Notes for "GRAVITATION" - MTW

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abc

1. GEOMETRODYNAMICS IN BRIEF

1.1. The parable of the apple

Space tells matter how to move and matter tells space how to curve.

1.2. Spacetime with and without coordinates

1.2.1. Hint for possible different characterization ($P.6$)

"But with all the daring in the world, how is one to drive a nail into spacetime to mark a point? Happily, nature provides its own way to localize a point in spacetime, as Einstein was the first to emphasize. Characterize the point by what happens there!"

1.2.2. Hint from idealized limit ($P.10$)

"A more detailed diagram would show a maze of world line and of light rays and the intersections between them. From such a picture, one can in imagination step to the *idealized* limit: an infinitely dense collection of light rays and of world lines of infinitesimal test particles."

1.2.3. Hint of breakdown of manifold description ($P.12$)

"Not so quantum general relativity or 'quantum geometrodynamics'. It predicts violent fluctuations in the geometry at distances on the order of the Planck length,... As nearly as one can estimate these fluctuations give space at small distances a 'multiply connected' or 'foam-like' character."

1.3. Weightlessness

1.3.1. Box 1.2 ($P.16$)

a. Lorand von Eötvös ($P.16$) The forces mentioned are shown in Fig. 1, from which we directly observe that

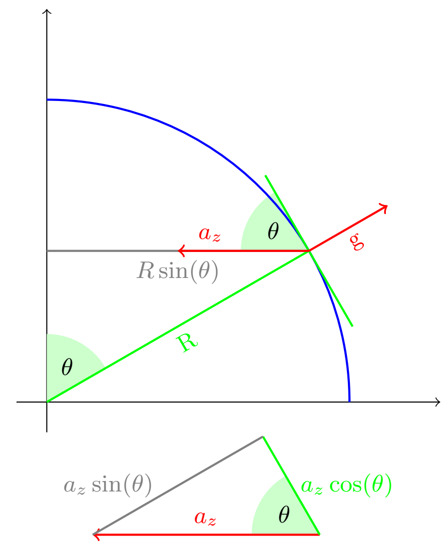

Figure 1: Forces mentioned in Box 1.2, *Lorand von Eötvös*, where R is the radius of the earth and θ is the angle measured from the north pole

the centripetal acceleration is in total

$$a_z = \omega^2 R\sin(\theta),$$

which is *independent* of g in this way of viewing forces (ω is the rotation speed of the earth). From this we get the northward directed part of a_z to be

$$a_z\cos(\theta) = \omega^2 R\sin(\theta)\cos(\theta).$$

b. Beall ($P.17$) We know the highest observed energies of the myon to be

$$E_\mu = 1\cdot 10^{13}\,\text{eV}.$$

While the upper threshold is mentioned to be

$$E_{\text{thresh}} = 1\cdot 10^3 mc^2$$

. If now the myon were to be "too light" we would have

$$E_\mu > E_{\text{thresh}}.$$

On the other hand we know the highest observed energies of photons to be

$$E_\gamma = 1\cdot 10^{13}\,\text{eV}.$$

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The transferred energy (to a photon) mentioned results in

$$E_\gamma \geq 2mc^2.$$

These two observations combined put an upper limit (Not "too heavy") on m.

1.3.2. "This assumption of exact physical equivalence makes it impossible for us to speak of the absolute acceleration of the system of reference, . . . " (P.17)

This because any part of the acceleration could be due to a gravitational field.

1.4. Local Lorentz Geometry, with and without Coordinates

1.4.1. Hint : What about breakdown at small scales? (P.21)

"The geometry of spacetime is locally Lorentzian everywhere."

1.5. Time

Time is defined so that motion looks simple.

1.6. Curvature

1.6.1. Box 1.6 (P.32)

a. Figure C (P.33) The formula

$$r = \frac{d^2}{8a},$$

where r is the radius of curvature, d is the "direct" distance between start and end point and a is the raise of the track of e.g. the ball, can be derived as follows:

Denote by α the angle inscribed by the shown cone under consideration, then

$$\sin(\alpha) = \frac{d}{r} \tag{1}$$

Further denote by x the distance between the center of curvature and the intersection point of the radial line at $\alpha/2$ with the "direct" line of connection between start and end point. Or in formulas

$$x = r - a.$$

Then we know

$$\begin{aligned}
\cos\left(\frac{\alpha}{2}\right) &= \frac{x}{r} \\
\Rightarrow a &= r - x \\
&= r\left(1 - \cos\left(\frac{\alpha}{2}\right)\right) \\
&\overset{Eq.\ 1}{\approx} r\left(1 - \sqrt{1 - \left(\frac{d}{2r}\right)^2}\right) \\
&\approx r\left(1 - \left(1 - \frac{d^2}{8r^2}\right)\right) \\
&= \frac{d^2}{8r} \\
\Rightarrow r &= \frac{d^2}{8a}
\end{aligned}$$

1.6.2. Eq. (1.13) (P.37)

The summation only happens over the spatial components i.e. j because we are here considering $v \ll c$, such that the separation in the time coordinate is negligible.

1.7. Effect of Matter on Geometry

1.7.1. Box 1.9 (P.38)

a. Galileo Galilei (1638) What is meant here is uniform acceleration leading to

$$s = \frac{1}{2}at^2 \Rightarrow s \propto t^2.$$

1.7.2. Hint : GR only being valid for larger (average) scales? (P.42)

"the field equation shows how the stress-energy of matter generates an *average* curvature in its neighborhood."

1.7.3. Exercise 1.1 (P.44)

By unrolling the cylinder it is obvious that geodesics, which are parallel at one point, are parallel at any other point, i.e. they suffer no geodesic deviation:

$$\begin{aligned}
\frac{\mathrm{d}^2\xi}{\mathrm{d}s^2} &= 0 \overset{Eq.\ (1.6)}{=} -R\xi \\
\Rightarrow R &= 0
\end{aligned}$$

Given the formula

$$R = \frac{1}{\varrho_1 \varrho_2},$$

where ϱ_i are the principal radii of curvature at the point in question. We then get for a cylinder

$$\varrho_1 = \infty \qquad \varrho_2 = r \quad \Rightarrow R = 0,$$

where r is the radius of the cylinder in the 3-dimensional euclidean embedding space.

1.7.4. *Exercise* 1.2 (*P*.44)

Using Eq. (1.14) the values in question can be calculated.

1.7.5. *Exercise* 1.3 (*P*.44)

First fix some notation:

$$\begin{aligned} M &= \text{Mass of satellite} \\ \omega &= \text{Angular frequency of satellite} \\ r &= \text{Radius of orbit of satellite} \\ m &= \text{Mass of central object} \\ \varrho_{\text{Kepler}} &= \frac{m}{\frac{4\pi}{3} r^3} = \text{Kepler density} \end{aligned}$$

By setting the centripetal acceleration of the satellite equal to the gravitational acceleration (in Newtonian mechanics), we get:

$$\begin{aligned} M\omega^2 r &= G\frac{mM}{r^2} \\ \Rightarrow \omega^2 &= G\frac{m}{r^3} = \frac{4\pi G}{3}\varrho_{\text{Kepler}} \end{aligned}$$

2. FOUNDATIONS OF SPECIAL RELATIVITY

2.1. Overview

Assumed background.

2.2. Geometric Objects

Geometric objects exist independently of coordinates.

2.3. Vectors

2.3.1. *"...and only thereafter draw the straight line $\mathcal{P}(0) + \lambda \left(\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}\lambda}\right)_0 \ldots$"* (*P*.49)

Strictly speaking this addition is not possible, because the point $\mathcal{P}(0)$ lives in the manifold and the tangent vector $\left(\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}\lambda}\right)_0$ live in tangent space at $\mathcal{P}(0)$. In case of flat space, these spaces should be isomorphic, s.t. one can symbolically write the addition like it is done here.

2.3.2. *"Relative to the origin $\mathcal{O}$ of this frame, the world line has a coordinate description..."* (*P*.50)

Such a direct description is only possible in flat space, as there we can define a complete set of basis vectors.

2.4. The metric tensor

Useful.

2.5. Differential forms

2.5.1. *Fig* 2.5 (*P*.56)

a. "At different events, $\tilde{\boldsymbol{k}} = \mathbf{d}\phi$ is different..." What is described here is the concept of a form field.

2.5.2. *"The surface of $\tilde{\boldsymbol{k}}$ that passes through $\mathcal{P}_0$ contains points $\mathcal{P}$ for which $\left\langle \tilde{\boldsymbol{k}}, \mathcal{P} - \mathcal{P}_0 \right\rangle = 0$"* (*P*.56)

This can't be so easily written down in GR as $\mathcal{P} \in \mathcal{M}$ and *not* $\mathcal{P} \in T\mathcal{M}$.

2.5.3. *Eq.* (2.13) (*P*.57)

TODO

2.5.4. *Fig* 2.7 (*P*.58)

a. Direction of $\boldsymbol{A}$ and $\tilde{\boldsymbol{A}}$ Assume components of $\boldsymbol{A}$ to be $(a, 0, 0, 0)$. From this we get for any vector $\boldsymbol{v}$:

$$\begin{aligned} \boldsymbol{A} \cdot \boldsymbol{v} &\overset{(2.11)}{=} -av^0 \\ &\overset{(2.14)}{=} \left\langle \tilde{\boldsymbol{A}}, \boldsymbol{v} \right\rangle \\ &\overset{(2.22)}{=} \tilde{A}_\alpha v^\alpha \\ \Rightarrow \tilde{A}_i &= 0 \\ \Rightarrow \tilde{A}_0 &= -a \end{aligned}$$

2.5.5. *"Consider a 1-form representing the march of Lorentz coordinate time toward the future. The corresponding vector points toward the past ..."* (*P*.59)

From 2 2.5 2.5.4 a we observe for $a = -1$:

$$\boldsymbol{A} = -\boldsymbol{e}_0 \qquad \tilde{\boldsymbol{A}} = \mathbf{d}x^0$$

2.5.6. "Such practice is justified by the unique correspondence between $\tilde{\boldsymbol{p}}$ and $\boldsymbol{p}$." (P.59)

This is the metric duality in TODO.

2.5.7. Exercise 2.1 (P.59)

Set $\mathcal{P}_0 = 0 \Rightarrow x = \mathcal{P} - \mathcal{P}_0 = \mathcal{P}$, s.t.:

$$\begin{aligned} \langle \tilde{\boldsymbol{p}}, \boldsymbol{x} \rangle &= \hbar \left(\Phi(\mathcal{P}) - \Phi(\mathcal{P}_0) \right) \\ &= \hbar \left(\Phi(x) - \Phi(0) \right) \\ &= \hbar \left(k \cdot x - \omega t \right) \\ &= \boldsymbol{p} \cdot \boldsymbol{x} \end{aligned}$$

Where the first equality holds since $\Phi(x)$ is linear in x an therefore all terms of higher order in x in Eq. (2.13) are actually zero.

2.6. Gradients and directional derivatives

2.6.1. Eq. (2.17) (P.60)

This is the analog of defining how the basis of the cotangent space acts on the basis of the tangent space (see Eq. (2.19)), because $\partial_{\boldsymbol{v}} f$ can be defined in a coordinate free manner.

2.7. Coordinate representation of geometric objects

2.7.1. Exercise 2.2 (P.62)

$$\begin{aligned} u^\gamma \eta_{\gamma\alpha} &= u^\gamma \eta_{\gamma\beta} \delta^\beta_\alpha \\ &= \boldsymbol{u} \cdot \boldsymbol{e}_\alpha \\ &= \langle \tilde{\boldsymbol{u}}, \boldsymbol{e}_\alpha \rangle \\ &= u_\alpha \end{aligned}$$

2.7.2. Exercise 2.3 (P.62)

Trivial since $\|\eta^{\alpha\beta}\|$ is the inverse of $\|\eta_{\alpha\beta}\|$:

$$u^\alpha = \delta^\alpha_\gamma u^\gamma = \eta^{\alpha\beta} \eta_{\beta\gamma} u^\gamma = \eta^{\alpha\beta} u_\beta$$

2.7.3. Exercise 2.4 (P.62)

By definition, we have:

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) &= u^\alpha v^\beta \eta_{\alpha\beta} \\ &\overset{2.7.1}{=} u^\alpha v_\alpha \\ &\overset{2.7.2}{=} u_\alpha v_\beta \eta^{\alpha\beta} \end{aligned}$$

2.8. The centrifuge and the photon

2.8.1. "... and the magnitudes - but not the directions - of u_e and u_a are equal." (P.64)

This follows from Eq. (2.2).

2.8.2. "From the geometry of Fig 2.9, one sees that u_e makes the same angle with p as does u_a." (P.64)

TODO

2.8.3. Exercise 2.5 (P.65)

a. In the rest frame of the observer we have $u^\mu = (1, \vec{0})$ and $p^\mu = (E, \vec{p})$, s.t.

$$-\boldsymbol{p} \cdot \boldsymbol{u} = -(-E) = E.$$

Since this was computed in the rest frame of the observer E is indeed the energy which the observer would measure.

b. In the rest frame of the particle we have $p^\mu = (m, \vec{0})$, s.t.

$$\boldsymbol{p}^2 = -m^2$$

which has to equal the $\boldsymbol{p}^2 = -E^2 + \vec{p}^2$ measured by the observer.

c.

$$\begin{aligned} |\vec{p}|^2 &\overset{b)}{=} E^2 + \boldsymbol{p}^2 \\ \Rightarrow |\vec{p}| &\overset{a)}{=} \sqrt{(\boldsymbol{p} \cdot \boldsymbol{u})^2 + \boldsymbol{p} \cdot \boldsymbol{p}} \end{aligned}$$

d. From Eq. (2.2) we know $p^\mu = m \left(u_p \right)^\mu = (m\gamma, m\gamma\vec{v})$, s.t.:

$$\begin{aligned} \frac{|\vec{p}|^2}{E^2} &\overset{a),c)}{=} \frac{(\boldsymbol{p} \cdot \boldsymbol{u})^2}{(\boldsymbol{p} \cdot \boldsymbol{u})^2} + \frac{\boldsymbol{p} \cdot \boldsymbol{p}}{E^2} \\ &\overset{b)}{=} 1 - \frac{m^2}{E^2} \\ &= 1 - \frac{m^2}{\gamma^2 m^2} \\ &\overset{(2.2)}{=} \vec{v}^2 \\ \Rightarrow |\vec{v}| &= \frac{|\vec{p}|}{E} \end{aligned}$$

e. Work in the rest frame of the observer:

$$\begin{aligned}\left(\frac{\boldsymbol{p}+(\boldsymbol{p}\cdot\boldsymbol{u})\boldsymbol{u}}{-\boldsymbol{p}\cdot\boldsymbol{u}}\right)^{\mu} &\overset{a)}{=} \left(\frac{\boldsymbol{p}-E\boldsymbol{u}}{E}\right)^{\mu} \\ &= \left(\frac{E-E}{E},\frac{\vec{p}}{E}\right) \\ &= \left(0,\frac{\vec{v}|\vec{p}|}{|\vec{v}|E}\right) \\ &\overset{d)}{=} (0,\vec{v})\end{aligned}$$

2.8.4. *Exercise* 2.6 (*P*.65)

In any local Lorentz-frame we have:

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}\tau} &= \frac{\mathrm{d}x^{\alpha}}{\mathrm{d}\tau}\frac{\partial T}{\partial x^{\alpha}} \\ &= u^{\alpha}\frac{\partial T}{\partial x^{\alpha}} \\ &= \partial_{\boldsymbol{u}}T \\ &= \langle \mathbf{d}T,\boldsymbol{u}\rangle\end{aligned}$$

In the local Lorentz-frame of the cosmic ray we have $u^{\mu}=(1,\vec{0})$, s.t. this really *is* the time derivative as measured by the clock of the cosmic ray. This result is reasonable as due to the movement of the cosmic ray the spatial change of T must be factored in.

2.9. Lorentz transformations

2.9.1. *Eq.* (2.40) (*P*.66)

$$\begin{aligned}\boldsymbol{u} &= \frac{\mathrm{d}x^{\beta}}{\mathrm{d}\tau}\boldsymbol{e}_{\beta} \\ &\overset{(2.39)}{=} \frac{\mathrm{d}x^{\alpha'}}{\mathrm{d}\tau}\underbrace{\Lambda^{\beta}{}_{\alpha'}\boldsymbol{e}_{\beta}}_{=\boldsymbol{e}_{\alpha'}}\end{aligned}$$

2.9.2. *Box* 2.4 (*P*.67)

a. "Aberration, incoming photon:..."(*P*.68) See Wikipedia "Relativistic aberration", general setup of definitions of angles, etc. is not clear. TODO

b. "Effect of transformation on other quantities:..."(*P*.68) One has to be careful in these equations about which system (primed or un-primed) is written on what side and remember the notation from Eqs. (2.38/39).

2.9.3. *Exercise* 2.7 (*P*.69)

$$\|\Lambda^{\nu'}{}_{\mu}\| = \begin{pmatrix} \gamma & -\beta\gamma n^1 & -\beta\gamma n^2 & -\beta\gamma n^3 \\ -\beta\gamma n^1 & (\gamma-1)\left(n^1\right)^2+1 & (\gamma-1)n^1n^2 & (\gamma-1)n^1n^3 \\ -\beta\gamma n^2 & (\gamma-1)n^2n^1 & (\gamma-1)\left(n^2\right)^2+1 & (\gamma-1)n^2n^3 \\ -\beta\gamma n^3 & (\gamma-1)n^3n^1 & (\gamma-1)n^3n^2 & (\gamma-1)\left(n^3\right)^2+1 \end{pmatrix} \tag{2}$$

a.

$$\Lambda^{\nu'}{}_{\alpha}\eta_{\nu'\mu'}\Lambda^{\mu'}{}_{\beta} = -\Lambda^{0'}{}_{\alpha}\Lambda^{0'}{}_{\beta}+\Lambda^{i'}{}_{\alpha}\Lambda^{i'}{}_{\beta} \overset{!}{=} \eta_{\alpha\beta}$$

For the $\alpha=\beta=0$ component we get:

$$\Lambda^{\nu'}{}_{0}\eta_{\nu'\mu'}\Lambda^{\mu'}{}_{0} = -\gamma^2+\beta^2\gamma^2n^in^i = -\gamma^2\left(1-\beta^2\right) = -1$$

For the $\alpha=0,\beta=j$ components we get:

$$\Lambda^{\nu'}{}_{0}\eta_{\nu'\mu'}\Lambda^{\mu'}{}_{j} = -\gamma\left(-\beta\gamma n^j\right)+\left(-\beta\gamma n^i\right)\left((\gamma-1)n^in^j+\delta^i_j\right) = \beta\gamma^2n^j-\beta\gamma^2n^j+\beta\gamma n^j-\beta\gamma n^j = 0$$

For the $\alpha=j,\beta=k$ components we get:

$$\begin{aligned}\Lambda^{\nu'}{}_{j}\eta_{\nu'\mu'}\Lambda^{\mu'}{}_{k} &= -\beta^2\gamma^2n^jn^k+\left((\gamma-1)n^in^j+\delta^i_j\right)\left((\gamma-1)n^in^k+\delta^i_k\right) \\ &= -\beta^2\gamma^2n^jn^k+(\gamma-1)^2n^jn^k+2(\gamma-1)n^jn^k+\delta_{jk} = n^jn^k\left(-\beta^2\gamma^2+\gamma^2-2\gamma+1+2\gamma-2\right)+\delta_{jk} = \delta_{jk}\end{aligned}$$

b. The 4-velocity of the primed frame as seen in the primed frame is $u^{\nu'} = \left(1, \vec{0}\right)$, and as seen in the un-primed frame it is:

$$u^{\mu} = \Lambda^{\mu}{}_{\nu'} u^{\nu'} = \Lambda^{\mu}{}_{0'} = (\gamma, \beta\gamma\vec{n}) \Rightarrow \vec{v} = \beta\vec{n}$$

c. Same as b) with $\beta \to -\beta$.
d. For motion in z-direction we have $n^3 = 1, n^1 = n^2 = 0$, with this and Eq. 2 we obtain Eq. (2.45).

2.10. Collisions

See e.g. Weinberg QFT.

3. THE ELECTROMAGNETIC FIELD

3.1. "In" and "Out" States

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