The Grassmannian as a Quotient of $\mathrm{GL}_n(\mathbb{k})$

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Introduction

These notes are my summary of the realization of the Grassmanian $\mathrm{Gr}(n,k)$ as a quotient of the Lie group $\mathrm{GL}_n(\mathbb{k})$. In particular the focus will be on $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$.

Vista: Where We're Headed

The idea that GL_n acts on vector subspaces of a Euclidean space should be unsurprising and natural, but the upshot to considering this viewpoint is that is allows us to consider this object from the perspective of smooth manifold theory. This gives us some great machinery to grasp onto to prove some nice properties about $\mathrm{Gr}(n,k)$.

1 The Group Action

Let $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$ and let $V = \mathbb{k}^n$. Let $G = \mathrm{GL}(V)$, considered as a Lie group. The first thing to notice is that there is a natural action of G on the set of k-dimensional subspaces of V in the following way:

2 Smooth Manifold Structure