General Exam Paper

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Abstract

We begin by going through a considerable amount of domain knowledge concerning representations of GL_n, representations of $\mathfrak{S}_n$, and strict polynomial functors all in service of understanding the Schur-Weyl functor that relates several of these categories. From there, we investigate recent work on the part of Krause and his students Aquilino and Reischuk on this functor and the fact that it is monoidal under reasonably natural monoidal structures on the categories in question. Finally we ask some questions about whether the monoidal structure on strict polynomial functors extends meaningfully to pathologies that arise in positive characteristic.

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1 Introduction

1.1 Schur and Polynomial Representations

The story of this project (more-or-less) begins with Schur's doctoral thesis [Sch01] in which he defined polynomial representations of GL_n—a theory which he developed more completely in his later paper *Über die rationalen Darstellungen der allgemeinen linearen Gruppe*[1] [Sch73]. In these papers, Schur develops the idea of a **polynomial representation of** GL_n, meaning a (finite dimensional) representation where the coefficient functions of the representing map

$$\rho : \mathrm{GL}_n \to \mathrm{GL}(V)$$

is polynomial in each coordinate. That is, if $V = \oplus_{i=1}^n k v_i$, then for every $1 \le i, j \le n$, we have the map $r_{v_i v_j} : \mathrm{GL}_n \to k$ such that

$$\rho(g) \cdot v_i = \sum_{i=1}^n r_{v_i v_j} v_j.$$

A result in [Sch01] gives us a nice simplifying observation: if V is a polynomial representation, then V decomposes as a direct sum of representations

$$V = \bigoplus_\delta V_\delta$$

where each V_δ is a polynomial representation where the coefficient functions are *homogeneous degree* δ. This allows us to focus our attention to the structure of these V_δ as the fundamental building blocks of the theory.

One of the more surprising connections made in this theory comes from the observation that the vector space (recall $V \cong k^n$)

$$E = V^{\otimes r}$$

is made into a $(\mathrm{GL}_n(k), \mathfrak{S}_r)$-bimodule in a very natural way, and that this bimodule gives us a way to relate **mod**-$\mathfrak{S}_r$ with (a subcategory of) $\mathrm{GL}_n(k)$-**mod** via the so-called **Schur-Weyl functor.**

1.2 The Schur-Weyl Functor

Clearly a connection between representations of two groups that are so ubiquitous in group theory and math in general is a stunning observation, and much effort has been expended since the late 20th century to study this functor and its properties—especially in how it relates the representation theory of these two groups.

For instance, Friedlander and Suslin [FS97] originally discussed the idea of **strict polynomial functors** and showed that the category of repesentations of the Schur algebra $S(n,d)$ was equivalent to the category $\mathscr{P}_d$ of homogeneous degree d strict polynomial functors.

[1] English: *On the rational representations of the general linear group*

In later work, Krause [Kra13] used an alternative construction of $\mathscr{P}_d$ as the category of of reprsentations of the d-divided powers of the category of finitely generated projective k-modules. The upshot being that the latter object $\Gamma^d P_k$ has an obvious monoidal structure which $\mathscr{P}_d$ inherits in a natural way. This new concrete monoidal structure opens up the field to discussing several notions of duality defined in different contexts and solidifying connections between them.

Krause's students Aquilino and Reischuk, in their paper [AR17], prove, among other facts, that under these natural monoidal structures the Schur-Weyl functor is in fact monoidal. This puts the theory of representations of these groups and algebras firmly in the realm of monoidal categories, opening up the area to new questions using tools from category theory.

2 Representations of GL_n and of $\mathfrak{S}_n$

Through the following discussion, let $\Gamma = \Gamma_k = \mathrm{GL}_n(k)$ for some field k. Let $\rho : \Gamma \to \mathrm{GL}(V)$ be a representation of Γ.

3 The Schur-Weyl Functor

From the discussion in the last section it is evident that the combinatorics behind the representation theory of $S(n,r)$ and $\mathfrak{S}(r)$ have some intersections in their use of Young tableaux and this connection is more than superficial. In fact, there is a functor relating the representations of these two objects in the following way:

3.1 Construction of the functor

Let $V \in M_k(n,r)$ be a $S(n,r)$-representation and select any weight $\alpha \in \Lambda(n,r)$. Then the weight space

$$V^\alpha = \xi_\alpha V$$

becomes a $S(\alpha) \stackrel{\text{def}}{=} \xi_\alpha S(n,r)\xi_\alpha$-module using the action from $S(n,r)$. Now if we allow $r \leq n$ and let

$$\omega = (1,\ldots,1,0,\ldots,0) \in \Lambda(n,r)$$

notice that $S(\omega)$ is spanned by the elements

$$\xi_\omega \xi_{i,j} \xi_\omega, \quad i,j \in I(n,r)$$

but by the multiplication rules established in the definition of $S(n,r)$, these are nonzero precisely when i and j are both of shape ω. So then since $\xi_{i,j} = \xi_{i\sigma,j\sigma}$ for all $\sigma \in \mathfrak{S}_r$, we can take as a basis of $S(\omega)$ the set

$$\{\xi_{u\pi,u} | \pi \in \mathfrak{S}_r\}$$

where $u = (1,2,\cdots,r) \in I(n,r)$.

To prove the next statement we require a computational result.

3.1.1 Lemma
If $u=(1,2,\ldots,r)\in I(n,r)$, then for all $\pi,\sigma\in\mathfrak{S}_r$,

$$\xi_{u\pi,u}\cdot\xi_{u\sigma,u}=\xi_{u\pi\sigma,u}.$$

Proof
Using the formulas for multiplication in $S(n,r)$, recall that

$$\xi_{u\pi,u}\cdot\xi_{u\sigma,u}=\sum Z_{i,j}\xi_{i,j} \tag{1}$$

where
$$Z_{i,j}=\#\{s\in I(n,r)|(u\pi,u)\sim(i,s)\text{ and }(u\sigma,u)\sim(s,j)\}.$$
Then for each i,j, since $u=(1,2,\ldots,r)$ has no stabilizer in $\mathfrak{S}_r$, there is a unique g such that $u\pi g=i$, meaning that $s=ug$.

But then this fixes (again a unique) $h\in\mathfrak{S}_r$ such that $u\sigma h=s=ug$ whence $\sigma h=g$. One computes that
$$u\pi\sigma h=u\pi g=i\quad\text{and}\quad uh=j$$
therefore since in the above computation s was completely determined by i, we have

$$Z_{i,j}=\begin{cases}1, & (i,j)\sim(u\pi\sigma,u)\\ 0, & \text{otherwise}\end{cases}$$

and the result follows. ♠

Using this result, we prove a more obviously useful statement:

3.1.2 Lemma
$S(\omega)\cong k\mathfrak{S}(r)$.

Proof
Define the map $\varphi:S(\omega)\to k\mathfrak{S}_r$ on the basis above to be

$$\varphi(\xi_{u\pi,u})=\pi$$

and extending k-linearly.

This is a homomorphism since

$$\varphi(\xi_{u\pi,u}\xi_{u\sigma,u})=\varphi(\xi_{u\pi\sigma,u})=\pi\sigma$$

and it is bijective since it is bijective on the respective bases. ♠

The upshot of these lemmas is that one can define the functor

$$\mathscr{F}:M_k(n,r)\to\mathbf{Rep}(\mathfrak{S}_r)$$

via the map that sends any representation V to its ω weight space.

3.2

4 Strict Polynomial Functors

5 Questions and Extensions

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