

General Exam Paper

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Abstract

We begin by going through a considerable amount of domain knowledge concerning representations of GL_n , representations of \mathfrak{S}_n , and strict polynomial functors all in service of understanding the Schur-Weyl functor that relates several of these categories. From there, we investigate recent work on the part of Krause and his students Aquilino and Reischuk on this functor and the fact that it is monoidal under reasonably natural monoidal structures on the categories in question. Finally we ask some questions about whether the monoidal structure on strict polynomial functors extends meaningfully to pathologies that arise in positive characteristic.

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1 Introduction

1.1 Schur and Polynomial Representations

The story of this project (more-or-less) begins with Schur's doctoral thesis [Sch01] in which he defined polynomial representations of GL_n —a theory which he developed more completely in his later paper *Über die rationalen Darstellungen der allgemeinen linearen Gruppe*¹ [Sch73]. In these papers, Schur develops the idea of a **polynomial representation** of GL_n , meaning a (finite dimensional) representation where the coefficient functions of the representing map

$$\rho : \mathrm{GL}_n \rightarrow \mathrm{GL}(V)$$

is polynomial in each coordinate. That is, if $V = \bigoplus_{i=1}^n k v_i$, then for every $1 \leq i, j \leq n$, we have the map $r_{v_i v_j} : \mathrm{GL}_n \rightarrow k$ such that

$$\rho(g) \cdot v_i = \sum_{j=1}^n r_{v_i v_j}(g) v_j.$$

A result in [Sch01] gives us a nice simplifying observation: if V is a polynomial representation, then V decomposes as a direct sum of representations

$$V = \bigoplus_{\delta} V_{\delta}$$

where each V_{δ} is a polynomial representation where the coefficient functions are *homogeneous degree* δ . This allows us to focus our attention to the structure of these V_{δ} as the fundamental building blocks of the theory.

One of the more surprising connections made in this theory comes from the observation that the vector space (recall $V \cong k^n$)

$$E = V^{\otimes r}$$

is made into a $(\mathrm{GL}_n(k), \mathfrak{S}_r)$ -bimodule in a very natural way, and that this bimodule gives us a way to relate $\mathbf{mod}\text{-}\mathfrak{S}_r$ with (a subcategory of) $\mathrm{GL}_n(k)\text{-}\mathbf{mod}$ via the so-called **Schur-Weyl functor**.

1.2 The Schur-Weyl Functor

Clearly a connection between representations of two groups that are so ubiquitous in group theory and math in general is a stunning observation, and much effort has been expended since the late 20th century to study this functor and its properties—especially in how it relates the representation theory of these two groups.

For instance, Friedlander and Suslin [FS97] originally discussed the idea of **strict polynomial functors** and showed that the category of representations of the Schur algebra $S(n, d)$ was equivalent to the category \mathcal{P}_d of homodegree d strict polynomial functors.

¹English: *On the rational representations of the general linear group*

In later work, Krause [Kra13] used an alternative construction of \mathcal{P}_d as the category of representations of the d -divided powers of the category of finitely generated projective k -modules. The upshot being that the latter object $\Gamma^d P_k$ has an obvious monoidal structure which \mathcal{P}_d inherits in a natural way. This new concrete monoidal structure opens up the field to discussing several notions of duality defined in different contexts and solidifying connections between them.

Krause’s students Aquilino and Reischuk, in their paper [AR17], prove, among other facts, that under these natural monoidal structures the Schur-Weyl functor is in fact monoidal. This puts the theory of representations of these groups and algebras firmly in the realm of monoidal categories, opening up the area to new questions using tools from category theory.

2 Representations of GL_n and of \mathfrak{S}_n

3 The Schur-Weyl Functor

4 Strict Polynomial Functors

5 Questions and Extensions

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