Esercizi

BIBLIOGRAFIA: [6], [12], [28], [30], [61], [68]

A) Funzioni irrazionali

Calcolare gli integrali del tipo

$$R\left(x, \sqrt[p]{\left(\frac{ax+b}{cx+d}\right)^{\alpha}}, \sqrt[q]{\left(\frac{ax+b}{cx+d}\right)^{\beta}}, \ldots\right)$$

provando che è:

1. $\displaystyle\int \frac{dx}{x(\sqrt{x}+\sqrt[5]{x^2})} = \ln \frac{x}{(1+\sqrt[10]{x})^{10}} + \frac{10}{\sqrt[10]{x}} - \frac{5}{\sqrt[5]{x}} + \frac{10}{3\sqrt[10]{x^3}} - \frac{5}{2\sqrt[5]{x^2}} + c.$

2. $\displaystyle\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}+2\sqrt[4]{x}} = 2\sqrt{x} - 3\sqrt[3]{x} - 8\sqrt[4]{x} + 6\sqrt[6]{x} + 48\sqrt[12]{x} +$

$+ 3\ln(1+\sqrt[12]{x}) + \frac{33}{2}\ln(\sqrt[6]{x} - \sqrt[12]{x} + 12) - \frac{171}{\sqrt{7}}\operatorname{arctg}\frac{2\sqrt[12]{x}-1}{\sqrt{7}} + c.$

3. $\displaystyle\int \frac{x\,dx}{(x+1)^{\frac{1}{2}}(x+1)^{\frac{1}{3}}} = 6\left[\frac{1}{9}(x+1)^{\frac{3}{2}} - \frac{1}{8}(x+1)^{\frac{4}{3}} + \frac{1}{7}(x+1)^{\frac{7}{6}} - \frac{1}{6}(x+1) + \frac{1}{5}(x+1)^{\frac{5}{6}} - \frac{1}{4}(x+1)^{\frac{2}{3}}\right] + c.$

4. $\displaystyle\int \sqrt{\frac{1-x}{1+x}} \cdot \frac{dx}{x} = \ln\left|\frac{\sqrt{x+1}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right| + 2\operatorname{arctg}\sqrt{\frac{1-x}{1+x}} + c.$

5. $\displaystyle\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx = (\sqrt{x}-2)\sqrt{1-x} - \operatorname{arcsen}\sqrt{x} + c.$

6. $\displaystyle\int \frac{x^2+\sqrt{1+x}}{\sqrt[3]{1+x}}\,dx = 6\sqrt[3]{(1+x)^2}\left[\frac{(1+x)^2}{16} - \frac{1+x}{5} + \frac{\sqrt{1+x}}{7} + \frac{1}{4}\right] + c.$

7. $\displaystyle\int \sqrt[3]{\frac{1-x}{1+x}} \cdot \frac{dx}{x} = \ln\frac{|u^2-1|}{\sqrt{u^4+u^2+1}} + \sqrt{3}\operatorname{arctg}\frac{1+2u^2}{\sqrt{3}} + c$, con $u = \sqrt[3]{\frac{1-x}{1+x}}$.

8. $\displaystyle\int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}} = \frac{4}{3}\sqrt{\frac{x-1}{x+2}} + c$, (Moltiplicare numeratore e denominatore per $\sqrt[4]{x-1}$ ...).

Calcolare gli integrali del tipo $R(x, \sqrt{ax^2+bx+c})$, *provando che si ha:*

9. $\displaystyle\int \frac{dx}{x\sqrt{x^2+x+1}} = \ln\frac{|cx|}{2+x+2\sqrt{x^2+x+1}}$, $\left(\text{Porre } x = \frac{1}{t}\right)$.

10. $\displaystyle\int \frac{dx}{x\sqrt{x^2+4x-4}} = \frac{1}{2}\arccos\frac{2-x}{x\sqrt{2}} + c.$

11. $\displaystyle\int \frac{dx}{x\sqrt{x^2+2x-1}} = \operatorname{arcsen}\frac{x-1}{x\sqrt{2}} + c.$

12. $\int \frac{dx}{x\sqrt{2+x-x^2}} = c - \frac{1}{\sqrt{2}} \ln\left|\frac{\sqrt{2+x-x^2}+\sqrt{2}}{x} + \frac{1}{2\sqrt{2}}\right|.$

13. $\int \frac{\sqrt{2x+x^2}}{x^2}\,dx = \ln|x+1+\sqrt{2x+x^2}| - \frac{4}{x+\sqrt{2x+x^2}} + c.$

14. $\int \frac{dx}{(x-1)\sqrt{x^2+x+1}} = c - \frac{1}{\sqrt{3}} \ln\left|\frac{3+3x+2\sqrt{3(x^2+x+1)}}{x-1}\right|.$

15. $\int \frac{dx}{(2x-3)\sqrt{4x-x^2}} = c - \frac{1}{\sqrt{15}} \ln\left|\frac{x+6+\sqrt{60x-15x^2}}{2x-3}\right|.$

16. $\int \sqrt{x^2-2x-1}\,dx = \frac{1}{2}(x-1)\sqrt{x^2-2x-1} - \ln|x-1+\sqrt{x^2-2x-1}| + c.$

17. $\int \sqrt{3x^2-3x+1}\,dx = \frac{1}{2}\left(x-\frac{1}{2}\right)\sqrt{3x^2-3x+1} + \frac{1}{8\sqrt{3}} \ln\left|\sqrt{3x^2-3x+1} + \frac{\sqrt{3}}{2}(2x-1)\right| + c.$

18. $\int \sqrt{1-4x-x^2}\,dx = \frac{1}{2}\left[(x+2)\sqrt{1-4x-x^2} + 5 \operatorname{arcsen}\frac{x+2}{\sqrt{5}}\right] + c.$

19. $\int \frac{dx}{x-\sqrt{x^2-x+1}} = c - \frac{3}{2(2x-1-2\sqrt{x^2-x+1})} - \frac{3}{2}\ln|2x-1-2\sqrt{x^2-x+1}| + 2\ln|x-\sqrt{x^2-x+1}|.$

20. $\int \frac{dx}{x^2(x+\sqrt{1+x^2})} = \ln\left|\frac{x+\sqrt{x^2+1}}{x}\right| - \frac{\sqrt{1+x^2}}{x} + c.$

21. $\int \frac{dx}{1+\sqrt{x^2+2x+2}} = \frac{1-\sqrt{x^2+2x+2}}{x+1} + \ln(x+1+\sqrt{x^2+2x+2}) + c.$

22. $\int \frac{x^2\,dx}{\sqrt{1-2x-x^2}} = \frac{1}{2}(3-x)\sqrt{1-2x-x^2} + 2\operatorname{arcsen}\frac{x+1}{\sqrt{2}} + c.$

23. $\int \frac{2x^2-3x}{\sqrt{x^2-2x+5}}\,dx = x\sqrt{x^2-2x+5} - 5\ln(x-1+\sqrt{x^2-2x+5}) + c.$

24. $\int \frac{3x^2-5x}{\sqrt{3-2x-x^2}}\,dx = c - \frac{1}{2}(3x-19)\sqrt{3-2x-x^2} + 14\operatorname{arcsen}\frac{x+1}{2}.$

25. $\int \frac{3x^3}{\sqrt{x^2+4x+5}}\,dx = (x^2-5x+20)\sqrt{x^2+4x+5} - 15\ln(x+2+\sqrt{x^2+4x+5}) + c.$

26. $\int \frac{x^3-x+1}{\sqrt{x^2+2x+2}}\,dx = \left(\frac{1}{3}x^2 - \frac{5}{6}x + \frac{1}{6}\right)\sqrt{x^2+2x+2} + \frac{5}{2}\ln(x+1+\sqrt{x^2+2x+2}) + c.$

27. $\int \frac{3x^3-8x+5}{\sqrt{x^2-4x-7}}\,dx = (x^2+5x+36)\sqrt{x^2-4x-7} + 112\ln|x-2+\sqrt{x^2-4x-7}| + c.$

28. $\int \frac{x^4}{\sqrt{x^2+4x+5}}\,dx = \left(\frac{1}{4}x^3 - \frac{7}{6}x^2 + \frac{95}{24}x - \frac{145}{12}\right)\sqrt{x^2+4x+5} + \frac{35}{8}\ln(x+2+\sqrt{x^2+4x+5}) + c.$

29. $\displaystyle\int \frac{dx}{(x^3+3x^2+3x+1)\sqrt{x^2+2x-3}} = \frac{\sqrt{x^2+2x-3}}{8(x+1)^2} + \frac{1}{16}\arccos\frac{2}{x+1} + c.$

30. $\displaystyle\int \frac{\sqrt{1+x^2}}{2+x^2}\,dx = \frac{1}{2\sqrt{2}}\ln\frac{\sqrt{2+2x^2}-x}{\sqrt{2+2x^2}+x} + \ln(x+\sqrt{x^2+1}) + c.$

31. $\displaystyle\int \frac{x-1}{x^2\sqrt{2x^2-2x+1}}\,dx = \frac{\sqrt{2x^2-2x+1}}{x} + c.$

32. $\displaystyle\int \frac{2x+3}{(x^2+2x+3)\sqrt{x^2+2x+4}}\,dx = \ln\frac{\sqrt{x^2+2x+4}-1}{\sqrt{x^2+2x+4}+1} - \frac{1}{\sqrt{2}}\operatorname{arctg}\frac{\sqrt{2(x^2+2x+4)}}{x+1} + c.$

Calcolare gli integrali dei seguenti **differenziali binomi,** *provando che è:*

33. $\displaystyle\int \sqrt{x}(1+\sqrt[3]{x})^4\,dx = \frac{2}{3}x\sqrt{x} + \frac{24}{11}x\sqrt[6]{x^5} + \frac{36}{13}x^2\sqrt[6]{x} + \frac{8}{5}x^2\sqrt{x} + \frac{6}{17}x^2\sqrt[6]{x^5} + c.$

34. $\displaystyle\int x^{-1}(1+x^{\frac{1}{3}})^{-3}\,dx = 3\left[\ln\left|\frac{\sqrt[3]{x}}{1+\sqrt[3]{x}}\right| + \frac{2\sqrt[3]{x}+3}{2(1+\sqrt[3]{x})^2}\right] + c.$

35. $\displaystyle\int \frac{dx}{x\sqrt[3]{x^2+1}} = \frac{1}{2}\ln(\sqrt[3]{x^2+1}-1) - \frac{1}{4}\ln[\sqrt[3]{(x^2+1)^2}+\sqrt[3]{x^2+1}+1] + \frac{\sqrt{3}}{2}\operatorname{arctg}\frac{2\sqrt[3]{x^2+1}+1}{\sqrt{3}} + c.$

36. $\displaystyle\int x^5\sqrt[3]{(1+x^3)^2}\,dx = \frac{1}{8}\sqrt[3]{(1+x^3)^8} - \frac{1}{5}\sqrt[3]{(1+x^3)^5} + c.$

37. $\displaystyle\int \frac{dx}{\sqrt[3]{1+x^3}} = \frac{1}{6}\ln\frac{u^2+u+1}{(u-1)^2} - \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2u+1}{\sqrt{3}} + c,$ con $u = \dfrac{\sqrt[3]{x^3+1}}{x}.$

38. $\displaystyle\int \frac{dx}{\sqrt[4]{1+x^4}} = \frac{1}{4}\ln\frac{\sqrt[4]{1+x^4}+x}{\sqrt[4]{1+x^4}-x} - \frac{1}{2}\operatorname{arctg}\frac{\sqrt[4]{1+x^4}}{x} + c.$

39. $\displaystyle\int \frac{\sqrt{1-x^4}}{x^5}\,dx = \frac{1}{4}\ln\frac{\sqrt{1-x^4}+1}{x^2} - \frac{1}{4}\frac{\sqrt{1-x^4}}{x^4} + c.$

40. $\displaystyle\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx = \frac{3}{7}(4\sqrt{x}+\sqrt[4]{x}-3)\sqrt[3]{1+\sqrt[4]{x}} + c.$

41. $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx = 6u + 2\ln\frac{u-1}{\sqrt{u^2+u+1}} - 2\sqrt{3}\operatorname{arctg}\frac{2u+1}{\sqrt{3}} + c,$ con $u = \sqrt[3]{1+\sqrt{x}}.$

42. $\displaystyle\int \frac{\sqrt[3]{1+x^3}}{x^2}\,dx = c - \frac{\sqrt[3]{1+x^3}}{x} + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2\sqrt[3]{1+x^3}+x}{x\sqrt{3}} - \frac{1}{2}\ln\left|\frac{\sqrt[3]{1+x^3}+x}{\sqrt[3]{(1+x^3)^2}+x\sqrt[3]{1+x^3}+x^2}\right|.$

43. $\displaystyle\int \frac{dx}{x\sqrt[3]{1+x^5}} = \frac{1}{5}\ln\frac{|u-1|}{\sqrt{u^2+u+1}} + \frac{\sqrt{3}}{5}\operatorname{arctg}\frac{1+2u}{\sqrt{3}} + c,$ con $u = \sqrt[3]{1+x^5}.$

44. $\displaystyle\int \frac{dx}{x^{11}\sqrt{1+x^4}} = c - \frac{1}{10}\sqrt{\left(\frac{1+x^4}{x^4}\right)^5} + \frac{1}{3}\sqrt{\left(\frac{1+x^4}{x^4}\right)^3} - \frac{1}{2}\sqrt{\frac{1+x^4}{x^4}}.$

45. $\displaystyle\int \sqrt[3]{x(1-x^2)}\,dx = \frac{u}{2(u^3+1)} - \frac{1}{6}\ln\frac{u+1}{\sqrt{u^2-u+1}} - \frac{1}{2\sqrt{3}}\operatorname{arctg}\frac{2u-1}{\sqrt{3}} + c,$ con $u = \sqrt[3]{\dfrac{1-x^2}{x^2}}.$

46. $\displaystyle\int \sqrt[3]{1+\sqrt[4]{x}}\,dx = 12\left[\frac{\sqrt[3]{u^{13}}}{13} - \frac{3\sqrt[3]{u^{10}}}{10} + \frac{3\sqrt[3]{u^{7}}}{7} - \frac{\sqrt[3]{u^{4}}}{4}\right] + c, \text{con } u = 1+\sqrt[4]{x}.$

Calcolare i seguenti integrali indefiniti di **funzioni irrazionali,** *dimostrando che risulta:*

47. $\displaystyle\int \frac{1}{x-\sqrt{x}}\,dx = 2\ln|\sqrt{x}-1| + c.$

48. $\displaystyle\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx = 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln(1+\sqrt[6]{x}) + c.$

49. $\displaystyle\int \frac{1-\sqrt[4]{x}}{1+\sqrt{x}}\,dx = -\frac{4}{3}\sqrt[4]{x^3} + 2\sqrt{x} + 4\sqrt[4]{x} - 4\operatorname{arctg}\sqrt[4]{x} - 2\ln(1+\sqrt{x}) + c.$

50. $\displaystyle\int \frac{\sqrt{1+x}}{x^2}\,dx = \frac{1}{2}\ln\left|\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}\right| - \frac{\sqrt{1+x}}{x} + c.$

51. $\displaystyle\int \frac{dx}{x\sqrt{x^2-x+3}} = \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{x^2-x+3}-\sqrt{3}}{3} + \frac{1}{2\sqrt{3}}\right| + c.$

52. $\displaystyle\int \frac{dx}{\sqrt{(2x-x^2)^3}} = \frac{x-1}{\sqrt{2x-x^2}} + c.$

53. $\displaystyle\int \sqrt{2x-x^2}\,dx = \frac{1}{2}[(x-1)\sqrt{2x-x^2} + \operatorname{arcsen}(x-1)] + c.$

54. $\displaystyle\int \frac{dx}{x-\sqrt{x^2-1}} = \frac{x^2}{2} + \frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ln|x+\sqrt{x^2-1}| + c.$

55. $\displaystyle\int \frac{dx}{(1+x)\sqrt{1+x+x^2}} = \ln\left|\frac{x+\sqrt{1+x+x^2}}{2+x+\sqrt{1+x+x^2}}\right| + c.$

56. $\displaystyle\int \frac{(x+1)}{(2x+x^2)\sqrt{2x+x^2}}\,dx = -\frac{1}{\sqrt{2x+x^2}} + c.$

57. $\displaystyle\int \frac{1-\sqrt{1+x+x^2}}{x\sqrt{1+x+x^2}}\,dx = \ln\left|\frac{2+x-2\sqrt{1+x+x^2}}{x^2}\right| + c.$

58. $\displaystyle\int \frac{\sqrt{x^2+4x}}{x^2}\,dx = -\frac{8}{x+\sqrt{x^2+4x}} + \ln|x+2+\sqrt{x^2+4x}| + c.$

59. $\displaystyle\int \frac{x}{\sqrt{x+2}}\,dx = \frac{2}{3}(x+2)^{\frac{3}{2}} - 4\sqrt{x+2} + c.$

60. $\displaystyle\int \frac{x+\sqrt[4]{x-2}}{\sqrt[3]{x-2}}\,dx = \frac{3}{5}\sqrt{(x-2)^5} + \frac{12}{11}\cdot\sqrt[12]{(x-2)^{11}} + 3\sqrt[3]{(x-2)^2} + c.$

61. $\displaystyle\int \frac{1}{2\sqrt{x}-\sqrt[3]{x}}\,dx = \frac{1}{8}\left[8\sqrt{x} + 6\sqrt[3]{x} + 6\sqrt[6]{x} + 3\ln\left|\sqrt[6]{x} - \frac{1}{2}\right|\right] + c.$

62. $\displaystyle\int \left(\sqrt{\frac{x}{1-x}}\right)^3 dx = (3-x)\sqrt{\frac{x}{1-x}} - 3\operatorname{arctg}\sqrt{\frac{x}{1-x}} + c.$

63. $\displaystyle\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\,dx = 2\sqrt{x} - 4\sqrt[4]{x} + 4\ln(1+\sqrt[4]{x}) + c.$

64. $\displaystyle\int \frac{\sqrt{x+1}}{x}\,dx = 2\sqrt{x+1} + \ln\frac{(\sqrt{x+1}-1)^2}{|x|} + c.$

65. $\displaystyle\int \frac{1}{\sqrt{x^2-3x+2}}\,dx = -\ln(3+2\sqrt{x^2-3x+2}-2x) + c.$

66. $\displaystyle\int \frac{1}{x+\sqrt{x^2-4x-3}}\,dx = -\frac{3}{8}\ln(x+\sqrt{x^2-4x-3}) + \frac{7}{8}\ln|x+\sqrt{x^2-4x-3}-2| +$
$\displaystyle+\frac{7}{(4x-2+\sqrt{x^2-4x-3})} + c.$

67. $\displaystyle\int \frac{1}{(x+2)\sqrt{x^2+2x+2}}\,dx = \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{x^2+2x+2}-x-2-\sqrt{2}}{\sqrt{x^2+2x+2}-x-2+\sqrt{2}}\right| + c.$

68. $\displaystyle\int \frac{\sqrt{x}}{\sqrt[4]{x^3}+1}\,dx = \frac{4}{3}[\sqrt[4]{x^3} - \ln(\sqrt[4]{x^3}+1)] + c,$

69. $\displaystyle\int \frac{\sqrt{x^3}-\sqrt[3]{x}}{6\sqrt[4]{x}}\,dx = \frac{2}{27}\sqrt[4]{x^9} - \frac{2}{13}\sqrt[12]{x^{13}} + c.$

70. $\displaystyle\int \frac{\sqrt[6]{x}+1}{\sqrt[6]{x^7}+\sqrt[4]{x^5}}\,dx = -\frac{6}{\sqrt[6]{x}} + \frac{12}{\sqrt[12]{x}} + 2\ln|x| - 24\ln(\sqrt[12]{x}+1) + c.$

71. $\displaystyle\int \frac{2+\sqrt[3]{x}}{\sqrt[6]{x}+\sqrt[3]{x}+\sqrt{x}+1}\,dx = \frac{6}{5}\sqrt[6]{x^5} - \frac{3}{2}\sqrt[6]{x^4} + 4\sqrt[6]{x^3} - 6\sqrt[6]{x^2} +$
$+6\sqrt[6]{x} - 9\ln(\sqrt[6]{x}+1) + \frac{3}{2}\ln(\sqrt[6]{x^2}+1) + 3\operatorname{arctg}\sqrt[6]{x} + c.$

72. $\displaystyle\int \sqrt{\frac{1-x}{1+x}}\,\frac{dx}{x^2} = \ln\left|\frac{\sqrt{1-x}+\sqrt{1+x}}{\sqrt{1-x}-\sqrt{1+x}}\right| - \frac{\sqrt{1-x^2}}{x} + c.$

73. $\displaystyle\int \frac{1}{\sqrt{x^2+x+1}}\,dx = -\ln|1+2x-2\sqrt{x^2+x+1}| + c.$

74. $\displaystyle\int \frac{1}{\sqrt{4x^2-2x+1}}\,dx = -\frac{1}{2}\ln|2\sqrt{4x^2-2x+1}-4x+1| + c.$

75. $\displaystyle\int \frac{1}{\sqrt{-x^2+3x-2}}\,dx = 2\operatorname{arctg}\sqrt{\frac{x-1}{2-x}} + c.$

76. $\displaystyle\int \frac{1}{(x-1)\sqrt{-x^2+1}}\,dx = -\sqrt{\frac{1+x}{1-x}} + c.$

77. $\displaystyle\int \frac{x}{\sqrt{-x^2+x+2}}\,dx = -\sqrt{-x^2+x+2} + \operatorname{arctg}\sqrt{\frac{x+1}{2-x}} + c.$

78. $\displaystyle\int \frac{\sqrt{-x^2+2x}}{x^3}\,dx = -\frac{1}{3}\sqrt{\left(\frac{2-x}{x}\right)^3} + c.$

79. $\displaystyle\int \frac{1}{(1+x)\sqrt{x-x^2}}\,dx = \sqrt{2}\cdot\operatorname{arctg}\sqrt{\frac{2x}{1-x}} + c.$

80. $\displaystyle\int \frac{1}{(x-1)\sqrt{-x^2+x+2}}\,dx = \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{x+1}-\sqrt{2(2-x)}}{\sqrt{x+1}+\sqrt{2(2-x)}}\right| + c.$

81. $\displaystyle\int \frac{3x+2}{\sqrt{x^2+2x+2}}\,dx = 3\sqrt{x^2+2x+2} - \ln\left(x+1+\sqrt{x^2+2x+2}\right) + c.$

82. $\displaystyle\int \frac{1}{x\sqrt{x^2+1}}\,dx = \ln\left|\frac{x+\sqrt{x^2+1}-1}{x+\sqrt{x^2+1}+1}\right| + c.$

B) Funzioni goniometriche e iperboliche

Calcolare i seguenti integrali di funzioni goniometriche, provando che si ha:

1. $\displaystyle\int \frac{dx}{4\operatorname{sen}x + 3\cos x + 5} = -\frac{1}{\operatorname{tg}\frac{x}{2}+2} + c.$

2. $\displaystyle\int \frac{dx}{3+5\operatorname{sen}x+3\cos x} = \frac{1}{5}\ln\left|5\operatorname{tg}\frac{x}{2}+3\right| + c.$

3. $\displaystyle\int \frac{dx}{1-\operatorname{sen}x} = -\frac{2}{\operatorname{tg}\frac{x}{2}-1} + c.$

4. $\displaystyle\int \frac{\cos^2 x}{\operatorname{sen}^2 x + 4\operatorname{sen}x\cos x}\,dx = -\frac{x}{17} + \frac{1}{4}\ln|\operatorname{sen}x| - \frac{1}{68}\ln|\operatorname{sen}x + 4\cos x| + c.$ (Porre: $\operatorname{ctg}x = t$).

5. $\displaystyle\int \frac{\cos^3 x}{\operatorname{sen}^2 x + \operatorname{sen}x}\,dx = \ln|\operatorname{sen}x| - \operatorname{sen}x + c.$

6. $\displaystyle\int \frac{\operatorname{sen}2x}{\cos^3 x - \operatorname{sen}^2 x - 1}\,dx = -\frac{2}{5}\ln|1-\cos x| + \frac{1}{5}\ln(\cos^2 x + 2\cos x + 2) - \frac{6}{5}\operatorname{arctg}(1+\cos x) + c.$

7. $\displaystyle\int \operatorname{sen}^3 x\,dx = \frac{1}{3}\cos^3 x - \cos x + c.$

8. $\displaystyle\int \frac{\cos^5 x}{\operatorname{sen}x}\,dx = \ln|\operatorname{sen}x| - \operatorname{sen}^2 x + \frac{1}{4}\operatorname{sen}^4 x + c.$

9. $\displaystyle\int \operatorname{sen}^2\frac{x}{4}\cdot\cos^2\frac{x}{4}\,dx = \frac{1}{8}x - \frac{1}{8}\operatorname{sen}x + c.$

10. $\displaystyle\int \cos^4 x\,dx = \frac{3}{8}x + \frac{1}{4}\operatorname{sen}2x + \frac{1}{32}\operatorname{sen}4x + c.$

11. $\displaystyle\int \operatorname{tg}^4 x\,dx = \frac{1}{3}\operatorname{tg}^3\frac{x}{2} - 2\operatorname{tg}\frac{x}{2} + x + c.$

12. $\displaystyle\int \operatorname{ctg}^3 3x\,dx = \frac{1}{6}\operatorname{ctg}^2 3x - \frac{1}{3}\ln|\operatorname{sen}3x| + c.$

13. $\displaystyle\int \operatorname{sen}^6 x\,dx = \operatorname{tg}x + \frac{2}{3}\operatorname{tg}^3 x + \frac{1}{5}\operatorname{tg}^5 x + c.$

14. $\displaystyle\int \frac{\cos^2 x}{\operatorname{sen}^4 x}\,dx = -\frac{1}{3}\operatorname{ctg}^3 x + c.$

15. $\displaystyle\int \sec^3 x\,dx = \frac{1}{2}\operatorname{tg}x\cdot\sec x + \frac{1}{2}\ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + c.$

16. $\int \operatorname{ctg}^2 x \cdot \operatorname{cosec} x\,dx = -\frac{1}{2}\operatorname{ctg} x \cdot \operatorname{cosec} x - \frac{1}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right| + c.$

17. $\int \operatorname{sen} 3x \cdot \operatorname{sen} x\,dx = \frac{1}{4}\operatorname{sen} 2x - \frac{1}{8}\operatorname{sen} 4x + c.$

18. $\int \cos\frac{x}{2}\cos\frac{x}{3}\,dx = \frac{3}{5}\operatorname{sen}\frac{5}{6}x + 3\operatorname{sen}\frac{x}{6} + c.$

19. $\int \frac{dx}{\operatorname{sen}^2 x + 2\operatorname{sen} x\cos x - \cos^2 x} = \frac{1}{2\sqrt{2}}\ln\left|\frac{\operatorname{tg} x + 1 - \sqrt{2}}{\operatorname{tg} x + 1 + \sqrt{2}}\right| + c.$

20. $\int \frac{\operatorname{sen} x + \operatorname{sen}^3 x}{\cos 2x}\,dx = \frac{1}{2}\cos x - \frac{3}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}\cos x - 1}{\sqrt{2}\cos x + 1}\right| + c.$

21. $\int \frac{\cos^3 x + \cos^5 x}{\operatorname{sen}^2 x + \operatorname{sen}^4 x}\,dx = \operatorname{sen} x - \frac{2}{\operatorname{sen} x} - 6\operatorname{arctg}(\operatorname{sen} x) + c.$

22. $\int \operatorname{sen}^2 x\cos^2 x\,dx = \frac{1}{8}x - \frac{1}{32}\operatorname{sen} 4x + c.$

23. $\int \cos^6 x\,dx = \frac{5}{16}x + \frac{1}{4}\operatorname{sen} 2x + \frac{3}{64}\operatorname{sen} 4x - \frac{1}{48}\operatorname{sen}^3 2x + c.$

24. $\int \operatorname{sen}^2 x\cos^4 x\,dx = \frac{1}{16}x - \frac{1}{64}\operatorname{sen} 4x + \frac{1}{48}\operatorname{sen}^3 2x + c.$

25. $\int \operatorname{tg}^7 x\,dx = \frac{\operatorname{tg}^6 x}{6} - \frac{\operatorname{tg}^4 x}{4} + \frac{\operatorname{tg}^2 x}{2} + \ln|\cos x| + c.$

26. $\int \operatorname{ctg}^6 x\,dx = -\frac{\operatorname{ctg}^5 x}{5} + \frac{\operatorname{ctg}^3 x}{3} - \operatorname{ctg} x - x + c.$

27. $\int \operatorname{sen} 2x \cdot \cos 5x\,dx = -\frac{1}{14}\cos 7x + \frac{1}{6}\cos 3x + c.$

28. $\int \operatorname{sen}^5 x\,dx = -\cos x + \frac{2}{3}\cos^3 x - \frac{\cos^5 x}{5} + c.$

29. $\int \cos^4 x\operatorname{sen}^3 x\,dx = -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + c.$

30. $\int \frac{\cos^3 x}{\operatorname{sen}^4 x}\,dx = \operatorname{cosec} x - \frac{1}{3}\operatorname{cosec}^3 x + c.$

31. $\int \operatorname{sen}^4 x\cos^4 x\,dx = \frac{1}{128}\left(3x - \operatorname{sen} 4x + \frac{\operatorname{sen} 8x}{8}\right) + c.$

Calcolare i seguenti integrali di funzioni iperboliche o di funzioni non iperboliche con sostituzioni iperboliche, provando che è:

32. $\int \operatorname{ch} x\,dx = \operatorname{sh} x + c.$

33. $\int \operatorname{sh} x\,dx = \operatorname{ch} x + c.$

34. $\int \frac{1}{\operatorname{ch}^2 x}\,dx = \operatorname{th} x + c.$

35. $\int \frac{e^x}{\operatorname{ch} x + \operatorname{sh} x}\,dx = x + c.$

36. $\int (\mathrm{ch}^2 ax + \mathrm{sh}^2 ax)\,dx = \frac{1}{2a}\,\mathrm{sh}\,2ax + c.$

37. $\int sh^2 x\,dx = \frac{\mathrm{sh}x \cdot \mathrm{ch}x - x}{2} + c.$

38. $\int \mathrm{th}^2 x\,dx = x - \mathrm{tg}x + c.$

39. $\int \mathrm{cth}^2 x\,dx = x - \mathrm{cth}x + c.$

40. $\int \mathrm{sh}^3 x\,dx = \frac{1}{3}\,\mathrm{ch}^3 x - \mathrm{ch}x + c.$

41. $\int \mathrm{ch}^3 x\,dx = \mathrm{sh}x + \frac{1}{3}\,\mathrm{sh}^3 x + c.$

42. $\int \mathrm{th}^4 x\,dx = x - \mathrm{th}x - \frac{1}{3}\,\mathrm{th}^3 x + c.$

43. $\int \mathrm{sh}^2 x \mathrm{ch}^3 x\,dx = \frac{1}{3}\,\mathrm{sh}^3 x + \frac{1}{5}\,\mathrm{sh}^5 x + c.$

44. $\int \mathrm{cth}^5 x\,dx = \ln|\,\mathrm{sen}x\,| - \frac{1}{2}\,\mathrm{cth}^2 x - \frac{1}{4}\,\mathrm{cth}^4 x + c.$

45. $\int \frac{dx}{\mathrm{sh}x \cdot \mathrm{ch}x} = \ln|\,\mathrm{th}x\,| + c.$

46. $\int \frac{dx}{\mathrm{sh}x} = \ln\left|\mathrm{tg}\,\frac{x}{2}\right| + c.$

47. $\int \frac{dx}{(1+\mathrm{ch}x)^2} = \frac{1}{2}\,\mathrm{th}\,\frac{x}{2} - \frac{1}{6}\,\mathrm{th}^3\,\frac{x}{2} + c.$

48. $\int \sqrt{\mathrm{th}x}\,dx = \frac{1}{2}\ln\frac{1+\sqrt{\mathrm{th}x}}{|\,1-\sqrt{\mathrm{th}x}\,|} + c.$

49. $\int \frac{x}{\mathrm{ch}^2 x}\,dx = x\,\mathrm{th}x - \ln\mathrm{ch}x + c.$

50. $\int \frac{e^{2x}}{\mathrm{sh}^4 x}\,dx = c - \frac{e^{3x}}{2\,\mathrm{sh}^3 x}.$

51. $\int \frac{dx}{(x+1)^2\sqrt{x^2+2x+2}} = -\frac{\sqrt{x^2+2x+2}}{x+1} + c.$ [Porre: $x+1=t$ e poi $t=\mathrm{sh}z$]

52. $\int x^2\sqrt{x^2-1}\,dx = -\frac{1}{8}\ln(x+\sqrt{x^2-1}) + \frac{1}{8}\,x(2x^2-1)\sqrt{x^2-1} + c.$

53. $\int \frac{\sqrt{x^2+1}}{x^2}\,dx = \ln(x+\sqrt{x^2+1}) - \frac{\sqrt{x^2+1}}{x} + c.$

54. $\int \sqrt{(x^2-1)^3}\,dx = \frac{1}{8}\,x(2x^2-1)\sqrt{x^2-1} - \frac{1}{2}\,x\sqrt{x^2-1} + \frac{3}{8}\ln(x+\sqrt{x^2-1}) + c.$

Calcolare i seguenti integrali indefiniti di funzioni goniometriche, dimostrando che risulta:

55. $\int \frac{1}{1+\mathrm{sen}^2 x}\,\mathrm{cos}x\,dx = \mathrm{arctgsen}x + c.$

56. $\int \frac{\mathrm{sen}^5 x}{\cos^3 x}\,dx = -\frac{1}{2}\cos^2 x + 2\ln|\,\mathrm{cos}x\,| + \frac{1}{2\cos^2 x} + c.$

57. $\int \frac{\mathrm{sen}^3 x \mathrm{cos}x}{1-3\cos^2 x}\,dx = \frac{\mathrm{sen}^2 x}{6} + \frac{1}{9}\ln|\,3\mathrm{sen}^2 x - 2\,| + c.$

58. $\int \frac{2+\mathrm{cos}x}{\mathrm{sen}x(1+2\mathrm{cos}x)}\,dx = \ln\left|\frac{\mathrm{sen}x}{1+2\mathrm{cos}x}\right| + c.$

59. $\int \frac{3\,\mathrm{sen}^2 x}{4-3\cos^2 x}\,dx = x - \frac{1}{2}\,\mathrm{arctg}(2\,\mathrm{tg}x) + c.$

60. $\int \frac{\mathrm{sen}^2 x}{1-\cos^2 x}\,dx = x - \frac{\sqrt{3}}{2}\,\mathrm{arctg}\,\frac{2\,\mathrm{tg}x}{\sqrt{3}} + c.$

61. $\int \frac{\mathrm{tg}x}{\mathrm{sen}^2 x + 1}\,dx = \frac{1}{4}\ln(2\,\mathrm{tg}^2 x + 1) + c.$

62. $\int \frac{\operatorname{tg}^3 x + \operatorname{tg} x}{\operatorname{tg} x + 4}\, dx = \operatorname{tg} x - 4 \ln |\operatorname{tg} x + 4| + c.$

63. $\int \frac{\operatorname{tg}^2 x + 1}{\operatorname{tg}^2 x - \operatorname{tg} x - \operatorname{tg} x - 2}\, dx = \ln \left| \sqrt[3]{\frac{\operatorname{tg} x - 2}{\operatorname{tg} x + 1}} \right| + c.$

64. $\int \operatorname{sen}^3 x \cos^2 x\, dx = \frac{1}{15} \cos^3 x (3 \cos^2 x - 5) + c.$

65. $\int \frac{\operatorname{sen}^3 x}{\cos^4 x}\, dx = \frac{1}{3 \cos^3 x} - \frac{1}{\cos x} + c.$

66. $\int \frac{dx}{\cos x \cdot \operatorname{sen}^3 x} = \ln |\operatorname{tg} x| - \frac{1}{2 \operatorname{sen}^2 x} + c.$

67. $\int \frac{\operatorname{sen}^4 x}{\cos^2 x}\, dx = \operatorname{tg} x + \frac{1}{4} \operatorname{sen} 2x - \frac{3}{2} x + c.$

68. $\int \frac{dx}{\cos^3 x \cdot \operatorname{sen}^3 x} = \frac{1}{2} (\operatorname{tg}^2 x - \operatorname{ctg}^2 x) + 2 \ln |\operatorname{tg} x| + c.$

69. $\int \frac{dx}{\operatorname{sen}^4 x \cos^4 x} = \frac{(\operatorname{tg}^2 x - 1)(\operatorname{tg}^4 x + 10 \operatorname{tg}^2 x + 1)}{3 \operatorname{tg}^3 x} + c.$

70. $\int \frac{\operatorname{sen} x}{(1 - \cos x)^2}\, dx = \frac{1}{\cos x - 1} + c.$

71. $\int \frac{\cos x}{(1 - \cos x)^2}\, dx = \frac{1}{2} \operatorname{ctg} \frac{x}{2} - \frac{1}{6} \operatorname{ctg}^3 \frac{x}{2} + c.$

72. $\int \frac{1}{\operatorname{tg}^8 x}\, dx = x - \frac{1}{7} \operatorname{ctg}^7 x + \frac{1}{5} \operatorname{ctg}^5 x - \frac{1}{3} \operatorname{ctg}^3 x + \operatorname{ctg} x + c.$

73. $\int \frac{1}{\operatorname{sen}^3 x}\, dx = c - \frac{\cos x}{2 \operatorname{sen}^2 x} + \frac{1}{2} \ln \left| \operatorname{tg} \frac{x}{2} \right| + c.$

74. $\int \frac{\cos^4 x + \operatorname{sen}^4 x}{\cos^2 x - \operatorname{sen}^2 x}\, dx = \frac{1}{4} \ln \left| \frac{1 + \operatorname{tg} x}{1 - \operatorname{tg} x} \right| + \frac{1}{2} \operatorname{sen} x \cos x + c.$

75. $\int \frac{dx}{(\operatorname{sen} x + \cos x)^2} = c - \frac{1}{1 + \operatorname{tg} x}.$

76. $\int \frac{1}{\operatorname{sen} x + \cos x}\, dx = \frac{\sqrt{2}}{2} \ln \left| \operatorname{tg} \left(\frac{\pi}{8} + \frac{x}{2} \right) \right| + c.$

77. $\int \frac{1}{a \cos x + b \operatorname{sen} x}\, dx = \frac{1}{\sqrt{a^2 + b^2}} \ln \left| \operatorname{tg} \frac{x + \operatorname{arctg} \frac{a}{b}}{2} \right| + c.$

78. $\int \frac{1}{\operatorname{tg} x \cos 2x}\, dx = \ln \left| \frac{c \operatorname{sen} x}{\sqrt{\cos 2x}} \right|.$

79. $\int \frac{\cos^2 x}{\operatorname{sen} x \cdot \cos 3x}\, dx = \ln \frac{|c \operatorname{sen} x|}{\sqrt{1 - 4 \operatorname{sen}^2 x}}.$

80. $\int \frac{dx}{1 + \operatorname{tg} x} = \frac{1}{2} (x + \ln |\operatorname{sen} x + \cos x|) + c.$

81. $\displaystyle\int \frac{1}{3\operatorname{sen}x - 4\cos x}\,dx = \frac{1}{5}\ln\left|\frac{2\operatorname{tg}\frac{x}{2} - 1}{\operatorname{tg}\frac{x}{2} + 2}\right| + c.$

82. $\displaystyle\int \frac{1-\cos x}{1+\cos x}\,dx = 2\operatorname{tg}\frac{x}{2} - x + c.$

83. $\displaystyle\int \frac{1}{3 + \cos x - 3\operatorname{sen}x}\,dx = \ln\left|\frac{\operatorname{tg}\frac{x}{2} - 2}{\operatorname{tg}\frac{x}{2} - 1}\right| + c.$

84. $\displaystyle\int \frac{5+\cos x}{(5+3\cos x)\cos x}\,dx = \ln\left|\frac{1+\operatorname{tg}\frac{x}{2}}{1-\operatorname{tg}\frac{x}{2}}\right| - \operatorname{arctg}\left(\frac{1}{2}\operatorname{tg}\frac{x}{2}\right) + c.$

85. $\displaystyle\int \frac{1}{\cos x}\,dx = -\ln\left|\operatorname{tg}\left(\frac{\pi}{4} - \frac{x}{2}\right)\right| + c.$

86. $\displaystyle\int \frac{dx}{5 - 4\operatorname{sen}x + 3\cos x} = \frac{1}{2 - \operatorname{tg}\frac{x}{2}} + c.$

87. $\displaystyle\int \frac{dx}{4 - 3\cos^2 x + 5\operatorname{sen}^2 x} = \frac{1}{3}\operatorname{arctg}(3\operatorname{tg}x) + c.$

88. $\displaystyle\int \frac{dx}{1 + \operatorname{sen}^2 x} = \frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\operatorname{tg}x) + c.$

89. $\displaystyle\int \frac{dx}{1 - \operatorname{sen}^4 x} = \frac{1}{2}\operatorname{tg}x + \frac{1}{2\sqrt{2}}\operatorname{arctg}(\sqrt{2}\operatorname{tg}x) + c.$

90. $\displaystyle\int \frac{dx}{a^2\operatorname{sen}^2 x + b^2\cos^2 x} = \frac{1}{ab}\operatorname{arctg}\frac{a\operatorname{tg}x}{b} + c.$

91. $\displaystyle\int \frac{dx}{\operatorname{sen}^2 x + \operatorname{tg}^2 x} = c - \frac{1}{2}\left[\operatorname{ctg}x + \frac{1}{\sqrt{2}}\operatorname{arctg}\left(\frac{\operatorname{tg}x}{\sqrt{2}}\right)\right].$

92. $\displaystyle\int \frac{\cos x}{\operatorname{sen}^3 x - \cos^3 x}\,dx = \ln\frac{|\sqrt[3]{\operatorname{tg}x - 1}|}{\sqrt[6]{\operatorname{tg}^2 x + \operatorname{tg}x + 1}} - \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2\operatorname{tg}x + 1}{\sqrt{3}} + c.$

93. $\displaystyle\int \sqrt{1 + \operatorname{sen}x}\,dx = \begin{cases} 2\left(\operatorname{sen}\frac{x}{2} - \cos\frac{x}{2}\right) + c, \text{ per tutti gli } x \text{ per cui: } \operatorname{sen}\frac{x}{2} + \cos\frac{x}{2} \geqslant 0, \\ -2\left(\operatorname{sen}\frac{x}{2} - \cos\frac{x}{2}\right) + c, \text{ per tutti gli } x \text{ per cui: } \operatorname{sen}\frac{x}{2} + \cos\frac{x}{2} \leqslant 0. \end{cases}$

94. $\displaystyle\int \frac{\sqrt{\operatorname{tg}x}}{\operatorname{sen}x\cos x}\,dx = 2\sqrt{\operatorname{tg}x} + c.$

95. $\displaystyle\int \frac{\sqrt{\operatorname{sen}^3 2x}}{\operatorname{sen}^5 x}\,dx = c - \frac{4\sqrt{2}}{5}\sqrt{\operatorname{ctg}^5 x}$, (porre: $t = \operatorname{ctg}x$).

96. $\int \frac{dx}{\sqrt[4]{\operatorname{sen}^3 x \cos^5 x}} = 4\sqrt[4]{\operatorname{tg} x} + c.$

97. $\int \frac{dx}{\sqrt{1-\operatorname{sen}^4 x}} = \frac{1}{\sqrt{2}} \ln\left(\sqrt{2}\operatorname{tg} x + \sqrt{1+2\operatorname{tg}^2 x}\right) + c.$

98. $\int \frac{dx}{5-3\cos x} = \frac{1}{2}\operatorname{arctg}\left(2\operatorname{tg}\frac{x}{2}\right) + c.$

99. $\int \frac{dx}{5+4\operatorname{sen} x} = \frac{2}{3}\operatorname{arctg}\frac{5\operatorname{tg}\frac{x}{2}+4}{3} + c.$

100. $\int \frac{2-\operatorname{sen} x}{2+\cos x}\,dx = \ln(2+\cos x) + \frac{4}{\sqrt{3}}\operatorname{arctg}\left(\frac{1}{\sqrt{3}}\operatorname{tg}\frac{x}{2}\right) + c.$

101. $\int \frac{\operatorname{sen}^2 x}{1-\operatorname{tg} x}\,dx = \frac{\cos x(\cos x - \operatorname{sen} x)}{4} - \frac{1}{4}\ln|\cos x - \operatorname{sen} x| + c.$

102. $\int \frac{dx}{4+\operatorname{tg} x + 4\operatorname{ctg} x} = \frac{4}{25}x - \frac{3}{25}\ln|\operatorname{tg} x + 2| + \frac{5}{5(\operatorname{tg} x + 2)} - \frac{3}{25}\ln|\cos x| + c.$

103. $\int \frac{dx}{(\operatorname{sen} x + 2\sec x)^2} = \frac{\cos 2x - 15}{15(4+\operatorname{sen} 2x)} + \frac{4}{15\sqrt{15}}\operatorname{arcsen}\frac{4\operatorname{sen} 2x + 1}{4+\operatorname{sen} 2x} + c.$

104. $\int \sqrt{1+\operatorname{cosec} x}\,dx = 2\operatorname{arcsen}\sqrt{\operatorname{sen} x} + c.$

105. $\int \frac{\cos 2x - 3}{\cos^4 x\sqrt{4-\operatorname{ctg}^2 x}}\,dx = c - \frac{1}{3}\operatorname{tg} x(2+\operatorname{tg}^2 x)\sqrt{4-\operatorname{ctg}^2 x}.$

106. $\int \frac{dx}{\operatorname{sen}\frac{x}{2}\sqrt{\cos^3\frac{x}{2}}} = \frac{4}{\sqrt{\cos\frac{x}{2}}} + 2\operatorname{arctg}\sqrt{\cos\frac{x}{2}} - \ln\frac{1+\sqrt{\cos\frac{x}{2}}}{1-\sqrt{\cos\frac{x}{2}}} + c.$

107. $\int \sqrt{\operatorname{tg} x}\,dx = \frac{1}{\sqrt{2}}\left[\ln(\operatorname{sen} x + \cos x - \sqrt{\operatorname{sen} 2x}) + \operatorname{arcsen}(\operatorname{sen} x - \cos x)\right] + c.$

Esercizi di riepilogo

Calcolare i seguenti integrali dimostrando che si ha:

1. $\int \operatorname{sen}^2 x\cos 3x\,dx = -\frac{1}{6}\cos 3x + \frac{1}{20}\cos 5x + \frac{1}{4}\cos x + c.$

2. $\int (2x^2 - 2x + 1)e^{-\frac{x}{2}}\,dx = -2(2x^2 + 6x + 13)e^{-\frac{x}{2}} + c.$

3. $\int \frac{\ln x}{x^3}\,dx = -\frac{1}{4x^2}(2\ln x + 1) + c.$

4. $\int \frac{x^2-2}{x^2+1}\cdot \operatorname{arctg} x\,dx = x\cdot\operatorname{arctg} x - \frac{3}{2}(\operatorname{arctg} x)^2 - \frac{1}{2}\ln(1+x^2) + c.$

5. $\int (2x^2-1)\cos 2x\,dx = (x^2-1)\operatorname{sen}2x + x\cos 2x + c.$

6. $\int x\ln^2 x\,dx = \frac{1}{4}x^2(2\ln^2 x - 2\ln x + 1) + c$

7. $\int \frac{2e^{2x}-e^x-3}{e^{2x}-2e^x-3}\,dx = x + \ln|e^x-3| + c.$

8. $\int \operatorname{arctg}\sqrt{x}\,dx = (x+1)\operatorname{arctg}\sqrt{x} - \sqrt{x} + c.$

9. $\int \sqrt{2^x-1}\,dx = \frac{2}{\ln 2}\left(\sqrt{2^x-1} - \operatorname{arctg}\sqrt{2^x-1}\right) + c.$

10. $\int \frac{dx}{\sqrt[4]{1+x^4}} = \frac{1}{4}\ln\left|\frac{t+1}{t-1}\right| - \frac{1}{2}\operatorname{arctg}t + c,$ con $t^4 = 1 + x^4.$

11. $\int \sqrt{6+4x-2x^2}\,dx = \sqrt{2}\left(\frac{x-1}{2}\sqrt{3+2x-x^2} + 2\operatorname{arcsen}\frac{x-1}{2}\right) + c.$

12. $\int e^x \operatorname{sen}(e^x)\,dx = \operatorname{sen}(e^x) - e^x\cos(e^x) + c.$

13. $\int \frac{dx}{\cos^2 x\sqrt{2+5\operatorname{tg}^2 x}} = \frac{1}{\sqrt{5}}\ln\left|\operatorname{tg}x + \sqrt{\operatorname{tg}^2 x + \frac{2}{5}}\right| + c.$

14. $\int \operatorname{sen}2x \cdot \ln\cos x\,dx = \frac{1}{2}\cos^2 x(1 - 2\ln\cos x) + c.$

15. $\int (x+2)\cos(x^2+4x+1)\,dx = \frac{1}{2}\operatorname{sen}(x^2+4x+1) + c.$

16. $\int \frac{x\cos x}{\operatorname{sen}^3 x}\,dx = -\frac{1}{2}\left(\frac{x}{\operatorname{sen}^2 x} + \operatorname{ctg}x\right) + c.$

17. $\int \frac{xe^x}{\sqrt{1+e^x}}\,dx = 2(x-2)\sqrt{1+e^x} - 2\ln\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1} + c.$

18. $\int \ln(x^2+x)\,dx = x\ln(x^2+x) + \ln|x+1| - x + c.$

19. $\int \frac{dx}{x^4+x^2} = -\frac{1}{x} - \operatorname{arctg}x + c.$

20. $\int \cos(\ln x)\,dx = \frac{x}{2}(\cos\ln x + \operatorname{sen}\ln x) + c.$

21. $\int \frac{1+\sqrt[6]{x}}{(\sqrt[3]{x}-\sqrt[4]{x})\sqrt[4]{x^3}}\,dx = 12\left[\sqrt[12]{x} + \ln\frac{(\sqrt[12]{x}-1)^2}{\sqrt[12]{x}}\right] + c.$

22. $\int \frac{dx}{a^2\cos^2 x + b^2\operatorname{sen}^2 x} = \frac{1}{ab}\operatorname{arctg}\left(\frac{b}{a}\operatorname{tg}x\right) + c.$

23. $\int \frac{e^x}{e^{2x}-7e^x+10}\,dx = \frac{1}{3}\ln\left|\frac{e^x-5}{e^x-2}\right| + c.$

24. $\int \frac{x}{x^3+1}\,dx = -\frac{1}{3}\ln|x+1| + \frac{1}{6}\ln(x^2-x+1) + \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2x-1}{\sqrt{3}} + c.$

25. $\displaystyle\int \frac{x^4+4x^3+11x^2+12x+8}{(x^2+2x+3)^2(x+1)}\,dx = \ln|x+1| - \frac{x+2}{2(x^2+2x+3)} - \frac{1}{2\sqrt{2}}\operatorname{arctg}\frac{x+1}{\sqrt{2}} + c.$

26. $\displaystyle\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}\,dx = \frac{3}{2}x^{\frac{2}{3}} + 6\operatorname{arctg}\sqrt[6]{x} + c.$

27. $\displaystyle\int \frac{(2x-3)^{\frac{1}{2}}}{(2x-3)^{\frac{1}{3}}+1}\,dx = \frac{3}{7}t^7 - \frac{3}{5}t^5 + t^3 - 3t + 2\operatorname{arctg}t + c,$ con $t^6 = 2x-3.$

28. $\displaystyle\int \frac{2}{(2-x)^2}\sqrt[3]{\frac{2-x}{2+x}}\,dx = \frac{3}{4}\sqrt[3]{\left(\frac{2+x}{2-x}\right)^2} + c.$

29. $\displaystyle\int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}} = \frac{4}{3}\sqrt[4]{\frac{x-1}{x+2}} + c.$

30. $\displaystyle\int \frac{dx}{x+\sqrt{x^2-x+1}} = 2\ln|t| - \frac{1}{2}\ln|t-1| + \frac{3}{t+1} - \frac{3}{2}\ln|t+1| + c,$ con $t = \frac{\sqrt{x^2-x+1}+1}{x}.$

31. $\displaystyle\int \frac{x\,dx}{(\sqrt{7x-10-x^2})^3} = -\frac{2}{9}\left(-\frac{5}{t}+2t\right) + c,$ con $t = \frac{\sqrt{7x-10-x^2}}{x-2}.$

32. $\displaystyle\int \frac{x+3}{\sqrt{4x^2+4x-3}}\,dx = \frac{1}{4}\sqrt{4x^2+4x-3} + \frac{5}{4}\ln|2x+1+\sqrt{4x^2+4x-3}| + c.$

33. $\displaystyle\int x^{-11}(1+x^4)^{-\frac{1}{2}}dx = -\frac{1}{10x^{10}}\sqrt{(1+x^4)^5} + \frac{1}{3x^6}\sqrt{(1+x^4)^3} - \frac{1}{2x^2}\sqrt{1+x^4} + c.$

34. $\displaystyle\int \frac{\operatorname{sen}^3 x}{\sqrt[3]{\cos^2 x}}\,dx = 3\sqrt[3]{\cos x}\left(\frac{1}{7}\cos^2 x - 1\right) + c.$

35. $\displaystyle\int \frac{\operatorname{sen}^2 x}{\cos^6 x}\,dx = \frac{1}{3}\operatorname{tg}^3 x + \frac{1}{5}\operatorname{tg}^5 x + c.$

36. $\displaystyle\int \frac{dx}{\sqrt[3]{\operatorname{sen}^{11} x\cos x}} = -\frac{3(1+4\operatorname{tg}^2 x)}{8\operatorname{tg}^2 x\sqrt[3]{\operatorname{tg}^2 x}} + c.$

37. $\displaystyle\int \frac{dx}{\operatorname{sen}x(2+\cos x-2\operatorname{sen}x)} = \frac{1}{3}\ln\left|\operatorname{tg}\frac{x}{2}\right| + \frac{5}{3}\ln\left|\operatorname{tg}\frac{x}{2}-3\right| - \ln\left|\operatorname{tg}\frac{x}{2}-1\right| + c.$

38. $\displaystyle\int \frac{dx}{\operatorname{sen}x(2\cos^2 x-1)} = \frac{1}{\sqrt{2}}\ln\left|\frac{1+\sqrt{2}\cos x}{1-\sqrt{2}\cos x}\right| + \ln\left|\operatorname{tg}\frac{x}{2}\right| + c.$

39. $\displaystyle\int \frac{\operatorname{sen}^2 x\cos x}{\operatorname{sen}x+\cos x}\,dx = \frac{1}{4}\ln|\operatorname{sen}x+\cos x| - \frac{1}{4}\cos x(\operatorname{sen}x+\cos x) + c.$

40. $\displaystyle\int \frac{2\operatorname{tg}x+3}{\operatorname{sen}^2 x+2\cos^2 x}\,dx = \ln(\operatorname{tg}^2 x+2) + \frac{3}{\sqrt{2}}\operatorname{arctg}\frac{\operatorname{tg}x}{\sqrt{2}} + c.$

41. $\displaystyle\int \frac{dx}{(1+\sqrt{x})\sqrt{x-x^2}} = \frac{2(\sqrt{x}-1)}{\sqrt{1-x}} + c.$

42. $\displaystyle\int \frac{\ln x}{\sqrt{1-x}}\,dx = 4\sqrt{1-x} + 2\ln(2-x-2\sqrt{1-x}) - 2(1+\sqrt{1-x})\ln x + c.$

43. $\displaystyle\int \frac{e^x}{(1+e^{2x})^2}\,dx = \frac{e^x}{2(1+e^{2x})} + \frac{1}{2}\operatorname{arctg}e^x + c.$

44. $\int e^{-x}\ln(e^x+1)dx = -e^{-x}\ln(1+e^x)+x-\ln(1+e^x)+c.$

45. $\int \frac{e^{\alpha\,\mathrm{arctg}x}}{(1+x^2)^{\frac{3}{2}}}dx = e^{\alpha t}\frac{\alpha\cos t+\mathrm{sen}t}{\alpha^2+1}+c,$ con $t=\mathrm{arctg}x.$

46. $\int \frac{x\,\mathrm{arctg}x}{\sqrt{1+x^2}}dx = \sqrt{1+x^2}\,\mathrm{arctg}x-\ln(x+\sqrt{x^2+1})+c.$

47. $\int \frac{\mathrm{sh}2x+2\mathrm{ch}x}{\mathrm{ch}^2x+\mathrm{sh}x-\frac{1}{2}}dx = \ln\left|t^2+t+\frac{1}{2}\right|+2\,\mathrm{arctg}\frac{t+\frac{1}{2}}{\frac{1}{2}}+c,$ con $t=\mathrm{sh}x.$

48. $\int \mathrm{sen}2x\cdot\mathrm{arctg}\sqrt{1-\mathrm{sen}x}dx = \mathrm{arctg}t(t^4-2t^2-3)-\frac{t^3}{3}+3t+c,$ con $t^2=1-\mathrm{sen}x.$

Calcolare i seguenti integrali indefiniti[1]:

49. $\int \frac{x^3}{(x-1)^{12}}dx.$

50. $\int \frac{x}{x-\sqrt{x^2-1}}dx.$

51. $\int x\sqrt[3]{a+x}dx.$

52. $\int \frac{dx}{ae^{mx}+be^{-mx}}.$

53. $\int \frac{x\sqrt{1+x}}{\sqrt{1-x}}dx.$

54. $\int \frac{x^4}{(x^2-1)(x+2)}dx.$

55. $\int \frac{dx}{1-x^4}.$

56. $\int \frac{dx}{(x^4-1)^2}.$

57. $\int \frac{\ln(x+1)}{\sqrt{x+1}}dx.$

58. $\int (x^2+3x+5)\cos 2x\,dx.$

59. $\int x^2\mathrm{sh}x\,dx.$

60. $\int \mathrm{arctg}(1+\sqrt{x}\,)dx.$

61. $\int \frac{\mathrm{arcsen}x}{x^2}dx.$

62. $\int e^{\sqrt[3]{x}}dx.$

63. $\int xe^{\sqrt[3]{x}}dx.$

64. $\int (x^3-2x^2+5)e^{3x}dx.$

65. $\int \sin\sqrt{x}\,dx.$

66. $\int \frac{dx}{x^3(x-1)^{\frac{1}{2}}}.$

67. $\int \frac{dx}{x-\sqrt{x^2-1}}$

68. $\int \frac{\sqrt{(1+x^2)^5}}{x^6}dx.$

69. $\int \frac{x^4}{\sqrt{x^2+1}}dx.$

70. $\int \sqrt{\frac{1-\sqrt[3]{x}}{1+\sqrt[3]{x}}}\frac{dx}{x}.$

71. $\int \frac{dx}{x^3\sqrt{(1+x)^3}}.$

72. $\int \frac{\sqrt{2x+1}}{x^2}dx.$

73. $\int \frac{x^4}{x^{15}-1}dx.$

74. $\int \frac{dx}{\mathrm{sen}2x-2\mathrm{sen}x}.$

75. $\int \frac{dx}{1+\cos^2 x}.$

76. $\int \frac{dx}{a^2-b^2\cos^2 x}.$

77. $\int x\ln(1+x^3)dx.$

78. $\int \frac{\ln x-1}{\ln^2 x}dx.$

79. $\int \frac{x\ln x}{\sqrt{(x^2-1)^3}}dx.$

80. $\int x^e e^x\cos x\,dx.$

81. $\int xe^{x^2}(x^2+1)dx.$

82. $\int \frac{dx}{\sqrt{\mathrm{sen}^3x\cos^5 x}}.$

83. $\int \frac{dx}{\mathrm{sen}^5x\cos^5 x}.$

84. $\int \frac{\mathrm{sen}2x\,dx}{\cos^4x+\mathrm{sen}^4x}$

[1] Il risultato si trova alla fine del gruppo di esercizi.

85. $\int \frac{dx}{1+\operatorname{sen}x+\cos x}$.

86. $\int \sqrt{\operatorname{tg}^2 x+2}\,dx$.

87. $\int \frac{(x^2-1)}{x\sqrt{x^4+3x^2+1}}\,dx$.

88. $\int \frac{dx}{(2x-3)\sqrt{4x-x^2}}$.

89. $\int \frac{xe^x}{(1+x)^2}\,dx$.

90. $\int \frac{xe^x}{\sqrt{1+e^x}}\,dx$.

91. $\int \frac{\operatorname{arctg}x}{x^4}\,dx$.

92. $\int \frac{x\operatorname{arctg}x}{(1+x^2)^2}\,dx$.

93. $\int \frac{\operatorname{arctg}x}{(1+x)^3}\,dx$.

94. $\int \frac{dx}{(1-2^x)^4}$.

95. $\int \frac{(e^{3x}+e^x)}{e^{4x}-e^{2x}+1}\,dx$.

96. $\int \frac{dx}{\sqrt{1+e^x+e^{2x}}}$.

97. $\int \frac{\operatorname{tg}x}{1+\operatorname{tg}x+\operatorname{tg}^2 x}\,dx$.

98. $\int \operatorname{sen}^8 x\,dx$.

99. $\int \frac{(3+x^2)^2x^3}{(1+x^2)^3}\,dx$.

100. $\int \frac{x^2-8x+7}{(x^2-3x-10)^2}\,dx$.

101. $\int \frac{dx}{\operatorname{sen}^4 x+\cos^4 x}$.

102. $\int \frac{x+\operatorname{sen}x}{1+\cos x}\,dx$.

103. $\int \frac{x^2-1}{x^2+1}\cdot\frac{dx}{\sqrt{1+x^4}}$.

104. $\int e^{\operatorname{sen}x}\,\frac{x\cos^3 x-\operatorname{sen}x}{\cos^2 x}\,dx$.

Risultati

49. $c-\frac{1}{8(x-1)^8}-\frac{1}{3(x-1)^9}-\frac{3}{10(x-1)^{10}}-\frac{1}{11(x-1)^{11}}$. **50.** $\frac{1}{3}\left[x^3+\sqrt{(x^2-1)^3}\right]+c$,

51. $\frac{3(4x-3a)\sqrt[3]{(a+x)^4}}{28}+c$. **52.** $\frac{1}{m\sqrt{ab}}\operatorname{arctg}\left(e^{mx}\sqrt{\frac{a}{b}}\right)+c$. **53.** $\frac{1}{2}\operatorname{arcsen}x-\frac{x+2}{2}\sqrt{1-x^2}+c$.

54. $\frac{x^2}{2}-2x+\frac{1}{6}\ln\frac{|x-1|(x+2)^{32}}{|x+1|^3}+c$. **55.** $\frac{1}{4}\ln\left|\frac{1+x}{1-x}\right|+\frac{1}{2}\operatorname{arctg}x+c$.

56. $\frac{3}{8}\operatorname{arctg}x-\frac{x}{4(x^4-1)}-\frac{3}{16}\ln\left|\frac{x-1}{x+1}\right|+c$. **57.** $2\sqrt{x+1}(\ln|x+1|-2)+c$.

58. $\left(\frac{1}{2}x+\frac{3}{4}\right)\cos 2x+\left(\frac{1}{2}x^2+\frac{3}{2}x+\frac{9}{4}\right)\operatorname{sen}2x+c$. **59.** $x^2\operatorname{ch}x-2x\operatorname{sh}x+2\operatorname{ch}x+c$.

60. $x\operatorname{arctg}(1+\sqrt{x})-\sqrt{x}+\ln|x+2\sqrt{x}+2|+c$. **61.** $\ln\left|\frac{1-\sqrt{1-x^2}}{x}\right|-\frac{\operatorname{arcsen}x}{x}+c$.

62. $3e^{\sqrt[3]{x}}(\sqrt[3]{x^2}-2\sqrt[3]{x}+2)+c$ **63.** $3e^{\sqrt[3]{x}}(\sqrt[3]{x^5}-5\sqrt[3]{x^4}+20x-60\sqrt[3]{x^2}+120\sqrt[3]{x}-120)+c$.

64. $e^{3x}\left(\frac{1}{3}x^3-x^2+\frac{2}{3}x+\frac{13}{9}\right)+c$. **65.** $2(\operatorname{sen}\sqrt{x}-\sqrt{x}\cos\sqrt{x})+c$.

66. $\frac{\sqrt{x-1}(3x+2)}{4x^2}+\frac{3}{4}\operatorname{arctg}\sqrt{x-1}+c$. **67.** $\frac{x^2}{2}+\frac{x}{2}\sqrt{x^2-1}-\frac{1}{2}\ln|x+\sqrt{x^2-1}|+c$.

68. $\ln(x+\sqrt{1+x^2})-\frac{\sqrt{(1+x^2)^5}}{5x^5}-\frac{\sqrt{(1+x^2)^3}}{3x^3}-\frac{\sqrt{1+x^2}}{x}+c$.

69. $\left(\frac{1}{4}x^3-\frac{3}{8}x\right)\sqrt{x^2+1}+\frac{3}{8}\ln(x+\sqrt{x^2+1})+c$. **70.** $3[\ln|u|-\ln(1+\sqrt{1-u^2})-\operatorname{arcsen}u]+c$, con $u=\sqrt[3]{x}$.

71. $\dfrac{15x^2+5x-2}{4x^2\sqrt{1+x}}+\dfrac{15}{8}\ln\left|\dfrac{\sqrt{1+x}-1}{\sqrt{1+x}+1}\right|+c$. **72.** $c-\dfrac{\sqrt{2x+1}}{x}+\ln\left|\dfrac{\sqrt{2x+1}-1}{\sqrt{2x+1}+1}\right|$.

73. $\dfrac{1}{15}\left[\dfrac{1}{2}\ln\dfrac{(z-1)^2}{z^2+z+1}-\sqrt{3}\,\operatorname{arctg}\dfrac{2z+1}{\sqrt{3}}\right]+c$, con $z=x^5$. **74.** $c-\dfrac{1}{4}\ln\left|\operatorname{tg}\dfrac{x}{2}\right|+\dfrac{1}{8\operatorname{sen}^2\dfrac{x}{2}}$.

75. $\dfrac{1}{\sqrt{2}}\operatorname{arctg}\dfrac{\operatorname{tg}x}{\sqrt{2}}+c$. **76.** $\dfrac{2}{b^2\operatorname{sen}2\alpha}\ln\left|\dfrac{\operatorname{sen}(\alpha-x)}{\operatorname{sen}(\alpha+x)}\right|+c$, con $\alpha=\arccos\dfrac{a}{b}$, se $a^2<b^2$;

$\dfrac{1}{a^2\operatorname{sen}\alpha}\operatorname{arctg}\dfrac{\operatorname{tg}x}{\operatorname{sen}\alpha}+c$, con $\alpha=\arccos\dfrac{b}{a}$, se $a^2>b^2$.

77. $\dfrac{1}{2}x^2\ln(1+x^3)-\dfrac{3}{4}x^2+\dfrac{1}{4}\ln(x^2-x+1)-\dfrac{1}{2}\ln(x+1)+\dfrac{\sqrt{3}}{2}\operatorname{arctg}\dfrac{2x-1}{\sqrt{3}}+c$. **78.** $\dfrac{x}{\ln x}+c$.

79. $\operatorname{arctg}\sqrt{x^2-1}-\dfrac{\ln x}{\sqrt{x^2-1}}+c$. **80.** $\dfrac{1}{2}e^x[(x^2-1)\cos x+(x-1)^2\operatorname{sen}x]+c$. **81.** $\dfrac{x^2e^{x^2}}{2}+c$.

82. $\dfrac{2}{3}\dfrac{\operatorname{tg}^2x-3}{\sqrt{\operatorname{tg}x}}+c$. **83.** $\dfrac{1}{4}(\operatorname{tg}^4x-\operatorname{ctg}^4x)+2(\operatorname{tg}^2x-\operatorname{ctg}^2x)+6\ln|\operatorname{tg}x|+c$. **84.** $\operatorname{arctg}(\operatorname{tg}^2x)+c$.

85. $\ln\left|1+\operatorname{tg}\dfrac{x}{2}\right|+c$. **86.** $\operatorname{arctg}\dfrac{\operatorname{tg}x}{\sqrt{2+\operatorname{tg}^2x}}+\ln(\sqrt{2+\operatorname{tg}^2x}+\operatorname{tg}x)+c$. **87.** $\ln\dfrac{x^2+1+\sqrt{x^4+3x^2+1}}{x}+c$.

88. $c-\dfrac{1}{\sqrt{15}}\ln\left|\dfrac{x+6+\sqrt{60x-15x^2}}{2x-3}\right|$. **89.** $\dfrac{e^x}{1+x}+c$. **90.** $2x\sqrt{1+e^x}-4\sqrt{1+e^x}-2\ln\dfrac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}+c$,

91. $\dfrac{1}{6}\ln\dfrac{1+x^2}{x^2}-\dfrac{\operatorname{arctg}x}{3x^3}-\dfrac{1}{6x^2}+c$. **92.** $c-\dfrac{\operatorname{arctg}x}{2(1+x^2)}+\dfrac{\operatorname{arctg}x}{4}+\dfrac{x}{4(1+x)^2}$.

93. $\dfrac{1}{4}\ln\dfrac{|x+1|}{\sqrt{x^2+1}}-\dfrac{\operatorname{arctg}x}{2(x+1)^2}-\dfrac{1}{4(x+1)}+c$.

94. $x-\log_2|1-2^x|+\dfrac{1}{\ln 2}\left[\dfrac{1}{1-2^x}+\dfrac{1}{2(1-2^x)^2}+\dfrac{1}{3(1-2^x)^3}\right]+c$. **95.** $\operatorname{arctg}(e^x-e^{-x})+c$.

96. $\ln\dfrac{1+e^x-\sqrt{1+e^x+e^{2x}}}{1-e^x+\sqrt{1+e^x+e^{2x}}}+c$. **97.** $x-\dfrac{2}{\sqrt{3}}\operatorname{arctg}\dfrac{1+2\operatorname{tg}x}{\sqrt{3}}+c$.

98. $\dfrac{35}{128}x-\dfrac{1}{4}\operatorname{sen}2x+\dfrac{7}{128}\operatorname{sen}4x+\dfrac{1}{24}\operatorname{sen}^32x+\dfrac{1}{1024}\operatorname{sen}8x+c$.

99. $\dfrac{1}{2}x^2+\dfrac{3}{2}\ln(1+x^2)+\dfrac{1}{(1+x^2)^2}+c$. **100.** $\dfrac{8}{49(x-5)}-\dfrac{27}{49(x+2)}+\dfrac{30}{343}\ln\left|\dfrac{x-5}{x+2}\right|+c$.

101. $c-\dfrac{\sqrt{2}}{2}\operatorname{arctg}(\sqrt{2}\operatorname{ctg}2x)$. **102.** $x\operatorname{tg}\dfrac{x}{2}+c$. **103.** $\dfrac{1}{\sqrt{2}}\arccos\dfrac{x\sqrt{2}}{x^2+1}+c$.

(Dividere numeratore e denominatore per x^2 e porre: $x+\dfrac{1}{x}=t$). **104.** $e^{\operatorname{sen}x}(x-\sec x)+c$.