General Exam Paper

Nico Courts[*]

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Abstract

We begin by going through a considerable amount of domain knowledge concerning representations of GL_n, representations of $\mathfrak{S}_n$, and strict polynomial functors all in service of understanding the Schur-Weyl functor that relates several of these categories. From there, we investigate recent work on the part of Krause and his students Aquilino and Reischuk on this functor and the fact that it is monoidal under reasonably natural monoidal structures on the categories in question. Finally we ask some questions about whether the monoidal structure on strict polynomial functors extends meaningfully to pathologies that arise in positive characteristic.

An up-to-date version of this paper can be downloaded at the following link: `https://github.com/NicoCourts/General-Exam-Paper/raw/master/General-Paper.pdf`

[*]University of Washington, Seattle. Email: ncourts@uw.edu

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1 Introduction

1.1 Issai Schur and polynomial representations

The story of this project (more-or-less) begins with Schur's doctoral thesis [Sch01] in which he defined polynomial representations of GL_n—a theory which he developed more completely in his later paper *Über die rationalen Darstellungen der allgemeinen linearen Gruppe*[1] [Sch73]. In these papers, Schur develops the idea of a **polynomial representation of** GL_n, meaning a (finite dimensional) representation where the coefficient functions of the representing map

$$\rho : \mathrm{GL}_n \to \mathrm{GL}(V)$$

is polynomial in each coordinate. For example, the map sending

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2d - abc & acd - c^2b & 0 \\ abd - b^2c & ad^2 - bcd & 0 \\ 0 & 0 & ad - bc \end{pmatrix} = \rho(A)$$

is a three-dimensional polynomial representation of GL_2.

The block-diagonal form above demonstrates a direct sum decomposition of our representation into two parts: one two-dimensional homogeneous degree 3 and one one-dimensional homogeneous degree 2 (in the entries of A). A result in [Sch01] tells us that, in fact, this can always be done: if V is a polynomial representation of GL_n, then V decomposes as a direct sum of representations

$$V = \bigoplus_{\delta} V_\delta$$

where each V_δ is a polynomial representation where the coefficient functions are *homogeneous degree* δ. This allows us to focus our attention to the structure of these V_δ as the fundamental building blocks of the theory.

The key insight made in this theory comes from the observation that the vector space (recall $V \cong k^n$)

$$E = V^{\otimes r}$$

is made into a $(\mathrm{GL}_n(k), \mathfrak{S}_r)$-bimodule in a very natural way, and that this bimodule gives us a way to relate **mod**-$\mathfrak{S}_r$ with (a subcategory of) $\mathrm{GL}_n(k)$-**mod** via the so-called **Schur-Weyl functor.**

1.2 The Schur-Weyl functor

Clearly a connection between representations of two groups that are so ubiquitous in group theory and math in general is a stunning observation, and much effort has been expended since the late 20th century to study this functor and its properties—especially in how it relates the representation theory of these two groups.

[1] English: *On the rational representations of the general linear group*

For instance, Friedlander and Suslin [FS97] originally discussed the idea of **strict polynomial functors** and showed that the category of repesentations of the Schur algebra $S(n,d)$ was equivalent to the category $\mathcal{P}_d$ of homogeneous degree d strict polynomial functors.

In later work, Krause [Kra13] used an alternative construction of $\mathcal{P}_d$ as the category of of reprsentations of the d-divided powers of the category of finitely generated projective k-modules. The upshot being that the latter object $\Gamma^d P_k$ has an obvious monoidal structure which $\mathcal{P}_d$ inherits in a natural way. This new concrete monoidal structure opens up the field to discussing several notions of duality defined in different contexts and solidifying connections between them.

Krause's students Aquilino and Reischuk, in their paper [AR17], prove, among other facts, that under these natural monoidal structures the Schur-Weyl functor is in fact monoidal. This puts the theory of representations of these groups and algebras firmly in the realm of monoidal categories, opening up the area to new questions using tools from category theory.

1.3 Notation and conventions

Throughout this paper we will define k to be a field (not necessarily of characteristic zero or algebraically closed unless otherwise noted).

We will use $\Gamma = \Gamma_k = \mathrm{GL}_n = \mathrm{GL}_n(k)$ to denote the general linear group, $\mathrm{Aut}_k(k^n)$. Let $\mathfrak{S}_r$ denote the symmetric group on r letters.

When speaking of the (k) vector space spanned by elements $v_1, \ldots, v_n$, we will use the notation

$$\langle v_1, \ldots, v_n \rangle.$$

When the rest of the paper is finished, the intro will be rewritten to reflect the actual content.

2 Representations of Γ and of $\mathfrak{S}_n$

We begin by detailing the theory behind the (polynomial) representations of GL_n as well as the representations of $\mathfrak{S}_n$ to familiarize ourselves with the classical representation theory associated to these groups.

2.1 Representations of $\mathfrak{S}_n$

The representation theory for $\mathfrak{S}_n$ over the complex numbers is a subject that has been widely studied by representation theorists and combinatorialists alike for over a century. Before we dive into specifics, we write down the idea originally worked out by Frobenius [Fro73] in his work in 1900 on the characters of $\mathfrak{S}_n$:

> **2.1.1 Theorem**
> The conjugacy classes (and thus isomorphism classes of irreducible representations over $\mathbb{C}$) of $\mathfrak{S}_n$ are in bijection with partitions of n.

> 2.1.2 Remark: In what follows we attempt to give a tangible, minimalistic overview of the nicest case of representations of $\mathfrak{S}_n$. Some of the arguments below appeal more to intuition and examples than rigor, but we feel this better prepares the reader for computations in $\mathfrak{S}_n$ without being weighed down by unnecessary details. This can all be made rigorous, of course, at the expense of some clarity and conciseness.

Let's get some sense first about how we can relate these two ideas by recalling some easy lemmas from group theory. Recall that each element of $\mathfrak{S}_n$ can be written as a product of disjoint cycles and that this representation is unique up to reordering the cycles. We can make this representation unique by writing each cycle as one starting at its least element and then ordering the cycles by these least elements. For instance, the permutation (in two-line notation)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 7 & 5 & 3 & 8 & 4 & 6 \end{pmatrix} \in \mathfrak{S}_8$$

is represented uniquely in this way as the product of cycles:

$$\sigma = (1\,2)(3\,7\,4\,5)(6\,8).$$

The next observation to recover: if $\tau, \eta \in \mathfrak{S}_n$ and $\tau = (\tau_1\,\tau_2\,\cdots\,\tau_k)$ is a cycle,

$$\eta^{-1}\tau\eta = (\eta(\tau_1)\,\eta(\tau_2)\,\cdots\,\eta(\tau_k)).$$

We can see this demonstrated in the computation

$$\begin{aligned}(1\,3\,5)\sigma(1\,3\,5)^{-1} &= (1\,3\,5)(1\,2)(1\,5\,3)(1\,3\,5)(3\,7\,4\,5)(1\,5\,3)(1\,3\,5)(6\,8)(1\,5\,3)\\ &= (3\,2)(5\,7\,4\,1)(6\,8)\\ &= (1\,5\,7\,4)(2\,3)(6\,8)\end{aligned}$$

The important observation here is that the "shape" (the lengths of the cycles when written as a product of disjoint cycles) is preserved under conjugation. In fact,

> **2.1.3 Lemma**
> The conjugacy classes of $\mathfrak{S}_n$ are in one-to-one correspondence with the partitions of n.

Proof
Let $\mathscr{P}_n$ denote the partitions of n and let C_n denote the conjugacy classes in $\mathfrak{S}_n$. We construct the set map

$$\varphi : C_n \to \mathscr{P}_n$$

by sending a conjugacy class to the weakly-decreasing list of cycle lengths (including trivial cycles, if necessary). For instance in $\mathfrak{S}_8$,

$$(1\,5\,3)(2\,7) \qquad \text{cooresponds to} \qquad (3,2,1,1,1).$$

The results cited and demonstrated above shows that this map is well defined—conjugation preserves the cycle length in the disjoint cycle representation of an element. Furthermore if $p \in \mathscr{P}_n$ is a partition, the adjoint action of $\mathfrak{S}_n$ on $\varphi^{-1}(p)$ is transitive, since if two elements have the same cycle lengths when written as disjoint cycles, we can line the cycles up according to length and act by the permutation that "puts labels in the right place". If we look at σ and the element we found by conjugation above, we have

$$\begin{gathered}(1\,2)(3\,7\,4\,5)(6\,8)\\ (3\,2)(5\,7\,4\,1)(6\,8)\end{gathered}$$

where we notice that $1 \mapsto 3$, $3 \mapsto 5$, and $5 \mapsto 1$, meaning that the cycle that takes the top element to the bottom is $(1\,3\,5)$—although of course we already knew that. Another example are the elements $(1\,4\,5)$ and $(3\,2\,1)$. Here we want $1 \mapsto 3$, $4 \mapsto 2$ and $5 \mapsto 1$. Thus one element that takes the first to the second is $(5\,1\,3)(2\,4)$. This demonstrates that the action is not faithful since we could also act by $(5\,1\,3\,7)(2\,4)(6\,8)$ and get the same element. the important fact here is that $\mathfrak{S}_n$ acts transitively on the elements of $1, \ldots, n$, so there is always such an element.

The surjectivity of this map is clear since we can write from any partition of n a product of disjoint cycles corresponding to this partition (which then must map to it) and injectivity is clear since the disjoint cycle representation is unique (up to reordering cycles, which doesn't affect the image $\varphi(x)$). This proves the lemma. ♠

From here, the standard result that (again, over $\mathbb{C}$) the conjugacy classes of a group are in bijection with the irreducible representations finishes demonstrating how theorem 2.1.1 is true. But a simple set bijection belies the depth of the connection here.

Construction of the irreducible representations

It is possible, through the idea of a Young symmetrizer, to directly link a Young diagram to the corresponding irreducible representation. Throughout this subsection, we will be relying on facts developed in [FH91], although there is also a more complete combinatorial picture painted in Fulton's book *Young Tableaux* [Ful97].

To begin our discussion, consider the trivial representation within the left regular representation $\mathbb{C}\mathfrak{S}_n$: it is a one-dimensional subspace spanned by the element

$$x_1 = \sum_{\sigma \in \mathfrak{S}_n} \sigma$$

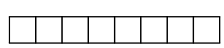
where you can see that this element is fixed by left multiplication, demonstrating that it has the trivial $\mathfrak{S}_n$ action. The subspace spanned by the element

$$x_{-1} = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\mathrm{sign}(\sigma)} \sigma$$

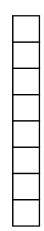
is the sign representation, where an element with sign 1 acts by -1. This is because

$$\mathrm{sign}(\tau\sigma) = \mathrm{sign}(\tau) + \mathrm{sign}(\sigma) \pmod 2.$$

It ends up that these two representations form the two "endpoints" of the representation theory of $\mathfrak{S}_n$. The exact sense in which this is true is captured through Young diagrams! For the purposes of illustration, let us return to our example above of $\mathfrak{S}_8$. Here the trivial and sign representations correspond (repsectively) to the tableaux



and



which, in turn, correspond to partitions (8) and $(1,1,1,1,1,1,1,1)$ of 8. The way to make this connection is through the definition of a *Young symmetrizer:*

2.1.4 Definition: Fix an $n \geq 1$ and let λ be a partition of n. Then define two elements of

$\mathbb{C}\mathfrak{S}_n$, a_λ and b_λ in the following way:

$$a_\lambda = \sum_{\sigma\in R(T_\lambda)} \sigma \qquad \text{and} \qquad b_\lambda = \sum_{\sigma\in C(T_\lambda)} (-1)^{\text{sign}(\sigma)}\sigma$$

where T_λ is the Young diagram corresponding to λ and given some labeling (say the canonical one that labels boxes left-to-right and top-to-bottom) $R(T_\lambda)$ (resp. $C(T_\lambda)$) denote the subgroups of $\mathfrak{S}_n$ stabilizing the rows (resp. columns) of T_λ under the action of $\mathfrak{S}_n$ on the labels.

Then the **Young centralizer** of λ is

$$c_\lambda = a_\lambda b_\lambda \in \mathbb{C}\mathfrak{S}_n.$$

The canonical fillings of the diagrams above are[2]

$$\boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5}\boxed{6}\boxed{7}\boxed{8} \qquad \text{and} \qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline 5\\\hline 6\\\hline 7\\\hline 8\\\hline\end{array}$$

and so since the column stabilizer of the first diagram is trivial and the row stablizer is everything,

$$c_{(8)} = \left(\sum_{\sigma\in R(T_{(8)})} \sigma\right)\left(\sum_{\sigma\in C(T_{(8)})} (-1)^{\text{sign}(\sigma)}\sigma\right) = \sum_{\sigma\in\mathfrak{S}_n} \sigma = x_1$$

and since the roles of the column and row stabilizing elements are reversed for the sign representation, we get

$$c_{(1,1,1,1,1,1,1,1)} = \left(\sum_{\sigma\in R(T_{(1,1,1,1,1,1,1,1)})} \sigma\right)\left(\sum_{\sigma\in C(T_{(1,1,1,1,1,1,1,1)})} (-1)^{\text{sign}(\sigma)}\sigma\right) = \sum_{\sigma\in\mathfrak{S}_n} (-1)^{\text{sign}(\sigma)}\sigma = x_{-1}.$$

That the Young symmetrizers correspond with the elements spanning the corresponding

[2]Here you can see yet another connection to disjoint cycle representations. Notice, under the map defined in lem. 2.1.3, that the conjugacy class corresponding to the trivial representation is the one consisting of "long" (length n) cycles. Using the unique ordering on products of disjoint cycles described after the statement of thm. 2.1.1, we can identify fillings with long cycles and we see that the cycle $(1\,2\,3\,4\,5\,6\,7\,8)$ is the only one in "standard form" in that it gives us a standard Young tableau. The complexity of the Young diagram (meaning how many different standard fillings it admits) gives us some information about the dimensionality of the corresponding irreducible representation, as we will see later.

representations of is no coincidence!

2.1.5 Definition: The module V_λ is a $\mathbb{C}\mathfrak{S}_n$-module generated by the Young symmetrizer c_λ.

Notice that the dimension of each V_λ is determined by number of linearly-independent elements that lie in the orbit of c_λ. We compute another example that gives a general pattern:

Example 2.1
Let $\lambda=(2,1,1)$ be the partition of 5, so

$$T_\lambda = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

Given the canonical filling of T_λ,

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array},$$

we have

$$a_\lambda = e + (1\,2) \qquad \text{and} \qquad b_\lambda = e - (1\,3) - (1\,4) - (3\,4) + (1\,3\,4) + (1\,4\,3)$$

and so we can compute that the Young symmetrizer for this partition is

$$c_\lambda = e - (1\,3) - (1\,4) - (3\,4) + (1\,2) + (1\,3\,4) + (1\,4\,3) - (2\,1\,4) - (1\,2)(3\,4) + (1\,3\,4\,2) + (1\,4\,3\,2)$$

and one can show (c.f. [FH91, p. 48]) that this is the representation $V \wedge V$ where V is the standard representation (the complement of copy of the trivial representation spanned by the vector $(1,1,1,1) \in \mathbb{C}^4$ under the usual embedding of $\mathfrak{S}_4$ in GL_4 as permutation matrices).

This completes the description of the representations of $\mathfrak{S}_n$ over $\mathbb{C}$, but in fact everything we have done here holds over the splitting field of $\mathfrak{S}_n$, that is, the minimal field such that representations don't split further under field extension. We haven't proved here that

(a) the V_λ are irreducible; or

(b) the V_λ are pairwise nonisomorphic,

but one can look up any of the standard texts (including the ones cited in this section) for more rigorous and thorough treatments of these facts.

2.2 Polynomial representations of Γ

Let Γ be the affine group scheme defined in section 1.3. Then

2.2.1 Definition: A (finite dimensional) **representation** of Γ is a (finite dimensional) vector space V along with a map

$$\rho : \Gamma \to \mathrm{GL}(V) \stackrel{\text{def}}{=} \mathrm{Aut}_k(V)$$

where ρ is a group homomorphism.

> 2.2.2 Remark: Often in representation theory one combines the map ρ and vector space V into a single object: a $k\Gamma$-module. This simultaneously encodes the vector space structure (via k-linearity) and the action by Γ.

Representations of Γ can be, in general, "analytic." One can check that the map

$$\rho : k^\times = \mathrm{GL}_1(k) \to \mathrm{GL}(k^2) \qquad \text{via} \qquad x \mapsto \begin{pmatrix} 1 & \ln|x| \\ 0 & 1 \end{pmatrix}$$

gives a group homomorphism (and thus representation) between these two groups, but the logarithm makes this representation decidedly *not algebraic*. To narrow our focus somewhat and ensure we stay within the realm of algebra, we make the following definition:

2.2.3 Definition: A **polynomial representation** of Γ is a representation ρ such that the structure maps of ρ are polynomials in the functions $c_{ij} : \Gamma \to k$ that extract the $(i,j)^{th}$ entry.

> 2.2.4 Remark: Recall (or learn for the first time!) that the *structure maps* of a representation (ρ, V) are a collection of maps r_{ij} for $1 \leq i, j, \leq n$ from Γ to k such that for all $g \in \Gamma$:
>
> $$g \cdot v_i = \sum_{j=1}^{n} r_{ij}(g) v_j$$
>
> where we have picked a basis $\{v_1, \ldots, v_n\}$ for V. Of course changing basis may change our r_{ij}, but an invariant of the representation is the **span** $\langle r_{ij} \rangle$.

2.2.5 Remark: If all r_{ij} are homogeneous polynomials of the same degree, we say that ρ is a **homogeneous polynomial representation** of Γ.

2.2.6 Definition: Let $M_k(n) = M(n)$ be the collection of all polynomial representations of GL_n and let $M_k(n,r) = M(n,r)$ be the collection of all degree r polynomial representations of GL_n.

Generally speaking, we will identify both of these with an appropriate subcategory of $k\Gamma$-**mod**.

It is the *polynomial* representations that we will concern ourselves with in the following sections.

2.2.1 Reducing scope

Using some of our familiar friends from representation theory (as well as some clever twists), we can simplify this picture considerably by proving the following structural result:

2.2.7 Theorem ([Sch01, pp.7-10])
Every polynomial representation V over an infinite field k decomposes as a direct sum

$$V \cong \bigoplus_{\delta\in\mathbb{N}} V_\delta$$

where V_δ is a *homogeneous* polynomial representation of degree δ.

Clearly, then, it suffices to understand the *homogeneous degree r* polynomial representations of Γ if we are looking to understand the larger structure.

We begin with a useful lemma extracted from a proof in [Sch01] echoing the general theory of orthogonal decomposition of Artinian algebras.

2.2.8 Lemma
Let $C_0,\dots,C_m \in M_n(k)$ be mutually orthogonal idempotent matrices that sum to the identity. That is,

$$I_n = \sum_i C_i \quad \text{and} \quad C_iC_j = \delta_{ij}C_i$$

for all $0 \le i,j \le m$. Then there exists an invertible matrix P such that for some positive integers $d_0,\dots,d_m$ with $\sum_k d_k = n$ and for all i,

$$P^{-1}C_iP = \begin{pmatrix} 0_{N_i} & & \\ & I_{d_i} & \\ & & 0_{M_i} \end{pmatrix}$$

Where $N_i = \sum_{0 \le j < i} d_j$ and $M_i = n - d_i - N_i$

PROOF (OF LEM 2.2.8)

We set $S_k = \{C_0, C_1, \dots, C_k\}$ and we proceed by induction on k. When $k = 0$, $S_k = \{C_0\}$. Now since $C_0^2 = C_0$, we get that 1 and 0 are the only eigenvalues of C_0, so there is an $r \times r$ matrix P_0 and a positive integer d_0 such that

$$P_0^{-1} C_0 P_0 = \begin{pmatrix} I_{d_0} & \\ & 0_{n-d_0} \end{pmatrix}.$$

which establishes the base case.

Now assume that we have a matrix P_{k-1} such that this property holds for all elements of S_{k-1}. Define, for each $0 \le i \le k$,

$$C_i' \stackrel{\text{def}}{=} P_{k-1}^{-1} C_i P_{k-1}$$

and since the C_k is assumed to be orthogonal to all other C_i,

$$C_k' = \begin{pmatrix} 0_{N_k} & \\ & D_k \end{pmatrix}$$

for some D_k.

Now by properties of block diagonal matrices, we have

$$D_k^2 = D_k$$

so the eigenvalues of D_k are again one and zero. Thus there is an invertible $Q \in \mathrm{GL}_{n-N_k}$ such that

$$Q^{-1} D_k Q = \begin{pmatrix} I_{d_k} & \\ & 0_{M_k} \end{pmatrix}$$

and so by setting

$$P_k \stackrel{\text{def}}{=} P_{k-1} \begin{pmatrix} I_{N_k} & \\ & Q \end{pmatrix}$$

we can define

$$C_i'' \stackrel{\text{def}}{=} P_k^{-1} C_i P_k = \begin{pmatrix} I_{N_k} & \\ & Q \end{pmatrix}^{-1} C' \begin{pmatrix} I_{N_k} & \\ & Q \end{pmatrix}$$

for $0 \le i \le k$, we see immediately that $C_i' = C_i''$ for $0 \le i < k$ and furthermore

$$C_k'' = \begin{pmatrix} 0_{N_k} & \\ & Q^{-1} D_k Q \end{pmatrix} = \begin{pmatrix} 0_{N_k} & & \\ & I_{d_k} & \\ & & 0_{M_k} \end{pmatrix}$$

completing the inductive step. This this result holds for all S_i and in particular for S_m, so the result is proven. ♠

As well as another result on a special class of commuting block diagonal matrices:

> **2.2.9 Lemma**
> Let k be an infinite field and let A be a block diagonal matrix over k of the form
>
> $$A = \operatorname{diag}(x^m I_{d_m}, x^{m-1} I_{d_{m-1}}, \dots, I_{d_0})$$
>
> where d_i is (clearly) the dimension of the $(m-i)^{th}$ block and let B be any matrix that commutes with A for every choice of $x \in k$. Then B is block diagonal of the same shape as A.

Proof (of lem 2.2.9)
We proceed by comparing the entries in AB and BA: notice that

$$(AB)_{ij} = \sum_k A_{ik} B_{kj} = A_{ii} B_{ij} = x^a B_{ij}$$

and

$$(BA)_{ij} = \sum_k B_{ik} A_{kj} = B_{ij} A_{jj} = x^b B_{ij}.$$

We will show that if the $(i,j)^{th}$ postion is not in one of the blocks of A, then it is zero.

But if (i,j) is not in one of the blocks of A, then the nonzero element in the i^{th} row and the nonzero element in the j^{th} column (x^a and x^b in the above equations) are not the same! Since x is arbitrary, this forces $B_{ij} = 0$, so B is block diagonal with blocks the same as A. ♠

> 2.2.10 Remark: Notice that in the above proof we used implicitly that there is an $x \in k$ such that for all a, b
>
> $$x^a = x^b \quad \Rightarrow \quad a = b$$
>
> which is true if (and only if!) k is infinite. This is because any such x must be a root of $x^n = x$ for some n, which has finitely many roots over any field (but every element in $\mathbb{F}_p$ satisfies $x^p = x$).

And finally using these two lemmas allows us to prove our main result:

Proof (of thm 2.2.7)
We recreate the argument in Schur's thesis, translated from German and reinterpreted in more modern parlance.

Let (ρ, V) be a polynomial representation of Γ with $\dim_k V = r$. Then let $x \in k^\times$ be arbitrary (thought of as an indeterminate) and consider the matrix $xI_n \in \Gamma$. The image of this matrix under ρ is a matrix

$$\rho(A) = \begin{pmatrix} p_{11}(x) & \cdots & p_{1r}(x) \\ \vdots & \ddots & \vdots \\ p_{r1}(x) & \cdots & p_{rr}(x) \end{pmatrix}$$

where each p_{ij} is a polynomial in x. Let $m = \max_{i,j} \deg p_{ij}$, and this gives us a decomposition

$$\rho(A) = x^m C_0 + x^{m-1} C_1 + \cdots + x C_{m-1} + C_m$$

where each C_i is an $r \times r$ matrix.

Let y be another indeterminate and $B = yI_n$. By virtue of being a representation of Γ, we get

$$\rho(A)\rho(B) = \rho(xI_n)\rho(yI_n) = \rho(xyI_n) = \rho(AB)$$

and using this setup we prove the following result:

$$\text{For all } 0 \le i, j \le m, \text{ with the } C_l \text{ as above,} \quad C_i C_j = \delta_{ij} C_i$$

That this is true can be established by comparing coefficients in the equation

$$\rho(AB) = \rho(A)\rho(B)$$
$$C_0(xy)^m + \cdots + C_i(xy)^{m-i} + \cdots + C_m = C_0^2 x^m y^m + \cdots + C_i C_j x^{m-i} y^{m-j} + \cdots + C_m^2$$

Indeed, we immediately get that $C_i = C_i^2$ and furthermore the coefficients on $x^i y^j$ when $i \neq j$ give us

$$0 = C_{m-i} C_{m-j}.$$

Thus we have shown that the C_i form a set of orthogonal idempotent matrices and evaluating our original equation at $x = 1$, we get (since ρ is a homomorphism)

$$I_r = 1C_0 + \cdots + 1C_m = \sum C_i$$

so the result from lemma 2.2.8 applies: we get a matrix P such that

$$P^{-1}\rho(xI_n)P = \begin{pmatrix} x^m I_{d_0} & & & & \\ & x^{m-1} I_{d_1} & & & \\ & & \ddots & & \\ & & & x I_{d_{m-1}} & \\ & & & & I_{d_m} \end{pmatrix}$$

Now let $\rho'(g) = P^{-1}\rho(g)P$ for all $g \in \Gamma$. This is a representation of Γ since it it differs from ρ by an automorphism of $\mathrm{GL}(V)$. Since matrix multiplication is an algebraic operation, ρ' is still a polynomial representation of Γ. But notice that for all $g \in \Gamma$

$$\rho'(g)\rho'(xI_n) = \rho'(xg) = \rho'(xI_n)\rho'(g)$$

Then lemma 2.2.8 gives us that $\rho'(g)$ decomposes in the same way for all $g \in \Gamma$, so we know that

ρ' decomposes as a direct sum of representations

$$\rho' = \sum_{i=0}^{m} \rho'_i$$

where for each i and $\lambda \in k$,

$$\rho'_i(\lambda g) = \rho'_i(\lambda I_{d_i})\rho'_i(g) = \lambda^i \rho'_i(g)$$

so each ρ'_i is a homogeneous degree i polynomial representation of Γ.

But of course the decomposition of a representation is independent of choice of basis, so we get a decomposition of ρ into homogenous pieces, as desired. ♠

2.2.2 Monomials and multi-indices

All of the discussion up to this point has revolved around polynomials in n^2 variables, which quickly gets unwieldy unless one uses some better notation. To that end,

2.2.11 Definition: An (n, r)-**multi-index** i is an r-tuple $(i_1, \ldots, i_r)$ where each $i_j \in \underline{n} \stackrel{\text{def}}{=} \{1, \ldots, n\}$. The collection of all (n, r)-multi-indices is denoted $I(n, r)$.

> 2.2.12 Remark: One can also think of an element $i \in I(n, r)$ as a (set) map
>
> $$i : \underline{r} \to \underline{n}.$$

The idea here is to associate to each monomial in a polynomial ring in many variables a tuple indicating its multidegree. That is we think of

$$(i_1, \ldots, i_r) \quad \leftrightsquigarrow \quad x_{i_1} \cdots x_{i_r}$$

as corresponding to the same object. Which is wonderful except for one small flaw: polynomials are commutative and multi-indices (as we have defined them) aren't! For example, in $I(3, 4)$,

$$(2, 2, 1, 3) \quad \leftrightsquigarrow \quad x_1 x_2^2 x_3 \quad \leftrightsquigarrow \quad (3, 2, 1, 2).$$

To handle this disparity, we define an equivalence relation on $I(n, r)$ where we say that $i \sim j$ if they are in the same orbit under the natural $\mathfrak{S}_r$ action. That is, if there exists $\sigma \in \mathfrak{S}_r$ such that

$$(i_1, \ldots, i_r) = (j_{\sigma(1)}, \ldots, j_{\sigma(r)})$$

In the context of polynomial representations of Γ, we want to consider polynomials in the coordinate functions c_{ij}, so as a matter of notation if $i, j \in I(n,r)$, let $c_{i,j}$ denote the monomial

$$c_{i,j} = c_{i_1 j_1} \cdots c_{i_r j_r}.$$

Again, we want to take into account that we can permute the order on the right hand side, but now we need that i_k and j_k remain linked to the same function. To deal with this, we define an equivalence relation $\sim$ on $I(n,r) \times I(n,r)$ such that

$$(a,c) \sim (b,d)$$

if there exists a $\sigma \in \mathfrak{S}_r$ such that

$$(a_1, \dots, a_r) = (b_{\sigma(1)}, \dots, b_{\sigma(r)}) \quad \text{and} \quad (c_1, \dots, c_r) = (d_{\sigma(1)}, \dots, d_{\sigma(r)}).$$

The upshot of this work is that it gives us a bijection between (total) degree r monomials in the c_{ij} and the set

$$I(n,r) \times I(n,r) / \sim$$

2.2.3 $A_k(n,r)$

Notice that if $V \in M(n,r)$, each of its structure maps are homogeneous degree r polynomials. As the first object of study, consider

2.2.13 Definition: Let $A_k(n,r) = A(n.r)$ denote the collection of all homogeneous degree r polynomials in the coordinate functions $c_{ij} : \Gamma \to k$.

It is not too hard to see that

> **2.2.14 Proposition**
> $A_k(n,r)$ is spanned by the elements
>
> $$\{c_{i,j} | (i,j) \in I(n,r) \times I(n,r)\}$$

however it takes a short argument to see

> **2.2.15 Lemma**
> The dimension of $A_k(n,r)$ over k is $\binom{n^2+r-1}{n^2-1} = \binom{n^2+r-1}{r}$.

Proof

The following is a "stars and bars" argument that is pervasive in combinatorics. See for example [Sta12] if unfamiliar with these techniques.

Fix an ordering of the c_{ij} (say the dictionary order) and relabel them $\{\gamma_1, \ldots, \gamma_m\}$ (here $m = n^2$) according to this order. Then the degree r monomials are in bijection with m-tuples $(a_1, \ldots, a_m) \in \mathbb{N}^m$ such that $\sum_i a_i = r$ via the map which sends

$$(a_1, \ldots, a_m) \mapsto \gamma_1^{a_1} \cdots \gamma_m^{a_m}.$$

But choosing such an element is the same as inserting $m-1$ bars into a line of r stars (that is an ordered partition of r into m parts, where parts are allowed to be zero). But this is equivalent to choosing $m-1$ bars in a field of $m+r-1$ symbols. This is just

$$\binom{m+r-1}{m-1}$$

and a well-known identity for binomial coefficients gets us the final equality. ♠

Example 2.2
In case the reader is unfamiliar with this kind of reasoning, consider the case when $n = 5$ and $r = 4$. Then the composition $(1, 0, 0, 2, 1)$ corresponding to $\gamma_1\gamma_4^2\gamma_5$ corresponds to the stars-and-bars diagram

$$*|||**|*$$

where there are $m+r-1 = 8$ symbols, $r = 4$ of which are stars.

2.2.4 A dip into affine group schemes and category theory

$A(n, r)$ lies within $k^\Gamma = k(\Gamma)$, the k-algebra of functions $\Gamma \to k$, which has the structure of a Hopf algebra induced from the group structure on Γ. More precisely, the functor $\mathrm{GL}_n : \mathbf{Alg}_k \to \mathbf{Grp}$ that assigns to every k-algebra A the group $\mathrm{GL}_n(A)$ is representable. In other words,

$$\mathrm{GL}_n(-) \simeq \mathrm{Hom}_{\mathbf{Alg}_k}(R, -)$$

where $R \cong k[c_{ij} | 1 \leq i, j \leq n]_{\det}$.

The anti-equivalence of the categories of affine group schemes over k and finite dimensional commutative k-Hopf algebras follows from Yoneda lemma (c.f. [Wat79, chp. 1]), and along with this equivalence comes a way to translate the group structure on Γ into a coalgebra structure on R: we have maps μ, ϵ, the multiplication and unit maps on Γ satisfying the diagrams

$$\begin{tikzcd} \Gamma\times\Gamma\times\Gamma \arrow[r, "\mu\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\mu"] & \Gamma\times\Gamma \arrow[d, "\mu"] \\ \Gamma\times\Gamma \arrow[r, "\mu"] & \Gamma \end{tikzcd} \qquad \begin{tikzcd} *\times G \arrow[r, "\epsilon\times\mathrm{id}"] \arrow[dr, leftrightarrow, "\sim"] & G\times G \arrow[d, "\mu"] & G\times * \arrow[l, "\mathrm{id}\times\epsilon"'] \arrow[dl, leftrightarrow, "\sim"'] \\ & G & \end{tikzcd}$$

(where $*$ is the trivial group) giving us associativity and identity. Yoneda gives us that the maps between functors (group schemes!)

$$\mu : \Gamma \times \Gamma \to \Gamma \quad \text{and} \quad \epsilon : * \to \Gamma$$

give rise to maps in $\mathbf{Alg}_k$:

$$\Delta \stackrel{\text{def}}{=} \mu^* : R \to R \otimes_k R \quad \text{and} \quad \varepsilon \stackrel{\text{def}}{=} \epsilon^* : R \to k$$

satisfying diagrams

$$\begin{tikzcd} R \otimes R \otimes R & R \otimes R \arrow[l, "\Delta\otimes\mathrm{id}"'] \\ R \otimes R \arrow[u, "\mathrm{id}\otimes\Delta"] & R \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} k \otimes R \arrow[dr, "\sim"'] & R \otimes R \arrow[l, "\varepsilon\otimes\mathrm{id}"'] \arrow[r, "\mathrm{id}\otimes\varepsilon"] & R \otimes k \\ & R \arrow[u, "\Delta"] \arrow[ur, "\sim"'] & \end{tikzcd}$$

2.2.16 Proposition
The maps Δ and ε give R a coalgebra structure. In coordinates, if $1 \le i, j \le n$,

$$\Delta(c_{ij}) = \sum_k c_{ik} \otimes c_{kj} \quad \text{and} \quad \varepsilon(c_{ij}) = \delta_{ij}$$

2.2.17 Remark: One can easily check that these maps make R into a bialgebra by checking that Δ and ε are algebra morphisms, but what is not immediately obvious is why *these particular maps* are the ones we use on R. To see this, one must dig into the Yoneda correspondence a bit to see what happens to the multiplication map.

In service of this, let's translate matrix multiplication into a statement about representable functors. We want to define m as a map

$$m : \mathrm{Hom}(R, -) \times \mathrm{Hom}(R, -) \to \mathrm{Hom}(R, -)$$

and to see what m should do in this context, we evaluate at a k-algebra

$$m_A : \mathrm{Hom}(R, A) \times \mathrm{Hom}(R, A) \to \mathrm{Hom}(R, A)$$

where we interpret each map $R \to A$ as a matrix with entries in A by saying a map f corresponds to a matrix A_f such that

$$(A_f)_{ij} = f(c_{ij}).$$

Then if $(f, g) \in \mathrm{Hom}(R, A) \times \mathrm{Hom}(R, A)$, we want that the algebra structure is the usual matrix multiplication, so

$$m_A(f,g)=A_fA_g$$

and by computing the $(i,j)^{th}$ entry everywhere, we get

$$m_A(f,g)(c_{ij})=(A_fA_g)_{ij}=\sum_{k=1}^{n}(A_f)_{ik}(A_g)_{kj}=\sum_k f(c_{ik})g(c_{kj}).$$

This gives us the values of our component maps everywhere, so this defines the natural transformation m. Then (the proof of) Yoneda tells us that we can compute the corresponding algebra morphism as

$$\mu(c_{ij})=m_{R\otimes R}(\iota_l\otimes\iota_r)(c_{ij})=\sum_k\iota_l(c_{ik})\iota_r(c_{kj})=\sum_k c_{ij}\otimes c_{kj}.$$

Above we call ι_l (resp. ι_r) to be the map $R\to R\otimes R$ which embeds R into the left (resp. right) tensor factor. Notice that $\iota_l\otimes\iota_r=\mathrm{id}_{R\otimes R}$.

Using the same identification between maps and matrices over A, let $*:k\to A$ be the unique map sending $1_k\mapsto 1_A$. Then we want

$$u_A(*)=f:R\to A$$

corresponding to the identity $(n\times n)$ matrix over A. So

$$u_A(*)(c_{ij})=f(c_{ij})=(I_n)_{ij}=\delta_{ij}\cdot 1_A.$$

Again applying Yoneda, we have

$$\varepsilon(c_{ij})=u_k(\mathrm{id}_k)(c_{ij})=\delta_{ij}1_k$$

and we have our counit map.

In fact, as mentioned before, R becomes a bialgebra (a Hopf algebra even, although we won't need the antipode here). This means that Δ and ε are algebra morphisms for the natural algebra structure given by multiplication m on R. In diagrams:

$$\begin{tikzcd} R^{\otimes 4} \arrow[r, "\mathrm{id}\otimes\tau\otimes 1"] & R^{\otimes 4} \arrow[r, "m\otimes m"] & R\otimes R \\ R\otimes R \arrow[u, "\Delta\otimes\Delta"] \arrow[rr, "m"] & & R \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} R\otimes R \arrow[r, "m"] \arrow[d, "\varepsilon\otimes\varepsilon"] & R \arrow[d, "\varepsilon"] \\ k\otimes k \arrow[r, "m"] & k \end{tikzcd}$$

where $\tau:R\otimes R\to R\otimes R$ is the twist map $a\otimes b\mapsto b\otimes a$. Chasing an element through the diagram on the left, we get

$$\widetilde{m}\circ(\Delta\otimes\Delta)(c_{ij}\otimes c_{ab})=\sum_{1\le k,l\le n}c_{ik}c_{al}\otimes c_{kj}c_{lb}=\Delta(c_{ij}c_{ab})$$

or using our multi-index notation,

$$\Delta(c_{(i,a),(j,b)}) = \sum_{(k,l)\in I(n,2)} c_{(i,a),(k,l)} \otimes c_{(k,l),(j,b)}.$$

Written more simply, the fact that Δ is an algebra morphism can be written

$$\Delta(a \cdot b) = \Delta(a) * \Delta(b)$$

under suitable definitions of $\cdot$ and $*$. In a way that can be made precise, this means in particular that

$$\Delta(a \cdot b \cdot c) = \Delta(a) * \Delta(b \cdot c) = \Delta(a) * \Delta(b) * \Delta(c)$$

and so on (since multiplication everywhere is associative) and therefore we can define this for arbitrary monomials and extend k-linearly:

> **2.2.18 Proposition**
> If $i, j \in A(n, r)$, then
>
> $$\Delta(c_{i,j}) = \sum_{k\in I(n,r)} c_{i,k} \otimes c_{k,j} \quad \text{and} \quad \varepsilon(c_{i.j}) = \delta_{i,j}$$

One can easily see that degree is preserved by Δ, meaning that

> **2.2.19 Proposition**
> Δ and ε descend to a coalgebra structure on $A(n, r)$. That is, $A(n, r)$ is a (k-)coalgebra.

2.2.5 The Schur algebra

Finally we get to the actual object of study:

2.2.20 Definition: A **Schur algebra** is an element of the two-parameter family $\{S(n, r)\} = \{S_k(n, r)\}$ where n and r are any positive integers. As a set, $S(n, r)$ is the linear dual of $A(n, r)$:

$$S(n, r) = A(n, r)^* = \mathrm{Hom}_k(A(n, r), k)$$

Let $\xi_{i,j}$ denote the element dual to $c_{i.j} \in A(n, r)$. In other words:

$$\xi_{(a,b)}(c_{i,j}) = \begin{cases} 1, & (a, b) \sim (i, j) \\ 0, & \text{otherwise} \end{cases}$$

> **2.2.21 Lemma**
> The coalgebra structure (Δ, ε) on $A(n,r)$ define an algebra structure on $S(n,r)$.

Proof

Since k is an initial object in $\mathbf{Alg}_k$, there is a unique map $u : k \hookrightarrow S(n,r)$ sending 1 to the unit function $\mathbb{1}$, which is given by

$$\mathbb{1}(c_{i,j}) = c_{i,j}(I_n) = \delta_{i,j}$$

Define multiplication $(\cdot)$ in $S(n,r)$ as follows: if $f, g \in S(n,r)$ then for any $x \in A(n,r)$ define

$$(f \cdot g)(x) = m_k \circ (f \otimes g) \circ \Delta(x) = \sum f(x_{(1)}) g(x_{(2)})$$

where $m_k : k \otimes k \to k$ denotes multiplication in k and $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$ in Sweedler notation.

Then we must just confirm that these maps satisfy the properties of a k-algebra. $(\cdot)$ is k-bilinear because (for instance)

$$\begin{aligned}((af + bg) \cdot h)(x) &= \sum (af + bg)(x_{(1)}) \otimes h(x_{(2)}) \\ &= \sum a(f(x_{(1)}) \otimes h(x_{(2)})) + b(g(x_{(1)}) \otimes h(x_{(2)})) \\ &= a \sum f(x_{(1)}) \otimes h(x_{(2)}) + b \sum g(x_{(1)}) \otimes h(x_{(2)}) \\ &= (a(f \cdot h) + b(g \cdot h))(x).\end{aligned}$$

By k-linearity, it suffices to show that the unit $\mathbb{1}$ acts as it should on the spanning set $\xi_{i,j}$ for a basis element $c_{a,b}$:

$$(\mathbb{1} \cdot \xi_{i,j})(c_{a,b}) = \sum_{k=1}^{n} \mathbb{1}(c_{a,k}) \cdot \xi_{i,j} = \mathbb{1}(c_{a,a}) \cdot \xi_{i,j}(c_{a,b}) = \xi_{i,j}(c_{a,b})$$

and a similar identity holds on the right.

Then it remains to show that this multiplication is associative. Again by linearity it suffices to check that this works on the spanning set $\{c_{i,j}\}$:

$$\begin{aligned}((\alpha \cdot \beta) \cdot \gamma)(c_{i,j}) &= \sum_{k \in I(n,r)} (\alpha \cdot \beta)(c_{i,k}) \gamma(c_{k,j}) \\ &= \sum_k \left(\sum_{l \in I(n,r)} \alpha(c_{i,l}) \beta(c_{l,k}) \right) \gamma(c_{k,j}) \\ &= \sum_l \alpha(c_{i,l}) \left(\sum_k \beta(c_{l,k}) \gamma(c_{k,j}) \right) \\ &= \sum_l \alpha(c_{i,l}) (\beta \cdot \gamma)(c_{l,j}) \\ &= (\alpha \cdot (\beta \cdot \gamma))(c_{i,j}).\end{aligned}$$

Thus since we have k-linear maps $\mathbb{1}$ and $m = (\cdot)$ satisfying the usual identity and associativity diagrams, $S(n,k)$ is a k-algebra with $\mathbb{1}$ and m as its unit and multiplication. ♠

Why have we done all this work to construct Schur algebras, one may ask? Well the idea is that there is a map

$$e : k\Gamma \to S(n,r)$$

where it sends

$$\sum_i k_i g_i \mapsto \sum_i k_i e_{g_i}$$

where e_g is the "evaluation at g" map–that is, for all $x \in A(n,r)$,

$$e_g(x) = x(g).$$

> **2.2.22 Lemma**
> The map $e : k\Gamma \to S(n,r)$ is surjective.

Proof

Let ξ be any element orthogonal to $W = \operatorname{Im} e$, if one exists. Say this element is $\sum_{i,j} a_{i,j}\xi_{i,j}$. But then the element $c = \sum a_{i,j} c_{i,j}$ is zero on every element in the image of e. In other words, for all $\varkappa \in k\Gamma$,

$$c(\varkappa) = e_\varkappa(c) = 0.$$

But the only function in k^Γ that is zero on all of Γ is the zero function. Thus $c = 0$, whence its coefficients $a_{i,j}$ are zero. So ξ is zero, so $W = S(n,r)$ ♠

In this way, e induces a map between categories

$$f : S_k(n,r)\text{-}\mathbf{mod} \to M_k(n,r)$$

where $f(V) = V$ as far as underlying sets are concerned, but we are given a new action: for any $\sum k_i g_i \in k\Gamma$ and $c \in V \in S(n,r)$-**mod**,

$$\left(\sum k_i g_i\right) \cdot v = \left(\sum k_i e_{g_i}\right) \cdot v.$$

Notice that

> **2.2.23 Lemma**
> f as above defines a functor that sends any $\alpha : V \to W \in S(n,r)$-**mod** to the map that is identical as a map of sets.

Proof

Since we are working in a concrete category[3], and since the V and $f(V)$ are identical as sets and since any morphism is sent to the same map on underlying sets, commutativity of the functorality diagram

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[d, "f"] & W \arrow[d, "f"] \\ f(V)=V \arrow[r, "f(\alpha)=\alpha"] & f(W)=W \end{tikzcd}$$

is trivial to check. It remains only to check that $f(\alpha): V \to W$ is a morphism in $M(n,r)$, so that this diagram makes sense.

To check this, notice that for any $v \in V \in M(n,r)$ and $g \in k\Gamma$,

$$f(\alpha)(g \cdot v) = \alpha(e_g \cdot v) = e_g \cdot \alpha(v) = g \cdot f(\alpha)(v)$$

so f is a functor. ♠

The upshot here is

> **2.2.24 Theorem**
> The functor f above is an equivalence of categories.

Proof

That f is faithful is easy enough to see since f is the identity functor on the level of underlying sets. Let $g: V \to W$ be a morphism in $M(n,r)$ and consider the same (set) map in $\widetilde{g} \in S(n,r)$-**mod**. For any $\xi \in S(n,r)$, let $\varkappa \in k\Gamma$ be an element such that $e_\varkappa = \xi$ (which exists due to lem 2.2.22). Then

$$\widetilde{g}(\xi \cdot v) = g(\varkappa \cdot v) = \varkappa \cdot g(v) = \xi \cdot \widetilde{g}(v)$$

But then $f(\widetilde{g}) = g$, so f is full.

It remains to see that f is essentially surjective. But again this is not too hard to see since for any $V \in M(n,r)$ the same object setwise with the action given by

$$\xi \cdot v = e_\varkappa \cdot v$$

(where again $\varkappa$ was chosen using lem 2.2.22) maps to $V \in M(n,r)$ and we are done. ♠

[3] There is a faithful functor to **Set**; equivalently, any morphism is determined by where it sends the "set of elements" comprising the object.

2.2.25 Remark: Actually, the above proof can be modified slightly to show that f has a functorial inverse–that is, f is an *isomorphism of categories*. Since we are only interested in representations up to isomorphism, however, equivalence is just as good.

2.2.26 Remark: Using this equivalence, we identify $S(n,r)$-**mod** with $M(n,r)$ whenever it suits us.

One of a representation theorist's favorite kinds of results follows:

2.2.27 Corollary
If $\operatorname{char} k = 0$, the algebra $S(n,r)$ is semisimple.

Proof
$k\Gamma$ is semisimple since $\operatorname{char} k = 0$, so every element in $k\Gamma$-**mod** splits into a direct sum of simple modules. But the irreducible objects in $M(n,r)$ and those in $S(n,r)$-**mod** are the same and decompositons in one category pull back to the other, so every element in $S(n,k)$-**mod** is also completely reducible. ♠

2.2.6 Weights and characters

The discussion in section 2.2.2 highlights an important idea: while we care about the *quantities* in which each c_{ij} occurs in a monomial, we are not particularly interested in the *order*. Sometimes it is easier, then, to simply regard these as weak compositions:

2.2.28 Definition: Let n and r be integers as usual. Then denote by $[a_1,\dots,a_n]$ the **weight** corresponding to $(i_1,\dots,i_r)\in I(n,r)$ where for each i,

$$a_i = \#\{k \in \underline{r} | i_k = i\}$$

Denote by $\Lambda(n,r)$ the collection of all weights.

2.2.29 Remark: Another way to realize $\Lambda(n,r)$ is in the presentation

$$\Lambda(n,r) = \left\{ [a_1,\dots,a_n] \middle| \sum_i a_i = r \right\},$$

or as the set of compositions of r into n parts (allowing zeros).

Yet another is to think of $\Lambda(n,r)$ as the set of $\mathfrak{S}_r$ orbits in $I(n,r)$ (where now two objects are distinguished only if their "contents" vary).

Recall (c.f. 2.2.20) that we had that $\xi_{i,j}(c_{a,b}) = 1$ if and only if $(i,j) \sim (a,b)$. Because of this, it makes sense (if α is the weight of i) to write

$$\xi_\alpha \overset{\text{def}}{=} \xi_{\alpha,\alpha} \overset{\text{def}}{=} \xi_{i,i}$$

since the action is the same irrespective of the choice of representative i of α.

Notice that the weights admit a $\mathfrak{S}_n$ action

$$\sigma \cdot [a_1, \dots, a_n] = [a_{\sigma(1)}, \dots, a_{\sigma(n)}]$$

then

2.2.30 Definition: $\Lambda_+(n,r)$ is the orbit space of $\Lambda(n,r)$ under the above $\mathfrak{S}(n)$ action.

2.2.31 Remark: The above are called the **dominant weights** in $M(n,r)$. Since each orbit α contains an element $[a_1, \dots, a_n] \in \alpha$ such that

$$a_1 \geq a_2 \geq \cdots \geq a_n$$

we will often identify weights with their weakly-decreasing representative.

Sometimes we will refer to the dominant weight representing the orbit of $i \in I(n,r)$ as the **shape of i.**

The theory of weights in representations of Γ closely mirrors similar decompositions in other Artinian algebras: first we identify a family of (mutually orthogonal) idempotents:

2.2.32 Lemma
For $\alpha \in \Lambda(n,r)$ and $i,j \in I(n,r)$,

$$\xi_\alpha \xi_{i,j} = \begin{cases} \xi_{i,j}, & i \in \alpha \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \xi_{i,j}\xi_\alpha = \begin{cases} \xi_{i,j}, & j \in \alpha \\ 0, & \text{otherwise} \end{cases}$$

Proof
We can compute the image of these on the $c_{a,b} \in A(n,r)$:

$$\begin{aligned} \xi_\alpha \cdot \xi_{i,j}(c_{a,b}) &= \sum_k \xi_\alpha(c_{a,k})\xi_{i,j}(c_{k,b}) \\ &= \xi_\alpha(c_{a,a})\xi_{i,j}(c_{a,b}) \end{aligned}$$

where above we used that $\xi_\alpha(c_{i,j}) = 0$ unless $i = j$. But

$$\xi_\alpha(c_{a,a}) = \begin{cases} 1, & a \in \alpha \\ 0, & \text{otherwise} \end{cases}$$

so

$$\xi_\alpha \cdot \xi_{i,j}(c_{a,b}) = \begin{cases} \xi_{i,j}(c_{a,b}), & a \in \alpha \\ 0, & \text{otherwise} \end{cases}$$

but in the case where $a \in \alpha$ and $\xi_{i,j}(c_{a,b}) \neq 0$, this implies that $i \sim a$, so $i \in \alpha$. So finally,

$$\xi_\alpha \cdot \xi_{i,j}(c_{a,b}) = \begin{cases} \xi_{i,j}(c_{a,b}), & i \in \alpha \\ 0, & \text{otherwise} \end{cases}$$

and since this holds for any $c_{a,b}$, the left-hand side is proven. A symmetric argument goes through for the right-hand side. ♠

For the next step, we decompose the identity into a sum of these idempotents:

> **2.2.33 Lemma**
> We have the decomposition
> $$\mathbb{1} = \sum_{\alpha \in \lambda(n,r)} \xi_\alpha.$$

Proof
On the one hand, for any $c_{a,b} \in A(n,r)$, $\mathbb{1}(c_{a,b}) = \delta_{a,b}$. On the other hand, for any α,

$$\xi_\alpha(c_{a,b}) = 0$$

when $a \neq b$ *or when* $a \notin \alpha$.

Therefore when $a = b$, there is precisely one α (the orbit of $a = b$) such that $\xi_\alpha(c_{a,b}) = 1$, so putting this all together,

$$\sum_{\alpha \in \Lambda(n,r)} \xi_\alpha(c_{a,b}) = \delta_{a,b}$$

whence these two functions are equal. ♠

> 2.2.34 Remark: Using lemma 2.2.33, we can then decompose any $V \in M(n,r)$ into weight spaces:
> $$V = \mathbb{1} \cdot V = \sum_{\alpha \in \Lambda(n,r)} \xi_\alpha V$$
> which we will denote
> $$\xi_\alpha V = V^\alpha.$$

2.2.35 Definition: The **formal character** of a representation $V \in M(n,r)$ is a polynomial

$$\Phi_V(X_1,\ldots,X_n) = \sum_{\alpha\in\Lambda(n,r)} (\dim V^\alpha) X_1^{\alpha_1}\cdots X_n^{\alpha_n} = \sum_{\alpha\in\Lambda_+(n,r)} (\dim V^\alpha) m_\alpha(X_1,\ldots,X_n)$$

where m_α is the *monomial symmetric polynomial*

$$m_\alpha(X_1,\ldots,X_n) = \sum_{\sigma\in\mathfrak{S}_n} X_{\sigma(1)}^{\alpha_1}\cdots X_{\sigma(n)}^{\alpha_n}.$$

2.2.7 Irreducible representations

The irreducible representations in $M(n,r)$ are given by a couple of results by some of the big names in representation theory: the original proof for $k=\mathbb{C}$ was proven in [Sch01, p.37] and then generalized in a later paper by Weyl [Wey25] and in work by Chevalley[4]:

> **2.2.36 Theorem**
> Fix the usual lexicographical ordering on monomials in $k[X_1,\ldots,X_n]$. Let n and r be given integers with $n \geq 1$ and $r \geq 0$ Let k be an infinite field. Then
>
> (a) For each $\lambda \in \Lambda_+(n,r)$, there exists an (absolutely) irreducible module $F_{\lambda,k}$ in $M_k(n,r)$ whose character $\Phi_{\lambda,k}$ has leading term $X_1^{\lambda_1}\cdots X_n^{\lambda_n}$.
>
> (b) Every irreducible $V \in M_k(n,r)$ is isomorphic to $F_{\lambda,k}$ for exactly one $\lambda \in \Lambda_+(n,r)$.

So then the problem of classifying the simple modules (the "basic building blocks" in the semisimple case) is completely solved for infinite fields. It remains to demonstrate a way to construct $F_{\lambda,k}$.

2.2.37 Definition: Fix some $\lambda \in \Lambda_+(n,r)$. Notice that this corresponds to a Young diagram with r boxes. Fix any labeling $1,\ldots,r$ of the boxes in the Young diagram corresponding to λ. Let T denote the diagram for λ along with this labeling.

Let $i : \underline{r} \to \underline{n}$ be any map. Then denote by T_i the λ-**tableau**, which is T with the k^{th} entry consisting of $i(k) \in \underline{r}$.

[4] Green [Gre07] mentions a paper by Serre: *Groupes de Grothendieck des Schémas en Groupes Réductifs Déployés* [Ser68], which makes mention to Chevalley's contributions in proving the existence of modules with prescribed characters. This author was unable to find Chevalley's work.

> 2.2.38 Remark: This notation varies slightly (but not in spirit) from the notation in Green's book. He denotes the Young diagram by $[\lambda]$ and lets T^λ be the labelling of the boxes in $[\lambda]$–a bijection $[\lambda] \to \underline{r}$.

Example 2.3
Let $\lambda = (3,1,1) \in \Lambda_+(3,5)$. Thus T is of shape

$$\ydiagram{3,1,1}$$

Then if we fix the left-to-right/top-to-bottom ordering of the boxes in T and let i : $\{1,2,3,4,5\} \to \{1,2,3\}$ be given by $(2,1,3,3,2)$, we get the λ-tableau

$$T_i = \young(213,3,2)$$

The core tool in constructing (a basis for) the irreducible modules is in the following definiton:

2.2.39 Definition: Let $\lambda \in \Lambda_+(n,r)$ be some shape with a fixed labeling and let $i, j : \underline{r} \to \underline{n}$. Then the **bideterminant of T_i and T_j** is

$$(T_i : T_j) = \sum_{\sigma \in C(T)} \operatorname{sgn}(\sigma) c_{i, j\sigma} \in A_k(n,r)$$

where $C(T)$ is the column stabilizer of T.

This definition can be a bit difficult to unpack, so we give some examples:

Example 2.4
(a) $\lambda = (2,1,0) \in \Lambda_+(3,3)$

$$\left(\young(12,3) : \young(31,2) \right) = \begin{vmatrix} c_{13} & c_{12} \\ c_{33} & c_{32} \end{vmatrix} c_{21} = (c_{13}c_{32} - c_{12}c_{33})c_{2,1} = c_{(1,2,3),(3,1,2)} - c_{(1,2,3),(2,1,3)}$$

(b) $\lambda = (n, 0, \ldots, 0) \in \Lambda_+(m,n)$

$$\left(\boxed{a_1}\boxed{a_2}\boxed{a_3}\cdots\boxed{a_n} : \boxed{b_1}\boxed{b_2}\boxed{b_3}\cdots\boxed{b_n}\right) = c_{a_1 b_1}\cdots c_{a_n b_n}$$

(c) $\lambda = (1, \ldots, 1, 0, \ldots) \in \Lambda_+(m,n)$ where $n \geq m$

$$\left(\begin{matrix}\boxed{a_1}\\ \boxed{a_2}\\ \vdots \\ \boxed{a_n}\end{matrix} : \begin{matrix}\boxed{b_1}\\ \boxed{b_2}\\ \vdots \\ \boxed{b_n}\end{matrix}\right) = \begin{vmatrix} c_{a_1 b_1} & \cdots & c_{a_1 b_n} \\ \vdots & \ddots & \vdots \\ c_{a_n b_1} & \cdots & c_{a_n b_n} \end{vmatrix}$$

In the following, let $l : \underline{r} \to \underline{n}$ be $(1, \ldots, 1, 2, \ldots, 2, 3, \ldots)$ such that for any shape λ the λ-tableau T_l is

$$\begin{array}{|c|c|cc|c|} \hline 1 & 1 & \cdots & \cdots & 1 \\ \hline 2 & 2 & \cdots & \multicolumn{1}{|c|}{2} & \\ \cline{1-4} \vdots & & & & \\ \cline{1-1} k & & & & \\ \cline{1-1} \end{array}$$

with i in every box on the i^{th} row from the top.

2.2.40 Definition: Define, for every shape $\lambda \in \Lambda_+(n, r)$, the module

$$D_{\lambda,k} = \langle (T_l : T_i) \rangle_{i \in I(n,r)}$$

where l is the filling defined above.

According to [Gre07], these modules were originally called "Weyl modules", while he (and we) reserve this name for the contravariant dual of these objects. To construct them, define the map

$$\pi : E^{\otimes r} \to D_{\lambda,k}, \tag{1}$$

and we get objects originally defined in Carter and Lusztig's treatment of modular representations of GL_n [CL74] and tweaked by Green in [Gre07]:

2.2.41 Definition: Given a shape λ, the **Weyl module of shape λ over k** is $V_{\lambda,k} \overset{\text{def}}{=} N^{\perp}$ where

$$N \overset{\text{def}}{=} \ker\pi \hookrightarrow E^{\otimes r} \to D_{\lambda,k}$$

and the orthogonal complement of N is taken with respect to the canonical contravariant form on $E^{\otimes r}$ that has the property $\langle e_i, e_j \rangle = \delta_{ij}$.

In their original paper [CL74, p.218], Carter and Lusztig showed that these modules are, in fact, generated as $S(n,r)$-modules by a single element:

> **2.2.42 Theorem**
> Let $\lambda \in \Lambda_+(n,r)$ and T the Young diagram corresponding to λ. Let l be the labelling above. Then the element
> $$f_l = e_l \cdot \sum_{g \in C(T) \subset \mathfrak{S}_n} \operatorname{sign}(\sigma)\sigma$$
> generates $V_{\lambda,k}$ as a $S(n,r)$-module.

Proof (sketch.)
We refer the reader to Green's [Gre07, p.46] proof for the details, but the idea is as follows: he relies on an earlier result that the modules $D_{\lambda,k}$ have a basis consisting of the bideterminants

$$(T_l : T_i)$$

such that T_i is in "standard form" (meaning that it forms a valid Young tableau). One can define a nondegenerate contravariant form

$$(\cdot,\cdot) : V_{\lambda,l} \times D_{\lambda,k} \to k$$

by pulling back any element in $D_{\lambda,k}$ to a representative in $E^{\otimes r}$ under the map $\pi : E^{\otimes r} \to D_{\lambda,k}$.

Recall that $V_{\lambda,k}$ is defined as the orthogonal complement (under the canonical form $\langle \cdot,\cdot \rangle$ on $E^{\otimes r}$) of $\ker\pi$. This gives us that $(\cdot,\cdot)$ is indeed well-defined. From there, Green does some computation to show that one can bootstrap the independence of the $(T_l : T_i)$ to prove that of the set

$$\{\xi_{jl} f_l | j \in I(n,r), T_j \text{ standard}\}$$

forms a (k—)basis for $V_{\lambda,k}$, and therefore f_l generates the entire module under the $S(n,r)$ action. ♠

> **2.2.43 Lemma**
> The modules $V_{\lambda,k}$ have a unique maximal submodule $V_{\lambda,k}^{\max}$

Proof ([Gre07, p.47])
Begin by noticing that the weight space $V_{\lambda,k}^{\lambda}$ is spanned by the single element f_l. This is because

$$\xi_l \cdot \xi_{il} f_l = \delta_{il} f_l$$

so the only nonzero basis vector from the proof of thm. 2.2.42 is f_l itself. Since f_l generates all of $V_{\lambda,k}$ as an $S(n,r)$-module, however, any proper submodule M of $V_{\lambda,k}$ must be contained in the complement of $V_{\lambda,k}^{\lambda}$. Thus the sum of all proper submodules is contained in the complement of this weight space, and is therefore proper! This sum is our $V_{\lambda,k}^{\max}$ ♠

We are finally in good shape to compute the irreducible modules promised to us in thm. 2.2.36. We define

$$F_{\lambda,k} = V_{\lambda,k}/V_{\lambda,k}^{\max}$$

where $V_{\lambda,k}^{\max}$ is the unique maximal submodule guaranteed to us by lemma 2.2.43. It remains to show that the $F_{\lambda,k}$ have the requisite characters $\Phi_{\lambda,k}$. But notice that $V_{\lambda,k}^{\lambda}$ is one-dimensional, so the character (c.f. definition 2.2.35) of $V_{\lambda,k}$ is of the form

$$m_\lambda(X_1,\ldots,X_n) + \sum_{\lambda\neq\alpha\in\Lambda_+(n,r)} \dim V_{\lambda,l}^{\alpha} m_\alpha(X_1,\ldots,X_n)$$

but since each $V_{\lambda,k}^{\alpha}$ is contained in $V_{\lambda,k}^{\max}$, it occurs as a weight space of this maximal submodule with the same multiplicity. Therefore the character of $V_{\lambda,k}^{\max}$ is

$$\sum_{\lambda\neq\alpha\in\Lambda_+(n,r)} \dim V_{\lambda,l}^{\alpha} m_\alpha(X_1,\ldots,X_n)$$

so we can conclude that

$$\Phi_{V_{\lambda,k}}(X_1,\ldots,X_n) = m_\lambda(X_1,\ldots,X_n) = X_1^{\lambda_1}\cdots X_n^{\lambda_n} + \cdots$$

which has leading term (under the lexicographic ordering) precisely what we wanted.

2.3 Explicit examples for comparison

To demonstrate the theory developed above, we begin a computation (in a simple case) of the isomorphism classes of irreducible representations of both $S_{\mathbb{C}}(2,2)$ and $\mathfrak{S}_2$.

2.3.1 The symmetric group on two letters

The representation theory (over $k = \mathbb{C}$) of $\mathfrak{S}_2$ is as simple as it comes: of course $\mathfrak{S}_2 \cong \mathbb{Z}/2\mathbb{Z}$ and we know that there are $|G|$ nonisomorphic irreducible representations of an abelian group G over $\mathbb{C}$. Since we are talking about a symmetric group, we can realize these as the trivial and sign representations, represented by the Young diagrams:

$$\begin{array}{|c|c|}\hline & \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$$

As submodules of the regular representation $k\mathfrak{S}_2 = ke \oplus k(12)$, we can construct these as $\langle e + (12)\rangle$ (trivial representation) and $\langle e - (12)\rangle$ (sign representation).

2.3.2 The Schur algebra $S_{\mathbb{C}}(2,2)$

Since $\operatorname{char}\mathbb{C} = 0$, corollary 2.2.27 implies that $S_{\mathbb{C}}(2,2)$ is semisimple, so it suffices to identify the irreducible submodules therein. We know

$$S = S_{\mathbb{C}}(2,2) \cong \mathbb{C}^2 \otimes \mathbb{C}^2$$

so $\dim_{\mathbb{C}} S = 4$. The theory outlined above gives us that isomorphism types of irreducible modules are in bijection with compositions of 2 of length 2, meaning we have two isomorphism types: one corresponding to $\lambda_1 = (1,1)$ and one corresponding to $\lambda_2 = (2,1)$.

Using the construction of $D_{\lambda,\mathbb{C}}$ from above, we can compute these two irreducible modules explicitly:

Example 2.5 ($\lambda_1 = (1,1)$)
In this case our shape is $(1,1)$, corresponding to the Young diagram

$$\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$$

and then $D_{\lambda_1,\mathbb{C}}$ is spanned by the element

$$(T_l : T_{(2,1)}) = \left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} : \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array} \right) = c_{12}c_{21} - c_{11}c_{22} = c_{(1,2),(2,1)} - c_{(1,2),(1,2)} \in A_{\mathbb{C}}(2,2)$$

since all other bideterminants of this shape are zero or linearly dependent. Thus this is a one-dimensional irreducible representation.

Example 2.6 ($\lambda_2 = (2,0)$)
Now our shape is $(2,0)$, corresponding to the diagram

$$\square\square.$$

The bideterminants here are

$$\begin{aligned}(T_l : T_{(1,1)}) = \left(\boxed{1}\boxed{1} : \boxed{1}\boxed{1}\right) &= c_{11}^2 \\ (T_l : T_{(1,2)}) = (T_l : T_{(2,1)}) &= c_{11}c_{12} \\ (T_l : T_{(2,2)}) &= c_{12}^2\end{aligned}$$

So we have a three-dimensional irreducible representation spanned by $\langle c_{11}^2, c_{11}c_{12}, c_{12}^2\rangle$.

Since these are the only two Young diagrams of size two, these examples form a complete list of isomorphism classes of irreducible representations of $S_{\mathbb{C}}(2,2)$.

If we prefer instead to recognize our irreducibles as submodules of $E^{\otimes 2} = (ke_1 \oplus ke_2)^{\otimes 2}$ (giving us a more obvious action by our algebras), we can use the short exact sequence

$$0 \to N \hookrightarrow E^{\otimes 2} \twoheadrightarrow D_{\lambda,\mathbb{C}} \to 0$$

to define the $N = \ker \pi$, where π is the map defined in equation (1) above. Then we can compute the orthogonal complement to N to get $V_{\lambda,\mathbb{C}}$. We can compute:

$$V_{\lambda_1,\mathbb{C}} = \langle e_1 \otimes e_2 - e_2 \otimes e_1\rangle$$

and

$$V_{\lambda_2,\mathbb{C}} = \langle e_1 \otimes e_1,\, e_1 \otimes e_2 + e_2 \otimes e_1,\, e_2 \otimes e_2\rangle.$$

3 The Schur-Weyl Functor

From the discussion in the last section it is evident that the combinatorics behind the representation theory of $S(n,r)$ and $\mathfrak{S}_r$ have some intersections in their use of Young tableaux and this connection is more than superficial. In fact, there is a functor relating the representations of these two objects in the following way:

3.1 Construction of the functor $\mathcal{F}$

Let $V \in M_k(n,r)$ be a $S(n,r)$-representation and select any weight $\alpha \in \Lambda(n,r)$. Then the weight space (cf. rmk 2.2.34)

$$V^\alpha = \xi_\alpha V$$

becomes a $S(\alpha) \stackrel{\text{def}}{=} \xi_\alpha S(n,r)\xi_\alpha$-module using the action from $S(n,r)$. Now if we allow $r \leq n$ and let

$$\omega = (1,\dots,1,0,\dots,0) \in \Lambda(n,r)$$

notice that $S(\omega)$ is spanned by the elements

$$\xi_\omega \xi_{i,j} \xi_\omega, \quad i,j \in I(n,r)$$

but by the multiplication rules established in the definition of $S(n,r)$, these are nonzero precisely when i and j are both of shape ω. So then since $\xi_{i,j} = \xi_{i\sigma,j\sigma}$ for all $\sigma \in \mathfrak{S}_r$, we can take as a basis of $S(\omega)$ the set

$$\{\xi_{u\pi,u} | \pi \in \mathfrak{S}_r\}$$

where $u = (1,2,\cdots,r) \in I(n,r)$.

To prove the next statement we require a computational result.

> **3.1.1 Lemma**
> If $u = (1,2,\dots,r) \in I(n,r)$, then for all $\pi, \sigma \in \mathfrak{S}_r$,
>
> $$\xi_{u\pi,u} \cdot \xi_{u\sigma,u} = \xi_{u\pi\sigma,u}.$$

Proof

Using the formulas for multiplication in $S(n,r)$, recall that

$$\xi_{u\pi,u} \cdot \xi_{u\sigma,u} = \sum Z_{i,j}\xi_{i,j} \tag{2}$$

where

$$Z_{i,j} = \#\{s \in I(n,r) | (u\pi,u) \sim (i,s) \text{ and } (u\sigma,u) \sim (s,j)\}.$$

Then for each i,j, since $u = (1,2,\dots,r)$ has no stabilizer in $\mathfrak{S}_r$, there is a unique g such that $u\pi g = i$, meaning that $s = ug$.

But then this fixes (again a unique) $h \in \mathfrak{S}_r$ such that $u\sigma h = s = ug$ whence $\sigma h = g$. One computes that

$$u\pi\sigma h = u\pi g = i \quad \text{and} \quad uh = j$$

therefore since in the above computation s was completely determined by i, we have

$$Z_{i,j} = \begin{cases} 1, & (i,j) \sim (u\pi\sigma, u) \\ 0, & \text{otherwise} \end{cases}$$

and the result follows. ♠

Using this result, we prove a more obviously useful statement:

> **3.1.2 Lemma**
> $S(\omega) \cong k\mathfrak{S}(r)$.

Proof
Define the map $\varphi : S(\omega) \to k\mathfrak{S}_r$ on the basis above to be

$$\varphi(\xi_{u\pi,u}) = \pi$$

and extending k-linearly.

This is a homomorphism since

$$\varphi(\xi_{u\pi,u}\xi_{u\sigma,u}) = \varphi(\xi_{u\pi\sigma,u}) = \pi\sigma = \varphi(\xi_{u\pi,u})\varphi(\xi_{u\sigma,u})$$

and it is bijective since it is bijective on the respective bases and is thus bijective as a linear map. ♠

The upshot of these lemmas is that one can define the **Schur-Weyl functor**

$$\mathcal{F} : M_k(n,r) \to \mathbf{Rep}(\mathfrak{S}_r)$$

via the map that sends any representation V to its ω weight space $V^\omega \in S(\omega)$-**mod** $\simeq \mathbf{Rep}(\mathfrak{S}_r)$.

3.2 The general theory

The idea of the Schur functor fits into a larger context: Let S be a k-algebra and let $M \in S$-**mod**. Furthermore, let $e \in S$ be a (nonzero) idempotent. Then one can define a functor

$$\mathcal{F} : S\text{-}\mathbf{mod} \to eSe\text{-}\mathbf{mod} \quad \text{via} \quad V \mapsto eV.$$

An important property of this functor is

3.2.1 Proposition
The image of an irreducible S module under the functor $\mathcal{F}$ above is zero or irreducible.

Proof
Let $e \in S$ be the idempotent in the discussion above and let $W \subseteq eV$ be any nonzero eSe-submodule. Then notice that eW is a nonzero S-module contained in $e^2V = eV$, so $eW = eV$. But since $eW \subseteq W$, this forces $W = eV$, so $\mathcal{F}(V)$ is irreducible. ♠

Next, a discussion in Green [Gre07, p. 56] gives us a natural thought process to follow in constructing a partial inverse to this functor. Let $\mathcal{G} : eSe\text{-}\mathbf{mod} \to S\text{-}\mathbf{mod}$ be an extension of scalars: specifically, if $M \in eSe\text{-}\mathbf{mod}$, then

$$\widetilde{\mathcal{G}}(M) = Se \otimes_{eSe} M.$$

This is clearly functorial and furthermore satisfies the property that

$$\mathcal{F} \circ \widetilde{\mathcal{G}}(M) = \mathcal{F}(Se \otimes_{eSe} M) = e(Se \otimes_{eSe} M) = eSe \otimes_{eSe} M \cong e \otimes_{eSe} M \cong M$$

so it is a right inverse (up to isomorphism) to $\mathcal{F}$—a good candidate for our purposes.

3.2.2 Remark: It is easy to prove the fact, which I glossed over above, that $M \cong e \otimes M$ via the eSe-isomorphism $m \mapsto e \otimes m$.

What we are really looking for, however, is a functor that sends irreducible modules to irreducibles. It can be shown that $\widetilde{G}$ *does not* satisfy this property, so we define

3.2.3 Definition: If $M \in S\text{-}\mathbf{mod}$ and $e \in S$ is an idempotent, denote by $M_{(e)}$ the largest S-submodule of $(1-e)M$.

which enables us to define the functor

$$\mathcal{G} : eSe\text{-}\mathbf{mod} \to S\text{-}\mathbf{mod} \quad \text{via} \quad M \mapsto \widetilde{\mathcal{G}}(M)/\widetilde{\mathcal{G}}(M)_{(e)}.$$

This leads to the result:

3.2.4 Proposition
If $M \in eSe\text{-}\mathbf{mod}$ is irreducible, then so is $\mathcal{G}(M)$.

Proof
Let W be an S-module such that

$$\widetilde{\mathcal{G}}(M)_{(e)} \subseteq W \subseteq \widetilde{\mathcal{G}}(M)$$

Then consider multiplying by e in the above inculsions: we get

$$0 = e\widetilde{\mathcal{G}}(M)_{(e)} \subseteq eW \subseteq e\widetilde{\mathcal{G}}(M) = \mathcal{F} \circ \widetilde{\mathcal{G}}(M) \simeq M$$

which, by the irreducibility of M, forces either $eW = 0$ (in which case $W \subseteq \widetilde{\mathcal{G}}(M)_{(e)}$ and we are done) or else $eW = e\widetilde{\mathcal{G}}(M)$.

In this latter case, we find

$$\widetilde{\mathcal{G}}(M) = Se \otimes M \simeq Se \otimes eSeM = S(eSe \otimes M) = S(e\widetilde{\mathcal{G}}(M)) = SeW \subseteq W$$

Thus we can conclude that $W = \widetilde{\mathcal{G}}(M)$, so $\mathcal{G}(M)$ has no nontrivial proper submodules, so it is simple. ♠

3.3 Properties of $\mathcal{F}$ and $\mathcal{G}$

Returning to the specific case of $S = S(n, r)$ and $eSe \cong \mathfrak{S}_r$, the theory developed in the last part gives us a pair of functors

$$\mathcal{F} : M(n, r) \to \mathfrak{S}_r\textbf{-mod}, \qquad \mathcal{G} : \mathfrak{S}_r\textbf{-mod} \to M(n, r),$$

each of which preserve irreducibility. We also have that

> **3.3.1 Proposition**
> If $M \in M(n, r)$ is irredicible and if $eM \neq 0$, then $\mathcal{G} \circ \mathcal{F}(M) = \mathcal{G}(eM) \cong M$.

Proof

Notice by prop. 3.2.1 and the following discussion that eM is irreducible and (by assumption) nonzero, so

$$\mathcal{F} \circ \mathcal{G}(eM) \cong eM$$

and since

$$0 \neq eM \subseteq M$$

and M is irreducible, $eM = M$. ♠

This leads us to the following realization:

> **3.3.2 Corollary**
> If $\{M_\alpha\}_{\alpha \in \mathcal{I}}$ is a complete collection of irreducible modules over $S(n, r)$, then there exists a subset $\mathcal{J} \subseteq \mathcal{I}$ such that $\{\mathcal{F}(M_\alpha)\}_{\alpha \in \mathcal{J}}$ is a complete (and irredunant) set of irreducible $\mathfrak{S}_r$-modules.
>
> Furthermore, any index $j \in \mathcal{J}$ satisfies $\mathcal{G} \circ \mathcal{F}(M_j) \cong M_j$.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\mathcal{F}\eta"] \arrow[dr, "1_{\mathcal{F}}"'] & \mathcal{F}\mathcal{G}\mathcal{F} \arrow[d, "\varepsilon\mathcal{F}"] \\ & \mathcal{F} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{G} \arrow[r, "\eta\mathcal{G}"] \arrow[dr, "1_{\mathcal{G}}"'] & \mathcal{G}\mathcal{F}\mathcal{G} \arrow[d, "\mathcal{G}\varepsilon"] \\ & \mathcal{G} \end{tikzcd}$$

Figure 1: The triangle axioms

Whenever we have a pair of opposing functors like this, we hope that we can prove that they tie the two categories together in a nice way, and in fact we can! A standard categorical fact, reprinted here from Emily Riehl's book, empowers us to make our last proof:

3.3.3 Proposition ([Rie16, prop. 4.2.6])
Given a pair of opposing functors $\mathcal{F} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathcal{G}$, is an adjoint pair (with $\mathcal{F} \dashv \mathcal{G}$) if and only if there exist unit and counit maps

$$\eta : 1_{\mathcal{C}} \to \mathcal{G}\mathcal{F} \qquad \text{and} \qquad \varepsilon : \mathcal{F}\mathcal{G} \to 1_{\mathcal{D}}$$

satisfying the *triangle axioms* illustrated in figure 1.

3.3.4 Theorem
The functors are adjoints.

Figure out what direction the adjunction goes and prove this!

4 Strict polynomial functors

The theory of strict polynomial functors has its genesis in the idea of *polynomial maps between vector spaces*, or equivalently the rational maps between the schemes they represent. The category of vector spaces with these polynomial maps—and more specifically, the representaion category associated to it—gives the category $\mathbf{Rep}\,\Gamma_k^d$ of strict polynomial functors.

Originally definied by Friedlander and Suslin in [FS97], the authors there showed that the category $S(n,r)$-**mod** is equivalent to this category, introducing the language of polynomial functors as a way to understand the structure of representations of the Schur algebras.

This process was carried out by Krause [Kra13] and his students Aquilino and Reischuk [AR17]. In the former, Krause identifies projective generators $\Gamma^{d,V}$ for $\mathbf{Rep}\,\Gamma_k^d$ and defines the tensor product by defining it for projectives and taking the appropriate colimits. In the latter paper, the construction is further elucidated and it is proven that the Schur-Weyl functor $\mathcal{F}$ is monoidal.

4.1 Polynomial maps

TODO: Rewrite this to account for the fact I am now using the algebraic definition of polynomial maps instead of the geometric one.

Let V, W be vector spaces over a field k. There are many equivalent formulations of polynomial maps between such spaces, but one that this author of this paper finds particulaly motivating is the following:

4.1.1 Definition: Let V, W be as above. Then the set of **polynomial maps from V to W** is defined to be

$$\mathrm{Hom}_{\mathrm{Pol}}(V,W) \overset{\mathrm{def}}{=} \mathrm{Hom}_{\mathbf{Sch}/k}(V,W).$$

To make sense of this definition, one recalls that every $V \in \mathbf{Vect}_{\mathrm{k}}$ corresponds to an affine k-scheme $\operatorname{Spec} S^*(V^\vee) = V \otimes_k -$ (which we, through an abuse of notation, again denote V) represented by the symmetric algebra of the dual of V. Thus the polynomial maps are precisely the rational maps one considers between these objects in their algebro-geometric realizations.

> 4.1.2 Remark: In Friedlander and Suslin's original paper, they define these maps abstractly as $S^*(V^\vee) \otimes W$. That this agrees with our definition (assuming that V and W

are finite dimensional) follows from the following series of isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Pol}}(V,W) &\overset{\text{def}}{=} \mathrm{Hom}_{\mathbf{Sch}/k}(V,W)\\ &\simeq \mathrm{Hom}_{\mathbf{Alg}_k}(S^*(W^\vee), S^*(V^\vee))\\ &\simeq \mathrm{Hom}_{\mathbf{Alg}_k}(W^\vee, S^*(V^\vee))\\ &\simeq W\otimes S^*(V^\vee)\end{aligned}$$

where we used above properties of affine schemes and standard facts of the linear algebra of finite dimensional vector spaces as well as the fact that a map from $S^*(V)$ is determined uniquely by its images on V.

For reasons that will become apparent shortly, it is easier to use the above remarks to define a polynomial map in the following way:

4.1.3 Definition: If $V, W \in \mathbf{Vect}_\mathrm{k}$ are finite dimensional, a **polynomial map** $f : V \to W$ can be alternatively defined as an element

$$f \in W \otimes S^*(V^\vee) \cong \mathrm{Hom}_{\mathbf{Sch}/k}(V,W)$$

through the identifications above.

The upshot to this seemingly more *ad hoc* definition is that, while it introduces the restriction of finite dimensionality (which will suffice for our definitions anyways), it enables us to make more simple the following idea:

4.1.4 Definition: Let V and W be vector spaces. Then a map $f \in \mathrm{Hom}_{\mathrm{Pol}}(V,W)$ is called **homogeneous degree** d if it corresponds (under the isomorphisms above) to an element

$$f \in W \otimes S^d(V^\vee).$$

This is clearly a tangible and sensible way to define a degree d map and it is less obvious how to define a property on the map of corresponding varieties that achieves the same goal. We will see in the next subsection other ways to define this notion that may appeal more to representation theorists.

Example 4.1
Here are some examples of polynomial maps:

- The identity (scheme) map $\mathrm{id} : V \to V$ is a (homogeneous degree 1) polynomial map. This corresponds to the element
$$\sum_{i=1}^{n} v_i \otimes v_i^\vee \in V \otimes S^*(V^\vee)$$
where $v_1, \ldots, v_n$ is a basis for V.
- If $V = \langle v_1, \ldots, v_n \rangle$ and $W = \langle w_1, \ldots, w_m \rangle$, the element
$$\sum_{1}^{m} w_i \otimes (v_i^\vee \otimes v_i^\vee)$$
gives rise to a map of algebras that sends basis element
$$\sum_{\sigma \in \mathfrak{S}_k} w^\vee_{i_{\sigma(1)}} \otimes \cdots \otimes w^\vee_{i_{\sigma(k)}} \mapsto \sum_{\sigma \in \mathfrak{S}_k} v^\vee_{i_{\sigma(1)}} \otimes v^\vee_{i_{\sigma(1)}} \otimes \cdots \otimes v^\vee_{i_{\sigma(k)}} \otimes v^\vee_{i_{\sigma(k)}}$$
which corresponds to a homogeneous degree 2 polynomial (scheme) map $V \to W$.

4.2 The categories $\mathcal{P}_k$ and $\mathbf{Rep}\,\Gamma_k^d$

Before we define these categories we should describe the objects in question!

4.2.1 Definition: A **strict polynomial functor** is a functor $T : \mathbf{Vect}_k \to \mathbf{Vect}_k$ such that for any $V, W \in \mathbf{Vect}_k$, the map on Homs
$$T_{V,W} : \mathrm{Hom}_k(V, W) \to \mathrm{Hom}_k(T(V), T(W))$$
is a polynomial map. That is,
$$T_{V,W} \in \mathrm{Hom}_{\mathrm{Pol}}\big(\mathrm{Hom}_k(V, W), \mathrm{Hom}_k(T(V), T(W))\big)$$

Earlier I promised that we would have a more representation-theoretic interpretation of the homogeneous degree of a strict polynomial functor. I am nothing if I am not true to my word:

4.2.2 Lemma (Lem. 2.2 in [FS97])
Let T be a strict polynomial functor and let $n \geq 0$ be an integer. Then the following conditions are equivalent:

(a) For any $V \in \mathbf{Vect}_k$, any field extension k'/k and any $0 \neq \lambda \in k'$, the k'-linear map $T_{k'}(\lambda \cdot 1_{V_{k'}}) \in \mathrm{End}_{k'}(T(V)_{k'})$ coincides with $\lambda^n 1_{T(V)_{k'}}$.

(b) For any $V \in \mathbf{Vect}_k$, n is the only weight of the representation of the algebraic group $\mathbb{G}_m$ in $T(V)$ obtained by applying T to the evident representation of $\mathbb{G}_m$ in V.

(c) For any $V, W \in \mathbf{Vect}_k$, the polynomial map

$$T_{V,W} : \mathrm{Hom}_k(V, W) \to \mathrm{Hom}_k(T(V), T(W))$$

is homogeneous of degree n (in the sense of 4.1.4).

Proof
Spell this out. ♠

4.2.3 Definition: The category $\mathcal{P}_d$ is the full subcategory

$$\mathcal{P}_d \subset \mathbf{Func}(\mathbf{Vect}_k, \mathbf{Vect}_k)$$

whose objects are the **strict polynomial functors of degree d.**

4.2.4 Theorem
The map

$$\Psi : \mathcal{P}_d \to S(n, d)\text{-}\mathbf{mod}$$

given by evaluation at k^n:

$$T \mapsto T(k^n)$$

is an equivalence of categories.

Proof
refactor the proof from Friedlander and Suslin. ♠

4.2.1 Yet another category

Just when you thought you had enough categories to consider, Krause developed a new category that more succinctly captures the stucture of homogeneous degree d polynomial maps: there the author changes the domain of these functors to encode the desired properties automatically.

4.2.5 Definition: Let k be any commutative ring. Then $P_k \subset \mathbf{Vect}_k$ is the full subcategory of finitely-generated projective k-modules.

Define $\Gamma^d P_k$ to be the category of **divided powers**—the objects are the same as those of P_k, but such that

$$\mathrm{Hom}_{\Gamma^d P_k}(V, W) = \Gamma^d \mathrm{Hom}_{P_k}(V, W)$$

where $\Gamma^d X = (X^{\otimes d})^{\mathfrak{S}_d}$ denotes the d^{th} divided powers of the k-module X.

Finally, as a matter of notation, let

$$\mathbf{Rep}\,\Gamma_k^d = \mathbf{Rep}\,\Gamma^d P_k = \mathbf{Func}(\Gamma^d P_k, k\text{-}\mathbf{mod})$$

which we (suggestively) call the **category of homogeneous degree d strict polynomial functors.**

4.2.6 Remark: Of course, when k is a field, we get that $P_k = \mathbf{Vect}_\mathrm{k}$ and an element

$$T \in \mathbf{Rep}\,\Gamma_k^d = \mathbf{Func}(\Gamma^d\,\mathbf{Vect}_\mathrm{k}, \mathbf{Vect}_\mathrm{k}),$$

is a functor that, on objects, is a map $\mathbf{Vect}_\mathrm{k} \to \mathbf{Vect}_\mathrm{k}$ and on morphisms is of the form

$$T_{VW} : \mathrm{Hom}_{\Gamma^d\,\mathbf{Vect}_\mathrm{k}}(V, W) = \Gamma^d\,\mathrm{Hom}_k(V, W) \to \mathrm{Hom}_k(T(V), T(W))$$

which, while not exactly our definition of a polynomial map, suggests some level of similarity.

Finish up proving that this category is also equivalent. Maybe try to see if I can cook up an equivlence directly between $\mathcal{P}_d$ and $\mathbf{Rep}\,\Gamma_k^d$?

4.3 Monoidicity of $\mathcal{F}$

Define Henning's monoidal structure on $\mathbf{Rep}\,\Gamma_k^d$ and go through Aquilino and Reischuk's proof the SW functor is monoidal under this definition.

4.4 A dictionary

Spell out how one can translate between the three different categories: irreducibles and tensor structure.

5 Tensor products in the derived category $\mathbf{D}^{\mathbf{b}}(S(n,r))$

Do some definitions here for derived category stuff and talk about how the induced tensor product from before compares with the one naturally on $\mathbf{D}^{\mathrm{b}}(S(n,r))$

5.1 Derived categories

5.2 Compatibility of monoidal structures

6 The (Balmer) spectrum of a tensor triangulated category

In Paul Balmer's 2005 paper [Bal05], he developed a general framework for understanding the structure of certain kinds of categories that arose from the original constructions in algebraic geometry. Serving as a source of inspiration for Balmer, in [FP07] Friedlander and Pevtsova proved that the projective geometry of the cohomology ring of a finite group scheme can be recovered by looking at "ideals" in the category **stmod** G of stable G modules.

Using this as a springboard, Balmer ported the definitions of ideals and prime ideals to tensor-triangulated categories (see below) and proved a broader result that gives some tools for better analyzing familes of representaitons of finite groups (among other things).

6.1 Some motivation and a definition

Let $\mathcal{C}$ be a symmetric monoidal (i.e. tensor) category with tensor product $\otimes$ and unit object $\mathbb{1}$. After giving some thought to the matter, one realizes that a ring is given by putting a "compatible" monoidal structure on top of an abelian group, and to that end, one may consider the case when $\mathcal{C}$ is also additive.

This perspective gives us an interesting analogy between (unital, commutative) rings in algebra and category theory. Since every triangulated category is also additive, we can further specify that $\mathcal{C}$ be triangulated:

6.1.1 Definition: A **tensor-triangulated** category $\mathcal{C}$ is both a moniodal category and a triangulated category such that the monoidal structure preserves the triangluated structure.

As a reminder, such a category is equipped with a tensor product $-\otimes- : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and unit object $\mathbb{1}$, along with a collecton distinguished triangles $\mathcal{T}$ comprised of objects in $\mathcal{C}$ and shift functor (an auto-equivalence) $(-)[1] : \mathcal{C}\to\mathcal{C}$ such that: $-\otimes-$ is a triangulated (or exact) functor in each entry (it takes $\mathcal{T}$ to itself).

6.1.1 Aside: Why triangulation?

Look into and figure out why we need stability here to make sense of things.

6.2 Construction of the spectrum

Once the appropriate context is identified (which is the real ingenuity of Balmer's paper), the construction very closely mirrors the construction seen in elementary algebraic geometry:

6.2.1 Definition: Let $\mathcal{C}$ be a tensor-triangulated category (TTC). Then a **(thick tensor) ideal** $I \subseteq \mathcal{C}$ is a full triangulated subcategory with the following conditons:

- *(2-of-3 rule/Triangulation)* If A, B, and $C \in \mathcal{C}$ are objects that fit into a distinguished triangle
$$A \to B \to C \to A[1]$$
in $\mathcal{C}$, and if any two of the three are objects in I, then so is the third.

- *(Thickness)* If $A \in I$ is an object that splits as $A \cong B \oplus C$ in $\mathcal{C}$, then both A and B belong to I.

- *(Tensor Ideal)* If $A \in I$ and $B \in \mathcal{C}$ then $A \otimes B = B \otimes A \in I$.

> 6.2.2 Remark: The first condition just ensures that our ideals respect the triangulated structure (and thus stability) in the parent category $\mathcal{C}$. The final condition is the most direct analog of an ideal and is central in the analogy between this theory and classical AG.

From here the rest of the picture is relatively straightforward:

6.2.3 Definition: Let $\mathcal{C}$ be a TTC as before. Then an ideal $I \subseteq \mathcal{C}$ is called a **prime ideal** if, whenever $A \otimes B \in I$ for some $A, B \in \mathcal{C}$, either A or B is in I.
We call the collection of all primes the **spectrum** of $\mathcal{C}$ and write $\mathrm{Spc}(\mathcal{C})$.

Here the construction varies slightly from the traditional construction of Spec: we define
$$Z(S) \stackrel{\text{def}}{=} \{\mathcal{P} \in \mathrm{Spc}(\mathcal{C}) | S \cap \mathcal{P} = \varnothing\}$$
and define sets (for any $S \subseteq \mathcal{C}$ and $A \in \mathcal{C}$):
$$U(S) \stackrel{\text{def}}{=} \mathrm{Spc}(\mathcal{C}) \setminus Z(S) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{C}) | S \cap \mathcal{P} \neq \varnothing\}$$
and
$$\mathrm{supp}(A) \stackrel{\text{def}}{=} Z(\{A\}) = \{\mathcal{P} \in \mathrm{Spc}\,\mathcal{C} | A \notin \mathcal{P}\}$$
A routine check of the axioms shows us

> **6.2.4 Lemma (2.6 of [Bal05])**
> The sets $U(S)$ for all $S \subseteq \mathcal{C}$ form a basis for a topology on $\mathrm{Spc}\,\mathcal{C}$.

which we call the **Zariski topology**, giving $\mathrm{Spc}\,\mathcal{C}$ the structure of a topological space.

6.3 As a locally-ringed space

The above discussion mentions how we can construct a topological space from the set of prime thick tensor ideals in a TTC, but there is even more we can get: the structure of a locally-ringed space.

To get this, we need to define the structure sheaf:

6.3.1 Definition: Let $\mathcal{C}$ be a tensor-triangulated category and let $\operatorname{Spc}\mathcal{C}$ be the construction discussed above. Then the structure sheaf on $\operatorname{Spc}\mathcal{C}$ is given by the sheafification $\mathcal{O}_{\mathcal{C}}$ of the presheaf

$$\widetilde{\mathcal{O}}_{\mathcal{C}} : \operatorname{Open}(\operatorname{Spc}\mathcal{C})^{\mathrm{op}} \to \mathbf{Ring}$$

given by

$$\widetilde{\mathcal{O}}_{\mathcal{C}}(U) \stackrel{\text{def}}{=} \operatorname{End}_{\mathcal{C}/\mathcal{C}_Z}(\mathbb{1}_U)$$

where $U \subseteq \operatorname{Spc}\mathcal{C}$ is an open set and $\mathcal{C}_Z$ is the thick tensor ideal in $\mathcal{C}$ supported on $Z = \operatorname{Spc}\mathcal{C} \setminus U$.

6.3.2 Remark: That $\mathcal{C}_Z$ is a thick tensor ideal requires some work, but it follows from work that Balmer does to define a support data (X, σ) on a tensor-triangulated category and showing that for any subset $Y \subset X$ of its associated topological space, the following set

$$\{A \in \mathcal{C} | \sigma(A) \subseteq Y\}$$

is a thick tensor ideal of $\mathcal{C}$ (c.f. lem. 3.4).

7 Questions and extensions

The representation theory of $S(n,r)$ in positive characteristic

Computing the spectrum of $\mathbf{D}^{\mathbf{b}}(S(n,r))$

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