Algebraic Groups

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Abstract

The topic of algebraic groups is a rich subject combining both group-theoretic and algebro-geometric-theoretic techniques. Examples include the general linear group GL_n, the special orthogonal group SO_n or the symplectic group Sp_n. Algebraic groups play an important role in algebraic geometry, representation theory and number theory.

In this course, we will take the functorial approach to the study of linear algebraic groups (more generally, affine group schemes) equivalent to the study of Hopf algebras. The classical view of an algebraic group as a variety will come up as a special case of a smooth algebraic group scheme. Our algebraic approach will be independent (even complementary) to the analytic approach taken in the course on Lie groups.

1 September 25, 2019

1.1 Group objects

Let $\mathscr{C}$ be a category with a final object and finite products.

1.1.1 Definition: A **group object G in $\mathscr{C}$** is an object in $\mathscr{C}$ along with multiplication, identity, and inverse morphisms satisfying the usual axioms.

One thing is that we are using that there is a final object $*$ along with our identity morphism $e : * \to G$. Here Jarrod explictly used the fact that there is a unique map to $*$.

Example 1.1
If $\mathscr{C}$ is Set, then G is a group. If $\mathscr{C} = \text{Top}$, then G is a topological group, smooth manifolds give Lie groups, and finally (interesting to us):

1.1.2 Definition: Let S be a scheme and let $\mathscr{C}$ be the category of schemes over S. Then a group object G in $\mathscr{C}$ is a **group scheme over S.**

WHen k is a field and $\mathscr{C}$ is schemes of finite type over k, we get a group scheme of finite type over k. There is not a great consensus on what makes an **algebraic group**, but this is what we will use.

When we instead restrict to *affine schemes* we get an affine groupe scheme of finite tipe over k, or a **linear algebraic group.**

1.2 Examples

$\mathbb{G}_m = \operatorname{Spec} k[t]_t$ is one.

If we consider the map $f : \mathbb{G}_m \to \mathbb{G}_m$ which on the level of elements sends $t \mapsto t^p$, the kernel is

$$\mu_p = \ker(f) = \operatorname{Spec} k[t]/(t^p - 1)$$

and that's great, but when $\operatorname{char} k = p$, this causes the group scheme to be **unreduced**. This is (apparently) a case when you need to use schemes.

1.3 The Functorial Approach

Let $\mathscr{C}$ be a category with object X. Define the functor $h_X : \mathscr{C}^{op} \to \mathbf{Set}$ where

$$h_X(Y) = \operatorname{Hom}_{\mathscr{C}}(Y, X).$$

Then we have

1.3.1 Lemma (Yoneda)
Let $G : \mathscr{C}^{op} \to \mathbf{Set}$ be a functor. There is a natural bijection

$$G(X) \simeq \operatorname{Nat}(h_X, G).$$

1.3.2 Proposition
A group object G in $\mathscr{C}$ is the same as an aobject $X \in \mathscr{C}$ together with a choice of factorization of $h_X : \mathscr{C} \to \mathbf{Set}$ through $\mathbf{Grp}$.

1.4 Exercises

(a) Spell out all the details of the proof of the above propositon.

(b) Given a group object G, define in two ways what it means for it to act on another object. (In coordinates and functorially).

1.5 Some Interesting Facts

If we had to write down five results that we'd like to get out of this class:

1.5.1 Proposition
Every affine group scheme of finite type over a field embeds into GL_n as a closed subgroup.

1.5.2 Theorem (Chevalley's Theorem)
Let G be a finite type group scheme over a field. Then it factors as

$$1 \to H \to G \to A \to 1$$

where A is abelian and H is affine (linear algebraic).

1.5.3 Proposition
If G is an affine group scheme of finite type over k, then we have af actorization

$$1 \to U \to G \to R \to 1$$

where U is unipotent and R is reductive.

1.5.4 Proposition
$H \subseteq G$ a subgroup scheme. Then G/H is a projective scheme.

Finally we want to talk about Tanakka duality and how the representations of G define G itself.

2 September 27th, 2019

Last time we defined a group scheme (a group object in the category of schemes over a base scheme). We also mentioned that You could define it as a map $h_G : \mathbf{Sch}/S \to \mathbf{Set}$ along with a factorization through **Grp**.

We defined an **algebraic group** over k as a group scheme over $\operatorname{Spec} k$ of finite type and a **linear algebraic group** to be an *affine* group scheme over k of finite type.

2.1 Hopf Algebras

Let $G = \operatorname{Spec} A$ be a linear algebraic group over k. I have seen most of these before (see Waterhouse or my Hopf algebra notes)

2.1.1 Remark: One think I haven't seen explicitly before: Notice that the augmentation ideal $\ker \varepsilon$, where ε is the counit, is the (maximal!) ideal corresponding in the algebro-geometric sense to the identity element in G.

2.1.2 Definition: A **Hopf algebra** is ...

2.1.3 Definition: Let G be an algebraic group over k. Then if h_G factors through **Ab**, G is called **commutative.**

2.2 Some Examples

2.2.1 REMARK: Note that to define a functor from schemes over k, is suffices to define it on affine schemes, thereby defining the (Zariski) local behavior of any such map. Thus we really only need to consider maps in **Alg**.

- $\mathbb{G}_a$. Here we can define it as a functor that sends $S \mapsto \Gamma(S, \mathscr{O}_S)$. Geometrically, $\mathbb{G}_a = \mathbb{A}^1$ where the multiplication is addition, inverses send $x \mapsto -x$ and the unit is the zero map. The Hopf algebraic picture is the usual dual thing.
- $\mathbb{G}_m$ as a scheme isthe map $S \mapsto \Gamma(S, \mathscr{O}_S)^*$. In the geometric picture, $\mathbb{A}^1 \setminus \{0\}$ and the algebra structure comes from multiplciation. Hopf is pretty easy.
- GL_n is a scheme that sends

$$S \mapsto \left\{ A = (a_{ij}) : a_{ij} \in \Gamma(S, \mathscr{O}_S), \det(A) \in \Gamma(S, \mathscr{O}_S)^* \right\}$$

  the algebra is $\mathbb{A}^{n\times n} \setminus \{\det = 0\}$ with the usual multiplication. The coalgebra structure can be seen in the book.

This one requires some more explaination so I am setting it apart.

Example 2.1
Let V be a finite dimensionatl vector space over k. Then we can define the algebraic group V_a which sends

$$S \mapsto \Gamma(S, \mathscr{O}_S) \otimes_k V.$$

Geometrically we are looking at $\mathbb{A}(V) = \operatorname{Spec} \operatorname{Sym}^* V^\vee \simeq \operatorname{Spec} k[x_1, \ldots, x_n]$ where $n = \dim V$.

What about finite groups? As a scheme, we want $G = \sqcup_{g \in G} \operatorname{Spec} k$. The functor sends $S \mapsto \operatorname{Mor}_{\mathrm{Set}}(\pi_0(S), G)$, or maps from the connected components into G.

Example 2.2
Now consider the n^{th} roots of unity: as a scheme, $\mu_n = \operatorname{Spec} k[t]/(t^n - 1) \subseteq \mathbb{G}_m$. If both $k = \bar{k}$ and $\operatorname{char} k \nmid n$, then $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$.

But if (e.g.) $k = \mathbb{Q}$, then μ_3 is $\mathbb{Q}[t]/(t^3 - 1) = \operatorname{Spec}\mathbb{Q} \sqcup \operatorname{Spec}\mathbb{Q}(\xi)$ where ξ is a primitive third root of unity.

If, on the other hand, $k = \bar{\mathbb{F}}_3$ and consider μ_3, we get a single point with residye field $\bar{\mathbb{F}}_3$.

Example 2.3
If we are in the case of positive characteristic, then we get an algebraic group α_p. Here the scheme is $\operatorname{Spec} k[x]/x^p$ and functorially it is the map $S \mapsto \{F \in \Gamma(S, \mathscr{O}_S) | f^p = 0\}$.

2.3 Matrix Groups

We already defined GL_n, but we can also define

$$\mathrm{SL}_n : S \mapsto \{A = (a_{ij}) | \det A = 1\}$$

with scheme $\operatorname{Spec} k[x_{ij}]/(\det - 1)$.

We also have the (upper) triangular matrices T_n and unitary group U_n and diagonal group D_n

2.3.1 Definition: Let G be a linear algebraic group. Then

- G is a **vector group** if $G \cong V_a$ for some finite dimensional V.
- G is a **split torus** if $G \cong \mathbb{G}_m^n$.
- G is a **torus** if there is a field extention $k \to k'$ such that

$$G \times_{\operatorname{Spec} k} \operatorname{Spec} k' \cong \mathbb{G}_{m,k'}^n$$

3 September 30th, 2019

Another example to consider:

Example 3.1
Let $G = \mathrm{PGL}_n$, the projective linear group. Recall we want to define this as GL_n / k^* (from group theory). To do this for algebraic groups, we define

$$\mathrm{PGL}_n = \mathrm{Proj}\, k[x_{ij}]_{det} := \mathrm{Spec}(k[x_{ij}]_{det})_0$$

The geometric picture is difficult since we haven't yet defined quotients, but as a functor we say PGL_n is $\mathrm{Aut}(\mathbb{P}^n)$, the functor that sends $S \mapsto \mathrm{Aut}(\mathbb{P}^n_S)$ where $\mathbb{P}^n_S = \mathbb{P}^n_k \times_{\mathrm{Spec}\, k} S$.

3.1 Non-affine group schemes

Example 3.2
Let $\lambda \neq 0, 1$ be an element in k. Then we can define the elliptic curve

$$E_\lambda = V(y^2 z - x(x-z)(x-\lambda z)) \subset \mathbb{P}^2$$

Which gives us a double cover over $(0,1)$ and (λ, ∞) with singleton fiber (ramified) over $0, 1$, and λ.

Then for any $\lambda \neq 0, 1$, E_λ is a **projective** group scheme.

3.1.1 Remark: If you look at the $\mathbb{C}$-points, you get $E_\lambda(\mathbb{C}) = \Lambda_\lambda$, giving you a torus. Recall (from e.g. complex analysis) that the moduli here is $\mathrm{SL}_2(\mathbb{Z})$ of all elliptic curves.

3.2 Abelian Varieties

3.2.1 Definition: An **abelian variety over** k is asmooth, geometrically connected ($A \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \bar{k}$ is connected), proper group scheme A over k.

Example 3.3
Over $\mathbb{C}$, $\mathbb{C}^g / \Lambda$ where $\Lambda \cong \mathbb{Z}^{2g} \subseteq \mathbb{C}^g$ gives us a genus g example.

3.2.2 Theorem
Any abelian variety over k is commutative and projective.

3.2.3 Theorem (Chevalley)
If G is any group scheme, then the sequence

$$1 \to H \to G \to A \to 1$$

is exact, where H is a linear algebraic group (affine!) and A is an abelian variety.

Example 3.4
Let $X \to \operatorname{Spec} k$ be a geometrically integral projective scheme (proper may suffice). The idea here is that over $\mathbb{C}$ the rings over every open set are integral domains.

Now consider the **Picard functor** $\operatorname{Pic}_X : \operatorname{Pic} : \mathbf{Sch/k} \to \mathbf{Grp}$ sending

$$S \mapsto \operatorname{Pic}(X_S = X \times_k S)/p^k \operatorname{Pic(S)}$$

3.2.4 Theorem
Pic_X is represented by a scheme locally of finite type, thus Pic^0_X, the connected component of the identity in $[\mathscr{O}_X] \in \operatorname{Pic}_X$ is an abelian variety.

3.3 Relative Group Schemes

Example 3.5
Consider $\mathbb{G}_{m,\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[t]_t$. Then $G_{m,S} = \mathbb{G}_{m,\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} S$. In the case that $S = \operatorname{Spec} R$, $\mathbb{G}_{m,S} = \operatorname{Spec} R[t]_t$.

Example 3.6
Let $\mathbb{A}^1 = \operatorname{Spec} k[x]$ and define $G = \operatorname{Spec} k[x,y]_{xy+1} \subseteq \mathbb{A}^2$. Notice this is the plane minus a hyperbola.

Define $\cdot : G \times_{\mathbb{A}^1} G \to G$ to be given by

$$(x,y)\cdot(x,y') = (x, xyy' + y + y')$$

Then the thing here is the fiber (think vertical line in the plane!) over 0 is $\mathbb{G}_a$ and is isomorphic to $\mathbb{G}_m$ otherwise.

Example 3.7
Let $\mathscr{E}_\lambda = V(y^2z - x(x-z)(x-\lambda z))$ over $\operatorname{Spec} k[\lambda]$. Then when $\lambda = 0$, we get the nodal cubic given by $y^2z - x^2(x-z)$ (node at the origin).

Now if you look at the connected component around 0 of $\operatorname{Aut}(\mathscr{E}_\lambda)/\mathbb{A}_\lambda$, you actually find (when $\lambda = 0$) that $\mathbb{G}_m \cong \operatorname{Aut}(\mathscr{E}_0)^0$.

3.4 Some definitions

3.4.1 Definition: A **homomorphism** $\phi : G \to G$ of group schemes over S is a map $\phi : H \to G$ of schemes such that

$$\begin{array}{ccc} H \times_S H & \xrightarrow{m_H} & H \\ \downarrow{\scriptstyle \phi\times\phi} & & \downarrow{\scriptstyle \phi} \\ G \times_S G & \xrightarrow{m_G} & G \end{array}$$

Problem 3.1
Show that this automoatically imples that the identity and inversion maps are respected as well (automatically).

3.4.2 Definition: A **subgroup of** $G \to S$ is a subscheme $H \subseteq G$ such that $H(T) \leq G(T)$ for all T over S.

Problem 3.2
Show that $\ker(\phi) \subseteq H$ *is a subgroup.*

3.4.3 Remark: This gives you a nice way to construct new group schemes. For example, the following are exact:

$$1 \to \mathrm{SL}_n \to GL_n \xrightarrow{\det} \mathbb{G}_m \to 1$$

and

$$1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$$

3.4.4 Proposition
Let $G \to S$ be a group scheme. Then $G \to S$ is separated if andy only if $e : S \to G$ is a closed immersion.

PROOF
The idea here is that $S \to G$ is a closed immersion. Then we consider the map $m \circ (\mathrm{id}, S) : G \times_S G \to G$ and consider this along with the diagonal map $\Delta : G \to G \times_S G$ and this is a pullback square! ♠

3.4.5 Corollary
Any group scheme over k is separated.

The idea is going to be that if X is any scheme over k, then any point $X \in X(k)$ is closed.

4 October 2, 2019

Notice that a **relative group scheme** (referred to in last lecture) refers to a groups scheme over an arbitrary base scheme S.

4.1 Properties of schemes

Today we are going to be talking about reducedness, connectedness, irreduciblility, regularity, and smoothness.

Recall that a scheme X is **reduced** if and only if $\forall x \in X$, $\mathcal{O}_{X,x}$ is reduced. An example of a non-reduced scheme is $\operatorname{Spec} k[x]/(x^2)$.

4.1.1 Definition: We say a scheme X over k is **geometrically reduced** if for all field extensions k'/k,

$$X_{k'} = X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$$

is reduced.

4.1.2 REMARK: It is equivalent that $X_{\bar{k}}$ is reduced if and only if every k'/k is purely inseperable (I think).

4.1.3 REMARK: If k is perfect, then X is reduced if and only if X is geometrically reduced.

4.1.4 Definition: A local ring $(A, \mathfrak{m})$ is **regular** if $\dim_{\mathrm{Krull}} A = \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$

4.1.5 Definition: A scheme X is regular if for all $x \in X$, $\mathcal{O}_{X,x}$ is regular.

4.1.6 Remark: If $X \to \operatorname{Spec} k$ and $x \in X(k)$, the tangent space at x is

$$T_{X,x} = (\mathfrak{m}/\mathfrak{m}^2)^\vee = \{f : \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \to X | 0 \mapsto x\}$$

4.1.7 Remark: Notice that if $X \to \operatorname{Spec} k$ is regular and k'/k is a field extension, then $X_{k'}$ is not necessarily regular.

4.1.8 Definition: A Scheme $X \to \operatorname{Spec} k$ of finite type is **smooth** if $X_{\bar{k}}$ is regular.

4.2 Facts about algebraic groups

Then we can return to the proposition we want to prove:

4.2.1 Proposition
Let $G \to \operatorname{Spec} k$ be an algebraic group. Then G is geometrically reduced if and only if G is smooth over $\operatorname{Spec} k$.

Proof
Smoothness over k implies reducedness. Now since we are only interested in the algebraic closure of k, we can say $k = \bar{k}$. Because G is reduced, there exists a nonempty open $U \subseteq G$ that is smooth. Then since $G(k) \subseteq |G|$ is dense in G (as a topological space) and Then $G = \cup_{g \in G(k)} m_g(U)$ for our smooth U, and this gives us a smooth cover of G. ♠

We will see next itme that all linear algebraic groups over k where $\operatorname{char} k = 0$ are all geometrically reduced (and thus smooth).

4.3 Connectedness

Let G be an algebraic group over k. Then we have our maps $e : \operatorname{Spec} k \to G$, so consider it as $e \in G(k)$. Let $G^0 \subseteq G$ be the connected component of e. It is both open and closed.

4.3.1 Remark: If $X \to \operatorname{Spec} k$ is of finite type and $x \in X(k)$, then X being connected implies that X is geometrically connected.

This establishes that G^0 is actually geometrically connected! We actually will see

4.3.2 Proposition
$G^0 \subseteq G$ is an (open and closed) algebraic subgroup.

The idea here is that $G^0 \times G^0$ is connected, so the image of the multipication map on this set lands in a connected component (since it is connected). Since $e \in G^0$, and $m(e,e) = e \in G^0$, this shows that the multiplication map restricts to a well-defined map $G^0 \times G^0 \to G^0$. A similar argument goes through for the inverset map, etc.

The upshot here is that if G is an algebraic group, then there exists a factorization

$$1 \to G^0 \to G \to \pi_0(G) \to 1$$

where $\pi_0(G)$ is given the structure of a discrete group.

4.3.3 Remark: Now we also have that $(G^0)_{k'} = (G_{k'})^0$ for all k'/k. The idea is to get a map of of one into the other and then use clopenness and connectedness to show they are equal.

4.3.4 Proposition
A connected algebraic group over k is irreducible.

Proof
We can assume $k = \bar{k}$. Suppose $G = X \cup Y$, where both are closed, X is irreducible, and $X \cap Y \neq \varnothing$. Thus there exists an element $g \in X \setminus Y$. That is, g lies in a single irreducible component.

But then using the multiplication by h map on G, we get to every other point in G, so every point is in a single irreducible component. But the intersection was nontrivial! Or something. ♠

4.3.5 Proposition
If G_{red} is geometrically connected, then $G_{\text{red}} \subseteq G$ is a subgroup. In particular, if k is perfect, then G_{red} is a subgroup of G.

4.3.6 Remark: X is geometrically reduced implies that $X \times X$ is geometrically reduced.

5 October 4, 2019

Some review. Let G be an algebraic group and denote $e : \operatorname{Spec} k \to G$ be the identity. We saw a lot of propositions last time.

Now let k be a nonperfect field and take $t \in k \setminus k^p$. Then define

$$G \overset{\text{def}}{=} V(x^{p^2} - t x^p) \subseteq \mathbb{G}_a$$

which Milne claims is not reduced. We can see why it's not geometrically reduced, but we're missing the details here.

5.1 Another special case

5.1.1 Theorem
When $k = \bar{k}$, G is smooth if and only if

$$\dim T_e G_{\text{red.}} = \dim T_e G.$$

5.1.2 Remark: When G is smooth, it is reduced, so the equality is clear. For the other direction, we get that k is pefect, so $G_{\text{red.}}$ which is geometrically reduced if and only if G is smooth. But

$$\dim G \leq \dim T_e G = \dim T_e G_{\text{red.}} = \dim G_{\text{red.}} = \dim G$$

5.1.3 Theorem

If G is a linear algebraic group over k and $\operatorname{char} k = 0$, G is smooth,

Proof

We can assume $k = \bar{k}$. Then set $G = \operatorname{Spec} A$ where A is a Hopf algebra. THen we get Hopf algebra maps m^* and e^*. Notice that the augmentation ideal $\mathfrak{m} = \ker(e^*)$ is a maximal ideal.

Then we want to prove

(a) $A \cong \mathfrak{m} \oplus k$ as a k-vector space (obvious).

(b) $\forall a \in \mathfrak{m}$, $m^*(a) - a \otimes 1 - 1 \otimes a \in \mathfrak{m} \otimes \mathfrak{m}$.

To see the second, notice that $m^*(a) - a \otimes 1 - 1 \otimes a$ is in the kernel of

$$e^* \otimes \mathrm{id} : A \otimes A \to k \otimes A.$$

This is clear from the commutative diagram

$$\begin{tikzcd} k \otimes A & A \otimes A \arrow[l, "e^*\otimes\mathrm{id}"'] \\ & A \arrow[u, "m^*"'] \arrow[ul, "\sim"] \end{tikzcd}$$

Then we conclude $f \in \ker(e^* \otimes \mathrm{id}) \cap \ker(\mathrm{id} \otimes e^*) = A \otimes \mathfrak{m} \cap \mathfrak{m} \otimes A$ by a symmetric argument. Finally we notice that $A \otimes \mathfrak{m} \cap \mathfrak{m} \otimes A = \mathfrak{m} \otimes \mathfrak{m}$, and so f lies in this ideal.

Now we want to show that $\dim T_e G = \dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim_k \mathfrak{m}/(\sqrt{0} + \mathfrak{m}^2) = \dim T_e G_{\text{red.}}$. It suffices to show that for all $a \in \sqrt{0}$, $a \in \mathfrak{m}^2$. Suppose the opposite–so let $a \in \sqrt{0} \setminus \mathfrak{m}^2$. Consider the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_{\mathfrak{m}} \arrow[d] \\ A/\mathfrak{m}^2 \arrow[r, "\sim"] & A_{\mathfrak{m}}/(\mathfrak{m}A_{\mathfrak{m}})^2 \end{tikzcd}$$

Now the image of a in $A_{\mathfrak{m}}$ is nonzero, so there exists $n > 0$ such that $a^n \in A_{\mathfrak{m}}$ but $a^{n-1} \neq 0$ in $A_{\mathfrak{m}}$. Thus there exists $f \notin \mathfrak{m}$ such that $a^n f = 0 \in A$. Subsititute af for a, and thus there is an $a \in \sqrt{0}$ such that $a^n = 0$ in A but $a^{n-1} \neq 0$ in $A_{\mathfrak{m}}$.

Then by fact 2,

$$m^*(a) = 1 \otimes a + a \otimes 1 + r, \quad r \in \mathfrak{m} \otimes \mathfrak{m}$$

and since m^* is a ring homomorphism,

$$0 = m^*(a^n) = (m^*(a))^n = (a \otimes 1 + (1 \otimes a + r))^n = a^n \otimes 1 + n(a^{n-1} \otimes a + (a^{n-1} \otimes 1)r) + X$$

where $X \in A \otimes \mathfrak{m}^2$. But since $a^n = 0$, we get

$$n(a^{n-1} \otimes a + (a^{n-1} \otimes 1)r) \in A \otimes \mathfrak{m}^2$$

Now since $(a^{n-1} \otimes 1)r \in (a^{n-1})\mathfrak{m} \otimes A$, so

$$n(a^{n-1} \otimes a) \in (a^{n-1})\mathfrak{m} \otimes A + A \otimes \mathfrak{m}^2$$

and since $\operatorname{char} k = 0$, we get that n is a unit, so

$$a^{n-1} \otimes a \in (a^{n-1})\mathfrak{m} \otimes A + A \otimes \mathfrak{m}^2$$

Now since this lives in $A \otimes A$, consider the image of the quotient map $A \otimes A \to A \otimes A/\mathfrak{m}^2$. Then

$$a^{n-1} \otimes \bar{a} \in (a^{n-1})\mathfrak{m} \otimes A/\mathfrak{m}^2 \subseteq A \otimes A/\mathfrak{m}^2$$

And note that $a^{n-1} \notin a^{n-1}\mathfrak{m}$ because otherwise $a^{n-1} = a^{n-1}q$ for $q \in \mathfrak{m}$. Then $a^{n-1}(1-q) = 0 \in A_{\mathfrak{m}}$, which implies that $a^{n-1} = 0 \in A_{\mathfrak{m}}$ (since $1-q$ is a unit here).

Then somehow we get that $\bar{a} = 0 \in A/\mathfrak{m}^2$, so $a \in \mathfrak{m}^2$. ♠

6 October 7, 2019

Today we will be primarly concerned with

6.1 Group actions

Let G be an algebraic group over k.

6.1.1 Definition: A **group action** of G on a scheme X over k is the data of a morphism

$$\mu : G \times X \to X$$

such that the usual axioms hold. That is,

$$\begin{tikzcd} G \times G \times X \arrow[r, "m \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \mu"] & G \times X \arrow[d, "\mu"] \\ G \times X \arrow[r, "\mu"] & X \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec} k \times x \arrow[r, "e \times \mathrm{id}"] \arrow[dr, "\sim"] & G \times X \arrow[d, "\mu"] \\ & X \end{tikzcd}$$

6.1.2 Remark: Apparently it was an exercise already to show that this is equivalent to an action of h_G on h_X.

6.1.3 Remark: The map $(g,x)\mapsto(g,gx)$ is an automorphism of $G\times X$, so if $p_2:G\times X\to X$ is projection,

$$\begin{tikzcd} G\times X \arrow[rr, "\sim"] \arrow[dr, "\mu"'] & & G\times X \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

commutes.

6.1.4 Definition: Let X and Y be schemes over k with a G action. Then a G-**equivariant morphism** $f:X\to Y$ is one such that for all $g\in G$, the following commutes:

$$\begin{tikzcd} G\times X \arrow[r, "\mathrm{id}\times f"] \arrow[d, "\mu_X"] & G\times Y \arrow[d, "\mu_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

6.1.1 Some examples

- G actions on itself by multiplication or conjugation.
- $\mathbb{G}_m$ acts on $\mathbb{A}^1$. Geometrically, we are just looking at k^* acting on k by scaling. Algebraically, we want a map $\mu\mathbb{G}_m\times\mathbb{A}^1\to\mathbb{A}^1$ given by the map of algebras:

  $$k[x]\xrightarrow{\mu^*}k[t]_t\otimes k[x]\quad\text{via}\quad x\mapsto tx$$

  Functorially, if S is a scheme over k, then $\mathbb{G}_m(S)=\Gamma(S,\mathscr{O}_S)^*$ which acts on $\mathbb{A}^1(S)=\Gamma(S,\mathscr{O}_S)$, again by scaling.
- You can consider GL_n action on $\mathbb{A}^n$ by multiplicaiton or on $\mathbb{A}^{n\times n}$ via multiplication or conjugation.

6.1.2 Orbits and Stabilizers

Let G be an algebraic group over k action on a scheme X over k. Let $x\in X(k)$. Then we have a map

$$\mu_x:G\times\operatorname{Spec}k\xrightarrow{\mathrm{id}\times x}G\times X\xrightarrow{\mu}X$$

where

$$g \mapsto (g,x) \mapsto gx.$$

6.1.5 Definition: The **orbit of** x is $Gx = \mu_x(G) \subseteq |X|$ set-theoretically. The **stabilizer of** x **in** G is $G_x = \mu_x^{-1}(x) \subseteq G$.

6.1.6 Remark: G_x is always a closed algebraic subgroup of G.

6.1.7 Proposition
$\mu_x(G)$ is open in its closure in $|X|$.

Recall first the following:

6.1.8 Theorem (Chevalley's Theorem (different?))
If $f : X \to Y$ is a map of schemes of finite type over k, then $f(X) \subseteq Y$ is contstructible (i.e. is a disjoint union of finitely many locally closed subsets).
Recall that locally closed means closed in an open subspace.

6.1.9 Corollary
Maybe a definition: The orbit $Gx \subseteq \mathrm{im}(\mu_x) \subseteq X$. If G is reduced, then Gx is reduced.

6.1.3 Applications

Say that $\operatorname{char} k = p$. Then μ_p acts on $\mathbb{G}_m$ by multiplication.

$\mathbb{G}_m$ acts on $\mathbb{A}^1$ with two orbits: $\mathbb{G}_m \cdot 1 = \{x \neq 0\}$ and $\mathbb{G}_m \cdot 0 = \{0\}$. The stabilizers are $G_1 = 1$ and $G_0 \cong \mathbb{G}_m$.

Consider G acting on $\mathbb{A}^2$ via $t(x,y) = (tx, t^{-1}y)$. Then the orbits are hyperbolas! There is also a notion of closed orbits that I didn't quite catch. Also apparently the orbit-stabilizer statement is easy to see in geometry via a fiber bundle G over Gx where the fiber over x is G_x.

6.1.10 Proposition
If $\phi : G \to H$ is a homomorphism of algebraic groups, then $\phi(G) \subseteq |H|$ is closed.

6.1.11 Remark: The proof included reducing first to $k = \bar{k}$. The trick here is to consider the group action induced by ϕ and then consider the map μ_{e_H} of this action. Then $\mu_{e_H}(G) = \phi(G)$ and one can prove that this is closed.

In particular, we have that **subgroups of an algebraic group are always closed.** Note that this stands in stark contrast to Lie theory where you get non-closed subgroups.

7 October 9, 2019

Recall that last time we were considering actions of algebraic groups on schemes of finite type over k. We discussed the orbit and stabilizer of an element $x \in X$ and showed that $G \cdot x$ is open in its closure. We also saw that G_x is a closed subgroup.

We also say that $\phi(H)$ (as a set) is always closed! None of these facts are true for Lie groups or relative group schemes (the base scheme is not $\operatorname{Spec} k$ for k a field).

7.1 Cartier Duality

Let $G \to \operatorname{Spec} k$ be a **finite** group scheme (so $G = \operatorname{Spec} A$ and A is a finite dimensional Hopf algebra). Some examples of finite group schemes are:

- G a finite group. Then $G = \sqcup_{g\in G} \operatorname{Spec} k = \operatorname{Spec}\left(\prod_{g\in G} k\right)$
- $\mu_n = \operatorname{Spec} k[t]/(t^n - 1)$
- $\operatorname{char} k = p$ and $\alpha_p = \operatorname{Spec} k[t]/t^p$

7.1.1 Remark: Recall all the maps and diagrams that A has as a Hopf algebra.

A question one may ask: what if we apply the idea of dualizing $(-)^\vee = \operatorname{Hom}_{\mathbf{Alg}}(-, k)$ to A? Do we get another Hopf algebra?

The short and sweet of it is yes! But notice that we are coming from the commutative world, so we expect A to be commutative. But in general, A is not cocommutative (in fact, it is if and only if G itself was commutative as a group).

Thus $A^\vee$ is indeed a (cocommutative) Hopf algebra, and when G is commutative, $A^\vee$ is as well. So

7.1.2 Proposition
If $G = \operatorname{Spec} A$ is a commutative group scheme, then the Cartier dual $G^D = \operatorname{Spec} A^\vee$ is a commutative group scheme as well.

7.1.3 Remark: The above observations gives us an anti-autoequivalence of the category of commutative affine group schemes. Furthermore $(G^D)^D = G$.

Example 7.1
Consider $\mu_n = \operatorname{Spec} A - \operatorname{Spec} \oplus_0^{n-1} k \cdot t^i$. So then if we let $\{e_i\}$ be the basis for $A^\vee$ dual to $\{t_i\}$. we can compute comultiplication

$$e_i \mapsto \sum_{j=0}^{n-1} e_j \otimes e_{i-j}$$

and multipication

$$e_i \otimes e_j \mapsto \delta_{ij} e_i$$

Then it can be shown that $G^D \cong \mathbb{Z}/n\mathbb{Z}$.

Now given G, an algebraic group over k, define

$$\underline{\mathrm{Hom}}(G, \mathbb{G}_m) : \mathbf{Sch}/k \to \mathbf{Set}$$

which takes

$$T \mapsto \mathrm{Hom}_{\mathbf{AlgGrp}}(G_T, \mathbb{G}_{mT}).$$

7.1.4 Theorem

If G is a commutative finite group scheme over k, then

$$G^D \cong \underline{\mathrm{Hom}}(G, \mathbb{G}_m)$$

Let $H = \operatorname{Spec} B \to \operatorname{Spec} R$ be a group scheme. Then

$$H_{\mathbf{GrpSch}/R}(H, \mathbb{G}_{mR}) \subseteq \mathrm{Mor}_{\mathbf{Sch}/R}(H, \mathbb{G}_{mR})$$

But the left hand side is equivalent to the grouplike elements of B and the right hand side is equivalent to $\mathrm{Hom}_{\mathbf{Alg}_R}(R[t]_t, B)$.

This leads to a proof of thm. 7.1.4:

Proof

Let $G = \operatorname{Spec} A$ and $G^D = \operatorname{Spec} A^\vee$. First look at the k-points:

$$\begin{aligned} G^D(k) &= \mathrm{Mor}_{\mathbf{Sch}/k}(\operatorname{Spec} k, G^D) \\ &= \mathrm{Hom}_{\mathbf{Alg}_k}(A^\vee, k) = \{f \in A | m^*(f) = f \otimes f\} \hookrightarrow \mathrm{Hom}_k A^\vee, k) \\ &= \mathrm{Hom}(G, \mathbb{G}_m) \\ &= \underline{\mathrm{Hom}}(G, \mathbb{G}_m)(k) \end{aligned}$$

If we then look at R points for a general R, most things just change over, but we see

$$\{f \in A \otimes R | m_R^*(f) = R \otimes R\} = \mathrm{Hom}_{\mathbf{Alg}_k}(A^\vee, R) = \mathrm{Hom}(G_R, \mathbb{G}_m)$$

and the rest follows. ♠

A question one may ask: what is $\mathrm{Hom}_{\mathbf{AlgGrp}}(\mathbb{G}_m, \mathbb{G}_m)$? It ends up it is $\mathbb{Z}$. You can send $t \mapsto t^n$ for all $n \in \mathbb{Z}$. But then $\underline{\mathrm{Hom}}(\mathbb{G}_m, \mathbb{G}_m)$ is $\mathbb{Z}$ as a group scheme over k, which is not quasicompact. There was more but I am le tired.

8 October 11, 2019

Today we're doing problems and stuff. Forgot about that.

8.1 Casey's Presentation

Let G and H be objects in a category $\mathscr{C}$ with finite products. Let $h : H \to G$ be a group homomorphism. That is,

Then we get similar diagrams for the identity and inverse maps (they are respected by h). Then there is a bunch of diagram work. It's too hard to do a diagram without knowing the shape ahead of time.

Oh hey I used Adam's site!

$$\begin{tikzcd}
* \arrow[dr, "{(e,e,e)}"] & & & \\
& H \times H \times H \arrow[r, "(i_g \circ h)\times h \times h"] \arrow[d, "1\times m_H"'] & G \times G \times G \arrow[d, "1\times m_g"] & \\
& H \times H \arrow[r, "(i_G \circ h)\times h"] & G \times G \arrow[dr, "m_G"] & \\
& & & G
\end{tikzcd}$$

$$\begin{tikzcd}
* \arrow[r, "e"] \arrow[dr, "{(e,e,e)}"] \arrow[ddddr, bend right, "he"'] & H \arrow[dr, "{(h,h)}"] & & \\
& H \times H \times H \arrow[d, "(i_g \circ h)\times h \times h"'] & G \times G \arrow[dl, "{(*,1)\times 1}"] \arrow[dr, "*\times 1"] & \\
& G \times G \times G \arrow[d, "1\times m"'] \arrow[dr, "m\times 1"] & & * \times G \arrow[dl, "e\times 1"'] \arrow[ddll, bend left, "\pi_1"] \\
& G \times G \arrow[d, "m"'] & G \times G \arrow[dl, "m"] & \\
& G & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & * \arrow[dl, "{(e,e,e)}"'] \arrow[d, "{(e,e,1)}"'] \arrow[dd, bend left, "{(e,e)}"] \arrow[dr, dashed, "he"] \arrow[r, bend left, "*"] & * \arrow[r, "e"] & G \\
H\times H\times H \arrow[dr, "1\times m"'] & H\times H\times * \arrow[l, "1\times 1\times e"'] \arrow[d, "1\times\pi_2"'] & G \arrow[u, "*"'] \arrow[d, "{(1,1)}"'] \arrow[dr, bend left, "{(i,1)}"] & \\
 & H\times H \arrow[r, "h\times h"] \arrow[rr, bend right, "(ih)\times h"'] & G\times G \arrow[r, "i\times 1"] & G\times G \arrow[uu, "m"']
\end{tikzcd}$$

8.1.1 Remark: The idea above is we want to show the first diagram commutes. That is captured in the paths of the second diagram which commutes by the axioms of a group object. The third diagram shows a similar commutativity for the unit e.

9 October 14th, 2019

Let G be a finite group. Recall the definiton of a **representation** (a linear action of G on a vector space V/k). This is the same data as a group homomorphism to $\mathrm{GL}(V)$.

9.1 Representations of Algebraic Groups

Now what if G is an algebraic group over k? Now we have some exra structure of G as a variety.

9.1.1 Definition: A (finite dimensional) **representation** of an algebraic group G/k is a homomorphism $\rho : G \to \mathrm{GL}(V)$ of algebraic groups.

9.1.2 Remark: Notice that when V is infinite-dimensional, $\mathrm{GL}(V)$ is no longer of finite type, so we have to let ρ be a morphism of group schemes.

We have the standard representation of GL_n acting on k^n in the natural way. We also have the regular representation G action on $\Gamma(G, \mathscr{O}_G)$. When $G = \mathbb{G}_m$, we get over $\mathbb{C}$ an action of $\mathbb{G}_m$ on $\mathrm{GL}(V)$ in the usual way (scaling by $\mathbb{C}^*$).

Observe that

$$\rho : G \to \mathrm{GL}(V) = \mathrm{Spec}(\mathrm{Sym}^*(V \otimes V^\vee))_{\det}$$

corresponds to a ring morphism

$$\mathrm{Sym}^*(V \otimes V^\vee)_{\det} \to \Gamma(G, \mathscr{O}_G) \stackrel{\text{def}}{=} \Gamma(G)$$

which corresponds to a map

$$V \otimes V^\vee \to \Gamma(G)$$

and then tensoring with V, this gives us a map

$$V \xrightarrow{\sigma} \Gamma(G) \otimes V$$

So any group action gives us a **coaction** of $\Gamma(G)$ on V.

9.1.3 Definition: A representation of G is a k-vector space V along with a coaction

$$\sigma : V \to \Gamma(G) \otimes V$$

satisfying the usual dual diagrams to actions.

9.1.4 Remark: As a matter of notation, recall that if $G = \operatorname{Spec} A$, then A is a Hopf algebra. So we call V an A-comodule.

9.2 Reps of diagonalizable group schemes

Let k be a field (or even a ring!) and let A be a finitely-generated abelian group. Define $D(A)$ to be

$$D(A) = \oplus_{a \in A} k \cdot t^a \stackrel{\text{def}}{=} \operatorname{Spec} R$$

Then we get a multiplication

$$R \otimes R \to R \qquad t^a \otimes t^b \mapsto t^{a+b}$$

and comultiplication

$$R \to R \otimes R \qquad t^a \to t^a \otimes t^a$$

and counit ε sending $t^a \to 1$ (all t^a are primitive).

9.2.1 Proposition
R is a Hopf algebra. In particular, $D(A) \to \operatorname{Spec} k$ is a linear algebraic group.

As an example, consider $A = \mathbb{Z}$. Then $R \cong k[t]_t$. Thus $D(A) = \mathbb{G}_m$.
If instead $A = \mathbb{Z}/n$, then $R \cong l[t]/(t^n - 1)$, so $D(A) \cong \mu_n$.
Finally when $A \cong \mathbb{Z}^r \oplus \mathbb{Z}/n_1 \oplus \cdots \oplus \mathbb{Z}/n_k$, then

$$D(A) = \mathbb{G}_m^r \times \mu_{n_1} \times \cdots \times \mu_{n_k}$$

9.2.2 Definition: An algebraic group over k is **diagonalizable** if $G \cong D(A)$ for some A.

Recall the definiton of irreduciblility.

9.2.3 Theorem

Let A be af initely generated abelian group and $G = D(A)$. Then

- Every irreducible representation of G is one-dimensional and isomorphic to I_a, corresponding to $k \to \Gamma(G) \otimes k$ where $1 \mapsto t^a \otimes 1$ for some $a \in A$.
- Every representation decomposes as a direct sum of irreducibles.

Proof

Let $\sigma : V \to \Gamma(G) \otimes V$ be a representation of a diagonalizable group. For $a \in A$, define

$$V_a \stackrel{\text{def}}{=} \{v \in V | \sigma(v) = t^a \otimes v\} \subseteq V$$

Now the claim is that $V_a \cap V_b = 0$ if $a \neq b$ and furthermore $\sum V_a = V$. The first isn't too hard to see.

The second follows by considering $v \in V$ and looking at the image of it under σ. That is,

$$\sigma(v) = \sum_1^N t^{\alpha_i} \otimes v_i$$

where $\alpha_i \in A$ and $v_i \in V$. Then a very simple argument shows $v = \sum v_i$ (using linearity). Then it remains to show that $v_i \in V_{a_i}$, but this will make things work. (use the other axiom of a coaction). ♠

9.2.4 Remark: When $A = \mathbb{Z}$, $G = D(\mathbb{Z}) = \mathbb{G}_m$, which tells us that representations of $\mathbb{G}_m$ are in bijection with $\mathbb{Z}$-gradings of $V \cong \oplus_{n\in\mathbb{Z}} V_n$!

9.2.5 Definition: A linear algebraic group $G \to \operatorname{Spec} k$ is called **linear reductive** if every representation decomposes as a direct sum of irreducibles.

Problem 9.1

Show the above is equivalent to the statements

- *for each G-representations $W \subseteq V$, there exists $W' \subseteq V$ subrepresentations such that $V \cong W \oplus W'$.*
- $0 \to W \to V \to W' \to 0$ *is exact.*

9.2.6 Remark: Notice that this says that $D(A)$ is linear reducible. In particular, $\mathbb{G}_m$ and $|mu_n$ are in **any characteristic**. This runs counter to Maschke in finite groups.

Consider $\mathbb{Z}/p$ in char p. We get an action $\mathbb{Z}/p$ on k^2 via

$$1 \cdot (x,y) = (x+y,y).$$

But notice that $k \overset{y=0}{\hookrightarrow} k^2$ is a subrepresentation, but has no complement! Thus this group is not linearly reductive!

As another example, consider

$$\mathbb{G}_a \cong \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \right\} \subset \mathrm{GL}_2(k)$$

where $\mathbb{G}_a$ acts on k^2 by $\alpha(x,y) = (a + \alpha y, y)$. Then it can be easily seen not to be a linear representation.

10 October 16th, 2019

Last time we talked about representations! Woot.

Notice that if G is linear (i.e. affine), then the multiplication map induces comultiplication

$$\Gamma(G) \to \Gamma(G) \otimes \Gamma(G)$$

so $\Gamma(G)$ is the **regular** representation with coaction given by multiplication.

We also saw some equivalent contitions similar to Machke for linear reductive groups. Finally we say some examples and diagonalizable groups.

10.1 New Stuff

Given a G-representation V, let V^G be

$$\{v \in V | \sigma(v) = 1 \otimes v\} = \mathrm{Eq}\{V \overset{\xrightarrow{1\otimes -}}{\underset{\sigma}{\longrightarrow}} \Gamma(G) \otimes V\} = \mathrm{Hom}^G(k,V) \subseteq V.$$

10.1.1 Remark: I need to figure out the TEXfor equalizers/parallel maps.

Example 10.1
Given the representation $\mathbb{G}_m$ action on $V = \oplus V_d$, $V^G = V_0$.

Problem 10.1
If $G(k)$ is dense in G, then $V^G = V^{G(k)}$.

10.1.2 Proposition
A linear algebraic group G over k is linearly reductive if and only if the functor from G-representations to k-vector spaces given by $V \mapsto V^G$ is exact.

Proof
If $V \cong W \oplus W' \twoheadrightarrow W$ is a G representation, then

$$W^G \oplus (W')^G = V^G \twoheadrightarrow W^G$$

is also surjective.

Suppose that we have a short exact sequence

$$0 \to W' \to V \to W \to 0$$

and that this functor is exact. Then we want to show we get a section $\sigma : W \to V$. To do this, consider the functor $\mathrm{Hom}^G(W,-) = \mathrm{Hom}^G(k, W^\vee \otimes -) = (W^\vee \otimes -)^G$, so by the assumption this is exact and we can lift the identity on W to a map in $\mathrm{Hom}^G(W,V)$, giving us our section. ♠

10.1.3 Proposition
Let G be a linear algebraic group over k and V a G-representation. Let $W \subseteq V$ be a finite dimensional k-subspace (not necessarily G-invariant). Then there exists $W \subseteq W' \subseteq V$ such that W' is a finite dimensional representation of G.

Proof
We can assume that $W = \langle w \rangle$ for $w \in V$. Apply $\sigma : V \to \Gamma(G) \otimes V$. Then if $\{t_i\}$ is a basis for $\Gamma(G)$, we get

$$w \mapsto \sum t_i \otimes w_i.$$

Then we claim that $w \in \langle w_i \rangle$ and $\langle w_i \rangle \subseteq V$ is a subrepresentation.

For the first, consider the diagrams:

$$\begin{tikzcd} k \otimes V & \Gamma(G) \otimes V \arrow[l, "e^*\otimes \mathrm{id}"'] \\ & V \arrow[u, "\sigma"'] \arrow[ul, "\sim"] \end{tikzcd} \qquad \begin{tikzcd} \sum e^*(t_i) w_i & \sum t_i \otimes w_i \arrow[l] \\ & w \arrow[u] \end{tikzcd}$$

so w is in the span of the w_i.

For the second claim, we need to show that

$$\sigma(w_i) \in \Gamma(G) \otimes \langle w_i \rangle.$$

To see this consider the diagram

$$\begin{tikzcd}
\Gamma(G)\otimes\Gamma(G)\otimes V & \Gamma(G)\otimes V \arrow[l, "m^*\otimes\mathrm{id}"'] \\
\Gamma(G)\otimes V \arrow[u, "\mathrm{id}\otimes\sigma"] & V \arrow[l, "\sigma"] \arrow[u, "\sigma"']
\end{tikzcd}$$

And tracing through $w\in V$, we get that

$$\sum t_i\otimes\sigma(w_i)=\sum_{i,j,k}\alpha_{i,j,k}t_i\otimes t_j'\otimes w_k$$

and so by looking at coefficients of $t_i\otimes\Gamma(G)\otimes V=\sigma(w_i)$ (look closer here), we see it is

$$\sum_{j,k}\alpha_{i,j,k}t_j'\otimes w-k\in\Gamma(G)\otimes\langle w_i\rangle.$$

♠

10.1.4 Corollary

If V is a G representation, then

$$W=\bigcup_{W\subset V}W$$

where the union is over all finite dimensional subgroups.

10.1.5 Corollary

If G is a linear algebraic group (affine finite type over k), then for some n $G\subseteq\mathrm{GL}_n$ is a closed subgroup. In other workds, there exists a faithful representation V of G.

Now consider the regular representation $\Gamma(G)\to\Gamma(G)\otimes\Gamma(G)$. Notice that $\Gamma(G)$ is a k-algebra of finite type. Choose generators $g_1,\dots,g_n$ for $\Gamma(G)$. Take a subrepresentation of $\Gamma(G)$ spanned by $\langle h_1,\dots,h_N\rangle$ containing the span of the g_i.

We have a map $G\to\mathrm{GL}(W)$ where $W=\langle h_i\rangle$ and we aske whether the induced map

$$\mathrm{Sym}^*(W\otimes W^\vee)_{\det}$$

is surjective.

Let's say that $h_i\mapsto\sum_j\gamma_{i,j}\otimes h_j$ under σ. Then using this map and the natural pairing between W and $W^\vee$, we get a map

$$\mathrm{Sym}^*(W\otimes W^\vee)\supset W\otimes W^\vee\to\Gamma(G)\otimes W\otimes W^\vee\to\Gamma(G)$$

where we send

$$h_i\otimes h_j^*\mapsto\gamma_{i,j}$$

So using the counit identiy we can write

$$h_i=\sum_j e^*(\gamma_{i,j})h_j$$

but we really want to write h_i as a linear combination of the $\gamma_{i,j}$ (since we have shown they all lie in the image of this map). We don't know how to finish up.

11 October 18th, 2019

Last time we say that any linear algebraic group G embeds into GL_n. The argument was basically that you look at the global functions $\Gamma(G)$ and doing cool stuff. Right at the end Taffy and Tuomas figured out that we just needed to use the other "side" of the counit diagram.

11.1 An example

Consider

$$\mathrm{PGL}_2 = (\operatorname{Proj} k[a,b,c,d])_{ad-bc]} = \operatorname{Spec}(k[a,b,c,d]_{\det})_0$$

Then consider the representation spanned by

$$\left\langle \frac{a^2}{\det}, \frac{ab}{\det}, \dots, \frac{d^2}{\det} \right\rangle$$

which has dimension 10 in $\Gamma(\mathrm{PGL}_2)$.

Thus we have a representation $\mathrm{PGL}_2 \to \mathrm{GL}_{10}$. We can compute the matrix representing a matrix (whose determinant can be assumed to be 1 since we are modding out by scalars).

Problem 11.1
Do this! In Sage or something.

11.2 Special Linear Groups

Lets discuss SL_2.

11.2.1 Theorem
If $\operatorname{char} k = 0$, then

- SL_2 is linearly reductive.
- Every irreducible representation of SL_2 is isomorphic to $\operatorname{Sym}^d k^2$ for some d.

where k^2 is the standard representation of SL_2.

Proof
Sketch: Recall that in the proof of Maschke one takes a surjection $V \twoheadrightarrow W$ of G representations and we want to show it has a section. We pick a section s in terms of vector spaces and then "average" it:

$$\widetilde{s} : W \to V \qquad w \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot s(g^{-1} w)$$

which is our section.

Then using the Harr measure on the group, we get from the inclusion of (compact) SU_2 in SL_2 with quotient $\mathbb{C}$ and we can construct the section via

$$w \mapsto \int_G g \cdot s(g^{-1}w)\,\mathrm{d}g$$

and we get $T_e\,\mathrm{SL}_2 = (T_e\,\mathrm{SU}_2 \otimes_{\mathbb{R}} \mathbb{C})$ and then there is a bit more Lie theory needed to show this makes full sense over $\mathbb{C}$. ♠

Now consider $\mathrm{Sym}^d(k^2)^\vee$, the degree d polynomials on two variables. Then we get an action of SL_2 via

$$g \cdot f(x,y) = f(g^{-1}\begin{pmatrix} x \\ y \end{pmatrix}) - f(dx - by, -cx + ay)$$

Then, as many arguments in linear algebraic groups, we can reduce to a so-called *maximal torus* of matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \cong \mathbb{G}_m$. Then we can use techniques on the lie algebra $T_e\,\mathrm{SL}_2 = \mathfrak{sl}_2$.

Whoa coool. The short exact sequence

$$1 \to \mathrm{SL}_2 \to \mathrm{GL}_n \xrightarrow{\det} \mathbb{G}_m \to 1$$

gives us that any representation of SL_2 lifts via $- \otimes \det^i$ to a representation of GL_2. Sean asked whether this actually gets all the representations or just the polynomial ones. I feel like I should know the answer to this. Jarod seems to think this is all of them. I think this stack post says something about that.

I got caught up in thinking and googling and missed an example.

Let $\operatorname{char} k = p$ and let $\alpha \in k^\times$. Then define $\alpha \cdot f = \alpha^p f$. This gives us a map

$$\mathrm{Sym}^d\, k^n \to \mathrm{Sym}^{dp}\, k^n$$

that is additive taking p^{th} powers.

So then $\mathrm{Sym}^N\, k^n$ is **not simple** if $p|N$.

12 October 21st, 2019

Recall that we had that any linear algebraic group over a field k injects into GL_n as a closed subgroup for some n.

An open question is as follows: if $G \to \operatorname{Spec} k[\varepsilon]/\varepsilon^2$ is flat affine group scheme of finite type Then is $G \subseteq \mathrm{GL}_{n,k[\varepsilon]}$ for some n? This question was asked (as far as Jarod knows) by Brian Conrad on Stack Overflow and is still open.

Our goal is to answer the following: if $\phi : H \to H$ then $K = \ker \varphi = H \times_G \operatorname{Spec} k \subseteq H$ is a closed subgroup. What about its image H/K?

12.1 Torsors

Today we are going to be talking about G-torsors.

12.1.1 Definition: If G is a group, a **torsor under** G is a set P with a free and transitive group action.

12.1.2 Remark: So then by fixing a point $p \in P$, we get $G \xrightarrow{\sim} P$ by sending $g \mapsto pg$. In this way, it is like thinking about a group without the identity.

An example is by taking a Galois extension $K(\alpha)/K$ with minimal polynomial f of α. Then $G = \mathrm{Gal}(K(\alpha)/K)$ acts on $\{x | f(x) = 0\}$, which is a G-torsor.

12.1.3 Definition: A G**-torsor over a set** S is a set P with a free right G-action such that $P \to S$ is G-invariant and $S \cong P/G$.

12.1.4 Remark: Notice that a torsor under G is a specialization of this definition by requiring that $S = \{*\}$, the singleton set.

Example 12.1
Let $H \subseteq G$. THen H acting on $H \to H \setminus G$ (left cosets gH of H) is an H-torsor.

12.2 Flatness

12.2.1 Definition: A map of rings $A \to B$ is **flat** if $- \otimes_A B$ is exact.

12.2.2 Remark: Equivalently: for all $p \in \operatorname{Spec} A$, $A_p \to B_p$ is flat. Also: for all $q \in \operatorname{Spec} B$. $A_{\phi^{-1}(q)} \to B_q$ is flat.

12.2.3 Definition: $A \to B$ is **faithfully flat** if and only if $- \otimes_A B$ is **faithfully exact** (exactness and its converse).

12.2.4 Remark: Other equivalence to faithful flatness: $A \to B$ is flat and $\operatorname{Spec} B \twoheadrightarrow \operatorname{Spec} A$; or $A \to B$ is flat and for any A-module M, $M = 0 \longleftrightarrow m \otimes_A B = 0$.

12.2.5 Remark: If $\operatorname{Spec} B \twoheadrightarrow \operatorname{Spec} A$ is faithfully flat and finite presented, then $\operatorname{Spec} A$ has the quotient topology.

12.2.6 Proposition
Let $S = \operatorname{Spec} A$ be Noetherian. Let $G \to S$ be an affine group scheme of flat and finite type over S. Let $P \to S$ be a scheme over S with a right G-action $P \times_S G \xrightarrow{\sigma} P$. Then the following are equivalent:

(a) $P \to S$ is affine, (faithfully?)[1] flat, finite type and $(\sigma, \pi_P) : (p, g) \mapsto (pg, p)$ is an isomorphism.

(b) There exists a faithfully flat S' such that

$$\begin{tikzcd} P_{S'} \arrow[r] \arrow[d] & P \arrow[d] \\ S' \arrow[r, "\sigma"] & S \end{tikzcd}$$

And $P_{S'} \cong G_{S'}$ as $G_{S'}$-modules.

12.2.7 Remark: Note that the above says exactly that P is a G-torsor. Another name that has been mentioned and Jarod seems to like is **principal G bundle.**

12.3 Descent

Along the way of proving the above propositon, we use the idea of descent.

12.3.1 Lemma
Consider $X' = \operatorname{Spec} B' \overset{\text{f.flat, f.type}}{\twoheadrightarrow} X = \operatorname{Spec} B \to Y = \operatorname{Spec} A$ where X', X, and Y are Noetherian. Then

(a) $X' \to Y$ is flat implies that $X \to Y$ is flat.

(b) $X' \to Y$ is faithfully flat implies that $X \to Y$ is faithfully flat.

(c) $X' \to Y$ is finite type implies that $X \to Y$ is.

12.3.2 Remark: The idea for the first two is just looking at the functors using that $B \to B'$ is faithfully flat. For the third, if $B = \cup_\lambda B_\lambda$, then $A \to B_\lambda$ is finitely generated. Then tensoring over B with B' gets us $B' = \cup_\lambda B_\lambda \otimes_B B'$.

But since A is finitely generated over $B_\lambda <$ eventually $B_\lambda \otimes_B B' = B'$. Then consider

$$0 \to B_\lambda \to B \to B/B_\lambda.$$

[1]We tried to prove this in class and it seemed not to be true if we don't say this

After tensoring with (faithfully flat!) B' over B, since for some λ

$$0 \to B_\lambda \otimes_B B' \xrightarrow{\sim} B' \to B/B_\lambda \otimes_B B' \to 0$$

is exact, forcing the rightmost term to be zero. But by faithfulness this implies $B/B_\lambda = 0$ and we are done.

12.3.3 Proposition
Consider

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ S' = \operatorname{Spec} A' \arrow[r, two heads, "\text{f.type, f.flat}"'] & S = \operatorname{Spec} A \end{tikzcd}$$

which is a Cartesian square. Then

(a) $X' \to S'$ is an isomorphism iff $X \to S$ is.

(b) $X' \to S'$ is affine iff $X \to S$ is.

13 October 23rd, 2019

Recall the following definition/proposition:

13.0.1 Definition: Let $S = \operatorname{Spec} R$ be Noetherian. Let $G \to S$ be an affine group scheme that is flat and of finite type. Let G be a scheme over S with a right G-action.

Then the following are equiavlent:

- $P \to S$ is a G-**torsor**
- $P \to S$ is faithfully flat and of finite type and $P \times_S G \xrightarrow{(\sigma,\pi_1} P \times_S P$ is an isomorphism.
- There exists $S' \twoheadrightarrow S$ faithfully flat such that

$$\begin{tikzcd} G \times_S S' \cong P \times_S S' \arrow[r] \arrow[d] & P \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

commutes (where the isomorphism shown is $G \times_S S'$-equivariant.)

13.1 Some Examples

We have the trival torsor $P = G \to S$. It is a propositoon that $P \to S$ is trivial iff there exists a section $s : S \to P$.

Let L/K be a finite Galois extension. Then we get $\mathrm{Gal}(L/K)$ acting on $P = \operatorname{Spec} L \to \operatorname{Spec} K$ is a $G = \mathrm{Gal}(L/K)$-torsor. Then in the diagram in the definition above, $\operatorname{Spec} L$ plays the part of S'. Then we get that $L \otimes L \cong L[x]/f \cong \prod_{g \in G} L$ and

$$G \times_{\operatorname{Spec} K} \operatorname{Spec} L \cong \operatorname{Spec}(L \otimes L) \cong \sqcup_{g \in G} \operatorname{Spec} L.$$

Now let X be a scheme and let $\mathbb{G}_m$ act on a line bundle $L \to X$ with section $o : X \to L$. Then $(L \setminus o(X)) \to X$ is a $\mathbb{G}_m$ torsor. It is a result, although we don't have the machinery yet, that

13.1.1 Proposition
There is a bijection between line bundles on X and $\mathbb{G}_m$-torsors.

13.1.2 Remark: We can do womething similar with any vector bundle over X: if $V \to X$ is one, then V_x over $x \in X$ is a vector space. We just send V to $\mathrm{Frame}(V)$, which over any $x \in X$ we have the set of orderd bases of V_x. This gives us a GL_n-torsor.

If we have a subgroup (say of an algebraic group) $H \subseteq G$, recall that we wanted to show the existence of an H-torsor $G \to G/H$.

We begin by talking about *abstract groups*. Assume we have an exact sequence

$$1 \to K \to G \xrightarrow{\pi} Q.$$

Then $G \times K \xrightarrow{\sim} G \times_Q G$ via the map $(g, k) \mapsto (g, gk)$. The proof isn't too bad.

So now consider a geometric group. If we have the same exact sequence of *algebraic groups over k*, we get $K = G \times_Q \operatorname{Spec} k$(??) and then evaluating at any scheme S, we get

$$1 \to K(S) \to G(S) \to Q(S)$$

and consider the map $G(S) \times K(S) \to G(S) \times_{Q(S)} G(S)$ as above. Then by Yoneda we get an isomorphism $G \times K \cong G \times_Q G$ of schemes.

13.1.3 Corollary
If $\pi : G \to Q$ is a faithfully flat map of linear algebraic groups over k, then $G \to Q$ is a torsor under $K = \ker \pi$

Let's do even better!

13.1.4 Corollary
Let $\pi : G \to Q$ be dominant (i.e. the image is dense in Q) and furthermore that Q is reduced. Then $G \to Q$ is faithfully flat and in particular $G \to Q$ is a K-torsor.

Proof

(Of the second corollary): We use the idea of "generic flatness". That is there exists a $U \subseteq Q$ such that $\pi^{-1}(U) \to U$ is flat. Then we can translate this by the G-action (after passing to the algebraic closure of k so that the points are dense) and then flat descent gives us the result we want. :) ♠

13.1.5 Theorem

Let G be alinearly algebraic group over k. Let X be a scheme over k of finite type. Then

$$\{G\text{-bundles on } X\} \cong H^1_{fl}(X, G)$$

where H_{fl} is flat cohomology.

The idea here is clear when $G = \mathbb{G}_m$: you get connections between line bundles on X (i.e. the Picard group of X) and $\mathbb{G}_m$-torsors and similarly between the line bundles and $H^1_{Zar}(X, \mathcal{O}_X^\times)$, using the (usual) Zariski sheaf cohomology.