Notes on the Schur-Weyl Functor

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Abstract

These notes are my summary of the development of the Schur-Weil functor from representations of the Schur algebra $S(n,d)$ to representations of $\mathfrak{S}_d$. After developing the theory that arose beginning in Schur's 1901 thesis, we will establish that this functor is exact and behaves nicely with respect to simple modules, as well as being a monoidal functor (under the correct monoidal structure).

1 Background and Introduction

The primary reference for this paper is J.A. Green's *Polynomial Representations of* GL_n [Gre07]. Other sources will be used as well, and will be introduced as needed.

The basic idea for this theory begins with Schur's thesis in 1901 [Sch01]. Here he developed the theory underlying the representation theory of the general linear group GL_n. As a tool, he recognized that the irreducible representations of GL_n took on a particular form that made them amenable to admitting actions by symmetric groups. Then Schur could rely on Frobenius' development of the representation theory for $\mathfrak{S}_d$ to say something about GL_n.

1.1 Notation

We always use $\mathfrak{S}_d$ to denote the symmetric group on d letters, and let k be an infinite field (unless otherwise noted).

1.2 Representations of Semigroups

Recall that a **semigroup** is a group where we relax the inverse requirement. If we are interested in studying the representation theory of a semigroup Γ, there are some natural modules to consider:

Example 1.1
$k\Gamma = \bigoplus_{g\in\Gamma} kg$, the **semigroup algebra** for Γ over k. Notice that if Γ is infinite, we only take the elements that are supported at finitely many g (that is, only finitely many coefficients are nonzero).

Example 1.2
k^Γ, the set of all maps $\Gamma \to k$, which is naturally a commutative k-algebra. It inherits a $k\Gamma$-bimodule structure by considering the left- and right-translation maps $L_g : k^\Gamma \to k$ and $R_g : k^\Gamma \to k$ defined for all $g \in \Gamma$ by

$$L_g(f)(h) = f(gh) \qquad R_g(f)(h) = f(hg)$$

Due to the fact that L_g defines the *right* action and R_g the left, we write for simplicity:

$$g \circ f = R_g(f) \quad \text{and} \quad f \circ g = L_g(f).$$

It is in the context of the $k\Gamma$ module k^Γ that we get our first definition:

1.2.1 Definition: Let k^Γ be as above. Then an element $f : \Gamma \to k$ is called a **finitary element** or **representative function** if f satisfies one of the following equivalent conditions:

- $k^\Gamma f$ is finite dimensional over k.
- $f k^\Gamma$ is finite dimensional over k.
- $\Delta f \in k^\Gamma \otimes k^\Gamma$ – in other words, the comultiplication Δf is finitely supported.

1.2.2 Remark: The collection of all finitary elements of k^Γ will be written $F = F(k^\Gamma)$ and is a k-bialgebra – in particular, a sub-bialgebra of k^Γ.

1.2.3 Remark: The reason such elements are called representative functions is that if we fix a finite dimensional V over k and a basis $\mathscr{B}$ for V, we can write down the structure maps of a particular Γ action by noticing

$$g \cdot \alpha = \sum_{\beta \in \mathscr{B}} r_{\alpha\beta}(g)\beta$$

where $r_{\alpha\beta} : \Gamma \to k$ are called the **coefficient functions** of the representation.

Then the k-span $\mathrm{cf}(V) = \sum_{\alpha,\beta} k \cdot r_{\alpha\beta}$ of coefficient functions for a particular (finite-dimensional) representation V ends up being a subcoalgebra of k^Γ and in fact every finitary function lies in $\mathrm{cf}(V)$ for some finite-dimensional V. Here we can just take $V = k^\Gamma f$ which is finite dimensional by defn 1.2.1.

An important idea that comes from this that one can define an **algebraic representation theory** of Γ over k. Take any subcoalgebra A of $F = F(k^\Gamma)$ and define the A**-representation theory** of Γ to be the full subcategory $\mathbf{mod}_A(k\Gamma)$ of $\mathbf{mod}$-$(k\Gamma)$ to be the one whose objects are the (finite-dimensional) $k\Gamma$ modules V such that $\mathrm{cf}(V) \subseteq A$.

A nice result is that the (left) A-rational $k\Gamma$-modules are equivalent to the category $\mathbf{com}(A)$ of finite (right) A-comodules.

1.2.4 Proposition
$\mathbf{mod}_A(k\Gamma) \simeq \mathbf{com}(A)$

Proof
Sketch: Take an A-rational $k\Gamma$ module V with action $\tau : \Gamma \to V$. Fix a basis $\{v_i\}_1^n$ and as before extract the structure maps $r_{\alpha\beta}$:

$$\tau(v_\alpha) = \sum_\beta r_{\alpha\beta} v_\beta.$$

Then define the comodule (V, γ) with coaction γ via

$$\gamma(v_\alpha) = \sum_\beta v_\beta \otimes r_{\alpha\beta}.$$

One can check this defines a comodule structure on V.

The process reverses very easily, where one extracts the $r_{\alpha\beta}$ from the coaction. By construction we will have that $r_{\alpha\beta} \in A$, so we get the A-rationality for free.

Some mopping up shows that these are equivalences of categories, which is pretty believable. ♠

1.2.5 Remark: One of the most immediate consequences of this realization (it is barely a proof) is that we can define a left $A^* = \mathrm{Hom}_k(A, k)$-module (note that A^* is a k-algebra with the convolution product) structure on any right A-comodule. We do this by writing for any $f \in A^*$

$$f \cdot v = (\mathbb{1}_V \otimes f)(\gamma(v))$$

or in coordinates

$$f \cdot v_\alpha = \sum_\beta f(r_{\alpha\beta}) v_\beta$$

2 Polynomial representations of GL_n and the Schur Algebra

From now on, we specialize the theory in the section above to the case when $\Gamma = GL_n$ (we can think of this as the group scheme).

2.1 Generators and Bases

There are very natural choices of generators (as (co)algebras and as k spaces). Starting off, we can define

$$A_k(n) \cong k[\omega_{ij}]$$

for $1 \leq i, j \leq n$, where we can think of $\omega_{ij} : \Gamma \to k$ as the map that "extracts" from $g \in \Gamma$ the $(i,j)^{th}$ entry.

Then $A_k(n)$ is the set of **polynomial maps** $\Gamma \to k$. Then we define $A_k(n,r)$ to be the k-space spanned by all homogeneous degree r polynomials in $A_k(n)$. A stars-and-bars argument gets us that there are $\binom{n^2+r-1}{r}$ ways to choose a monomial of degree r from among the n^2 generators, so this gives us the dimension of $A_k(n,d)$.

2.2 Index Notation

We need to use multi-indicies to rigorously discuss monomials so let $I(n,r)$ denote the set of length r multi-indices drawn from $\underline{n}$. Then we can impose the equivalence relation: if $\alpha, \beta \in I(n,r)$,

$$\alpha \sim \beta \quad \Leftrightarrow \quad \alpha = (n_{i_1}, \ldots, n_{i_r}) = (n_{i_{\sigma(1)}}, \ldots, n_{i_{\sigma(r)}}) = \beta$$

for some $\sigma \in \mathfrak{S}_r$. That is, two multi-indices are equivalent if they contain the same indicies with the same multiplicities. Sometimes it is useful to think of this as a group action of $\mathfrak{S}_r$ on $I(n,r)$, and we say two elements are equivalent if they are in the same orbit.

Extend this group action (and thus relation) to $I(n,r) \times I(n,r)$ where G acts diagonally:

$$g \cdot (i,j) = (g \cdot i, g \cdot j).$$

2.3 Module Categories

The module categories[1] $\mathbf{mod}_A(k\Gamma)$ of (homogeneous degree r) polynomial representations of $GL_n(k)$ (when $A = A_k(n,r)$ and $A = A_k(n)$ respectively) are precisely what you'd think. We will sometimes use the notation $M_k(n)$ and $M_k(n,r)$ following the text.

One of the big realizations of Schur's was that any polynomial representation of GL_n splits into a direct sum of homogeneous polynomial representations:

2.3.1 Theorem ([Sch01, p.5])
Let $V \in M_k(n)$. Then

$$V \cong \bigoplus_{r \geq 0} V_r$$

where $V_r \in M_k(n,r)$ for all r.

2.3.2 Corollary
The indecomposable modules in $M_k(n)$ are all in $M_k(n,r)$ for some r.

[1] Equivalently the appropriate comodule categories in light of prop. 1.2.4

But then if we are interested in the structure of polynomial representations of Γ, we can restrict our attention entirely to the homogeneous modules of any particular degree. The upshot here is that $A_k(n,r)$ is finite dimensional, so very computationally amenable.

2.4 The Schur Algebra

2.4.1 Definition: The **Schur algebra** $S_k(n,d)$ is the dual of $A=A_k(n,d)$:

$$S_k(n,d)=A^*=\mathrm{Hom}_k(A(n,d),k).$$

We can write down an explicit basis for $S_k(n,r)$ as a dual basis for the basis $\{c_{ij} | i,j\in I(n,r)\}$ for $A_k(n,r)$. We denote this basis ξ_{ij}, where

$$\xi_{ij}(c_{kl})=\begin{cases}1, & (i,j)\sim(k,l)\\ 0, & \text{otherwise}\end{cases}$$

and we can define multiplication via the coproduct Δ on $A_k(s,r)$:

$$\xi\eta(c)=\sum\xi(c_{(1)})\eta(c_{(2)})$$

where we recall that $\Delta(c_{ij})=\sum_k c_{ik}\otimes c_{kj}$ and $\varepsilon(c)=c(I_n)$ are the coalgebra structure inherited on $A_k(n,r)$ from $A_k(n)$ and thus from k^Γ.

2.4.2 Proposition
The product of two basis elements in $S_k(n,r)$ is given by

$$\xi_{ab}\xi_{cd}=\sum_{(p,q)\in I(n,r)^2} Z_{a,b,c,d,p,q}\xi_{pq}$$

where

$$Z_{a,b,c,d,p,q}=\#\{s\in I(n,r) | (a,b)\sim(p,s)\text{ and }(c,d)\sim(s,q)\}$$

2.4.3 Corollary
$\xi_{ab}\xi_{cd}=0$ if $b\not\sim c$.

2.4.4 Corollary
$\xi_{ab}\xi_{bb}=\xi_{ab}=\xi_{aa}\xi_{ab}$.

The last thing here is to notice that $\varepsilon=\sum_{I(n,r)^2}\xi_{ab}$ where equality can be seen by evaluating at the basis c_{ab} of $A_k(n,r)$.

2.5 The Evaluation Map

There is a natural map $e : k\Gamma \to S_k(n,r)$ defined in the following way: if $f \in A_k(n,r) \subset k^\Gamma$ (which can be uniquely extended k-linearly to a map on $k\Gamma$),

$$e(x)(f) = f(x)$$

where $x \in k\Gamma$.

The use of this map is that (since it can be shown to be surjective and that things in the kernel of this map must act by zero on $M_k(n,r)$), that the category of $S_k(n,r)$ modules and the category of $A_k(n,r)$-rational $k\Gamma$ representations are equivalent. The evaluation map gives us a direct translation between the two actions: $x \cdot v = e(x) \cdot v$ and if $k\Gamma$ acts on V with structure maps (r_{ab}), then for $\xi \in S_k(n,r)$,

$$\xi \cdot v_b = \sum_a \xi(r_{ab}) v_a$$

2.6 Modular Theory

We can extract the $\mathbb{Z}$ forms $A_{\mathbb{Z}}(n)$ and $A_{\mathbb{Z}}(n,r)$ as the $\mathbb{Z}$ span of the basis elements $c_{ij}^{\mathbb{Q}}$ (the basis with respect to $k = \mathbb{Q}$). These are closed under Δ and have the further property that for instance $\varepsilon(A_{\mathbb{Z}}(n,r)) \subseteq \mathbb{Z}$.

Then we can extend scalars in the way we'd like: there is a k-colagebra isomorphism

$$A_{\mathbb{Z}}(n,r) \otimes_{\mathbb{Z}} k \cong A_k(n,r),$$

so we can recover the larger algebra from this $\mathbb{Z}$-form. But this actually transfers to the Schur algebra as well!

Define $S_{\mathbb{Z}}(n,r)$ to be the collection of $\xi \in S_{\mathbb{Q}}(n,r)$ such that $\xi(A_{\mathbb{Z}}(n,r)) \subseteq \mathbb{Z}$. Then we get a k-algebra isomorphism

$$S_{\mathbb{Z}}(n,r) \otimes_{\mathbb{Z}} k \cong S_k(n,r).$$

This gives us a $\mathbb{Z}$-form for the scheme $S(n,r)$ – or that "the Schur scheme is defined over $\mathbb{Z}$."

We can also define a $\mathbb{Z}$-form of a ($A_{\mathbb{Q}}(n,r)$-rational) $\mathbb{Q}\Gamma$-module V to be denoted $V_{\mathbb{Z}}$ and to be the $\mathbb{Z}$-span of a $\mathbb{Q}$-basis for V and to furthermore be closed under the $S_{\mathbb{Z}}(n,r)$-action (leveraging the equivalence of categories in the previous section).

This process of generating V_k modules by extending scalars from a $\mathbb{Z}$-form is called **modular reduction**, and has a small caveat: the $\mathbb{Z}$-form we pick needn't be unique and, in general, using different $\mathbb{Z}$-forms will yield non-isomorphic extensions to k-modules. The upshot here, following the theory developed in [Bra39] and [Gre76], is that the multiplicity with which simple modules occur in a composition series of V_k are not dependent on the choice of $\mathbb{Z}$-form.

2.7 The Trick

To do some magic we notice that there is a duality at play between the $A_k(n,r)$-regular representations of GL_n and the representations of $\mathfrak{S}_r$, and to see this we consider the natural action of these groups on $E^{\otimes} r$, where $E \cong k^n$. The left action by Γ is just the diagonal action on each k^n and $\mathfrak{S}_r$ acts on $E^{\otimes r}$ on the right by permuting the tensor factors. The critical thing to notice here is that these two actions commute with one another:

$$(g \cdot v) \cdot \sigma = g \cdot (v \cdot \sigma),\ \forall g \in \Gamma, \sigma \in \mathfrak{S}_r.$$

References

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