# Notes and Problems from My Research

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# Part I
# Autumn 2018

## 1 Problems

**Problem 1.1** *Assume that $k$ is a field and let $K = k(t)$ (notice $K$ is a transcendental extension). Prove that $\operatorname{Hom}_k(K, k) \ncong K$.*

**Solution:**
This is basically just a cardinality argument. I don't think it's particularly worth doing at this juncture. ♠

**Problem 1.2** *Let $G$ be a finite group scheme (actually we need only assume that $G$ is a Frobenius algebra so that a module is injective if and only if it is projective). Prove that unless $M$ is projective, its projective dimension is infinite. Conclude that $H^n(G, M) = 0$ for $n > N$ implies that $M$ is projective.*

**Solution:**
Assume $M$ itself is not projective so that its minimal projective resolution is nontrivial and furthermore that it is finite. That is, let $P_i$ be projective modules such that

$$0 \to P_n \xrightarrow{f_n} P_{n-1} \to \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

is a minimal length projective resolution of $M$ (notice here that $n \geq 1$).

Next consider the short exact sequence

$$0 \to P_n \xrightarrow{f_n} P_{n-1} \to \operatorname{coker} f_n \to 0$$

since $P_n$ is projective (and thus injective!) this sequence splits and therefore $P_{n-1} \cong P_n \oplus \operatorname{coker} f_n$. But then consider the sequence

$$0 \to P_n \xrightarrow{g} P_{n-2} \to \cdots \xrightarrow{f_0} M \to 0$$

where above we are using $P_{n-1} \supseteq P_n \cong f_n(P_n)$ and that $g = f_{n-1}|_{f_n(P_n)}$. This map is injective since $\ker f_{n-1} = \operatorname{coker} f_n$, which is disjoint from $f_n(P_n) \cong P_n$. Exactness everywhere else is evident since the maps are not effectively changed.

But then the existence of this sequence contradicts the minimality of the original sequence, so no finite sequence can exist.

The last statement (as discussed with Julia) is actually false. ♠

**Problem 1.3** *Establish the five-term exact sequence for spectral sequences.*

**Solution:**
I plan to return to this problem in the future. I have other priorities at the moment, but I will eventually return to cohomology and spectral sequences and this will be a good exercise at that point. ♠

**Problem 1.4 (Waterhouse 1.1)**

*(a) If $R$ and $S$ are two $k$ algebras and $F$ is a representable functor, show $F(R \times S) \cong F(R) \times F(S)$.*

*(b) Show there is no representable functor $R$ such that every $F(R)$ has exactly two elements.*

*(c) Let $F$ be the functor represented by $k \times k$. Show that $F(R)$ has two elements exactly when $R$ has no idempotents besides 0 and 1.*

**Solution:**
**(a)**

Let $A$ be the $k$-algebra representing $F$. Thus $F(R)$ is naturally isomorphic to $\mathrm{Hom}_k(A, R)$ and $F(S) \simeq \mathrm{Hom}(A, S)$. Then define the map $\Phi : \mathrm{Hom}(A, R \times S) \to \mathrm{Hom}(A, R) \times \mathrm{Hom}(A, S)$ via

$$\Phi(\varphi) = (\pi_R \circ \varphi, \pi_S \circ \varphi)$$

where $\pi_X$ is the canonical projection onto $X$.

This is surjective since (by the universal property of products) any pair of maps $\varphi_R : A \to R$ and $\varphi_S : A \to S$ factors through the product $R \times S$ and furthermore it does so *uniquely*, giving us injectivity. Thus this map (which is clearly a homomorphism since $\pi_X$ is) is a bijection.

**(b)**

By the last problem this is impossible since if $|F(k)| = 2$ then

$$|F(k \times k)| = |F(k) \times F(k)| = 4.$$

**(c)**

Let $F$ be such a functor. Consider any $\varphi \in \mathrm{Hom}(k \times k, R) \simeq F(R)$. Assume first that $F(R) \cong \mathbb{Z}/2$ and let $r$ be an idempotent in $R$. ♠

**Problem 1.5 (Waterhouse 1.2)** *Let $E$ be a functor represented by $A$ and let $F$ be any functor. Show that the natural maps $\eta : E \to F$ correspond to elements in $F(A)$.*

**Solution:**
Consider the map $\Phi$ from natural maps $E \to F$ to elements in $F(A)$ defined by (again leveraging the representability of $E$)

$$\eta \mapsto \eta(\mathrm{id}_A) \in F(A).$$

Conversely, consider the map $\Psi$ from $F(A)$ to the natural maps $E \to F$ via

$$x \mapsto \xi_x$$

where $\xi_x$ where for any $Y$ and $y \in E(Y) \cong \mathrm{Hom}(A, Y)$ we define the $Y^{th}$ component of $\xi_x$ as

$$\xi_x(y) = F(y)(x) \in F(Y)$$

where (for clarity while I get a grasp here) $F(y) : F(A) \to F(Y)$.

Since we are only looking for a bijection, we only need that these maps are inverses. Consider that for all $Y$ and $y \in E(Y)$,

$$\begin{aligned} \Psi \circ \Phi(\eta)(y) &= \Psi\left(\eta(\mathrm{id}_A)\right)(y) \\ &= \xi_{\eta(\mathrm{id}_A)}(y) \\ &= F(y) \circ \eta(\mathrm{id}_A) \\ &= \eta \circ E(y)(\mathrm{id}_A) \\ &= \eta(y \circ \mathrm{id}_A) = \eta(y) \end{aligned}$$

where above we used the naturality of $\eta$ along with the fact that $E(y)$ is just precomposition with $y$. Thus $\Psi \circ \Phi(\eta) = \eta$.

But then for any $x \in F(A)$,

$$\begin{aligned} \Phi \circ \Psi(x) &= \Phi \circ \xi_x \\ &= \xi_x(\mathrm{id}_A) \\ &= F(\mathrm{id}_A)(x) \\ &= \mathrm{id}_{F(A)}(x) = x \end{aligned}$$

completing the proof. ♠

**Problem 1.6 (Waterhouse 1.3)** *Let $E$ be a functor represented by $A$, and let $F$ be any functor. Let $\Psi : F \to E$ be a natural map with surjective component maps. Show there is a natural map $\Phi : E \to F$ with $\Psi \circ \Phi = \mathrm{id}_E$.*

**Solution:**
Since in particular $\Psi_A$ is surjective, there is an $x \in F(A)$ such that $\Psi(x) = \mathrm{id}_A$. Then using the map from the last problem, let $\Phi = \xi_x$. Then we can compute for any $R$ and $g \in E(R)$

$$\begin{aligned}\Psi \circ \Phi(g) &= \Psi \circ F(g)(x)\\ &= E(g) \circ \Psi(x)\\ &= E(g)(\mathrm{id}_A)\\ &= g \circ \mathrm{id}_A = g\end{aligned}$$

since $g : A \to R$, so $E(g) : E(A) \to E(R)$, which is just composition with $g$. ♠

**Problem 1.7 (Waterhouse 1.5)** *Write out $\Delta, \varepsilon$, and $S$ for the Hopf algebras representing $\mathbf{SL_2}, \mu_\mathbf{n}$, and $\alpha_\mathbf{p}$.*

**Solution:**
**$\mathbf{SL_2}$:**

Notice $SL_2$ is represented by $A = k[X_1, X_2, X_3, X_4]/(X_1X_4 - X_3X_2 - 1)$ so take two elements $f, g \in \mathrm{Hom}(A, R)$ where $f(X_i) = a_i \in R$ and $g(X_i) = b_i \in R$ and notice that we want

$$(f, g)\Delta = h$$

where since

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$$

we want to have that $h(X_i) = c_i$.

So then if $\Delta : A \to A \otimes A$ is defined as follows:

$$\begin{aligned} X_1 &\mapsto X_1 \otimes X_1 + X_2 \otimes X_3\\ X_2 &\mapsto X_1 \otimes X_2 + X_2 \otimes X_4\\ X_3 &\mapsto X_3 \otimes X_1 + X_4 \otimes X_3\\ X_4 &\mapsto X_3 \otimes X_2 + X_4 \otimes X_4 \end{aligned}$$

Where one can compute

$$\begin{aligned}\mathrm{id} \otimes \Delta \circ \Delta(X_1) &= (\mathrm{id} \otimes \Delta)(X_1 \otimes X_1 + X_2 \otimes X_3)\\ &= X_1 \otimes X_1 \otimes X_1 + X_1 \otimes X_2 \otimes X_3 + X_2 \otimes X_3 \otimes X_1 + X_2 \otimes X_4 \otimes X_3\\ &= (\Delta \otimes \mathrm{id})(X_1 \otimes X_1 + X_2 \otimes X_3) = \Delta \otimes \mathrm{id} \circ \Delta(X_1)\end{aligned}$$

and similar equality holds for the other $X_i$, so this is $\Delta$.

Using that we want $\varepsilon \otimes \mathrm{id} \circ \Delta(X_i) = 1 \otimes X_i$, we see that the map $\varepsilon$ sending $X_1$ and $X_4$ to 1 and $X_2$ and $X_3$ to zero is the map we want.

Notice that as a sanity check we get that

$$\begin{pmatrix} \varepsilon(X_1) & \varepsilon(X_2) \\ \varepsilon(X_3) & \varepsilon(X_4) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Finally using that $(S, \mathrm{id}) \circ \Delta(X_i) = \varepsilon(X_i)$ and the fact that in $A$, $\det = X_1X_4 - X_3X_2 = 1$, we can define $S$ such that

$$\begin{pmatrix} S(X_1) & S(X_2) \\ S(X_3) & S(X_4) \end{pmatrix} = \begin{pmatrix} X_4 & -X_2 \\ -X_3 & X_1 \end{pmatrix} = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}^{-1}$$

and one can verify that this satisfies the relation above.

$\mu_{\mathbf{n}}$ :

For this scheme, $A = k[X]/(X^n - 1)$ is the representing algebra. If $f, g \in \mathrm{Hom}(A, k)$ with $f(X) = r$ and $g(X) = s$, then we want $(f, g)\Delta(X) = \sum f(X_{(1)})g(X_{(2)}) = m(r, s) = rs$ where $\Delta(X) = \sum X_{(1)} \otimes X_{(2)}$. An obvious choice is the diagonal map.

Then choosing $\varepsilon(X) = 1$ satisfies the diagrams (as before in $G_m$) and using our intuition, $S(X) = X^5$ which also works.

$\alpha_{\mathbf{p}}$**:**

This time we are working with $A = \mathbb{Z}/p[X]/(X^p)$. This time (since the group is additive) we want $\Delta(X) = X \otimes 1 + 1 \otimes X$, which we can see works with associativity immediately.

Following suit with the other additive group scheme $G$, setting $\varepsilon(X) = 0$ and $S(X) = -X$ we can quickly check these still satisfy the given axioms. ♠

**Problem 1.8 (Waterhouse 1.6)** *In $A = k[X_{11}, \ldots, X_{nn}, 1/\det]$ representing $GL_n$, show that $\Delta(X_{ij}) = \sum X_{ik} \otimes X_{kj}$. What is $\varepsilon(X_{ij})$?*

**Solution:**
Due to the uniqueness of $\Delta, \varepsilon$, and $S$, we need only find maps satisfying the diagrams. I claim that $\varepsilon(X_{ij}) = \delta_{ij}$. In this case, notice

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta(X_{ij}) = \varepsilon \otimes \mathrm{id}\left(\sum X_{ik} \otimes X_{kj}\right) = \sum \delta_{ik} \otimes X_{kj} = 1 \otimes X_{ij}$$

exactly as we want.

For associativity, notice

$$(\Delta \otimes \mathrm{id}) \circ \Delta(X_{ij}) = \Delta \otimes \mathrm{id}\left(\sum_k X_{ik} \otimes X_{kj}\right) = \sum_k \left(\sum_l X_{il} \otimes X_{lk}\right) \otimes X_{kj}$$

and then the associativity of $\Delta$ follows simply from the associativity of the tensor product.

For the last axiom, we compute $S$ such that $(S, \mathrm{id}) \circ \Delta = \iota \circ \varepsilon$ where $\iota : K \to A$ is the map sending $k \mapsto k \cdot 1_A$. That is, we define $S : A \to A$ so that

$$\sum_k S(X_{ik})X_{kj} = \delta_{ij}.$$

We want to leverage the fact that for a fixed $i$ and $j$, the determinant is

$$\begin{aligned}
\det &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_l X_{\sigma(l)l} \\
&= \sum_{\sigma} \operatorname{sgn}(\sigma) X_{\sigma j j} \prod_{l \neq j} X_{\sigma(l)l} \\
&= \sum_i X_{ij} \left( \sum_{\sigma(j)=i} \operatorname{sgn}(\sigma) \prod_{l \neq j} X_{\sigma(l)l} \right)
\end{aligned}$$

and so we want that

$$S(X_{ik}) = \frac{1}{\det} \sum_{\sigma(i)=k} \operatorname{sgn}(\sigma) \prod_{l \neq k} X_{\sigma(l)l}$$

so that when $i = j$,

$$\sum_k S(X_{ik})X_{kj} = \frac{1}{\det} \sum_k X_{kj} \sum_{\sigma(j)=k} \operatorname{sgn}(\sigma) \prod_{l \neq k} X_{\sigma(l)l} = 1 = \delta_{ij}$$

whenever $i \neq j$, however, this equation is the determinant of the matrix where we have replaced the $j^{th}$ column with a copy of the $i^{th}$ column. This is linearly dependent, so

$$\frac{1}{\det} \sum_k S(X_{ik})S_{kj} = 0 = \delta_{ij}.$$

Thus these are precisely the maps we desire. ♠

**Problem 1.9 (Waterhouse 1.10)** *Prove the following Hopf algebra facts by interpreting them as statements about group functors:*

*(a)* $S \circ S = \mathrm{id}$

*(b)* $\Delta \circ S = (\mathrm{twist}) \circ (S \otimes S)\Delta$

*(c)* $\varepsilon \circ S = \varepsilon$

*(d) The map $A \otimes A \to A \otimes A$ sending $a \otimes b$ to $(a \otimes 1)\Delta(b)$ is an algebra isomorphism.*

**Solution:**

**(a)**

Dualizing, we get

$$\begin{tikzcd} A \arrow[rr, equal] \arrow[dr, "S"'] & & A & A \arrow[rr, equal] & & A \arrow[dl, "inv"] \\ & A \arrow[ur, "S"'] & & & A \arrow[ul, "inv"] & \end{tikzcd}$$

so this statement is equivalent to the group (scheme) fact that $(g^{-1})^{-1} = g$.

**(b)**

Using a similar duality argument, this is equivalent to saying

$$\text{inv} \circ m = m \circ (\text{inv} \times \text{inv}) \circ (\text{twist})$$

but if we consider arbitrary elements $g, h \in G(R)$, this means

$$(gh)^{-1} = m \circ (\text{inv} \times \text{inv})(h, g) = m(h^{-1}, g^{-1}) = h^{-1}g^{-1}$$

which is clearly true.

**(c)**

This one is equivalent to $(\text{inv}) \circ i = i$ where if $g \in G(R)$,

$$(\text{inv}) \circ i(g) = (\text{inv})(e) = e = i(g)$$ ♠

or in other words $e^{-1} = e$.

**(d)**

# Part II
# Winter 2019

This is my first official quarter as Julia's student! My plan for now is to continue working on Waterhouse as well as learn about algebraic geometry (alongside my usual classes, of course).

## 2 Preparation, Waterhouse, and Görtz & Weddhorn

To give a sense of direction, Julia recommended that I take a look at the following regularity theorem:

**2.0.1 Theorem (Smoothness Theorem)**
Let $G$ be an algebraic affine group scheme over a field $k$. Then $k[G] \otimes \bar{k}$ is reduced if and only if $\dim G = \operatorname{rank} \Omega_{k[G]}$.

I have quite a bit of information to process before this, so I will get started!

**2.0.2 Definition (Closed Embedding)**
If $G$ and $H$ are affine group schemes represented by $A$ and $B$, respectively and if $\psi : H \to G$ is a homomorphism of affine group schemes (locally a group homomorphism) then if the corresponding algebra map $A \to B$ is surjective, then $\psi$ is called a **closed embedding.**

As its name suggests, this means that $\psi$ is an isomorphism onto a **closed subgroup** $H'$ of $G$. This is, in fact, a definition of this property. One can also think about it in the following ways: a group scheme $H$ is closed in $G$ if

- $H$ is defined by the relations imposed by $G$ plus some additional ones.
- $H = V(I)$ for some ideal $I \subset k[G]$.

Thinking back to our algebraic geometry, these are not too hard to see as equivalent. For instance, there is a closed embedding of $\mu_\mathbf{n}$ in $\mathbf{G_m}$ (simply adding in $x^n - 1$) and of $\mathbf{SL_n}$ in $\mathbf{GL_n}$ (adding $\det = 1$).

### 2.1 Hopf Ideals

One problem with the above characterization that one cannot choose $I$ arbitrarily and end up with a group scheme. This is equivalent to arbitrarily adding relations to a group, which is not always guaranteed to work out well (think adding $\det = 2$ to $\mathbf{GL_n}$).

Actually, we can exactly categorize the closed embeddings of subgroups in $G$ by considering certain ideals of the algebra $A$ which represents it.

**2.1.1 Definition**
Let $A$ be an algebra and $I \lhd A$. Then if

- $\Delta(I)$ goes to zero under the map $A \otimes A \to A/I \otimes A/I$,
- $S(I) \subseteq I$
- $\varepsilon(I) = 0$

then $I$ is called a **Hopf Ideal of $A$.**

2.1.2 Remark: That these ideals exactly characterize closed subgroups follows since any group (scheme) represented by $A'$ has the property that $\Delta'(A') \subseteq A' \otimes A'$. Since the new comultiplication map $\Delta'$ is derived from the old one $\Delta$, we have the diagram

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d] & A \otimes A \arrow[d] \\ A/I \arrow[r, "\Delta'"] & A/I \otimes A/I \end{tikzcd}$$

from which we see that $\Delta(I)$ must go to zero in the map on the right. Similarly, we want that $S$ and $\varepsilon$ satisfy similar diagrams.

Note that an example of such an ideal is $I = \ker(\varepsilon)$.

**2.1.3 Definition**
A **character** is a scheme morphism $G \to \mathbf{G_m}$.

Let $\Phi$ be a character of $G$ and notice that the corresponding hopf algebra map $\varphi : k[X, 1/X] \to A$ is defined by $\varphi(X) = b$, which is automatically invertible in $A$. Furthermore since for any $a, b \in G(R)$ we have

$$\Phi \circ m(a, b) = \Phi(ab) = \Phi(a)\Phi(b) = m \circ (\Phi, \Phi)(a, b)$$

we get the diagram

$$\begin{tikzcd} G \times G \arrow[r, "\Phi\times\Phi"] \arrow[d, "m"] & \mathbf{G_m} \times \mathbf{G_m} \arrow[d, "m"] \\ G \arrow[r, "\Phi"] & \mathbf{G_m} \end{tikzcd}$$

and then by dualizing we get that

$$\begin{tikzcd} A \otimes A & k[X, X^{-1}] \otimes k[X, X^{-1}] \arrow[l, "\varphi\otimes\varphi"'] \\ A \arrow[u, "\Delta"] & k[X, X^{-1}] \arrow[l, "\varphi"] \arrow[u, "\Delta"'] \end{tikzcd}$$

where in both diagrams we are abusing notation by using the same symbols for (co)multiplication in the different schemes. So we conclude that

$$\Delta(b) = \Delta \circ \varphi(X) = (\varphi \otimes \varphi) \circ \Delta(X) = (\varphi \otimes \varphi)(X \otimes X) = b \otimes b.$$

Then we can easily compute that $\varepsilon(b) = 1$ and $S(b) = b^{-1}$. This is precisely:

**2.1.4 Definition**
Let $A$ be a Hopf algebra and $a \in A$ such that

- $a$ is invertible
- $\Delta(a) = a \otimes a$
- $\varepsilon(a) = 1$
- $S(a) = a^{-1}$

then $a$ is called a **group-like** element of $A$.

2.1.5 Remark: As we saw above, every character of an affine group scheme $G$ corresponds to a group-like element of its representing algebra.

2.1.6 Remark: Using a parallel construction with any morphism $\Phi : G \to \mathbf{G_a}$, we get an element $b \in A$ such that $\Delta(b) = b \otimes 1 + 1 \otimes b$, $\varepsilon(b) = 0$ and $S(b) = -b$. These elements are called **primitive**

# 3 Problems

## 3.1 Week 1

## 3.2 Waterhouse