# General Exam Paper

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**Abstract**

We begin by going through a considerable amount of domain knowledge concerning representations of $\mathrm{GL}_n$, representations of $\mathfrak{S}_n$, and strict polynomial functors all in service of understanding the Schur-Weyl functor that relates several of these categories. From there, we investigate recent work on the part of Krause and his students Aquilino and Reischuk on this functor and the fact that it is monoidal under reasonably natural monoidal structures on the categories in question. Finally we ask some questions about whether the monoidal structure on strict polynomial functors extends meaningfully to pathologies that arise in positive characteristic.

An up-to-date version of this paper can be downloaded at the following link: `https://github.com/NicoCourts/General-Exam-Paper/raw/master/General-Paper.pdf`

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# 1 Introduction

## 1.1 Issai Schur and polynomial representations

The story of this project (more-or-less) begins with Schur's doctoral thesis [Sch01] in which he defined polynomial representations of $\mathrm{GL}_n$—a theory which he developed more completely in his later paper *Über die rationalen Darstellungen der allgemeinen linearen Gruppe*[1] [Sch73]. In these papers, Schur develops the idea of a **polynomial representation of** $\mathrm{GL}_n$, meaning a (finite dimensional) representation where the coefficient functions of the representing map

$$\rho : \mathrm{GL}_n \to \mathrm{GL}(V)$$

is polynomial in each coordinate. For example, the map sending

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2d - abc & acd - c^2b & 0 \\ abd - b^2c & ad^2 - bcd & 0 \\ 0 & 0 & ad - bc \end{pmatrix} = \rho(A)$$

is a three-dimensional polynomial representation of $\mathrm{GL}_2$.

The block-diagonal form above demonstrates a direct sum decomposition of our representation into two parts: one two-dimensional homogeneous degree 3 and one one-dimensional homogeneous degree 2 (in the entries of $A$). A result in [Sch01] tells us that, in fact, this can always be done: if $V$ is a polynomial representation of $\mathrm{GL}_n$, then $V$ decomposes as a direct sum of representations

$$V = \bigoplus_{\delta} V_\delta$$

where each $V_\delta$ is a polynomial representation where the coefficient functions are *homogeneous degree* $\delta$. This allows us to focus our attention to the structure of these $V_\delta$ as the fundamental building blocks of the theory.

The key insight made in this theory comes from the observation that the vector space (recall $V \cong k^n$)

$$E = V^{\otimes r}$$

is made into a $(\mathrm{GL}_n(k), \mathfrak{S}_r)$-bimodule in a very natural way, and that this bimodule gives us a way to relate **mod**-$\mathfrak{S}_r$ with (a subcategory of) $\mathrm{GL}_n(k)$-**mod** via the so-called **Schur-Weyl functor.**

## 1.2 The Schur-Weyl functor

Clearly a connection between representations of two groups that are so ubiquitous in group theory and math in general is a stunning observation, and much effort has been expended since the late 20th century to study this functor and its properties—especially in how it relates the representation theory of these two groups.

[1] English: *On the rational representations of the general linear group*

For instance, Friedlander and Suslin [FS97] originally discussed the idea of **strict polynomial functors** and showed that the category of repesentations of the Schur algebra $S(n,d)$ was equivalent to the category $\mathcal{P}_d$ of homogeneous degree $d$ strict polynomial functors.

In later work, Krause [Kra13] used an alternative construction of $\mathcal{P}_d$ as the category of of reprsentations of the $d$-divided powers of the category of finitely generated projective $k$-modules. The upshot being that the latter object $\Gamma^d P_k$ has an obvious monoidal structure which $\mathcal{P}_d$ inherits in a natural way. This new concrete monoidal structure opens up the field to discussing several notions of duality defined in different contexts and solidifying connections between them.

Krause's students Aquilino and Reischuk, in their paper [AR17], prove, among other facts, that under these natural monoidal structures the Schur-Weyl functor is in fact monoidal. This puts the theory of representations of these groups and algebras firmly in the realm of monoidal categories, opening up the area to new questions using tools from category theory.

## 1.3 Notation and conventions

Throughout this paper we will define $k$ to be a field (not necessarily of characteristic zero or algebraically closed unless otherwise noted).

We will use $\Gamma = \Gamma_k = \mathrm{GL}_n = \mathrm{GL}_n(k)$ to denote the general linear group, $\mathrm{Aut}_k(k^n)$. Let $\mathfrak{S}_r$ denote the symmetric group on $r$ letters.

When speaking of the $(k)$ vector space spanned by elements $v_1, \ldots, v_n$, we will use the notation

$$\langle v_1, \ldots, v_n \rangle.$$

When the rest of the paper is finished, the intro will be rewritten to reflect the actual content.

# 2 The classical theory: Representations of $\mathrm{GL}_n$ and of $\mathfrak{S}_n$

We begin by detailing the theory behind the (polynomial) representations of $\mathrm{GL}_n$ as well as the representations of $\mathfrak{S}_n$ to familiarize ourselves with the classical representation theory associated to these groups.

## 2.1 Representations of $\mathfrak{S}_n$

The representation theory for $\mathfrak{S}_n$ over the complex numbers is a subject that has been widely studied by representation theorists and combinatorialists alike for over a century. Before we dive into specifics, we write down the idea originally worked out by Frobenius [Fro73] in his work in 1900 on the characters of $\mathfrak{S}_n$:

> **2.1.1 Theorem**
> The conjugacy classes (and thus isomorphism classes of irreducible representations over $\mathbb{C}$) of $\mathfrak{S}_n$ are in bijection with partitions of $n$.

> 2.1.2 Remark: In what follows we attempt to give a tangible, minimalistic overview of the nicest case of representations of $\mathfrak{S}_n$. Some of the arguments below appeal more to intuition and examples than rigor, but we feel this better prepares the reader for computations in $\mathfrak{S}_n$ without being weighed down by unnecessary details. This can all be made rigorous, of course, at the expense of some clarity and conciseness.

Let's get some sense first about how we can relate these two ideas by recalling some easy lemmas from group theory. Recall that each element of $\mathfrak{S}_n$ can be written as a product of disjoint cycles and that this representation is unique up to reordering the cycles. We can make this representation unique by writing each cycle as one starting at its least element and then ordering the cycles by these least elements. For instance, the permutation (in two-line notation)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 7 & 5 & 3 & 8 & 4 & 6 \end{pmatrix} \in \mathfrak{S}_8$$

is represented uniquely in this way as the product of cycles:

$$\sigma = (1\,2)(3\,7\,4\,5)(6\,8).$$

The next observation to recover: if $\tau, \eta \in \mathfrak{S}_n$ and $\tau = (\tau_1\,\tau_2\,\cdots\,\tau_k)$ is a cycle,

$$\eta^{-1}\tau\eta = (\eta(\tau_1)\,\eta(\tau_2)\,\cdots\,\eta(\tau_k)).$$

We can see this demonstrated in the computation

$$\begin{aligned}(1\,3\,5)\sigma(1\,3\,5)^{-1} &= (1\,3\,5)(1\,2)(1\,5\,3)(1\,3\,5)(3\,7\,4\,5)(1\,5\,3)(1\,3\,5)(6\,8)(1\,5\,3)\\ &= (3\,2)(5\,7\,4\,1)(6\,8)\\ &= (1\,5\,7\,4)(2\,3)(6\,8)\end{aligned}$$

The important observation here is that the "shape" (the lengths of the cycles when written as a product of disjoint cycles) is preserved under conjugation. In fact,

> **2.1.3 Lemma**
> The conjugacy classes of $\mathfrak{S}_n$ are in one-to-one correspondence with the partitions of $n$.

Proof

Let $\mathscr{P}_n$ denote the partitions of $n$ and let $C_n$ denote the conjugacy classes in $\mathfrak{S}_n$. We construct the set map

$$\varphi : C_n \to \mathscr{P}_n$$

by sending a conjugacy class to the weakly-decreasing list of cycle lengths (including trivial cycles, if necessary). For instance in $\mathfrak{S}_8$,

$$(1\,5\,3)(2\,7) \qquad \text{cooresponds to} \qquad (3,2,1,1,1).$$

The results cited and demonstrated above shows that this map is well defined—conjugation preserves the cycle length in the disjoint cycle representation of an element. Furthermore if $p \in \mathscr{P}_n$ is a partition, the adjoint action of $\mathfrak{S}_n$ on $\varphi^{-1}(p)$ is transitive, since if two elements have the same cycle lengths when written as disjoint cycles, we can line the cycles up according to length and act by the permutation that "puts labels in the right place". If we look at $\sigma$ and the element we found by conjugation above, we have

$$\begin{gathered}(1\,2)(3\,7\,4\,5)(6\,8)\\(3\,2)(5\,7\,4\,1)(6\,8)\end{gathered}$$

where we notice that $1 \mapsto 3$, $3 \mapsto 5$, and $5 \mapsto 1$, meaning that the cycle that takes the top element to the bottom is $(1\,3\,5)$—although of course we already knew that. Another example are the elements $(1\,4\,5)$ and $(3\,2\,1)$. Here we want $1 \mapsto 3$, $4 \mapsto 2$ and $5 \mapsto 1$. Thus one element that takes the first to the second is $(5\,1\,3)(2\,4)$. This demonstrates that the action is not faithful since we could also act by $(5\,1\,3\,7)(2\,4)(6\,8)$ and get the same element. the important fact here is that $\mathfrak{S}_n$ acts transitively on the elements of $1, \ldots, n$, so there is always such an element.

The surjectivity of this map is clear since we can write from any partition of $n$ a product of disjoint cycles corresponding to this partition (which then must map to it) and injectivity is clear since the disjoint cycle representation is unique (up to reordering cycles, which doesn't affect the image $\varphi(x)$). This proves the lemma. ♠

From here, the standard result that (again, over $\mathbb{C}$) the conjugacy classes of a group are in bijection with the irreducible representations finishes demonstrating how theorem 2.1.1 is true. But a simple set bijection belies the depth of the connection here.

### 2.1.1 Construction of the irreducible representations

It is possible, through the idea of a Young symmetrizer, to directly link a Young diagram to the corresponding irreducible representation. Throughout this subsection, we will be relying on facts developed in [FH91], although there is also a more complete combinatorial picture painted in Fulton's book *Young Tableaux* [Ful97].

To begin our discussion, consider the trivial representation within the left regular representation $\mathbb{C}\mathfrak{S}_n$: it is a one-dimensional subspace spanned by the element

$$x_1 = \sum_{\sigma \in \mathfrak{S}_n} \sigma$$

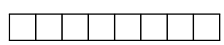
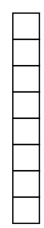
where you can see that this element is fixed by left multiplication, demonstrating that it has the trivial $\mathfrak{S}_n$ action. The subspace spanned by the element

$$x_{-1} = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\mathrm{sign}(\sigma)} \sigma$$

is the sign representation, where an element with sign 1 acts by -1. This is because

$$\mathrm{sign}(\tau\sigma) = \mathrm{sign}(\tau) + \mathrm{sign}(\sigma) \pmod 2.$$

It ends up that these two representations form the two "endpoints" of the representation theory of $\mathfrak{S}_n$. The exact sense in which this is true is captured through Young diagrams! For the purposes of illustration, let us return to our example above of $\mathfrak{S}_8$. Here the trivial and sign representations correspond (repsectively) to the tableaux

 and 

which, in turn, correspond to partitions $(8)$ and $(1,1,1,1,1,1,1,1)$ of 8. The way to make this connection is through the definition of a *Young symmetrizer:*

**2.1.4 Definition:** Fix an $n \geq 1$ and let $\lambda$ be a partition of $n$. Then define two elements of $\mathbb{C}\mathfrak{S}_n$, $a_\lambda$ and $b_\lambda$ in the following way:

$$a_\lambda = \sum_{\sigma \in R(T_\lambda)} \sigma \qquad \text{and} \qquad b_\lambda = \sum_{\sigma \in C(T_\lambda)} (-1)^{\mathrm{sign}(\sigma)} \sigma$$

where $T_\lambda$ is the Young diagram corresponding to $\lambda$ and given some labeling (say the canonical one that labels boxes left-to-right and top-to-bottom) $R(T_\lambda)$ (resp. $C(T_\lambda)$) denote the subgroups of $\mathfrak{S}_n$ stabilizing the rows (resp. columns) of $T_\lambda$ under the action of $\mathfrak{S}_n$ on the labels.

Then the **Young centralizer** of $\lambda$ is

$$c_\lambda = a_\lambda b_\lambda \in \mathbb{C}\mathfrak{S}_n.$$

The canonical fillings of the diagrams above are[2]

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&2&3&4&5&6&7&8\\ \hline\end{array} \qquad \text{and} \qquad \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline 4\\ \hline 5\\ \hline 6\\ \hline 7\\ \hline 8\\ \hline\end{array}$$

and so since the column stabilizer of the first diagram is trivial and the row stablizer is everything,

$$c_{(8)} = \left(\sum_{\sigma\in R(T_{(8)})} \sigma\right)\left(\sum_{\sigma\in C(T_{(8)})} (-1)^{\text{sign}(\sigma)}\sigma\right) = \sum_{\sigma\in\mathfrak{S}_n} \sigma = x_1$$

and since the roles of the column and row stabilizing elements are reversed for the sign representation, we get

$$c_{(1,1,1,1,1,1,1,1)} = \left(\sum_{\sigma\in R(T_{(1,1,1,1,1,1,1,1)})} \sigma\right)\left(\sum_{\sigma\in C(T_{(1,1,1,1,1,1,1,1)})} (-1)^{\text{sign}(\sigma)}\sigma\right) = \sum_{\sigma\in\mathfrak{S}_n}(-1)^{\text{sign}(\sigma)}\sigma = x_{-1}.$$

That the Young symmetrizers correspond with the elements spanning the corresponding representations of is no coincidence!

**2.1.5 Definition:** The module $V_\lambda$ is a $\mathbb{C}\mathfrak{S}_n$-module generated by the Young symmetrizer $c_\lambda$.

Notice that the dimension of each $V_\lambda$ is determined by number of linearly-independent elements that lie in the orbit of $c_\lambda$. We compute another example that gives a general pattern:

**Example 2.1**
Let $\lambda = (2,1,1)$ be the partition of 5, so

$$T_\lambda = \begin{array}{|c|c|}\hline \phantom{0} & \phantom{0} \\ \hline \phantom{0} \\ \cline{1-1} \phantom{0} \\ \cline{1-1}\end{array}.$$

[2]Here you can see yet another connection to disjoint cycle representations. Notice, under the map defined in lem. 2.1.3, that the conjugacy class corresponding to the trivial representation is the one consisting of "long" (length $n$) cycles. Using the unique ordering on products of disjoint cycles described after the statement of thm. 2.1.1, we can identify fillings with long cycles and we see that the cycle $(1\,2\,3\,4\,5\,6\,7\,8)$ is the only one in "standard form" in that it gives us a standard Young tableau. The complexity of the Young diagram (meaning how many different standard fillings it admits) gives us some information about the dimensionality of the corresponding irreducible representation, as we will see later.

Given the canonical filling of $T_\lambda$,

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array},$$

we have

$$a_\lambda = e + (1\,2) \qquad \text{and} \qquad b_\lambda = e - (1\,3) - (1\,4) - (3\,4) + (1\,3\,4) + (1\,4\,3)$$

and so we can compute that the Young symmetrizer for this partition is

$$c_\lambda = e - (1\,3) - (1\,4) - (3\,4) + (1\,2) + (1\,3\,4) + (1\,4\,3) - (2\,1\,4) - (1\,2)(3\,4) + (1\,3\,4\,2) + (1\,4\,3\,2)$$

and one can show (c.f. [FH91, p. 48]) that this is the representation $V \wedge V$ where $V$ is the standard representation (the complement of copy of the trivial representation spanned by the vector $(1,1,1,1) \in \mathbb{C}^4$ under the usual embedding of $\mathfrak{S}_4$ in $\mathrm{GL}_4$ as permutation matrices).

This completes the description of the representations of $\mathfrak{S}_n$ over $\mathbb{C}$, but in fact everything we have done here holds over the splitting field of $\mathfrak{S}_n$, that is, the minimal field such that representations don't split further under field extension. We haven't proved here that

(a) the $V_\lambda$ are irreducible; or

(b) the $V_\lambda$ are pairwise nonisomorphic,

but one can look up any of the standard texts (including the ones cited in this section) for more rigorous and thorough treatments of these facts.

## 2.2 Polynomial representations of $\Gamma$

Let $k$ be an infinite field[3] and $\Gamma$ be the affine group scheme $\mathrm{GL}_n$. This can be thought of as the functor

$$\Gamma : \mathbf{Alg}_k \to \mathbf{Grp} \quad \text{sending} \quad A \mapsto \mathrm{GL}_n(A).$$

Then

**2.2.1 Definition:** A (finite dimensional) **representation** of $\Gamma$ is a (finite dimensional) vector space $V$ along with a group scheme homomorphism

$$\rho : \Gamma \to \mathrm{GL}(V) \overset{\text{def}}{=} \mathrm{Aut}(V \otimes_k -)$$

[3] In some cases we will be able to allow $k$ to be a ring, but we will still need that $k$ be infinite so that polynomials over it are determined by their values.

2.2.2 Remark: Representations of (the group, which can be thought of as the $k$ points of the $k$-scheme) $\mathrm{GL}_n(k)$ can be, in general, "analytic." One can check that the map

$$\rho : k^\times = \mathrm{GL}_1(k) \to \mathrm{GL}(k^2) \qquad \text{via} \qquad x \mapsto \begin{pmatrix} 1 & \ln|x| \\ 0 & 1 \end{pmatrix}$$

gives a group homomorphism (and thus representation) between these two groups, but the logarithm makes this representation decidedly *not algebraic.*

This leads to slightly more awkward definitions in more classical treatments of the theory (e.g. [Gre07]), where one has to specifically rule these out. The upshot to using a more algebro-geometric approach is that we start off in the world of rational maps where such a representation doesn't make sense.

Recall that the affine group scheme $\mathrm{GL}_n$ is represented by the algebra

$$k[x_{ij}]_{\det}$$

where $1 \le i, j \le n$ and det is the polynomial corresponding to the determinant of the matrix $A = (x_{ij})$. Since $\mathrm{GL}_n$ is an affine scheme, we know that the global functions are

$$k[\Gamma] = k^\Gamma \cong k[x_{ij}]_{\det}$$

where we will (for clarity) use the notation $c_{ij} : \Gamma \to k$ to denote the function corresponding to $x_{ij}$.

**2.2.3 Definition:** A **polynomial representation** of $\Gamma$ is a representation $\rho : \Gamma \to \mathrm{GL}(V)$ (where $\dim_k V = m$) such that (on points) the structure maps (2.2.4) of

$$\rho_A : \Gamma(A) \to \mathrm{GL}(V)(A) \cong \mathrm{GL}_m(A)$$

are polynomials in the functions $c_{ij} : \Gamma(A) \to A$ that extract the $(i,j)^{th}$ entry.

If all the structure maps are homogeneous of degree $r$ for some fixed $r$, we say that $\gamma$ is a **homogeneous degree $r$ polynomial representation of $\Gamma$.**

2.2.4 Remark: Recall (or learn for the first time!) that the *structure maps* of a representation $(\rho, V)$ are a collection of maps $r_{ij}$ for $1 \le i, j, \le n$ from $\Gamma$ to $k$ such that for all $g \in \Gamma$:

$$g \cdot v_i = \sum_{j=1}^{n} r_{ij}(g) v_j$$

where we have picked a basis $\{v_1, \dots, v_n\}$ for $V$. Of course changing basis may change our $r_{ij}$, but their **span** $\langle r_{ij} \rangle$ is an invariant of the representation.

**2.2.5 Definition:** Let $\mathrm{Pol}_k(n) = \mathrm{Pol}(n)$ be the collection of all polynomial representations of $\mathrm{GL}_n$ and let $\mathrm{Pol}_k(n,r) = \mathrm{Pol}(n,r)$ be the collection of all homogeneous degree $r$ polynomial representations of $\mathrm{GL}_n$.

It is the *polynomial* representations that we will concern ourselves with in the following sections.

### 2.2.1 Reducing scope

In what follows we (temporarily) restrict to the case of considering the $R$-points of the scheme, where $R \in \mathbf{Alg}_k$. Using some of our familiar friends from representation theory (as well as some clever twists), we can simplify this picture considerably by proving the following structural result:

**2.2.6 Theorem ([Sch01, pp.7-10])**
Every polynomial representation $V$ of the group $\mathrm{GL}_n(R)$ (where $R$ is an algebra over an infinite field $k$) decomposes as a direct sum

$$V \cong \bigoplus_{\delta \in \mathbb{N}} V_\delta$$

where $V_\delta$ is a *homogeneous* polynomial representation of degree $\delta$.

Clearly, then, it suffices to understand the *homogeneous degree $r$* polynomial representations of $\Gamma(A)$ if we are looking to understand the larger structure.

We begin with a useful lemma extracted from a proof in [Sch01] echoing the general theory of orthogonal decomposition of Artinian algebras.

**2.2.7 Lemma**
Let $C_0, \ldots, C_m \in M_n(R)$ be mutually orthogonal idempotent matrices that sum to the identity. That is,

$$I_n = \sum_i C_i \quad \text{and} \quad C_i C_j = \delta_{ij} C_i$$

for all $0 \le i, j \le m$. Then there exists an invertible matrix $P$ such that for some positive integers $d_0, \ldots, d_m$ with $\sum_k d_k = n$ and for all $i$,

$$P^{-1} C_i P = \begin{pmatrix} 0_{N_i} & & \\ & I_{d_i} & \\ & & 0_{M_i} \end{pmatrix}$$

> Where $N_i = \sum_{0\le j<i} d_j$ and $M_i = n - d_i - N_i$

Proof (of lem 2.2.7)
We set $S_k = \{C_0, C_1, \ldots, C_k\}$ and we proceed by induction on $k$. When $k=0$, $S_k = \{C_0\}$. Now since $C_0^2 = C_0$, we get that 1 and 0 are the only eigenvalues of $C_0$, so there is an $r\times r$ matrix $P_0$ and a positive integer $d_0$ such that

$$P_0^{-1} C_0 P_0 = \begin{pmatrix} I_{d_0} & \\ & 0_{n-d_0} \end{pmatrix}.$$

which establishes the base case.

Now assume that we have a matrix $P_{k-1}$ such that this property holds for all elements of $S_{k-1}$. Define, for each $0 \le i \le k$,

$$C_i' \stackrel{\text{def}}{=} P_{k-1}^{-1} C_i P_{k-1}$$

and since the $C_k$ is assumed to be orthogonal to all other $C_i$,

$$C_k' = \begin{pmatrix} 0_{N_k} & \\ & D_k \end{pmatrix}$$

for some $D_k$.

Now by properties of block diagonal matrices, we have

$$D_k^2 = D_k$$

so the eigenvalues of $D_k$ are again one and zero. Thus there is an invertible $Q \in \mathrm{GL}_{n-N_k}$ such that

$$Q^{-1} D_k Q = \begin{pmatrix} I_{d_k} & \\ & 0_{M_k} \end{pmatrix}$$

and so by setting

$$P_k \stackrel{\text{def}}{=} P_{k-1} \begin{pmatrix} I_{N_k} & \\ & Q \end{pmatrix}$$

we can define

$$C_i'' \stackrel{\text{def}}{=} P_k^{-1} C_i P_k = \begin{pmatrix} I_{N_k} & \\ & Q \end{pmatrix}^{-1} C' \begin{pmatrix} I_{N_k} & \\ & Q \end{pmatrix}$$

for $0 \le i \le k$, we see immediately that $C_i' = C_i''$ for $0 \le i < k$ and furthermore

$$C_k'' = \begin{pmatrix} 0_{N_k} & \\ & Q^{-1} D_k Q \end{pmatrix} = \begin{pmatrix} 0_{N_k} & & \\ & I_{d_k} & \\ & & 0_{M_k} \end{pmatrix}$$

completing the inductive step. This this result holds for all $S_i$ and in particular for $S_m$, so the result is proven. ♠

As well as another result on a special class of commuting block diagonal matrices:

**2.2.8 Lemma**
Let $R \in \mathbf{Alg}_k$ ($k$ be an infinite field) and let $A$ be a block diagonal matrix over $k$ of the form

$$A = \operatorname{diag}(x^m I_{d_m}, x^{m-1} I_{d_{m-1}}, \dots, I_{d_0})$$

where $d_i$ is (clearly) the dimension of the $(m-i)^{th}$ block and let $B$ be any matrix that commutes with $A$ for every choice of $x \in k$. Then $B$ is block diagonal of the same shape as $A$.

Proof (of lem 2.2.8)
We proceed by comparing the entries in $AB$ and $BA$: notice that

$$(AB)_{ij} = \sum_k A_{ik} B_{kj} = A_{ii} B_{ij} = x^a B_{ij}$$

and

$$(BA)_{ij} = \sum_k B_{ik} A_{kj} = B_{ij} A_{jj} = x^b B_{ij}.$$

We will show that if the $(i,j)^{th}$ postion is not in one of the blocks of $A$, then it is zero.

But if $(i,j)$ is not in one of the blocks of $A$, then the nonzero element in the $i^{th}$ row and the nonzero element in the $j^{th}$ column ($x^a$ and $x^b$ in the above equations) are not the same! Since $x$ is arbitrary, this forces $B_{ij} = 0$, so $B$ is block diagonal with blocks the same as $A$. ♠

2.2.9 Remark: Notice that in the above proof we used implicitly that there is an $x \in R$ such that for all $a, b$

$$x^a = x^b \quad \Rightarrow \quad a = b$$

which is true since $k$ is infinite. This can cause a problem for finite fields since, for instance, every element in $\mathbb{F}_p$ satisfies $x^p = x$.

And finally using these two lemmas allows us to prove our main result:

Proof (of thm 2.2.6)
We recreate the argument in Schur's thesis, translated from German and reinterpreted in more modern parlance.

Let $(\rho, V)$ be a polynomial representation of $\mathrm{GL}_n(R)$ with $\dim_k V = r$. Then let $x \in R^\times$ be arbitrary (thought of as an indeterminate) and consider the matrix $xI_n \in \Gamma$. The image of this matrix under $\rho$ is a matrix

$$\rho(A) = \begin{pmatrix} p_{11}(x) & \cdots & p_{1r}(x) \\ \vdots & \ddots & \vdots \\ p_{r1}(x) & \cdots & p_{rr}(x) \end{pmatrix}$$

where each $p_{ij}$ is a polynomial in $x$. Let $m = \max_{i,j} \deg p_{ij}$, and this gives us a decomposition

$$\rho(A) = x^m C_0 + x^{m-1} C_1 + \cdots + x C_{m-1} + C_m$$

where each $C_i$ is an $r \times r$ matrix.

Let $y$ be another indeterminate and $B = yI_n$. By virtue of being a representation of $\mathrm{GL}_n(R)$, we get

$$\rho(A)\rho(B) = \rho(xI_n)\rho(yI_n) = \rho(xyI_n) = \rho(AB)$$

and using this setup we prove the following result:

$$\text{For all } 0 \leq i, j \leq m, \text{ with the } C_l \text{ as above,} \quad C_i C_j = \delta_{ij} C_i$$

That this is true can be established by comparing coefficients in the equation

$$\rho(AB) = \rho(A)\rho(B)$$
$$C_0(xy)^m + \cdots + C_i(xy)^{m-i} + \cdots + C_m = C_0^2 x^m y^m + \cdots + C_i C_j x^{m-i} y^{m-j} + \cdots + C_m^2$$

Indeed, we immediately get that $C_i = C_i^2$ and furthermore the coefficients on $x^i y^j$ when $i \neq j$ give us

$$0 = C_{m-i} C_{m-j}.$$

Thus we have shown that the $C_i$ form a set of orthogonal idempotent matrices and evaluating our original equation at $x = 1$, we get (since $\rho$ is a homomorphism)

$$I_r = 1C_0 + \cdots + 1C_m = \sum C_i$$

so the result from lemma 2.2.7 applies: we get a matrix $P$ such that

$$P^{-1}\rho(xI_n)P = \begin{pmatrix} x^m I_{d_0} & & & & \\ & x^{m-1} I_{d_1} & & & \\ & & \ddots & & \\ & & & x I_{d_{m-1}} & \\ & & & & I_{d_m} \end{pmatrix}$$

Now let $\rho'(g) = P^{-1}\rho(g)P$ for all $g \in \mathrm{GL}_n(R)$. This is a representation of $\Gamma$ since it it differs from $\rho$ by an automorphism of $\mathrm{GL}(V)$. Since matrix multiplication is an algebraic operation, $\rho'$ is still a polynomial representation of $\mathrm{GL}_n(R)$. But notice that for all $g \in \mathrm{GL}_n(R)$

$$\rho'(g)\rho'(xI_n) = \rho'(xg) = \rho'(xI_n)\rho'(g)$$

Then lemma 2.2.7 gives us that $\rho'(g)$ decomposes in the same way for all $g \in \mathrm{GL}_n(R)$, so we know that $\rho'$ decomposes as a direct sum of representations

$$\rho' = \sum_{i=0}^{m} \rho'_i$$

where for each $i$ and $\lambda \in k$,

$$\rho'_i(\lambda g) = \rho'_i(\lambda I_{d_i})\rho'_i(g) = \lambda^i \rho'_i(g)$$

so each $\rho'_i$ is a homogeneous degree $i$ polynomial representation of $\Gamma$.

But of course the decomposition of a representation is independent of choice of basis, so we get a decomposition of $\rho$ into homogenous pieces, as desired. ♠

This is wonderful if one is just interested in the representation theory of $\mathrm{GL}_n(A)$ for a specific $A$, but here we are interested in group *scheme* representations. The result above effectively tells us how the scheme splits up on points, but how about the global structure?

What we need is that this splitting is functorial. That is, if $\rho : \Gamma \to \mathrm{GL}(V)$ is a polynomial representation, there is a subscheme $\rho_r$ that, for all $A$, $(\rho_r)_A$ is the homogeneous degree $r$ part of $\rho_A$. Let $\rho_r$ be such a map and let $\varphi : A \to B$ be an algebra morphism. Then we want that this induces a map

$$\hat{\varphi} : (\rho_r)_A \to (\rho_r)_B.$$

Fix a basis $v_1, \ldots, v_n$ for $V$ and let $g \in \Gamma(A)$. Then since $(\rho_r)_A$ is homogeneous degree $r$, for all $v_i$,

$$g \cdot_A v_i = \sum_j f_{ij}(g) v_i$$

where the $f_{ij}$ is a homogeneous degree $r$ polynomial in the $c_{ij}$. Now if we write $(\rho_r)_A(g) = M_{g,A} = (m_{ij})_{i,j}$, this means that if $\lambda \in k$,

$$M_{\lambda g,A} = (\lambda^r m_{ij})_{i,j} = \lambda^r M_{g,A}$$

The map $\hat{\varphi}$ is such that

$$\hat{\varphi}(M_{g,A}) = (\varphi(m_{ij}))_{i,j} = M_{g,B}$$

and so

$$M_{\lambda g,B} = \hat{\varphi}(M_{\lambda g,A}) = (\varphi(\lambda^r m_{ij}))_{i,j} = (\lambda^r \varphi(m_{ij}))_{i,j} = \lambda^r M_{g,B}$$

which tells us that the map $\rho_r$ is indeed a functor $\Gamma \to \mathrm{GL}(V)$, so is a subgroup scheme of $\rho$. Thus the pointwise splitting shown above lifts to a splitting of the entire representation $\rho$.

### 2.2.2 Monomials and multi-indices

All of the discussion up to this point has revolved around polynomials in $n^2$ variables, which quickly gets unwieldy unless one uses some better notation. To that end,

**2.2.10 Definition:** An $(n,r)$-**multi-index** $i$ is an $r$-tuple $(i_1, \ldots, i_r)$ where each $i_j \in \underline{n} \overset{\text{def}}{=} \{1, \ldots, n\}$. The collection of all $(n,r)$-multi-indices is denoted $I(n,r)$.

2.2.11 Remark: One can also think of an element $i \in I(n,r)$ as a (set) map

$$i : \underline{r} \to \underline{n}.$$

The idea here is to associate to each monomial in a polynomial ring in many variables a tuple indicating its multidegree. That is we think of

$$(i_1, \ldots, i_r) \quad \leftrightsquigarrow \quad x_{i_1} \cdots x_{i_r}$$

as corresponding to the same object. Which is wonderful except for one small flaw: polynomials are commutative and multi-indices (as we have defined them) aren't! For example, in $I(3,4)$,

$$(2,2,1,3) \quad \leftrightsquigarrow \quad x_1x_2^2x_3 \quad \leftrightsquigarrow \quad (3,2,1,2).$$

To handle this disparity, we define an equivalence relation on $I(n,r)$ where we say that $i \sim j$ if they are in the same orbit under the natural $\mathfrak{S}_r$ action. That is, if there exists $\sigma \in \mathfrak{S}_r$ such that

$$(i_1,\dots,i_r)=(j_{\sigma(1)},\dots,j_{\sigma(r)})$$

In the context of polynomial representations of $\Gamma$, we want to consider polynomials in the coordinate functions $c_{ij}$, so as a matter of notation if $i,j \in I(n,r)$, let $c_{i,j}$ denote the monomial

$$c_{i,j}=c_{i_1j_1}\cdots c_{i_rj_r}.$$

Again, we want to take into account that we can permute the order on the right hand side, but now we need that $i_k$ and $j_k$ remain linked to the same function. To deal with this, we define an equivalence relation $\sim$ on $I(n,r)\times I(n,r)$ such that

$$(a,c)\sim(b,d)$$

if there exists a $\sigma \in \mathfrak{S}_r$ such that

$$(a_1,\dots,a_r)=(b_{\sigma(1)},\dots,b_{\sigma(r)}) \quad \text{and} \quad (c_1,\dots,c_r)=(d_{\sigma(1)},\dots,d_{\sigma(r)}).$$

The upshot of this work is that it gives us a bijection between (total) degree $r$ monomials in the $c_{ij}$ and the set

$$I(n,r)\times I(n,r)/\sim$$

### 2.2.3 $A_k(n,r)$

Notice that if $V \in M(n,r)$, each of its structure maps are homogeneous degree $r$ polynomials. As the first object of study, consider

**2.2.12 Definition:** Let $A_k(n,r)=A(n.r)$ denote the collection of all homogeneous degree $r$ polynomials in the coordinate functions $c_{ij}:\Gamma\to k$.

It is not too hard to see that

> **2.2.13 Proposition**
> $A_k(n,r)$ is spanned by the elements
>
> $$\{c_{i,j} | (i,j)\in I(n,r)\times I(n,r)\}$$

however it takes a short argument to see

**2.2.14 Lemma**
The dimension of $A_k(n,r)$ over $k$ is $\binom{n^2+r-1}{n^2-1}=\binom{n^2+r-1}{r}$.

Proof
The following is a "stars and bars" argument that is pervasive in combinatorics. See for example [Sta12] if unfamiliar with these techniques.

Fix an ordering of the $c_{ij}$ (say the dictionary order) and relabel them $\{\gamma_1,\dots,\gamma_m\}$ (here $m=n^2$) according to this order. Then the degree $r$ monomials are in bijection with $m$-tuples $(a_1,\dots,a_m)\in\mathbb{N}^m$ such that $\sum_i a_i=r$ via the map which sends

$$(a_1,\dots,a_m)\mapsto \gamma_1^{a_1}\cdots\gamma_m^{a_m}.$$

But choosing such an element is the same as inserting $m-1$ bars into a line of $r$ stars (that is an ordered partition of $r$ into $m$ parts, where parts are allowed to be zero). But this is equivalent to choosing $m-1$ bars in a field of $m+r-1$ symbols. This is just

$$\binom{m+r-1}{m-1}$$

and a well-known identity for binomial coefficients gets us the final equality. ♠

**Example 2.2**
In case the reader is unfamiliar with this kind of reasoning, consider the case when $n=5$ and $r=4$. Then the composition $(1,0,0,2,1)$ corresponding to $\gamma_1\gamma_4^2\gamma_5$ corresponds to the stars-and-bars diagram

$$*|||**|*$$

where there are $m+r-1=8$ symbols, $r=4$ of which are stars.

### 2.2.4 Hopf algebras and group schemes

$A(n,r)$ lies within $k^\Gamma=k[\Gamma]$, which has the structure of a Hopf algebra induced from the group structure on $\Gamma$. More precisely, the functor $\Gamma:\mathbf{Alg}_k\to\mathbf{Grp}$ that assigns to every $k$-algebra $A$ the group $\mathrm{GL}_n(A)$ is representable. In other words,

$$\mathrm{GL}_n(-)\simeq\mathrm{Hom}_{\mathbf{Alg}_k}(R,-)$$

where $R=k[\Gamma]$.

The anti-equivalence of the categories of affine group schemes over $k$ and finite dimensional commutative $k$-Hopf algebras (of which this is a particular instance) follows from Yoneda lemma (c.f. [Wat79, chp. 1]). The resulting Hopf algebra will be (as an algebra) $R$, and along with a coalgebra structure induced the group structure on $\Gamma$: we have maps $\mu, \epsilon$, the multiplication and unit maps on $\Gamma$ satisfying the diagrams

$$\begin{tikzcd} \Gamma\times\Gamma\times\Gamma \arrow[r, "\mu\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\mu"] & \Gamma\times\Gamma \arrow[d, "\mu"] \\ \Gamma\times\Gamma \arrow[r, "\mu"] & \Gamma \end{tikzcd} \qquad \begin{tikzcd} *\times G \arrow[r, "\epsilon\times\mathrm{id}"] \arrow[dr, leftrightarrow, "\sim"] & G\times G \arrow[d, "\mu"] & G\times * \arrow[l, "\mathrm{id}\times\epsilon"'] \arrow[dl, leftrightarrow, "\sim"'] \\ & G & \end{tikzcd}$$

(where $*$ is the trivial group and initial object in the category of group schemes) giving us associativity and identity. Yoneda tells us that the maps between schemes

$$\mu : \Gamma\times\Gamma \to \Gamma \quad \text{and} \quad \epsilon : * \to \Gamma$$

give rise to maps in $\mathbf{Alg}_k$:

$$\Delta \stackrel{\text{def}}{=} \mu^* : R \to R\otimes_k R \quad \text{and} \quad \varepsilon \stackrel{\text{def}}{=} \epsilon^* : R \to k$$

satisfying diagrams

$$\begin{tikzcd} R\otimes R\otimes R & R\otimes R \arrow[l, "\Delta\otimes\mathrm{id}"'] \\ R\otimes R \arrow[u, "\mathrm{id}\otimes\Delta"] & R \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} k\otimes R & R\otimes R \arrow[l, "\varepsilon\otimes\mathrm{id}"'] \arrow[r, "\mathrm{id}\otimes\varepsilon"] & R\otimes k \\ & R \arrow[u, "\Delta"] \arrow[ul, leftrightarrow, "\sim"] \arrow[ur, leftrightarrow, "\sim"'] & \end{tikzcd}$$

**2.2.15 Proposition**
The maps $\Delta$ and $\varepsilon$ which, in coordinates, for $1 \le i, j \le n$, are

$$\Delta(c_{ij}) = \sum_k c_{ik}\otimes c_{kj} \quad \text{and} \quad \varepsilon(c_{ij}) = \delta_{ij}$$

give a coalgebra structure on $R$

That these maps satisfy the diagrams above is a straightforward computation. That, furthermore, these maps make $R$ into a bialgebra amounts to checking that $\Delta$ and $\varepsilon$ are algebra morphisms. But what is not immediately obvious is why *these particular maps* are the ones we use on $R$. To see this, one must dig into the Yoneda correspondence a bit to see what happens to the multiplication and unit morphisms.

In service of this, let's translate matrix multiplication into a statement about representable functors. We want to define $m$ as a map

$$m : \mathrm{Hom}(R, -)\times\mathrm{Hom}(R, -) \to \mathrm{Hom}(R, -)$$

and to see what $m$ should do in this context, we evaluate at a $k$-algebra

$$m_A : \mathrm{Hom}(R,A) \times \mathrm{Hom}(R,A) \to \mathrm{Hom}(R,A)$$

where we interpret each map $f : R \to A$ as a matrix with entries in $A$ by saying $f$ corresponds to a matrix $A_f$ such that

$$(A_f)_{ij} = f(c_{ij}).$$

Then if $(f,g) \in \mathrm{Hom}(R,A) \times \mathrm{Hom}(R,A)$, we want that the algebra structure is the usual matrix multiplication, so

$$m_A(f,g) = A_f A_g$$

and by computing the $(i,j)^{th}$ entry everywhere, we get

$$m_A(f,g)(c_{ij}) = (A_f A_g)_{ij} = \sum_{k=1}^{n} (A_f)_{ik}(A_g)_{kj} = \sum_k f(c_{ik})g(c_{kj}).$$

This gives us the values of our component maps everywhere, so this defines the natural transformation $m$. Then (the proof of) Yoneda tells us that we can compute the corresponding algebra morphism as

$$\mu(c_{ij}) = m_{R\otimes R}(\iota_l \otimes \iota_r)(c_{ij}) = \sum_k \iota_l(c_{ik})\iota_r(c_{kj}) = \sum_k c_{ij} \otimes c_{kj}.$$

Above we call $\iota_l$ (resp. $\iota_r$) to be the map $R \to R \otimes R$ which embeds $R$ into the left (resp. right) tensor factor. Notice that $\iota_l \otimes \iota_r = \mathrm{id}_{R\otimes R}$.

Using the same identification between maps and matrices over $A$, let $* : k \to A$ be the unique map sending $1_k \mapsto 1_A$. Then we want

$$u_A(*) = f : R \to A$$

corresponding to the identity $(n \times n)$ matrix over $A$. So

$$u_A(*)(c_{ij}) = f(c_{ij}) = (I_n)_{ij} = \delta_{ij} \cdot 1_A.$$

Again applying Yoneda, we have

$$\varepsilon(c_{ij}) = u_k(\mathrm{id}_k)(c_{ij}) = \delta_{ij} 1_k$$

and we have our counit map.

In fact, as mentioned before, $R$ becomes a bialgebra (a Hopf algebra even, although we won't need the antipode here). This means that $\Delta$ and $\varepsilon$ are algebra morphisms for the natural algebra structure given by multiplication $m$ on $R$. In diagrams:

$$\begin{tikzcd} R^{\otimes 4} \arrow[r, "\mathrm{id}\otimes\tau\otimes 1"] & R^{\otimes 4} \arrow[r, "m\otimes m"] & R\otimes R \\ R\otimes R \arrow[u, "\Delta\otimes\Delta"] \arrow[rr, "m"] & & R \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} R\otimes R \arrow[r, "m"] \arrow[d, "\varepsilon\otimes\varepsilon"] & R \arrow[d, "\varepsilon"] \\ k\otimes k \arrow[r, "m"] & k \end{tikzcd}$$

where $\tau : R\otimes R \to R\otimes R$ is the twist map $a\otimes b \mapsto b\otimes a$. Chasing an element through the diagram on the left, we get

$$\widetilde{m}\circ(\Delta\otimes\Delta)(c_{ij}\otimes c_{ab}) = \sum_{1\le k,l\le n} c_{ik}c_{al}\otimes c_{kj}c_{lb} = \Delta(c_{ij}c_{ab})$$

or using our multi-index notation,

$$\Delta(c_{(i,a),(j,b)}) = \sum_{(k,l)\in I(n,2)} c_{(i,a),(k,l)}\otimes c_{(k,l),(j,b)}.$$

Written more simply, the fact that $\Delta$ is an algebra morphism can be written

$$\Delta(a\cdot b) = \Delta(a)*\Delta(b)$$

under suitable definitions of $\cdot$ and $*$. In a way that can be made precise, this means in particular that

$$\Delta(a\cdot b\cdot c) = \Delta(a)*\Delta(b\cdot c) = \Delta(a)*\Delta(b)*\Delta(c)$$

and so on (since multiplication everywhere is associative) and therefore we can define this for arbitrary monomials and extend $k$-linearly:

> **2.2.16 Proposition**
> If $i,j\in A(n,r)$, then
>
> $$\Delta(c_{i,j}) = \sum_{k\in I(n,r)} c_{i,k}\otimes c_{k,j} \quad\text{and}\quad \varepsilon(c_{i.j}) = \delta_{i,j}$$

One can easily see that degree is preserved by $\Delta$, meaning that

> **2.2.17 Proposition**
> $\Delta$ and $\varepsilon$ descend to a coalgebra structure on $A(n,r)$. That is, $A(n,r)$ is a ($k$-)coalgebra.

### 2.2.5 The structure maps of $\rho$

This context empowers us to better understand what is meant by the structure maps of a representation. At the moment, we define a homogeneous polynomial representation by how it looks on points and simply use the fact it coalesces to a functor. A natural question to ask is how the structure morphisms relate to the entries of the matrices $\rho_A(g) = M_{g,A}$.

To understand the answer to this question, we need to uncover how the entries of a matrix come about. We have been thinking of an element of $\Gamma(A)$ as a matrix, but what it actually is is a morphism

$$k[x_{ij}]_{\det}\to A$$

and then thinking of $\operatorname{Aut}(V\otimes A)\cong \mathrm{GL}_m(A)$ in the same way, we interpret a representation as a map

$$\rho : \operatorname{Hom}(k[x_{ij}]_{\det}, -)\to \operatorname{Hom}(k[y_{kl}]_{\det}, -)$$

where $i$ and $j$ run from 1 to $n$ and $k$ and $l$ run from 1 to $m$.

Then Yoneda lemma tells us that $\rho$ corresponds to an algebra map

$$\rho^* : k[y_{lk}]_{\det} \to k[x_{ij}]_{\det}$$

This map is the one such that if $f \in \mathrm{Hom}(k[x_{ij}]_{\det}, A)$,

$$\rho_A(f) = f \circ \rho^* : k[y_{lk}]_{\det} \to k[x_{ij}]_{\det} \to A.$$

What are the structure maps for this representation? We can compute for any $g \in \mathrm{GL}_n(A)$ (which we, though a mild abuse of notation, think of as a map $g : k[x_{ij}]_{\det} \to A$ where $g(x_{ij}) = g_{ij}$)

$$\rho_A(g) \cdot v_i = \rho_A(g(x_{ij}))_{i,j} v_i = ((g \circ \rho^*)(x_{ij}))_{i,j} v_i = \sum_j g(\rho^*(x_{ij}))v_j$$

from which we can see

**2.2.18 Lemma**
Let $\rho : \Gamma \to \mathrm{GL}(V)$ be a polynomial representation. Then for any $A \in \mathbf{Alg}_k$, the structure maps $f_{ij}$ of the group representation $\rho_A : \mathrm{GL}_n(A) \to \mathrm{GL}_m(A)$ are completely determined by $\rho^*$.

Since any $g : k[x_{ij}]_{\det} \to A$ defining a matrix is a linear map, this implies that the homogeneous degree of the structure morphisms is also due completely to $\rho^*$. This enables us to re-define polynomial representations in the following way (compare with definition 2.2.3):

**2.2.19 Definition:** A finite dimensional **polynomial representation of $\Gamma$ of degree $r$** is a finite dimensional vector space $V$ over $k$ along with a scheme map $\rho : \Gamma \to \mathrm{GL}(V)$ such that the associated algebra map $\rho^* : k[\mathrm{GL}(V)] \to k[\Gamma] \cong k[x_{ij}]_{\det}$ is homogeneous degree $r$. In other words, the image of $\rho^*$ is entirely contained within the degree $r$ graded piece of $k[\Gamma]$.

2.2.20 REMARK: In the following section, we freely identify $k[x_{ij}]_{\det}$ with the ring of functions on $\Gamma$, and $k[y_{lk}]_{\det}$ with the ring of functions on $\mathrm{GL}(V)$ (where $m = \dim V$).

### 2.2.6 Comodules

In this section let $A = A(n, r)$, which we have just established is a coalgebra with $\Delta$ and $\varepsilon$ defined above.

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "\phi"] & A\otimes V \arrow[d, "\mathrm{id}\otimes\phi"] \\ A\otimes V \arrow[r, "\Delta\otimes\mathrm{id}"'] & A\otimes A\otimes V \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\phi"] \arrow[dr, "\sim"'] & A\otimes V \arrow[d, "\varepsilon\otimes\mathrm{id}"] \\ & k\otimes V \end{tikzcd}$$

Figure 1: The coassociative and counital axioms

**2.2.21 Definition:** A (left) $A$-**comodule** is a vector space $V$ over $k$ along with a (left) $A$-**coaction** given by a ($k$-)morphism

$$\phi : V \to A \otimes_k V$$

that is both **coassociative and counital** in the sense that the diagrams in Figure 1 commute.

Given two $A$-comodules $V$ and $W$, a comodule morphism $\varphi : V \to W$ is one that preserves the coaction. That is

$$(\mathrm{id}\otimes\varphi)\phi_V(v) = \phi_W(\varphi(v))$$

for all $v \in V$.

**2.2.22 Definition:** Let $A$ be any coalgebra. Then $A$-**comod** denote the category of (finite-dimensional, left) $A$-comodules along with comodule morphisms.

Sometimes a more useful way to think of polynomial representations is as comodules. That idea is made more formal in the following theorem:

**2.2.23 Theorem**
Every homogeneous degree $r$ polynomial representation of $\mathrm{GL}_n$ gives rise to an $A(n,r)$-comodule in the following way: the underlying vector space is the same and the $A(n,r)$ coaction is given by

$$\phi(v_i) = \sum_j \rho^*(y_{ij}) \otimes v_j$$

where in the above we identify the algebra $k[y_{ij}]_{\det}$ with the ring of functions on $\mathrm{GL}(V)$.

Proof
By definition this gives us a map into $k[\Gamma]$-**comod**, but we can see that the image is entirely contained within $A(n,r)$-**comod** since, in light of definition 2.2.19, $\rho^*(y_{ij})$ is homogeneous degree $r$. Then it remains to show that the given map is legitimately a coaction. We can compute

(identifing the map $\varepsilon : k[\Gamma] \to k$ as the matrix $I_n$ over $k$)

$$(\varepsilon \otimes \mathrm{id}) \circ \phi(v_i) = \sum_j \varepsilon(\rho^*(y_{ij})) \otimes v_j = \sum_j (\varepsilon \circ \rho^*)(y_{ij}) \otimes v_j = \sum_j \rho_k(I_n)_{ij} \otimes v_j = 1_k \otimes v_i$$

so $\phi$ satisfies the counit identity. For coassociativity, identify the morphism $k[\Gamma] \to k[\Gamma] \otimes k[\Gamma]$ with the matrix $D$ whose entries are $\Delta(x_{ij})$. Then

$$\begin{aligned}(\Delta \otimes \mathrm{id}) \circ \phi(v_i) &= \sum_j \Delta(\rho^*(y_{ij})) \otimes v_j \\ &= \sum_j \rho(D)_{ij} \otimes v_j \\ &= \sum_j (\text{MAGIC}) \otimes v_j \\ &= \sum_j \left( \sum_k \rho^*(y_{ik}) \otimes \rho^*(y_{kj}) \right) \otimes v_j \\ &= \sum_k \rho^*(y_{ik}) \otimes \left( \sum_j \rho^*(y_{kj}) \otimes v_j \right) \\ &= (\mathrm{id} \otimes \phi) \sum_k \rho^*(y_{ik}) \otimes v_k \\ &= (\mathrm{id} \otimes \phi) \circ \phi(v_i)\end{aligned}$$

which shows that $\phi$ gives a $A(n,r)$-comodule structure on the underlying vector space of a representation $\rho$ of $\Gamma$. ♠

Still need to show that this gives an equivalence of categories. Also be sure to change all instances below of $M(n,r)$ to $\mathrm{Pol}(n,r)$.

### 2.2.7 The Schur algebra

Finally we get to the actual object of study:

**2.2.24 Definition:** A **Schur algebra** is an element of the two-parameter family $\{S(n,r)\} = \{S_k(n,r)\}$ where $n$ and $r$ are any positive integers. As a set, $S(n,r)$ is the linear dual of $A(n,r)$:

$$S(n,r) = A(n,r)^* = \mathrm{Hom}_k(A(n,r), k)$$

Let $\xi_{i,j}$ denote the element dual to $c_{i.j} \in A(n,r)$. In other words:

$$\xi_{(a,b)}(c_{i,j}) = \begin{cases} 1, & (a,b) \sim (i,j) \\ 0, & \text{otherwise} \end{cases}$$

**2.2.25 Lemma**
The coalgebra structure $(\Delta, \varepsilon)$ on $A(n,r)$ defines an algebra structure on $S(n,r)$.

Proof
Since $k$ is an initial object in $\mathbf{Alg}_k$, there is a unique map $u : k \hookrightarrow S(n,r)$ sending 1 to the unit function $\mathbb{1}$, which is given by

$$\mathbb{1}(c_{i,j}) = c_{i,j}(I_n) = \delta_{i,j}$$

Define multiplication $(\cdot)$ in $S(n,r)$ as follows: if $f, g \in S(n,r)$ then for any $x \in A(n,r)$ define

$$(f \cdot g)(x) = m_k \circ (f \otimes g) \circ \Delta(x) = \sum f(x_{(1)}) g(x_{(2)})$$

where $m_k : k \otimes k \to k$ denotes multiplication in $k$ and $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$ in Sweedler notation.

Then we must just confirm that these maps satisfy the properties of a $k$-algebra. $(\cdot)$ is $k$-bilinear because (for instance)

$$\begin{aligned}((af + bg) \cdot h)(x) &= \sum (af + bg)(x_{(1)}) \otimes h(x_{(2)}) \\ &= \sum a(f(x_{(1)}) \otimes h(x_{(2)})) + b(g(x_{(1)}) \otimes h(x_{(2)})) \\ &= a \sum f(x_{(1)}) \otimes h(x_{(2)}) + b \sum g(x_{(1)}) \otimes h(x_{(2)}) \\ &= (a(f \cdot h) + b(g \cdot h))(x).\end{aligned}$$

By $k$-linearity, it suffices to show that the unit $\mathbb{1}$ acts as it should on the spanning set $\xi_{i,j}$ for a basis element $c_{a,b}$:

$$(\mathbb{1} \cdot \xi_{i,j})(c_{a,b}) = \sum_{k=1}^{n} \mathbb{1}(c_{a,k}) \cdot \xi_{i,j} = \mathbb{1}(c_{a,a}) \cdot \xi_{i,j}(c_{a,b}) = \xi_{i,j}(c_{a,b})$$

and a similar identity holds on the right.

Then it remains to show that this multiplication is associative. Again by linearity it suffices to check that this works on the spanning set $\{c_{i,j}\}$:

$$\begin{aligned}((\alpha \cdot \beta) \cdot \gamma)(c_{i,j}) &= \sum_{k \in I(n,r)} (\alpha \cdot \beta)(c_{i,k}) \gamma(c_{k,j}) \\ &= \sum_k \left( \sum_{l \in I(n,r)} \alpha(c_{i,l}) \beta(c_{l,k}) \right) \gamma(c_{k,j}) \\ &= \sum_l \alpha(c_{i,l}) \left( \sum_k \beta(c_{l,k}) \gamma(c_{k,j}) \right) \\ &= \sum_l \alpha(c_{i,l}) (\beta \cdot \gamma)(c_{l,j}) \\ &= (\alpha \cdot (\beta \cdot \gamma))(c_{i,j}).\end{aligned}$$

Thus since we have $k$-linear maps $\mathbb{1}$ and $m = (\cdot)$ satisfying the usual identity and associativity diagrams, $S(n,k)$ is a $k$-algebra with $\mathbb{1}$ and $m$ as its unit and multiplication. ♠

Why have we done all this work to construct Schur algebras, one may ask? Well the idea is that there is a map

$$e : k\Gamma \to S(n,r)$$

where it sends

$$\sum_i k_i g_i \mapsto \sum_i k_i e_{g_i}$$

where $e_g$ is the "evaluation at $g$" map–that is, for all $x \in A(n,r)$,

$$e_g(x) = x(g).$$

> **2.2.26 Lemma**
> The map $e : k\Gamma \to S(n,r)$ is surjective.

PROOF
Let $\xi$ be any element orthogonal to $W = \operatorname{Im} e$, if one exists. Say this element is $\sum_{i,j} a_{i,j}\xi_{i,j}$. But then the element $c = \sum a_{i,j} c_{i,j}$ is zero on every element in the image of $e$. In other words, for all $\varkappa \in k\Gamma$,

$$c(\varkappa) = e_\varkappa(c) = 0.$$

But the only function in $k^\Gamma$ that is zero on all of $\Gamma$ is the zero function. Thus $c = 0$, whence its coefficients $a_{i,j}$ are zero. So $\xi$ is zero, so $W = S(n,r)$ ♠

In this way, $e$ induces a map between categories

$$f : S_k(n,r)\text{-}\mathbf{mod} \to M_k(n,r)$$

where $f(V) = V$ as far as underlying sets are concerned, but we are given a new action: for any $\sum k_i g_i \in k\Gamma$ and $c \in V \in S(n,r)$-**mod**,

$$\left(\sum k_i g_i\right) \cdot v = \left(\sum k_i e_{g_i}\right) \cdot v.$$

Notice that

> **2.2.27 Lemma**
> $f$ as above defines a functor that sends any $\alpha : V \to W \in S(n,r)$-**mod** to the map that is identical as a map of sets.

PROOF
Since we are working in a concrete category[4], and since the $V$ and $f(V)$ are identical as sets and since any morphism is sent to the same map on underlying sets, commutativity of the functorality diagram

[4] There is a faithful functor to **Set**; equivalently, any morphism is determined by where it sends the "set of elements" comprising the object.

$$\begin{array}{ccc} V & \xrightarrow{\quad\alpha\quad} & W \\ \Big\downarrow f & & \Big\downarrow f \\ f(V)=V & \xrightarrow{f(\alpha)=\alpha} & f(W)=W \end{array}$$

is trivial to check. It remains only to check that $f(\alpha): V \to W$ is a morphism in $M(n,r)$, so that this diagram makes sense.

To check this, notice that for any $v \in V \in M(n,r)$ and $g \in k\Gamma$,

$$f(\alpha)(g\cdot v) = \alpha(e_g \cdot v) = e_g \cdot \alpha(v) = g \cdot f(\alpha)(v)$$

so $f$ is a functor. ♠

The upshot here is

**2.2.28 Theorem**
The functor $f$ above is an equivalence of categories.

Proof
That $f$ is faithful is easy enough to see since $f$ is the identity functor on the level of underlying sets. Let $g : V \to W$ be a morphism in $M(n,r)$ and consider the same (set) map in $\widetilde{g} \in S(n,r)$-**mod**. For any $\xi \in S(n,r)$, let $x \in k\Gamma$ be an element such that $e_x = \xi$ (which exists due to lem 2.2.26). Then

$$\widetilde{g}(\xi\cdot v) = g(x\cdot v) = x \cdot g(v) = \xi \cdot \widetilde{g}(v)$$

But then $f(\widetilde{g}) = g$, so $f$ is full.

It remains to see that $f$ is essentially surjective. But again this is not too hard to see since for any $V \in M(n,r)$ the same object setwise with the action given by

$$\xi \cdot v = e_x \cdot v$$

(where again $x$ was chosen using lem 2.2.26) maps to $V \in M(n,r)$ and we are done. ♠

2.2.29 Remark: Actually, the above proof can be modified slightly to show that $f$ has a functorial inverse–that is, $f$ is an *isomorphism of categories*. Since we are only interested in representations up to isomorphism, however, equivalence is just as good.

2.2.30 Remark: Using this equivalence, we identify $S(n,r)$-**mod** with $M(n,r)$ whenever it suits us.

One of a representation theorist's favorite kinds of results follows:

**2.2.31 Corollary**
If $\operatorname{char} k = 0$, the algebra $S(n,r)$ is semisimple.

Proof

$k\Gamma$ is semisimple since $\operatorname{char} k = 0$, so every element in $k\Gamma$-**mod** splits into a direct sum of simple modules. But the irreducible objects in $M(n,r)$ and those in $S(n,r)$-**mod** are the same and decompositons in one category pull back to the other, so every element in $S(n,k)$-**mod** is also completely reducible. ♠

### 2.2.8 Weights and characters

The discussion in section 2.2.2 highlights an important idea: while we care about the *quantities* in which each $c_{ij}$ occurs in a monomial, we are not particularly interested in the *order*. Sometimes it is easier, then, to simply regard these as weak compositions:

**2.2.32 Definition:** Let $n$ and $r$ be integers as usual. Then denote by $[a_1,\dots,a_n]$ the **weight** corresponding to $(i_1,\dots,i_r)\in I(n,r)$ where for each $i$,

$$a_i = \#\{k \in \underline{r} | i_k = i\}$$

Denote by $\Lambda(n,r)$ the collection of all weights.

2.2.33 Remark: Another way to realize $\Lambda(n,r)$ is in the presentation

$$\Lambda(n,r) = \left\{ [a_1,\dots,a_n] \middle| \sum_i a_i = r \right\},$$

or as the set of compositions of $r$ into $n$ parts (allowing zeros).

Yet another is to think of $\Lambda(n,r)$ as the set of $\mathfrak{S}_r$ orbits in $I(n,r)$ (where now two objects are distinguished only if their "contents" vary).

Recall (c.f. 2.2.24) that we had that $\xi_{i,j}(c_{a,b}) = 1$ if and only if $(i,j)\sim(a,b)$. Because of this, it makes sense (if $\alpha$ is the weight of $i$) to write

$$\xi_\alpha \overset{\text{def}}{=} \xi_{\alpha,\alpha} \overset{\text{def}}{=} \xi_{i,i}$$

since the action is the same irrespective of the choice of representative $i$ of $\alpha$.

Notice that the weights admit a $\mathfrak{S}_n$ action

$$\sigma\cdot[a_1,\dots,a_n] = [a_{\sigma(1)},\dots,a_{\sigma(n)}]$$

then

**2.2.34 Definition:** $\Lambda_+(n,r)$ is the orbit space of $\Lambda(n,r)$ under the above $\mathfrak{S}(n)$ action.

2.2.35 Remark: The above are called the **dominant weights** in $M(n,r)$. Since each orbit $\alpha$ contains an element $[a_1,\dots,a_n]\in\alpha$ such that

$$a_1 \geq a_2 \geq \cdots \geq a_n$$

we will often identify weights with their weakly-decreasing representative.

Sometimes we will refer to the dominant weight representing the orbit of $i\in I(n,r)$ as the **shape of** $i$**.**

The theory of weights in representations of $\Gamma$ closely mirrors similar decompositions in other Artinian algebras: first we identify a family of (mutually orthogonal) idempotents:

**2.2.36 Lemma**
For $\alpha\in\Lambda(n,r)$ and $i,j\in I(n,r)$,

$$\xi_\alpha\xi_{i,j} = \begin{cases}\xi_{i,j}, & i\in\alpha\\ 0, & \text{otherwise}\end{cases} \quad\text{and}\quad \xi_{i,j}\xi_\alpha = \begin{cases}\xi_{i,j}, & j\in\alpha\\ 0, & \text{otherwise}\end{cases}$$

Proof
We can compute the image of these on the $c_{a,b}\in A(n,r)$:

$$\begin{aligned}\xi_\alpha\cdot\xi_{i,j}(c_{a,b}) &= \sum_k \xi_\alpha(c_{a,k})\xi_{i,j}(c_{k,b})\\ &= \xi_\alpha(c_{a,a})\xi_{i,j}(c_{a,b})\end{aligned}$$

where above we used that $\xi_\alpha(c_{i,j})=0$ unless $i=j$. But

$$\xi_\alpha(c_{a,a}) = \begin{cases}1, & a\in\alpha\\ 0, & \text{otherwise}\end{cases}$$

so

$$\xi_\alpha\cdot\xi_{i,j}(c_{a,b}) = \begin{cases}\xi_{i,j}(c_{a,b}), & a\in\alpha\\ 0, & \text{otherwise}\end{cases}$$

but in the case where $a\in\alpha$ and $\xi_{i,j}(c_{a,b})\neq 0$, this implies that $i\sim a$, so $i\in\alpha$. So finally,

$$\xi_\alpha\cdot\xi_{i,j}(c_{a,b}) = \begin{cases}\xi_{i,j}(c_{a,b}), & i\in\alpha\\ 0, & \text{otherwise}\end{cases}$$

and since this holds for any $c_{a,b}$, the left-hand side is proven. A symmetric argument goes through for the right-hand side. ♠

For the next step, we decompose the identity into a sum of these idempotents:

**2.2.37 Lemma**
We have the decomposition
$$\mathbb{1} = \sum_{\alpha \in \lambda(n,r)} \xi_\alpha.$$

Proof
On the one hand, for any $c_{a,b} \in A(n,r)$, $\mathbb{1}(c_{a,b}) = \delta_{a,b}$. On the other hand, for any $\alpha$,
$$\xi_\alpha(c_{a,b}) = 0$$
when $a \neq b$ *or when* $a \notin \alpha$.

Therefore when $a = b$, there is precisely one $\alpha$ (the orbit of $a = b$) such that $\xi_\alpha(c_{a,b}) = 1$, so putting this all together,
$$\sum_{\alpha \in \Lambda(n,r)} \xi_\alpha(c_{a,b}) = \delta_{a,b}$$
whence these two functions are equal. ♠

2.2.38 Remark: Using lemma 2.2.37, we can then decompose any $V \in M(n,r)$ into weight spaces:
$$V = \mathbb{1} \cdot V = \sum_{\alpha \in \Lambda(n,r)} \xi_\alpha V$$
which we will denote
$$\xi_\alpha V = V^\alpha.$$

**2.2.39 Definition:** The **formal character** of a representation $V \in M(n,r)$ is a polynomial
$$\Phi_V(X_1, \dots, X_n) = \sum_{\alpha \in \Lambda(n,r)} (\dim V^\alpha) X_1^{\alpha_1} \cdots X_n^{\alpha_n} = \sum_{\alpha \in \Lambda_+(n,r)} (\dim V^\alpha) m_\alpha(X_1, \dots, X_n)$$
where $m_\alpha$ is the *monomial symmetric polynomial*
$$m_\alpha(X_1, \dots, X_n) = \sum_{\sigma \in \mathfrak{S}_n} X_{\sigma(1)}^{\alpha_1} \cdots X_{\sigma(n)}^{\alpha_n}.$$

### 2.2.9 Irreducible representations

The irreducible representations in $M(n,r)$ are given by a couple of results by some of the big names in representation theory: the original proof for $k=\mathbb{C}$ was proven in [Sch01, p.37] and then generalized in a later paper by Weyl [Wey25] and in work by Chevalley[5]:

> **2.2.40 Theorem**
> Fix the usual lexicographical ordering on monomials in $k[X_1,\ldots,X_n]$. Let $n$ and $r$ be given integers with $n \geq 1$ and $r \geq 0$ Let $k$ be an infinite field. Then
>
> (a) For each $\lambda \in \Lambda_+(n,r)$, there exists an (absolutely) irreducible module $F_{\lambda,k}$ in $M_k(n,r)$ whose character $\Phi_{\lambda,k}$ has leading term $X_1^{\lambda_1}\cdots X_n^{\lambda_n}$.
>
> (b) Every irreducible $V \in M_k(n,r)$ is isomorphic to $F_{\lambda,k}$ for exactly one $\lambda \in \Lambda_+(n,r)$.

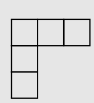
So then the problem of classifying the simple modules (the "basic building blocks" in the semisimple case) is completely solved for infinite fields. It remains to demonstrate a way to construct $F_{\lambda,k}$.

**2.2.41 Definition:** Fix some $\lambda \in \Lambda_+(n,r)$. Notice that this corresponds to a Young diagram with $r$ boxes. Fix any labeling $1,\ldots,r$ of the boxes in the Young diagram corresponding to $\lambda$. Let $T$ denote the diagram for $\lambda$ along with this labeling.

Let $i : \underline{r} \to \underline{n}$ be any map. Then denote by $T_i$ the **$\lambda$-tableau**, which is $T$ with the $k^{th}$ entry consisting of $i(k) \in \underline{r}$.

> 2.2.42 Remark: This notation varies slightly (but not in spirit) from the notation in Green's book. He denotes the Young diagram by $[\lambda]$ and lets $T^\lambda$ be the labelling of the boxes in $[\lambda]$–a bijection $[\lambda] \to \underline{r}$.

**Example 2.3**
Let $\lambda = (3,1,1) \in \Lambda_+(3,5)$. Thus $T$ is of shape



[5] Green [Gre07] mentions a paper by Serre: *Groupes de Grothendieck des Schémas en Groupes Réductifs Déployés* [Ser68], which makes mention to Chevalley's contributions in proving the existence of modules with prescribed characters. This author was unable to find Chevalley's work.

Then if we fix the left-to-right/top-to-bottom ordering of the boxes in $T$ and let $i : \{1,2,3,4,5\} \to \{1,2,3\}$ be given by $(2,1,3,3,2)$, we get the $\lambda$-tableau

$$T_i = \begin{array}{|c|c|c|}\hline 2 & 1 & 3 \\ \hline 3 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$$

The core tool in constructing (a basis for) the irreducible modules is in the following definiton:

**2.2.43 Definition:** Let $\lambda \in \Lambda_+(n,r)$ be some shape with a fixed labeling and let $i, j : \underline{r} \to \underline{n}$. Then the **bideterminant of $T_i$ and $T_j$** is

$$(T_i : T_j) = \sum_{\sigma \in C(T)} \operatorname{sgn}(\sigma) c_{i, j\sigma} \in A_k(n,r)$$

where $C(T)$ is the column stabilizer of $T$.

This definition can be a bit difficult to unpack, so we give some examples:

**Example 2.4**

(a) $\lambda = (2,1,0) \in \Lambda_+(3,3)$

$$\left( \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} : \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \right) = \begin{vmatrix} c_{13} & c_{12} \\ c_{33} & c_{32} \end{vmatrix} c_{21} = (c_{13}c_{32} - c_{12}c_{33})c_{2,1} = c_{(1,2,3),(3,1,2)} - c_{(1,2,3),(2,1,3)}$$

(b) $\lambda = (n, 0, \dots, 0) \in \Lambda_+(m,n)$

$$\left( \begin{array}{|c|c|c|}\hline a_1 & a_2 & a_3 \\ \hline\end{array} \dots \begin{array}{|c|}\hline a_n \\ \hline\end{array} : \begin{array}{|c|c|c|}\hline b_1 & b_2 & b_3 \\ \hline\end{array} \dots \begin{array}{|c|}\hline b_n \\ \hline\end{array} \right) = c_{a_1 b_1} \cdots c_{a_n b_n}$$

(c) $\lambda = (1, \dots, 1, 0, \dots) \in \Lambda_+(m,n)$ where $n \geq m$

$$\left( \begin{array}{c} \boxed{a_1} \\ \boxed{a_2} \\ \vdots \\ \boxed{a_n} \end{array} : \begin{array}{c} \boxed{b_1} \\ \boxed{b_2} \\ \vdots \\ \boxed{b_n} \end{array} \right) = \begin{vmatrix} c_{a_1 b_1} & \cdots & c_{a_1 b_n} \\ \vdots & \ddots & \vdots \\ c_{a_n b_1} & \cdots & c_{a_n b_n} \end{vmatrix}$$

In the following, let $l : \underline{r} \to \underline{n}$ be $(1, \ldots, 1, 2, \ldots, 2, 3, \ldots)$ such that for any shape $\lambda$ the $\lambda$-tableau $T_l$ is

$$\begin{array}{|c|c|ccc|c|}
\hline
1 & 1 & \ldots & \ldots & & \multicolumn{1}{|c|}{1} \\
\hline
2 & 2 & \ldots & \multicolumn{1}{|c|}{2} & & \\
\cline{1-2}\cline{4-4}
\vdots & & & & & \\
\cline{1-1}
k & & & & & \\
\cline{1-1}
\end{array}$$

with $i$ in every box on the $i^{th}$ row from the top.

**2.2.44 Definition:** Define, for every shape $\lambda \in \Lambda_+(n,r)$, the module

$$D_{\lambda,k} = \langle (T_l : T_i) \rangle_{i \in I(n,r)}$$

where $l$ is the filling defined above.

According to [Gre07], these modules were originally called "Weyl modules", while he (and we) reserve this name for the contravariant dual of these objects. To construct them, define the map

$$\pi : E^{\otimes r} \to D_{\lambda,k}, \tag{1}$$

and we get objects originally defined in Carter and Lusztig's treatment of modular representations of $\mathrm{GL}_n$ [CL74] and tweaked by Green in [Gre07]:

**2.2.45 Definition:** Given a shape $\lambda$, the **Weyl module of shape $\lambda$ over $k$** is $V_{\lambda,k} \stackrel{\text{def}}{=} N^{\perp}$ where

$$N \stackrel{\text{def}}{=} \ker \pi \hookrightarrow E^{\otimes r} \to D_{\lambda,k}$$

and the orthogonal complement of $N$ is taken with respect to the canonical contravariant form on $E^{\otimes r}$ that has the property $\langle e_i, e_j \rangle = \delta_{ij}$.

In their original paper [CL74, p.218], Carter and Lusztig showed that these modules are, in fact, generated as $S(n,r)$-modules by a single element:

**2.2.46 Theorem**

Let $\lambda \in \Lambda_+(n,r)$ and $T$ the Young diagram corresponding to $\lambda$. Let $l$ be the labelling above. Then the element

$$f_l = e_l \cdot \sum_{g \in C(T) \subset \mathfrak{S}_n} \operatorname{sign}(\sigma)\sigma$$

generates $V_{\lambda,k}$ as a $S(n,r)$-module.

Proof (sketch.)
We refer the reader to Green's [Gre07, p.46] proof for the details, but the idea is as follows: he relies on an earlier result that the modules $D_{\lambda,k}$ have a basis consisting of the bideterminants

$$(T_l : T_i)$$

such that $T_i$ is in "standard form" (meaning that it forms a valid Young tableau). One can define a nondegenerate contravariant form

$$(\cdot,\cdot) : V_{\lambda,l} \times D_{\lambda,k} \to k$$

by pulling back any element in $D_{\lambda,k}$ to a representative in $E^{\otimes r}$ under the map $\pi : E^{\otimes r} \to D_{\lambda,k}$.

Recall that $V_{\lambda,k}$ is defined as the orthogonal complement (under the canonical form $\langle\cdot,\cdot\rangle$ on $E^{\otimes r}$) of $\ker\pi$. This gives us that $(\cdot,\cdot)$ is indeed well-defined. From there, Green does some computation to show that one can bootstrap the independence of the $(T_l : T_i)$ to prove that of the set

$$\{\xi_{jl} f_l | j \in I(n,r), T_j \text{ standard}\}$$

forms a ($k$—)basis for $V_{\lambda,k}$, and therefore $f_l$ generates the entire module under the $S(n,r)$ action. ♠

**2.2.47 Lemma**
The modules $V_{\lambda,k}$ have a unique maximal submodule $V_{\lambda,k}^{\max}$

Proof ([Gre07, p.47])
Begin by noticing that the weight space $V_{\lambda,k}^{\lambda}$ is spanned by the single element $f_l$. This is because

$$\xi_l \cdot \xi_{il} f_l = \delta_{il} f_l$$

so the only nonzero basis vector from the proof of thm. 2.2.46 is $f_l$ itself. Since $f_l$ generates all of $V_{\lambda,k}$ as an $S(n,r)$-module, however, any proper submodule $M$ of $V_{\lambda,k}$ must be contained in the complement of $V_{\lambda,k}^{\lambda}$. Thus the sum of all proper submodules is contained in the complement of this weight space, and is therefore proper! This sum is our $V_{\lambda,k}^{\max}$ ♠

We are finally in good shape to compute the irreducible modules promised to us in thm. 2.2.40. We define

$$F_{\lambda,k} = V_{\lambda,k} / V_{\lambda,k}^{\max}$$

where $V_{\lambda,k}^{\max}$ is the unique maximal submodule guaranteed to us by lemma 2.2.47. It remains to show that the $F_{\lambda,k}$ have the requisite characters $\Phi_{\lambda,k}$. But notice that $V_{\lambda,k}^{\lambda}$ is one-dimensional, so the character (c.f. definition 2.2.39) of $V_{\lambda,k}$ is of the form

$$m_\lambda(X_1,\ldots,X_n) + \sum_{\lambda\neq\alpha\in\Lambda_+(n,r)} \dim V_{\lambda,l}^{\alpha} m_\alpha(X_1,\ldots,X_n)$$

but since each $V_{\lambda,k}^{\alpha}$ is contained in $V_{\lambda,k}^{\max}$, it occurs as a weight space of this maximal submodule with the same multiplicity. Therefore the character of $V_{\lambda,k}^{\max}$ is

$$\sum_{\lambda \neq \alpha \in \Lambda_+(n,r)} \dim V_{\lambda,l}^{\alpha} m_\alpha(X_1,\dots,X_n)$$

so we can conclude that

$$\Phi_{V_{\lambda,k}}(X_1,\dots,X_n) = m_\lambda(X_1,\dots,X_n) = X_1^{\lambda_1}\cdots X_n^{\lambda_n} + \cdots$$

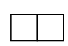
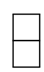
which has leading term (under the lexicographic ordering) precisely what we wanted.

## 2.3 Explicit examples for comparison

To demonstrate the theory developed above, we begin a computation (in a simple case) of the isomorphism classes of irreducible representations of both $S_{\mathbb{C}}(2,2)$ and $\mathfrak{S}_2$.

### 2.3.1 The symmetric group on two letters

The representation theory (over $k = \mathbb{C}$) of $\mathfrak{S}_2$ is as simple as it comes: of course $\mathfrak{S}_2 \cong \mathbb{Z}/2\mathbb{Z}$ and we know that there are $|G|$ nonisomorphic irreducible representations of an abelian group $G$ over $\mathbb{C}$. Since we are talking about a symmetric group, we can realize these as the trivial and sign representations, represented by the Young diagrams:

 and 

As submodules of the regular representation $k\mathfrak{S}_2 = ke \oplus k(12)$, we can construct these as $\langle e + (12)\rangle$ (trivial representation) and $\langle e - (12)\rangle$ (sign representation).

### 2.3.2 The Schur algebra $S_{\mathbb{C}}(2,2)$

Since $\operatorname{char}\mathbb{C} = 0$, corollary 2.2.31 implies that $S_{\mathbb{C}}(2,2)$ is semisimple, so it suffices to identify the irreducible submodules therein. We know

$$S = S_{\mathbb{C}}(2,2) \cong \mathbb{C}^2 \otimes \mathbb{C}^2$$

so $\dim_{\mathbb{C}} S = 4$. The theory outlined above gives us that isomorphism types of irreducible modules are in bijection with compositions of 2 of length 2, meaning we have two isomorphism types: one corresponding to $\lambda_1 = (1,1)$ and one corresponding to $\lambda_2 = (2,1)$.

Using the construction of $D_{\lambda,\mathbb{C}}$ from above, we can compute these two irreducible modules explicitly:

**Example 2.5 ($\lambda_1 = (1,1)$)**
In this case our shape is $(1,1)$, corresponding to the Young diagram

$$\begin{array}{|c|}\hline \phantom{1} \\ \hline \phantom{1} \\ \hline \end{array}$$

and then $D_{\lambda_1,\mathbb{C}}$ is spanned by the element

$$(T_l : T_{(2,1)}) = \left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} : \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}\right) = c_{12}c_{21} - c_{11}c_{22} = c_{(1,2),(2,1)} - c_{(1,2),(1,2)} \in A_{\mathbb{C}}(2,2)$$

since all other bideterminants of this shape are zero or linearly dependent. Thus this is a one-dimensional irreducible representation.

**Example 2.6 ($\lambda_2 = (2,0)$)**
Now our shape is $(2,0)$, corresponding to the diagram

$$\begin{array}{|c|c|}\hline \phantom{1} & \phantom{1} \\ \hline \end{array}.$$

The bideterminants here are

$$\begin{aligned} (T_l : T_{(1,1)}) &= \left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array} : \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}\right) = c_{11}^2 \\ (T_l : T_{(1,2)}) = (T_l : T_{(2,1)}) &= c_{11}c_{12} \\ (T_l : T_{(2,2)}) &= c_{12}^2 \end{aligned}$$

So we have a three-dimensional irreducible representation spanned by $\langle c_{11}^2, c_{11}c_{12}, c_{12}^2 \rangle$.

Since these are the only two Young diagrams of size two, these examples form a complete list of isomorphism classes of irreducible representations of $S_{\mathbb{C}}(2,2)$.

If we prefer instead to recognize our irreducibles as submodules of $E^{\otimes 2} = (ke_1 \oplus ke_2)^{\otimes 2}$ (giving us a more obvious action by our algebras), we can use the short exact sequence

$$0 \to N \hookrightarrow E^{\otimes 2} \twoheadrightarrow D_{\lambda,\mathbb{C}} \to 0$$

to define the $N = \ker \pi$, where $\pi$ is the map defined in equation (1) above. Then we can compute the orthogonal complement to $N$ to get $V_{\lambda,\mathbb{C}}$. We can compute:

$$V_{\lambda_1,\mathbb{C}} = \langle e_1 \otimes e_2 - e_2 \otimes e_1 \rangle$$

and

$$V_{\lambda_2,\mathbb{C}} = \langle e_1 \otimes e_1,\, e_1 \otimes e_2 + e_2 \otimes e_1,\, e_2 \otimes e_2 \rangle.$$

# 3 The Schur-Weyl Functor

From the discussion in the last section it is evident that the combinatorics behind the representation theory of $S(n,r)$ and $\mathfrak{S}_r$ have some intersections in their use of Young tableaux and this connection is more than superficial. In fact, there is a functor relating the representations of these two objects in the following way:

## 3.1 Construction of the functor $\mathcal{F}$

Let $V \in M_k(n,r)$ be a $S(n,r)$-representation and select any weight $\alpha \in \Lambda(n,r)$. Then the weight space (cf. rmk 2.2.38)

$$V^\alpha = \xi_\alpha V$$

becomes a $S(\alpha) \stackrel{\text{def}}{=} \xi_\alpha S(n,r)\xi_\alpha$-module using the action from $S(n,r)$. Now if we allow $r \leq n$ and let

$$\omega = (1,\dots,1,0,\dots,0) \in \Lambda(n,r)$$

notice that $S(\omega)$ is spanned by the elements

$$\xi_\omega \xi_{i,j} \xi_\omega, \quad i,j \in I(n,r)$$

but by the multiplication rules established in the definition of $S(n,r)$, these are nonzero precisely when $i$ and $j$ are both of shape $\omega$. So then since $\xi_{i,j} = \xi_{i\sigma,j\sigma}$ for all $\sigma \in \mathfrak{S}_r$, we can take as a basis of $S(\omega)$ the set

$$\{\xi_{u\pi,u} | \pi \in \mathfrak{S}_r\}$$

where $u = (1,2,\cdots,r) \in I(n,r)$.

To prove the next statement we require a computational result.

> **3.1.1 Lemma**
> If $u = (1,2,\dots,r) \in I(n,r)$, then for all $\pi,\sigma \in \mathfrak{S}_r$,
>
> $$\xi_{u\pi,u} \cdot \xi_{u\sigma,u} = \xi_{u\pi\sigma,u}.$$

Proof

Using the formulas for multiplication in $S(n,r)$, recall that

$$\xi_{u\pi,u} \cdot \xi_{u\sigma,u} = \sum Z_{i,j} \xi_{i,j} \tag{2}$$

where

$$Z_{i,j} = \#\{s \in I(n,r) | (u\pi,u) \sim (i,s) \text{ and } (u\sigma,u) \sim (s,j)\}.$$

Then for each $i,j$, since $u = (1,2,\dots,r)$ has no stabilizer in $\mathfrak{S}_r$, there is a unique $g$ such that $u\pi g = i$, meaning that $s = ug$.

But then this fixes (again a unique) $h \in \mathfrak{S}_r$ such that $u\sigma h = s = ug$ whence $\sigma h = g$. One computes that

$$u\pi\sigma h = u\pi g = i \quad \text{and} \quad uh = j$$

therefore since in the above computation $s$ was completely determined by $i$, we have

$$Z_{i,j} = \begin{cases} 1, & (i,j) \sim (u\pi\sigma, u) \\ 0, & \text{otherwise} \end{cases}$$

and the result follows. ♠

Using this result, we prove a more obviously useful statement:

> **3.1.2 Lemma**
> $S(\omega) \cong k\mathfrak{S}(r)$.

PROOF
Define the map $\varphi : S(\omega) \to k\mathfrak{S}_r$ on the basis above to be

$$\varphi(\xi_{u\pi,u}) = \pi$$

and extending $k$-linearly.

This is a homomorphism since

$$\varphi(\xi_{u\pi,u}\xi_{u\sigma,u}) = \varphi(\xi_{u\pi\sigma,u}) = \pi\sigma = \varphi(\xi_{u\pi,u})\varphi(\xi_{u\sigma,u})$$

and it is bijective since it is bijective on the respective bases and is thus bijective as a linear map.♠

The upshot of these lemmas is that one can define the **Schur-Weyl functor**

$$\mathcal{F} : M_k(n,r) \to \mathbf{Rep}(\mathfrak{S}_r)$$

via the map that sends any representation $V$ to its $\omega$ weight space $V^\omega \in S(\omega)$-**mod** $\simeq \mathbf{Rep}(\mathfrak{S}_r)$.

## 3.2 The general theory

The idea of the Schur functor fits into a larger context: Let $S$ be a $k$-algebra and let $M \in S$-**mod**. Furthermore, let $e \in S$ be a (nonzero) idempotent. Then one can define a functor

$$\mathcal{F} : S\text{-}\mathbf{mod} \to eSe\text{-}\mathbf{mod} \quad \text{via} \quad V \mapsto eV.$$

An important property of this functor is

**3.2.1 Proposition**
The image of an irreducible $S$ module under the functor $\mathcal{F}$ above is zero or irreducible.

PROOF
Let $e \in S$ be the idempotent in the discussion above and let $W \subseteq eV$ be any nonzero $eSe$-submodule. Then notice that $eW$ is a nonzero $S$-module contained in $e^2V = eV$, so $eW = eV$. But since $eW \subseteq W$, this forces $W = eV$, so $\mathcal{F}(V)$ is irreducible. ♠

Next, a discussion in Green [Gre07, p. 56] gives us a natural thought process to follow in constructing a partial inverse to this functor. Let $\mathcal{G} : eSe\text{-}\mathbf{mod} \to S\text{-}\mathbf{mod}$ be an extension of scalars: specifically, if $M \in eSe\text{-}\mathbf{mod}$, then

$$\widetilde{\mathcal{G}}(M) = Se \otimes_{eSe} M.$$

This is clearly functorial and furthermore satisfies the property that

$$\mathcal{F} \circ \widetilde{\mathcal{G}}(M) = \mathcal{F}(Se \otimes_{eSe} M) = e(Se \otimes_{eSe} M) = eSe \otimes_{eSe} M \cong e \otimes_{eSe} M \cong M$$

so it is a right inverse (up to isomorphism) to $\mathcal{F}$—a good candidate for our purposes.

3.2.2 REMARK: It is easy to prove the fact, which I glossed over above, that $M \cong e \otimes M$ via the $eSe$-isomorphism $m \mapsto e \otimes m$.

What we are really looking for, however, is a functor that sends irreducible modules to irreducibles. It can be shown that $\widetilde{G}$ *does not* satisfy this property, so we define

**3.2.3 Definition:** If $M \in S\text{-}\mathbf{mod}$ and $e \in S$ is an idempotent, denote by $M_{(e)}$ the largest $S$-submodule of $(1-e)M$.

which enables us to define the functor

$$\mathcal{G} : eSe\text{-}\mathbf{mod} \to S\text{-}\mathbf{mod} \quad \text{via} \quad M \mapsto \widetilde{\mathcal{G}}(M)/\widetilde{\mathcal{G}}(M)_{(e)}.$$

This leads to the result:

**3.2.4 Proposition**
If $M \in eSe\text{-}\mathbf{mod}$ is irreducible, then so is $\mathcal{G}(M)$.

PROOF
Let $W$ be an $S$-module such that

$$\widetilde{\mathcal{G}}(M)_{(e)} \subseteq W \subseteq \widetilde{\mathcal{G}}(M)$$

Then consider multiplying by $e$ in the above inculsions: we get

$$0 = e\widetilde{\mathcal{G}}(M)_{(e)} \subseteq eW \subseteq e\widetilde{\mathcal{G}}(M) = \mathcal{F} \circ \widetilde{\mathcal{G}}(M) \simeq M$$

which, by the irreducibility of $M$, forces either $eW = 0$ (in which case $W \subseteq \widetilde{\mathcal{G}}(M)_{(e)}$ and we are done) or else $eW = e\widetilde{\mathcal{G}}(M)$.

In this latter case, we find

$$\widetilde{\mathcal{G}}(M) = Se \otimes M \simeq Se \otimes eSeM = S(eSe \otimes M) = S(e\widetilde{\mathcal{G}}(M)) = SeW \subseteq W$$

Thus we can conclude that $W = \widetilde{\mathcal{G}}(M)$, so $\mathcal{G}(M)$ has no nontrivial proper submodules, so it is simple. ♠

## 3.3 Properties of $\mathcal{F}$ and $\mathcal{G}$

Returning to the specific case of $S = S(n,r)$ and $eSe \cong \mathfrak{S}_r$, the theory developed in the last part gives us a pair of functors

$$\mathcal{F} : M(n,r) \to \mathfrak{S}_r\text{-}\mathbf{mod}, \qquad \mathcal{G} : \mathfrak{S}_r\text{-}\mathbf{mod} \to M(n,r),$$

each of which preserve irreducibility. We also have that

> **3.3.1 Proposition**
> If $M \in M(n,r)$ is irredicible and if $eM \neq 0$, then $\mathcal{G} \circ \mathcal{F}(M) = \mathcal{G}(eM) \cong M$.

Proof

Notice by prop. 3.2.1 and the following discussion that $eM$ is irreducible and (by assumption) nonzero, so

$$\mathcal{F} \circ \mathcal{G}(eM) \cong eM$$

and since

$$0 \neq eM \subseteq M$$

and $M$ is irreducible, $eM = M$. ♠

## 3.4 In positive characteristic

Schur's classical work dealt only with the case when $k$ is a field of characteristic zero. In Aquilino and Reischuk's paper on the monoidal structure of $S(n,d)$-**mod**, the authors mention that (in general),

$$S(n,d)\text{-}\mathbf{mod} \ncong \mathfrak{S}_d\text{-}\mathbf{mod}.$$

To fix this problem, the authors restrict attention to the "nicely behaved ones". Let $M^\lambda$ denote the $\lambda \in \Lambda(n,d)$ weight space of $E^{\otimes d} = (k^n)^{\times d}$. Then one can define

**3.4.1 Definition:** Let $M = \{M^\lambda | \lambda \in \Lambda(n,d)\}$ and let the category $\mathbf{add}(M)$ be the full subcategory of $\mathfrak{S}_d$-**mod** consisting of modules that are summands of finite direct sums of weight modules $M^\lambda \in M$.

One can define an analogous subcategory $\mathbf{add}(S(n,d))$, and the usual Schur-Weyl functor

$$\mathcal{F}(M) = \xi_\omega M = M^\omega$$

restricts to an equivalence between the categories $\mathbf{add}(M)$ and $\mathbf{add}(S(n,d))$.

# 4 Strict polynomial functors

The theory of strict polynomial functors has its genesis in the idea of *polynomial maps between vector spaces*, or equivalently the rational maps between the schemes they represent. The category of vector spaces with these polynomial maps—and more specifically, the representaion category associated to it—gives the category $\mathbf{Rep}\,\Gamma_k^d$ of strict polynomial functors.

Originally definied by Friedlander and Suslin in [FS97], the authors there showed that the category $S(n,r)$-**mod** is equivalent to this category, introducing the language of polynomial functors as a way to understand the structure of representations of the Schur algebras.

This process was carried out by Krause [Kra13] and his students Aquilino and Reischuk [AR17]. In the former, Krause identifies projective generators $\Gamma^{d,V}$ for $\mathbf{Rep}\,\Gamma_k^d$ and defines the tensor product by defining it for projectives and taking the appropriate colimits. In the latter paper, the construction is further elucidated and it is proven that the Schur-Weyl functor $\mathcal{F}$ is monoidal.

## 4.1 Polynomial maps

TODO: Rewrite this to account for the fact I am now using the algebraic definition of polynomial maps instead of the geometric one.

Let $V, W$ be vector spaces over a field $k$. There are many equivalent formulations of polynomial maps between such spaces, but one that this author of this paper finds particulaly motivating is the following:

**4.1.1 Definition:** Let $V, W$ be as above. Then the set of **polynomial maps from $V$ to $W$** is defined to be

$$\mathrm{Hom}_{\mathrm{Pol}}(V,W) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathbf{Sch}/k}(V,W).$$

To make sense of this definition, one recalls that every $V \in \mathbf{Vect}_{\mathrm{k}}$ corresponds to an affine $k$-scheme $\operatorname{Spec} S^*(V^\vee) = V \otimes_k -$ (which we, through an abuse of notation, again denote $V$) represented by the symmetric algebra of the dual of $V$. Thus the polynomial maps are precisely the rational maps one considers between these objects in their algebro-geometric realizations.

> 4.1.2 Remark: In Friedlander and Suslin's original paper, they define these maps abstractly as $S^*(V^\vee) \otimes W$. That this agrees with our definition (assuming that $V$ and $W$

are finite dimensional) follows from the following series of isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Pol}}(V,W) &\overset{\text{def}}{=} \mathrm{Hom}_{\mathbf{Sch}/k}(V,W)\\ &\simeq \mathrm{Hom}_{\mathbf{Alg}_k}(S^*(W^\vee),S^*(V^\vee))\\ &\simeq \mathrm{Hom}_{\mathbf{Alg}_k}(W^\vee,S^*(V^\vee))\\ &\simeq W\otimes S^*(V^\vee)\end{aligned}$$

where we used above properties of affine schemes and standard facts of the linear algebra of finite dimensional vector spaces as well as the fact that a map from $S^*(V)$ is determined uniquely by its images on $V$.

For reasons that will become apparent shortly, it is easier to use the above remarks to define a polynomial map in the following way:

**4.1.3 Definition:** If $V,W\in\mathbf{Vect}_k$ are finite dimensional, a **polynomial map** $f:V\to W$ can be alternatively defined as an element

$$f\in W\otimes S^*(V^\vee)\cong \mathrm{Hom}_{\mathbf{Sch}/k}(V,W)$$

through the identifications above.

The upshot to this seemingly more *ad hoc* definition is that, while it introduces the restriction of finite dimensionality (which will suffice for our definitions anyways), it enables us to make more simple the following idea:

**4.1.4 Definition:** Let $V$ and $W$ be vector spaces. Then a map $f\in\mathrm{Hom}_{\mathrm{Pol}}(V,W)$ is called **homogeneous degree** $d$ if it corresponds (under the isomorphisms above) to an element

$$f\in W\otimes S^d(V^\vee).$$

This is clearly a tangible and sensible way to define a degree $d$ map and it is less obvious how to define a property on the map of corresponding varieties that achieves the same goal. We will see in the next subsection other ways to define this notion that may appeal more to representation theorists.

**Example 4.1**
Here are some examples of polynomial maps:

- The identity (scheme) map $\mathrm{id} : V \to V$ is a (homogeneous degree 1) polynomial map. This corresponds to the element

$$\sum_{i=1}^{n} v_i \otimes v_i^\vee \in V \otimes S^*(V^\vee)$$

where $v_1, \ldots, v_n$ is a basis for $V$.

- If $V = \langle v_1, \ldots, v_n \rangle$ and $W = \langle w_1, \ldots, w_m \rangle$, the element

$$\sum_{1}^{m} w_i \otimes (v_i^\vee \otimes v_i^\vee)$$

gives rise to a map of algebras that sends basis element

$$\sum_{\sigma \in \mathfrak{S}_k} w_{i_{\sigma(1)}}^\vee \otimes \cdots \otimes w_{i_{\sigma(k)}}^\vee \mapsto \sum_{\sigma \in \mathfrak{S}_k} v_{i_{\sigma(1)}}^\vee \otimes v_{i_{\sigma(1)}}^\vee \otimes \cdots \otimes v_{i_{\sigma(k)}}^\vee \otimes v_{i_{\sigma(k)}}^\vee$$

which corresponds to a homogeneous degree 2 polynomial (scheme) map $V \to W$.

## 4.2 The category $\mathcal{P}_k$ of strict polynomial functors

Before we define these categories we should describe the objects in question!

**4.2.1 Definition:** A **strict polynomial functor** is a functor $T : \mathbf{Vect}_k \to \mathbf{Vect}_k$ such that for any $V, W \in \mathbf{Vect}_k$, the map on Homs

$$T_{V,W} : \mathrm{Hom}_k(V, W) \to \mathrm{Hom}_k(T(V), T(W))$$

is a polynomial map. That is,

$$T_{V,W} \in \mathrm{Hom}_{\mathrm{Pol}}\big(\mathrm{Hom}_k(V, W), \mathrm{Hom}_k(T(V), T(W))\big)$$

Earlier I promised that we would have a more representation-theoretic interpretation of the homogeneous degree of a strict polynomial functor. I am nothing if I am not true to my word:

**4.2.2 Lemma (Lem. 2.2 in [FS97])**
Let $T$ be a strict polynomial functor and let $n \geq 0$ be an integer. Then the following conditions are equivalent:

(a) For any $V \in \mathbf{Vect}_k$, any field extension $k'/k$ and any $0 \neq \lambda \in k'$, the $k'$-linear map $T_{k'}(\lambda \cdot 1_{V_{k'}}) \in \mathrm{End}_{k'}(T(V)_{k'})$ coincides with $\lambda^n 1_{T(V)_{k'}}$.

(b) For any $V \in \mathbf{Vect}_k$, $n$ is the only weight of the representation of the algebraic group $\mathbb{G}_m$ in $T(V)$ obtained by applying $T$ to the evident representation of $\mathbb{G}_m$ in $V$.

(c) For any $V, W \in \mathbf{Vect}_k$, the polynomial map

$$T_{V,W} : \mathrm{Hom}_k(V, W) \to \mathrm{Hom}_k(T(V), T(W))$$

is homogeneous of degree $n$ (in the sense of 4.1.4).

Proof
sketch out ideas here. Maybe just important ones. ♠

**4.2.3 Definition:** The category $\mathcal{P}_d$ is the full subcategory

$$\mathcal{P}_d \subset \mathbf{Func}(\mathbf{Vect}_k, \mathbf{Vect}_k)$$

whose objects are the **strict polynomial functors of degree $d$.**

We refer the reader to [FS97, Thm. 3.2] for a proof of the following fact:

**4.2.4 Theorem**
Let $n \geq d$. Then the map

$$\Psi : \mathcal{P}_d \to S(n,d)\text{-}\mathbf{mod}$$

given by evaluation at $k^n$:

$$T \mapsto T(k^n)$$

is an equivalence of categories with quasi-inverse

$$M \mapsto \Gamma^{d,n} \otimes_{S(n,d)} M$$

where $\Gamma^{d,n} = \Gamma^d \circ \mathrm{Hom}_k(k^n, -)$ (c.f. 4.3.1 below).

The important idea in this proof is that, for any polynomial functor $T$ and any finite-dimensional $V, W \in \mathbf{Vect}_\mathrm{k}$, we get

$$\begin{aligned} T_{VW} &\in \mathrm{Hom}(T(V), T(W)) \otimes S^d(\mathrm{Hom}(V,W)^\vee) \\ &\cong \mathrm{Hom}(S^d(\mathrm{Hom}(V,W)^\vee)^\vee, \mathrm{Hom}(T(V), T(W))) \\ &\cong \mathrm{Hom}(S^d(\mathrm{Hom}(V,W)^\vee)^\vee \otimes T(V), T(W)) \end{aligned}$$

and by using that $\Gamma^d(X) \cong S^d(X^\vee)^\vee$ and letting $V = W = k^n$, we can identify a canonical map

$$T_{k^n k^n} : \Gamma^d(\mathrm{End}(k^n)) \otimes T(k^n) \to T(k^n)$$

which gives us an action of $\Gamma^d(\mathrm{End}(k^n))$ on $T(k^n)$ and one can see without too much trouble that

$$\Gamma^d(\mathrm{End}(k^n)) \cong S(n,d).$$

The rest of the proof is showing that these maps do what we want them to do.

## 4.3 Strict polynomial functors... again

Just when you thought you had enough categories to consider, Krause developed a new category that more succinctly captures the stucture of homogeneous degree $d$ polynomial maps: there the author changes the domain of these functors to encode the desired properties into the functors, rather than take a subcategory of objects satisfying a condition (which is inherently more difficult to work with).

**4.3.1 Definition:** Let $k$ be any commutative ring. Then $P_k \subset \mathbf{Vect}_k$ is the full subcategory of finitely-generated projective $k$-modules.

Define $\Gamma^d P_k$ to be the category of **divided powers**—the objects are the same as those of $P_k$, but such that

$$\mathrm{Hom}_{\Gamma^d P_k}(V, W) = \Gamma^d \mathrm{Hom}_{P_k}(V, W)$$

where $\Gamma^d X = (X^{\otimes d})^{\mathfrak{S}_d}$ denotes the $d^{\mathbf{th}}$ divided powers of the $k$-module $X$.

Finally, as a matter of notation, let

$$\mathbf{Rep}\,\Gamma^d_k = \mathbf{Rep}\,\Gamma^d P_k = \mathbf{Func}(\Gamma^d P_k, k\text{-}\mathbf{mod})$$

which we (suggestively) call the **category of homogeneous degree $d$ strict polynomial functors.**

4.3.2 Remark: Of course, when $k$ is a field, we get that $P_k = \mathbf{Vect}_\mathrm{k}$ and an element

$$T \in \mathbf{Rep}\,\Gamma^d_k = \mathbf{Func}(\Gamma^d\,\mathbf{Vect}_\mathrm{k}, \mathbf{Vect}_\mathrm{k}),$$

is a functor that, on objects, is a map $\mathbf{Vect}_\mathrm{k} \to \mathbf{Vect}_\mathrm{k}$ and on morphisms is of the form

$$T_{VW} : \mathrm{Hom}_{\Gamma^d \mathbf{Vect}_\mathrm{k}}(V, W) = \Gamma^d \mathrm{Hom}_k(V, W) \to \mathrm{Hom}_k(T(V), T(W))$$

which, leveraging ⊗-Hom adjunction, gives us a map

$$\Gamma^d(V, W) \otimes T(V) \to T(W)$$

just as we got in the discussion following thm. 4.2.4.

Finish up proving that this category is also equivalent. Maybe try to see if I can cook up an equivalence directly between $\mathcal{P}_d$ and $\mathbf{Rep}\,\Gamma_k^d$?

## 4.4 The monoidal structure on $\mathbf{Rep}\,\Gamma_k^d$

One of the upshots of Krause's reformulation of strict polynomial functors is that it admits a more obvious monoidal structure. His construction of the tensor product on $\mathbf{Rep}\,\Gamma_k^d$ takes the following tack: notice that the Yoneda embedding is a map

$$y : (\Gamma^d P_k)^{\mathrm{op}} \to \mathbf{Rep}\,\Gamma_k^d$$

sending each object $V \mapsto \mathrm{Hom}_{\Gamma^d P_k}(V, -)$. Furthermore, the embedding $y$ is dense!

**4.4.1 Lemma**
Given a small category $\mathcal{C}$, let $y$ be the Yoneda embedding

$$y : (\mathcal{C}) \to \mathbf{Func}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}) = \mathbf{PreSh}(\mathcal{C}).$$

Then every element in $\mathbf{PreSh}(\mathcal{C})$ is (in a canonical way) a colimit of elements in the image of $y$. That is, for some collection of $C_i \in \mathcal{C}$,

$$X = \varinjlim_i y(C_i)$$

To prove this lemma, let us remind you of the construction called the **category of elements** of a functor $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$. It elements are pairs $(C, x)$ where $C \in \mathcal{C}$ and $x \in \mathcal{F}(C)$ is a point.

Morphisms between two objects

$$f : (C, x) \to (C', y)$$

are honest morphisms $f : C \to C'$ in $\mathcal{C}$ such that the set morphism

$$\mathcal{F}(f) : \mathcal{F}(C') \to \mathcal{F}(C)$$

has the property that

$$\mathcal{F}(f)(y) = x.$$

This category gives us a way to work with elements of a category "locally" even if the category $\mathcal{C}$ is not concrete.

Proof (of 4.4.1)
The setup (but not the details) for following proof comes from one in *Sheaves in Geometry and Logic* [MM92, pp. 41–43].

Define the functor

$$R : \mathbf{PreSh}(\mathcal{C}) \to \mathbf{PreSh}(\mathcal{C}) \quad \text{via} \quad E \mapsto \mathrm{Hom}_{\hat{\mathcal{C}}}(y(-), E).$$

Define also the opposing functor

$$L : \mathbf{PreSh}(\mathcal{C}) \to \mathbf{PreSh}(\mathcal{C}) \quad \text{via} \quad F \mapsto \operatorname{colim} \mathcal{D}_F$$

where $\mathcal{D}_F$ is the diagram

$$\int_{\mathcal{C}} F \xrightarrow{\pi_{\mathcal{C}}} \mathcal{C} \xrightarrow{y} \hat{\mathcal{C}}$$

(this is makes sense since **Set** is cocomplete).

Now we claim that $L \dashv R$ are a pair of adjoint functors. To prove this, it suffices to show that

$$\mathrm{Hom}_{\mathbf{PreSh}(\mathcal{C})}(F, R(E)) \cong \mathrm{Hom}_{\mathbf{PreSh}(\mathcal{C})}(L(F), E)$$

for all $F, E \in \mathbf{PreSh}(\mathcal{C})$.

But notice that maps from the colimit of a diagram to $E$ are in bijection with cones under the diagram (i.e. cocones) with nadir $E$ by the universal property of colimits! So

$$\mathrm{Hom}(L(F), E) = \mathrm{Hom}(\operatorname{colim} \mathcal{D}_F, E) = \operatorname{cocone}(\mathcal{D}_F, E)$$

where an element of $\operatorname{cocone}(\mathcal{D}_F, E)$ is a collection of maps $(\varphi_{(C,x)})_{(C,x)\in\int_{\mathcal{C}} F}$ such that for all morphisms $\alpha : (C', x') \to (C, x)$ in $\int_{\mathcal{C}} F$, the following diagram commutes

$$\begin{tikzcd} \mathcal{D}_F(C,x) \arrow[rr, "\mathcal{D}_F(\alpha)"] \arrow[dr, "\varphi_{(C,x)}"'] & & \mathcal{D}_F(C',y) \arrow[dl, "\varphi_{(C',y)}"] \\ & E & \end{tikzcd}$$

Using these maps, we can construct a natural transformation $\eta : F \to \mathrm{Hom}(y(-), E)$ in the following way: for each $C \in \mathcal{C}$, let

$$\eta_C : F(C) \to \mathrm{Hom}(y(C), E) \quad \text{such that} \quad \eta_C(x) = \varphi_{(C,x)} \in \mathrm{Hom}(y(C), E).$$

This assembles to an honest natural transformation since for each $C, C' \in \mathcal{C}$, and morphism $f : C \to C'$, we have a diagram

$$\begin{tikzcd} F(C) \arrow[r, "x \mapsto \varphi_{(C,x)}"] & \mathrm{Hom}(y(C), E) \\ F(C') \arrow[u, "F(f)"] \arrow[r, "x' \mapsto \varphi_{(C',x')}"'] & \mathrm{Hom}(y(C'), E) \arrow[u, "\mathrm{Hom}(y(f),E)"'] \end{tikzcd}$$

which commutes since (by the commutativity of the colimit diagram above),

$$\varphi_{(C,x)} = \varphi_{(C',x')} \circ \mathrm{Hom}(-, \alpha|_C)$$

where $\alpha|_C : C \to C'$ denotes the underlying map in $\mathcal{C}$ (instead of in $\int_{\mathcal{C}} F$). Then fixing $x' \in F(C')$—and therefore $x = F(f)(x') \in F(C)$—we can see that naturality of $\eta$ means that

$$\mathrm{Hom}(y(f), E) \circ \eta_{C'}(x') = \mathrm{Hom}(y(f), E)(\varphi_{(C',x')}) = \varphi_{(C',x')} \circ y(f)$$

and (continuing in the other direction)

$$\eta_C \circ F(f)(x') = \varphi_{(C,x)}$$

must be the same. But in this case the map $f : C \to C'$ lifts to a map $\hat{f} : (C', x') \to (C, x)$ in $\int_{\mathcal{C}} F$, and $\hat{f}|_C = f$ we get that the equality of these two expressions is precisely the compatibility condition of the structural morphisms of the cocone.

Thus we have showed that there is a well-defined map

$$\Psi_{E,F} : \mathrm{cocone}(\mathcal{D}_F, E) \to \mathrm{Hom}(F, R(E))$$

since a natural transformation is defined by its structural maps and a cocone by its legs, this map is injective. It is surjective because for every $\eta : F \to R(E)$, we can define legs for a cocone:

$$\varphi_{(C,x)} = \eta_C(x) \in \mathrm{Hom}(y(C), E) = \mathrm{Hom}(\mathcal{D}_F(C, x), E).$$

Next, we aim to show is natural in $F$ and $E$. If $\epsilon : E \to E'$ is a morphism, we have the diagram

$$\begin{tikzcd}
\mathrm{cocone}(\mathcal{D}_F, E) \arrow[r, "\Psi_{E.F}"] \arrow[d, "(\varphi_a)\mapsto(\epsilon\circ\varphi_a)"'] & \mathrm{Hom}(F, R(E)) \arrow[d, "\mathrm{Hom}(F,R(\epsilon))"] \\
\mathrm{cocone}(\mathcal{D}_F, E') \arrow[r, "\Psi_{E',F}"'] & \mathrm{Hom}(F, R(E'))
\end{tikzcd}$$

But tracing along the bottom left, a cocone $(\varphi_a)_a$ under $\mathcal{D}_F$ with nadir $E$ maps to the cocone $(\epsilon \circ \varphi_a)_a$ with nadir $E'$. Under the map $\Psi_{E',F}$ just defined, this has as its image the natural transformation

$$\eta : F \to \mathrm{Hom}(-, E') \quad \text{via} \quad \eta_C(x) = \varphi_{(C,x)} \circ \epsilon \in \mathrm{Hom}(C, E')$$

Proceeding along the top right, the same cocone with nadir $E$ is mapped to the natural transformation

$$\hat{\eta} : F \to \mathrm{Hom}(-, E) \quad \text{via} \quad \hat{\eta}_C(x) = \varphi_{(C,x)} \in \mathrm{Hom}(C, E)$$

which is then mapped to $\varphi_{(C,x)} \circ \epsilon \in \mathrm{Hom}(C, E')$. This gives us naturality in $E$.

To show naturality in $F$, let $\beta : F \to F'$ be a natural map between presheaves. Then this induces a map

$$\int_{\mathcal{C}} \beta : \int_{\mathcal{C}} F \to \int_{\mathcal{C}} F' \quad \text{via} \quad (C, x) \mapsto (C, \beta_C(x))$$

which, in turn, induces a map between diagrams

$$\mathcal{D}_\beta : \mathcal{D}_{F'} \to \mathcal{D}_F$$

where, on points, this is the map (if $x \in F(C)$)

$$\left(\mathcal{D}_\beta(\mathcal{D}_{F'})\right)_C (C,x) = (\mathcal{D}_{F'})_C(C, \beta_C(x))$$

which, in turn, induces the map

$$\hat{\beta} : \mathrm{cocone}(\mathcal{D}_{F'}, E) \to \mathrm{cocone}(\mathcal{D}_F, E)$$

such that, if $(\varphi_{(C,x)})_{\int_C F'}$ is a cone under $\mathcal{D}_{F'}$ with nadir $E$, the image $\rho = \hat{\beta}(\varphi_{(C,x)})$ is such that

$$\rho_{(C,x)} = \varphi_{(C,\beta_C(x))}.$$

So naturality in $F$ is equivalent to the commutivity of

$$\begin{tikzcd}
\mathrm{cocone}(\mathcal{D}_F, E) \arrow[r, "\Psi_{E,F}"] & \mathrm{Hom}(F, R(E)) \\
\mathrm{cocone}(\mathcal{D}_{F'}, E) \arrow[u, "\hat{\beta}"] \arrow[r, "\Psi_{E,F'}"'] & \mathrm{Hom}(F', R(E)) \arrow[u, "\mathrm{Hom}(\beta, R(E))"']
\end{tikzcd}$$

which follows from the statement that the two natural transformations below take the same values:

$$\left(\Psi_{E,F} \circ \hat{\beta}(\varphi_\alpha)\right)_C (x) = \varphi_{(C,\beta_C(x))}$$

and

$$\left(\mathrm{Hom}(\beta, R(E)) \circ \Psi_{E,F'}(\varphi_\alpha)\right)_C (x) = (\Psi_{E,F'}(\varphi_\alpha) \circ \beta)_C(x) = (\Psi_{E,F'}(\varphi_{(C,x)}))_C(\beta_C(x)) = \varphi_{(C,\beta_C(x))}$$

This completes the proof that $L \dashv R$.

But by the Yoneda lemma, we know that

$$R(E)(C) = \mathrm{Hom}(y(C), E) \cong E(C)$$

which implies that $R$ is naturally isomorphic to $\mathrm{id}_{\mathbf{PreSh}(\mathcal{C})}$. But adjoints, when they exist, are unique! Therefore $L \simeq \mathrm{id}_{\mathbf{PreSh}(\mathcal{C})}$, or in other words for all $F \in \mathbf{PreSh}(\mathcal{C})$,

$$F = \mathrm{id}_{\mathbf{PreSh}(\mathcal{C})}(F) \simeq L(F) = \mathrm{colim}\, D_F = \mathrm{colim}_i \mathrm{Hom}(-, C_i)$$

proving the result. ♠

Notice that since

$$\mathrm{Hom}_{\mathcal{C}}(-,X)=\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,-)$$

we can replace $\mathcal{C}$ with $\mathcal{C}^{\mathrm{op}}$ in the above argument and get that any functor in $\mathbf{Func}(\mathcal{C},\mathbf{Set})$ is a colimit of (covariant) representable functors of the form $\mathrm{Hom}(C_i,-)$. Thus **all** the elements in our category $\mathbf{Rep}\,\Gamma_k^d$ can be written as a colimit of elements of the form

$$\Gamma^{d,V}\overset{\text{def}}{=}\mathrm{Hom}_{\Gamma^d P_k}(V,-)$$

Let $0\to X\to Y\to Z\to 0$ be an exact sequence of elements in $\mathbf{Rep}\,\Gamma_k^d$. Then by applying $\Gamma^{d,V}$ and applying the Yoneda isomorphism $\mathrm{Hom}(\Gamma^{d,V},X)\simeq X(V)$, we get that

$$0\to X(V)\to Y(V)\to Z(V)\to 0$$

is exact whence

$$0\to\mathrm{Hom}(\Gamma^{d,V},X)\to\mathrm{Hom}(\Gamma^{d,V},Y)\to\mathrm{Hom}(\Gamma^{d,V},Z)\to 0$$

is. This proves the fact that

> **4.4.2 Lemma**
> For all $V\in\Gamma^d P_k$, $\Gamma^{d,V}$ is a projective object.

From these reasonably simple objects, one defines (letting $\Gamma_k^d=\mathbf{Rep}\,\Gamma_k^d$ in what follows)

$$\Gamma^{d,V}\otimes_{\Gamma_k^d}\Gamma^{d,W}\overset{\text{def}}{=}\Gamma^{d,V\otimes W}$$

and leveraging the facts above, for each $Y\in\Gamma^d P_k$,

$$\Gamma^{d,V}\otimes_{\Gamma_k^d}Y\overset{\text{def}}{=}\operatorname*{colim}_{\Gamma^{d,W}\to Y}\Gamma^{d,V\otimes W}$$

and finally for each $X\in\Gamma^d P_k$,

$$X\otimes_{\Gamma_k^d}Y\overset{\text{def}}{=}\operatorname*{colim}_{\Gamma^{d,V}\to X}\Gamma^{d,V}\otimes Y.$$

One can similarly define internal hom:

$$\mathscr{H}\mathrm{om}_{\Gamma_k^d}(X,Y)\overset{\text{def}}{=}\lim_{\Gamma^{d,V}\to X}\operatorname*{colim}_{\Gamma^{d,W}\to Y}\Gamma^{d,\mathrm{Hom}(V,W)}$$

which in [Kra13, prop 2.4] is shown to satisfy the usual adjunction:

> **4.4.3 Proposition (Krause)**
> If $X,Y,Z\in\Gamma^d P_k$,
>
> $$\mathrm{Hom}_{\Gamma_k^d}(X\otimes_{\Gamma_k^d}Y,Z)\cong\mathrm{Hom}_{\Gamma_k^d}(X,\mathscr{H}\mathrm{om}_{\Gamma_k^d}(Y,Z))$$

## 4.5 Monoidicity of the Schur-Weyl functor $\mathcal{F}$

In [AR17], the authors show that this is the "correct" monoidal structure. This is summed up in the primary result of their paper:

**4.5.1 Theorem ([AR17, thm. 4.4])**
The functor

$$\mathcal{F} = \mathrm{Hom}(\Gamma^\omega, -) : \mathbf{Rep}\,\Gamma_k^d \to k\mathfrak{S}_d\text{-}\mathbf{mod}$$

preserves the monoidal structure defined on strict polynomial functors, i.e.

$$\mathcal{F}(X \otimes_{\Gamma_k^d} Y) \cong \mathcal{F}(X) \otimes_k \mathcal{F}(Y)$$

for all $X$ and $Y$ and if $\mathbb{1}$ is the tensor unit,

$$\mathcal{F}(\mathbb{1}_{\mathbf{Rep}\,\Gamma_k^d}) = \mathbb{1}_{k\mathfrak{S}_d}.$$

The key observation in their proof of this result is that strict polynomial functors can be computed as limits of representable presheaves where the representing objects are *free*. This is believable enough if we (as they do) allow $k$ to be any commutative ring. Since we are only interested in the case when $k$ is a field, however, we have to make no such reduction.

Then a combinatorial argument connecting weights in $\Lambda(mn,d)$ to the collection of all matrices $A_\mu^\lambda$ with $\lambda \in \Lambda(n,d)$ and $\mu \in \Lambda(m,d)$ such that the $i^{th}$ column sums to $\lambda_i$ and the $j^{th}$ row sums to $\mu_j$. Then we observe that

$$\mathrm{Hom}(\Gamma^\omega, \Gamma^{d,n} \otimes \Gamma^{d_m}) \cong \bigoplus_{\lambda \in \Lambda(n,d),\, \mu \in \Lambda(m,d)} \bigoplus_{A \in A_\mu^\lambda} \mathrm{Hom}(\Gamma^\omega, \Gamma^A) = \bigoplus_{\lambda,\mu} \bigoplus_A {}^\lambda M$$

and by a decomposition result (their lemma 3.1), then have that

$$\bigoplus_A {}^\lambda M \cong {}^\lambda M \otimes_k {}^\mu M$$

which is the crucial step in separating into a tensor product of $\mathfrak{S}_d$ modules.

## 4.6 A dictionary

Spell out how one can translate between the three different categories: irreducibles and tensor structure.

# 5 Tensor products in the derived category $\mathbf{D}^{\mathrm{b}}(S(n,r))$

(Co)homology is a powerful tool in analyzing the composition of objects and their actions. This is evidenced by the sheer number of cohomology theories that are in use across many different fields. Homological computations are, in their nature, lossy—one is reducing the object to its signature and then we play the game of gleaning what we can from the structure that remains.

It is a well-known fact of homological algebra that the cohomology of an $R$-module is independent of resolution by projective objects. Because of this fact, if we are interested in the homological properties of modules over a ring $R$, it isn't useful to look at the (abelian) category $R$-**mod**, but rather its "homologically-distilled" analog, $\mathbf{D}(R)$. Throughout this section we will be relying on Weibel [Wei94] and his discussion on chain, homotopy, and derived categories.

## 5.1 Derived categories

In what follows, let $\mathcal{A}$ denote any abelian category. If it helps, the reader can relatively safely assume that $\mathcal{A}$ is $R$-**mod**, the category of (left) $R$-modules.[6] Denote by $\mathbf{Ch}(\mathcal{A})$ (or $\mathbf{Ch}(R)$ when $\mathcal{A} = R$-**mod**) the category of chain complexes $(C_\bullet, \partial)$ such that each $C_i \in \mathcal{A}$ and $\partial \circ \partial = 0$. Let $\mathbf{Ch}^{\mathrm{b}}(\mathcal{A})$ denote the full subcategory of $\mathbf{Ch}(\mathcal{A})$ consisting of the complexes that are bounded—that is, $C_i = 0$ for all $i > N$ and $i < M$ for some $N, M$.

Recall that a chain complex morphism[7] $f_\bullet : C_\bullet \to D_\bullet$ is a **chain nullhomotopic** in $\mathbf{Ch}(\mathcal{A})$ if there exist maps $\sigma_i : C_i \to D_{i+1}$ such that we have following (non-commuting) diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "\partial"] \arrow[d, "f_n"] \arrow[dl, "\sigma_n"] & C_{n-1} \arrow[r, "\partial"] \arrow[d, "f_{n-1}"] \arrow[dl, "\sigma_{n-1}"] & \cdots \\
\cdots \arrow[r, "\partial"'] & D_{n+1} \arrow[r, "\partial"'] & D_n \arrow[r, "\partial"'] & D_{n-1} \arrow[r, "\partial"'] & \cdots
\end{tikzcd}$$

with the condition that (for all $n$)

$$f_n = \partial \circ \sigma_n + \sigma_{n-1} \circ \partial.$$

**5.1.1 Definition:** Two chain maps $f, g : C_\bullet \to D_\bullet$ in $\mathbf{Ch}(\mathcal{A})$ are said to be **chain homotopic** if their difference is chain nullhomotopic. That is, if there exists maps $\sigma_i : C_i \to D_{i+1}$ such that

$$f_n - g_n = \partial \circ \sigma_n + \sigma_{n-1} \circ \partial.$$

[6] That one can do this is the subject of the *Freyd-Mitchell embedding theorem*, which tells us that any small Abelian category can be embedded faithfully in $R$-**mod** for some ring $R$. Even if $\mathcal{A}$ isn't small (a set), one can study it via this embedding by restricting attention to small abelian subcategories.

[7] A morphism that commutes with the differential.

A well-known lemma is the following:

**5.1.2 Lemma**
If $f$ and $g$ are chain homotopic maps, then they induce the same maps on (co)homology.

Chain homotopies play the role of **homotopy equivalences** (keeping in mind the example of topological spaces with simplicial homology for intuition) and the fact we have nontrivial homotopy equivalences is the first indication that we aren't in the right category to study homology. A natural thing to do, then, is to attempt to pass to a category where we identify equivalent morphisms.

**5.1.3 Definition:** Given the category $\mathbf{Ch}(\mathcal{A})$, we define the **homotopy category $\mathbf{K}(\mathcal{A})$** to be the category whose objects are the same as those in $\mathbf{Ch}(\mathcal{A})$ and whose morphisms between any two chains $C_\bullet$ and $D_\bullet$ are

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(C_\bullet, D_\bullet) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathbf{Ch}(\mathcal{A})}(C_\bullet, D_\bullet)/H$$

where $H$ consists of all chain nullhomotopic maps from $C_\bullet$ to $D_\bullet$.

5.1.4 Remark: We can analogously define the category $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$ that is formed through the same process after first restricting to the subcategory $\mathbf{Ch}^{\mathrm{b}}(\mathcal{A})$ of bounded chain complexes.

The upshot here is that we are now closer to our (until now only implicit) goal: to find a category that captures the information in $\mathbf{Ch}(\mathcal{A})$ *up to quasi-isomorphism*. One can show that, however, that in general there are quasi-isomoprhisms that are not homotopic to the identity map! So our job is only partially complete.

A result of great importance to reaching our goal is that $\mathbf{K}(\mathcal{A})$ is *triangulated* with distinguished triangles given by the mapping cones

$$A \xrightarrow{u} B \to \mathrm{cone}(u) \to A[1]$$

and all triangles equivalent to them[8]

The importance of triangluated categories cannot be understated (it is critical, e.g. in the construction of the Balmer spectrum in sec. 6). Many people, including Verdier ([Ver67]), and Neeman ([Nee96], [Nee10]) have put considerable time and effort into developing a framework within the context of triangulated categories to enable examination and manipulation. One of the tools that we will now use is *Verdier localization*. It closely mirrors the idea of localization of a ring at a multiplicative subset (a parallel that will be extended further in the following section).

[8]We say a triangle $X \to Y \to Z \to X[1]$ is equivalent to a mapping cone if $X, Y, Z \in \mathbf{K}(\mathcal{A})$ and there exists isomorphisms (equivalently, homotopy equivalences when considered as maps in $\mathbf{Ch}(\mathcal{A})$) $f, g, h$ such that the diagram in fig. 2 commutes (for some $A, B$ and $u$):

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\
A \arrow[r, "u"] & B \arrow[r] & \mathrm{cone}(u) \arrow[r] & A[1]
\end{tikzcd}$$

Figure 2: Equivalence of triangles in $\mathbf{K}(\mathcal{A})$

$$\begin{tikzcd}
W \arrow[r, "f"] \arrow[d, "s"] & Z \arrow[d, "t"] \\
X \arrow[r, "g"] & Y
\end{tikzcd}$$

Figure 3: Ore condition in a multiplicative system

**5.1.5 Definition:** Given a triangulated category $\mathcal{T}$, a **multiplicative system** $S$ in $\mathcal{T}$ is a collection of morphisms in $\mathcal{T}$ satisfying the following properties:

- If $s, s' \in S$, so are $s \circ s'$ and $s' \circ s$ (whenever either of these make sense).
- $\mathrm{id}_X \in S$ for all $X \in \mathcal{T}$
- (**Ore condition**) If $t \in S$ with $t : Z \to Y$ then for every $g : X \to Y$ there are maps $f$ and $s$ (with $s \in S$) such that the diagram in figure 3 commutes. The symmetric statement also holds.
- (**Cancellation**) If $f, g : X \to Y$ are two morphisms, then there is an $s \in S$ with $sf = sg$ if and only if there is a $t \in S$ with $ft = gt$.

5.1.6 Remark: Under the foresight we will eventually be inverting the elements in $S$, the Ore condition translates into the following idea: for all $g : X \to Y$ and $t : Z \to Y$ in $S$,

$$t^{-1}g = fs^{-1}$$

for some maps $s \in S$ and $f$. This fixes the inherent noncommutativity of function composition.

### 5.1.1 The calculus of fractions

We can finally construct the Verdier localization of $\mathbf{K}(\mathcal{A})$ using a generalization of the calculus of fractions in localization of a ring. We will call a diagram of the form

$$f s^{-1} : X \xleftarrow{s} X_1 \xrightarrow{f} Y$$

where $s \in S$ a **fraction** and say that two fractions $f s^{-1}$ and $g t^{-1}$ are equivalent if there exists an element $X_3$ fitting into the commutative diagram below:

$$\begin{tikzcd} & X_1 \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & X_3 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y \\ & X_2 \arrow[ul, "t"] \arrow[ur, "g"'] & \end{tikzcd}$$

Then from this we can define

**5.1.7 Definition:** Let $\mathcal{T}$ be a triangulated category and $S$ be a multiplicative system for $\mathcal{T}$. Then the **Verdier localization of $\mathcal{T}$ at $S$**, $\mathcal{T}[S^{-1}]$ is a category whose objects are the same as those of $\mathcal{T}$ and whose morphisms are equivalence classes of fractions of maps, as defined above.

From this more general framework, we can very simply define the **derived category of an abelian category** $\mathcal{A}$ to be

$$\mathbf{D}(\mathcal{A}) = \mathbf{K}(\mathcal{A})[W^{-1}]$$

where $W$ is the collection of weak homotopy equivalences (quasi-isomorphisms). For our purposes, it will suffice to restrict to the full triangulated subcategory $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$, giving us the **bounded derived category**

$$\mathbf{D}^{\mathrm{b}}(\mathcal{A}) = \mathbf{K}^{\mathrm{b}}(\mathcal{A})[W^{-1}].$$

### 5.1.2 Tensor products in $\mathbf{D}^{\mathrm{b}}(R)$

In the context of $R$ (where $R$ is a $k$ algebra) modules, there is a tensor bifunctor

$$- \otimes_R - : \mathbf{mod}\text{-}R \times R\text{-}\mathbf{mod} \to \mathbf{Vect}_{\mathrm{k}}$$

and since it is right exact, but not exact, we can take the left derived functor

$$- \otimes_R^{\mathbf{L}} - \stackrel{\text{def}}{=} \mathbf{L}(- \otimes_R -) : \mathbf{D}(\mathbf{mod}\text{-}R) \times \mathbf{D}(R\text{-}\mathbf{mod}) \to \mathbf{D}(\mathbf{Vect}_{\mathrm{k}})$$

which we call **the derived tensor product**. This can defined via a Kan extension:

**5.1.8 Definition:** Let $\mathcal{F}:\mathcal{A}\to\mathcal{B}$ be an additive functor between abelian categories. Then since $\mathcal{F}$ preserves chain homotopies, it descends to a functor $\mathbf{K}\mathcal{F}:\mathbf{K}(\mathcal{A})\to\mathbf{K}(\mathcal{B})$.

We define the **right derived functor** (if it exists) to be a functor $\mathbf{R}\mathcal{F}:\mathbf{D}(\mathcal{A})\to\mathbf{D}(\mathcal{B})$ along with a natural transformation $\xi: q\circ\mathbf{K}\mathcal{F}\to\mathbf{R}\mathcal{F}\circ q$ such that for any $\mathcal{G}:\mathbf{D}(\mathcal{A})\to\mathbf{D}(\mathcal{B})$ and $\zeta: q\circ\mathbf{K}\mathcal{F}\Rightarrow\mathcal{G}\circ q$ fitting into the diagram

$$\begin{tikzcd}
\mathbf{K}(\mathcal{A}) \arrow[r, "\mathbf{K}\mathcal{F}"] \arrow[dr, "q"'] & \mathbf{K}(\mathcal{B}) \arrow[r, "q"] \arrow[d, Rightarrow, "\zeta"] & \mathbf{D}(\mathcal{B}) \\
& \mathbf{D}(\mathcal{A}) \arrow[ur, "\mathcal{G}"] \arrow[ur, bend right=50, "\mathbf{R}\mathcal{F}"'] \arrow[ur, phantom, shift right=3ex, "\Uparrow\eta"] &
\end{tikzcd}$$

there exists a unique $\eta:\mathbf{R}\mathcal{F}\Rightarrow\mathcal{G}$ such that $\eta q\circ\xi=\zeta$. In other words,

$\mathbf{R}\mathcal{F}$ is the *right Kan extension of $q\circ\mathbf{K}\mathcal{F}$ along the localization map $q$.*

Similarly, the left derived functor $\mathbf{L}\mathcal{F}$ can be defined as the left Kan extension of $q\circ\mathbf{K}\mathcal{F}$ along $q$, satisfying the same universal property with the natural transformations reversed.

This gives us a property characterizing the functor, but in practice one usually computes this via resolutions. Let $M,N\in\mathcal{A}$ for some abelian monoidal category $\mathcal{A}$. Then

$$M[0]\otimes_R^{\mathbf{L}}N[0]=F_\bullet\otimes G_\bullet$$

where $F_\bullet$ and $G_\bullet$ are chain complexes quasi-isomorphic to $M[0]$ and $N[0]$, respectively (e.g. flat resolutions).

More generally, if $\mathcal{A}$ has enough projectives, so does $\mathbf{Ch}(\mathcal{A})$. So if $A_\bullet,B_\bullet\in\mathbf{D}(\mathcal{A})$, one can take projective (*a fortiori* flat) resolutions of them to compute This is still weird.

# 6 The (Balmer) spectrum of a tensor triangulated category

In Paul Balmer's 2005 paper [Bal05], he developed a general framework for understanding the structure of certain kinds of categories that arose from the original constructions in algebraic geometry. Serving as a source of inspiration for Balmer, in [FP07] Friedlander and Pevtsova proved that the projective geometry of the cohomology ring of a finite group scheme can be recovered by looking at "ideals" in the category **stmod** $G$ of stable $G$ modules.

Using this as a springboard, Balmer ported the definitions of ideals and prime ideals to tensor-triangulated categories (see below) and proved a broader result that gives some tools for better analyzing familes of representaitons of finite groups (among other things).

## 6.1 Some motivation and a definition

Let $\mathcal{C}$ be a symmetric monoidal (i.e. tensor) category with tensor product $\otimes$ and unit object $\mathbb{1}$. After giving some thought to the matter, one realizes that a ring is given by putting a "compatible" monoidal structure on top of an abelian group, and to that end, one may consider the case when $\mathcal{C}$ is also additive.

This perspective gives us an interesting analogy between (unital, commutative) rings in algebra and category theory. Since every triangulated category is also additive, we can further specify that $\mathcal{C}$ be triangulated:

**6.1.1 Definition:** A **tensor-triangulated** category $\mathcal{C}$ is both a symmetric moniodal category and a triangulated category such that the monoidal structure preserves the triangluated structure.

As a reminder, such a category is equipped with a tensor product $-\otimes- : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and unit object $\mathbb{1}$, along with a collecton distinguished triangles $\mathcal{T}$ comprised of objects in $\mathcal{C}$ and shift functor (an auto-equivalence) $(-)[1] : \mathcal{C}\to\mathcal{C}$ such that: $-\otimes-$ is a triangulated (or exact) functor in each entry (it takes $\mathcal{T}$ to itself).

### 6.1.1 Aside: Why triangulation?

In the construction of the spectrum, we will see that the triangulated structure isn't explicitly necessary. It appears that one only needs a symmetric (or not!) monoidal category with all sums (at least if we are just relying on analogy to rings). A question, which may not have an answer yet (fully or in part) is whether changing these requirements significantly changes things. For instance, what happens when one tries to compute the spectrum of the abelian (symmetric monoidal) category **Rep** $G$?

## 6.2 Construction of the spectrum

Once the appropriate context is identified (which is the real ingenuity of Balmer's paper), the construction very closely mirrors the construction seen in elementary algebraic geometry:

**6.2.1 Definition:** Let $\mathcal{C}$ be a tensor-triangulated category (TTC). Then a **(thick tensor) ideal** $I \subseteq \mathcal{C}$ is a full triangulated subcategory with the following conditons:

- *(2-of-3 rule/Triangulation)* If $A, B$, and $C \in \mathcal{C}$ are objects that fit into a distinguished triangle
$$A \to B \to C \to A[1]$$
in $\mathcal{C}$, and if any two of the three are objects in $I$, then so is the third.
- *(Thickness)* If $A \in I$ is an object that splits as $A \cong B \oplus C$ in $\mathcal{C}$, then both $B$ and $C$ belong to $I$.
- *(Tensor Ideal)* If $A \in I$ and $B \in \mathcal{C}$ then $A \otimes B = B \otimes A \in I$.

> 6.2.2 Remark: The first condition just ensures that our ideals respect the triangulated structure (and thus stability) in the parent category $\mathcal{C}$. The final condition is the most direct analog of an ideal and is central in the analogy between this theory and classical AG.

From here the rest of the picture is relatively straightforward:

**6.2.3 Definition:** Let $\mathcal{C}$ be a TTC as before. Then an ideal $I \subseteq \mathcal{C}$ is called a **prime ideal** if, whenever $A \otimes B \in I$ for some $A, B \in \mathcal{C}$, either $A$ or $B$ is in $I$.

We call the collection of all primes the **spectrum** of $\mathcal{C}$ and write $\mathrm{Spc}(\mathcal{C})$.

Here the construction varies slightly from the traditional construction of Spec: we define

$$Z(S) \stackrel{\text{def}}{=} \{\mathcal{P} \in \mathrm{Spc}(\mathcal{C}) | S \cap \mathcal{P} = \varnothing\}$$

and define sets (for any $S \subseteq \mathcal{C}$ and $A \in \mathcal{C}$):

$$U(S) \stackrel{\text{def}}{=} \mathrm{Spc}(\mathcal{C}) \setminus Z(S) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{C}) | S \cap \mathcal{P} \neq \varnothing\}$$

and

$$\mathrm{supp}(A) \stackrel{\text{def}}{=} Z(\{A\}) = \{\mathcal{P} \in \mathrm{Spc}\,\mathcal{C} | A \notin \mathcal{P}\}$$

A routine check of the axioms shows us

> **6.2.4 Lemma (2.6 of [Bal05])**
> The sets $U(S)$ for all $S \subseteq \mathcal{C}$ form a basis for a topology on $\operatorname{Spc}\mathcal{C}$.

which we call the **Zariski topology**, giving $\operatorname{Spc}\mathcal{C}$ the structure of a topological space.

## 6.3 As a locally-ringed space

The above discussion mentions how we can construct a topological space from the set of prime thick tensor ideals in a TTC, but there is even more we can get: the structure of a locally-ringed space.

To get this, we need to define the structure sheaf:

**6.3.1 Definition:** Let $\mathcal{C}$ be a tensor-triangulated category and let $\operatorname{Spc}\mathcal{C}$ be the construction discussed above. Then the structure sheaf on $\operatorname{Spc}\mathcal{C}$ is given by the sheafification $\mathcal{O}_{\mathcal{C}}$ of the presheaf

$$\widetilde{\mathcal{O}}_{\mathcal{C}} : \operatorname{Open}(\operatorname{Spc}\mathcal{C})^{\mathrm{op}} \to \mathbf{Ring}$$

given by

$$\widetilde{\mathcal{O}}_{\mathcal{C}}(U) \overset{\text{def}}{=} \operatorname{End}_{\mathcal{C}/\mathcal{C}_Z}(\mathbb{1}_U)$$

where $U \subseteq \operatorname{Spc}\mathcal{C}$ is an open set and $\mathcal{C}_Z$ is the thick tensor ideal in $\mathcal{C}$ supported on $Z = \operatorname{Spc}\mathcal{C} \setminus U$. The ringed space $(\operatorname{Spc}\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ is denoted $\operatorname{Spec}_{\mathrm{Bal}}\mathcal{C}$.

> 6.3.2 Remark: That $\mathcal{C}_Z$ is a thick tensor ideal requires some work, but it follows from work that Balmer does to define a support data $(X, \sigma)$ on a tensor-triangulated category and showing that for any subset $Y \subset X$ of its associated topological space, the following set
> $$\{A \in \mathcal{C} | \sigma(A) \subseteq Y\}$$
> is a thick tensor ideal of $\mathcal{C}$ (c.f. lem. 3.4).

Balmer emphasizes that this is the "correct" ringed space structure to put on $\operatorname{Spc}\mathcal{C}$. To do so, one defines an abstract support datum:

**6.3.3 Definition:** A **support datum** for a TTC $\mathcal{C}$ is a pair

$$(X, \sigma)$$

where $X$ is a topological space and $\sigma : \mathcal{C} \to \operatorname{closed}(X)$ is a map sending $a \mapsto \sigma_a$ such that

(a) $\sigma(0) = \varnothing$ and $\sigma(1) = X$,

(b) $\sigma(a \oplus b) = \sigma(a) \cup \sigma(b)$,

(c) $\sigma(a[1]) = \sigma(a)$,

(d) $\sigma(a) \subseteq \sigma(b) \cup \sigma(c)$ for any triangle $a \to b \to c \to a[1]$,

(e) $\sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$.

Using this definition, Balmer shows

**6.3.4 Theorem ([Bal05, thm. 3.2])**
$(\operatorname{Spc}\mathcal{C}, \operatorname{supp})$ is a support datum for $\mathcal{C}$ and furthermore this support datum is terminal in the category of support data for $\mathcal{C}$. That is, for any other $(X, \sigma)$, there exists a unique continuous map $f : X \to \operatorname{Spc}\mathcal{C}$ such that

$$\sigma(a) = f^{-1}(\operatorname{supp}(a)).$$

To finish up the discussion of tensor-triangulated geometry, we state a couple of results originally proven in different contexts but used by Balmer to motivate the utility of this construction. In [Tho97], the author classifies the triangulated tensor subcategories of $\mathbf{D}_{\text{perf}}(X)$, thereby defining the set $\operatorname{Spc}\mathbf{D}_{\text{perf}}(X)$. Applying Balmer's language and structure, he proved that

**6.3.5 Theorem ([Bal05, thm. 6.3(a)])**
If $X$ is a topologically Noetherian scheme, then (as ringed spaces)

$$\operatorname{Spec}_{\text{Bal}} \mathbf{D}_{\text{perf}}(X) \simeq X.$$

Furthermore another result from Friedlander and Pevtsova [FP07] showed (again using the language of $\operatorname{Spec}_{\text{Bal}}$):

**6.3.6 Theorem ([FP07, thm. 3.6],[Bal05, thm. 6.3(b)])**
Let $G$ be a finite group scheme over a field $k$. Then

$$\operatorname{Spec}_{\text{Bal}}(\mathbf{stmod}(kG)) \simeq \operatorname{Proj}(H^\bullet(G,k))$$

where, $\mathbf{stmod}(kG)$ is the full subcategory of the stable module category consisting of the finitely generated modules and $H^\bullet(G,k) = \operatorname{Ext}^\bullet_G(k,k)$ is the cohomology ring of $G$.

# 7 Questions and extensions

## 7.1 The representation theory of $S(n, r)$ in positive characteristic

## 7.2 Computing the spectrum of $\mathbf{D}^{\mathrm{b}}(S(n, r))$

## 7.3 The Schur-Weyl functor and adjoints

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