General Exam Paper

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Abstract

We begin by going through a considerable amount of domain knowledge concerning representations of GL_n, representations of $\mathfrak{S}_n$, and strict polynomial functors all in service of understanding the Schur-Weyl functor that relates several of these categories. From there, we investigate recent work on the part of Krause and his students Aquilino and Reischuk on this functor and the fact that it is monoidal under reasonably natural monoidal structures on the categories in question. Finally we ask some questions about whether the monoidal structure on strict polynomial functors extends meaningfully to pathologies that arise in positive characteristic.

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1 Introduction

1.1 Schur and Polynomial Representations

The story of this project (more-or-less) begins with Schur's doctoral thesis [Sch01] in which he defined polynomial representations of GL_n—a theory which he developed more completely in his later paper *Über die rationalen Darstellungen der allgemeinen linearen Gruppe*[1] [Sch73]. In these papers, Schur develops the idea of a **polynomial representation of** GL_n, meaning a (finite dimensional) representation where the coefficient functions of the representing map

$$\rho : \mathrm{GL}_n \to \mathrm{GL}(V)$$

is polynomial in each coordinate. For example, the map sending

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2d - abc & acd - c^2b & 0 \\ abd - b^2c & ad^2 - bcd & 0 \\ 0 & 0 & ad - bc \end{pmatrix} = \rho(A)$$

is a three-dimensional polynomial representation of GL_2.

The block-diagonal form above demonstrates a direct sum decomposition of our representation into two parts: one two-dimensional homogeneous degree 3 and one one-dimensional homogeneous degree 2 (in the entries of A). A result in [Sch01] tells us that, in fact, this can always be done: if V is a polynomial representation of GL_n, then V decomposes as a direct sum of representations

$$V = \bigoplus_{\delta} V_\delta$$

where each V_δ is a polynomial representation where the coefficient functions are *homogeneous degree* δ. This allows us to focus our attention to the structure of these V_δ as the fundamental building blocks of the theory.

The key insight made in this theory comes from the observation that the vector space (recall $V \cong k^n$)

$$E = V^{\otimes r}$$

is made into a $(\mathrm{GL}_n(k), \mathfrak{S}_r)$-bimodule in a very natural way, and that this bimodule gives us a way to relate **mod**-$\mathfrak{S}_r$ with (a subcategory of) $\mathrm{GL}_n(k)$-**mod** via the so-called **Schur-Weyl functor.**

1.2 The Schur-Weyl Functor

Clearly a connection between representations of two groups that are so ubiquitous in group theory and math in general is a stunning observation, and much effort has been expended since the late 20th century to study this functor and its properties—especially in how it relates the representation theory of these two groups.

[1] English: *On the rational representations of the general linear group*

For instance, Friedlander and Suslin [FS97] originally discussed the idea of **strict polynomial functors** and showed that the category of repesentations of the Schur algebra $S(n,d)$ was equivalent to the category $\mathscr{P}_d$ of homogeneous degree d strict polynomial functors.

In later work, Krause [Kra13] used an alternative construction of $\mathscr{P}_d$ as the category of of reprsentations of the d-divided powers of the category of finitely generated projective k-modules. The upshot being that the latter object $\Gamma^d P_k$ has an obvious monoidal structure which $\mathscr{P}_d$ inherits in a natural way. This new concrete monoidal structure opens up the field to discussing several notions of duality defined in different contexts and solidifying connections between them.

Krause's students Aquilino and Reischuk, in their paper [AR17], prove, among other facts, that under these natural monoidal structures the Schur-Weyl functor is in fact monoidal. This puts the theory of representations of these groups and algebras firmly in the realm of monoidal categories, opening up the area to new questions using tools from category theory.

1.3 Notation and Conventions

Throughout this paper we will define k to be a field (not necessarily of characteristic zero or algebraically closed unless otherwise noted).

We will use $\Gamma = \Gamma_k = \mathrm{GL}_n = \mathrm{GL}_n(k)$ to denote the general linear group, $\mathrm{Aut}_k(k^n)$. Let $\mathfrak{S}_r$ denote the symmetric group on r letters.

When speaking of the (k) vector space spanned by elements $v_1, \ldots, v_n$, we will use the notation

$$\langle v_1, \ldots, v_n \rangle.$$

2 Representations of Γ and of $\mathfrak{S}_n$

We begin by detailing the theory behind the (polynomial) representations of GL_n as well as the representations of $\mathfrak{S}_n$ to familiarize ourselves with the classical representation theory associated to these groups.

2.1 Polynomial representations of Γ

Let Γ be the affine group scheme defined in section 1.3. Then

2.1.1 Definition: A (finite dimensional) **representation** of Γ is a (finite dimensional) vector space V along with a map

$$\rho : \Gamma \to \mathrm{GL}(V) \stackrel{\text{def}}{=} \mathrm{Aut}_k(V)$$

where ρ is a group homomorphism.

2.1.2 Remark: Often in representation theory one combines the map ρ and vector space V into a single object: a $k\Gamma$-module. This simultaneously encodes the vector space structure (via k-linearity) and the action by Γ.

Representations of Γ can be, in general, "analytic." One can check that the map

$$\rho : k^\times = \mathrm{GL}_1(k) \to \mathrm{GL}(k^2) \qquad \text{via} \qquad x \mapsto \begin{pmatrix} 1 & \ln|x| \\ 0 & 1 \end{pmatrix}$$

gives a group homomorphism (and thus representation) between these two groups, but the logarithm makes this representation decidedly *not algebraic*. To narrow our focus somewhat and ensure we stay within the realm of algebra, we make the following definition:

2.1.3 Definition: A **polynomial representation** of Γ is a representation ρ such that the structure maps of ρ are polynomials in the functions $c_{ij} : \Gamma \to k$ that extract the $(i,j)^{th}$ entry.

2.1.4 Remark: Recall (or learn for the first time!) that the *structure maps* of a representation (ρ, V) are a collection of maps r_{ij} for $1 \le i, j, \le n$ from Γ to k such that for all $g \in \Gamma$:

$$g \cdot v_i = \sum_{j=1}^{n} r_{ij}(g) v_j$$

where we have picked a basis $\{v_1, \ldots, v_n\}$ for V. Of course changing basis may change our r_{ij}, but an invariant of the representation is the **span** $\langle r_{ij} \rangle$.

2.1.5 Remark: If all r_{ij} are homogeneous polynomials of the same degree, we say that ρ is a **homogeneous polynomial representation** of Γ.

2.1.6 Definition: Let $M_k(n) = M(n)$ be the collection of all polynomial representations of GL_n and let $M_k(n,r) = M(n,r)$ be the collection of all degree r polynomial representations of GL_n.

Generally speaking, we will identify both of these with an appropriate subcategory of $k\Gamma$-**mod**.

It is the *polynomial* representations that we will concern ourselves with in the following sections.

2.1.1 Reducing scope

Using some of our familiar friends from representation theory (as well as some clever twists), we can simplify this picture considerably by proving the following structural result:

2.1.7 Theorem ([Sch01, pp.7-10])
Every polynomial representation V over an infinite field k decomposes as a direct sum

$$V \cong \bigoplus_{\delta \in \mathbb{N}} V_\delta$$

where V_δ is a *homogeneous* polynomial representation of degree δ.

Clearly, then, it suffices to understand the *homogeneous degree r* polynomial representations of Γ if we are looking to understand the larger structure.

We begin with a useful lemma extracted from a proof in [Sch01] echoing the general theory of orthogonal decomposition of Artinian algebras.

2.1.8 Lemma
Let $C_0, \ldots, C_m \in M_n(k)$ be mutually orthogonal idempotent matrices that sum to the identity. That is,

$$I_n = \sum_i C_i \quad \text{and} \quad C_i C_j = \delta_{ij} C_i$$

for all $0 \leq i, j \leq m$. Then there exists an invertible matrix P such that for some positive

> integers $d_0, \ldots, d_m$ with $\sum_k d_k = n$ and for all i,
>
> $$P^{-1}C_iP = \begin{pmatrix} 0_{N_i} & & \\ & I_{d_i} & \\ & & 0_{M_i} \end{pmatrix}$$
>
> Where $N_i = \sum_{0 \le j < i} d_j$ and $M_i = n - d_i - N_i$

Proof (of lem 2.1.8)
We set $S_k = \{C_0, C_1, \ldots, C_k\}$ and we proceed by induction on k. When $k = 0$, $S_k = \{C_0\}$. Now since $C_0^2 = C_0$, we get that 1 and 0 are the only eigenvalues of C_0, so there is an $r \times r$ matrix P_0 and a positive integer d_0 such that

$$P_0^{-1}C_0P_0 = \begin{pmatrix} I_{d_0} & \\ & 0_{n-d_0} \end{pmatrix}.$$

which establishes the base case.

Now assume that we have a matrix P_{k-1} such that this property holds for all elements of S_{k-1}. Define, for each $0 \le i \le k$,

$$C_i' \stackrel{\text{def}}{=} P_{k-1}^{-1}C_iP_{k-1}$$

and since the C_k is assumed to be orthogonal to all other C_i,

$$C_k' = \begin{pmatrix} 0_{N_k} & \\ & D_k \end{pmatrix}$$

for some D_k.

Now by properties of block diagonal matrices, we have

$$D_k^2 = D_k$$

so the eigenvalues of D_k are again one and zero. Thus there is an invertible $Q \in \mathrm{GL}_{n-N_k}$ such that

$$Q^{-1}D_kQ = \begin{pmatrix} I_{d_k} & \\ & 0_{M_k} \end{pmatrix}$$

and so by setting

$$P_k \stackrel{\text{def}}{=} P_{k-1}\begin{pmatrix} I_{N_k} & \\ & Q \end{pmatrix}$$

we can define

$$C_i'' \stackrel{\text{def}}{=} P_k^{-1}C_iP_k = \begin{pmatrix} I_{N_k} & \\ & Q \end{pmatrix}^{-1} C' \begin{pmatrix} I_{N_k} & \\ & Q \end{pmatrix}$$

for $0 \le i \le k$, we see immediately that $C_i' = C_i''$ for $0 \le i < k$ and furthermore

$$C_k'' = \begin{pmatrix} 0_{N_k} & \\ & Q^{-1}D_kQ \end{pmatrix} = \begin{pmatrix} 0_{N_k} & & \\ & I_{d_k} & \\ & & 0_{M_k} \end{pmatrix}$$

completing the inductive step. This this result holds for all S_i and in particular for S_m, so the result is proven. ♠

As well as another result on a special class of commuting block diagonal matrices:

> **2.1.9 Lemma**
> Let k be an infinite field and let A be a block diagonal matrix over k of the form
>
> $$A = \operatorname{diag}(x^m I_{d_m}, x^{m-1} I_{d_{m-1}}, \dots, I_{d_0})$$
>
> where d_i is (clearly) the dimension of the $(m-i)^{th}$ block and let B be any matrix that commutes with A for every choice of $x \in k$. Then B is block diagonal of the same shape as A.

Proof (of lem 2.1.9)
We proceed by comparing the entries in AB and BA: notice that

$$(AB)_{ij} = \sum_k A_{ik}B_{kj} = A_{ii}B_{ij} = x^a B_{ij}$$

and

$$(BA)_{ij} = \sum_k B_{ik}A_{kj} = B_{ij}A_{jj} = x^b B_{ij}.$$

We will show that if the $(i,j)^{th}$ postion is not in one of the blocks of A, then it is zero.

But if (i,j) is not in one of the blocks of A, then the nonzero element in the i^{th} row and the nonzero element in the j^{th} column (x^a and x^b in the above equations) are not the same! Since x is arbitrary, this forces $B_{ij} = 0$, so B is block diagonal with blocks the same as A. ♠

2.1.10 Remark: Notice that in the above proof we used implicitly that there is an $x \in k$ such that for all a, b

$$x^a = x^b \quad \Rightarrow \quad a = b$$

which is true if (and only if!) k is infinite. This is because any such x must be a root of $x^n = x$ for some n, which has finitely many roots over any field (but every element in $\mathbb{F}_p$ satisfies $x^p = x$).

And finally using these two lemmas allows us to prove our main result:

PROOF (OF THM 2.1.7)
We recreate the argument in Schur's thesis, translated from German and reinterpreted in more modern parlance.

Let (ρ, V) be a polynomial representation of Γ with $\dim_k V = r$. Then let $x \in k^\times$ be arbitrary (thought of as an indeterminate) and consider the matrix $xI_n \in \Gamma$. The image of this matrix under ρ is a matrix

$$\rho(A) = \begin{pmatrix} p_{11}(x) & \cdots & p_{1r}(x) \\ \vdots & \ddots & \vdots \\ p_{r1}(x) & \cdots & p_{rr}(x) \end{pmatrix}$$

where each p_{ij} is a polynomial in x. Let $m = \max_{i,j} \deg p_{ij}$, and this gives us a decomposition

$$\rho(A) = x^m C_0 + x^{m-1} C_1 + \cdots + x C_{m-1} + C_m$$

where each C_i is an $r \times r$ matrix.

Let y be another indeterminate and $B = yI_n$. By virtue of being a representation of Γ, we get

$$\rho(A)\rho(B) = \rho(xI_n)\rho(yI_n) = \rho(xyI_n) = \rho(AB)$$

and using this setup we prove the following result:

$$\text{For all } 0 \le i, j \le m \text{, with the } C_l \text{ as above,} \quad C_i C_j = \delta_{ij} C_i$$

That this is true can be established by comparing coefficients in the equation

$$\rho(AB) = \rho(A)\rho(B)$$
$$C_0(xy)^m + \cdots + C_i(xy)^{m-i} + \cdots + C_m = C_0^2 x^m y^m + \cdots + C_i C_j x^{m-i} y^{m-j} + \cdots + C_m^2$$

Indeed, we immediately get that $C_i = C_i^2$ and furthermore the coefficients on $x^i y^j$ when $i \neq j$ give us

$$0 = C_{m-i} C_{m-j}.$$

Thus we have shown that the C_i form a set of orthogonal idempotent matrices and evaluating our original equation at $x = 1$, we get (since ρ is a homomorphism)

$$I_r = 1C_0 + \cdots + 1C_m = \sum C_i$$

so the result from lemma 2.1.8 applies: we get a matrix P such that

$$P^{-1}\rho(xI_n)P = \begin{pmatrix} x^m I_{d_0} & & & & \\ & x^{m-1} I_{d_1} & & & \\ & & \ddots & & \\ & & & x I_{d_{m-1}} & \\ & & & & I_{d_m} \end{pmatrix}$$

Now let $\rho'(g) = P^{-1}\rho(g)P$ for all $g \in \Gamma$. This is a representation of Γ since it it differs from ρ by an automorphism of $\mathrm{GL}(V)$. Since matrix multiplication is an algebraic operation, ρ' is still a polynomial representation of Γ. But notice that for all $g \in \Gamma$

$$\rho'(g)\rho'(xI_n) = \rho'(xg) = \rho'(xI_n)\rho'(g)$$

Then lemma 2.1.8 gives us that $\rho'(g)$ decomposes in the same way for all $g \in \Gamma$, so we know that ρ' decomposes as a direct sum of representations

$$\rho' = \sum_{i=0}^{m} \rho'_i$$

where for each i and $\lambda \in k$,

$$\rho'_i(\lambda g) = \rho'_i(\lambda I_{d_i})\rho'_i(g) = \lambda^i \rho'_i(g)$$

so each ρ'_i is a homogeneous degree i polynomial representation of Γ.

But of course the decomposition of a representation is independent of choice of basis, so we get a decomposition of ρ into homogenous pieces, as desired. ♠

2.1.2 Monomials and multi-indices

All of the discussion up to this point has revolved around polynomials in n^2 variables, which quickly gets unwieldy unless one uses some better notation. To that end,

2.1.11 Definition: An (n,r)-**multi-index** i is an r-tuple $(i_1, \ldots, i_r)$ where each $i_j \in \underline{n} \overset{\text{def}}{=} \{1, \ldots, n\}$. The collection of all (n,r)-multi-indices is denoted $I(n,r)$.

2.1.12 Remark: One can also think of an element $i \in I(n,r)$ as a (set) map

$$i : \underline{r} \to \underline{n}.$$

The idea here is to associate to each monomial in a polynomial ring in many variables a tuple indicating its multidegree. That is we think of

$$(i_1, \ldots, i_r) \quad \leftrightsquigarrow \quad x_{i_1} \cdots x i_r$$

as corresponding to the same object. Which is wonderful except for one small flaw: polynomials are commutative and multi-indices (as we have defined them) aren't! For example, in $I(3,4)$,

$$(2,2,1,3) \quad \leftrightsquigarrow \quad x_1 x_2^2 x_3 \quad \leftrightsquigarrow \quad (3,2,1,2).$$

To handle this disparity, we define an equivalence relation on $I(n,r)$ where we say that $i \sim j$ if they are in the same orbit under the natural $\mathfrak{S}_r$ action. That is, if there exists $\sigma \in \mathfrak{S}_r$ such that

$$(i_1,\ldots,i_r) = (j_{\sigma(1)},\ldots,j_{\sigma(r)})$$

In the context of polynomial representations of Γ, we want to consider polynomials in the coordinate functions c_{ij}, so as a matter of notation if $i,j \in I(n,r)$, let $c_{i,j}$ denote the monomial

$$c_{i,j} = c_{i_1 j_1} \cdots c_{i_r j_r}.$$

Again, we want to take into account that we can permute the order on the right hand side, but now we need that i_k and j_k remain linked to the same function. To deal with this, we define an equivalence relation $\sim$ on $I(n,r) \times I(n,r)$ such that

$$(a,b) \sim (c,d)$$

if there exists a $\sigma \in \mathfrak{S}_r$ such that

$$(a_1,\ldots,a_r) = (b_{\sigma(1)},\ldots,b_{\sigma(r)}) \quad \text{and} \quad (c_1,\ldots,c_r) = (d_{\sigma(1)},\ldots,d_{\sigma(r)}).$$

The upshot of this work is that it gives us a bijection between (total) degree r monomials in the c_{ij} and the set

$$I(n,r) \times I(n,r)/\sim$$

2.1.3 $A_k(n,r)$

Notice that if $V \in M(n,r)$, each of its structure maps are homogeneous degree r polynomials. As the first object of study, consider

2.1.13 Definition: Let $A_k(n,r) = A(n.r)$ denote the collection of all homogeneous degree r polynomials in the coordinate functions $c_{ij} : \Gamma \to k$.

It is not too hard to see that

> **2.1.14 Proposition**
> $A_k(n,r)$ is spanned by the elements
>
> $$\{c_{i,j} | (i,j) \in I(n,r) \times I(n,r)\}$$

however it takes a short argument to see

2.1.15 Lemma
The dimension of $A_k(n,r)$ over k is $\binom{n^2+r-1}{n^2-1} = \binom{n^2+r-1}{r}$.

Proof
The following is a "stars and bars" argument that is pervasive in combinatorics. See for example [Sta12] if unfamiliar with these techniques.

Fix an ordering of the c_{ij} (say the dictionary order) and relabel them $\{\gamma_1,\dots,\gamma_m\}$ (here $m=n^2$) according to this order. Then the degree r monomials are in bijection with m-tuples $(a_1,\dots,a_m)\in\mathbb{N}^m$ such that $\sum_i a_i = r$ via the map which sends

$$(a_1,\dots,a_m)\mapsto \gamma_1^{a_1}\cdots\gamma_m^{a_m}.$$

But choosing such an element is the same as inserting $m-1$ bars into a line of r stars (that is an ordered partition of r into m parts, where parts are allowed to be zero). But this is equivalent to choosing $m-1$ bars in a field of $m+r-1$ symbols. This is just

$$\binom{m+r-1}{m-1}$$

and a well-known identity for binomial coefficients gets us the final equality. ♠

Example 2.1
In case the reader is unfamiliar with this kind of reasoning, consider the case when $n=5$ and $r=4$. Then the partition $(1,0,0,2,1)$ corresponding to $\gamma_1\gamma_4^2\gamma_5$ corresponds to the stars-and-bars diagram

$$*|||**|*$$

where there are $m+r-1=8$ symbols, $r=4$ of which are stars.

2.1.4 A dip into affine group schemes and category theory

$A(n,r)$ lies within $k^\Gamma = k(\Gamma)$, the k-algebra of functions $\Gamma\to k$, which has the structure of a Hopf algebra induced from the group structure on Γ. More precisely, the functor $\mathrm{GL}_n : \mathbf{Alg}_k \to \mathbf{Grp}$ that assigns to every k-algebra A the group $\mathrm{GL}_n(A)$ is representable. In other words,

$$\mathrm{GL}_n(-)\simeq \mathrm{Hom}_{\mathbf{Alg}_k}(R,-)$$

where $R\cong k[x_{ij}|1\le i,j\le n]_{\det}$.

The anti-equivalence of the categories of affine group schemes over k and finite dimensional commutative k-Hopf algebras follows from Yoneda lemma (c.f. [Wat79, chp. 1]), and along with this equivalence comes a way to translate the group structure on Γ into a coalgebra structure on R: we have maps μ, ϵ, the multiplication and unit maps on Γ satisfying the diagrams

$$\begin{tikzcd} \Gamma\times\Gamma\times\Gamma \arrow[r, "\mu\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\mu"] & \Gamma\times\Gamma \arrow[d, "\mu"] \\ \Gamma\times\Gamma \arrow[r, "\mu"] & \Gamma \end{tikzcd} \qquad \begin{tikzcd} *\times G \arrow[r, "\epsilon\times\mathrm{id}"] \arrow[dr, leftrightarrow, "\sim"'] & G\times G \arrow[d, "\mu"] & G\times * \arrow[l, "\mathrm{id}\times\epsilon"'] \arrow[dl, leftrightarrow, "\sim"] \\ & G & \end{tikzcd}$$

(where $*$ is the trivial group) giving us associativity and identity. Yoneda gives us that the maps between functors (group schemes!)

$$\mu : \Gamma\times\Gamma \to \Gamma \quad \text{and} \quad \epsilon : * \to \Gamma$$

give rise to maps in $\mathbf{Alg}_k$:

$$\Delta \stackrel{\text{def}}{=} \mu^* : R \to R\otimes_k R \quad \text{and} \quad \varepsilon \stackrel{\text{def}}{=} \epsilon^* : R \to k$$

satisfying diagrams

$$\begin{tikzcd} R\otimes R\otimes R & R\otimes R \arrow[l, "\Delta\otimes\mathrm{id}"'] \\ R\otimes R \arrow[u, "\mathrm{id}\otimes\Delta"] & R \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} k\otimes R & R\otimes R \arrow[l, "\varepsilon\otimes\mathrm{id}"'] \arrow[r, "\mathrm{id}\otimes\varepsilon"] & R\otimes k \\ & R \arrow[ul, leftrightarrow, "\sim"] \arrow[u, "\Delta"] \arrow[ur, leftrightarrow, "\sim"'] & \end{tikzcd}$$

2.1.16 Proposition
The maps Δ and ε give R a coalgebra structure. In coordinates, if $1 \le i, j \le n$,

$$\Delta(c_{ij}) = \sum_k c_{ik}\otimes c_{kj} \quad \text{and} \quad \varepsilon(c_{ij}) = \delta_{ij}$$

In fact, as mentioned before, R becomes a bialgebra (a Hopf algebra even, although we won't need the antipode here). This means that Δ and ε are algebra morphisms for the natural algebra structure given by multiplication m on R. In diagrams:

$$\begin{tikzcd} R^{\otimes 4} \arrow[r, "\mathrm{id}\otimes\tau\otimes 1"] & R^{\otimes 4} \arrow[r, "m\otimes m"] & R\otimes R \\ R\otimes R \arrow[u, "\Delta\otimes\Delta"] \arrow[rr, "m"] & & R \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} R\otimes R \arrow[r, "m"] \arrow[d, "\varepsilon\otimes\varepsilon"] & R \arrow[d, "\varepsilon"] \\ k\otimes k \arrow[r, "m"] & k \end{tikzcd}$$

where $\tau : R\otimes R \to R\otimes R$ is the twist map $a\otimes b \mapsto b\otimes a$. Chasing an element through the diagram on the left, we get

$$\widetilde{m}\circ(\Delta\otimes\Delta)(c_{ij}\otimes c_{ab}) = \sum_{1\le k,l\le n} c_{ik}c_{al}\otimes c_{kj}c_{lb} = \Delta(c_{ij}c_{ab})$$

or using our multi-index notation,

$$\Delta(c_{(i,a),(j,b)}) = \sum_{(k,l)\in I(n,2)} c_{(i,a),(k,l)} \otimes c_{(k,l),(j,b)}.$$

Written more simply, the fact that Δ is an algebra morphism can be written

$$\Delta(a\cdot b) = \Delta(a) * \Delta(b)$$

under suitable definitions of $\cdot$ and $*$. In a way that can be made precise, this means in particular that

$$\Delta(a\cdot b\cdot c) = \Delta(a) * \Delta(b\cdot c) = \Delta(a) * \Delta(b) * \Delta(c)$$

and so on (since multiplication everywhere is associative) and therefore we can define this for arbitrary monomials and extend k-linearly:

> **2.1.17 Proposition**
> If $i,j \in A(n,r)$, then
>
> $$\Delta(c_{i,j}) = \sum_{k\in I(n,r)} c_{i,k} \otimes c_{k,j} \quad \text{and} \quad \varepsilon(c_{i.j}) = \delta_{i,j}$$

One can easily see that degree is preserved by Δ, meaning that

> **2.1.18 Proposition**
> Δ and ε descend to a coalgebra structure on $A(n,r)$. That is, $A(n,r)$ is a (k-)coalgebra.

2.1.5 The Schur Algebra

Finally we get to the actual object of study:

2.1.19 Definition: A **Schur algebra** is an element of the two-parameter family $\{S(n,r)\} = \{S_k(n,r)\}$ where n and r are any positive integers. As a set, $S(n,r)$ is the linear dual of $A(n,r)$:

$$S(n,r) = A(n,r)^* = \operatorname{Hom}_k(A(n,r),k)$$

Let $\xi_{i,j}$ denote the element dual to $c_{i.j} \in A(n,r)$. In other words:

$$\xi_{(a,b)}(c_{i,j}) = \begin{cases} 1, & (a,b)\sim(i,j) \\ 0, & \text{otherwise} \end{cases}$$

> **2.1.20 Lemma**
> The coalgebra structure (Δ, ε) on $A(n,r)$ define an algebra structure on $S(n,r)$.

PROOF
Since k is an initial object in $\mathbf{Alg}_k$, there is a unique map $u : k \hookrightarrow S(n,r)$ sending 1 to the constant one function $\mathbb{1}$, which is our unit map. Define multiplication $(\cdot)$ in $S(n,r)$ as follows: if $f, g \in S(n,r)$ then for any $x \in A(n,r)$ define

$$(f \cdot g)(x) = m_k \circ (f \otimes g) \circ \Delta(x) = \sum f(x_{(1)}) g(x_{(2)})$$

where $m_k : k \otimes k \to k$ denotes multiplication in k and $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$ in Sweedler notation.

Then we must just confirm that these maps satisfy the properties of a k-algebra. $(\cdot)$ is k-bilinear because (for instance)

$$\begin{aligned}((af + bg) \cdot h)(x) &= \sum (af + bg)(x_{(1)}) \otimes h(x_{(2)}) \\ &= \sum a(f(x_{(1)}) \otimes h(x_{(2)})) + b(g(x_{(1)}) \otimes h(x_{(2)})) \\ &= a \sum f(x_{(1)}) \otimes h(x_{(2)}) + b \sum g(x_{(1)}) \otimes h(x_{(2)}) \\ &= (a(f \cdot h) + b(g \cdot h))(x).\end{aligned}$$

It suffices to show that the unit $\mathbb{1}$ acts as it should on the spanning set $\xi_{i,j}$ for a basis element $c_{a,b}$:

$$(\mathbb{1} \cdot \xi_{i,j})(c_{a,b}) = \sum_{k=1}^{n} 1 \cdot \xi_{i,j}(c_{k,b})$$

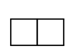
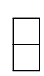
2.2 Representations of $\mathfrak{S}_n$

2.3 Explicit Examples

To demonstrate the theory developed above, we begin a computation (in a simple case) of the isomorphism classes of irreducible representations of both $S_{\mathbb{C}}(2,2)$ and $\mathfrak{S}_2$.

2.3.1 The symmetric group on two letters

The representation theory (over $k = \mathbb{C}$) of $\mathfrak{S}_2$ is as simple as it comes: of course $\mathfrak{S}_2 \cong \mathbb{Z}/2\mathbb{Z}$ and we know that there are $|G|$ nonisomorphic irreducible representations of an abelian group G over $\mathbb{C}$. Since we are talking about a symmetric group, we can realize these as the trivial and sign representations, represented by the Young diagrams:

 and 

As submodules of the regular representation $k\mathfrak{S}_2 = ke \oplus k(12)$, we can construct these as $\langle e + (12) \rangle$ (trivial representation) and $\langle e - (12) \rangle$ (sign representation).

2.3.2 The Schur algebra $S_{\mathbb{C}}(2,2)$

Since $\operatorname{char}\mathbb{C}=0$, (by e.g. [Sch01]) we know immediately that $S_{\mathbb{C}}(2,2)$ is semisimple, so it suffices to identify the irreducible submodules therein. Now we know

$$S=S_{\mathbb{C}}(2,2)\cong\mathbb{C}^2\otimes\mathbb{C}^2$$

so $\dim_{\mathbb{C}} S=4$. The theory outlined above give us that isomorphism types of irreducible modules are in bijection with partitions of 2 into two parts, meaning we have two isomorphism types: one corresponding to $\lambda_1=(1,1)$ and one corresponding to $\lambda_2=(2,1)$.

Using the construction of $D_{\lambda,\mathbb{C}}$ from above, we can compute these two irreducible modules explicitly:

Example 2.2 ($\lambda_1=(1,1)$)
In this case our shape is $(1,1)$, corresponding to the Young diagram

$$\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$$

and then $D_{\lambda_1,\mathbb{C}}$ is spanned by the element

$$(T_l:T_{(2,1)})=\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}:\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}\right)=c_{12}c_{21}-c_{11}c_{22}=c_{(1,2),(2,1)}-c_{(1,2),(1,2)}\in A_{\mathbb{C}}(2,2)$$

since all other bideterminants of this shape are zero or linearly dependent. Thus this is a one-dimensional irreducible representation.

Example 2.3 ($\lambda_2=(2,0)$)
Now our shape is $(2,0)$, corresponding to the diagram

$$\begin{array}{|c|c|}\hline & \\ \hline\end{array}.$$

The bideterminants here are

$$\begin{aligned}(T_l:T_{(1,1)})&=\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}:\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}\right)=c_{11}^2\\ (T_l:T_{(1,2)})=(T_l:T_{(2,1)})&=c_{11}c_{12}\\ (T_l:T_{(2,2)})&=c_{12}^2\end{aligned}$$

So we have a three-dimensional irreducible representation spanned by $\langle c_{11}^2,c_{11}c_{12},c_{12}^2\rangle$.

Since these are the only two Young diagrams of size two, these examples form a complete list of isomorphism classes of irreducible representations of $S_{\mathbb{C}}(2,2)$.

If we prefer instead to recognize our irreducibles as submodules of $E^{\otimes 2}$ (giving us a more obvious action by our algebras), we can use the short exact sequence

$$0 \to N \hookrightarrow E^{\otimes 2} \twoheadrightarrow D_{\lambda,\mathbb{C}} \to 0$$

to define the $n = \ker \pi$, where π is defined to be the map in REFERENCE HERE. Then we can compute the orthogonal complement to N to get $V_{\lambda,\mathbb{C}}$. We can compute:

$$V_{\lambda_1,\mathbb{C}} = \langle e_1 \otimes e_2 - e_2 \otimes e_1 \rangle$$

and

$$V_{\lambda_2,\mathbb{C}} = \langle e_1 \otimes e_1,\, e_1 \otimes e_2 + e_2 \otimes e_1,\, e_2 \otimes e_2 \rangle$$

where $\{e_1, e_2\}$ is a basis for $E \cong k^2$.

3 The Schur-Weyl Functor

From the discussion in the last section it is evident that the combinatorics behind the representation theory of $S(n,r)$ and $\mathfrak{S}_r$ have some intersections in their use of Young tableaux and this connection is more than superficial. In fact, there is a functor relating the representations of these two objects in the following way:

3.1 Construction of the functor $\mathscr{F}$

Let $V \in M_k(n,r)$ be a $S(n,r)$-representation and select any weight $\alpha \in \Lambda(n,r)$. Then the weight space

$$V^\alpha = \xi_\alpha V$$

becomes a $S(\alpha) \overset{\text{def}}{=} \xi_\alpha S(n,r)\xi_\alpha$-module using the action from $S(n,r)$. Now if we allow $r \leq n$ and let

$$\omega = (1,\dots,1,0,\dots,0) \in \Lambda(n,r)$$

notice that $S(\omega)$ is spanned by the elements

$$\xi_\omega \xi_{i,j} \xi_\omega, \quad i,j \in I(n,r)$$

but by the multiplication rules established in the definition of $S(n,r)$, these are nonzero precisely when i and j are both of shape ω. So then since $\xi_{i,j} = \xi_{i\sigma,j\sigma}$ for all $\sigma \in \mathfrak{S}_r$, we can take as a basis of $S(\omega)$ the set

$$\{\xi_{u\pi,u} | \pi \in \mathfrak{S}_r\}$$

where $u = (1,2,\cdots,r) \in I(n,r)$.

To prove the next statement we require a computational result.

> **3.1.1 Lemma**
> If $u = (1,2,\dots,r) \in I(n,r)$, then for all $\pi,\sigma \in \mathfrak{S}_r$,
>
> $$\xi_{u\pi,u} \cdot \xi_{u\sigma,u} = \xi_{u\pi\sigma,u}.$$

Proof

Using the formulas for multiplication in $S(n,r)$, recall that

$$\xi_{u\pi,u} \cdot \xi_{u\sigma,u} = \sum Z_{i,j}\xi_{i,j} \tag{1}$$

where

$$Z_{i,j} = \#\{s \in I(n,r) | (u\pi,u) \sim (i,s) \text{ and } (u\sigma,u) \sim (s,j)\}.$$

Then for each i,j, since $u = (1,2,\dots,r)$ has no stabilizer in $\mathfrak{S}_r$, there is a unique g such that $u\pi g = i$, meaning that $s = ug$.

But then this fixes (again a unique) $h \in \mathfrak{S}_r$ such that $u\sigma h = s = ug$ whence $\sigma h = g$. One computes that

$$u\pi\sigma h = u\pi g = i \quad \text{and} \quad uh = j$$

therefore since in the above computation s was completely determined by i, we have

$$Z_{i,j} = \begin{cases} 1, & (i,j) \sim (u\pi\sigma, u) \\ 0, & \text{otherwise} \end{cases}$$

and the result follows. ♠

Using this result, we prove a more obviously useful statement:

3.1.2 Lemma
$S(\omega) \cong k\mathfrak{S}(r)$.

Proof
Define the map $\varphi : S(\omega) \to k\mathfrak{S}_r$ on the basis above to be

$$\varphi(\xi_{u\pi,u}) = \pi$$

and extending k-linearly.

This is a homomorphism since

$$\varphi(\xi_{u\pi,u}\xi_{u\sigma,u}) = \varphi(\xi_{u\pi\sigma,u}) = \pi\sigma = \varphi(\xi_{u\pi,u})\varphi(\xi_{u\sigma,u})$$

and it is bijective since it is bijective on the respective bases and is thus bijective as a linear map. ♠

The upshot of these lemmas is that one can define the functor

$$\mathscr{F} : M_k(n,r) \to \mathbf{Rep}(\mathfrak{S}_r)$$

via the map that sends any representation V to its ω weight space $V^\omega \in S(\omega)$-**mod** $\simeq \mathbf{Rep}(\mathfrak{S}_r)$.

3.2 Properties of $\mathscr{F}$

4 Strict Polynomial Functors

5 Questions and Extensions

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