Algebraic Geometry

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Autumn 2019/ Winter and Spring 2020

Abstract

A three-quarter sequence covering the basic theory of affine and projective varieties, rings of functions, the Hilbert Nullstellensatz, localization, and dimension; the theory of algebraic curves, divisors, cohomology, genus, and the Riemann-Roch theorem; and related topics.

1 September 25, 2019

The first thing that one asks is "what is geometry?" One needs to be able to answer this question before they define AG. One idea is that geometry is topology + structure.

1.1 What is Geometry?

Example 1.1
Exotic differentiable structures on a sphere. There are many different smooth structures, all of which are independent of the topology,

$S^1 \times S^1$ has infinitely many complex structures (remember the parallelograms)!

How to you go about defining the geometry of a thing? One idea from manifolds: charts. These describe the local models and the interesting part is how this comes together to a whole space.

There is another idea to capture the "local" model of geometry that underlies modern algebraic geometry: consider the map $\varphi : W \to W' \in \mathbb{CP}^n$ and then say that this map is C^∞ if and only if its coordinate functions are. But the coordinate functions are problematic, so we can replace it with the following idea:

$\phi : W \to W'$ is $C\infty$ if and only if for all C^∞ functions $f : W' \to \mathbb{R}$, the composition

$$\varphi^* f = f \circ \varphi : W \to \mathbb{R}$$

is C^∞.

To capture the manifold structure on M, it is equivalent to know the set of C^∞ fucnctions $U \to \mathbb{R}$ for every open $U \subseteq M$.

1.2 The Big Idea

So then the idea we are talking away here is that *geometry is in the functions* that exist on a particular space!

Fix a field k.

1.2.1 Definition: A **space with functions** is a topological space X along with a collection (a k-algebra!) $\mathscr{O}(U)$ of maps $U \to k$ for each open $U \subseteq X$.

$\mathscr{O}(U)$ are called **regular functions** and must satisfy:

- Given an open cover U_α of U, a function is regular if and only if its restrictions to each element of the cover is regular.
- If $f : U \to k$ is regular, then $D(f) = \{u \in U | f(u) \neq 0\}$ is an open set and $\frac{1}{f} \in \mathscr{O}(D(f))$.

For the next time, try to think of as many examples of this as you can. Next time will be a mind blowing example of a variety.

2 September 27, 2019

Problem 2.1
Do all the exercises in Kempf chapter 1!

For now we assume that k is algebraically closed.

2.1 Examples of spaces with functions

There were lots of suggestions, but here are a couple:

Example 2.1
Let $X = \mathbb{S}^2$ and let $\mathscr{O}_X^{cts}$ be the continuous $\mathbb{C}$-valued functions. Alternatively we could consider a different sheaf $\mathscr{O}_X^{an}$, the holomorphic functions. Or we could consider $\mathscr{O}_X^{\infty}$, the C^∞ functions (under some smooth structure).

2.1.1 Definition: A **morphism** of spaces with functions between $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ is a continuous map $\varphi : X \to Y$ such that for all $U \subseteq Y$ open and $f \in \mathscr{O}_Y(U)$, the function

$$\phi^* f = f \circ \phi|_{\phi^{-1}(U)} : \phi^{-1}(U) \to k \in \mathscr{O}_X(\phi^{-1}(U))$$

In other words, a morphism of spaces with functions is a map of spaces that *respects the regular functions.*

Example 2.2
Let X, Y be topological spaces and let $\mathscr{O}_X$ and $\mathscr{O}_Y$ be the continuous functions. Then morphisms are just continuous maps.

Example 2.3
When X and Y are manifolds and $\mathscr{O}_\bullet$ are complex-valued functions, then the maorphisms are maps of manifolds.

So now we return to the examples we saw before: $(\mathbb{S}^2, \mathscr{O}^\infty)$, $(\mathbb{S}^2, \mathscr{O}^{cts})$, and $(\mathbb{S}^2, \mathscr{O}^{an})$. A natural question to ask is when we have morphisms between these spaces to see if there exist ones that are the identity on $\mathbb{S}^2$.

Consider the identity map from the continuous to the analytic functions. Then take any map $f \in \mathscr{O}^{an}$ and consider that

$$f = f \circ id_{id^{-1}(U)} : U \to k \in \mathscr{O}^{cts}(U)$$

and there is no map in the other direction.

2.1.2 Remark: Notice that since we are pulling functions back, the maps go in the opposite direction as you may think at first.

We can also talk about **open subspaces**. If $V \subseteq X$ is an open subset, we can let the induced space with functions be $(V, \mathscr{O}_V)$ where if $U \subseteq V$ then $\mathscr{O}_V(U) := \mathscr{O}_X(U)$.

2.2 Varieties

2.2.1 Definition: An **affine k-variety** is a space with functions $(Y, \mathscr{O}_Y)$ such that for every space with functions $(X, \mathscr{O}_X)$, the natural map

$$\mathrm{Hom}((X,\mathscr{O}_X),(Y,\mathscr{O}_Y)) \to \mathrm{Hom}_{\mathbf{Alg}_k}(\mathscr{O}_Y(Y), \mathscr{O}_X(X))$$

is a bijection and furthermore $\mathscr{O}_Y(Y) =: k[Y]$ is a finitely generated k-algebra.

2.2.2 Remark: The idea here is that the algebra maps (on the right) are precisely the same as the geometry maps (on the left). Algebraic geometry, baby.

So then this leads to a very simple (loose) definition:

2.2.3 Definition: A **variety** is something that is covered by affine varieties.

Example 2.4
$\mathbb{A}^1 = k$. Give this space the cofinite topology. Then if we have $U = k\setminus\{x_1, \dots, x_n\} \subset \mathbb{A}^1$,

$$\mathscr{O}_{\mathbb{A}^1}(U) = \{f(t) \in k(t) | \text{poles are in } \{x_i\}\}$$

Problem 2.2
Show that $\mathbb{A}^1$ is an affine variety!

2.2.4 Remark: Notice that this staetment is equivalent to saying that any morphism of spaces with functions gives us a regular map $X \to k$.

3 September 30th, 2019

One question that was asked: if we have fixed the underlying topological space in a space with functions, must there be a morphism between them somehow? Might there instead be a common cover of the two?

Example 3.1
Let k be a field with some topology on it such that every point is closed (you could do the discrete topology). Let $\widetilde{\mathcal{O}}(U)$ be the continuous funcitons $U \to k$. In other words, these functions are locally constant.

Locally constant functions behave nicely under restrictions to opens, of course. The other axioms are also great.

Have we really found an initial object in our category? This would be enough to establish a "tent" (as in localization of categories). Try this out and see what happens!

3.1 The question of affine space

Recall the question about whether $\mathbb{A}^1$ is an affine variety. The idea here is that $\phi : X \to k$ is a morphism of spaces with functions if and only if it is regular (that is, in $\mathcal{O}_{\mathbb{A}^1}$).

One direction is tautological (a morphism to $\mathbb{A}^1$ has a polynomial underlying it), so let ϕ be regular. Then to see that ϕ is continuous can be checked by pulling back all closed sets. The important observation is that $D(f-a) = X \setminus \phi^{-1}(a)$, which is closed (an axiom for spaces with functions).

The last thing to check is where ϕ pulls back regular functions to regular functions. This relies on the facts that $\mathcal{O}_X$ is a k-algebra and that $\phi(x) - b_j$ is regular on U when $b_j \notin U$.

3.2 Algebra maps

Notice that since we have a condition that $\mathcal{O}_X(X)$ must be finitely generated as a k-algebra, this means that

$$\mathrm{Hom}(X,Y) = \mathrm{Hom}_k(\mathcal{O}_Y(Y), \mathcal{O}_X(X)) = \mathrm{Hom}_k(k[x_1,\dots,x_n]/(f_1,\dots,f_m), O_X(X))$$

and

$$\mathrm{Hom}(X,Y) = \{(\gamma_1,\dots,\gamma_n) \in (\mathcal{O}_X(X))^n : f_j(\gamma_i) = 0, \forall j = 1,\dots,m\}$$

In other words, we are looking at maps that factor through Z:

$$\begin{tikzcd} (\gamma_1,\dots,\gamma_n) : X \arrow[r] \arrow[dr] & k^n \\ & Z = Z(f_i) \arrow[u, hook] \end{tikzcd}$$

Now what we want to say is that $Y = Z$. That is, *affine varieties are closed subsets of affine spaces.*

Now this is all good, but the problem is that we had to *choose* a presentation of $\mathcal{O}_Y(Y)$ to get this picture. of course we want something more canonical! We will see in this class (and in Kempf) that this can be done.