Notes on the Grassmannian

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Abstract

These notes are my summary of the development of the Schur-Weil functor from representations of the Schur algebra $S(n,d)$ to representations of $\mathfrak{S}_d$. After developing the theory that arose beginning in Schur's 1901 thesis, we will establish that this functor is exact and behaves nicely with respect to simple modules, as well as being a monoidal functor (under the correct monoidal structure).

1 Background and Introduction

The primary reference for this paper is J.A. Green's *Polynomial Representations of* GL_n [Gre07]. Other sources will be used as well, and will be introduced as needed.

The basic idea for this theory begins with Schur's thesis in 1901 [Sch01]. Here he developed the theory underlying the representation theory of the general linear group GL_n. As a tool, he recognized that the irreducible representations of GL_n took on a particular form that made them amenable to admitting actions by symmetric groups. Then Schur could rely on Frobenius' development of the representation theory for $\mathfrak{S}_d$ to say something about GL_n.

1.1 Notation

We always use $\mathfrak{S}_d$ to denote the symmetric group on d letters, and let k be an infinite field (unless otherwise noted).

1.2 Representations of Semigroups

Recall that a **semigroup** is a group where we relax the inverse requirement. If we are interested in studying the representation theory of a semigroup Γ, there are some natural modules to consider:

Example 1.1
$k\Gamma = \bigoplus_{g\in\Gamma} kg$, the **semigroup algebra** for Γ over k. Notice that if Γ is infinite, we only take the elements that are supported at finitely many g (that is, only finitely many coefficients are nonzero).

Example 1.2
k^Γ, the set of all maps $\Gamma \to k$, which is naturally a commutative k-algebra. It inherits a $k\Gamma$-bimodule structure by considering the left- and right-translation maps $L_g : k^\Gamma \to k$ and $R_g : k^\Gamma \to k$ defined for all $g \in \Gamma$ by

$$L_g(f)(h) = f(gh) \qquad R_g(f)(h) = f(hg)$$

Due to the fact that L_g defines the *right* action and R_g the left, we write for simplicity:

$$g \circ f = R_g(f) \quad \text{and} \quad f \circ g = L_g(f).$$

It is in the context of the $k\Gamma$ module k^Γ that we get our first definition:

1.2.1 Definition: Let k^Γ be as above. Then an element $f : \Gamma \to k$ is called a **finitary element** or **representative function** if f satisfies one of the following equivalent conditions:

- $k^\Gamma f$ is finite dimensional over k.
- $f k^\Gamma$ is finite dimensional over k.
- $\Delta f \in k^\Gamma \otimes k^\Gamma$ – in other words, the comultiplication Δf is finitely supported.

1.2.2 Remark: The collection of all finitary elements of k^Γ will be written $F = F(k^\Gamma)$ and is a k-bialgebra – in particular, a sub-bialgebra of k^Γ.

1.2.3 Remark: The reason such elements are called representative functions is that if we fix a finite dimensional V over k and a basis $\mathscr{B}$ for V, we can write down the structure maps of a particular Γ action by noticing

$$g \cdot \alpha = \sum_{\beta\in\mathscr{B}} r_{\alpha\beta}(g)\beta$$

where $r_{\alpha\beta} : \Gamma \to k$ are called the **coefficient functions** of the representation.

Then the k-span $\mathrm{cf}(V) = \sum_{\alpha,\beta} k \cdot r_{\alpha\beta}$ of coefficient functions for a particular (finite-dimensional) representation V ends up being a subcoalgebra of k^Γ and in fact every finitary function lies in $\mathrm{cf}(V)$ for some finite-dimensional V. Here we can just take $V = k^\Gamma f$ which is finite dimensional by defn 1.2.1.

An important idea that comes from this that one can define an **algebraic representation theory** of Γ over k. Take any subcoalgebra A of $F = F(k^\Gamma)$ and define the A**-representation theory** of Γ to be the full subcategory $\mathbf{mod}_A(k\Gamma)$ of $\mathbf{mod}$-$(k\Gamma)$ to be the one whose objects are the (finite-dimensional) $k\Gamma$ modules V such that $\mathrm{cf}(V) \subseteq A$.

A nice result is that the (left) A-rational $k\Gamma$-modules are equivalent to the category $\mathbf{com}(A)$ of finite (right) A-comodules.

1.2.4 Proposition
$\mathbf{mod}_A(k\Gamma) \simeq \mathbf{com}(A)$

Proof
Sketch: Take an A-rational $k\Gamma$ module V with action $\tau : \Gamma \to V$. Fix a basis $\{v_i\}_1^n$ and as before extract the structure maps $r_{\alpha\beta}$:

$$\tau(v_\alpha) = \sum_\beta r_{\alpha\beta} v_\beta.$$

Then define the comodule (V, γ) with coaction γ via

$$\gamma(v_\alpha) = \sum_\beta v_\beta \otimes r_{\alpha\beta}.$$

One can check this defines a comodule structure on V.

The process reverses very easily, where one extracts the $r_{\alpha\beta}$ from the coaction. By construction we will have that $r_{\alpha\beta} \in A$, so we get the A-rationality for free.

Some mopping up shows that these are equivalences of categories, which is pretty believable. ♠

1.2.5 Remark: One of the most immediate consequences of this realization (it is barely a proof) is that we can define a left $A^* = \mathrm{Hom}_k(A, k)$-module (note that A^* is a k-algebra with the convolution product) structure on any right A-comodule. We do this by writing for any $f \in A^*$

$$f \cdot v = (\mathbb{1}_V \otimes f)(\gamma(v))$$

or in coordinates

$$f \cdot v_\alpha = \sum_\beta f(r_{\alpha\beta}) v_\beta$$

2 GL_n and the Schur Algebra

From now on, we specialize the theory in the section above to the case when $\Gamma = GL_n$ (we can think of this as the group scheme). Then we can set A_k to be the k-algebra generated by the function c_{ij} which extracts the $(i,j)^{th}$ entry. This gives us precisely the polynomial representations of Γ and restricting to the homogenous polynomials in A of degree d gives us $A(n,d)$.

2.0.1 Definition: The **Schur algebra** $S(n,d)$ is the dual of $A = A(n,d)$:

$$S(n,d) = A^* = \mathrm{Hom}_k(A(n,d),k).$$

References

[Gre07] J. A. (James Alexander) Green. *Polynomial representations of GL_n*. eng. 2nd corr. and augm. ed. Lecture notes in mathematics (Springer-Verlag) ; 830. Berlin ; New York: Springer, 2007. ISBN: 9783540469445.

[Sch01] Issai Schur. *Über eine Klasse von Matrizen, die sich einer gegebenen Matrix zuordnen lassen*. s.n., 1901. URL: http://hdl.handle.net/2027/hvd.32044091874271.