Local Duality Theorems

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Introduction

These notes are my attempt to understand the current state of the art in "local duality" theorems, especially of the kind investigated by Benson, Iyengar, Krause, and Pevtsova (herein abbreviated BKIP) in [Ben+18] and [Ben+19].

Furthermore I will continue my investigation into quantum groups and other Hopf algebras and see if some of the methods developed earlier can be applied to this new area.

1 Current State of the Art

1.1 Ideas to flush out

From meeting with Julia, the idea that I have gathered is that these two papers constitute two (slightly) different approaches to showing that the notion of Serre duality (which by itself is a "global" phenomenon) restricts ($\mathfrak{p}$)-locally (more on this later) to an analogous result.

The original result in [Ben+18] proves that such a duality exists for finite group schemes, but the objects of study here are much simpler (for instance the algebras that arise are Frobenius). Furthermore (this part may be sketchy), the results that are proven rely on the (relatively simply) monoidal structure of G-modules (where G is a finite group scheme).

The newer result in [Ben+19] reconstructs this result in the context of Gorenstein rings, which are decidedly less degenerate than the case of finite group schemes. What is important here is that this is done although the structure theory of these algebras is rather poorly understood, so what this represents is the formation of the idea that this duality somehow can be understood at a higher level, meaning that it may apply to broader classes of algebras.

1.2 Background results and definitions

I have seen a good deal of these results already, but it will be useful to put everything into one place so I can reference them as needed.

First, Serre duality, which comes from [Har77]:

1.2.1 Theorem
Let XS be a projective Cohen-Macaulay scheme of equidimension n over k. Then for any locally free sheaf $\mathscr{F}$ on X, there are natural isomorphisms

$$H^i(X, \mathscr{F}) \cong H^{n-i}(X, \mathscr{F}^\vee \otimes \omega_X^\circ)'$$

where $(-)'$ indicates taking a vector space dual, $(-)^\vee$ indicates taking the dual of a locally free sheaf: $\mathscr{H}om(-, \mathcal{O}_X)$ (sheaf hom) and ω_X° denotes a dualizing sheaf on X.

1.2.2 Definition: Let X be a proper scheme of dimension n. Recall that a dualizing sheaf is a (coherent) sheaf ω_X° such that there is an isomorphism

$$\mathrm{Hom}(\mathscr{F}, \omega_X^\circ) \cong H^n(X, \mathscr{F})'.$$

1.2.3 REMARK: In fact, there is something more: that this isomorphism is induced from a natural pairing of Hom and H^n. I won't worry about this too much for now, but it can be found in [Har77, p. 240]

1.3 BKIP Paper 1—Finite Group Schemes

Let's start off with the primary result. We will need some definitions and other results to get us there, but it will give us a sense of where we're headed.

1.3.1 Theorem (BKIP) '18)

References

[Ben+18] Dave Benson et al. "Local duality for representations of finite group schemes". In: *Compositio Mathematica* (Nov. 2018).

[Ben+19] Dave Benson et al. "Local duality for the singularity category of a finite dimensional Gorenstein algebra". In: *arXiv preprint arXiv:1905.01506* (2019).

[Har77] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1977. ISBN: 9780387902449.