

# Notes for “The Quantum Theory of Fields 1, Foundations” - Weinberg

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## I. INTRODUCTION

4. Eq. (2.2.22) (P.54)

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## II. RELATIVISTIC QUANTUM MECHANICS

$$\begin{aligned} -if_{bc}^a t_a - t_b t_c &\stackrel{(2.2.21)}{=} t_{bc} = t_{cb} \stackrel{(2.2.21)}{=} -if_{cb}^a t_a - t_c t_b \\ \Leftrightarrow [t_b, t_c] &= i(f_{cb}^a - f_{bc}^a) t_a \end{aligned}$$

### A. Quantum Mechanics

### B. Symmetries

1. “For this to be unitary and linear,  $t$  must be Hermitian and linear” (P.51)

Linearity is trivial and hermiticity follow from the following observation:

$$\begin{aligned} \langle U\Psi|U\Phi\rangle &= \langle (1+i\varepsilon t)\Psi|(1+i\varepsilon t)\Phi\rangle \\ &= \langle \Psi|\Phi\rangle + \varepsilon i(\langle \Psi|t\Phi\rangle - \langle t\Psi|\Phi\rangle) + \mathcal{O}(\varepsilon^2) \end{aligned}$$

$$\text{Eq. (2.2.2)} \quad \Leftrightarrow \langle \Psi|t\Phi\rangle = \langle t\Psi|\Phi\rangle$$

$$\text{Eq. (2.1.5)} \quad t^\dagger = t$$

2. Eq. (2.2.19) (P.54)

$f_{bc}^a$  and  $f^a$  have to be real as  $\theta^a$  are real.

3. Eq. (2.2.21) (P.54)

From Eq. (2.2.20) we have up to  $\mathcal{O}(\theta^2, \bar{\theta}^2)$

$$\begin{aligned} 1 + i(\theta^a + \bar{\theta}^a + f_{bc}^a \bar{\theta}^b \theta^c) t_a + \frac{1}{2}(\theta^b + \bar{\theta}^b)(\theta^c + \bar{\theta}^c) t_{bc} \\ = \left[ 1 + i\bar{\theta}^a t_a + \frac{1}{2}\bar{\theta}^b \bar{\theta}^c t_{bc} \right] \cdot \left[ 1 + i\theta^a t_a + \frac{1}{2}\theta^b \theta^c t_{bc} \right] \\ = 1 + i\theta^a t_a + \frac{1}{2}\theta^b \theta^c t_{bc} + i\bar{\theta}^a t_a - \bar{\theta}^b t_b \theta^c t_c + \frac{1}{2}\bar{\theta}^b \bar{\theta}^c t_{bc} \\ \Leftrightarrow if_{bc}^a \bar{\theta}^b \theta^c t_a + \frac{1}{2}(\theta^b \bar{\theta}^c + \bar{\theta}^b \theta^c) t_{bc} = -\bar{\theta}^b \theta^c t_b t_c \\ \Leftrightarrow \bar{\theta}^b \theta^c [t_{bc} + if_{bc}^a t_a + t_b t_c] = 0 \\ \Leftrightarrow t_{bc} = -if_{bc}^a t_a - t_b t_c \end{aligned}$$

### C. Quantum Lorentz Transformations

1. “ $\Lambda^\mu{}_\nu$  has an *inverse*” (P.57)

This is true because

$$\det(\Lambda) = \pm 1 \neq 0$$

2. “ $(\bar{\Lambda}\Lambda)^0{}_0 \geq \bar{\Lambda}^0{}_0 \Lambda^0{}_0 - \sqrt{(\Lambda^0{}_0)^2 - 1} \sqrt{(\bar{\Lambda}^0{}_0)^2 - 1} \geq 1$ ” (P.58)

The first inequality is trivial from the preceding considerations. For the second inequality assume the contrary holds, then:

$$\begin{aligned} 0 &\leq \bar{\Lambda}^0{}_0 \Lambda^0{}_0 - 1 < \sqrt{(\Lambda^0{}_0)^2 - 1} \sqrt{(\bar{\Lambda}^0{}_0)^2 - 1} \\ &\Rightarrow (\bar{\Lambda}^0{}_0)^2 (\Lambda^0{}_0)^2 - 2\bar{\Lambda}^0{}_0 \Lambda^0{}_0 + 1 \\ &< (\bar{\Lambda}^0{}_0)^2 (\Lambda^0{}_0)^2 - (\bar{\Lambda}^0{}_0)^2 - (\Lambda^0{}_0)^2 + 1 \\ &\Rightarrow (\Lambda^0{}_0 + \bar{\Lambda}^0{}_0)^2 = (\bar{\Lambda}^0{}_0)^2 - 2\bar{\Lambda}^0{}_0 \Lambda^0{}_0 + (\Lambda^0{}_0)^2 < 0 \end{aligned}$$

Which is a contradiction as  $\Lambda^0{}_0 + \bar{\Lambda}^0{}_0 \geq 1 + 1 = 2$  and therefore completes the proof.

### D. The Poincaré Algebra

1. “In order for  $U(1+\omega, \varepsilon)$  to be unitary, the operators  $J^{\rho\sigma}$  and  $P^\rho$  must be **Hermitian**” (P.59)

Analog to IIB 1.

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## 2. Eqs. (2.4.8/9) (P.60)

$$\begin{aligned}
& \frac{1}{2} \omega_{\rho\sigma} U J^{\rho\sigma} U^{-1} - \varepsilon_\rho U P^\rho U^{-1} \\
& \stackrel{(2.4.7)}{=} \frac{1}{2} (\Lambda \omega \Lambda^{-1})_{\mu\nu} J^{\mu\nu} - (\Lambda \varepsilon - \Lambda \omega \Lambda^{-1} a)_\mu P^\mu \\
& = \frac{1}{2} \Lambda_\mu{}^\rho \omega_{\rho\sigma} (\Lambda^{-1})^\sigma{}_\nu J^{\mu\nu} \\
& - \left( \Lambda_\mu{}^\rho \varepsilon_\rho - \Lambda_\mu{}^\rho \omega_{\rho\sigma} (\Lambda^{-1})^\sigma{}_\nu a^\nu \right) P^\mu \\
& \stackrel{(2.3.10)}{=} \frac{1}{2} \Lambda_\mu{}^\rho \omega_{\rho\sigma} \Lambda_\nu{}^\sigma J^{\mu\nu} \\
& - \left( \Lambda_\mu{}^\rho \varepsilon_\rho - \Lambda_\mu{}^\rho \omega_{\rho\sigma} \Lambda_\nu{}^\sigma a^\nu \right) P^\mu \\
& = \frac{1}{2} \omega_{\rho\sigma} (\Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma J^{\mu\nu} + \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma a^\nu P^\mu) \\
& - \varepsilon_\rho \Lambda_\mu{}^\rho P^\mu
\end{aligned}$$

In order to be able to compare coefficients in this, the coefficient of  $\omega_{\rho\sigma}$  has to be anti symmatrized. With this Eqs. (2.4.8/9) follow immediately.

## 3. Eqs. (2.4.10/11) (P.60)

Up to  $\mathcal{O}(\omega, \varepsilon)$  one can identify

$$U^{-1}(1 + \omega, \varepsilon) = U(1 - \omega, -\varepsilon)$$

,since

$$U(1 + \omega, \varepsilon)U(1 - \omega, -\varepsilon) = U(1 - \omega + \omega, -\varepsilon + \varepsilon) = U(1, 0).$$

With this we have up to  $\mathcal{O}(\omega, \varepsilon)$

$$\begin{aligned}
& i \left[ \frac{1}{2} \omega_{\mu\nu} J^{\mu\nu} - \varepsilon_\mu P^\mu, J^{\rho\sigma} \right] \\
& = \left( 1 + \frac{1}{2} i \omega_{\mu\nu} J^{\mu\nu} - i \varepsilon_\mu P^\mu \right) J^{\rho\sigma} \\
& \cdot \left( 1 - \frac{1}{2} i \omega_{\mu\nu} J^{\mu\nu} + i \varepsilon_\mu P^\mu \right) - J^{\rho\sigma} \\
& = U J^{\rho\sigma} U^{-1} - J^{\rho\sigma} \\
& \stackrel{(2.4.8)}{=} (1 + \omega)_\mu{}^\rho (1 + \omega)_\nu{}^\sigma \\
& \cdot (J^{\mu\nu} - \varepsilon^\mu P^\nu + \varepsilon^\nu P^\mu) - J^{\rho\sigma} \\
& = -\varepsilon^\rho P^\sigma + \varepsilon^\sigma P^\rho + \omega_\nu{}^\sigma J^{\rho\nu} + \omega_\mu{}^\rho J^{\mu\sigma}
\end{aligned}$$

and also

$$\begin{aligned}
& i \left[ \frac{1}{2} \omega_{\mu\nu} J^{\mu\nu} - \varepsilon_\mu P^\mu, P^\rho \right] \\
& = \left( 1 + \frac{1}{2} i \omega_{\mu\nu} J^{\mu\nu} - i \varepsilon_\mu P^\mu \right) P^\rho \\
& \cdot \left( 1 - \frac{1}{2} i \omega_{\mu\nu} J^{\mu\nu} + i \varepsilon_\mu P^\mu \right) - P^\rho \\
& = U P^\rho U^{-1} - P^\rho \\
& \stackrel{(2.4.9)}{=} \omega_\mu{}^\rho P^\mu
\end{aligned}$$

## 4. Eqs. (2.4.12/13/14) (P.60)

The only difficulty lies in the derivation of Eq. (2.4.12) because for this one has to anti symmatrize the coefficient of  $\omega_{\mu\nu}$  in Eq. (2.4.10) in order to be able to compare coefficients, similar to II D 2. (Similarly if one derives Eq. (2.4.13) from Eq. (2.4.11))

## 5. Eqs. (2.4.18 – 24) (P.61)

First observe

$$J^{ij} = \varepsilon_{ijk} J_k$$

and

$$J_i = \frac{1}{2} \varepsilon_{lmi} J^{lm}$$

$$\begin{aligned}
[J_i, J_j] &= \frac{\varepsilon_{lmi} \varepsilon_{kpj}}{4} [J^{lm}, J^{kp}] \\
&\stackrel{(2.4.12)}{=} -i \frac{\varepsilon_{lmi} \varepsilon_{kpj}}{4} [\eta^{mk} J^{lp} - \eta^{lk} J^{mp} - \eta^{pl} J^{km} + \eta^{pm} J^{kl}] \\
&= -\frac{i}{2} [\varepsilon_{kil} \varepsilon_{kmj} J^{lm} + \varepsilon_{kim} \varepsilon_{kjl} J^{lm}] \\
&= -\frac{i}{2} [J^{ji} - J^{ij}] \\
&= i J^{ij} = i \varepsilon_{ijk} J_k \\
[J_i, K_j] &= [J^{lm}, J^{0j}] \frac{\varepsilon_{lmi}}{2} \\
&\stackrel{(2.4.12)}{=} -i \frac{\varepsilon_{lmi}}{2} [\eta^{m0} J^{lj} - \eta^{l0} J^{mj} - \eta^{jl} J^{0m} + \eta^{jm} J^{0l}] \\
&= -i \frac{\varepsilon_{lmi}}{2} [\delta_{jm} K_l - \delta_{jl} K_m] = i \varepsilon_{ijl} K_l \\
[K_i, K_j] &= [J^{0i}, J^{0j}] \\
&\stackrel{(2.4.12)}{=} -i [\eta^{i0} J^{0j} - \eta^{00} J^{ij} - \eta^{j0} J^{0i} + \eta^{ij} J^{00}] \\
&= -i J^{ij} = -i \varepsilon_{ijk} J_k \\
[J_i, P_j] &= \frac{\varepsilon_{lmi}}{2} [J^{lm}, P^j] \\
&\stackrel{(2.4.13)}{=} \frac{i \varepsilon_{lmi}}{2} [\eta^{jl} P^m - \eta^{jm} P^l] \\
&= \frac{i}{2} [\varepsilon_{jmi} P^m - \varepsilon_{mji} P^m] = i \varepsilon_{ijm} P_m \\
[K_i, P_j] &= [J^{0i}, P^j] \\
&\stackrel{(2.4.13)}{=} i [\eta^{j0} P^i - \eta^{ji} P^0] \\
&= -i \delta_{ji} P^0 = -i \delta_{ij} H \\
[J_i, H] &\stackrel{(2.4.13)}{=} \frac{i \varepsilon_{lmi}}{2} [\eta^{0l} P^m - \eta^{0m} P^l] = 0 \\
[P_i, H] &= [P^i, P^0] \stackrel{(2.4.14)}{=} 0 \\
[H, H] &= [P^0, P^0] \stackrel{(2.4.14)}{=} 0 \\
[K_i, H] &= [J^{0i}, P^0] \\
&\stackrel{(2.4.13)}{=} i [\eta^{00} P^i - \eta^{0i} P^0] = -i P_i
\end{aligned}$$

6. Eq. (2.4.27) (P.61)

When trying to check this for e.g. the standard representation in 4 dimensions, in terms of infinitesimal rotations, attention with the sign in front of sin for the different rotation axis (see remark at the end of I E 23).

7. “Inspection of Eqs. (2.4.18 – 24) shows that these commutation relations have a limit for  $v \ll 1$  of the form ...” (P.62)

Always equate same orders in  $v$  for this.

$$8. \quad \text{“exp}(-i\mathbf{K} \cdot \mathbf{v}) \text{exp}(-i\mathbf{P} \cdot \mathbf{a}) = \text{exp}(iM\mathbf{a} \cdot \mathbf{v}/2) \text{exp}(-i(\mathbf{K} \cdot \mathbf{v} + \mathbf{P} \cdot \mathbf{a})) \text{” (P.62)}$$

Use BCH Formula

$$\begin{aligned} & \exp(-iK_i v_i) \exp(-iP_j a_j) \\ &= \exp\left(-i(K_i v_i + P_j a_j) + \frac{1}{2}(-i)^2 [K_i, P_j] v_i a_j + 0\right) \\ &= \exp\left(-i(K_i v_i + P_j a_j) + \frac{1}{2}iM v_i a_i\right) \end{aligned}$$

## E. One-Particle States

1. Eq. (2.5.2) (P.63)

$\Lambda^{-1}$  shows up here since in Eq. (2.4.9)  $UPU^{-1}$  is given but here  $U^{-1}PU$  is being used.

2. “with  $\sigma$  within any one block by themselves furnish a representation of the inhomogeneous Lorentz group” (P.63)

In this case the Blocks do not mix with other blocks.

3. “and for  $p^2 \leq 0$ , also the sign of  $p^0$ ” (P.64)

For  $p^2 \leq 0$  we have

$$\begin{aligned} p^2 &= -(p^0)^2 + \vec{p}^2 \leq 0 \\ \Rightarrow |\vec{p}| &\leq |p^0| \end{aligned}$$

and from Eq. (2.3.13) we know

$$|\Lambda^0_0| \geq |\Lambda^0_{\vec{p}}|.$$

First suppose  $p^0 \geq 0$ :

$$\begin{aligned} p'^0 &= \Lambda^0_0 p^0 + \Lambda^0_{\vec{p}} p^i \\ &\geq \Lambda^0_0 p^0 - |\Lambda^0_{\vec{p}}| |\vec{p}| \\ &= |\Lambda^0_0| |p^0| - |\Lambda^0_{\vec{p}}| |\vec{p}| \\ &\geq |\Lambda^0_{\vec{p}}| (|p^0| - |\vec{p}|) \geq 0 \end{aligned}$$

Now suppose  $p^0 \leq 0$ :

$$\begin{aligned} p'^0 &= \Lambda^0_0 p^0 + \Lambda^0_{\vec{p}} p^i \\ &\leq \Lambda^0_0 p^0 + |\Lambda^0_{\vec{p}}| |\vec{p}| \\ &= -|\Lambda^0_0| |p^0| + |\Lambda^0_{\vec{p}}| |\vec{p}| \\ &\leq |\Lambda^0_0| (-|p^0| + |\vec{p}|) \leq 0 \end{aligned}$$

4. Eq. (2.5.12) “The delta function appears here because  $\Psi_{k,\sigma}$  and  $\Psi_{k',\sigma'}$  are eigenstates of a Hermitian operator with eigenvalues  $\mathbf{k}$  and  $\mathbf{k}'$ , respectively.” (P.66)

**Note:** From this point forward in this section of the book the states considered have standard momentum  $k^\mu$ ! (see Text in the Book preceding Eq. (2.5.12))

Suppose that for given values of  $\mathbf{k}$  and  $\mathbf{k}'$

$$\langle \Psi_{k',\sigma'} | \Psi_{k,\sigma} \rangle \neq 0$$

then from

$$\begin{aligned} & k^i \langle \Psi_{k',\sigma'} | \Psi_{k,\sigma} \rangle \\ &= \langle \Psi_{k',\sigma'} | P^i \Psi_{k,\sigma} \rangle \\ &= \langle P^i \Psi_{k',\sigma'} | \Psi_{k,\sigma} \rangle \\ &= k'^i \langle \Psi_{k',\sigma'} | \Psi_{k,\sigma} \rangle \end{aligned}$$

follows

$$k^i = k'^i.$$

On why there is only a 3 dimensional delta function showing up:

This is because both states are assumed to have standard momentum  $k^\mu$  and therefore their 0-th component is completely fixed from their spatial components:

$$k'^0 = \pm \sqrt{\mathbf{k}'^2 - k^2}$$

Choice between  $+$  and  $-$  comes from sign of  $k^0$ , i.e.  $+$  for cases (a) and (c) of Table 2.1.

5. Eq. (2.5.13) (P.67)

$$\begin{aligned}
& \langle U(W)\Psi_{k',\sigma'} | U(W)\Psi_{k,\sigma} \rangle \\
& \stackrel{(2.5.8)}{=} \sum_{\sigma''\sigma'''} \langle D_{\sigma''\sigma'} \Psi_{k',\sigma''} | D_{\sigma'''\sigma} \Psi_{k,\sigma'''} \rangle \\
& = \sum_{\sigma''\sigma'''} (D_{\sigma''\sigma'})^* D_{\sigma'''\sigma} \langle \Psi_{k',\sigma''} | \Psi_{k,\sigma'''} \rangle \\
& \stackrel{(2.5.12)}{=} \sum_{\sigma''} (D_{\sigma''\sigma'})^* D_{\sigma''\sigma} \delta^{(3)}(\mathbf{k}' - \mathbf{k}) \\
& \stackrel{(2.2.2)}{=} \langle \Psi_{k',\sigma'} | \Psi_{k,\sigma} \rangle \\
& \stackrel{(2.5.12)}{=} \delta_{\sigma'\sigma} \delta^{(3)}(\mathbf{k}' - \mathbf{k})
\end{aligned}$$

From this it follows

$$D^\dagger(W) = D^{-1}(W)$$

6. “ $\langle \Psi_{p',\sigma'} | \Psi_{p,\sigma} \rangle = \dots$ ” (P.67)

**Note:** What is meant by “arbitrary momenta” is that these momenta still have the standard momentum  $k^\mu$ , but now none of the states has exactly  $k^\mu$  as its momentum.

First define

$$k' := L^{-1}(p)p'$$

with this we get:

$$\begin{aligned}
& \langle \Psi_{p',\sigma'} | \Psi_{p,\sigma} \rangle \\
& \stackrel{(2.5.5)}{=} \langle \Psi_{p',\sigma'} | N(p) U(L(p)) \Psi_{k,\sigma} \rangle \\
& = N(p) \langle U(L^{-1}(p)) \Psi_{p',\sigma'} | \Psi_{k,\sigma} \rangle \\
& \stackrel{(2.5.11)}{=} N(p) \frac{N^*(p')}{N^*(k')} \\
& \cdot \sum_{\sigma''} D_{\sigma''\sigma'}^* (W(L^{-1}(p), p')) \langle \Psi_{k',\sigma''} | \Psi_{k,\sigma} \rangle \\
& \stackrel{(2.5.12)}{=} N(p) N^*(p') D_{\sigma\sigma'}^* (W(L^{-1}(p), p')) \delta^{(3)}(\mathbf{k}' - \mathbf{k})
\end{aligned}$$

Where in the last step we used

$$\frac{\delta^{(3)}(\mathbf{k}' - \mathbf{k})}{N^*(k')} = \frac{\delta^{(3)}(\mathbf{k}' - \mathbf{k})}{N^*(k)} \stackrel{(2.5.5)}{=} \delta^{(3)}(\mathbf{k}' - \mathbf{k})$$

This is valid as from Eq. (2.5.5) we know that  $N(p)$  is implicitly dependent on  $k^\mu$ . This therefor fixes  $p^2 = k^2$

and the sign of  $p^0$  (see end of II E 4), s.t.

$$N(p) = N(\mathbf{p}).$$

Further we are allowed to use Eq. (2.5.12) in the last step since  $k'$  also has  $k$  as its standard momentum:

$$\begin{aligned}
k' &= L^{-1}(p)p' = \underbrace{L^{-1}(p)L(p')}_{=:L(k')} k \\
7. \quad & “W(L^{-1}(p), p) = 1” \text{ (P.67)}
\end{aligned}$$

First note

$$L(k) = 1 = L^{-1}(k)$$

with this we get:

$$\begin{aligned}
W(L^{-1}(p), p) & \stackrel{(2.5.10)}{=} L^{-1}(L^{-1}(p)p) L^{-1}(p)L(p) \\
& = L^{-1}(k) = 1
\end{aligned}$$

8. “So we see that the **invariant delta function** is ...” (P.67)

Invariant in this case means w.r.t. proper orthochronous Lorentz transformations, which can be interpreted as change of variables under the  $\int d^4p$  integral.

9. Eq. (2.5.24) (P.68)

$$\begin{aligned}
p^0 &= L^0_0 k^0 + L^0_i k^i \\
&= \frac{\sqrt{\mathbf{p}^2 + M^2}}{M} M = \sqrt{\mathbf{p}^2 + M^2} \\
p^i &= L^i_0 k^0 + L^i_j k^j \\
&= \frac{p_i}{|\mathbf{p}|} \sqrt{\frac{\mathbf{p}^2 + M^2}{M^2}} - 1M \\
&= \frac{p_i}{|\mathbf{p}|} \sqrt{\frac{\mathbf{p}^2}{M^2}} M = p_i
\end{aligned}$$

10. “To see this, note that the boost Eq. (2.5.24) may be expressed as  $L(p) = R(\hat{\mathbf{p}})B(|\mathbf{p}|)R^{-1}(\hat{\mathbf{p}})$ ” (P.68)

First note that the columns and rows of the matrix  $B(|\mathbf{p}|)$  are counted in the order 1,2,3,0.

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Let

$$\hat{\mathbf{p}} = \begin{pmatrix} \sin(\theta) \cos(\phi) \\ \sin(\theta) \sin(\phi) \\ \cos(\theta) \end{pmatrix}$$

then we have

$$R(-\hat{\mathbf{p}}) = R_3(-\phi)R_2(\theta) = \begin{pmatrix} \cos(\phi)\cos(\theta) & -\sin(\phi)\cos(\theta) & \cos(\phi)\sin(\theta) \\ \sin(\phi)\cos(\theta) & \cos(\phi)\sin(\theta) & \sin(\phi)\sin(\theta) \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

(see IIE 23) and

$$\begin{aligned} R^{-1}(\hat{\mathbf{p}}) &= R_2(\theta)R_3(\phi) = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta)\cos(\phi) & \cos(\theta)\sin(\phi) & -\sin(\theta) \\ -\sin(\phi) & \cos(\phi) & 0 \\ \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \end{pmatrix} = R^\top(\hat{\mathbf{p}}) \end{aligned}$$

From Eq. (2.5.24) we know

$$L(p) = \begin{pmatrix} 1 + (\gamma - 1)s_\theta^2 c_\phi^2 & (\gamma - 1)s_\theta c_\phi s_\theta s_\phi & (\gamma - 1)s_\theta c_\phi c_\theta & s_\theta c_\phi \sqrt{\gamma^2 - 1} \\ (\gamma - 1)s_\theta c_\phi s_\theta s_\phi & 1 + (\gamma - 1)s_\theta^2 s_\phi^2 & (\gamma - 1)s_\theta s_\phi c_\theta & s_\theta s_\phi \sqrt{\gamma^2 - 1} \\ (\gamma - 1)s_\theta c_\phi c_\theta & (\gamma - 1)s_\theta s_\phi c_\theta & 1 + (\gamma - 1)c_\theta^2 & c_\theta \sqrt{\gamma^2 - 1} \\ s_\theta c_\phi \sqrt{\gamma^2 - 1} & s_\theta s_\phi \sqrt{\gamma^2 - 1} & c_\theta \sqrt{\gamma^2 - 1} & \gamma \end{pmatrix}$$

where  $s$  and  $c$  are shorthand for  $\sin$  and  $\cos$ . With this we can now check:

$$\begin{aligned} R(\hat{\mathbf{p}})B(|\mathbf{p}|)R^{-1}(\hat{\mathbf{p}}) &= \begin{pmatrix} c_\phi c_\theta & -s_\phi & c_\phi s_\theta & 0 \\ s_\phi c_\theta & c_\phi & s_\phi s_\theta & 0 \\ -s_\theta & 0 & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & \sqrt{\gamma^2 - 1} \\ 0 & 0 & \sqrt{\gamma^2 - 1} & \gamma \end{pmatrix} \begin{pmatrix} c_\phi c_\theta & s_\phi c_\theta & -s_\theta & 0 \\ -s_\phi & c_\phi & 0 & 0 \\ c_\phi s_\theta & s_\phi s_\theta & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_\phi c_\theta & -s_\phi & c_\phi s_\theta & 0 \\ s_\phi c_\theta & c_\phi & s_\phi s_\theta & 0 \\ -s_\theta & 0 & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_\phi c_\theta & s_\phi c_\theta & -s_\theta & 0 \\ -s_\phi & c_\phi & 0 & 0 \\ \gamma c_\phi s_\theta & \gamma s_\phi s_\theta & \gamma c_\theta & \sqrt{\gamma^2 - 1} \\ \sqrt{\gamma^2 - 1} c_\phi s_\theta & \sqrt{\gamma^2 - 1} s_\phi s_\theta & \sqrt{\gamma^2 - 1} c_\theta & \gamma \end{pmatrix} = L(p) \end{aligned}$$

11. “ $W(\mathbf{R}, p) = \mathbf{R}$ ” (P.69)

13. Eq. (2.5.26) (P.70)

To see this just substitute  $R(\theta)$  back into the previous result.

This is a Lorentz transformation, since

12. Eq. (2.5.25) (P.70)

$$\begin{aligned} S^\top \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} S &= \begin{pmatrix} 1 & 0 & \alpha & \alpha \\ 0 & 1 & \beta & \beta \\ -\alpha & -\beta & 1 - \zeta & -\zeta \\ \alpha & \beta & \zeta & 1 + \zeta \end{pmatrix} \\ &\cdot \begin{pmatrix} 1 & 0 & -\alpha & \alpha \\ 0 & 1 & -\beta & \beta \\ \alpha & \beta & 1 - \zeta & \zeta \\ -\alpha & -\beta & \zeta & -1 - \zeta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} -1 &= (Wt)^\mu (Wt)_\mu = \alpha^2 + \beta^2 + \zeta^2 - (1 + \zeta)^2 \\ &\Leftrightarrow \alpha^2 + \beta^2 = 2\zeta \end{aligned}$$

14. Eqs. (2.5.29/30) (P.70)

Eq. (2.5.30) is trivial since the rotations around the three axis satisfy this group multiplication and

$$S(0,0) = \mathbf{1}.$$

For Eq. (2.5.29) explicit calculation shows it (see **TODO**) together with

$$R(0) = \mathbf{1}.$$

15. Eq. (2.5.31) (P.70)

Explicit calculation shows Eq. (2.5.31) (see **TODO**) and the invariance follows then immediately, since:

$$W' S W'^{-1} = S' \underbrace{R' S R'^{-1}}_{=S''} S'^{-1} = S'''$$

16. “ $W(\theta, \alpha, \beta) = 1 + \omega$ ” (P.71)

From Eq. (2.5.28) we immediately get for infinitesimal  $\theta, \alpha, \beta$ :

$$(W(\theta, \alpha, \beta))^\mu{}_\nu = \delta^\mu_\nu + \begin{pmatrix} 0 & \theta & -\alpha & \alpha \\ -\theta & 0 & -\beta & \beta \\ \alpha & \beta & 0 & 0 \\ \alpha & \beta & 0 & 0 \end{pmatrix}^\mu{}_\nu$$

$$\Rightarrow \omega_{\mu\nu} = \begin{pmatrix} 0 & \theta & -\alpha & \alpha \\ -\theta & 0 & -\beta & \beta \\ \alpha & \beta & 0 & 0 \\ -\alpha & -\beta & 0 & 0 \end{pmatrix}_{\mu\nu}$$

17. Eq. (2.5.32) (P.71)

$$\begin{aligned} U(1 + \omega) &\stackrel{(2.4.3)}{=} 1 + \frac{1}{2} i \omega_{\rho\sigma} J^{\rho\sigma} \\ &= 1 + i\theta J^{12} + i\alpha(-J^{13} + J^{10}) + i\beta(-J^{23} + J^{20}) \\ &= 1 + i\theta J_3 + i\alpha(J_2 - K_1) + i\beta(-J_1 - K_2) \end{aligned}$$

18. Eqs. (2.5.35/36/37) (P.71)

From Eqs. (2.4.18/19/20) we get:

$$\begin{aligned} [J_3, A] &= [J_3, J_2 - K_1] \\ &= [J_3, J_2] - [J_3, K_1] \\ &= -iJ_1 - iK_2 = iB \\ [J_3, B] &= -[J_3, J_1] - [J_3, K_2] \\ &= -iJ_2 - i(-K_1) = -iA \\ [A, B] &= -[J_2, J_1] - [J_2, K_2] + [K_1, J_1] + [K_1, K_2] \\ &= iJ_3 - iJ_3 = 0 \end{aligned}$$

19. “Since  $A$  and  $B$  are commuting Hermitian operators they can be simultaneously diagonalized by states  $\Psi_{k,a,b}$ ” (P.71)

This is valid, including the label  $k$ , since

$$\begin{aligned} [A, P_1] \Psi_{k,a,b} &= ([J_2, P_1] - [K_1, P_1]) \Psi_{k,a,b} \\ &= (-iP_3 + iP^0) \Psi_{k,a,b} = 0 \\ [A, P_2] \Psi_{k,a,b} &= ([J_2, P_2] - [K_1, P_2]) \Psi_{k,a,b} \\ &= 0 \\ [A, P_3] \Psi_{k,a,b} &= ([J_2, P_3] - [K_1, P_3]) \Psi_{k,a,b} \\ &= (iP_1) \Psi_{k,a,b} = 0 \\ [A, P^0] \Psi_{k,a,b} &= ([J_2, P^0] - [K_1, P^0]) \Psi_{k,a,b} \\ &= (-iP_1) \Psi_{k,a,b} = 0 \\ [B, P_1] \Psi_{k,a,b} &= (-[J_1, P_1] - [K_2, P_1]) \Psi_{k,a,b} \\ &= 0 \\ [B, P_2] \Psi_{k,a,b} &= ([J_1, P_2] - [K_2, P_2]) \Psi_{k,a,b} \\ &= (iP_3 - iP^0) \Psi_{k,a,b} = 0 \\ [B, P_3] \Psi_{k,a,b} &= ([J_1, P_3] - [K_2, P_3]) \Psi_{k,a,b} \\ &= (-iP_2) \Psi_{k,a,b} = 0 \\ [B, P^0] \Psi_{k,a,b} &= ([J_1, P^0] - [K_2, P^0]) \Psi_{k,a,b} \\ &= (-iP_2) \Psi_{k,a,b} = 0 \end{aligned}$$

20. “ $\sigma$  gives the component of angular momentum in the direction of motion, or helicity” (P.72)

The derivation of Eq. (2.5.26) starts among other conditions from the explicit form of  $k$ . Which results in

$$\text{Eqs. (2.5.27/28)} \rightarrow \text{Eq. (2.5.32)} \rightarrow 2.5.39,$$

s.t. this is really connected to the direction of motion.

$$21. \quad “\mathcal{U}(W) \Psi_{k,\sigma} = \exp(i\theta\sigma) \Psi_{k,\sigma}” \text{ (P.72)}$$

Use Eqs. (2.5.38/39).

22. Eq. (2.5.44) (P.73)

With  $B(u)$  from Eq. (2.5.45) we get:

$$B\left(\frac{|\mathbf{p}|}{\kappa}\right)k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{(\frac{|\mathbf{p}|}{\kappa})^2+1}{2\frac{|\mathbf{p}|}{\kappa}} & \frac{(\frac{|\mathbf{p}|}{\kappa})^2-1}{2\frac{|\mathbf{p}|}{\kappa}} \\ 0 & 0 & \frac{(\frac{|\mathbf{p}|}{\kappa})^2-1}{2\frac{|\mathbf{p}|}{\kappa}} & \frac{(\frac{|\mathbf{p}|}{\kappa})^2+1}{2\frac{|\mathbf{p}|}{\kappa}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \kappa \\ \kappa \end{pmatrix} \\ = |\mathbf{p}| \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

23. “take  $R(\hat{\mathbf{p}})$  as a rotation by angle  $\theta$  around the two-axis followed by a rotation by angle  $\phi$  around the three-axis” (P.73)

Let

$$\hat{\mathbf{p}} = \begin{pmatrix} \sin(\theta) \cos(\phi) \\ \sin(\theta) \sin(\phi) \\ \cos(\theta) \end{pmatrix}$$

then

$$R(\hat{\mathbf{p}}) = R_3(-\phi)R_2(-\theta) \\ = \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \\ = \begin{pmatrix} \cos(\phi) \cos(\theta) & -\sin(\phi) \cos(\theta) & \sin(\theta) \\ \sin(\phi) \cos(\theta) & \cos(\phi) \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix},$$

(With this sign convention everything is consistent, see definition of  $R(\theta)$  after Eq. (2.5.27) and compare Eq. (2.5.47) to Eq. (2.4.27)) with this we have

$$\hat{\mathbf{p}} = R(\hat{\mathbf{p}})\hat{\mathbf{e}}_3.$$

## F. Space Inversion and Time-Reversal

1. Eqs. (2.6.7 – 12) (P.76)

$$\begin{aligned} \mathbf{P}J_i\mathbf{P}^{-1} &= \frac{1}{2}\varepsilon_{ijk}\mathbf{P}J^{jk}\mathbf{P}^{-1} \\ &= \frac{1}{2}\varepsilon_{ijk}\left(-\delta_l^j\right)\left(-\delta_m^k\right)J^{lm} \\ &= \frac{1}{2}\varepsilon_{ijk}J^{jk} = J_i \\ \mathbf{P}K_i\mathbf{P}^{-1} &= \mathbf{P}J^{0i}\mathbf{P}^{-1} \\ &= \delta_\mu^0\left(-\delta_\nu^j\right)J^{\mu\nu} \\ &= -J^{0i} = -K_i \\ \mathbf{P}P_i\mathbf{P}^{-1} &= \left(-\delta_\nu^i\right)P^\mu = -P_i \\ \mathbf{T}J_i\mathbf{T}^{-1} &= \frac{1}{2}\varepsilon_{ijk}\mathbf{T}J^{jk}\mathbf{T}^{-1} \\ &= -\frac{1}{2}\varepsilon_{ijk}\delta_l^j\delta_m^k J^{lm} \\ &= -\frac{1}{2}\varepsilon_{ijk}J^{jk} = -J_i \\ \mathbf{T}K_i\mathbf{T}^{-1} &= \mathbf{T}J^{0i}\mathbf{T}^{-1} \\ &= -\left(-\delta_\mu^0\right)\delta_\nu^j J^{\mu\nu} \\ &= J^{0i} = K_i \\ \mathbf{T}P_i\mathbf{T}^{-1} &= -\delta_\nu^i P^\mu = -P_i \end{aligned}$$

2. “ $\mathcal{P}L(p)\mathcal{P}^{-1} = L(\mathcal{P}p)$ ” (P.77)

From Eq. (2.5.24) we immediately see:

$$\begin{aligned} (\mathcal{P}L(p)\mathcal{P}^{-1})^i_k &= (L(p))^i_k = (L(\mathcal{P}p))^i_k \\ (\mathcal{P}L(p)\mathcal{P}^{-1})^i_0 &= -(L(p))^i_0 = (L(\mathcal{P}p))^i_0 \\ (\mathcal{P}L(p)\mathcal{P}^{-1})^0_0 &= (L(p))^0_0 = (L(\mathcal{P}p))^0_0 \end{aligned}$$

3. “Using Eq. (2.6.14) again on the left, we see that the square-root factors cancel...” (P.77)

$$\begin{aligned} (-J_1 \pm iJ_2)\zeta_\sigma\Psi_{k,-\sigma} &= -(J_1 \mp iJ_2)\zeta_\sigma\Psi_{k,-\sigma} \\ &\stackrel{(2.6.14)}{=} -\sqrt{(j \pm (-\sigma))(j \mp (-\sigma) + 1)}\zeta_\sigma\Psi_{k,-\sigma \mp 1} \\ &= -\sqrt{(j \mp \sigma)(j \pm \sigma + 1)}\zeta_\sigma\Psi_{k,-\sigma \mp 1} \end{aligned}$$

4. “The time-reversal phase  $\zeta$  has no physical significance.” (P.78)

This redefinition only works because  $\mathbf{T}$  is *anti-linear*.

5. “ $\mathcal{T}L(p)\mathcal{T}^{-1} = L(\mathcal{P}p)$ ” (P.78)

From Eq. (2.5.24) we immediately see:

$$\begin{aligned} (\mathcal{T}L(p)\mathcal{T}^{-1})^i_k &= (L(p))^i_k = (L(\mathcal{P}p))^i_k \\ (\mathcal{T}L(p)\mathcal{T}^{-1})^i_0 &= -(L(p))^i_0 = (L(\mathcal{P}p))^i_0 \\ (\mathcal{T}L(p)\mathcal{T}^{-1})^0_0 &= (L(p))^0_0 = (L(\mathcal{P}p))^0_0 \end{aligned}$$

6. “ $\mathcal{P}$  yields a state with four-momentum ...” (P.78)

What is meant by this is:

$$\begin{aligned} P^i \mathcal{P} \Psi_{k,\sigma} &\stackrel{(2.6.9)}{=} \mathcal{P} (-P^i) \Psi_{k,\sigma} = (-\delta^i_3 \kappa) \mathcal{P} \Psi_{k,\sigma} \\ H \mathcal{P} \Psi_{k,\sigma} &\stackrel{(2.6.13)}{=} \mathcal{P} H \Psi_{k,\sigma} = \kappa \mathcal{P} \Psi_{k,\sigma} \\ J_3 \mathcal{P} \Psi_{k,\sigma} &\stackrel{(2.6.7)}{=} \mathcal{P} J_3 \Psi_{k,\sigma} = \sigma \mathcal{P} \Psi_{k,\sigma} \end{aligned}$$

7. Eq. (2.6.20) (P.79)

We have

$$R_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = R_2^{-1}$$

from which we get (Is  $U(R_2^{-1}) = U^{-1}(R_2)$  ? **TODO**)

$$\begin{aligned} U^{-1}(R_2) J_3 U(R_2) &= U^{-1}(R_2) J^{12} U(R_2) \\ &\stackrel{(2.4.8)}{=} (R_2^{-1})^1_\mu (R_2^{-1})^2_\nu J^{\mu\nu} \\ &= -J^{12} = -J_3 \\ U^{-1}(R_2) P^\nu U(R_2) &\stackrel{(2.4.9)}{=} (R_2^{-1})^\nu_\mu P^\mu \end{aligned}$$

such that

$$\begin{aligned} J_3 U(R_2^{-1}) \mathcal{P} \Psi_{k,\sigma} &= -U(R_2^{-1}) J_3 \mathcal{P} \Psi_{k,\sigma} \\ &\stackrel{II F 6}{=} -\sigma U(R_2^{-1}) \mathcal{P} \Psi_{k,\sigma} \\ P^i U(R_2^{-1}) \mathcal{P} \Psi_{k,\sigma} &\stackrel{II F 6}{=} U(R_2^{-1}) (-(-\delta^i_3 \kappa)) \mathcal{P} \Psi_{k,\sigma} \\ &= \kappa^i U(R_2^{-1}) \mathcal{P} \Psi_{k,\sigma} \\ H U(R_2^{-1}) \mathcal{P} \Psi_{k,\sigma} &\stackrel{II F 6}{=} \kappa U(R_2^{-1}) \mathcal{P} \Psi_{k,\sigma} \\ &= \kappa^0 U(R_2^{-1}) \mathcal{P} \Psi_{k,\sigma}. \end{aligned}$$

Further we get

$$R_2^{-1} \mathcal{P} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

8. “ $\mathcal{P}$  commutes with the rotation  $R(\hat{\mathbf{p}})$ ” (P.79)

This is true because  $R(\hat{\mathbf{p}})$  only acts on space components non trivially, which all just get a “-” sign from  $\mathcal{P}$ .

9. “ $\mathcal{P} \Psi_{p,\sigma} = \sqrt{\frac{\kappa}{p^0}} \eta_\sigma U \left( R(\hat{\mathbf{p}}) R_2 B \left( \frac{|\mathbf{p}|}{\kappa} \right) \right) \Psi_{k,-\sigma}$ ” (P.79)

$$\begin{aligned} \mathcal{P} \Psi_{p,\sigma} &\stackrel{(2.5.5)}{=} N(p) \mathcal{P} U(L(p)) \Psi_{k,\sigma} \\ &\stackrel{(2.5.44)}{=} \sqrt{\frac{\kappa}{p^0}} \mathcal{P} U \left( R(\hat{\mathbf{p}}) B \left( \frac{|\mathbf{p}|}{\kappa} \right) \right) \Psi_{k,\sigma} \\ &= \sqrt{\frac{\kappa}{p^0}} U \left( \mathcal{P} R(\hat{\mathbf{p}}) B \left( \frac{|\mathbf{p}|}{\kappa} \right) \right) \Psi_{k,\sigma} \\ &= \sqrt{\frac{\kappa}{p^0}} U \left( R(\hat{\mathbf{p}}) \mathcal{P} B \left( \frac{|\mathbf{p}|}{\kappa} \right) \right) \Psi_{k,\sigma} \\ &= \sqrt{\frac{\kappa}{p^0}} U \left( R(\hat{\mathbf{p}}) R_2 R_2^{-1} \mathcal{P} B \left( \frac{|\mathbf{p}|}{\kappa} \right) \right) \Psi_{k,\sigma} \\ &= \sqrt{\frac{\kappa}{p^0}} U \left( R(\hat{\mathbf{p}}) R_2 B \left( \frac{|\mathbf{p}|}{\kappa} \right) R_2^{-1} \mathcal{P} \right) \Psi_{k,\sigma} \\ &= \sqrt{\frac{\kappa}{p^0}} U \left( R(\hat{\mathbf{p}}) R_2 B \left( \frac{|\mathbf{p}|}{\kappa} \right) \right) U(R_2^{-1}) \mathcal{P} \Psi_{k,\sigma} \\ &\stackrel{(2.6.20)}{=} \sqrt{\frac{\kappa}{p^0}} \eta_\sigma U \left( R(\hat{\mathbf{p}}) R_2 B \left( \frac{|\mathbf{p}|}{\kappa} \right) \right) \Psi_{k,-\sigma} \end{aligned}$$

10. “But a rotation of  $\pm 180^\circ$  around the three-axis reverses the sign of  $J_2, \dots$ ” (P.79)

Analogously to II F 6 ( $2 \leftrightarrow 3$ ).

11. Eq. (2.6.22) (P.79)

First note that

$$B \left( \frac{|\mathbf{p}|}{\kappa} \right)$$



and rotations around the three-axis commute (see Eq. (2.5.45)).

$$\begin{aligned}
P\Psi_{p,\sigma} &\stackrel{II F 9}{=} \sqrt{\frac{\kappa}{p^0}} \eta_\sigma U \left( R(\hat{\mathbf{p}}) R_2 B \left( \frac{|\mathbf{p}|}{\kappa} \right) \right) \Psi_{k,-\sigma} \\
&\stackrel{Eq. (2.6.21)}{=} \sqrt{\frac{\kappa}{p^0}} \eta_\sigma U (R(-\hat{\mathbf{p}})) \\
&\quad \cdot \exp(\pm i\pi J_3) U \left( B \left( \frac{|\mathbf{p}|}{\kappa} \right) \right) \Psi_{k,-\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}} \eta_\sigma U (R(-\hat{\mathbf{p}})) \\
&\quad \cdot U \left( B \left( \frac{|\mathbf{p}|}{\kappa} \right) \right) \exp(\pm i\pi J_3) \Psi_{k,-\sigma} \\
&= \eta_\sigma \exp(\pm i\pi(-\sigma)) \sqrt{\frac{\kappa}{p^0}} U (L(\mathcal{P}p)) \Psi_{k,-\sigma} \\
&= \eta_\sigma \exp(\mp i\pi\sigma) \Psi_{\mathcal{P}p,-\sigma}
\end{aligned}$$

12. “This peculiar change of sign in the operation of parity for mass-less particles of half-integer spin is due to the convention adopted in Eq. (2.5.47) for the rotation used to define mass-less particles states of arbitrary momentum.” (P.79)

This is based on the fact that the choice of  $R(\hat{\mathbf{p}})$ , which transforms the three axis into the unit vector  $\hat{\mathbf{p}}$ , is *not* unique. As mentioned in the text right after Eq. (2.5.47), one could always add an initial rotation around the three axis. This is also why the factor of

$$\exp(\pm i\pi J_3)$$

shows up in Eq. (2.6.21), despite  $R(\hat{\mathbf{p}}) R_2$  and  $R(-\hat{\mathbf{p}})$  both transforming the three axis into the unit vector  $-\hat{\mathbf{p}}$ .

13. “ $\mathbb{T}$  yields a state which has values ...” (P.79)

What is meant by this is:

$$\begin{aligned}
P^i \mathbb{T} \Psi_{k,\sigma} &\stackrel{(2.6.12)}{=} \mathbb{T} (-P^i) \Psi_{k,\sigma} = (-\delta_3^i \kappa) \mathbb{T} \Psi_{k,\sigma} \\
H \mathbb{T} \Psi_{k,\sigma} &\stackrel{(2.6.13)}{=} \mathbb{T} H \Psi_{k,\sigma} = \kappa \mathbb{T} \Psi_{k,\sigma} \\
J_3 \mathbb{T} \Psi_{k,\sigma} &\stackrel{(2.6.10)}{=} \mathbb{T} (-J_3) \Psi_{k,\sigma} = -\sigma \mathbb{T} \Psi_{k,\sigma}
\end{aligned}$$

14. Eq. (2.6.23) (P.80)

This is completely analog to Eq. (2.6.20) when using IIF 13 (see IIF 7). Further we get

$$R_2^{-1} \mathbb{T} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

15. Eq. (2.6.24) (P.80)

$$\begin{aligned}
\mathbb{T} \Psi_{p,\sigma} &\stackrel{(2.5.5)}{=} N(p) \mathbb{T} U (L(p)) \Psi_{k,\sigma} \\
&\stackrel{(2.5.44)}{=} \sqrt{\frac{\kappa}{p^0}} \mathbb{T} U \left( R(\hat{\mathbf{p}}) B \left( \frac{|\mathbf{p}|}{\kappa} \right) \right) \Psi_{k,\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}} U \left( \mathcal{T} R(\hat{\mathbf{p}}) B \left( \frac{|\mathbf{p}|}{\kappa} \right) \right) \Psi_{k,\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}} U \left( R(\hat{\mathbf{p}}) \mathcal{T} B \left( \frac{|\mathbf{p}|}{\kappa} \right) \right) \Psi_{k,\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}} U \left( R(\hat{\mathbf{p}}) R_2 R_2^{-1} \mathcal{T} B \left( \frac{|\mathbf{p}|}{\kappa} \right) \right) \Psi_{k,\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}} U \left( R(\hat{\mathbf{p}}) R_2 B \left( \frac{|\mathbf{p}|}{\kappa} \right) R_2^{-1} \mathcal{T} \right) \Psi_{k,\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}} U \left( R(\hat{\mathbf{p}}) R_2 B \left( \frac{|\mathbf{p}|}{\kappa} \right) \right) U (R_2^{-1}) \mathbb{T} \Psi_{k,\sigma} \\
&\stackrel{(2.6.23)}{=} \sqrt{\frac{\kappa}{p^0}} \zeta_\sigma U \left( R(\hat{\mathbf{p}}) R_2 B \left( \frac{|\mathbf{p}|}{\kappa} \right) \right) \Psi_{k,\sigma}
\end{aligned}$$

16. Eq. (2.6.25) (P.80)

First note that

$$B \left( \frac{|\mathbf{p}|}{\kappa} \right)$$

and rotations around the three-axis commute (see Eq. (2.5.45)).

$$\begin{aligned}
\mathbb{T} \Psi_{p,\sigma} &\stackrel{II F 15}{=} \sqrt{\frac{\kappa}{p^0}} \zeta_\sigma U \left( R(\hat{\mathbf{p}}) R_2 B \left( \frac{|\mathbf{p}|}{\kappa} \right) \right) \Psi_{k,\sigma} \\
&\stackrel{Eq. (2.6.21)}{=} \sqrt{\frac{\kappa}{p^0}} \zeta_\sigma U (R(-\hat{\mathbf{p}})) \\
&\quad \cdot \exp(\pm i\pi J_3) U \left( B \left( \frac{|\mathbf{p}|}{\kappa} \right) \right) \Psi_{k,\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}} \zeta_\sigma U (R(-\hat{\mathbf{p}})) \\
&\quad \cdot U \left( B \left( \frac{|\mathbf{p}|}{\kappa} \right) \right) \exp(\pm i\pi J_3) \Psi_{k,\sigma} \\
&= \zeta_\sigma \exp(\pm i\pi\sigma) \sqrt{\frac{\kappa}{p^0}} U (L(\mathcal{P}p)) \Psi_{k,\sigma} \\
&= \zeta_\sigma \exp(\pm i\pi\sigma) \Psi_{\mathcal{P}p,\sigma}
\end{aligned}$$

17. “...the total angular momentum  $j$  of any state of this system would have to be a half-integer...” (P.81)

This is true as all spins and helicities except for one half integer spin/helicity would couple to a an integer

angular momentum. And this would then couple with the remaining half integer spin/helicity to a half-integer total angular momentum  $j$ .

### G. Projective Representations

1. Eq. (2.7.6) (P.83)

From Eq. (2.2.20) together with the modification of Eq. (2.7.1) we have up to  $\mathcal{O}(\theta^2, \bar{\theta}^2)$

$$\begin{aligned}
& (1 + i f_{ab} \bar{\theta}^a \theta^b) \\
& \cdot \left[ 1 + i (\theta^a + \bar{\theta}^a + f_{bc}^a \bar{\theta}^b \theta^c) t_a + \frac{1}{2} (\theta^b + \bar{\theta}^b) (\theta^c + \bar{\theta}^c) t_{bc} \right] \\
& = \left[ 1 + i \bar{\theta}^a t_a + \frac{1}{2} \bar{\theta}^b \bar{\theta}^c t_{bc} \right] \cdot \left[ 1 + i \theta^a t_a + \frac{1}{2} \theta^b \theta^c t_{bc} \right] \\
& = 1 + i \theta^a t_a + \frac{1}{2} \theta^b \theta^c t_{bc} + i \bar{\theta}^a t_a - \bar{\theta}^b t_b \theta^c t_c + \frac{1}{2} \bar{\theta}^b \bar{\theta}^c t_{bc} \\
& \Leftrightarrow i f_{bc}^a \bar{\theta}^b \theta^c t_a + \frac{1}{2} (\theta^b \bar{\theta}^c + \bar{\theta}^b \theta^c) t_{bc} + i f_{ab} \bar{\theta}^a \theta^b = -\bar{\theta}^b \theta^c t_b t_c \\
& \Leftrightarrow \bar{\theta}^b \theta^c [t_{bc} + i f_{bc}^a t_a + t_b t_c + i f_{bc}] = 0 \\
& \Leftrightarrow t_{bc} = -i f_{bc}^a t_a - t_b t_c - i f_{bc}
\end{aligned}$$

Note that here the order of multiplication is adapted from Eq. (2.2.18) and not from Eq. (2.7.1), which does not matter for the result since  $\theta$  and  $\bar{\theta}$  are dropped in the end. With this we now get the analog of Eq. (2.2.22):

$$\begin{aligned}
& -i f_{bc}^a t_a - t_b t_c - i f_{bc} \\
& = t_{bc} = t_{cb} = -i f_{cb}^a t_a - t_c t_b - i f_{cb} \\
& \Leftrightarrow [t_b, t_c] = i (f_{cb}^a - f_{bc}^a) t_a + i (f_{cb} - f_{bc}) \mathbf{1}
\end{aligned}$$

2. Eq. (2.7.12) (P.83)

$$[\tilde{t}_b, \tilde{t}_c] = [t_b, t_c] = i C_{bc}^a t_a + i C_{bc}^a \phi_a \mathbf{1} = i C_{bc}^a \tilde{t}_a$$

3. Eqs. (2.7.23/24/25) (P.85)

Inserting Eqs. (2.7.14 – 16) into Eq. (2.7.20) we get:

$$\begin{aligned}
0 & \stackrel{(2.7.20)}{=} [J^{\mu\nu}, C^{\rho,\mu}] \\
& + [P^\sigma, \eta^{\nu\rho} P^\mu - \eta^{\mu\rho} P^\nu + C^{\rho,\mu\nu}] \\
& + [P^\rho, \eta^{\sigma\mu} P^\nu - \eta^{\sigma\nu} P^\mu + C^{\mu\nu,\sigma}] \\
& = \eta^{\nu\rho} [P^\sigma, P^\mu] - \eta^{\mu\rho} [P^\sigma, P^\nu] \\
& + \eta^{\sigma\mu} [P^\rho, P^\nu] - \eta^{\sigma\nu} [P^\rho, P^\mu] \\
& \stackrel{(2.7.16)}{\Rightarrow} 0 = \eta^{\nu\rho} C^{\mu,\sigma} - \eta^{\mu\rho} C^{\nu,\sigma} \\
& + \eta^{\sigma\mu} C^{\nu,\rho} - \eta^{\sigma\nu} C^{\mu,\rho}
\end{aligned}$$

Inserting Eqs. (2.7.13 – 15) into Eq. (2.7.21) we get:

$$\begin{aligned}
0 & \stackrel{(2.7.21)}{=} [J^{\lambda\eta}, \eta^{\nu\rho} P^\mu - \eta^{\mu\rho} P^\nu + C^{\rho,\mu\nu}] \\
& + [P^\rho, -\eta^{\nu\lambda} J^{\mu\eta} + \eta^{\mu\lambda} J^{\nu\eta} + \eta^{\eta\mu} J^{\lambda\nu} - \eta^{\eta\nu} J^{\lambda\mu} - C^{\lambda\eta,\mu\nu}] \\
& + [J^{\mu\nu}, \eta^{\rho\lambda} P^\eta - \eta^{\rho\eta} P^\lambda + C^{\lambda\eta,\rho}] \\
& = \eta^{\nu\rho} [J^{\lambda\eta}, P^\mu] - \eta^{\mu\rho} [J^{\lambda\eta}, P^\nu] \\
& - \eta^{\nu\lambda} [P^\rho, J^{\mu\eta}] + \eta^{\mu\lambda} [P^\rho, J^{\nu\eta}] \\
& + \eta^{\eta\mu} [P^\rho, J^{\lambda\nu}] - \eta^{\eta\nu} [P^\rho, J^{\lambda\mu}] \\
& + \eta^{\rho\lambda} [J^{\mu\nu}, P^\eta] - \eta^{\rho\eta} [J^{\mu\nu}, P^\lambda]
\end{aligned}$$

Using Eqs. (2.7.14/15) we finally get: **TODO**  
Rest analog **TODO**

4. Eq. (2.7.26) (P.85)

$$\begin{aligned}
0 & = 4C^{\mu,\sigma} - \delta_\nu^\mu C^{\nu,\sigma} \\
& - \delta_\rho^\sigma C^{\mu,\rho} + \eta^{\sigma\mu} C^{\nu,\rho} \eta_{\nu\rho} \\
& = 2C^{\mu,\sigma} + 0 = 2C^{\mu,\sigma}
\end{aligned}$$

5. Eqs. (2.7.27/28) (P.85)

$$\begin{aligned}
0 & = 4C^{\mu,\lambda\eta} - \delta_\nu^\mu C^{\nu,\lambda\eta} - \eta^{\mu\eta} C^{\rho,\lambda\nu} \eta_{\nu\rho} \\
& + \eta^{\lambda\mu} C^{\rho,\eta\nu} \eta_{\nu\rho} + \delta_\rho^\lambda C^{\rho,\mu\eta} - \delta_\rho^\eta C^{\rho,\mu\lambda} \\
& + \delta_\nu^\lambda C^{\eta,\mu\nu} - \delta_\nu^\eta C^{\lambda,\mu\nu} \\
& = 3C^{\mu,\lambda\eta} - \eta^{\mu\eta} C^{\rho,\lambda\nu} \eta_{\nu\rho} + \eta^{\lambda\mu} C^{\rho,\eta\nu} \eta_{\nu\rho} \\
& = 3(C^{\mu,\lambda\eta} - \eta^{\mu\eta} C^{\lambda\eta} + \eta^{\lambda\mu} C^{\eta\eta})
\end{aligned}$$

6. Eqs. (2.7.29/30) (P.85)

First note

$$C^{\rho\nu,\lambda\eta} \eta_{\nu\rho} = 0$$

from the antisymmetry of  $J^{\rho\sigma}$  in Eq. (2.7.13).

$$\begin{aligned}
0 & = 4C^{\mu\sigma,\lambda\eta} - \delta_\nu^\mu C^{\nu\sigma,\lambda\eta} - \eta^{\sigma\mu} C^{\rho\nu,\lambda\eta} \eta_{\nu\rho} + \delta_\rho^\sigma C^{\rho\mu,\lambda\eta} \\
& + \eta^{\eta\mu} C^{\lambda\nu,\rho\sigma} \eta_{\nu\rho} - \eta^{\lambda\mu} C^{\eta\nu,\rho\sigma} \eta_{\nu\rho} - \delta_\rho^\lambda C^{\mu\eta,\rho\sigma} + \delta_\rho^\eta C^{\mu\lambda,\rho\sigma} \\
& + \eta^{\sigma\lambda} C^{\rho\eta,\mu\nu} \eta_{\nu\rho} - \delta_\nu^\lambda C^{\sigma\eta,\mu\nu} - \delta_\nu^\rho C^{\lambda\sigma,\mu\nu} + \eta^{\eta\sigma} C^{\lambda\rho,\mu\nu} \eta_{\nu\rho} \\
& = 2C^{\mu\sigma,\lambda\eta} - 2\eta^{\eta\mu} C^{\lambda\sigma} + 2\eta^{\lambda\mu} C^{\eta\sigma} \\
& - 2\eta^{\sigma\lambda} C^{\eta\mu} + 2\eta^{\eta\sigma} C^{\lambda\mu} \\
& = 2(C^{\mu\sigma,\lambda\eta} - \eta^{\eta\mu} C^{\lambda\sigma} + \eta^{\lambda\mu} C^{\eta\sigma} - \eta^{\sigma\lambda} C^{\eta\mu} + \eta^{\eta\sigma} C^{\lambda\mu})
\end{aligned}$$

7. Eqs. (2.7.33/34/35) (P.86)

$$\begin{aligned}
i[\tilde{J}^{\mu\nu}, \tilde{J}^{\rho\sigma}] &= i[J^{\mu\nu}, J^{\rho\sigma}] \\
&= \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\sigma\mu} J^{\rho\nu} + \eta^{\sigma\nu} J^{\rho\mu} + C^{\rho\sigma, \mu\nu} \\
&\stackrel{(2.7.29)}{=} \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\sigma\mu} J^{\rho\nu} + \eta^{\sigma\nu} J^{\rho\mu} \\
&\quad + \eta^{\nu\rho} C^{\mu\sigma} - \eta^{\mu\rho} C^{\nu\sigma} + \eta^{\sigma\mu} C^{\rho\nu} - \eta^{\sigma\nu} C^{\rho\mu} \\
&= \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\sigma\mu} J^{\rho\nu} + \eta^{\sigma\nu} J^{\rho\mu} \\
&\quad + \eta^{\nu\rho} C^{\mu\sigma} - \eta^{\mu\rho} C^{\nu\sigma} - \eta^{\sigma\mu} C^{\rho\nu} + \eta^{\sigma\nu} C^{\rho\mu} \\
&= \eta^{\nu\rho} \tilde{J}^{\mu\sigma} - \eta^{\mu\rho} \tilde{J}^{\nu\sigma} - \eta^{\sigma\mu} \tilde{J}^{\rho\nu} + \eta^{\sigma\nu} \tilde{J}^{\rho\mu}
\end{aligned}$$

$$\begin{aligned}
i[\tilde{J}^{\mu\nu}, \tilde{P}^\rho] &= i[J^{\mu\nu}, P^\rho] \\
&= \eta^{\nu\rho} P^\mu - \eta^{\mu\rho} P^\nu + C^{\rho, \mu\nu} \\
&\stackrel{(2.7.27)}{=} \eta^{\nu\rho} P^\mu - \eta^{\mu\rho} P^\nu + \eta^{\rho\nu} C^\mu - \eta^{\rho\mu} C^\nu \\
&= \eta^{\nu\rho} \tilde{P}^\mu - \eta^{\mu\rho} \tilde{P}^\nu
\end{aligned}$$

$$i[\tilde{P}^\mu, \tilde{P}^\rho] = [P^\mu, P^\rho] = 0$$

## H. The Symmetry Representation Theorem

1. “But  $\langle \Psi'_k | \Psi'_k \rangle$  is automatically **real and positive**” (P.91)

This follows immediately from Eq. (2.1.1).

2. “From Eq. (2.A.1) we have  $|c_{kk}| = |c_{k1}| = \frac{1}{\sqrt{2}}$  and for  $l \neq k$  and  $l \neq 1$ :  $c_{kl} = 0$ ” (P.91)

$$\begin{aligned}
|c_{kl'}|^2 &\stackrel{(2.A.3)}{=} \left| \sum_l c_{kl}^* \langle \Psi'_l | \Psi'_{l'} \rangle \right|^2 = |\langle \Upsilon'_k | \Psi'_{l'} \rangle|^2 \\
&\stackrel{(2.A.1)}{=} |\langle \Upsilon_k | \Psi_{l'} \rangle|^2 = \begin{cases} \frac{1}{2} & \text{for } l' = 1, k \\ 0 & \text{for } l' \neq 1, k \end{cases}
\end{aligned}$$

3. Eq. (2.A.10) (P.92)

$$\begin{aligned}
|C_k|^2 + |C_1|^2 + 2 \operatorname{Re}(C_k C_1^*) &= |C_k + C_1|^2 \\
\stackrel{(2.A.9)}{=} |C'_k + C'_1|^2 &= |C'_k|^2 + |C'_1|^2 + 2 \operatorname{Re}(C'_k C'^*_1) \\
&\stackrel{\text{Eq. (2.A.8)}}{=} \operatorname{Re}(C_k C_1^*) = \operatorname{Re}(C'_k C'^*_1) \\
&\stackrel{\text{Eq. (2.A.8)}}{=} \operatorname{Re}\left(\frac{C_k}{C_1}\right) = \operatorname{Re}\left(\frac{C'_k}{C'_1}\right)
\end{aligned}$$

4. Eq. (2.A.11) (P.92)

$$\begin{aligned}
\left\{ \operatorname{Re}\left(\frac{C_k}{C_1}\right) \right\}^2 + \left\{ \operatorname{Im}\left(\frac{C_k}{C_1}\right) \right\}^2 &= \left| \frac{C_k}{C_1} \right|^2 \\
\stackrel{\text{Eq. (2.A.8)}}{=} \left| \frac{C'_k}{C'_1} \right|^2 &= \left\{ \operatorname{Re}\left(\frac{C'_k}{C'_1}\right) \right\}^2 + \left\{ \operatorname{Im}\left(\frac{C'_k}{C'_1}\right) \right\}^2 \\
&\stackrel{\text{Eq. (2.A.10)}}{\Rightarrow} \operatorname{Im}\left(\frac{C_k}{C_1}\right) = \pm \operatorname{Im}\left(\frac{C'_k}{C'_1}\right)
\end{aligned}$$

5. “This is only possible if  $\operatorname{Re}\left(\frac{C_k}{C_1} \frac{C_l^*}{C_1^*}\right) = \operatorname{Re}\left(\frac{C_k}{C_1} \frac{C_l}{C_1}\right)$  or, in other words, if  $\operatorname{Im}\left(\frac{C_k}{C_1}\right) \operatorname{Im}\left(\frac{C_l}{C_1}\right) = 0$ ” (P.93)

Define

$$\begin{aligned}
a &:= \frac{C_k}{C_1} \\
b &:= \frac{C_l}{C_1}
\end{aligned}$$

With this we have

$$|1 + a + b^*|^2 = |1 + a + b|^2 \quad (1)$$

$$\Leftrightarrow 1 + a^* + b + a + |a|^2 + ab + b^* + b^* a^* + |b|^2 \quad (2)$$

$$= 1 + a^* + b^* + a + |a|^2 + ab^* + b + ba^* + |b|^2 \quad (3)$$

$$\Leftrightarrow ab + b^* a^* = ab^* + ba^* \quad (4)$$

And further rewriting yields

$$\begin{aligned}
\operatorname{Re}\left(\frac{C_k}{C_1} \frac{C_l}{C_1}\right) &= \operatorname{Re}(ab) \stackrel{4}{=} \operatorname{Re}(ab^*) = \operatorname{Re}\left(\frac{C_k}{C_1} \frac{C_l^*}{C_1^*}\right) \\
\operatorname{Im}\left(\frac{C_k}{C_1}\right) \operatorname{Im}\left(\frac{C_l}{C_1}\right) &= \operatorname{Im}(a) \operatorname{Im}(b) \\
&= -\frac{1}{4}(ab - ab^* - a^* b + a^* b^*) \stackrel{4}{=} 0
\end{aligned}$$

6. “Then the invariance of transition probabilities requires

$$\text{that } \left| \sum_k B_k^* A_k \right|^2 = \left| \sum_k B_k A_k \right|^2 \text{” (P.93)}$$

$$\begin{aligned} \left| \sum_k B_k^* A_k \right|^2 &\stackrel{\text{Eq. (2.A.2)}}{=} \left| \sum_{kl} B_k^* A_l \langle \Psi_k | \Psi_l \rangle \right|^2 \\ &= \left| \left\langle \sum_k B_k \Psi_k \left| \sum_l A_l \Psi_l \right. \right\rangle \right|^2 \\ &\stackrel{\text{Eq. (2.A.1)}}{=} \left| \left\langle U \left( \sum_k B_k \Psi_k \right) \left| U \left( \sum_l A_l \Psi_l \right) \right. \right\rangle \right|^2 \\ &= \left| \left\langle \sum_k B_k^* U \Psi_k \left| \sum_l A_l U \Psi_l \right. \right\rangle \right|^2 \\ &= \left| \sum_{kl} B_k A_l \langle U \Psi_k | U \Psi_l \rangle \right|^2 \\ &\stackrel{\text{Eq. (2.A.3)}}{=} \left| \sum_k B_k A_k \right|^2 \end{aligned}$$

7. Eq. (2.A.16)(P.94)

$$\begin{aligned} &\sum_{kl} \text{Im}(B_k^* B_l) \text{Im}(A_k^* A_l) \\ &= \text{Im} \left( \sum_{kl} \text{Im}(B_k^* B_l) A_k^* A_l \right) \\ &= \frac{1}{2i} \left[ \sum_{kl} \text{Im}(B_k^* B_l) A_k^* A_l - \text{c.c.} \right] \\ &= \frac{1}{2i} \left[ \sum_{kl} \frac{1}{2i} (B_k^* B_l - B_k B_l^*) A_k^* A_l - \text{c.c.} \right] \\ &= \frac{1}{2i} \left[ \frac{1}{2i} \left( \sum_{kl} B_k^* B_l A_k^* A_l - \sum_{kl} B_k B_l^* A_k^* A_l \right) - \text{c.c.} \right] \\ &= \frac{1}{2i} \left[ \frac{1}{2i} \left( \left| \sum_k B_k A_k \right|^2 - \left| \sum_k B_k^* A_k \right|^2 \right) - \text{c.c.} \right] \\ &= \frac{1}{2i} \left[ \frac{1}{i} \left( \left| \sum_k B_k A_k \right|^2 - \left| \sum_k B_k^* A_k \right|^2 \right) \right] \\ &\stackrel{II \text{ H } 6}{=} 0 \end{aligned}$$

8. “However, for any pair of such state-vectors, with neither  $A_k$  nor  $B_k$  **all of the same phase**” (P.94)

If they were all of the same phase then

$$\forall k, l : \text{Im}\{A_k^* A_l\} = 0$$

or

$$\forall k, l : \text{Im}\{B_k^* B_l\} = 0$$

See Footnote j, for why this is relevant.

9. “We have thus shown that for a given symmetry transformation  $T$  either all state-vectors satisfy Eq. (2.A.14) or else they all satisfy Eq. (2.A.15)” (P.94)

Preceding this a contradiction was derived from the assumption that Eq. (2.A.14) applies for a state-vector

$$\sum_k A_k \Psi_k$$

while Eq. (2.A.15) applies for a state vector

$$\sum_k B_k \Psi_k.$$

Such that the statement is obvious.

## I. Group Operators and Homotopy Classes

### 1. Eq. (2.B.7) (P.97)

Taylor evolving Eq. (2.B.6) in  $\theta_3^c$  up to  $\mathcal{O}(\theta_3^2)$  yields:

$$\begin{aligned} &f^a(\theta_2, \theta_1) + [h^{-1}]^a_c (f(\theta_2, \theta_1)) \theta_3^c \\ &= f^a(0, f(\theta_2, \theta_1)) + \left[ \frac{\partial f^a(\bar{\theta}, f(\theta_2, \theta_1))}{\partial \bar{\theta}^c} \right]_{\bar{\theta}=0} \theta_3^c \\ &= f^a(\theta_3, f(\theta_2, \theta_1)) \\ &\stackrel{(2.B.6)}{=} f^a(f(\theta_3, \theta_2), \theta_1) \\ &= f^a \left( f(0, \theta_2) + \left[ \frac{\partial f^a(\bar{\theta}, \theta_2)}{\partial \bar{\theta}^c} \right]_{\bar{\theta}=0} \theta_3^c, \theta_1 \right) \\ &= f^a(\theta_2, \theta_1) + \left[ \frac{\partial f^a(\bar{\theta}, \theta_1)}{\partial \bar{\theta}^b} \right]_{\bar{\theta}=\theta_2} \left[ \frac{\partial f^b(\bar{\theta}, \theta_2)}{\partial \bar{\theta}^c} \right]_{\bar{\theta}=0} \theta_3^c \\ &= f^a(\theta_2, \theta_1) + \left[ \frac{\partial f^a(\bar{\theta}, \theta_1)}{\partial \bar{\theta}^b} \right]_{\bar{\theta}=\theta_2} [h^{-1}]^b_c (\theta_2) \theta_3^c \end{aligned}$$

Equating coefficients of  $\theta_3^c$ , we get:

$$\begin{aligned} [h^{-1}]^a_c (f(\theta_2, \theta_1)) &= \left[ \frac{\partial f^a(\bar{\theta}, \theta_1)}{\partial \bar{\theta}^b} \right]_{\bar{\theta}=\theta_2} [h^{-1}]^b_c (\theta_2) \\ &\Leftrightarrow h^c_b (\theta_2) = \left[ \frac{\partial f^a(\bar{\theta}, \theta_1)}{\partial \bar{\theta}^b} \right]_{\bar{\theta}=\theta_2} h^c_a (f(\theta_2, \theta_1)) \end{aligned}$$

2. “Along the second segment the differential equation Eq. (2.B.2) for  $U_{\mathcal{P}}(s)$  is thus the same as the differential equation for  $U_{\theta_2}(2s-1)$ .” (P.97)

In the second segment  $\left(\frac{1}{2} \leq s \leq 1\right)$  the differential equation for  $U_{\mathcal{P}}(s)$  reads:

$$\begin{aligned} & \frac{dU_{\mathcal{P}}(s)}{ds} \\ &= it_c U_{\mathcal{P}}(s) h^c_a (\Theta_{\mathcal{P}}(s)) \frac{d\Theta_{\mathcal{P}}^c(s)}{ds} \\ &= it_c U_{\mathcal{P}}(s) h^c_a (f(\Theta_{\theta_2}(2s-1), \theta_1)) \frac{df^c(\Theta_{\theta_2}(2s-1), \theta_1)}{ds} \\ &= it_c U_{\mathcal{P}}(s) h^c_a (f(\Theta_{\theta_2}(2s-1), \theta_1)) \left[ \frac{\partial f^a(\bar{\theta}, \theta_1)}{\partial \bar{\theta}^b} \right]_{\bar{\theta}=\Theta_{\theta_2}(2s-1)} \\ & \cdot \frac{d\Theta_{\theta_2}^b(2s-1)}{ds} \\ & \stackrel{(2.B.7)}{=} it_c U_{\mathcal{P}}(s) h^c_b (\Theta_{\theta_2}(2s-1)) \frac{d\Theta_{\theta_2}^b(2s-1)}{ds} \end{aligned}$$

Which is just the differential equation for  $U_{\theta_2}(2s-1)$ :

$$\begin{aligned} \frac{dU_{\theta_2}(2s-1)}{ds} &= \left[ \frac{dU_{\theta_2}(t)}{dt} \right]_{t=2s-1} 2 \\ &= it_c U_{\theta_2}(2s-1) h^c_b (\Theta_{\theta_2}(2s-1)) \left[ \frac{d\Theta_{\theta_2}^b(t)}{dt} \right]_{t=2s-1} 2 \\ &= it_c U_{\theta_2}(2s-1) h^c_b (\Theta_{\theta_2}(2s-1)) \frac{d\Theta_{\theta_2}^b(2s-1)}{ds} \end{aligned}$$

3. Eq. (2.B.9) (P.98)

First note

$$\frac{dU^{-1}}{ds} = \left( \frac{dU}{ds} \right)^\dagger \stackrel{(2.B.2)}{=} -i(t_a U)^\dagger h^a_b \frac{d\Theta^b}{ds} \quad (5)$$

$$= -iU^{-1} t_a h^a_b \frac{d\Theta^b}{ds}, \quad (6)$$

and

$$\frac{d}{ds} (U^{-1} t_a U h^a_b) \quad (7)$$

$$= \frac{dU^{-1}}{ds} t_e U h^e_b + U^{-1} t_e \frac{dU}{ds} h^e_b + U^{-1} t_a U \frac{dh^a_b}{ds} \quad (8)$$

$$\stackrel{(2.B.2)}{=} -iU^{-1} t_d h^d_c \frac{d\Theta^c}{ds} t_e U h^e_b \quad (9)$$

$$+ U^{-1} t_e i t_d U h^d_c \frac{d\Theta^c}{ds} h^e_b \quad (10)$$

$$+ U^{-1} t_a U h^a_{b,c} \frac{d\Theta^c}{ds} \quad (11)$$

$$= i \frac{d\Theta^c}{ds} h^d_c h^e_b U^{-1} [t_e, t_d] U + U^{-1} t_a U \frac{d\Theta^c}{ds} h^a_{b,c} \quad (12)$$

$$\stackrel{(2.2.22)}{=} U^{-1} t_a U \frac{d\Theta^c}{ds} (i h^d_c h^e_b i C^a_{ed} + h^a_{b,c}). \quad (13)$$

By using Eq. (2.2.22) we are making use of condition (a) of the Theorem. With this we get:

$$\begin{aligned} & \frac{dU^{-1} \delta U}{ds} \\ &= \frac{dU^{-1}}{ds} \delta U + U^{-1} \frac{d\delta U}{ds} \\ &\stackrel{6}{=} -iU^{-1} t_a h^a_b \frac{d\Theta^b}{ds} \delta U + U^{-1} \frac{d\delta U}{ds} \\ &= iU^{-1} t_a U h^a_{c,b} (\delta \Theta^b) \frac{d\Theta^c}{ds} \\ &+ iU^{-1} t_a U h^a_b \frac{d\delta \Theta^b}{ds} \\ &= iU^{-1} t_a U (\delta \Theta^b) \frac{d\Theta^c}{ds} h^a_{c,b} \\ &+ \frac{d}{ds} (iU^{-1} t_a U h^a_b \delta \Theta^b) - i(\delta \Theta^b) \frac{d}{ds} (U^{-1} t_a U h^a_b) \\ &\stackrel{13}{=} iU^{-1} t_a U (\delta \Theta^b) \frac{d\Theta^c}{ds} (h^a_{c,b} - (-h^d_c h^e_b C^a_{ed} + h^a_{b,c})) \\ &+ \frac{d}{ds} (iU^{-1} t_a U h^a_b \delta \Theta^b) \\ &= iU^{-1} t_a U (\delta \Theta^b) \frac{d\Theta^c}{ds} (h^a_{c,b} + h^d_c h^e_b C^a_{ed} - h^a_{b,c}) \\ &+ \frac{d}{ds} (iU^{-1} t_a U h^a_b \delta \Theta^b) \end{aligned}$$

Where we inserted the expression for  $\frac{d\delta U}{ds}$  in the third step.

4. Eq. (2.B.10) (P.98)

Taylor evolving Eq. (2.B.6) in  $\theta_3, \theta_2$  up to  $\mathcal{O}(\theta_3^3, \theta_2^3)$  yields:

$$\begin{aligned}
& f^a(0, \theta_1) + [h^{-1}]^a_b(\theta_1) (\theta_3^b + \theta_2^b + f^b_{ec} \theta_3^e \theta_2^c) \\
&= f^a(0, \theta_1) + \left[ \frac{\partial f^a(\bar{\theta}, \theta_1)}{\partial \bar{\theta}^b} \right]_{\bar{\theta}=0} (\theta_3^b + \theta_2^b + f^b_{ec} \theta_3^e \theta_2^c) \\
&= f^a(\theta_3 + \theta_2 + f^b_{ec} \theta_3^e \theta_2^c, \theta_1) \\
&\stackrel{(2.2.19)}{=} f^a(f(\theta_3, \theta_2), \theta_1) \\
&\stackrel{(2.B.6)}{=} f^a(\theta_3, f(\theta_2, \theta_1)) \\
&= f^a \left( \theta_3, f(0, \theta_1) + \left[ \frac{\partial f^a(\bar{\theta}, \theta_1)}{\partial \bar{\theta}^e} \right]_{\bar{\theta}=0} \theta_2^e \right) \\
&= f^a \left( \theta_3, \theta_1 + [h^{-1}]^e_e(\theta_1) \theta_2^e \right) \\
&= f^a(0, \theta_1) + \left[ \frac{\partial f^a(\bar{\theta}, \theta_1)}{\partial \bar{\theta}^b} \right]_{\bar{\theta}=0} \theta_3^b \\
&+ \left[ \frac{\partial f^a(0, \bar{\theta})}{\partial \bar{\theta}^b} \right]_{\bar{\theta}=\theta_1} [h^{-1}]^b_e(\theta_1) \theta_2^e \\
&+ \frac{1}{2} \left[ \frac{\partial}{\partial \bar{\theta}^c} \left[ \frac{\partial f^a(\bar{\theta}, \bar{\theta})}{\partial \bar{\theta}^b} \right]_{\bar{\theta}=0} \right]_{\bar{\theta}=\theta_1} \theta_3^b [h^{-1}]^c_e(\theta_1) \theta_2^e \\
&+ \frac{1}{2} \left[ \frac{\partial}{\partial \bar{\theta}^b} \left[ \frac{\partial f^a(\bar{\theta}, \bar{\theta})}{\partial \bar{\theta}^c} \right]_{\bar{\theta}=0} \right]_{\bar{\theta}=\theta_1} \theta_3^b [h^{-1}]^c_e(\theta_1) \theta_2^e \\
&= f^a(0, \theta_1) + [h^{-1}]^a_b(\theta_1) \theta_3^b + [h^{-1}]^a_e(\theta_1) \theta_2^e \\
&+ \frac{\partial}{\partial \theta_1^c} \left( [h^{-1}]^a_b(\theta_1) \right) \theta_3^b [h^{-1}]^c_e(\theta_1) \theta_2^e
\end{aligned}$$

No  $\theta_3^2, \theta_2^2$  terms show up and

$$\left[ \frac{\partial f^a(0, \bar{\theta})}{\partial \bar{\theta}^b} \right]_{\bar{\theta}=\theta_1} = \delta_b^a,$$

see Eq. (2.2.19). Equating coefficients of  $\theta_3^b \theta_2^c$ , we get:

$$\begin{aligned}
& [h^{-1}]^a_e(\theta_1) f^e_{bc} \\
&= \frac{\partial}{\partial \theta_1^d} \left( [h^{-1}]^a_b(\theta_1) \right) [h^{-1}]^d_c(\theta_1) \\
&\Leftrightarrow f^e_{bc} = h^e_a(\theta_1) \frac{\partial}{\partial \theta_1^d} \left( [h^{-1}]^a_b(\theta_1) \right) [h^{-1}]^d_c(\theta_1) \\
&= \left( 0 - h^e_{a,d} [h^{-1}]^a_b \right) [h^{-1}]^d_c \\
&\Leftrightarrow h^e_{a,d} = -f^e_{bc} h^b_a h^c_d
\end{aligned}$$

5. Eq. (2.B.11) (P.98)

$$\begin{aligned}
h^a_{c,b} - h^a_{b,c} &\stackrel{(2.B.10)}{=} -f^a_{de} h^d_c h^e_b + f^a_{de} h^d_b h^e_c \\
&= h^d_c h^e_b (-f^a_{de} + f^a_{ed}) \\
&\stackrel{(2.2.23)}{=} h^d_c h^e_b (-C^a_{ed})
\end{aligned}$$

6. “It follows that  $U_\theta(1)$  is stationary under any infinitesimal variation of the path that leaves the endpoints  $\Theta(0) = 0$  and  $\Theta(1) = \theta$  (and  $U_\theta(0) = \mathbf{1}$ ) fixed.” (P.98)

We have

$$U^{-1} \delta U - i U^{-1} t_a U h^a_b \delta \Theta^b = C = \text{const}$$

For  $s = 0$  this gives

$$0 = \delta U(0) = i t_a U(0) h^a_b(0) \underbrace{\delta \Theta^b(0)}_{=0} + U(0) C = \mathbf{1} C$$

such that  $C = 0$ . For  $s = 1$  we then get

$$\delta U(1) = i t_a U(1) h^a_b(1) \underbrace{\delta \Theta^b(1)}_{=0} + U(1) \underbrace{C}_{=0} = 0.$$

$$\begin{aligned}
7. \quad \mathfrak{V} &= \frac{\partial}{\partial \theta^b} \left\{ U[\theta]^{-1} \tilde{U}[\theta] \right\} + i \phi_b(\theta) U[\theta]^{-1} \tilde{U}[\theta] \text{ where} \\
&\phi_b(\theta) = h^a_b \left[ \frac{\partial \phi(\theta', \theta)}{\partial \theta'^a} \right]_{\theta'=\theta} \text{ ” (P.99)}
\end{aligned}$$

By taking the derivative of the previous equation w.r.t.  $\theta'^a$ , we obtain:

$$\begin{aligned}
0 &= U[\theta]^{-1} (-t_a + t_a) \tilde{U}[\theta] \\
&= \left[ \frac{\partial}{\partial \theta'^a} U[\theta]^{-1} U[\theta']^{-1} \tilde{U}[\theta'] \tilde{U}[\theta] \right]_{\theta'=0} \\
&= \left[ \frac{\partial}{\partial \theta'^a} U[f(\theta', \theta)]^{-1} \tilde{U}[f(\theta', \theta)] \exp(i\phi(\theta', \theta)) \right]_{\theta'=0} \\
&= \left[ \frac{\partial}{\partial \bar{\theta}^b} U[\bar{\theta}]^{-1} \tilde{U}[\bar{\theta}] \right]_{\bar{\theta}=f(0, \theta)=\theta} \left[ \frac{\partial f^b(\theta', \theta)}{\partial \theta'^a} \right]_{\theta'=0} \exp(i\phi(0, \theta)) \\
&+ U[\theta]^{-1} \tilde{U}[\theta] i \left[ \frac{\partial \phi(\theta', \theta)}{\partial \theta'^a} \right]_{\theta'=0} \exp(i\phi(0, \theta)) \\
&= \frac{\partial}{\partial \theta^b} \left\{ U[\theta]^{-1} \tilde{U}[\theta] \right\} [h^{-1}]^b_a(\theta) \\
&+ U[\theta]^{-1} \tilde{U}[\theta] i \left[ \frac{\partial \phi(\theta', \theta)}{\partial \theta'^a} \right]_{\theta'=0}
\end{aligned}$$

Multiplying by  $h^a_b(\theta)$  yields finally

$$0 = \frac{\partial}{\partial \theta^b} \left\{ U[\theta]^{-1} \tilde{U}[\theta] \right\} + i \phi_b(\theta) U[\theta]^{-1} \tilde{U}[\theta].$$

8.  $\mathfrak{0} = \frac{\partial \phi_b(\theta)}{\partial \theta^c} - \frac{\partial \phi_c(\theta)}{\partial \theta^b}$  (P.99)

Differentiating the result of III 7 w.r.t  $\theta^c$  yields

$$\begin{aligned} 0 &= \frac{\partial^2}{\partial \theta^c \partial \theta^b} \left\{ U[\theta]^{-1} \tilde{U}[\theta] \right\} + i \frac{\partial \phi_b(\theta)}{\partial \theta^c} U[\theta]^{-1} \tilde{U}[\theta] \\ &\quad + i \phi_b(\theta) \frac{\partial}{\partial \theta^c} \left\{ U[\theta]^{-1} \tilde{U}[\theta] \right\} \\ &\stackrel{III 7}{=} \frac{\partial^2}{\partial \theta^c \partial \theta^b} \left\{ U[\theta]^{-1} \tilde{U}[\theta] \right\} + i \frac{\partial \phi_b(\theta)}{\partial \theta^c} U[\theta]^{-1} \tilde{U}[\theta] \\ &\quad + \phi_b(\theta) \phi_c(\theta) U[\theta]^{-1} \tilde{U}[\theta]. \end{aligned}$$

Antisymmetrizing then gives

$$0 = \frac{\partial \phi_b(\theta)}{\partial \theta^c} - \frac{\partial \phi_c(\theta)}{\partial \theta^b}.$$

9. “Then  $U^{-1}(f(\theta_2, \theta_1))U(\theta_2)U(\theta_1)$  can be a phase factor  $\exp(i\phi(\theta_2, \theta_1)) \neq 1$ , but  $\phi$  will be the same for all other loops into which this can be continuously deformed.” (P.99)

This can be seen from the statement of III 6 by setting  $\theta = 0$ .

## J. Inversions and Degenerate Multiplets

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