The Grassmannian as a Quotient of $\mathrm{GL}_n(\Bbbk)$

Nico Courts

Summer 2019

Introduction

These notes are my summary of the realization of the Grassmanian $\mathrm{Gr}(n,k)$ as a quotient of the Lie group $\mathrm{GL}_n(\Bbbk)$. In particular the focus will be on $\Bbbk = \mathbb{R}$ or $\mathbb{C}$. I also talk about quantum deformations of $\mathrm{Gr}(n,k)$.

Vista: Where We're Headed

The idea that GL_n acts on vector subspaces of a Euclidean space should be unsurprising and natural, but the upshot to considering this viewpoint is that is allows us to consider this object from the perspective of smooth manifold theory. This gives us some great machinery to grasp onto to prove some nice properties about $\mathrm{Gr}(n,k)$.

Eventually I will lead us slightly astray by investigating a "quantum analog" of $\mathrm{Gr}(n,k)$. This q-deformation is related closely to the kinds of structures I have been thinking about recently: specifically quantum groups. One can think of these as "deformations" of groups in some controlled way.

1 The Grassmannian as a Lie group quotient

In the following, let $\Bbbk = \mathbb{R}$ or $\mathbb{C}$ and let $V = \Bbbk^n$. For a great introduction to this theory see [Lee13], (specifically chapters 1, 7, and 21), which serves as the primary source for this section.

1.1 Smooth Manifolds

Recall that a smooth manifold is a topological manifold equipped with a **smooth structure**: that is, an atlas comprised of smoothly-compatable charts (ones where the transition functions are smooth as maps of Euclidean spaces).

1.2 The general linear Lie group

Let $G = \mathrm{GL}(V)$, initially considered just as a group. In fact, one can apply to G a smooth structure very naturally by recognizing $\mathrm{GL}(V) \subseteq M_{n\times n}(\Bbbk)$, the latter of which is identified with the Euclidean space $\Bbbk^{n^2}$, whose smooth structure is the obvious one (the identity map).

That G itself is a smooth manifold comes from the fact that $\mathrm{GL}(V) = \det^{-1}(\Bbbk \setminus \{0\})$, which is itself an open set as the inverse image of an open set under a continuous map. So then G is a (codimension zero) open (smooth) submanifold of $M_{n\times n}(k)$ whose smooth structure is inherited from the ambient space.

In fact, the algebraic and geometric structure of G come together in a very nice way to define what is called a **Lie group:**

1.2.1 Definition: Let G be a smooth manifold. Then a **Lie group** is G along with a group structure given by maps $m : G \times G \to G$ and $i : G \to G$ (representing multiplication and inverse maps) such that m and i are smooth maps (smooth in charts).

1.2.2 Proposition
G is a Lie group.

PROOF
Since we have already defined the smooth and group structures on G, it suffices to show that m and i (matrix multiplication and the inverse map) are smooth maps.

Multiplication is smooth since if we consider the i^{th} component function of m, where $i = an + b$ for $1 \leq a, b \leq n$, we get (for $\mathbf{x}$ and $\mathbf{y}$ in $\Bbbk^{n^2}$)

$$m_i(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^{n} \mathbf{x}_{an+k}\mathbf{y}_{kn+b}$$

which is a polynomial in the x_i and y_i and is thus (clearly) smooth.

That the inverse map is also smooth follows via similar reasoning: since det is smooth (again a polynomial in the entries), computing the adjugate matrix is a smooth operation and so via Cramer's rule (since $\det A \neq 0$)

$$i(A) = \frac{1}{\det A} \operatorname{adj} A$$

is also a smooth operation. ♠

1.3 The action on $\mathrm{Gr}(n, k)$

The first thing to notice is that there is a natural action of G on the set of k-dimensional subspaces of V in the following way: let $A \in \mathrm{GL}(V)$ and let $W \leq V$ be a k-dimensional subspace. Then $A(W)$ is also k-dimensional since A is invertible[1].

1.3.1 Proposition
The action defined above gives a group action of $\mathrm{GL}(V)$ on $\mathrm{Gr}(n,k)$.

Proof
This is clear since the identity matrix I_n fixes subspaces and since matrix multiplication is associative. ♠

1.4 Defining a smooth structure on the Grassmanian

As we are working in the context of smooth manifolds, we need to make some sense of how this action fits into the theory of smooth group actions. Normally,

1.4.1 Definition: If G is a Lie group and M is a smooth manifold, we say G acts smoothly on M if G acts on M as a set and the map

$$\rho : G \times M \to M$$

given by

$$\rho(g,m) = g.m$$

is a smooth map.

In this case, we don't have a smooth structure (yet!) on $\mathrm{Gr}(n,k)$, so we will define one:

1.4.1 Stabilizers

Our first step is to show that the stabilizer of an arbitrary $W \in \mathrm{Gr}(n,k)$, denoted G_W, is a closed Lie subgroup (that is a Lie group and closed smooth submanifold). To this end:

1.4.2 Theorem (Closed Subgroup Theorem)
Suppose G is a Lie group and $H \subseteq G$ is a subgroup that is also a closed subset of G. Then H is an embedded Lie subgroup.

1.4.3 Remark: The proof of this theorem is two pages of Lie theory and is a little involved for this talk. Jack does a great job (as always) in his proof (cf. [Lee13, Thm. 20.12])

Using this theorem we start by proving something about a particular subset:

[1] If you are not convinced, remember that A being invertible means all minors are nonsingular, so by extending a basis for W to V and looking at A in this basis, the top-left $k \times k$ matrix defines an isomorphism of W onto some other subspace of V.

1.4.4 Proposition
Fix some basis $\mathcal{B}$ for V. Let $W \leq V$ be the subspace of $\Bbbk^n$ by setting the last $(n-k)$ coordinates in this basis to zero (which is isomorphic to $\Bbbk^k$). Then the stabilizer G_W of W is

$$G_W = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : A \in \mathrm{GL}_k(\Bbbk),\ D \in \mathrm{GL}_{n-k}(\Bbbk),\ B \in M_{k\times n-k}(\Bbbk) \right\}$$

and this set is a closed Lie subgroup of $\mathrm{GL}(V)$.

Proof
We begin by noticing that G_W as above is precisely the stabilizer of W. But because we picked W in the way we did, we know immediately that $\mathcal{A} \in \mathrm{GL}(V)$ fixes W if the i^{th} coordinate of $\mathcal{A}(\mathbf{x})$ is zero for all $\mathbf{x} = (x_1, \ldots, x_k, 0, \ldots, 0)$. Since that x_i are arbitrary, we get that $\mathcal{A}$ is block upper triangular as above. That A and D are invertible comes from the fact that

$$\det \mathcal{A} = \det A \det D \neq 0.$$

Therefore the stabilizer of W is contained in G_W. The reverse inclusion is clear.

That G_W is a subgroup of $\mathrm{GL}(V)$ then follows from group theory. In light of thm. 1.4.2, proving that G_W is a Lie subgroup amounts to showing that G_W forms a closed subset of $\mathrm{GL}(V)$. But this is clear because this is inverse image of all of $\mathrm{GL}_k(\Bbbk)$ under $F : \mathrm{GL}(V) \to \mathrm{GL}_k(\Bbbk)$ that maps a matrix to the $k \times k$ minor in the upper left corner.

Equivalently, G_W is defined by the closed condition $\mathcal{A}_{ij} = 0$ if $i \geq k_1$ and $j \leq k$. That every such matrix has the form of G_W is clear as argued above.

Thus G_W is closed, so is a closed Lie subgroup of G. ♠

So we have shown that a particular stabilizer $G_{\Bbbk^k}$ is a closed Lie subgroup, but in fact this holds for *any* $W \in \mathrm{Gr}(n,k)$*:*

1.4.5 Corollary
For any $W \in \mathrm{Gr}(n,k)$, G_W is a closed Lie subgroup of G.

Proof
Simply choose a new basis for V such that all vectors in W are of the form

$$(v_1, \ldots, v_k, 0, \ldots, 0)$$

and apply prop. 1.4.4. ♠

1.4.2 The Quotient Manifold Theorem

Recall from point-set topology that a proper map is one that pulls back compact sets to compact sets.

1.4.6 Definition: An action of a Lie group G on a smooth manifold M is **proper** if the map

$$G \times M \to M \times M \qquad \text{via} \qquad (g,p) \mapsto (g.p, p)$$

is a proper map.

1.4.7 Remark: Jack ([Lee13, Ex. 21.3]) says that this is a *strictly weaker* condition than saying $(g,p) \mapsto g.p$ is proper.

1.4.8 Theorem (Quotient Manifold Theorem)
Let G be a Lie group acting *smoothly, freely, and properly* on a smooth manifold M. Then the orbit space M/G is a topological manifold of dimension $\dim M - \dim G$ and has a unique smooth structure making the quotient map $\pi : M \to M/G$ a smooth submersion.

Proof
As before, see Jack's proof [Lee13, thm. 21.10]. ♠

1.4.9 Remark: In case this wasn't obvious, this is a rather high-powered result and is the primary facilitator of what follows.

1.4.3 Homogeneous Spaces

1.4.10 Definition: A smooth manifold M endowed with a *transitive, smooth* action by a Lie group G is called a **homogeneous space.**

Then we use the quotient manifold theorem to construct a homogeneous space for ourselves (following part of the proof of [Lee13, thm. 21.17]):

1.4.11 Proposition
Let W be an arbitrary element in $\mathrm{Gr}(n,k)$. Then if $G = \mathrm{GL}(V)$, G/G_W is a homogeneous space given by the action

$$g \cdot (g'H) = (gg')H$$

Proof
Consider the action of G_W on G by left multiplication. This action is free since if $hg = g$, we get $h = e$. The action is a restriction of G on itself by left multiplication, which is smooth by virtue of being a Lie group, so the action is smooth.

To see the action is proper, we use the fact (proved in [Lee13, prop. 21.5(a,b)]) that the action is proper if and only if (h_i) is a sequence in G_W and (g_i) are a sequence in G, if both (h_i) and $(h_i g_i)$ converge then the (g_i) converge.

Let (g_i) be a convergent sequence in G and (h_i) a sequence in G_W such that $(h_i g_i)$ converges. Then by the continuity of the G action on itself, $h_i = (h_i g_i) g_i^{-1}$ converges to a point in G. Since G_W is closed in G, this limit converges in G_W.

Thus we can apply thm 1.4.8 to say that G/G_W has a unique smooth manifold structure making the quotient map $\pi : G \to G/G_W$ a smooth submersion. The action

$$g \cdot g' G_W = (gg')G_W$$

is transitive and smooth, giving us our homogeneous space. ♠

1.4.4 Bringing it home

Now that we have that G/G_W is a smooth manifold, consider the map

$$F : G/G_W \to \mathrm{Gr}(n,k)$$

be defined by

$$F(A \cdot G_W) = A(W).$$

1.4.12 Lemma
F defined as above is a G-equivariant bijection.

Proof
This is a well-defined map since if $gG_W = g'G_W$, we get that $g = g'h$ for some $h \in G_W$, thus

$$F(gG_W) = g \cdot W = g'hW = g'W = F(g'G_W).$$

Then that this map is surjective follows since the action of G on $\mathrm{Gr}(n,k)$ has a single orbit. That the map is injective follows since if the images of the same, this means precisely that $gW = g'W$, so $g^{-1}g' \in G_W$, proving $gG_W = g'G_W$.

G-equivariance is a simple consequence of the fact

$$g \cdot F(g'G_W) = g \cdot (g'W) = (gg')W = F(gg'G_W) = F(g(g'G_W)).$$ ♠

Now we can give $\mathrm{Gr}(n,k)$ a smooth structure such that F is a diffeomorphism[2]. Notice now that the action of G on $\mathrm{Gr}(n,k)$ is smooth since we can write

$$g \cdot V = F(g \cdot F^{-1}(V))$$

since F and the action of G on G/G_W is smooth.

We still need to establish that such a smooth structure is, in some sense, *the* structure that makes the action provided into a smooth action. This will be the smooth structure we give the Grassmannian.

We need one more theorem from smooth manifolds to finish this off:

1.4.13 Theorem (Equivariant Rank Theorem)
Let M and N be smooth manifolds and G be a Lie group. If $F : M \to N$ is a smooth, bijective, G-equivariant map with respect to a smooth, transitive action of G on M and any smooth action of G on N, then F is a diffeomorphism.

[2]Open sets are precisely the images of open sets of G/G_V and the smooth structure is the one pulled back through F.

PROOF
See [Lee13, thm 7.25]. ♠

1.4.14 Lemma
There is only one smooth structure on $X = \mathrm{Gr}(n, k)$ such that the action of G on X is smooth.

PROOF
Let $\widetilde{X}$ be X with any other smooth structure. Then if G acts smoothly on $\widetilde{X}$, the (smooth) action $\theta^{(W)} : G \to \widetilde{X}$ given by $\theta^{(W)}(g) = g \cdot W$ descends to the quotient to give us a smooth map $\mathcal{F} : G/G_W \to \widetilde{X}$, which as before can be shown to be a G-equivariant bijection.

But then by the equivariant rank theorem since the action of G on G/G_W is smooth and transitive, and the action of G on $\widetilde{X}$ is smooth, the map $\mathcal{F}$ is a diffeomorphism. But then in particular since F gave us a diffeomorphism from G/G_W to X, we have a diffeomorphism $X \cong \widetilde{X}$, so the smooth structure on X is unique. ♠

1.5 Denouement: $\mathrm{Gr}(n, k)$ is compact.

I found a neat proof of this online here.

1.5.1 Proposition
$\mathrm{Gr}(n, k)$ is compact.

PROOF
Let $S = \{v \in V = \Bbbk^n : \|v\| = 1\}$. Notice that $S \cong \mathbb{S}^{n-1}$, the $n-1$ sphere in $\Bbbk^n$, which is compact. Then $S^d = S \times \cdots \times S$ is compact as well.

Considering S^d as some choice of d unit vectors in $\Bbbk^n$, we can consider the map

$$S^d \to \Bbbk^{\binom{d}{2}} \qquad \text{via} \qquad (v_1 \ldots, v_d) \mapsto (v_1 \cdot v_2, v_1 \cdot v_3, \ldots, v_1 \cdot v_n, v_2 \cdot v_3, \ldots, v_{n-1} \cdot v_n)$$

and notice that pulling back $\mathbf{0}$ in this map gets us a (closed whence) compact subspace C of S^d corresponding to the sets of d orthonormal vectors in $\Bbbk^n$.

Finally define the equivalence relation on C where $c_1 \sim c_2$ if $\operatorname{span} c_1 = \operatorname{span} c_2$. Then the quotient $C/\sim$ is precisely $\mathrm{Gr}(n, k)$ which, as a quotient of a compact space, is compact. ♠

2 Aside: Quantized Grassmanians

I will try to say something here about the quantum theory here.

References

[Lee13] John M. Lee. *Introduction to smooth manifolds*. Second. Vol. 218. Graduate Texts in Mathematics. Springer, New York, 2013, pp. xvi+708. ISBN: 978-1-4419-9981-8.