Lie Algebras and Groups

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Introduction

These notes are my best attempt at following along with our *Math 508 – Lie Algebras* course at UW. This is my first time trying to type my notes on-the-fly in class so we'll see how well this goes. The course reference is Humphreys' *Introduction to Lie Algebras and Representation Theory.*

The course description follows:

This is the second course in the Algebraic Structures sequence. I will classify finite-dimensional complex semisimple Lie algebras, also proving some structural results on general Lie algebras along the way. Although one usually first encounters Lie algebras in a manifolds course, the treatment (following the text) will be entirely algebraic.

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The homework is posted on Monty's website. :)

1.1 Lie algebras

This course will be studying Lie algebras, but as opposed to their treatment in manifolds, we will be studying them from a purely algebraic point of view. The book (Humphreys) actually never defines a Lie group.

1.1.1 Definition

A **Lie Algebra** L or $\mathfrak{g}$ over a field k is a k-vector space (usually f.d.) along with a *bracket operation* $[vw] : L \times L \to L$ such that $[\cdot\cdot]$ is

- anticommutative,
- bilinear,
- $[x[yz]] = [[xy]z] + [y[xz]]$

1.1.2 Remark: The last principle above is actually equivalent to the *Jacobi identity:*

$$[x[yz]] + [y[xz]] + [z[xy]].$$

This follows from bilinearity and anticommutativity of the bracket.

The most natural place for these to arise is as *derivations* on an algebra!

1.1.3 Definition
A **k-derivation** $d : A \to A$ on an algebra A over k is a k-linear map satisfying the Leibniz rule.

1.1.4 Remark: Some key facts about derivations (for us):

- Given a fixed $a \in A$, the map d_a sending $b \mapsto ab - ba$, the **commutator** $[ab]$ is a derivation.
- If d, e are derivations, then so is $[de] = de - ed$, where de is the *composite* of d and e as opposed to the product.

1.2 Examples

A main source of Lie algebras is (associative) algebras! *Any associative k-algebra A* becomes a Lie algebra over k, taking $[ab] = ab - ba$. In particular, one obvious choice for k-algebra is $M_n(k) = \mathfrak{gl}_n(k)$, the (Lie) algebra of $n \times n$ matrices over k.

Lie subalgebras are what you'd expect (including closure under brackets). Notice that if $L' \leq L$, then they **must both be over the same field.**

If L is a k-Lie algebra and $I \lhd L$ is an ideal of L, then the quotient space L/I becomes a Lie algebra with $[x + I, y + I] = [xy] + I$ as the bracket.

A **Lie algebra homomorphism** is a map $\varphi : L \to L'$ such that φ is k-linear and $\varphi([xy]) = [\varphi(x)\varphi(y)]$.

We get the usual first isomorphism theorem $L/\ker\varphi \cong \varphi(L)$.

Associative algebras are not the only source of Lie algebras, however! One example is $\mathfrak{sl}(n, k) = \{n \times n \text{ matrices over } k \text{ with trace zero}\}$

Note that this is **not closed under product** since $\operatorname{tr}(AB) \neq \operatorname{tr} A \operatorname{tr} B$ but $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ so $\operatorname{tr}(AB - BA) = \operatorname{tr}(AB) - \operatorname{tr}(BA) = 0$.

1.2.1 Definition
We call this algebra (or, in fact any subalgebra of $\mathfrak{gl}(n, k)$) **linear**. Think "Linear" means "of matrices."

We say that $\mathfrak{sl}(n,k)$ has **type** A_{n-1}. Eventually we will see seven types $A-G$ of semisimple Lie algebras. The shift in index will emerge later.

$\mathfrak{sl}(n,k)$ is, in fact, a simple Lie algebra: for $k=\mathbb{C}$, $\mathfrak{sl}(n,\mathbb{C})$ has no ideals apart from the trivial ones.

Other non-associative examples include k^n with a bilinear form $(\cdot,\cdot)$ which is either symmetric or skew-symmetric and (in either case) is nondegenerate.

1.2.2 Definition
$(\cdot,\cdot)$ is **nondegenerate** if the map $v \mapsto (v,\cdot)$ is injective. Equivalently there is no $v \in V$ such that $(v,w)=0$ for all $w \in V$.

Given $V = k^n$ and a bilinear form on V, we can look at all $X \in \mathfrak{gl}(n,k) = \mathfrak{gl}(V)$ such that $(Xv,w) = (v,Xw)$. Then X is **skew-adjoint** with respect to the form. One can check that $[XY]$ is skew-adjoint whenever both X and Y are.

1.3 Generating (skew) symmetric forms

It ends up that the dot product (which a symmetric form) is misleadingly simple – thus we will look elsewhere.

If $M \in \mathfrak{gl}(n,k)$ is symmetric, so that $M^t = M$, then $(v,w) = v^t M w$ is a symmetric. If instead M is skew-symmetric, then the same definition yields a skew-symmetric form. This actually induces a one-to-one correspondence between matrices and forms.

In both cases, if M is invertible, then the form will be nondegenerate. As a consequence, since skew-symmetric matrices are always singular in odd dimensions, we see that nondegenerate skew-symmetric forms (over $\operatorname{char} k \neq 2$ where the two families of forms coincide) exist only in even dimensions.

1.4 A peek at classifications

If we have a nondegenerate symmetric form where $n = 2m$ is going to give us an algebra of type D_m. If $n = 2m+1$, then it is of type B_m. Both of these cases are called **orthogonal.**

If instead we have a skew-symmetric form and $n = 2m$, then this is of type C_m, and we call this algebra **symplectic.**

We will make a particular choice for our matrix M and then study the resulting Lie algebras in much more detail next time. The choices will be:

- For type D_m:

$$\begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$$

- For type C_m:

$$\begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}$$

- For type B_m:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_m & 0 \end{pmatrix}$$