathrm{Hom}(A, Y)$ we define the $Y^{th}$ component of $\xi_x$ as

$$\xi_x(y) = F(y)(x) \in F(Y)$$

where (for clarity while I get a grasp here) $F(y) : F(A) \to F(Y)$.

Since we are only looking for a bijection, we only need that these maps are inverses. Consider that for all $Y$ and $y \in E(Y)$,

$$\begin{aligned}
\Psi \circ \Phi(\eta)(y) &= \Psi\left(\eta(\mathrm{id}_A)\right)(y) \\
&= \xi_{\eta(\mathrm{id}_A)}(y) \\
&= F(y) \circ \eta(\mathrm{id}_A) \\
&= \eta \circ E(y)(\mathrm{id}_A) \\
&= \eta(y \circ \mathrm{id}_A) = \eta(y)
\end{aligned}$$

where above we used the naturality of $\eta$ along with the fact that $E(y)$ is just precomposition with $y$. Thus $\Psi \circ \Phi(\eta) = \eta$.

But then for any $x \in F(A)$,

$$\begin{aligned}
\Phi \circ \Psi(x) &= \Phi \circ \xi_x \\
&= \xi_x(\mathrm{id}_A) \\
&= F(\mathrm{id}_A)(x) \\
&= \mathrm{id}_{F(A)}(x) = x
\end{aligned}$$

completing the proof. ♠

**Problem 1.6 (Waterhouse 1.3)** *Let $E$ be a functor represented by $A$, and let $F$ be any functor. Let $\Psi : F \to E$ be a natural map with surjective component maps. Show there is a natural map $\Phi : E \to F$ with $\Psi \circ \Phi = \mathrm{id}_E$.*

**Solution:**
Since in particular $\Psi_A$ is surjective, there is an $x \in F(A)$ such that $\Psi(x) = \mathrm{id}_A$. Then using the map from the last problem, let $\Phi = \xi_x$. Then we can compute for any $R$ and $g \in E(R)$

$$\begin{aligned}\Psi \circ \Phi(g) &= \Psi \circ F(g)(x)\\ &= E(g) \circ \Psi(x)\\ &= E(g)(\mathrm{id}_A)\\ &= g \circ \mathrm{id}_A = g\end{aligned}$$

since $g : A \to R$, so $E(g) : E(A) \to E(R)$, which is just composition with $g$. ♠

**Problem 1.7 (Waterhouse 1.5)** *Write out $\Delta, \varepsilon$, and $S$ for the Hopf algebras representing $\mathbf{SL_2}, \boldsymbol{\mu_n}$, and $\boldsymbol{\alpha_p}$.*

**Solution:**
**$\mathbf{SL_2}$:**

Notice $SL_2$ is represented by $A = k[X_1, X_2, X_3, X_4]/(X_1X_4 - X_3X_2 - 1)$ so take two elements $f, g \in \mathrm{Hom}(A, R)$ where $f(X_i) = a_i \in R$ and $g(X_i) = b_i \in R$ and notice that we want

$$(f, g)\Delta = h$$

where since

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$$

we want to have that $h(X_i) = c_i$.

So then if $\Delta : A \to A \otimes A$ is defined as follows:

$$\begin{aligned} X_1 &\mapsto X_1 \otimes X_1 + X_2 \otimes X_3\\ X_2 &\mapsto X_1 \otimes X_2 + X_2 \otimes X_4\\ X_3 &\mapsto X_3 \otimes X_1 + X_4 \otimes X_3\\ X_4 &\mapsto X_3 \otimes X_2 + X_4 \otimes X_4\end{aligned}$$

Where one can compute

$$\begin{aligned}\mathrm{id} \otimes \Delta \circ \Delta(X_1) &= (\mathrm{id} \otimes \Delta)(X_1 \otimes X_1 + X_2 \otimes X_3)\\ &= X_1 \otimes X_1 \otimes X_1 + X_1 \otimes X_2 \otimes X_3 + X_2 \otimes X_3 \otimes X_1 + X_2 \otimes X_4 \otimes X_3\\ &= (\Delta \otimes \mathrm{id})(X_1 \otimes X_1 + X_2 \otimes X_3) = \Delta \otimes \mathrm{id} \circ \Delta(X_1)\end{aligned}$$

and similar equality holds for the other $X_i$, so this is $\Delta$.

Using that we want $\varepsilon \otimes \mathrm{id} \circ \Delta(X_i) = 1 \otimes X_i$, we see that the map $\varepsilon$ sending $X_1$ and $X_4$ to 1 and $X_2$ and $X_3$ to zero is the map we want.

Notice that as a sanity check we get that

$$\begin{pmatrix} \varepsilon(X_1) & \varepsilon(X_2) \\ \varepsilon(X_3) & \varepsilon(X_4) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Finally using that $(S, \mathrm{id}) \circ \Delta(X_i) = \varepsilon(X_i)$ and the fact that in $A$, $\det = X_1X_4 - X_3X_2 = 1$, we can define $S$ such that

$$\begin{pmatrix} S(X_1) & S(X_2) \\ S(X_3) & S(X_4) \end{pmatrix} = \begin{pmatrix} X_4 & -X_2 \\ -X_3 & X_1 \end{pmatrix} = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}^{-1}$$

and one can verify that this satisfies the relation above.

$\boldsymbol{\mu_n}$ :

For this scheme, $A = k[X]/(X^n - 1)$ is the representing algebra. If $f, g \in \mathrm{Hom}(A, k)$ with $f(X) = r$ and $g(X) = s$, then we want $(f, g)\Delta(X) = \sum f(X_{(1)})g(X_{(2)}) = m(r, s) = rs$ where $\Delta(X) = \sum X_{(1)} \otimes X_{(2)}$. An obvious choice is the diagonal map.

Then choosing $\varepsilon(X) = 1$ satisfies the diagrams (as before in $G_m$) and using our intuition, $S(X) = X^5$ which also works.

$\boldsymbol{\alpha_p}$:

This time we are working with $A = \mathbb{Z}/p[X]/(X^p)$. This time (since the group is additive) we want $\Delta(X) = X \otimes 1 + 1 \otimes X$, which we can see works with associativity immediately.

Following suit with the other additive group scheme $G$, setting $\varepsilon(X) = 0$ and $S(X) = -X$ we can quickly check these still satisfy the given axioms. ♠

**Problem 1.8 (Waterhouse 1.6)** *In $A = k[X_{11}, \ldots, X_{nn}, 1/\det]$ representing $GL_n$, show that $\Delta(X_{ij}) = \sum X_{ik} \otimes X_{kj}$. What is $\varepsilon(X_{ij})$?*

**Solution:**
Due to the uniqueness of $\Delta, \varepsilon$, and $S$, we need only find maps satisfying the diagrams. I claim that $\varepsilon(X_{ij}) = \delta_{ij}$. In this case, notice

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta(X_{ij}) = \varepsilon \otimes \mathrm{id}\left(\sum X_{ik} \otimes X_{kj}\right) = \sum \delta_{ik} \otimes X_{kj} = 1 \otimes X_{ij}$$

exactly as we want.

For associativity, notice

$$(\Delta \otimes \mathrm{id}) \circ \Delta(X_{ij}) = \Delta \otimes \mathrm{id}\left(\sum_k X_{ik} \otimes X_{kj}\right) = \sum_k \left(\sum_l X_{il} \otimes X_{lk}\right) \otimes X_{kj}$$

and then the associativity of $\Delta$ follows simply from the associativity of the tensor product.

For the last axiom, we compute $S$ such that $(S, \mathrm{id}) \circ \Delta = \iota \circ \varepsilon$ where $\iota : K \to A$ is the map sending $k \mapsto k \cdot 1_A$. That is, we define $S : A \to A$ so that

$$\sum_k S(X_{ik})X_{kj} = \delta_{ij}.$$

We want to leverage the fact that for a fixed $i$ and $j$, the determinant is

$$\begin{aligned}\det &= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_l X_{\sigma(l)l} \\ &= \sum_\sigma \mathrm{sgn}(\sigma) X_{\sigma j j} \prod_{l \neq j} X_{\sigma(l)l} \\ &= \sum_i X_{ij} \left( \sum_{\sigma(j)=i} \mathrm{sgn}(\sigma) \prod_{l \neq j} X_{\sigma(l)l} \right)\end{aligned}$$

and so we want that

$$S(X_{ik}) = \frac{1}{\det} \sum_{\sigma(i)=k} \mathrm{sgn}(\sigma) \prod_{l \neq k} X_{\sigma(l)l}$$

so that when $i = j$,

$$\sum_k S(X_{ik})X_{kj} = \frac{1}{\det} \sum_k X_{kj} \sum_{\sigma(j)=k} \mathrm{sgn}(\sigma) \prod_{l \neq k} X_{\sigma(l)l} = 1 = \delta_{ij}$$

whenever $i \neq j$, however, this equation is the determinant of the matrix where we have replaced the $j^{th}$ column with a copy of the $i^{th}$ column. This is linearly dependent, so

$$\frac{1}{\det} \sum_k S(X_{ik})S_{kj} = 0 = \delta_{ij}.$$

Thus these are precisely the maps we desire. ♠

**Problem 1.9 (Waterhouse 1.10)** *Prove the following Hopf algebra facts by interpreting them as statements about group functors:*

*(a)* $S \circ S = \mathrm{id}$

*(b)* $\Delta \circ S = (\mathrm{twist}) \circ (S \otimes S)\Delta$

*(c)* $\varepsilon \circ S = \varepsilon$

*(d) The map* $A \otimes A \to A \otimes A$ *sending* $a \otimes b$ *to* $(a \otimes 1)\Delta(b)$ *is an algebra isomorphism.*

**Solution:**
**(a)**

Dualizing, we get

$$
\begin{tikzcd}
A \arrow[rr, equal] \arrow[dr, "S"'] & & A & A \arrow[rr, equal] & & A \arrow[dl, "inv"] \\
& A \arrow[ur, "S"'] & & & A \arrow[ul, "inv"] &
\end{tikzcd}
$$

so this statement is equivalent to the group (scheme) fact that $(g^{-1})^{-1} = g$.

**(b)**

Using a similar duality argument, this is equivalent to saying

$$\mathrm{inv} \circ m = m \circ (\mathrm{inv} \times \mathrm{inv}) \circ (\mathrm{twist})$$

but if we consider arbitrary elements $g, h \in G(R)$, this means

$$(gh)^{-1} = m \circ (\mathrm{inv} \times \mathrm{inv})(h, g) = m(h^{-1}, g^{-1}) = h^{-1}g^{-1}$$

which is clearly true.

**(c)**

This one is equivalent to $(\mathrm{inv}) \circ i = i$ where if $g \in G(R)$,

$$(\mathrm{inv}) \circ i(g) = (\mathrm{inv})(e) = e = i(g)$$ $\spadesuit$

or in other words $e^{-1} = e$.

**(d)**

# Part II
# Winter 2019

## 2 Preparation for the Quarter

This is my first official quarter as Julia's student! My plan for now is to continue working on Waterhouse as well as learn about algebraic geometry (alongside my usual classes, of course). To give a sense of direction, Julia recommended that I take a look at the following regularity theorem:

**2.0.1 Theorem (Smoothness Theorem)**
Let $G$ be an algebraic affine group scheme over a field $k$. Then $k[G] \otimes \bar{k}$ is reduced if and only if $\dim G = \operatorname{rank} \Omega_{k[G]}$.

## 3 Overarching Ideas and Notes

### 3.1 Week 2

Consider the representations of A over a field $k$ (or more generally $A$ modules, which we can make into a tensor category). Recall that in this category we have enough projectives and (with finitely generated Hopf algebras) we get that all projectives are injective.

The object of study here:

**3.1.1 Definition**
The **Hochschild cohomology** of a Hopf algebra $A$ over the field $k$ is

$$\operatorname{Ext}^*_A(k,k) = H^*_A(A,k)$$

.

The conjecture here is

**3.1.2 Conjecture**
when $A$ is a finite dimensional Hopf algebra, $H^*(A,k)$ (the Hochschild cohomology) is finitely generated as a $k$-algebra.

This is known in some special cases:

- Finite groups
- Finite group schemes (in positive characteristic) due to Friedlander and Suslin.

- In characteristic zero we have
  - Quantum groups
  - Hopf algebras that come from Nichols algebras

Papers to look at:

- Ginzburg and Kumar '94 "Cohomology of Quantum Groups at Roots of Unity"
- Mastnak, Pevtsova, Shauenburg, and Witherspoon '09 "Cohomology of Finite Dimensional Pointed Hopf Algebras"
- Witherspoon '17 "Varieties for Modules of Finite Dimensional Hopf Algebras"

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Sarah Witherspoon has a book she is writing on Hochschild cohomology that could be very good. It's on her website (Texas A& M). Check out chapter 9 and appendix A.

## 3.2 Week 3

We primarily discussed support varieties in the context of Sarah Witherspoon's *Varieties for Modules of Finite Dimensional Hopf Algebras.*

### 3.2.1 Questions

(a) On page 5, in the definition of the support variety of a Hopf module $M$ (see Definition 3.2.1), the construction seems to depend rather heavily on the choice (and existence!) of a graded subalgebra $H \leq \mathrm{HH}^*(A)$ satisfying (fg1) and (fg2). But the notation seems to imply there is no dependence there. What's going on?

(b) On page 9, why is $\mathcal{Z}(kG) = \mathrm{HH}^0(kG)$ a local ring (for $G$ a finite group)?

(c) Directly after the last question, why would this ring being local imply that, on the variety level, we can use $H^{ev}(G, k)$ instead of $H^{ev}(G, k) \cdot \mathrm{HH}^0(kG)$?

### 3.2.2 Answers/Hints

(a) Actually the definition of support variety, when properly generalized to triangulated tensor categories, doesn't rely on the choice of this algebra. But to see this, we need to dive deeper into that subject. Julia gave me a book by Dave Benson, Srikanth Iyengar, and Henning Krause on local cohomology and support. It was written as a compendium from a course at Oberwolfach that they gave. It is available on the Arxiv.

(b) Julia suggests that this is a "standard" result that one might find in Webb's book (or another book where the author has taken care to be thorough). I couldn't locate it there immediately, but I found a page online that gave me the maximal ideal, so I am going to try to prove it as lemma 3.2.3.

(c) The take-away here was that when computing the varieties the nilpotent elements are irrelevant. Since the only maximal ideal in $\mathrm{HH}^0(kG)$ is composed of nilpotents, it doesn't contribute to the support variety.

**3.2.1 Definition (Support Variety)**
Let $A$ be a finite dimensional algebra and assume ${}_AA$ is injective (notice that finite dimensional Hopf algebras are Frobenius whence self-injective). Assume further that there is a graded subalgebra $H \leq \mathrm{HH}^*(A)$ that satisfies the following finiteness conditions:

- **(fg1):** $H$ is finitely generated, commutative, and $H^0 := H \cap \mathrm{HH}^0(A) = \mathrm{HH}^0(A)$;
- **(fg2):** for all finite dimensional $A$-modules $M$, $\mathrm{Ext}_A^*(M, M)$ is finitely generated as an $H$-module.

Then if $I_A(M)$ is the annihilator of $\mathrm{Ext}_A^*(M, M)$ in $H$, the **support variety** is

$$V_A(M) = \mathrm{MaxSpec(H/I_A(M))}$$

3.2.2 Remark: In taking with Julia about this definition she mentioned that the use of the max spectrum instead of the entire spectrum is a bit watered-down and, while simpler, perhaps hearkens back to the beginnings of AG where that is what one considered.

**3.2.3 Lemma**
If $G$ is a finite $p$-group and $k$ a field of characteristic $p$, $\mathcal{Z}(kG)$ is local.

Proof: I claim that the maximal ideal is

$$I = \left\{ \sum_{g \in G} a_g g : \sum_g a_g = 0 \right\}$$

To see this, let $A = \mathcal{Z}(kG)$, and notice that $A = \ker \varphi$ where $\varphi(\sum a_g g) = \sum a_i$. Since $\varphi(kG) = k$, this gives us that $I$ is a maximal ideal in $A$.

Notice that if we prove that $I$ consists entirely of nilpotents, we will be done. This is because the nilradical of $A$ is the intersection of its prime ideals. But if $I$ is contained in the nilradical, it must, in particular, be contained in every maximal ideal of $A$. Thus it must be the only maximal ideal.

To see this, we proceed by induction on $n$ where $|G| = p^n$. When $n = 0$ this is obvious and when $n = 1$, this means $G = \mathbb{Z}/p\mathbb{Z} = \langle \alpha \rangle$. In this case each element is nilpotent of degree $p$ due to the fact that (since in this case our ring is commutative):

$$\left( \sum_1^p a_i \alpha^i \right)^p = \sum_1^p a_i^p \alpha^{pi} = \sum_1^p a_i 1_G = 0.$$

Now assume that $I$ consists of nilpotents for all $n \leq k$ and let $|G| = p^{k+1}$. Let $Z \leq G$ be the (necessarily nontrivial) center of $G$. Extend the canonical quotient map $q : G \to G/Z$ to a map between group rings $\widetilde{q} : kG \to k(G/Z)$. Let $I_Z$ be the augmentation ideal of $k(Z)$. Then $\widetilde{q}$ is surjective with kernel $I_Z \cdot kG$. That this ideal is contained within the kernel is clear since if $y$ is a $kG$-linear sum of elements of the form $\sum_Z a_z z$ where $\sum_Z a_z = 0$, then the image is just the sum of zeros.

To see that every element of the kernel is of this form, let $y = \sum k_i g_i$ and notice

$$\widetilde{q}\left(\sum k_i g_i\right) = \sum k_i \widetilde{q}(g_i) = \sum_j \left(\sum_{g_i - g_j \in Z} k_i\right) g_j$$

and this is zero precisely when all the sums are. But then the sum of the coefficients of $y$ restricted to any coset of $Z$ are zero.

Thus in particular $kG/(I_Z) \cong k(G/Z)$.

Then look at the augmentation ideal $I_{G/Z} \lhd k(G/Z)$ which, by the induction hypothesis, consists of nilpotents. Pull this back through $\widetilde{q}$ to the ideal $I = (I_{G/Z}) + (I_Z)$. Since some power of $p$ kills all elements of the two summands, this implies that some power of $p$ kills each element in $I$, completing the proof. ♠

## 3.3 Week 4

Recall that $\mathcal{U} : \mathbf{Lie} \to \mathbf{Ass}$ is a functor that assigns to each Lie algebra the universal enveloping algebra: the tensor algebra over $\mathfrak{g}$ modulo the relations $x \otimes y - y \otimes x - [x, y]$.

Then we have the PBW theorem:

**3.3.1 Theorem (Poincaré-Birkhoff-Witt)**
If $\mathfrak{g}$ is a finite dimensional Lie algebra, with basis $x_1, \ldots, x_n$, then $\mathcal{U}(\mathfrak{g})$ has a basis in the form $\prod x_i^{a_i}$ for $a_i \in \mathbb{Z}_{\geq 0}$.

3.3.2 Remark: A fancier way of saying this is to say that $\mathcal{U}(\mathfrak{g})$ admits a filtration such that $\operatorname{gr}\mathcal{U}(\mathfrak{g}) \cong S^*(\mathfrak{g})$.

Furthermore, there is a more general class of algebras called **PBW algebras** where you get commutation modulo lower-order terms. This idea is also what forms the basis of the proof for the above remark.

The idea here, then, is to take the center and see how it acts on the universal enveloping algebra.

### 3.3.1 The BGG category $\mathcal{O}$

The representations of finite dimensional Lie algebras are in correspondence with the dominant integral weights. This will be worth looking into, perhaps.

The general question we want to answer is what he structure of finite dimensional representations of $\mathfrak{g}$. In the case when $\mathfrak{g} = \mathfrak{sl}_2$, we have the representation $V = ke_1 \oplus ke_2$.

Here I need to work out some of the notation, but $k[x,y]$ is a representation and $S^d(V) = k_d[x,y]$ is as well. (F&H lecture 11 for more details there).

So we end up realizing that we can generate any such Lie algebra by taking a single highest-weight vector and applying $\mathcal{U}(\mathfrak{n}^-)$ to get the whole thing. This help motivate:

**3.3.3 Definition (BGG Category $\mathcal{O}$)**
$\mathcal{O}$ is the full subcategory of $\mathcal{U}(\mathfrak{g})$ modules such that

- $M$ is finitely generated over $\mathcal{U}(\mathfrak{g})$ – this is necessary since we want free modules and $\mathcal{U}(\mathfrak{g})$ itself to be representations under study and these are excluded from finite ($k$-) dimensional algebras.
- $M$ is $\mathfrak{h}$ semisimple.
- $M$ is $\mathfrak{n}$-locally finite – for all $v \in M$, $\mathcal{U}(\mathfrak{n})v$ is finite dimensional.

Ideas for things to understand/cover for next week:

- First things first, understand the representations of $\mathfrak{sl}(2,k)$.
- Then in Humphreys(I), read/do section §6.21 to understand the universal enveloping algebra.
- Read Humphreys(II) chapter 1 as much as possible.

## 3.4 Week 5

We spent most of the time deriving the fact that any simple $\mathfrak{sl}_2$ module is isomorphic to one of the “standard” highest weight modules.

Our assignment for this upcoming week: Write down the Serre relations. Then do some computations and read section 21 in Humphreys. Read those lemmas and theorems and start to describe decomposition of general modules of Lie algebras.

# 4 Lie Algebras

We're planning on digging into the category $\mathcal{O}$, the category of (reasonably finite) $\mathcal{U}(\mathfrak{g})$ modules. To begin, however, we are going to make sure we have our fundamentals vis a vis (semi) simple representations of Lie algebras.

## 4.1 $\mathfrak{sl}_2$

In the case when $L = \mathfrak{sl}_2$, everything is relatively nice. We read (and discussed) the fact that any simple, finite-dimensional $\mathfrak{sl}_2$-module is isomorphic to $V(\lambda)$ for some integer $\lambda$ (highest weight).

The action of $L$ on these modules are precisely as one would hope: $x$ and $y$ (the off-diagonal generators) act by raising and lowering the weight of a vector by 2 and the central generator fixes each eigenspace.

## 4.2 The Serre Relations

Let $L$ be a semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra with corresponding roots $\Phi$ with simple root system $\Delta$. Recall that

$$\langle \alpha_i, \alpha_j \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \alpha_i(h_{\alpha_j})$$

Fix generators $x_i$ and $y_i$ of $L_{\alpha_i}$ and $L_{-\alpha_i}$, respectively, so that $[x_i, y_i] = h_i = h_{\alpha_i}$ Recall that the $x_i$ and $y_i$ generate $L$ (as a Lie algebra).

Then in particular we have the relations:

$$[h_i, h_j] = 0 \tag{S1}$$

$$[x_i, y_j] = \delta_{ij} h_i \tag{S2}$$

$$[h_i, x_j] = \langle \alpha_j, \alpha_i \rangle x_j \text{ and } [h_i, y_j] = -\langle \alpha_j, \alpha_i \rangle y_j \tag{S3}$$

**4.2.1 Definition (Serre Relations)**
With the notation above, the relations S1-3 above are called the **Serre relations**.

These relations end up being crucial in making the connection between root systems and semisimple Lie algebras concrete: taking the abstract free Lie algebra on generators satisfying these properties, we get a finite dimensional semisimple Lie algebra with roots corresponding exactly to the $\alpha_i$. This also gets us uniqueness of such a root system.

## 4.3 The more general case.

This primarily borrows from §21 in Humphreys (I) but also relies on some results from earlier that I will cite as necessary.

Throughout this discussion, we rely on the theory developed in sections 15 and 16 regarding Cartan and Borel subalgebras:

**4.3.1 Definition (Cartan Subalgebra (CSA))**
If $L$ is a Lie algebra, then $C \subseteq L$ is called a **Cartan subalgebra** if $C$ is

(a) Nilpotent; and

(b) $N_L(C) = C$.

**4.3.2 Definition (Borel Subalgebra)**
If $L$ is a Lie algebra, $B \subseteq L$ is called a **Borel subalgebra** if $B$ is a maximal solvable subalgebra.

4.3.3 Remark: Since every nilpotent Lie algebra is solvable, each CSA lies in some Borel subalgebra $B$ of $L$. then by showing that any two Borel subalgebras are conjugate under $\mathcal{E}(L)$ (the subalgebra of $\operatorname{Int} L$ generated by the strongly ad-nilpotent elements of $L$), it suffices to show that two CSAs are $\mathcal{E}(L)$-conjugate in any *semisimple* Lie algebra.

In all that follows, we've fixed a CSA $\mathfrak{h}$ of $L$ and root system $\Phi$ with simple root system $\Delta = \alpha_1, \ldots, \alpha_l$ of positive roots with $\mathcal{W}$ the Weyl group.

## 4.4 Representations

Since $\mathfrak{h}$ is semisimple (since $L$ is), it acts diagonally on any finite dimensional $L$-module. Thus we can always do the same trick of decomposing $V$ into eigenspaces $V = \sqcup_{\lambda\in\mathfrak{h}^*} V_\lambda$ that we do for $\mathfrak{sl}_2$. Slightly more generally (when $V$ is infinite-dimensional), we can consider the sum of weight spaces $V_\lambda$ of $V$, which is necessarily direct[1], so there always exists a direct sum of simple modules contained in $V$. When $\dim V < \infty$, this is equal to $V$ itself.

### 4.4.1 Standard Cyclic Modules

Here

[1]Think: vectors can't have multiple eigenvalues.

# 5 Quantum Groups

I am mostly working out of Jantzen's *Lectures on Quantum Groups* with the goal of reading through Ginzburg & Kumar's (apparently influential!) paper *Cohomology of Quantum Groups at Roots of Unity.*

## 5.1 What they are

The core idea to keep in mind is that quantum groups (or quantum enveloping algebras) are "deformations" (or perhaps $q$-analogues) of regular enveloping algebras. Well, at least that is the kind we're mostly working with here. A general definition eludes me (and Jantzen).

Mostly we will be interested in $U = \mathcal{U}_q(\mathfrak{sl}_2)$. In this setting, we get the definition

**5.1.1 Definition (Quantum Enveloping Algebra of $\mathfrak{sl}_2$)**
$\mathcal{U}_q(\mathfrak{sl}_2)$ is the associative algebra $k\langle E, F, K, K^{-1}\rangle$ under the following relations:

$$KK^{-1} = K^{-1}K = 1 \tag{R1}$$
$$KEK^{-1} = q^2E \tag{R2}$$
$$KFK^{-1} = q^{-2}F \tag{R3}$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \tag{R4}$$

where $q$ is some nonzero element of $k$ such that $q^2 \neq 1$.

Some notation that may be helpful in the future:

- For $a \in \mathbb{Z}$, define $[a] = \frac{v^a - v^{-a}}{v - v^{-1}} \in \mathbb{Q}(v)$ – the $v$ analogue of $a$. Equivalently, (and perhaps suggestively):
$$[a] = v^{a-1} + v^{a-3} + \cdots + v^{-a+3} + v^{-a+1}.$$
- the $v$ analogue of the factorial is
$$[a]^! = [a][a-1]\cdots[1]$$
- Then you get the binomial coefficients:
$$\begin{bmatrix} a \\ n \end{bmatrix} = \frac{[a]^!}{[n]^![a-n]^!}$$
-
$$[K; a] = \frac{Kq^a - K^{-1}q^{-a}}{q - q^{-1}}$$

## 5.2 Getting a basis

The first part of the first chapter focuses on getting a PBW-style basis for $U$, which comes in the form $F^s, K^n, E^r$ for $r, s \in \mathbb{N}$, $n \in \mathbb{Z}$. This is mostly standard but I thought this proof had a neat idea in it:

PROOF: Let $A = k[X, Y, Z]$ and define $f, e, h \in \mathrm{Hom}_k(A, A)$ such that

$$\begin{aligned} f(Y^s Z^n X^r) &= Y^{s+1} Z^n X^r, \\ e(Y^s Z^n X^r) &= q^{-2n} Y^s Z^n X^{r+1} + [s] Y^{s-1} [Z; 1-s] Z^n X^r, \\ h(Y^s Z^n X^r) &= q^{-2s} Y^s Z^{n+1} X^r \end{aligned}$$

One can check that these vector space homomorphisms exactly mirror multiplication of these monomials by $F, E$, and $H$, respectively, so we get a map $\varphi : U \to \mathrm{Hom}_k(A, A)$. Then take any linear combination of $F^s K^n E^r$, mapped though to $A$ via $\varphi$ and then evaluation at 1, gives us a linear combination of the $Y^s Z^n X^r$, so it must be trivial. Sick. ♠

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After that, the book focuses on $U_0 = k\langle Z, Z^{-1}\rangle$ as well as $U^+ = k\langle E\rangle$ and $U^- = k\langle F\rangle$, the latter of two which are isomorphic to $k[x]$ by the linear independence above.

---

Here's another cool bit: By studying products of monomials (or just defining degree outright) we can impose a graded structure on $U$: let $\deg(E) = 1$, $\deg(F) = -1$ and $\deg(K) = \deg(K^{-1}) = 0$ and notice (R1-4) are homogeneous according to this degree. Thus the quotient by this homogeneous ideal gives a graded structure to $U$.

Furthermore if $u \in U$ is homogeneous of degree $i$,

$$KuK^{-1} = q^{2i} u$$

so when the powers of $q$ are distinct ($|q| \neq 1$), the graded pieces are precisely the of $K$ under the adjoint action.

## 5.3 Representations of $U$

# 6 Affine Group Schemes (Waterhouse and Jantzen)

I have quite a bit of information to process before this, so I will get started!

**6.0.1 Definition (Closed Embedding)**
If $G$ and $H$ are affine group schemes represented by $A$ and $B$, respectively and if $\psi : H \to G$ is a homomorphism of affine group schemes (locally a group homomorphism) then if the corresponding algebra map $A \to B$ is surjective, then $\psi$ is called a **closed embedding.**

As its name suggests, this means that $\psi$ is an isomorphism onto a **closed subgroup** $H'$ of $G$. This is, in fact, a definition of this property. One can also think about it in the following ways: a group scheme $H$ is closed in $G$ if

- $H$ is defined by the relations imposed by $G$ plus some additional ones.
- $H = V(I)$ for some ideal $I \subset k[G]$.

Thinking back to our algebraic geometry, these are not too hard to see as equivalent. For instance, there is a closed embedding of $\boldsymbol{\mu}_\mathbf{n}$ in $\mathbf{G_m}$ (simply adding in $x^n - 1$) and of $\mathbf{SL_n}$ in $\mathbf{GL_n}$ (adding $\det = 1$).

## 6.1 Hopf Ideals

One problem with the above characterization that one cannot choose $I$ arbitrarily and end up with a group scheme. This is equivalent to arbitrarily adding relations to a group, which is not always guaranteed to work out well (think adding $\det = 2$ to $\mathbf{GL_n}$).

Actually, we can exactly categorize the closed embeddings of subgroups in $G$ by considering certain ideals of the algebra $A$ which represents it.

**6.1.1 Definition**
Let $A$ be an algebra and $I \lhd A$. Then if

- $\Delta(I)$ goes to zero under the map $A \otimes A \to A/I \otimes A/I$,
- $S(I) \subseteq I$
- $\varepsilon(I) = 0$

then $I$ is called a **Hopf Ideal of $A$.**

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \downarrow & & \downarrow \\ A/I & \xrightarrow{\Delta'} & A/I \otimes A/I \end{array}$$

6.1.2 Remark: That these ideals exactly characterize closed subgroups follows since any group (scheme) represented by $A'$ has the property that $\Delta'(A') \subseteq A' \otimes A'$. Since the new comultiplication map $\Delta'$ is derived from the old one $\Delta$, we have the diagram
from which we see that $\Delta(I)$ must go to zero in the map on the right. Similarly, we want that $S$ and $\varepsilon$ satisfy similar diagrams.

Note that an example of such an ideal is $I = \ker(\varepsilon)$.

**6.1.3 Definition**
A **character** is a scheme morphism $G \to \mathbf{G_m}$.

Let $\Phi$ be a character of $G$ and notice that the corresponding hopf algebra map $\varphi : k[X, 1/X] \to A$ is defined by $\varphi(X) = b$, which is automatically invertible in $A$. Furthermore since for any $a, b \in G(R)$ we have

$$\Phi \circ m(a, b) = \Phi(ab) = \Phi(a)\Phi(b) = m \circ (\Phi, \Phi)(a, b)$$

we get the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\Phi \times \Phi} & \mathbf{G_m} \times \mathbf{G_m} \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\ G & \xrightarrow{\Phi} & \mathbf{G_m} \end{array}$$

and then by dualizing we get that

$$\begin{tikzcd} A \otimes A & k[X, X^{-1}] \otimes k[X, X^{-1}] \arrow[l, "\varphi\otimes\varphi"'] \\ A \arrow[u, "\Delta"] & k[X, X^{-1}] \arrow[l, "\varphi"] \arrow[u, "\Delta"'] \end{tikzcd}$$

where in both diagrams we are abusing notation by using the same symbols for (co)multiplication in the different schemes. So we conclude that

$$\Delta(b) = \Delta \circ \varphi(X) = (\varphi \otimes \varphi) \circ \Delta(X) = (\varphi \otimes \varphi)(X \otimes X) = b \otimes b.$$

Then we can easily compute that $\varepsilon(b) = 1$ and $S(b) = b^{-1}$. This is precisely:

**6.1.4 Definition**
Let $A$ be a Hopf algebra and $a \in A$ such that

- $a$ is invertible
- $\Delta(a) = a \otimes a$
- $\varepsilon(a) = 1$
- $S(a) = a^{-1}$

then $a$ is called a **group-like** element of $A$.

6.1.5 Remark: As we saw above, every character of an affine group scheme $G$ corresponds to a group-like element of its representing algebra.

6.1.6 Remark: Using a parallel construction with any morphism $\Phi : G \to \mathbf{G_a}$, we get an element $b \in A$ such that $\Delta(b) = b \otimes 1 + 1 \otimes b$, $\varepsilon(b) = 0$ and $S(b) = -b$. These elements are called **primitive**

**6.1.7 Definition**
A group scheme $G$ represented by $A$ that consists entirely of group-like elements is called **diagonalizable.**

6.1.8 Remark: Notice that an alternative definition is to begin with an Abelian group $M$ and to define $S, \Delta$, and $\varepsilon$ such that each element is group-like. The resulting Hopf algebra represents a diagonalizable group scheme.

Consider the group algebra $k[M]$ on the group in Remark 6.1.8. If this algebra is finitely generated, we get the following:

#### 6.1.9 Theorem

Let $G$ be diagonalizable and represented by $A$ and assume $A$ is finitely generated as a $k$-algebra. Then $G$ is a finite product of copies of $\mathbf{G_m}$ and $\boldsymbol{\mu}_\mathbf{n}$.

Proof: Let $x_1, \ldots, x_n$ be generators for $k[M] = A$. Each $x_i$ can be written as a (finite!) $k$-linear combination of elements in $M$, so we can instead use the finitely many $m_i$ which generate these generators, and this gives us a $k$-algebra basis for $k[M]$. Call this new generating set $U$.

Let $M'$ be the abelian group generated by $U$ and notice that $k[M']$ is a subalgebra of $k[M]$ containing $U$ so $k[M'] = k[M]$ and therefore $M' = M$. This establishes that $M$ is a finitely generated abelian group, so we can split up the algebra into a tensor product of $k[\mathbb{Z}]$ and $k[\mathbb{Z}/n\mathbb{Z}]$.

When $M = \mathbb{Z}$, $k[M] \cong k[X, X^{-1}]$, so $G = \mathbf{G_m}$[2] Similarly if $M = \mathbb{Z}/n$ then $G \cong \boldsymbol{\mu}_\mathbf{n}$ ♠

Looking at my progress over the last several days, I am thinking that I am getting too much in the weeds writing out all the lemmas and proofs. Perhaps it is better to read more quickly and only take note of theorems as I need/use them.

## 6.2 Cartier Duals

The gist here is that if $G$ is represented by $A$, then we can define the dual $A^D = \mathrm{Hom}(A, k)$ and then $A^D$ represents a new scheme $G^D$ called the **Cartier Dual.**

We do some work to show that, in fact, elements of $G^D$ are in correspondence with the group-like elements of $A$, or equivalently the character group of $G$, $X_G$.

It is easy enough to evaluate $G^D(k) \cong \mathrm{Hom}(A^D, k)$, but luckily one shows that "dualizing" commutes with tensor products (and thus with extension of scalars) and Hom. Thus if $G_R$ is the scheme represented by $A \otimes_k R$, then using that $\mathrm{Hom}_k(A^D, R) \cong \mathrm{Hom}_R(A^D \otimes R, R)$,

$$G^D(R) = (G^D)_R(R)$$

which is represented by $(A \otimes R)^D = A^D \otimes R$, so finally

$$G^D(R) = (G_R)^D(R) = \{\text{group-like elements in } A \otimes_k R\} \cong \mathrm{Hom}(G_R, (\mathbf{G_m})_R)$$

[2]Notice that here Waterhouse also uses that the basis elements $e_i$ are group-like and so $\Delta(e_i) = e_i \otimes e_i$. I am not quite sure why this is necessary.

# 7 Algebraic Geometry

# 8 Problems

## 8.1 Week 1

## 8.2 Waterhouse

**Problem 2.1 (Waterhouse 2.1)**

*(a) Show that there are no nontrivial homomorphisms from* $\mathbf{G_m}$ *to* $\mathbf{G_a}$.

*(b) If* $k$ *is reduced, show that there are no nontrivial homomorphisms from* $\mathbf{G_a}$ *to* $\mathbf{G_m}$.

*(c) For each nonzero* $b \in k$ *with* $b^2 = 0$*, find a nontrivial homomorphism* $\mathbf{G_a} \to \mathbf{G_m}$.

Hmm, there must be a small bug in my macro but I can't find it. I get an error about `missing '\item'` in the definition below. It seems to only occur when I use `\subsubsection*` and the like.

**Solution:**
**(a)**

Let $\Phi : \mathbf{G_m} \to \mathbf{G_a}$ and let $\varphi : k[X] \to k[X, X^{-1}]$ be the corresponding Hopf algebra map. But then we have that

$$\varphi \otimes \varphi \circ \Delta_a(X) = \Delta_m \circ \varphi(X)$$

and so using that $\Delta_m(a) = a \otimes a$ and $\Delta_a(b) = b \otimes 1 + 1 \otimes b$, we get

$$\varphi(X) \otimes \varphi(1) + \varphi(1) \otimes \varphi(X) = \varphi(X) \otimes \varphi(X)$$

♠

# 9 To be filed: