Parabolic cylinder functions

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Routines for computation of Weber's parabolic cylinder functions and their derivatives are provided for both moderate and great values of the argument. Standard, real solutions are considered. Tables of values are included.

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I. INTRODUCTION

The parabolic cylinder functions were introduced by Weber [1] in 1869. Standard solutions to Weber's equation were given by Miller [2] in 1952. These relations are also provided by Abramowitz and Stegun [3]. There are two standard forms of the Weber's equation,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - (\frac{1}{4}x^2 + a)y = 0, \tag{1}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (\frac{1}{4}x^2 - a)y = 0. \tag{2}$$

Equation (2) is obtained from (1) with changes a by $-ia$ and x by $xe^{i\pi/4}$. Thus, if $y(a,x)$ is a solution of (1), then (2) has solutions: $y(-ia, xe^{i\pi/4}), y(-ia, -xe^{i\pi/4}), y(ia, -xe^{-i\pi/4})$, and $y(ia, xe^{-i\pi/4})$. In the following we consider only real solutions of real equations.

II. SOLUTIONS OF EQUATION (1)

A. Standard solutions

There are two standard solutions of Eq. (1), $U(a,x)$ and $V(a,x)$, both of them expressed in terms of Whittaker's function $D_{-a-\frac{1}{2}}$,

$$U(a,x) = D_{-a-\frac{1}{2}}, \tag{3}$$

$$V(a,x) = \frac{1}{\pi}\Gamma(\frac{1}{2}+a)[sin(\pi a)U(a,x) + U(a,-x)]. \tag{4}$$

In a more symmetrical notation, these solutions are

$$U(a,x) = D_{-a-\frac{1}{2}} = Y_1 cos\beta - Y_2 sin\beta, \tag{5}$$

$$V(a,x) = \frac{1}{\Gamma(\frac{1}{2}-a)}(Y_1 sin\beta + Y_2 cos\beta), \tag{6}$$

where

$$\beta = \pi(\frac{a}{2}+\frac{1}{4}), \tag{7}$$

$$Y_1 = \frac{y_1\Gamma(\frac{1}{4}-\frac{a}{2})}{\sqrt{\pi}2^{\frac{a}{2}+\frac{1}{4}}}, \qquad Y_2 = \frac{y_2\Gamma(\frac{3}{4}-\frac{a}{2})}{\sqrt{\pi}2^{\frac{a}{2}-\frac{1}{4}}}, \tag{8}$$

$$y_1 = 1 + a\frac{x^2}{2!} + (a^2+\frac{1}{2})\frac{x^4}{4!} + (a^3+\frac{7a}{2})\frac{x^6}{6!} + (a^4+11a^2+\frac{15}{4})\frac{x^8}{8!} + \ldots, \tag{9}$$

$$y_2 = x + a\frac{x^3}{3!} + (a^2 + \frac{3}{2})\frac{x^5}{5!} + (a^3 + \frac{13a}{2})\frac{x^7}{7!} + (a^4 + 17a^2 + \frac{63}{4})\frac{x^9}{9!} + \ldots, \qquad (10)$$

in which the coefficients A_n of $\frac{x^n}{n!}$ obey the recurrence relation

$$A_{n+2} = aA_n + \frac{1}{4}n(n-1)A_{n-2}. \qquad (11)$$

Similarly to Eq. (4), there is relation

$$U(a,x) = \frac{\pi}{\Gamma(a+\frac{1}{2})cos^2(\pi a)}[V(a,-x) - sin(\pi a)V(a,x)]. \qquad (12)$$

At $x = 0$,

$$U(a,0) = \frac{\sqrt{\pi}}{2^{\frac{a}{2}+\frac{1}{4}}\Gamma(\frac{a}{2}+\frac{3}{4})},$$
$$U'(a,0) = -\frac{\sqrt{\pi}}{2^{\frac{a}{2}-\frac{1}{4}}\Gamma(\frac{a}{2}+\frac{1}{4})}, \qquad (13)$$

$$V(a,0) = \frac{2^{\frac{a}{2}+\frac{1}{4}}sin\pi(\frac{3}{4}-\frac{a}{2})}{\Gamma(\frac{3}{4}-\frac{a}{2})},$$
$$V'(a,0) = \frac{2^{\frac{a}{2}+\frac{3}{4}}sin\pi(\frac{1}{4}-\frac{a}{2})}{\Gamma(\frac{1}{4}-\frac{a}{2})}, \qquad (14)$$

B. Recurrence relations for $U(a,x)$ and $V(a,x)$

Standard solutions $U(a,x)$ and $V(a,x)$ obey the recurrence relations

$$xU(a,x) - U(a-1,x) + (a+\frac{1}{2})U(a+1,x) = 0,$$
$$U'(a,x) - \frac{1}{2}xU(a,x) + U(a-1,x) = 0, \qquad (15)$$

$$xV(a,x) - V(a+1,x) + (a-\frac{1}{2})V(a-1,x) = 0,$$
$$V'(a,x) - \frac{1}{2}xV(a,x) - (a-\frac{1}{2})V(a-1,x) = 0. \qquad (16)$$

C. Relations at large values of argument x

At large values of argument x, when $x \gg |a|$, there are relations

$$U(a,x) \sim x^{-a-\frac{1}{2}}e^{-\frac{x^2}{4}}[1 - \frac{(a+\frac{1}{2})(a+\frac{3}{2})}{2x^2} + \frac{(a+\frac{1}{2})(a+\frac{3}{2})(a+\frac{5}{2})(a+\frac{7}{2})}{2\cdot 4x^4} - \ldots], \qquad (17)$$

$$V(a,x) \sim \sqrt{\frac{2}{\pi}}x^{a-\frac{1}{2}}e^{\frac{x^2}{4}}[1 + \frac{(a-\frac{1}{2})(a-\frac{3}{2})}{2x^2} + \frac{(a-\frac{1}{2})(a-\frac{3}{2})(a-\frac{5}{2})(a-\frac{7}{2})}{2\cdot 4x^4} + \ldots]. \qquad (18)$$

III. SOLUTIONS OF EQUATION 2

A. Standard solution

The standard solution $W(a,x)$ of Eq. (2) is

$$W(a,\pm x) = 2^{-\frac{3}{4}}\left(\sqrt{\frac{G_1}{G_3}}y_1 \mp \sqrt{\frac{2G_3}{G_1}}y_2\right), \tag{19}$$

where

$$G_1 = |\Gamma(ia+\frac{1}{4})|, \qquad G_3 = |\Gamma(ia+\frac{3}{4})|, \tag{20}$$

$$y_1 = 1 + a\frac{x^2}{2!} + (a^2-\frac{1}{2})\frac{x^4}{4!} + (a^3-\frac{7a}{2})\frac{x^6}{6!} + (a^4-11a^2+\frac{15}{4})\frac{x^8}{8!} + \ldots, \tag{21}$$

$$y_2 = x + a\frac{x^3}{3!} + (a^2-\frac{3}{2})\frac{x^5}{5!} + (a^3-\frac{13a}{2})\frac{x^7}{7!} + (a^4-17a^2+\frac{63}{4})\frac{x^9}{9!} + \ldots, \tag{22}$$

in which the coefficients A_n of $\frac{x^n}{n!}$ obey the recurrence relation

$$A_{n+2} = aA_n - \frac{1}{4}n(n-1)A_{n-2}. \tag{23}$$

Relations for gamma function of complex argument are given in Appendix A. At $x=0$,

$$W(a,0) = 2^{-\frac{3}{4}}\sqrt{\frac{G_1}{G_3}},$$
$$W'(a,0) = -2^{-\frac{1}{4}}\sqrt{\frac{G_3}{G_1}}. \tag{24}$$

B. Relations at large values of argument x

At large values of the argument x, when $x \gg |a|$, there are relations

$$W(a,x) = \sqrt{\frac{2k}{x}}[s_1(a,x)\cos\gamma - s_2(a,x)\sin\gamma],$$
$$W(a,-x) = \sqrt{\frac{2}{kx}}[s_1(a,x)\sin\gamma + s_2(a,x)\cos\gamma], \tag{25}$$

where

$$k = \sqrt{1+e^{2\pi a}} - e^{\pi a}, \qquad k^{-1} = \sqrt{1+e^{2\pi a}} + e^{\pi a}, \tag{26}$$

$$\gamma = \frac{x^2}{4} - a\ln x + \frac{\pi}{4} + \frac{\phi}{2}, \tag{27}$$

with

$$\phi = \arg\Gamma(ia+\frac{1}{2}), \tag{28}$$

$$s_1(a,x) \sim 1 + \frac{v_2}{1!2x^2} - \frac{u_4}{2!2^2x^4} - \frac{v_6}{3!2^3x^6} + \frac{u_8}{4!2^4x^8} + \dots, \tag{29}$$

$$s_2(a,x) \sim -\frac{u_2}{1!2x^2} - \frac{v_4}{2!2^2x^4} + \frac{u_6}{3!2^3x^6} + \frac{v_8}{4!2^4x^8} - \dots, \tag{30}$$

with

$$u_m + iv_m = \frac{\Gamma(m + ia + \frac{1}{2})}{\Gamma(ia + \frac{1}{2})}, \qquad m = 2, 4, 6\dots \tag{31}$$

C. Analytic relations at $a = 0$

At $a = 0$ there are relations

$$W(0, \pm x) = 2^{-\frac{5}{4}}\sqrt{\pi x}\Big[J_{-\frac{1}{4}}\Big(\frac{x^2}{4}\Big) \mp J_{\frac{1}{4}}\Big(\frac{x^2}{4}\Big)\Big], \tag{32}$$

$$\frac{\mathrm{d}W(0,\pm x)}{\mathrm{d}x} = -2^{-\frac{9}{4}}x\sqrt{\pi x}\Big[J_{\frac{3}{4}}\Big(\frac{x^2}{4}\Big) \pm J_{-\frac{3}{4}}\Big(\frac{x^2}{4}\Big)\Big], \tag{33}$$

where $J_\nu(z)$ is the Bessel function of the first kind.

IV. IMPLEMENTATION OF PARABOLIC CYLINDER FUNCTIONS IN MATLAB

Routines provided for computation of parabolic cylinder functions are shortly described in Table I. For moderate values of argument x and parameter a, standard parabolic cylinder functions $U(a,x)$ and $V(a,x)$ are computed with routines "pu" and "pv", respectively, whereas $W(a,x)$ is computed with routine "pw". Differentiation with respect to argument x is computed with routines "dpu", "dpv", and "dpw". For large values of argument x, when $|x| \gg |a|$, functions $U(a,x)$, $V(a,x)$, and $W(a,x)$ are computed with routines "pulx", "pvlx", and "pwlx", and their derivatives with routines "dpulx", "dpvlx", and "dpwlx", respectively. Routine "cgamma" computes the gamma function of complex argument; using the function code kf, it computes either the logarithm of gamma function (when $kf = 0$) or gamma function (when $kf = 1$). Values of parabolic cylinder functions obtained by using these routines are shown in Tables II–VII.

APPENDIX A: RELATIONS FOR GAMMA FUNCTION OF COMPLEX ARGUMENT

If $|z| \gg 1$ and $|\arg z| \leq \pi - \epsilon$ with $\epsilon > 0$, there is relation

$$\ln \Gamma(z) \sim (z - \frac{1}{2}) \ln z - z + \frac{1}{2} \ln 2\pi + \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)} \frac{1}{z^{2n-1}}, \tag{A1}$$

where B_{2n} are the Bernoulli's numbers,

$$B_{2k} = (-1)^{k-1} \frac{2 \cdot (2k)!}{(2\pi)^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}}, \qquad k = 1, 2, \ldots \tag{A2}$$

Specific values,

$$\begin{aligned} B_2 &= 1/6 & B_{12} &= -691/2730 \\ B_4 &= -1/30 & B_{14} &= 7/6 \\ B_6 &= 1/42 & B_{16} &= -3617/510 \\ B_8 &= -1/30 & B_{18} &= 43867/798 \\ B_{10} &= 5/66 & B_{20} &= -174611/330 \end{aligned} \tag{A3}$$

Other useful relations,

$$\begin{aligned} \Gamma(z+n) &= z(z+1)\cdots(z+n-1)\Gamma(z), \\ \Gamma(z)\Gamma(-z) &= \frac{-\pi}{z \sin \pi z} \end{aligned} \tag{A4}$$

[1] H. F. Weber "Ueber die Integration der partiellen Differential-gleichung: $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + k^2 u = 0$," Math. Ann. **1**, 1–36 (1869).

[2] J. C. P. Miller "On the choice of standard solutions to Weber's equation," Proc. Cambridge Philos. Soc. **48**, 428–435 (1952).

[3] M. Abramowitz and I. Stegun *Handbook of Mathematical Functions* (New York, 1964).

TABLE I: Routines for parabolic cylinder functions

Name of routine	Routine call	What the routine computes
cgamma	$[gr, gi]$=cgamma(x, y, kf)	$\Gamma(z)$ with complex argument z (when $kf = 1$) or $\ln\Gamma(z)$, (when $kf = 0$); x and y are the real and imaginary parts of z gr and gi are the real and imaginary parts of $\ln\Gamma(z)$ or $\Gamma(z)$ [Eqs. (A1–A4)].
pu	u=pu(a, x)	Parabolic cylinder function $U(a, x)$ for moderate values of parameter a and argument x [Eqs. (3–12)].
dpu	du=dpu(a, x)	Derivative with respect to x of parabolic cylinder function $U(a, x)$ for moderate values of parameter a and argument x
pv	v=pv(a, x)	Parabolic cylinder function $V(a, x)$ for moderate values of parameter a and argument x [Eqs. (3–12)].
dpv	dv=dpv(a, x)	Derivative with respect to x of parabolic cylinder function $V(a, x)$ for moderate values of parameter a and argument x
pw	w=pw(a, x)	Parabolic cylinder function $W(a, x)$ for moderate values of parameter a and argument x [Eqs. (19–23)].
dpw	dw=dpw(a, x)	Derivative with respect to x of parabolic cylinder function $W(a, x)$ for moderate values of parameter a and argument x
pulx	u=pulx(a, x)	Parabolic cylinder function $U(a, x)$ for large values of parameter x ($\|x\| \gg \|a\|$) and moderate values of parameter a [Eq. (17)].
dpulx	du=dpulx(a, x)	Derivative with respect to x of parabolic cylinder function $U(a, x)$ for large values of parameter x ($\|x\| \gg \|a\|$) and moderate values of parameter a.
pvlx	v=pvlx(a, x)	Parabolic cylinder function $V(a, x)$ for large values of parameter x ($\|x\| \gg \|a\|$) and moderate values of parameter a [Eq. (18)].
dpvlx	dv=dpvlx(a, x)	Derivative with respect to x of parabolic cylinder function $V(a, x)$ for large values of parameter x ($\|x\| \gg \|a\|$) and moderate values of parameter a.
pwlx	w=pwlx(a, x)	Parabolic cylinder function $W(a, x)$ for large values of parameter x ($\|x\| \gg \|a\|$) and moderate values of parameter a [Eqs. (25–31)].
dpwlx	dw=dpwlx(a, x)	Derivative with respect to x of parabolic cylinder function $W(a, x)$ for large values of parameter x ($\|x\| \gg \|a\|$) and moderate values of parameter a.

TABLE II: Values of $U(a,x)$ with $a = -5, -3.5, -1, 1, 3.5, 5$ and $x = 0, 1, 3, 5$

$x\backslash a$	-5.0	-3.5	-1.0
0.0	3.052183664350372	-0.000000000000000	0.581368317019118
1.0	0.579926011661105	-1.557601566142810	0.842203244069839
3.0	3.202129097812791	1.897186042113549	0.184881790005045
5.0	1.879976816310843	0.212349954984646	0.004337473181400
$x\backslash a$	1.0	3.5	5.0
0.0	1.162736634038237	0.333333333333333	0.103354367470066
1.0	0.378262434740955	0.048971230815929	0.010659966828235
3.0	0.017224293634316	0.000610423938072	0.000070950238455
5.0	0.000161381143270	0.000002208878109	0.000000155227075

TABLE III: Values of $U(a,-x)$ with $a = -5, -3.5, -1, 1, 3.5, 5$ and $x = 0, 1, 3, 5$

$x\backslash a$	-5.0	-3.5	-1.0
0.0	3.052183664350372	-0.000000000000000	0.581368317019118
1.0	-4.332232266251285	1.557601566142810	-0.195001018223362
3.0	3.802753160685226	-1.897186042113549	-1.767855400724101
5.0	-9.615606269532364	-0.212349954984649	-35.754085404247576
$x\backslash a$	1.0	3.5	5.0
0.0	1.16273663404	0.33333333333	0.10335436747
1.0	3.27078479478	2.19468750736	0.97838806074
3.0	45.73101176423	142.69397188181	125.30190015651
5.0	3259.12460949910	30297.53050402874	45998.28922772748

TABLE IV: Values of $V(a,x)$ with $a = -5, -3.5, -1, 1, 3.5, 5$ and $x = 0, 1, 3, 5$

$x \backslash a$	-5.0	-3.5	-1.0
0.0	-0.058311457540778	0.265961520267622	-0.656003897333753
1.0	0.082766571619165	-0.076762147625440	0.220035086525655
3.0	-0.072650962016911	0.097154672861824	1.994811204614366
5.0	0.183704546768818	1.173350875864019	40.344165108706711
$x \backslash a$	1.0	3.5	5.0
0.0	0.3280019487	0	1.7220102305
1.0	0.9226713556	4.0980162226	16.3011422859
3.0	12.9004802412	272.5242458690	2087.6829809173
5.0	919.3820780818	57864.0209141053	766387.7838412275

TABLE V: Values of $V(a,-x)$ with $a = -5, -3.5, -1, 1, 3.5, 5$ and $x = 0, 1, 3, 5$

$x \backslash a$	-5.0	-3.5	-1.0
0.0	-0.058311457540778	0.265961520267622	-0.656003897333753
1.0	-0.011079389291262	-0.076762147625440	-0.950324595068664
3.0	-0.061176139925034	0.097154672861824	-0.208616760217021
5.0	-0.035916642101972	1.173350875864019	-0.004894314375732
$x \backslash a$	1.0	3.5	5.0
0.0	0.32800194867	0	1.72201023050
1.0	0.10670586276	-4.09801622261	0.17760809131
3.0	0.00485888353	-272.52424586904	0.00118211779
5.0	0.00004552478	-57864.02091410524	0.00000258678

TABLE VI: Values of $W(a, x)$ with $a = -5, -3, -1, 1, 3, 5$ and $x = 0, 1, 3, 5$

$x \backslash a$	-5.0	-3.0	-1.0
0.0	0.473478576486605	0.539330386270653	0.731481090245431
1.0	-0.657520526362908	-0.611126375982879	-0.184115556183355
3.0	-0.062604004232077	0.636305300554784	-0.053352644054153
5.0	0.089361847055232	0.437066960213013	-0.570254174032845
$x \backslash a$	1.0	3.0	5.0
0.0	0.731481090245431	0.539330386270653	0.473478576486605
1.0	0.315937643962764	0.101682226485666	0.052572013487910
3.0	0.016773032899024	0.009166528652640	0.001223742332881
5.0	0.022807516888135	-0.003844865237560	0.000115773464320

TABLE VII: Values of $W(a, -x)$ with $a = -5, -3, -1, 1, 3, 5$ and $x = 0, 1, 3, 5$

$x \backslash a$	-5.0	-3.0	-1.0
0.0	0.473478576486605	0.539330386270653	0.731481090245431
1.0	0.070610950611453	0.428801301530536	0.950916920458344
3.0	0.606270877302830	0.177268761402591	-0.757374330077355
5.0	0.538608396875686	-0.370945283780393	0.180907184885679
$x \backslash a$	1.0	3.0	5.0
0.0	0.731481090245	0.539330386271	0.473478576487
1.0	1.903689596383	3.001251077335	4.378212848013
3.0	6.183176599808	57.210355295947	253.398744868662
5.0	-4.359927574948	66.590129609337	2852.835947866653