Lie Algebras and Groups

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Introduction

These notes are my best attempt at following along with our *Math 508 – Lie Algebras* course at UW. This is my first time trying to type my notes on-the-fly in class so we'll see how well this goes. The course reference is Humphreys' *Introduction to Lie Algebras and Representation Theory.*

The course description follows:

This is the second course in the Algebraic Structures sequence. I will classify finite-dimensional complex semisimple Lie algebras, also proving some structural results on general Lie algebras along the way. Although one usually first encounters Lie algebras in a manifolds course, the treatment (following the text) will be entirely algebraic.

1 January 7, 2019

The homework is posted on Monty's website. :)

1.1 Lie algebras

This course will be studying Lie algebras, but as opposed to their treatment in manifolds, we will be studying them from a purely algebraic point of view. The book (Humphreys) actually never defines a Lie group.

1.1.1 Definition

A **Lie Algebra** L or $\mathfrak{g}$ over a field k is a k-vector space (usually f.d.) along with a *bracket operation* $[vw] : L \times L \to L$ such that $[\cdot\cdot]$ is

- anticommutative,
- bilinear,
- $[x[yz]] = [[xy]z] + [y[xz]]$

1.1.2 Remark: The last principle above is actually equivalent to the *Jacobi identity:*

$$[x[yz]] + [y[xz]] + [z[xy]].$$

This follows from bilinearity and anticommutativity of the bracket.

The most natural place for these to arise is as *derivations* on an algebra!

1.1.3 Definition
A **k-derivation** $d : A \to A$ on an algebra A over k is a k-linear map satisfying the Leibniz rule.

1.1.4 Remark: Some key facts about derivations (for us):

- Given a fixed $a \in A$, the map d_a sending $b \mapsto ab - ba$, the **commutator** $[ab]$ is a derivation.
- If d, e are derivations, then so is $[de] = de - ed$, where de is the *composite* of d and e as opposed to the product.

1.2 Examples

A main source of Lie algebras is (associative) algebras! *Any associative k-algebra A* becomes a Lie algebra over k, taking $[ab] = ab - ba$. In particular, one obvious choice for k-algebra is $M_n(k) = \mathfrak{gl}_n(k)$, the (Lie) algebra of $n \times n$ matrices over k.

Lie subalgebras are what you'd expect (including closure under brackets). Notice that if $L' \leq L$, then they **must both be over the same field.**

If L is a k-Lie algebra and $I \lhd L$ is an ideal of L, then the quotient space L/I becomes a Lie algebra with $[x + I, y + I] = [xy] + I$ as the bracket.

A **Lie algebra homomorphism** is a map $\varphi : L \to L'$ such that φ is k-linear and $\varphi([xy]) = [\varphi(x)\varphi(y)]$.

We get the usual first isomorphism theorem $L/\ker\varphi \cong \varphi(L)$.

Associative algebras are not the only source of Lie algebras, however! One example is $\mathfrak{sl}(n, k) = \{n \times n \text{ matrices over } k \text{ with trace zero}\}$

Note that this is **not closed under product** since $\operatorname{tr}(AB) \neq \operatorname{tr} A \operatorname{tr} B$ but $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ so $\operatorname{tr}(AB - BA) = \operatorname{tr}(AB) - \operatorname{tr}(BA) = 0$.

1.2.1 Definition
We call this algebra (or, in fact any subalgebra of $\mathfrak{gl}(n, k)$) **linear**. Think "Linear" means "of matrices."

We say that $\mathfrak{sl}(n,k)$ has **type** A_{n-1}. Eventually we will see seven types $A-G$ of semisimple Lie algebras. The shift in index will emerge later.

$\mathfrak{sl}(n,k)$ is, in fact, a simple Lie algebra: for $k=\mathbb{C}$, $\mathfrak{sl}(n,\mathbb{C})$ has no ideals apart from the trivial ones.

Other non-associative examples include k^n with a bilinear form $(\cdot,\cdot)$ which is either symmetric or skew-symmetric and (in either case) is nondegenerate.

1.2.2 Definition
$(\cdot,\cdot)$ is **nondegenerate** if the map $v\mapsto(v,\cdot)$ is injective. Equivalently there is no $v\in V$ such that $(v,w)=0$ for all $w\in V$.

Given $V=k^n$ and a bilinear form on V, we can look at all $X\in\mathfrak{gl}(n,k)=\mathfrak{gl}(V)$ such that $(Xv,w)=(v,Xw)$. Then X is **adjoint** with respect to the form. There is a similar definition for when X is **skew-adjoint.** One can check that $[XY]$ is skew-adjoint whenever both X and Y are.

1.3 Generating (skew) symmetric forms

It ends up that the dot product (which is a symmetric form) is misleadingly simple – thus we will look elsewhere.

If $M\in\mathfrak{gl}(n,k)$ is symmetric, so that $M^t=M$, then $(v,w)=v^tMw$ is a symmetric. If instead M is skew-symmetric, then the same definition yields a skew-symmetric form. This actually induces a one-to-one correspondence between matrices and forms.

In both cases, if M is invertible, then the form will be nondegenerate. As a consequence, since skew-symmetric matrices are always singular in odd dimensions, we see that nondegenerate skew-symmetric forms (over $\operatorname{char} k\neq 2$ where the two families of forms coincide) exist only in even dimensions.

1.4 A peek at classifications

If we have a nondegenerate symmetric form where $n=2m$ is going to give us an algebra of type D_m. If $n=2m+1$, then it is of type B_m. Both of these cases are called **orthogonal.**

If instead we have a skew-symmetric form and $n=2m$, then this is of type C_m, and we call this algebra **symplectic.**

We will make a particular choice for our matrix M and then study the resulting Lie algebras in much more detail next time. The choices will be:

- For type D_m:

$$\begin{pmatrix}0 & I_m\\ I_m & 0\end{pmatrix}$$

- For type C_m:

$$\begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}$$

- For type B_m:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_m & 0 \end{pmatrix}$$

2 January 9, 2019

Today we will be looking deeply into the stucture of linear Lie algebras of types A-D.

2.1 Linear Lie Algebras Revisited

Recall that the **matrix unit** e_{ij} is the matrix with 1 in the (i,j) entry and zero elsewhere. And then $e_{ij}e_{kl} = \delta_{jk}e_{il}$ and furthermore

$$[e_{ij}e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}.$$

This is especially nice when $j = i$, called the **diagonal matrix unit**.

Then we look at type A_{n-1} ($\mathfrak{sl}(n,k)$). Let D be the set of diagonal matrices in this algebra. Notice the dimension is $n-1$ since then n^{th} term on the diagonal is determined as the negative of the sum of the other $n-1$ terms. Let $A = \operatorname{diag}(d_1, \ldots, d_n)$. Then consider the eigenvalues associated with e_{ij}:

$$[Ae_{ij}] = (d_i - d_j)e_{ij} = (E_i - E_j)Ae_{ij}$$

where E_i is the linear functional selecting the i^{th} entry in A. Moreover, D is abelian as a Lie algebra, so D acts diagonally on $L = \mathfrak{sl}(n,k)$ by commutation with eigenvalues $E_i - E_j$ and zero for $1 \leq i,j \leq n$ and $i \neq j$.

In the other classical cases B-D, there is always a matrix M which defines the form $(v,w) = v^tMw$ as we saw yesterday. In all three cases, the Lie algebra exists consists of all skew-adjoint matrices X relative to the form. $B_m = \mathfrak{so}(2m+1,k)$ as well as $D_m = \mathfrak{so}(2m,k)$ and $C_m = \mathfrak{sp}(2m,k)$.

This condition translates to the form of matrices in the above Lie groups and the condition is always $Mx = -x^tM$ in all cases.

Type B:

$$\begin{pmatrix} 0 & b_1 & b_2 \\ c_1 & m & n \\ c_2 & p & q \end{pmatrix}$$

where $c_1 = -b_2^t$, $c_2 = -b_1^t$, $q = -m^t$, $n^t = -n$ and $p^t = -p$.

Type C:

$$\begin{pmatrix} m & n \\ p & q \end{pmatrix}$$

where $n^t = n, p^t = p$, and $m^t = -q$.

Type D:

$$\begin{pmatrix} m & n \\ p & q \end{pmatrix}$$

where $n^t = -n, p^t = -p$, and $m^t = -q$.

Looking at the eigenvalues of elements of D associated to vectors $e_{ij} - e_{m+i,m+j}$. Look at photos

Using a similar analysis, we can look at types B and D. We define the functions E_i similarly on the space of diagonalmatrices and gives a rise to the following collection of linear functions: in $B_m : \pm E_i$ and $\pm(E_i \pm E_j)$ and D_m gives us $\pm(E_i \pm E_j)$.

This collection of functions in each case is called the **root system of the Lie algebra**, Then any complex simple finite-dimensional Lie algebra is classified by its root system. The (perhaps surprising) fact is that this already encompasses all but finitely many of these things up to ismorphism: the classical Lie algebras. Eventually we will learn more about the **exceptional Lie groups.**

This section was a little hard to follow and the handling in Humphreys is easier to follow, but delays speaking about root systems and actually deriving the eigenfunctions (is that the right word?) until significantly later. Monty seemed to think it was acceptable to delay the understanding of this a bit.

2.2 Derivations and exp

Look an an arbitrary Lie algebra over a field k where $\operatorname{char} k = 0$ (which we will mostly be assuming from here on) . Let δ be a derivation of L, so that $[\delta x, y] + [x \delta y] = \delta[x, y]$. Assume that δ is nilpotent.

Then the "power series" (polynomial) is

$$\exp \delta = \sum_{i>0} \frac{\delta^i}{i!}$$

Problem 2.1 *This is a good exercise to go through: Check that*

$$[(\exp \delta)x, (\exp \delta)y) = [xy]$$

for each $x, y \in L$

2.2.1 Remark: This actually shows that $\exp \delta$ is an automorphism of L. Furthermore you'd find that

$$(\exp \delta)(\exp(-\delta)) = 1.$$

What if $k = \mathbb{R}$ or $\mathbb{C}$? Then the power series (even when δ is not nilpotent!) always converges and defines an automorphism as before.

2.2.2 Lemma
For all **complex** semisimple Lie algebras L it turns out that the group generated by $\exp \operatorname{ad} x$ ($\operatorname{ad} x(y) = [xy]$) coincides with the group generated by all nilpotent $\operatorname{ad} x$.

2.3 Adjoint group

The last thing for today is to define the adjoint group:

2.3.1 Definition
Let L be a Lie algebra, then
$$\operatorname{Int}(L) = \exp \operatorname{ad} L$$
is the **Adjoint group** of L. It is a subgroup (so we believe) of the Lie Group associated to L.

2.3.2 Remark: Actually after talking to Monty, $\exp \operatorname{ad} L$ *is* (essentially) the Lie group corresponding to L. Such a group is not unique, however.

Some examples of adjoint groups:

- If $L = \mathfrak{sl}(n, \mathbb{C})$, then $\operatorname{Int}(L) = PSL(n, \mathbb{C}) = SL(n, \mathbb{C})/\text{center}$
- If $L = \mathfrak{so}(n, \mathbb{C})$, then $\operatorname{Int}(L) = PSO(n, \mathbb{C})$
- If $L = \mathfrak{sp}(2n, \mathbb{C})$ then $\operatorname{Int}(L) = PSp(2n, \mathbb{C})$.

HW 1 – Due January 18

Do problems 1.9, 2.1, 3.8. 3.9, and 4.3.

Problem 1.9 *When* $\operatorname{char} k = 0$*, show that each classical Lie algebra* A_l, B_l, C_l *and* D_l *satisfies* $[LL] = L$*. (This shows again that each algebra consists of trace zero matrices.)*

Solution:
Type A_k:

Recall that a basis for A_l is the collection $\mathcal{B} = \{e_{ij} | i \neq j\} \cup \{e_{ii} - e_{i+1,i+1} | 1 \leq i \leq l\}$. Using linearity of the bracket, it then suffices to check that $\mathcal{B}$ is contained in the algebra generated by $[\mathcal{B}\mathcal{B}]$. To see this, we use that

$$[e_{ij}e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}.$$

First, notice we have

$$[e_{i,i+1}e_{i+1,i}] = e_{ii} - e_{i+1,i+1},$$

so it suffices to show we also have the e_{ij} for $i \neq j$. ♠

Problem 2.1 *Prove that the set of all inner derivations* $\operatorname{ad} x$ *for* $x \in L$ *is an ideal of* $\operatorname{Der} L$*.*

Solution:
Let $I = \operatorname{ad} L$. $I \subseteq \operatorname{Der} L$ since $\operatorname{ad} x$ is a derivation. This is a subspace since $\operatorname{ad} 0 = 0$ and since the bracket is bilinear

$$\operatorname{ad} x(ay + bz) = [x, ay + bz] = [x, ay] + [x, bz] = a[xy] + b[xz] = a \operatorname{ad} x(y) + b \operatorname{ad} x(z).$$

Take any $x \in I$ and let $f \in \operatorname{Der} L$ be arbitrary. But then for $y, z \in L$

$$\begin{aligned}
[\operatorname{ad} x, f](yz) &= (\operatorname{ad} x f - f \operatorname{ad} x)(yz) \\
&= [x, f(yz)] - f[x, yz] \\
&= [x, f(y)z + yf(z)] - f([x, y]z + y[x, z]) \\
&= [x, f(y)z] + [x, yf(z)] - f([x, y]z) - f(y[x, z]) \\
&= [x, f(y)]z + f(y)[x, z] + [xy]f(z) + y[x, f(z)] - f([x, y])z \\
&\qquad - [x, y]f(z) - f(y)[x, z] - yf[x, z] \\
&= ([x, f(y)] - f[x, y])z + y([x, f(z)] - f[x, z]) \\
&= ([\operatorname{ad} x, f](y))z + y([\operatorname{ad} x, f](z))
\end{aligned}$$

so $[IL] \subseteq I$ whence I is an ideal of L. ♠

Problem 3.8 *Let L be nilpotent. Prove that L has an ideal of codimension 1.*

Solution:
Assume that L is nilpotent. Then defining $L^0 = L$, and by induction $L^i = [LL^{i-1}]$, this means that $L^n = 0$. ♠

Problem 3.9 *Prove that every nilpotent Lie algebra L has an outer derivation (see 1.3) as follows: Write $L = K + Fx$ for some ideal K of codimension 1. Then $C_L(K) \neq 0$ (why?). Choose n so that $C_L(K) \subseteq L^n$, $C_L(K) \nsubseteq L^{n+1}$ and let $z \in C_L(K) - L^{n+1}$. Then the linear map δ sending K to zero and x to z is an outer derivation.*

Solution:

Problem 4.3 *This exercise illustrates the failure of Lie's theorem when F is allowed to have positive characteristic. Consider the $p \times p$ matrices*

$$x = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad y = \mathrm{diag}(0, 1, \ldots, p-1)$$

Check that $[xy] = x$, hence that x and y span a two dimensional solvable subalgebra L of $\mathfrak{gl}(p, F)$. Verify that x and y have no common eigenvector.

Solution:

3 January 11, 2019

A few more things about $\operatorname{Int} L$ in classical cases:

Begin with $n \times n$ matrices M over $\mathbb{R}$ or $\mathbb{C}$. Given any such M, we have

$$e^M = \sum_{i>0} \frac{M^i}{i!}$$

always converges (that is, the series for each entry converges). Futhermore $e^M e^{-M} = I$.

3.0.1 Remark: Notice that over *any* field over characteristic zero whenever M is nilpotent this still holds.

Now if M is skew-adjoint with respect to a bilinear form $(\cdot, \cdot)$, then

$$(M^i v, w) = (-1)^i (v, M^i w)$$

whence

$$(e^M v, w) = (v, e^{-M} w)$$

and so

$$(e^M v, e^M w) = (v, w)$$

for all $v, w \in \mathbb{R}^n$ or $\mathbb{C}^n$.

But notice that then e^M is an **isometry of** or **preserves** the form. Moreover, if we set $A = \operatorname{ad} M$, then A is given as the difference of two *commuting* linear maps, left and right multiplication by M.

Using the binomial theorem: $e^A N = e^M N e^{-M}$. This, in a nutshell, is why the formulas for $\operatorname{Int} L$ all involved modding out by the center, since conjugation by a scalar matrix is trivial! :)

3.1 Combining Ideals

Now we return to the case of general Lie algebras over arbitrary fields (no longer assuming $\operatorname{char} k = 0$). We have seen that a subspace $I \leq L$ is an ideal if $[IL] \subseteq I$.

If $I, J \lhd L$, then $I + J$ is an ideal, as well as

$$[IJ] = \{\sum_1^n [x_i y_i] | x_i \in I, y_i \in J\}.$$

We also are sometimes interested in the direct sum of ideals $\oplus L_i$ where $[L_i L_j] = 0$ when $i \neq j$.

3.2 Examples

0 and L are ideals as well as the **center** $Z(L)$.

3.3 Characterizing Lie Algebras as Linear

We now want to realize many Lie algebras as (isomorphic to) linear ones. For now, note that the adjoint map ad is a Lie algebra homomorphism by the Jacobi identity. Then $\ker \mathrm{ad} = Z(L)$, so when $Z = 0$, then L is isomorphic to a linear Lie algebra.

3.3.1 Definition
If $Z(L)$, and the map $\mathrm{ad} : L \to L$ is an isomorphism, we say L **acts faithfully** on itself (via ad).

3.3.2 Definition
If we have a homomorphism $L \to \mathfrak{gl}(n, k)$, then we call k^n an L-**module**. We define the action of L on K^n through the homomorphism.

Notice that L is always an L module over itself.

One can check, using the realization of $\mathfrak{sl}(n, k)$ we saw last time that (when $\mathrm{char}\, k \nmid n$) that $\mathfrak{sl}(n, k)$ is simple. This is actually the Lie algebra analogue of the fact that $M_n(K)$ is simple as a ring over any field K.

3.3.3 Remark: Notice that if $\mathrm{char}\, k|n$, then *any* scalar matrix $aI \in \mathfrak{sl}(n, k)$ so $Z(\mathfrak{sl}(n, k))$ is nontrivial.

In any event you can look up the proof for $\mathfrak{sl}(2, k)$ in the book. The thing to glean here is that

$$[xy] = h, \quad [hx] = 2x, \quad [hy] = -2y$$

for $h = e_{11} - e_{22}$, $x = e_{12}$ and $y = e_{21}$. This will keep popping up throughout the course.

3.4 Ideals of Linear Lie Algebras

Many subalgebras of $\mathfrak{sl}(n, k)$ or $\mathfrak{gl}(n, k)$ have many ideals (which happen to be ring-theoretic ideals).

3.4.1 Definition
Some notation:

$$\mathfrak{t}(n, k) = \{\text{upper-triangular matrices}\}$$

also

$$\mathfrak{n}(n, k) = \{\text{strictly-upper-triangular matrices}\}$$

3.5 Derived Series

For any Lie algebra, we can define

3.5.1 Definition
The **derived series** of a Lie algebra L is defined to be

$$L^{(0)} = L, \quad L^{(1)} = [LL], \quad L^{(n+1)} = [L^{(n)}L^{(n)}]$$

which form an ascending chain of ideals.

3.5.2 Definition
If $L^{(n)} = 0$ for some n, and thus for all sufficiently large n, we call L **solvable.**

3.5.3 Remark: Note that actually there is a cooresponding definition for groups using commutators $[G, G]$. This was defined first and ported to Lie algebras. It actually happened in the reverse direction for:

3.5.4 Definition
The **central series** of L is

$$L^0 = L, \quad L^1 = [LL], \quad L^2 = [L, L^1], \ldots$$

and if $L^n = 0$ for some n, then we call L **nilpotent.**

3.5.5 Remark: Equivalently, we can say that L is nilpotent if *any n-fold bracket* in L is zero.

3.5.6 Remark: Notice that every nilpotent Lie group is solvable, but the converse fails: $t(n)$ is solvable by not nilpotent.

3.6 Examples of solvable algebras

One can easily check that every ideal and homomorphic image of a solvable algebra is solvable. Therefore if I and J are solvable ideals of L, then so is $(I + J)/J \cong I/(I \cap J)$ is solvable, as is $I + J$. Furthermore if I and L/I are solvable, then L is.

3.6.1 Definition
$\operatorname{Rad} L$, the **radical of** L, is the unique largest solvable ideal of L.

3.6.2 Remark: Any finite dimensional algebra has a radical (taken to be the sum of all solvable ideals in L).

Using this definition, we are able to give two different characterizations:

3.6.3 Definition (Semisimple Lie Algebra)
L is semisimple if $\operatorname{Rad} L = 0$.

3.6.3′ Definition (Semisimple Lie Algebra)
L is semisimple if it is a direct sum of simple Lie algebras.

This equivalence will not become clear until we learn about the Killing form, but at least it jives with our experience with representation theory and semisimple modules.

4 January 14, 2019

I missed this day for an event. I will try to catch up if I have time.

5 January 16, 2019

We are continuing from last time when we were showing that as solvable Lie subalgebra of $\mathfrak{gl}(n, k)$ (when $\operatorname{char} k = 0$ and $\bar{k} = k$) necessarily admits a vector $v \in k^n \setminus 0$ that is a common eigenvector:

$$xv = \lambda(x)v$$

for some **linear** function $\lambda : L \to k$.

We argues (as in the parllel case of a Lie subalgebra of $\mathfrak{gl}(n, k)$ consisting of nilpotent matrices by induction on dimension and found the same result (except it is zero). In this case when $\dim L = 1$, since k is algebraically closed, L is spanned by asingle matrix x and any eigenvector of x does the trick.

Now assume that the result holds for solvable Lie algebras of dimension $< d$. Then since L is sovable, we know $[LL] \subsetneq L$. Let I be any subspace containing $[LL]$ and properly contained in L of codimension one. Then I is an ideal since $[LI] \subseteq [LL] \subseteq I$, so by the inductive hypothesis I has a common eigenvector, so that there is a linear function λ such that the **weight space**

$$V_\lambda = \{v \in k^n : xv = \lambda(x)v, \forall x \in I\}$$

is nonzero.

We know that $L = I + kx$ for some $x \in L$, so we must show that x preserves V_λ: then any eigenvector of x on V_λ will do the job. Given $v \in V_\lambda$, we must show that

$$yxv = xyv + [yx]v = \lambda(y)xv + \lambda([yx])v$$

so we must show that $\lambda([xy]) = 0$.

Given $v \in V_\lambda$, look at hte powers $v, xv, x^2v, \ldots$. Let m be the least power such that the $v, xv, \ldots, x^m v$ are linearly independent and let W_i be the span of the first i of these.

Then one checks by induction for an $y \in I$ that $yx^iv = \lambda(y)x^iv \pmod{v, xv, \ldots, x^iv}$ It follows that y was in the subspace W_m (oh no missed some stuff).

So the trave of y on W_m is $m\lambda(y)$. Now for any $y \in I$, the matrix $[xy]$ acts on W_m ascendingthe commutator of two matrices having trace zero, whence $m\lambda([xy]) = 0$, so since $\operatorname{char} k = 0$, $\lambda([xy]) = 0$, as desired.

As a consequence, given a solvable Lie subalgebra of $\mathfrak{gl}(n, k)$ here is a chain of subspaces

$$0 = V_0 \leq V_1 \leq \cdots \leq V_n = k^n$$

or a **flag** such that $LV_i \subseteq V_i$ for all i: L acts by upper triangular matrices with respect to a suitable basis.

Another way of putting this, using module language, is that given any **finite dimensional** module over a solvable Lie algebra (over an algebraically closed characteristic zero field) has a one-dimensional submodules N.

Equivalently, the only finite-dimensional irreducible modules over a solvable Lie algebra are one-dimensional. We will use this later in the context of semisimple Lie algebras with large solvable subalgebra.

5.0.1 Remark: For the record, Lie's theorem *definitely fails* when k has positive characteristic. We see one in our homework, and another example comes from $\mathfrak{sl}(2, k)$ where $\operatorname{char} k = 2$.

We know that $\mathfrak{sl}(2, k)$ has a basis: $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = I_2$, $x = e_{12}$, $y = e_{21}$. Here $[hx] = 0 = [hy]$ and $[xy] = h$. It is shown in the book and is easy to see that when $\operatorname{char} k \neq 2$, $\mathfrak{sl}(2, k)$ is simple, but here $\mathfrak{sl}(2, k)$ is nilpotent and its bracket relations become $[xy] = z$, $[xz] = [yz] = 0$.

It ends up we can define (actually over any field) the **Heisenberg** Lie algebra, spanned by $x = e_{12}, y = e_{23}, z = e_{13}$ or more simply the 3×3 strictly-upper-triangular matrices.

The matrices in $\mathfrak{sl}(2, k)$ for any k have no common eigenvector, so our proof of Lie's theorem must fail somewhere? We indeed have an ideal I of coimension one spanned by h and x and there is acommon eigenvector for I, namely e_1. Then I acts diagonally on

ke_1 via the linear function sending h to 1, x to 1. Here y plays the role of x in the proof, but is outside the ideal. That is,

$$\lambda[yx] = \lambda(h) \neq 0.$$

Returning to the good case of k, we can now apply our result on a linear solvable Lie algebras to arbitrary ones: uGIven any solvable Lie algebra L, there is a flag $0 = L_0 \leq L_1 \leq \cdots \leq L_n = L$ of ideas of L so that $[LL_i] \subseteq L_i$ Also $[LL]$ is nilpotent, since it acts on L by strictly upper triangular matrices.

Both results fail for general solvable Lie algebras when the basefield is not nice. If a solvable Lie algebra L **does** have a chain of ideals $0 = L_0 \leq L_1 \leq \cdots \leq L_n = L$ where $\dim L_i = i$, then we call L **completely solvable.**

5.1 Returning to Linear Algebra

At this point we will take some time to return to an important idea from Linear algebra, namely the **Jordan canonical form** of a square matrix. Recall Jordan blocks and similarity when eigenvalues lie in the field.

If M is already in this form, then we can write $M = M_s + M_n$ where M_s consists of the diagonal entries of each block (zeroes elsewhere) and M_n the off-diagonal entries. Then M_s is diagonal (or for general M diagonalizable), or **semisimple** and M_n is nilpotent and $[M_s M_n] = 0$ (they commute).

5.1.1 Definition
$M = M_s + M_n$ is called the **Jordan decomposition of M.** We call M_s and M_n the **semisimple** and **nilpotent** parts of M.

5.1.2 Remark: Each are uniquely determined for M and (and this is not generally proved in the first year algebra sequence) each are equal to polynomials in M with no constant term.x‘