Notes for "The Quantum Theory of Fields 1, Foundations" - Weinberg

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I. INTRODUCTION

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II. RELATIVISTIC QUANTUM MECHANICS

A. Quantum Mechanics

B. Symmetries

1. *"For this to be unitary and linear, t must be Hermitian and linear"* $(P.51)$

Linearity is trivial and hermiticity follow from the following observation:

$$\begin{aligned}\langle U\Psi|U\Phi\rangle &= \langle(1+i\varepsilon t)\Psi|(1+i\varepsilon t)\Phi\rangle\\ &= \langle\Psi|\Phi\rangle + \varepsilon i\left(\langle\Psi|t\Phi\rangle - \langle t\Psi|\Phi\rangle\right) + \mathcal{O}\left(\varepsilon^2\right)\\ &\overset{Eq.\,(2.2.2)}{\Leftrightarrow} \langle\Psi|t\Phi\rangle = \langle t\Psi|\Phi\rangle\\ &\overset{Eq.\,(2.1.5)}{\Leftrightarrow} t^\dagger = t\end{aligned}$$

2. *Eq.* (2.2.19) $(P.54)$

f^a_{bc} and f^a have to be real as θ^a are real.

3. *Eq.* (2.2.21) $(P.54)$

From Eq. (2.2.20) we have up to $\mathcal{O}\left(\theta^2,\bar\theta^2\right)$

$$\begin{aligned}&1 + i\left(\theta^a + \bar\theta^a + f^a{}_{bc}\bar\theta^b\theta^c\right)t_a + \frac{1}{2}\left(\theta^b+\bar\theta^b\right)\left(\theta^c+\bar\theta^c\right)t_{bc}\\ &= \left[1 + i\bar\theta^a t_a + \frac{1}{2}\bar\theta^b\bar\theta^c t_{bc}\right]\cdot\left[1 + i\theta^a t_a + \frac{1}{2}\theta^b\theta^c t_{bc}\right]\\ &= 1 + i\theta^a t_a + \frac{1}{2}\theta^b\theta^c t_{bc} + i\bar\theta^a t_a - \bar\theta^b t_b\theta^c t_c + \frac{1}{2}\bar\theta^b\bar\theta^c t_{bc}\\ \Leftrightarrow & i f^a{}_{bc}\bar\theta^b\theta^c t_a + \frac{1}{2}\left(\theta^b\bar\theta^c + \bar\theta^b\theta^c\right)t_{bc} = -\bar\theta^b\theta^c t_b t_c\\ \Leftrightarrow & \bar\theta^b\theta^c\left[t_{bc} + i f^a{}_{bc}t_a + t_b t_c\right] = 0\\ \Leftrightarrow & t_{bc} = -i f^a{}_{bc}t_a - t_b t_c\end{aligned}$$

4. *Eq.* (2.2.22) $(P.54)$

$$\begin{gathered}-i f^a{}_{bc}t_a - t_b t_c \overset{(2.2.21)}{=} t_{bc} = t_{cb} \overset{(2.2.21)}{=} -i f^a{}_{cb}t_a - t_c t_b\\ \Leftrightarrow [t_b, t_c] = i\left(f^a{}_{cb} - f^a{}_{bc}\right)t_a\end{gathered}$$

C. Quantum Lorentz Transformations

1. *"$\Lambda^\mu{}_\nu$ has an **inverse**"* $(P.57)$

This is true because

$$\det(\Lambda) = \pm 1 \neq 0$$

2. *"$\left(\bar\Lambda\Lambda\right)^0{}_0 \geq \bar\Lambda^0{}_0\Lambda^0{}_0 - \sqrt{\left(\Lambda^0{}_0\right)^2 - 1}\sqrt{\left(\bar\Lambda^0{}_0\right)^2 - 1} \geq 1$"* $(P.58)$

The first inequality is trivial from the preceding considerations. For the second inequality assume the contrary holds, then:

$$\begin{aligned}&0 \leq \bar\Lambda^0{}_0\Lambda^0{}_0 - 1 < \sqrt{\left(\Lambda^0{}_0\right)^2 - 1}\sqrt{\left(\bar\Lambda^0{}_0\right)^2 - 1}\\ &\Rightarrow \left(\bar\Lambda^0{}_0\right)^2\left(\Lambda^0{}_0\right)^2 - 2\bar\Lambda^0{}_0\Lambda^0{}_0 + 1\\ &\quad < \left(\bar\Lambda^0{}_0\right)^2\left(\Lambda^0{}_0\right)^2 - \left(\bar\Lambda^0{}_0\right)^2 - \left(\Lambda^0{}_0\right)^2 + 1\\ &\Rightarrow \left(\Lambda^0{}_0 + \bar\Lambda^0{}_0\right)^2 = \left(\bar\Lambda^0{}_0\right)^2 - 2\bar\Lambda^0{}_0\Lambda^0{}_0 + \left(\Lambda^0{}_0\right)^2 < 0\end{aligned}$$

Which is a contradiction as $\Lambda^0{}_0 + \bar\Lambda^0{}_0 \geq 1 + 1 = 2$ and therefore completes the proof.

D. The Poincaré Algebra

1. *"In order fo $U(1+\omega,\varepsilon)$ to be unitary, the operators $J^{\rho\sigma}$ and P^ρ must be **Hermitian**"* $(P.59)$

Analog to II B 1.

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2. *Eqs.* (2.4.8/9) (*P.*60)

$$\begin{aligned}
&\frac{1}{2}\omega_{\rho\sigma} U J^{\rho\sigma} U^{-1} - \varepsilon_\rho U P^\rho U^{-1} \\
&\overset{(2.4.7)}{=} \frac{1}{2}\left(\Lambda\omega\Lambda^{-1}\right)_{\mu\nu} J^{\mu\nu} - \left(\Lambda\varepsilon - \Lambda\omega\Lambda^{-1}a\right)_\mu P^\mu \\
&= \frac{1}{2}\Lambda_\mu{}^\rho \omega_{\rho\sigma}\left(\Lambda^{-1}\right)^\sigma{}_\nu J^{\mu\nu} \\
&- \left(\Lambda_\mu{}^\rho\varepsilon_\rho - \Lambda_\mu{}^\rho\omega_{\rho\sigma}\left(\Lambda^{-1}\right)^\sigma{}_\nu a^\nu\right) P^\mu \\
&\overset{(2.3.10)}{=} \frac{1}{2}\Lambda_\mu{}^\rho\omega_{\rho\sigma}\Lambda_\nu{}^\sigma J^{\mu\nu} \\
&- \left(\Lambda_\mu{}^\rho\varepsilon_\rho - \Lambda_\mu{}^\rho\omega_{\rho\sigma}\Lambda_\nu{}^\sigma a^\nu\right) P^\mu \\
&= \frac{1}{2}\omega_{\rho\sigma}\left(\Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma J^{\mu\nu} + \Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma a^\nu P^\mu\right) \\
&- \varepsilon_\rho \Lambda_\mu{}^\rho P^\mu
\end{aligned}$$

In order to be able to compare coefficients in this, the coefficient of $\omega_{\rho\sigma}$ has to be anti symmatrized. With this Eqs. (2.4.8/9) follow immediately.

3. *Eqs.* (2.4.10/11) (*P.*60)

Up to $\mathcal{O}(\omega,\varepsilon)$ one can identify

$$U^{-1}(1+\omega,\varepsilon) = U(1-\omega,-\varepsilon)$$

,since

$$U(1+\omega,\varepsilon)U(1-\omega,-\varepsilon) = U(1-\omega+\omega,-\varepsilon+\varepsilon) = U(1,0).$$

With this we have up to $\mathcal{O}(\omega,\varepsilon)$

$$\begin{aligned}
&i\left[\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - \varepsilon_\mu P^\mu, J^{\rho\sigma}\right] \\
&= \left(1 + \frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu} - i\varepsilon_\mu P^\mu\right) J^{\rho\sigma} \\
&\cdot \left(1 - \frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu} + i\varepsilon_\mu P^\mu\right) - J^{\rho\sigma} \\
&= U J^{\rho\sigma} U^{-1} - J^{\rho\sigma} \\
&\overset{(2.4.8)}{=} (1+\omega)_\mu{}^\rho (1+\omega)_\nu{}^\sigma \\
&\cdot \left(J^{\mu\nu} - \varepsilon^\mu P^\nu + \varepsilon^\nu P^\mu\right) - J^{\rho\sigma} \\
&= -\varepsilon^\rho P^\sigma + \varepsilon^\sigma P^\rho + \omega_\nu{}^\sigma J^{\rho\nu} + \omega_\mu{}^\rho J^{\mu\sigma}
\end{aligned}$$

and also

$$\begin{aligned}
&i\left[\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - \varepsilon_\mu P^\mu, P^\rho\right] \\
&= \left(1 + \frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu} - i\varepsilon_\mu P^\mu\right) P^\rho \\
&\cdot \left(1 - \frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu} + i\varepsilon_\mu P^\mu\right) - P^\rho \\
&= U P^\rho U^{-1} - P^\rho \\
&\overset{(2.4.9)}{=} \omega_\mu{}^\rho P^\mu
\end{aligned}$$

4. *Eqs.* (2.4.12/13/14) (*P.*60)

The only difficulty lies in the derivation of Eq. (2.4.12) because for this one has to anti symmatrize the coefficient of $\omega_{\mu\nu}$ in Eq. (2.4.10) in order to be able to compare coefficients, similar to II D 2. (Similarly if one derives Eq. (2.4.13) from Eq. (2.4.11))

5. *Eqs.* (2.4.18 − 24) (*P.*61)

First observe

$$J^{ij} = \varepsilon_{ijk} J_k$$

and

$$J_i = \frac{1}{2}\varepsilon_{lmi} J^{lm}$$

$$\begin{aligned}
\left[J_i, J_j\right] &= \frac{\varepsilon_{lmi}\varepsilon_{kpj}}{4}\left[J^{lm}, J^{kp}\right] \\
&\overset{(2.4.12)}{=} -i\frac{\varepsilon_{lmi}\varepsilon_{kpj}}{4}\left[\eta^{mk}J^{lp} - \eta^{lk}J^{mp} - \eta^{pl}J^{km} + \eta^{pm}J^{kl}\right] \\
&= -\frac{i}{2}\left[\varepsilon_{kil}\varepsilon_{kmj}J^{lm} + \varepsilon_{kim}\varepsilon_{kjl}J^{lm}\right] \\
&= -\frac{i}{2}\left[J^{ji} - J^{ij}\right] \\
&= iJ^{ij} = i\varepsilon_{ijk}J_k \\
\left[J_i, K_j\right] &= \left[J^{lm}, J^{0j}\right]\frac{\varepsilon_{lmi}}{2} \\
&\overset{(2.4.12)}{=} -i\frac{\varepsilon_{lmi}}{2}\left[\eta^{m0}J^{lj} - \eta^{l0}J^{mj} - \eta^{jl}J^{0m} + \eta^{jm}J^{0l}\right] \\
&= -i\frac{\varepsilon_{lmi}}{2}\left[\delta_{jm}K_l - \delta_{jl}K_m\right] = i\varepsilon_{ijl}K_l \\
\left[K_i, K_j\right] &= \left[J^{0i}, J^{0j}\right] \\
&\overset{(2.4.12)}{=} -i\left[\eta^{i0}J^{0j} - \eta^{00}J^{ij} - \eta^{j0}J^{0i} + \eta^{ij}J^{00}\right] \\
&= -iJ^{ij} = -i\varepsilon_{ijk}J_k \\
\left[J_i, P_j\right] &= \frac{\varepsilon_{lmi}}{2}\left[J^{lm}, P^j\right] \\
&\overset{(2.4.13)}{=} \frac{i\varepsilon_{lmi}}{2}\left[\eta^{jl}P^m - \eta^{jm}P^l\right] \\
&= \frac{i}{2}\left[\varepsilon_{jmi}P^m - \varepsilon_{mji}P^m\right] = i\varepsilon_{ijm}P_m \\
\left[K_i, P_j\right] &= \left[J^{0i}, P^j\right] \\
&\overset{(2.4.13)}{=} i\left[\eta^{j0}P^i - \eta^{ji}P^0\right] \\
&= -i\delta_{ji}P^0 = -i\delta_{ij}H \\
\left[J_i, H\right] &\overset{(2.4.13)}{=} \frac{i\varepsilon_{lmi}}{2}\left[\eta^{0l}P^m - \eta^{0m}P^l\right] = 0 \\
\left[P_i, H\right] &= \left[P^i, P^0\right] \overset{(2.4.14)}{=} 0 \\
\left[H, H\right] &= \left[P^0, P^0\right] \overset{(2.4.14)}{=} 0 \\
\left[K_i, H\right] &= \left[J^{0i}, P^0\right] \\
&\overset{(2.4.13)}{=} i\left[\eta^{00}P^i - \eta^{0i}P^0\right] = -iP_i
\end{aligned}$$

6. Eq. (2.4.27) (*P.*61)

When trying to check this for e.g. the standard representation in 4 dimensions, in terms of infinitesimal rotations, attention with the sign in front of sin for the different rotation axis (see remark at the end ofII E 23).

7. "Inspection of Eqs. (2.4.18 − 24) *shows that these commutation relations have a limit for* $v \ll 1$ *of the form* . . . *"* (*P.*62)

Always equate same orders in v for this.

8. "$\exp(-i\mathbf{K}\cdot\mathbf{v})\exp(-i\mathbf{P}\cdot\mathbf{a}) = \exp(iM\mathbf{a}\cdot\mathbf{v}/2)\exp(-i(\mathbf{K}\cdot\mathbf{v}+\mathbf{P}\cdot\mathbf{a}))$*"* (*P.*62)

Use BCH Formula

$$
\begin{aligned}
&\exp(-iK_i v_i)\exp(-iP_j a_j)\\
&= \exp\left(-i(K_i v_i + P_j a_j) + \frac{1}{2}(-i)^2[K_i, P_j]v_i a_j + 0\right)\\
&= \exp\left(-i(K_i v_i + P_j a_j) + \frac{1}{2}iMv_i a_i\right)
\end{aligned}
$$

E. One-Particle States

1. Eq. (2.5.2) (*P.*63)

Λ^{-1} shows up here since in Eq. (2.4.9) UPU^{-1} is given but here $U^{-1}PU$ is being used.

2. "with σ *within any one block by themselves furnish a representation of the inhomogeneous Lorentz group"* (*P.*63)

In this case the Blocks do not mix with other blocks.

3. "and for $p^2 \le 0$*, also the sign of* p^0*"* (*P.*64)

For $p^2 \le 0$ we have

$$
\begin{aligned}
p^2 &= -\left(p^0\right)^2 + \vec{p}^2 \le 0\\
\Rightarrow |\vec{p}| &\le \left|p^0\right|
\end{aligned}
$$

and from Eq. (2.3.13) we know

$$
\left|\Lambda^0{}_0\right| \ge \left|\Lambda^{\vec{0}}{}_{\cdot}\right|.
$$

First suppose $p^0 \ge 0$:

$$
\begin{aligned}
p'^0 &= \Lambda^0{}_0 p^0 + \Lambda^0{}_i p^i\\
&\ge \Lambda^0{}_0 p^0 - \left|\Lambda^{\vec{0}}{}_{\cdot}\right||\vec{p}|\\
&= \left|\Lambda^0{}_0\right|\left|p^0\right| - \left|\Lambda^{\vec{0}}{}_{\cdot}\right||\vec{p}|\\
&\ge \left|\Lambda^{\vec{0}}{}_{\cdot}\right|\left(\left|p^0\right| - |\vec{p}|\right) \ge 0
\end{aligned}
$$

Now suppose $p^0 \le 0$:

$$
\begin{aligned}
p'^0 &= \Lambda^0{}_0 p^0 + \Lambda^0{}_i p^i\\
&\le \Lambda^0{}_0 p^0 + \left|\Lambda^{\vec{0}}{}_{\cdot}\right||\vec{p}|\\
&= -\left|\Lambda^0{}_0\right|\left|p^0\right| + \left|\Lambda^{\vec{0}}{}_{\cdot}\right||\vec{p}|\\
&\le \left|\Lambda^0{}_0\right|\left(-\left|p^0\right| + |\vec{p}|\right) \le 0
\end{aligned}
$$

4. Eq. (2.5.12) *"The delta function appears here because* $\Psi_{k,\sigma}$ *and* $\Psi_{k',\sigma'}$ *are eigenstates of a Hermitian operator with eigenvalues* $\mathbf{k}$ *and* $\mathbf{k}'$*, respectively."* (*P.*66)

> Note: From this point forward in this section of the book the states considered have standard momentum k^μ! (see Text in the Book preceding Eq. (2.5.12))

Suppose that for given values of $\mathbf{k}$ and $\mathbf{k}'$

$$
\langle \Psi_{k',\sigma'} | \Psi_{k,\sigma} \rangle \ne 0
$$

then from

$$
\begin{aligned}
&k^i \langle \Psi_{k',\sigma'} | \Psi_{k,\sigma} \rangle\\
&= \left\langle \Psi_{k',\sigma'} \middle| P^i \Psi_{k,\sigma} \right\rangle\\
&= \left\langle P^i \Psi_{k',\sigma'} \middle| \Psi_{k,\sigma} \right\rangle\\
&= k'^i \langle \Psi_{k',\sigma'} | \Psi_{k,\sigma} \rangle
\end{aligned}
$$

follows

$$
k^i = k'^i.
$$

On why there is only a 3 dimensional delta function showing up:

This is because both states are assumed to have standard momentum k^μ and therefore their 0-th component is completely fixed from their spatial components:

$$
k'^0 = \pm\sqrt{\mathbf{k}'^2 - k^2}
$$

Choice between + and − comes from sign of k^0, i.e. + for cases (a) and (c) of Table 2.1.

5. Eq. (2.5.13) (P.67)

$$\begin{aligned}
&\langle U(W)\Psi_{k',\sigma'}|U(W)\Psi_{k,\sigma}\rangle \\
&\overset{(2.5.8)}{=} \sum_{\sigma''\sigma'''} \langle D_{\sigma''\sigma'}\Psi_{k',\sigma''}|D_{\sigma'''\sigma}\Psi_{k,\sigma'''}\rangle \\
&= \sum_{\sigma''\sigma'''} \left(D_{\sigma''\sigma'}\right)^\star D_{\sigma'''\sigma} \langle \Psi_{k',\sigma''}|\Psi_{k,\sigma'''}\rangle \\
&\overset{(2.5.12)}{=} \sum_{\sigma''} \left(D_{\sigma''\sigma'}\right)^\star D_{\sigma''\sigma}\delta^{(3)}(\mathbf{k}'-\mathbf{k}) \\
&\overset{(2.2.2)}{=} \langle \Psi_{k',\sigma'}|\Psi_{k,\sigma}\rangle \\
&\overset{(2.5.12)}{=} \delta_{\sigma'\sigma}\delta^{(3)}(\mathbf{k}'-\mathbf{k})
\end{aligned}$$

From this it follows

$$D^\dagger(W) = D^{-1}(W)$$

6. "$\langle\Psi_{p',\sigma'}|\Psi_{p,\sigma}\rangle = \ldots$" (P.67)

Note: What is meant by "arbitrary momenta" is that these momenta still have the standard momentum k^μ, but now none of the states has exactly k^μ as its momentum.

First define

$$k' := L^{-1}(p)p'$$

with this we get:

$$\begin{aligned}
&\langle\Psi_{p',\sigma'}|\Psi_{p,\sigma}\rangle \\
&\overset{(2.5.5)}{=} \langle\Psi_{p',\sigma'}|N(p)U(L(p))\Psi_{k,\sigma}\rangle \\
&= N(p)\left\langle U(L^{-1}(p))\Psi_{p',\sigma'}\middle|\Psi_{k,\sigma}\right\rangle \\
&\overset{(2.5.11)}{=} N(p)\frac{N^\star(p')}{N^\star(k')} \\
&\cdot \sum_{\sigma''} D^\star_{\sigma''\sigma'}\left(W\left(L^{-1}(p),p'\right)\right)\langle\Psi_{k',\sigma''}|\Psi_{k,\sigma}\rangle \\
&\overset{(2.5.12)}{=} N(p)N^\star(p')\,D^\star_{\sigma\sigma'}\left(W\left(L^{-1}(p),p'\right)\right)\delta^{(3)}(\mathbf{k}'-\mathbf{k})
\end{aligned}$$

Where in the last step we used

$$\frac{\delta^{(3)}(\mathbf{k}'-\mathbf{k})}{N^\star(k')} = \frac{\delta^{(3)}(\mathbf{k}'-\mathbf{k})}{N^\star(k)} \overset{(2.5.5)}{=} \delta^{(3)}(\mathbf{k}'-\mathbf{k})$$

This is valid as from Eq. (2.5.5) we know that $N(p)$ is implicitly dependent on k^μ. This therefor fixes $p^2 = k^2$ and the sign of p^0 (see end of II E 4), s.t.

$$N(p) = N(\mathbf{p}).$$

Further we are allowed to use Eq. (2.5.12) in the last step since k' also has k as its standard momentum:

$$k' = L^{-1}(p)p' = \underbrace{L^{-1}(p)L(p')}_{=:L(k')}k$$

7. "$W\left(L^{-1}(p),p\right) = 1$" (P.67)

First note

$$L(k) = 1 = L^{-1}(k)$$

with this we get:

$$\begin{aligned}
W\left(L^{-1}(p),p\right) &\overset{(2.5.10)}{=} L^{-1}\left(L^{-1}(p)p\right)L^{-1}(p)L(p) \\
&= L^{-1}(k) = 1
\end{aligned}$$

8. "So we see that the ***invariant delta function*** *is ..." (P.67)*

Invariant in this case means w.r.t. proper orthochronous Lorentz transformations, which can be interpreted as change of variables under the $\int d^4p$ integral.

9. Eq. (2.5.24) (P.68)

$$\begin{aligned}
p^0 &= L^0{}_0k^0 + L^0{}_ik^i \\
&= \frac{\sqrt{\mathbf{p}^2+M^2}}{M}M = \sqrt{\mathbf{p}^2+M^2} \\
p^i &= L^i{}_0k^0 + L^i{}_jk^j \\
&= \frac{p_i}{|\mathbf{p}|}\sqrt{\frac{\mathbf{p}^2+M^2}{M^2}-1}M \\
&= \frac{p_i}{|\mathbf{p}|}\sqrt{\frac{\mathbf{p}^2}{M^2}}M = p_i
\end{aligned}$$

10. "To see this, note that the boost Eq. (2.5.24) may be expressed as $L(p) = R(\hat{\mathbf{p}})B(|\mathbf{p}|)R^{-1}(\hat{\mathbf{p}})$" (P.68)

First note that the columns and rows of the matrix $B(|\mathbf{p}|)$ are counted in the order 1,2,3,0.

Let

$$\hat{\mathbf{p}} = \begin{pmatrix}\sin(\theta)\cos(\phi)\\ \sin(\theta)\sin(\phi)\\ \cos(\theta)\end{pmatrix}$$

then we have

$$R(-\hat{\mathbf{p}}) = R_3(-\phi)R_2(\theta) = \begin{pmatrix} \cos(\phi)\cos(\theta) & -\sin(\phi) & \cos(\phi)\sin(\theta) \\ \sin(\phi)\cos(\theta) & \cos(\phi) & \sin(\phi)\sin(\theta) \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

(see II E 23) and

$$\begin{aligned} R^{-1}(\hat{\mathbf{p}}) = R_2(\theta)R_3(\phi) &= \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta)\cos(\phi) & \cos(\theta)\sin(\phi) & -\sin(\theta) \\ -\sin(\phi) & \cos(\phi) & 0 \\ \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \end{pmatrix} = R^\top(\hat{\mathbf{p}}) \end{aligned}$$

From Eq. (2.5.24) we know

$$L(p) = \begin{pmatrix} 1+(\gamma-1)s_\theta^2 c_\phi^2 & (\gamma-1)s_\theta c_\phi s_\theta s_\phi & (\gamma-1)s_\theta c_\phi c_\theta & s_\theta c_\phi\sqrt{\gamma^2-1} \\ (\gamma-1)s_\theta c_\phi s_\theta s_\phi & 1+(\gamma-1)s_\theta^2 s_\phi^2 & (\gamma-1)s_\theta s_\phi c_\theta & s_\theta s_\phi\sqrt{\gamma^2-1} \\ (\gamma-1)s_\theta c_\phi c_\theta & (\gamma-1)s_\theta s_\phi c_\theta & 1+(\gamma-1)c_\theta^2 & c_\theta\sqrt{\gamma^2-1} \\ s_\theta c_\phi\sqrt{\gamma^2-1} & s_\theta s_\phi\sqrt{\gamma^2-1} & c_\theta\sqrt{\gamma^2-1} & \gamma \end{pmatrix}$$

where s and c are shorthand for sin and cos. With this we can now check:

$$\begin{aligned} R(\hat{\mathbf{p}})B(|\mathbf{p}|)R^{-1}(\hat{\mathbf{p}}) &= \begin{pmatrix} c_\phi c_\theta & -s_\phi & c_\phi s_\theta & 0 \\ s_\phi c_\theta & c_\phi & s_\phi s_\theta & 0 \\ -s_\theta & 0 & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & \sqrt{\gamma^2-1} \\ 0 & 0 & \sqrt{\gamma^2-1} & \gamma \end{pmatrix} \begin{pmatrix} c_\phi c_\theta & s_\phi c_\theta & -s_\theta & 0 \\ -s_\phi & c_\phi & 0 & 0 \\ c_\phi s_\theta & s_\phi s_\theta & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_\phi c_\theta & -s_\phi & c_\phi s_\theta & 0 \\ s_\phi c_\theta & c_\phi & s_\phi s_\theta & 0 \\ -s_\theta & 0 & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_\phi c_\theta & s_\phi c_\theta & -s_\theta & 0 \\ -s_\phi & c_\phi & 0 & 0 \\ \gamma c_\phi s_\theta & \gamma s_\phi s_\theta & \gamma c_\theta & \sqrt{\gamma^2-1} \\ \sqrt{\gamma^2-1}c_\phi s_\theta & \sqrt{\gamma^2-1}s_\phi s_\theta & \sqrt{\gamma^2-1}c_\theta & \gamma \end{pmatrix} = L(p) \end{aligned}$$

11. "$W(\mathbf{R},p)=\mathbf{R}$" (P.69)

To see this just substitute $R(\theta)$ back into the previous result.

12. Eq. (2.5.25) (P.70)

$$\begin{aligned} -1 = (Wt)^\mu (Wt)_\mu &= \alpha^2+\beta^2+\zeta^2-(1+\zeta)^2 \\ \Leftrightarrow \alpha^2+\beta^2 &= 2\zeta \end{aligned}$$

13. Eq. (2.5.26) (P.70)

This is a Lorentz transformation, since

$$\begin{aligned} S^\top \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} S &= \begin{pmatrix} 1 & 0 & \alpha & \alpha \\ 0 & 1 & \beta & \beta \\ -\alpha & -\beta & 1-\zeta & -\zeta \\ \alpha & \beta & \zeta & 1+\zeta \end{pmatrix} \\ &\cdot \begin{pmatrix} 1 & 0 & -\alpha & \alpha \\ 0 & 1 & -\beta & \beta \\ \alpha & \beta & 1-\zeta & \zeta \\ -\alpha & -\beta & \zeta & -1-\zeta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

14. *Eqs.* (2.5.29/30) (*P.*70)

Eq. (2.5.30) is trivial since the rotations around the three axis satisfy this group multiplication and

$$S(0,0) = \mathbf{1}.$$

For Eq. (2.5.29) explicit calculation shows it (see TODO) together with

$$R(0) = \mathbf{1}.$$

15. *Eq.* (2.5.31) (*P.*70)

Explicit calculation shows Eq. (2.5.31) (see TODO) and the invariance follows then immediately, since:

$$W'SW'^{-1} = S' \underbrace{R'SR'^{-1}}_{=S''} S'^{-1} = S'''$$

16. *"$W(\theta,\alpha,\beta) = 1 + \omega$"* (*P.*71)

From Eq. (2.5.28) we immediately get for infinitesimal θ, α, β :

$$(W(\theta,\alpha,\beta))^\mu{}_\nu = \delta^\mu_\nu + \begin{pmatrix} 0 & \theta & -\alpha & \alpha \\ -\theta & 0 & -\beta & \beta \\ \alpha & \beta & 0 & 0 \\ \alpha & \beta & 0 & 0 \end{pmatrix}^\mu{}_\nu$$

$$\Rightarrow \omega_{\mu\nu} = \begin{pmatrix} 0 & \theta & -\alpha & \alpha \\ -\theta & 0 & -\beta & \beta \\ \alpha & \beta & 0 & 0 \\ -\alpha & -\beta & 0 & 0 \end{pmatrix}_{\mu\nu}$$

17. *Eq.* (2.5.32) (*P.*71)

$$\begin{aligned} U(1+\omega) &\overset{(2.4.3)}{=} 1 + \frac{1}{2} i\omega_{\rho\sigma} J^{\rho\sigma} \\ &= 1 + i\theta J^{12} + i\alpha(-J^{13} + J^{10}) + i\beta(-J^{23} + J^{20}) \\ &= 1 + i\theta J_3 + i\alpha(J_2 - K_1) + i\beta(-J_1 - K_2) \end{aligned}$$

18. *Eqs.* (2.5.35/36/37) (*P.*71)

From Eqs. (2.4.18/19/20) we get:

$$\begin{aligned} [J_3, A] &= [J_3, J_2 - K_1] \\ &= [J_3, J_2] - [J_3, K_1] \\ &= -iJ_1 - iK_2 = iB \\ [J_3, B] &= -[J_3, J_1] - [J_3, K_2] \\ &= -iJ_2 - i(-K_1) = -iA \\ [A, B] &= -[J_2, J_1] - [J_2, K_2] + [K_1, J_1] + [K_1, K_2] \\ &= iJ_3 - iJ_3 = 0 \end{aligned}$$

19. *"Since A and B are commuting Hermitian operators they can be simultaneously diagonalized by states $\Psi_{k,a,b}$"* (*P.*71)

This is valid, including the label k, since

$$\begin{aligned} [A, P_1]\Psi_{k,a,b} &= ([J_2, P_1] - [K_1, P_1])\Psi_{k,a,b} \\ &= (-iP_3 + iP^0)\Psi_{k,a,b} = 0 \\ [A, P_2]\Psi_{k,a,b} &= ([J_2, P_2] - [K_1, P_2])\Psi_{k,a,b} \\ &= 0 \\ [A, P_3]\Psi_{k,a,b} &= ([J_2, P_3] - [K_1, P_3])\Psi_{k,a,b} \\ &= (iP_1)\Psi_{k,a,b} = 0 \\ \left[A, P^0\right]\Psi_{k,a,b} &= (\left[J_2, P^0\right] - \left[K_1, P^0\right])\Psi_{k,a,b} \\ &= (-iP_1)\Psi_{k,a,b} = 0 \\ [B, P_1]\Psi_{k,a,b} &= (-[J_1, P_1] - [K_2, P_1])\Psi_{k,a,b} \\ &= 0 \\ [B, P_2]\Psi_{k,a,b} &= ([J_1, P_2] - [K_2, P_2])\Psi_{k,a,b} \\ &= (iP_3 - iP^0)\Psi_{k,a,b} = 0 \\ [B, P_3]\Psi_{k,a,b} &= ([J_1, P_3] - [K_2, P_3])\Psi_{k,a,b} \\ &= (-iP_2)\Psi_{k,a,b} = 0 \\ \left[B, P^0\right]\Psi_{k,a,b} &= (\left[J_1, P^0\right] - \left[K_2, P^0\right])\Psi_{k,a,b} \\ &= (-iP_2)\Psi_{k,a,b} = 0 \end{aligned}$$

20. *"σ gives the component of angular momentum in the direction of motion, or* helicity*"* (*P.*72)

The derivation of Eq. (2.5.26) starts among other conditions from the explicit form of k. Which results in

$$\textit{Eqs.}\,(2.5.27/28) \to \textit{Eq.}\,(2.5.32) \to 2.5.39,$$

s.t. this is really connected to the direction of motion.

21. *"$U(W)\Psi_{k,\sigma} = \exp(i\theta\sigma)\Psi_{k,\sigma}$"* (*P.*72)

Use Eqs. (2.5.38/39).

22. *Eq.* (2.5.44) (*P.*73)

With $B(u)$ from Eq. (2.5.45) we get:

$$
\begin{aligned}
B\left(\frac{|\mathbf{p}|}{\kappa}\right)k &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\left(\frac{|\mathbf{p}|}{\kappa}\right)^2+1}{2\frac{|\mathbf{p}|}{\kappa}} & \frac{\left(\frac{|\mathbf{p}|}{\kappa}\right)^2-1}{2\frac{|\mathbf{p}|}{\kappa}} \\ 0 & 0 & \frac{\left(\frac{|\mathbf{p}|}{\kappa}\right)^2-1}{2\frac{|\mathbf{p}|}{\kappa}} & \frac{\left(\frac{|\mathbf{p}|}{\kappa}\right)^2+1}{2\frac{|\mathbf{p}|}{\kappa}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \kappa \\ \kappa \end{pmatrix} \\
&= |\mathbf{p}| \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}
\end{aligned}
$$

23. *"take $R(\hat{\mathbf{p}})$ as a rotation by angle θ around the two-axis followed by a rotation by angle ϕ around the three-axis"* (*P.*73)

Let

$$
\hat{\mathbf{p}} = \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix}
$$

then

$$
\begin{aligned}
R(\hat{\mathbf{p}}) &= R_3(-\phi)R_2(-\theta) \\
&= \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\phi)\cos(\theta) & -\sin(\phi) & \cos(\phi)\sin(\theta) \\ \sin(\phi)\cos(\theta) & \cos(\phi) & \sin(\phi)\sin(\theta) \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix},
\end{aligned}
$$

(With this sign convention everything is consistent, see definition of $R(\theta)$ after Eq. (2.5.27) and compare Eq. (2.5.47) to Eq. (2.4.27)) with this we have

$$
\hat{\mathbf{p}} = R(\hat{\mathbf{p}})\hat{e}_3.
$$

F. Space Inversion and Time-Reversal

1. *Eqs.* (2.6.7 − 12) (*P.*76)

$$
\begin{aligned}
\mathsf{P}J_i\mathsf{P}^{-1} &= \frac{1}{2}\varepsilon_{ijk}\mathsf{P}J^{jk}\mathsf{P}^{-1} \\
&= \frac{1}{2}\varepsilon_{ijk}\left(-\delta^j_l\right)\left(-\delta^k_m\right)J^{lm} \\
&= \frac{1}{2}\varepsilon_{ijk}J^{jk} = J_i \\
\mathsf{P}K_i\mathsf{P}^{-1} &= \mathsf{P}J^{0i}\mathsf{P}^{-1} \\
&= \delta^0_\mu\left(-\delta^j_\nu\right)J^{\mu\nu} \\
&= -J^{0i} = -K_i \\
\mathsf{P}P_i\mathsf{P}^{-1} &= \left(-\delta^i_\nu\right)P^\mu = -P_i \\
\mathsf{T}J_i\mathsf{T}^{-1} &= \frac{1}{2}\varepsilon_{ijk}\mathsf{T}J^{jk}\mathsf{T}^{-1} \\
&= -\frac{1}{2}\varepsilon_{ijk}\delta^j_l\delta^k_m J^{lm} \\
&= -\frac{1}{2}\varepsilon_{ijk}J^{jk} = -J_i \\
\mathsf{T}K_i\mathsf{T}^{-1} &= \mathsf{T}J^{0i}\mathsf{T}^{-1} \\
&= -\left(-\delta^0_\mu\right)\delta^j_\nu J^{\mu\nu} \\
&= J^{0i} = K_i \\
\mathsf{T}P_i\mathsf{T}^{-1} &= -\delta^i_\nu P^\mu = -P_i
\end{aligned}
$$

2. *"$\mathcal{P}L(p)\mathcal{P}^{-1} = L(\mathcal{P}p)$"* (*P.*77)

From Eq. (2.5.24) we immediately see:

$$
\begin{aligned}
\left(\mathcal{P}L(p)\mathcal{P}^{-1}\right)^i{}_k &= \left(L(p)\right)^i{}_k = \left(L(\mathcal{P}p)\right)^i{}_k \\
\left(\mathcal{P}L(p)\mathcal{P}^{-1}\right)^i{}_0 &= -\left(L(p)\right)^i{}_0 = \left(L(\mathcal{P}p)\right)^i{}_0 \\
\left(\mathcal{P}L(p)\mathcal{P}^{-1}\right)^0{}_0 &= \left(L(p)\right)^0{}_0 = \left(L(\mathcal{P}p)\right)^0{}_0
\end{aligned}
$$

3. *"Using Eq. (2.6.14) again on the left, we see that the square-root factors cancel,..."* (*P.*77)

$$
\begin{aligned}
&\left(-J_1 \pm iJ_2\right)\zeta_\sigma\Psi_{k,-\sigma} = -\left(J_1 \mp iJ_2\right)\zeta_\sigma\Psi_{k,-\sigma} \\
&\overset{(2.6.14)}{=} -\sqrt{(j \pm (-\sigma))(j \mp (-\sigma)+1)}\zeta_\sigma\Psi_{k,-\sigma\mp 1} \\
&= -\sqrt{(j \mp \sigma)(j \pm \sigma + 1)}\zeta_\sigma\Psi_{k,-\sigma\mp 1}
\end{aligned}
$$

4. *"The time-reversal phase ζ has no physical significance."* (*P.*78)

This redefinition only works because $\mathsf{T}$is *anti-linear*.

5. "$\mathcal{T}L(p)\mathcal{T}^{-1} = L(\mathcal{P}p)$" (P.78)

From Eq. (2.5.24) we immediately see:

$$\begin{aligned}
\left(\mathcal{T}L(p)\mathcal{T}^{-1}\right)^i{}_k &= (L(p))^i{}_k = (L(\mathcal{P}p))^i{}_k \\
\left(\mathcal{T}L(p)\mathcal{T}^{-1}\right)^i{}_0 &= -(L(p))^i{}_0 = (L(\mathcal{P}p))^i{}_0 \\
\left(\mathcal{T}L(p)\mathcal{T}^{-1}\right)^0{}_0 &= (L(p))^0{}_0 = (L(\mathcal{P}p))^0{}_0
\end{aligned}$$

6. "Pyields a state with four-momentum ..." (P.78)

What is meant by this is:

$$\begin{aligned}
P^i \mathsf{P}\Psi_{k,\sigma} &\overset{(2.6.9)}{=} \mathsf{P}\left(-P^i\right)\Psi_{k,\sigma} = \left(-\delta^i_3\kappa\right)\mathsf{P}\Psi_{k,\sigma} \\
H\mathsf{P}\Psi_{k,\sigma} &\overset{(2.6.13)}{=} \mathsf{P}H\Psi_{k,\sigma} = \kappa\mathsf{P}\Psi_{k,\sigma} \\
J_3\mathsf{P}\Psi_{k,\sigma} &\overset{(2.6.7)}{=} \mathsf{P}J_3\Psi_{k,\sigma} = \sigma\mathsf{P}\Psi_{k,\sigma}
\end{aligned}$$

7. Eq. (2.6.20) (P.79)

We have

$$R_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = R_2^{-1}$$

from which we get (Is $U(R_2^{-1}) = U^{-1}(R_2)$? TODO)

$$\begin{aligned}
U^{-1}(R_2)J_3U(R_2) &= U^{-1}(R_2)J^{12}U(R_2) \\
&\overset{(2.4.8)}{=} \left(R_2^{-1}\right)^1{}_\mu \left(R_2^{-1}\right)^2{}_\nu J^{\mu\nu} \\
&= -J^{12} = -J_3 \\
U^{-1}(R_2)P^\nu U(R_2) &\overset{(2.4.9)}{=} \left(R_2^{-1}\right)^\nu{}_\mu P^\mu
\end{aligned}$$

such that

$$\begin{aligned}
J_3U(R_2^{-1})\mathsf{P}\Psi_{k,\sigma} &= -U(R_2^{-1})J_3\mathsf{P}\Psi_{k,\sigma} \\
&\overset{II\,F\,6}{=} -\sigma U(R_2^{-1})\mathsf{P}\Psi_{k,\sigma} \\
P^iU(R_2^{-1})\mathsf{P}\Psi_{k,\sigma} &\overset{II\,F\,6}{=} U(R_2^{-1})\left(-\left(-\delta^i_3\kappa\right)\right)\mathsf{P}\Psi_{k,\sigma} \\
&= k^iU(R_2^{-1})\mathsf{P}\Psi_{k,\sigma} \\
HU(R_2^{-1})\mathsf{P}\Psi_{k,\sigma} &\overset{II\,F\,6}{=} \kappa U(R_2^{-1})\mathsf{P}\Psi_{k,\sigma} \\
&= k^0U(R_2^{-1})\mathsf{P}\Psi_{k,\sigma}.
\end{aligned}$$

Further we get

$$R_2^{-1}\mathcal{P} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

8. "$\mathcal{P}$commutes with the rotation $R(\hat{\mathbf{p}})$" (P.79)

This is true because $R(\hat{\mathbf{p}})$ only acts on space components non trivially, which all just get a "-" sign from $\mathcal{P}$.

9. "$\mathsf{P}\Psi_{p,\sigma} = \sqrt{\frac{\kappa}{p^0}}\eta_\sigma U\left(R(\hat{\mathbf{p}})R_2B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,-\sigma}$" (P.79)

$$\begin{aligned}
\mathsf{P}\Psi_{p,\sigma} &\overset{(2.5.5)}{=} N(p)\mathsf{P}U(L(p))\Psi_{k,\sigma} \\
&\overset{(2.5.44)}{=} \sqrt{\frac{\kappa}{p^0}}\mathsf{P}U\left(R(\hat{\mathbf{p}})B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}}U\left(\mathcal{P}R(\hat{\mathbf{p}})B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}}U\left(R(\hat{\mathbf{p}})\mathcal{P}B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}}U\left(R(\hat{\mathbf{p}})R_2R_2^{-1}\mathcal{P}B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}}U\left(R(\hat{\mathbf{p}})R_2B\left(\frac{|\mathbf{p}|}{\kappa}\right)R_2^{-1}\mathcal{P}\right)\Psi_{k,\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}}U\left(R(\hat{\mathbf{p}})R_2B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)U\left(R_2^{-1}\right)\mathsf{P}\Psi_{k,\sigma} \\
&\overset{(2.6.20)}{=} \sqrt{\frac{\kappa}{p^0}}\eta_\sigma U\left(R(\hat{\mathbf{p}})R_2B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,-\sigma}
\end{aligned}$$

10. "But a rotation of $\pm180°$ around the three-axis reverses the sign of J_2, ..." (P.79)

Analogously to II F 6 ($2 \leftrightarrow 3$).

11. Eq. (2.6.22) (P.79)

First note that

$$B\left(\frac{|\mathbf{p}|}{\kappa}\right)$$

and rotations around the three-axis commute (see Eq. (2.5.45)).

$$\begin{aligned}
\mathsf{P}\Psi_{p,\sigma} &\overset{II\,F\,9}{=} \sqrt{\frac{\kappa}{p^0}}\eta_\sigma U\left(R\left(\hat{\mathbf{p}}\right)R_2 B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,-\sigma}\\
&\overset{Eq.\,(2.6.21)}{=} \sqrt{\frac{\kappa}{p^0}}\eta_\sigma U\left(R\left(-\hat{\mathbf{p}}\right)\right)\\
&\cdot\exp(\pm i\pi J_3)U\left(B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,-\sigma}\\
&= \sqrt{\frac{\kappa}{p^0}}\eta_\sigma U\left(R\left(-\hat{\mathbf{p}}\right)\right)\\
&\cdot U\left(B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\exp(\pm i\pi J_3)\Psi_{k,-\sigma}\\
&= \eta_\sigma\exp(\pm i\pi(-\sigma))\sqrt{\frac{\kappa}{p^0}}U\left(L(\mathcal{P}p)\right)\Psi_{k,-\sigma}\\
&= \eta_\sigma\exp(\mp i\pi\sigma)\Psi_{\mathcal{P}p,-\sigma}
\end{aligned}$$

12. "This peculiar change of sign in the operation of parity for mass-less particles of half-integer spin is due to the convention adopted in Eq. (2.5.47) *for the rotation used to define mass-less particles states of arbitrary momentum."* (P.79)

This is based on the fact that the choice of $R\left(\hat{\mathbf{p}}\right)$, which transforms the three axis into the unit vector $\hat{\mathbf{p}}$, is *not* unique. As mentioned in the text right after Eq. (2.5.47), one could always add an initial rotation around the three axis. This is also why the factor of

$$\exp(\pm i\pi J_3)$$

shows up in Eq. (2.6.21), despite $R\left(\hat{\mathbf{p}}\right)R_2$ and $R\left(-\hat{\mathbf{p}}\right)$ both transforming the three axis into the unit vector $-\hat{\mathbf{p}}$.

13. "T yields a state which has values . . . " (P.79)

What is meant by this is:

$$\begin{aligned}
P^i\mathsf{T}\Psi_{k,\sigma} &\overset{(2.6.12)}{=} \mathsf{T}\left(-P^i\right)\Psi_{k,\sigma} = \left(-\delta^i_3\kappa\right)\mathsf{T}\Psi_{k,\sigma}\\
H\mathsf{T}\Psi_{k,\sigma} &\overset{(2.6.13)}{=} \mathsf{T}H\Psi_{k,\sigma} = \kappa\mathsf{T}\Psi_{k,\sigma}\\
J_3\mathsf{T}\Psi_{k,\sigma} &\overset{(2.6.10)}{=} \mathsf{T}\left(-J_3\right)\Psi_{k,\sigma} = -\sigma\mathsf{T}\Psi_{k,\sigma}
\end{aligned}$$

14. Eq. (2.6.23) (P.80)

This is completely analog to Eq. (2.6.20) when using II F 13 (see II F 7). Further we get

$$R_2^{-1}\mathsf{T} = \begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}.$$

15. Eq. (2.6.24) (P.80)

$$\begin{aligned}
\mathsf{T}\Psi_{p,\sigma} &\overset{(2.5.5)}{=} N(p)\mathsf{T}U(L(p))\Psi_{k,\sigma}\\
&\overset{(2.5.44)}{=} \sqrt{\frac{\kappa}{p^0}}\mathsf{T}U\left(R\left(\hat{\mathbf{p}}\right)B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma}\\
&= \sqrt{\frac{\kappa}{p^0}}U\left(\mathcal{T}R\left(\hat{\mathbf{p}}\right)B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma}\\
&= \sqrt{\frac{\kappa}{p^0}}U\left(R\left(\hat{\mathbf{p}}\right)\mathcal{T}B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma}\\
&= \sqrt{\frac{\kappa}{p^0}}U\left(R\left(\hat{\mathbf{p}}\right)R_2R_2^{-1}\mathcal{T}B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma}\\
&= \sqrt{\frac{\kappa}{p^0}}U\left(R\left(\hat{\mathbf{p}}\right)R_2B\left(\frac{|\mathbf{p}|}{\kappa}\right)R_2^{-1}\mathcal{T}\right)\Psi_{k,\sigma}\\
&= \sqrt{\frac{\kappa}{p^0}}U\left(R\left(\hat{\mathbf{p}}\right)R_2B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)U\left(R_2^{-1}\right)\mathsf{T}\Psi_{k,\sigma}\\
&\overset{(2.6.23)}{=} \sqrt{\frac{\kappa}{p^0}}\zeta_\sigma U\left(R\left(\hat{\mathbf{p}}\right)R_2B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma}
\end{aligned}$$

16. Eq. (2.6.25) (P.80)

First note that

$$B\left(\frac{|\mathbf{p}|}{\kappa}\right)$$

and rotations around the three-axis commute (see Eq. (2.5.45)).

$$\begin{aligned}
\mathsf{T}\Psi_{p,\sigma} &\overset{II\,F\,15}{=} \sqrt{\frac{\kappa}{p^0}}\zeta_\sigma U\left(R\left(\hat{\mathbf{p}}\right)R_2B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma}\\
&\overset{Eq.\,(2.6.21)}{=} \sqrt{\frac{\kappa}{p^0}}\zeta_\sigma U\left(R\left(-\hat{\mathbf{p}}\right)\right)\\
&\cdot\exp(\pm i\pi J_3)U\left(B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma}\\
&= \sqrt{\frac{\kappa}{p^0}}\zeta_\sigma U\left(R\left(-\hat{\mathbf{p}}\right)\right)\\
&\cdot U\left(B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\exp(\pm i\pi J_3)\Psi_{k,\sigma}\\
&= \zeta_\sigma\exp(\pm i\pi\sigma)\sqrt{\frac{\kappa}{p^0}}U\left(L(\mathcal{P}p)\right)\Psi_{k,\sigma}\\
&= \zeta_\sigma\exp(\pm i\pi\sigma)\Psi_{\mathcal{P}p,\sigma}
\end{aligned}$$

17. ". . . the total angular momentum j of any state of this system would have to be a half-integer,. . . " (P.81)

This is true as all spins and helicities except for one half integer spin/helicity would couple to a an integer angular momentum. And this would then couple with the remaining half integer spin/helicity to a half-integer total angular momentum j.

G. Projective Representations

1. Eq. (2.7.6) (P.83)

From Eq. (2.2.20) together with the modification of Eq. (2.7.1) we have up to $\mathcal{O}\left(\theta^2, \bar{\theta}^2\right)$

$$\begin{aligned}
&\left(1 + i f_{ab}\bar{\theta}^a\theta^b\right)\\
&\cdot\left[1 + i\left(\theta^a + \bar{\theta}^a + f^a{}_{bc}\bar{\theta}^b\theta^c\right)t_a + \frac{1}{2}\left(\theta^b + \bar{\theta}^b\right)\left(\theta^c + \bar{\theta}^c\right)t_{bc}\right]\\
&= \left[1 + i\bar{\theta}^a t_a + \frac{1}{2}\bar{\theta}^b\bar{\theta}^c t_{bc}\right]\cdot\left[1 + i\theta^a t_a + \frac{1}{2}\theta^b\theta^c t_{bc}\right]\\
&= 1 + i\theta^a t_a + \frac{1}{2}\theta^b\theta^c t_{bc} + i\bar{\theta}^a t_a - \bar{\theta}^b t_b\theta^c t_c + \frac{1}{2}\bar{\theta}^b\bar{\theta}^c t_{bc}\\
\Leftrightarrow & i f^a{}_{bc}\bar{\theta}^b\theta^c t_a + \frac{1}{2}\left(\theta^b\bar{\theta}^c + \bar{\theta}^b\theta^c\right)t_{bc} + i f_{ab}\bar{\theta}^a\theta^b = -\bar{\theta}^b\theta^c t_b t_c\\
\Leftrightarrow & \bar{\theta}^b\theta^c\left[t_{bc} + i f^a{}_{bc}t_a + t_b t_c + i f_{bc}\right] = 0\\
\Leftrightarrow & t_{bc} = -i f^a{}_{bc}t_a - t_b t_c - i f_{bc}
\end{aligned}$$

Note that here the order of multiplication is adapted from Eq. (2.2.18) and not from Eq. (2.7.1), which does not matter for the result since θ and $\bar{\theta}$ are dropped in the end. With this we now get the analog of Eq. (2.2.22):

$$\begin{aligned}
& - i f^a{}_{bc}t_a - t_b t_c - i f_{bc}\\
&= t_{bc} = t_{cb} = -i f^a{}_{cb}t_a - t_c t_b - i f_{cb}\\
\Leftrightarrow\ & \left[t_b, t_c\right] = i\left(f^a{}_{cb} - f^a{}_{bc}\right)t_a + i\left(f_{cb} - f_{bc}\right)\mathbf{1}
\end{aligned}$$

2. Eq. (2.7.12) (P.83)

$$\left[\tilde{t}_b, \tilde{t}_c\right] = \left[t_b, t_c\right] = iC^a{}_{bc}t_a + iC^a{}_{bc}\phi_a\mathbf{1} = iC^a{}_{bc}\tilde{t}_a$$

3. Eqs. (2.7.23/24/25) (P.85)

Inserting Eqs. (2.7.14 – 16) into Eq. (2.7.20) we get:

$$\begin{aligned}
0 &\overset{(2.7.20)}{=} \left[J^{\mu\nu}, C^{\rho,\mu}\right]\\
&+ \left[P^\sigma, \eta^{\nu\rho}P^\mu - \eta^{\mu\rho}P^\nu + C^{\rho,\mu\nu}\right]\\
&+ \left[P^\rho, \eta^{\sigma\mu}P^\nu - \eta^{\sigma\nu}P^\mu + C^{\mu\nu,\sigma}\right]\\
&= \eta^{\nu\rho}\left[P^\sigma, P^\mu\right] - \eta^{\mu\rho}\left[P^\sigma, P^\nu\right]\\
&+ \eta^{\sigma\mu}\left[P^\rho, P^\nu\right] - \eta^{\sigma\nu}\left[P^\rho, P^\mu\right]\\
\overset{(2.7.16)}{\Rightarrow} 0 &= \eta^{\nu\rho}C^{\mu,\sigma} - \eta^{\mu\rho}C^{\nu,\sigma}\\
&+ \eta^{\sigma\mu}C^{\nu,\rho} - \eta^{\sigma\nu}C^{\mu,\rho}
\end{aligned}$$

Inserting Eqs. (2.7.13 – 15) into Eq. (2.7.21) we get:

$$\begin{aligned}
0 &\overset{(2.7.21)}{=} \left[J^{\lambda\eta}, \eta^{\nu\rho}P^\mu - \eta^{\mu\rho}P^\nu + C^{\rho,\mu\nu}\right]\\
&+ \left[P^\rho, -\eta^{\nu\lambda}J^{\mu\eta} + \eta^{\mu\lambda}J^{\nu\eta} + \eta^{\eta\mu}J^{\lambda\nu} - \eta^{\eta\nu}J^{\lambda\mu} - C^{\lambda\eta,\mu\nu}\right]\\
&+ \left[J^{\mu\nu}, \eta^{\rho\lambda}P^\eta - \eta^{\rho\eta}P^\lambda + C^{\lambda\eta,\rho}\right]\\
&= \eta^{\nu\rho}\left[J^{\lambda\eta}, P^\mu\right] - \eta^{\mu\rho}\left[J^{\lambda\eta}, P^\nu\right]\\
&- \eta^{\nu\lambda}\left[P^\rho, J^{\mu\eta}\right] + \eta^{\mu\lambda}\left[P^\rho, J^{\nu\eta}\right]\\
&+ \eta^{\eta\mu}\left[P^\rho, J^{\lambda\nu}\right] - \eta^{\eta\nu}\left[P^\rho, J^{\lambda\mu}\right]\\
&+ \eta^{\rho\lambda}\left[J^{\mu\nu}, P^\eta\right] - \eta^{\rho\eta}\left[J^{\mu\nu}, P^\lambda\right]
\end{aligned}$$

Using Eqs. (2.7.14/15) we finally get: TODO

Rest analog TODO

4. Eq. (2.7.26) (P.85)

$$\begin{aligned}
0 &= 4C^{\mu,\sigma} - \delta^\mu_\nu C^{\nu,\sigma}\\
&- \delta^\sigma_\rho C^{\mu,\rho} + \eta^{\sigma\mu}C^{\nu,\rho}\eta_{\nu\rho}\\
&= 2C^{\mu,\sigma} + 0 = 2C^{\mu,\sigma}
\end{aligned}$$

5. Eqs. (2.7.27/28) (P.85)

$$\begin{aligned}
0 &= 4C^{\mu,\lambda\eta} - \delta^\mu_\nu C^{\nu,\lambda\eta} - \eta^{\mu\eta}C^{\rho,\lambda\nu}\eta_{\nu\rho}\\
&+ \eta^{\lambda\mu}C^{\rho,\eta\nu}\eta_{\nu\rho} + \delta^\lambda_\rho C^{\rho,\mu\eta} - \delta^\eta_\rho C^{\rho,\mu\lambda}\\
&+ \delta^\lambda_\nu C^{\eta,\mu\nu} - \delta^\eta_\nu C^{\lambda,\mu\nu}\\
&= 3C^{\mu,\lambda\eta} - \eta^{\mu\eta}C^{\rho,\lambda\nu}\eta_{\nu\rho} + \eta^{\lambda\mu}C^{\rho,\eta\nu}\eta_{\nu\rho}\\
&= 3\left(C^{\mu,\lambda\eta} - \eta^{\mu\eta}C^\lambda + \eta^{\lambda\mu}C^\eta\right)
\end{aligned}$$

6. Eqs. (2.7.29/30) (P.85)

First note

$$C^{\rho\nu,\lambda\eta}\eta_{\nu\rho} = 0$$

from the antisymmetry of $J^{\rho\sigma}$ in Eq. (2.7.13).

$$\begin{aligned}
0 &= 4C^{\mu\sigma,\lambda\eta} - \delta^\mu_\nu C^{\nu\sigma,\lambda\eta} - \eta^{\sigma\mu}C^{\rho\nu,\lambda\eta}\eta_{\nu\rho} + \delta^\sigma_\rho C^{\rho\mu,\lambda\eta}\\
&+ \eta^{\eta\mu}C^{\lambda\nu,\rho\sigma}\eta_{\nu\rho} - \eta^{\lambda\mu}C^{\eta\nu,\rho\sigma}\eta_{\nu\rho} - \delta^\lambda_\rho C^{\mu\eta,\rho\sigma} + \delta^\eta_\rho C^{\mu\lambda,\rho\sigma}\\
&+ \eta^{\sigma\lambda}C^{\rho\eta,\mu\nu}\eta_{\nu\rho} - \delta^\lambda_\nu C^{\sigma\eta,\mu\nu} - \delta^\eta_\nu C^{\lambda\sigma,\mu\nu} + \eta^{\eta\sigma}C^{\lambda\rho,\mu\nu}\eta_{\nu\rho}\\
&= 2C^{\mu\sigma,\lambda\eta} - 2\eta^{\eta\mu}C^{\lambda\sigma} + 2\eta^{\lambda\mu}C^{\eta\sigma}\\
&- 2\eta^{\sigma\lambda}C^{\eta\mu} + 2\eta^{\eta\sigma}C^{\lambda\mu}\\
&= 2\left(C^{\mu\sigma,\lambda\eta} - \eta^{\eta\mu}C^{\lambda\sigma} + \eta^{\lambda\mu}C^{\eta\sigma} - \eta^{\sigma\lambda}C^{\eta\mu} + \eta^{\eta\sigma}C^{\lambda\mu}\right)
\end{aligned}$$

7. Eqs. (2.7.33/34/35) (*P*.86)

$$
\begin{aligned}
&i\left[\tilde{J}^{\mu\nu},\tilde{J}^{\rho\sigma}\right] = i[J^{\mu\nu},J^{\rho\sigma}]\\
&= \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\sigma\mu}J^{\rho\nu} + \eta^{\sigma\nu}J^{\rho\mu} + C^{\rho\sigma,\mu\nu}\\
&\overset{(2.7.29)}{=} \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\sigma\mu}J^{\rho\nu} + \eta^{\sigma\nu}J^{\rho\mu}\\
&+ \eta^{\nu\rho}C^{\mu\sigma} - \eta^{\mu\rho}C^{\nu\sigma} + \eta^{\sigma\mu}C^{\nu\rho} - \eta^{\nu\sigma}C^{\mu\rho}\\
&= \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\sigma\mu}J^{\rho\nu} + \eta^{\sigma\nu}J^{\rho\mu}\\
&+ \eta^{\nu\rho}C^{\mu\sigma} - \eta^{\mu\rho}C^{\nu\sigma} - \eta^{\sigma\mu}C^{\rho\nu} + \eta^{\sigma\nu}C^{\rho\mu}\\
&= \eta^{\nu\rho}\tilde{J}^{\mu\sigma} - \eta^{\mu\rho}\tilde{J}^{\nu\sigma} - \eta^{\sigma\mu}\tilde{J}^{\rho\nu} + \eta^{\sigma\nu}\tilde{J}^{\rho\mu}
\end{aligned}
$$

$$
\begin{aligned}
&i\left[\tilde{J}^{\mu\nu},\tilde{P}^{\rho}\right] = i[J^{\mu\nu},P^{\rho}]\\
&= \eta^{\nu\rho}P^{\mu} - \eta^{\mu\rho}P^{\nu} + C^{\rho,\mu\nu}\\
&\overset{(2.7.27)}{=} \eta^{\nu\rho}P^{\mu} - \eta^{\mu\rho}P^{\nu} + \eta^{\rho\nu}C^{\mu} - \eta^{\rho\mu}C^{\nu}\\
&= \eta^{\nu\rho}\tilde{P}^{\mu} - \eta^{\mu\rho}\tilde{P}^{\nu}
\end{aligned}
$$

$$
i\left[\tilde{P}^{\mu},\tilde{P}^{\rho}\right] = [P^{\mu},P^{\rho}] = 0
$$

8. Eq. (2.7.42) (*P*.87)

This does not set the overall phase completely because

$$
1 = \det(\exp(i\theta)\lambda) = \exp(i2\theta)\det(\lambda) = \exp(i2\theta)
$$

...so for $\theta = \pi$, $\exp(i\theta)\lambda = -\lambda$ satisfies the condition if λ satisfies it.

9. "The group elements depend on $4 - 1 = 3$ complex parameters,..." (*P*.87)

This can easily been seen by calculating the determinant of a general 2×2 complex matrix:

$$
\begin{aligned}
&\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = 1\\
&\overset{d\neq 0}{\Rightarrow} a = \frac{1+bc}{d}\\
&\overset{d=0}{\Rightarrow} b = -\frac{1}{c}
\end{aligned}
$$

10. "...produces a Lorentz transformation $\Lambda(\lambda(\theta))$ which is just a rotation by an angle θ around the three-axis,..." (*P*.87)

$$
\begin{aligned}
&\lambda(\theta)v\lambda^{\dagger}(\theta)\\
&= \begin{pmatrix} \exp\left(i\frac{\theta}{2}\right) & 0 \\ 0 & \exp\left(-i\frac{\theta}{2}\right) \end{pmatrix}
\begin{pmatrix} V^0+V^3 & V^1-iV^2 \\ V^1+iV^2 & V^0-V^3 \end{pmatrix}
\begin{pmatrix} \exp\left(-i\frac{\theta}{2}\right) & 0 \\ 0 & \exp\left(i\frac{\theta}{2}\right) \end{pmatrix}\\
&= \begin{pmatrix} \exp\left(i\frac{\theta}{2}\right) & 0 \\ 0 & \exp\left(-i\frac{\theta}{2}\right) \end{pmatrix}
\begin{pmatrix} \exp\left(-i\frac{\theta}{2}\right)(V^0+V^3) & \exp\left(i\frac{\theta}{2}\right)(V^1-iV^2) \\ \exp\left(-i\frac{\theta}{2}\right)(V^1+iV^2) & \exp\left(i\frac{\theta}{2}\right)(V^0-V^3) \end{pmatrix}\\
&= \begin{pmatrix} V^0+V^3 & \exp(i\theta)(V^1-iV^2) \\ \exp(-i\theta)(V^1+iV^2) & V^0-V^3 \end{pmatrix}
\end{aligned}
$$

From this we can read of:

$$
\begin{aligned}
V'^0 &= V^0\\
V'^1 &= \mathrm{Re}\left\{\exp(i\theta)(V^1-iV^2)\right\} = \mathrm{Re}\left\{(\cos(\theta)+i\sin(\theta))(V^1-iV^2)\right\}\\
&= \cos(\theta)V^1 + \sin(\theta)V^2\\
V'^2 &= \mathrm{Im}\left\{\exp(-i\theta)(V^1+iV^2)\right\} = \mathrm{Im}\left\{(\cos(\theta)-i\sin(\theta))(V^1+iV^2)\right\}\\
&= -\sin(\theta)V^1 + \cos(\theta)V^2
\end{aligned}
$$

11. "...$\det(\exp(h)) = \exp(\mathrm{tr}(h))$ is real and positive" (*P*.88)

This is true because eigenvalues of an hermitian matrix are real, s.t.:

$$
\exp(tr(h)) = \exp\left(\sum_i e_i\right) > 0
$$

12. "... with d, e, f, g subject to the single non-linear constraint $d^2+e^2+f^2+g^2=1$,..." ($P.88$)

$$u^\dagger u = 1$$

and

$$\det(u) = 1$$

yield the same constraint.

13. "because $\exp(h)$ is always positive" ($P.89$)

The eigenvalues of $\exp(h)$ are positive, since

$$\exp(h) = \exp\bigl(u \operatorname{diag}(e_i)\, u^\dagger\bigr) = u \operatorname{diag}(\exp(e_i))\, u^\dagger$$

and $e_i \in \mathbb{R}$.

14. "$\left[U(\Lambda)U(\bar\Lambda)U^{-1}(\Lambda\bar\Lambda)\right]^2 = \mathbf{1}$" ($P.89$)

This follows from the discussion in Appendix B, more precisely see III 6, because a contraction of the double loop to a point is possible.

15. "These two cases correspond to the two irreducible representations of the first homotopy group Z_2" ($P.89$)

These are the trivial representation

$$1 \to 1 \qquad -1 \to 1$$

and the faithful representation

$$1 \to 1 \qquad -1 \to -1.$$

16. "We must not mix states of integer and half-integer spin." ($P.89$)

Because they are different representations of Z_2 or because some of their loops are contractable to a point compare Superselection rule in Section 2.2. TODO

*17. "..., so the factor $\exp(4\pi i\sigma)$ must be **unity**, and hence σ must be an integer or half-integer." ($P.90$)*

This can be seen from the transformation behavior of massless one particle states Eq. (2.5.42) and

$$\begin{aligned}\mathbf{1}\Psi_{p,\sigma} &= \left[U(R(2\pi))\right]^2 \Psi_{p,\sigma} \\ &= \left(\exp(i\sigma 2\pi)\right)^2 \Psi_{p,\sigma} \\ &= \exp(i\sigma 4\pi)\Psi_{p,\sigma}\end{aligned}$$

TODO

H. The Symmetry Representation Theorem

*1. "But $\langle\Psi'_k|\Psi'_k\rangle$ is automatically **real and positive**" ($P.91$)*

This follows immediately from Eq. (2.1.1).

2. "From Eq. (2.A.1) we have $|c_{kk}| = |c_{k1}| = \frac{1}{\sqrt{2}}$ and for $l \neq k$ and $l \neq 1$: $c_{kl} = 0$" ($P.91$)

$$\begin{aligned}|c_{kl'}|^2 &\overset{(2.A.3)}{=} \left|\sum_l c^\star_{kl}\,\langle\Psi'_l|\Psi'_{l'}\rangle\right|^2 = |\langle\Upsilon'_k|\Psi'_{l'}\rangle|^2 \\ &\overset{(2.A.1)}{=} |\langle\Upsilon_k|\Psi_{l'}\rangle|^2 = \begin{cases}\frac{1}{2} & \text{for } l' = 1, k \\ 0 & \text{for } l' \neq 1, k\end{cases}\end{aligned}$$

3. Eq. (2.A.10) ($P.92$)

$$\begin{aligned}|C_k|^2 + |C_1|^2 + 2\operatorname{Re}(C_k C_1^\star) &= |C_k + C_1|^2 \\ &\overset{(2.A.9)}{=} |C'_k + C'_1|^2 = |C'_k|^2 + |C'_1|^2 + 2\operatorname{Re}(C'_k C_1'^\star) \\ &\overset{Eq.\,(2.A.8)}{\Rightarrow} \operatorname{Re}(C_k C_1^\star) = \operatorname{Re}(C'_k C_1'^\star) \\ &\overset{Eq.\,(2.A.8)}{\Rightarrow} \operatorname{Re}\left(\frac{C_k}{C_1}\right) = \operatorname{Re}\left(\frac{C'_k}{C'_1}\right)\end{aligned}$$

4. Eq. (2.A.11) ($P.92$)

$$\begin{aligned}\left\{\operatorname{Re}\left(\frac{C_k}{C_1}\right)\right\}^2 + \left\{\operatorname{Im}\left(\frac{C_k}{C_1}\right)\right\}^2 &= \left|\frac{C_k}{C_1}\right|^2 \\ &\overset{Eq.\,(2.A.8)}{=} \left|\frac{C'_k}{C'_1}\right|^2 = \left\{\operatorname{Re}\left(\frac{C'_k}{C'_1}\right)\right\}^2 + \left\{\operatorname{Im}\left(\frac{C'_k}{C'_1}\right)\right\}^2 \\ &\overset{Eq.\,(2.A.10)}{\Rightarrow} \operatorname{Im}\left(\frac{C_k}{C_1}\right) = \pm\operatorname{Im}\left(\frac{C'_k}{C'_1}\right)\end{aligned}$$

5. "This is only possible if $\operatorname{Re}\left(\frac{C_k}{C_1}\frac{C_l^\star}{C_1^\star}\right) = \operatorname{Re}\left(\frac{C_k}{C_1}\frac{C_l}{C_1}\right)$ or, in other words, if $\operatorname{Im}\left(\frac{C_k}{C_1}\right)\operatorname{Im}\left(\frac{C_l}{C_1}\right) = 0$" ($P.93$)

Define

$$a := \frac{C_k}{C_1}$$
$$b := \frac{C_l}{C_1}$$

With this we have

$$
\begin{aligned}
|1+a+b^\star|^2 &= |1+a+b|^2 && (1)\\
\Leftrightarrow 1+a^\star+b+a+|a|^2+ab+b^\star+b^\star a^\star+|b|^2 & && (2)\\
=1+a^\star+b^\star+a+|a|^2+ab^\star+b+ba^\star+|b|^2 & && (3)\\
\Leftrightarrow ab+b^\star a^\star &= ab^\star+ba^\star && (4)
\end{aligned}
$$

And further rewriting yields

$$
\begin{aligned}
\mathrm{Re}\left(\frac{C_k}{C_1}\frac{C_l}{C_1}\right) &= \mathrm{Re}(ab) \overset{4}{=} \mathrm{Re}(ab^\star) = \mathrm{Re}\left(\frac{C_k}{C_1}\frac{C_l^\star}{C_1^\star}\right)\\
\mathrm{Im}\left(\frac{C_k}{C_1}\right)\mathrm{Im}\left(\frac{C_l}{C_1}\right) &= \mathrm{Im}(a)\,\mathrm{Im}(b)\\
&= -\frac{1}{4}(ab-ab^\star-a^\star b+a^\star b^\star) \overset{4}{=} 0
\end{aligned}
$$

6. "Then the invariance of transition probabilities requires that $\left|\sum_k B_k^\star A_k\right|^2 = \left|\sum_k B_k A_k\right|^2$*"* (*P*.93)

$$
\begin{aligned}
\left|\sum_k B_k^\star A_k\right|^2 &\overset{Eq.\,(2.A.2)}{=} \left|\sum_{kl} B_k^\star A_l \langle\Psi_k|\Psi_l\rangle\right|^2\\
&= \left|\left\langle \sum_k B_k\Psi_k \middle| \sum_l A_l\Psi_l\right\rangle\right|^2\\
&\overset{Eq.\,(2.A.1)}{=} \left|\left\langle U\left(\sum_k B_k\Psi_k\right) \middle| U\left(\sum_l A_l\Psi_l\right)\right\rangle\right|^2\\
&= \left|\left\langle \sum_k B_k^\star U\Psi_k \middle| \sum_l A_l U\Psi_l\right\rangle\right|^2\\
&= \left|\sum_{kl} B_k A_l \langle U\Psi_k|U\Psi_l\rangle\right|^2\\
&\overset{Eq.\,(2.A.3)}{=} \left|\sum_k B_k A_k\right|^2
\end{aligned}
$$

7. Eq. (2.*A*.16)(*P*.94)

$$
\begin{aligned}
&\sum_{kl}\mathrm{Im}(B_k^\star B_l)\,\mathrm{Im}(A_k^\star A_l)\\
&= \mathrm{Im}\left(\sum_{kl}\mathrm{Im}(B_k^\star B_l)A_k^\star A_l\right)\\
&= \frac{1}{2i}\left[\sum_{kl}\mathrm{Im}(B_k^\star B_l)A_k^\star A_l - \text{c.c.}\right]\\
&= \frac{1}{2i}\left[\sum_{kl}\frac{1}{2i}(B_k^\star B_l - B_k B_l^\star)A_k^\star A_l - \text{c.c.}\right]\\
&= \frac{1}{2i}\left[\frac{1}{2i}\left(\sum_{kl}B_k^\star B_l A_k^\star A_l - \sum_{kl}B_k B_l^\star A_k^\star A_l\right) - \text{c.c.}\right]\\
&= \frac{1}{2i}\left[\frac{1}{2i}\left(\left|\sum_k B_k A_k\right|^2 - \left|\sum_k B_k^\star A_k\right|^2\right) - \text{c.c.}\right]\\
&= \frac{1}{2i}\left[\frac{1}{i}\left(\left|\sum_k B_k A_k\right|^2 - \left|\sum_k B_k^\star A_k\right|^2\right)\right]\\
&\overset{II\,\underline{H}\,6}{=} 0
\end{aligned}
$$

8. "However, for any pair of such state-vectors, with neither A_k *nor* B_k ***all of the same phase****"* (*P*.94)

If they were all of the same phase then

$$\forall k,l : \mathrm{Im}\{A_k^\star A_l\} = 0$$

or

$$\forall k,l : \mathrm{Im}\{B_k^\star B_l\} = 0$$

See Footnote j, for why this is relevant.

9. "We have thus shown that for a given symmetry transformation T either all state-vectors satisfy Eq. (2.*A*.14) *or else they all satisfy Eq.* (2.*A*.15)*"* (*P*.94)

Preceding this a contradiction was derived from the assumption that Eq. (2.*A*.14) applies for a state-vector

$$\sum_k A_k\Psi_k$$

while Eq. (2.*A*.15) applies for a state vector

$$\sum_k B_k\Psi_k.$$

Such that the statement is obvious.

I. Group Operators and Homotopy Classes

1. *Eq.* (2.*B*.7) (*P*.97)

Taylor evolving Eq. (2.*B*.6) in θ_3^c up to $\mathcal{O}(\theta_3^2)$ yields:

$$
\begin{aligned}
& f^a(\theta_2,\theta_1) + \left[h^{-1}\right]^a{}_c\left(f(\theta_2,\theta_1)\right)\theta_3^c \\
&= f^a(0,f(\theta_2,\theta_1)) + \left[\frac{\partial f^a(\bar\theta, f(\theta_2,\theta_1))}{\partial\bar\theta^c}\right]_{\bar\theta=0}\theta_3^c \\
&= f^a(\theta_3, f(\theta_2,\theta_1)) \\
&\overset{(2.B.6)}{=} f^a(f(\theta_3,\theta_2),\theta_1) \\
&= f^a\left(f(0,\theta_2) + \left[\frac{\partial f^a(\bar\theta,\theta_2)}{\partial\bar\theta^c}\right]_{\bar\theta=0}\theta_3^c,\theta_1\right) \\
&= f^a(\theta_2,\theta_1) + \left[\frac{\partial f^a(\bar\theta,\theta_1)}{\partial\bar\theta^b}\right]_{\bar\theta=\theta_2}\left[\frac{\partial f^b(\bar\theta,\theta_2)}{\partial\bar\theta^c}\right]_{\bar\theta=0}\theta_3^c \\
&= f^a(\theta_2,\theta_1) + \left[\frac{\partial f^a(\bar\theta,\theta_1)}{\partial\bar\theta^b}\right]_{\bar\theta=\theta_2}\left[h^{-1}\right]^b{}_c(\theta_2)\,\theta_3^c
\end{aligned}
$$

Equating coefficients of θ_3^c, we get:

$$
\begin{aligned}
\left[h^{-1}\right]^a{}_c\left(f(\theta_2,\theta_1)\right) &= \left[\frac{\partial f^a(\bar\theta,\theta_1)}{\partial\bar\theta^b}\right]_{\bar\theta=\theta_2}\left[h^{-1}\right]^b{}_c(\theta_2) \\
\Leftrightarrow h^c{}_b(\theta_2) &= \left[\frac{\partial f^a(\bar\theta,\theta_1)}{\partial\bar\theta^b}\right]_{\bar\theta=\theta_2} h^c{}_a\left(f(\theta_2,\theta_1)\right)
\end{aligned}
$$

2. *"Along the second segment the differential equation Eq.* (2.*B*.2) *for* $U_{\mathcal{P}}(s)$ *is thus the same as the differential equation for* $U_{\theta_2}(2s-1)$*."* (*P*.97)

In the second segment $\left(\frac{1}{2}\leq s\leq 1\right)$ the differential equation for $U_{\mathcal{P}}(s)$ reads:

$$
\begin{aligned}
& \frac{\mathrm{d}U_{\mathcal{P}}(s)}{\mathrm{d}s} \\
&= it_cU_{\mathcal{P}}(s)h^c{}_a\left(\Theta_{\mathcal{P}}(s)\right)\frac{\mathrm{d}\Theta_{\mathcal{P}}^c(s)}{\mathrm{d}s} \\
&= it_cU_{\mathcal{P}}(s)h^c{}_a\left(f\left(\Theta_{\theta_2}(2s-1),\theta_1\right)\right)\frac{\mathrm{d}f^c\left(\Theta_{\theta_2}(2s-1),\theta_1\right)}{\mathrm{d}s} \\
&= it_cU_{\mathcal{P}}(s)h^c{}_a\left(f\left(\Theta_{\theta_2}(2s-1),\theta_1\right)\right)\left[\frac{\partial f^a(\bar\theta,\theta_1)}{\partial\bar\theta^b}\right]_{\bar\theta=\Theta_{\theta_2}(2s-1)} \\
&\cdot\frac{\mathrm{d}\Theta_{\theta_2}^b(2s-1)}{\mathrm{d}s} \\
&\overset{(2.B.7)}{=} it_cU_{\mathcal{P}}(s)h^c{}_b\left(\Theta_{\theta_2}(2s-1)\right)\frac{\mathrm{d}\Theta_{\theta_2}^b(2s-1)}{\mathrm{d}s}
\end{aligned}
$$

Which is just the differential equation for $U_{\theta_2}(2s-1)$:

$$
\begin{aligned}
\frac{\mathrm{d}U_{\theta_2}(2s-1)}{\mathrm{d}s} &= \left[\frac{\mathrm{d}U_{\theta_2}(t)}{\mathrm{d}t}\right]_{t=2s-1} 2 \\
&= it_cU_{\theta_2}(2s-1)h^c{}_b\left(\Theta_{\theta_2}(2s-1)\right)\left[\frac{\mathrm{d}\Theta_{\theta_2}^b(t)}{\mathrm{d}t}\right]_{t=2s-1} 2 \\
&= it_cU_{\theta_2}(2s-1)h^c{}_b\left(\Theta_{\theta_2}(2s-1)\right)\frac{\mathrm{d}\Theta_{\theta_2}^b(2s-1)}{\mathrm{d}s}
\end{aligned}
$$

3. *Eq.* (2.*B*.9) (*P*.98)

First note

$$
\begin{aligned}
\frac{\mathrm{d}U^{-1}}{\mathrm{d}s} &= \left(\frac{\mathrm{d}U}{\mathrm{d}s}\right)^\dagger \overset{(2.B.2)}{=} -i\left(t_aU\right)^\dagger h^a{}_b\frac{\mathrm{d}\Theta^b}{\mathrm{d}s} && (5) \\
&= -iU^{-1}t_ah^a{}_b\frac{\mathrm{d}\Theta^b}{\mathrm{d}s}, && (6)
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}s}\left(U^{-1}t_aUh^a{}_b\right) && (7) \\
&= \frac{\mathrm{d}U^{-1}}{\mathrm{d}s}t_eUh^e{}_b + U^{-1}t_e\frac{\mathrm{d}U}{\mathrm{d}s}h^e{}_b + U^{-1}t_aU\frac{\mathrm{d}h^a{}_b}{\mathrm{d}s} && (8) \\
&\overset{(2.B.2)}{=} -iU^{-1}t_dh^d{}_c\frac{\mathrm{d}\Theta^c}{\mathrm{d}s}t_eUh^e{}_b && (9) \\
&+ U^{-1}t_eit_dUh^d{}_c\frac{\mathrm{d}\Theta^c}{\mathrm{d}s}h^e{}_b && (10) \\
&+ U^{-1}t_aUh^a{}_{b,c}\frac{\mathrm{d}\Theta^c}{\mathrm{d}s} && (11) \\
&= i\frac{\mathrm{d}\Theta^c}{\mathrm{d}s}h^d{}_ch^e{}_bU^{-1}[t_e,t_d]U + U^{-1}t_aU\frac{\mathrm{d}\Theta^c}{\mathrm{d}s}h^a{}_{b,c} && (12) \\
&\overset{(2.2.22)}{=} U^{-1}t_aU\frac{\mathrm{d}\Theta^c}{\mathrm{d}s}\left(ih^d{}_ch^e{}_biC^a{}_{ed} + h^a{}_{b,c}\right). && (13)
\end{aligned}
$$

By using Eq. (2.2.22) we are making use of condition (a) of the Theorem. With this we get:

$$
\begin{aligned}
&\frac{\mathrm{d}U^{-1}\delta U}{\mathrm{d}s} \\
&= \frac{\mathrm{d}U^{-1}}{\mathrm{d}s}\delta U + U^{-1}\frac{\mathrm{d}\delta U}{\mathrm{d}s} \\
&\overset{6}{=} -iU^{-1}t_a h^a{}_b \frac{\mathrm{d}\Theta^b}{\mathrm{d}s}\delta U + U^{-1}\frac{\mathrm{d}\delta U}{\mathrm{d}s} \\
&= iU^{-1}t_a U h^a{}_{c,b}\left(\delta\Theta^b\right)\frac{\mathrm{d}\Theta^c}{\mathrm{d}s} \\
&+ iU^{-1}t_a U h^a{}_b \frac{\mathrm{d}\delta\Theta^b}{\mathrm{d}s} \\
&= iU^{-1}t_a U\left(\delta\Theta^b\right)\frac{\mathrm{d}\Theta^c}{\mathrm{d}s} h^a{}_{c,b} \\
&+ \frac{\mathrm{d}}{\mathrm{d}s}\left(iU^{-1}t_a U h^a{}_b \delta\Theta^b\right) - i\left(\delta\Theta^b\right)\frac{\mathrm{d}}{\mathrm{d}s}\left(U^{-1}t_a U h^a{}_b\right) \\
&\overset{13}{=} iU^{-1}t_a U\left(\delta\Theta^b\right)\frac{\mathrm{d}\Theta^c}{\mathrm{d}s}\left(h^a{}_{c,b} - \left(-h^d{}_c h^e{}_b C^a{}_{ed} + h^a{}_{b,c}\right)\right) \\
&+ \frac{\mathrm{d}}{\mathrm{d}s}\left(iU^{-1}t_a U h^a{}_b \delta\Theta^b\right) \\
&= iU^{-1}t_a U\left(\delta\Theta^b\right)\frac{\mathrm{d}\Theta^c}{\mathrm{d}s}\left(h^a{}_{c,b} + h^d{}_c h^e{}_b C^a{}_{ed} - h^a{}_{b,c}\right) \\
&+ \frac{\mathrm{d}}{\mathrm{d}s}\left(iU^{-1}t_a U h^a{}_b \delta\Theta^b\right)
\end{aligned}
$$

Where we inserted the expression for $\frac{\mathrm{d}\delta U}{\mathrm{d}s}$ in the third step.

4. Eq. (2.B.10) (P.98)

Taylor evolving Eq. (2.B.6) in θ_3, θ_2 up to $\mathcal{O}\left(\theta_3^3, \theta_2^3\right)$ yields:

$$
\begin{aligned}
&f^a(0,\theta_1) + \left[h^{-1}\right]^a{}_b(\theta_1)\left(\theta_3^b + \theta_2^b + f^b{}_{ec}\theta_3^e\theta_2^c\right) \\
&= f^a(0,\theta_1) + \left[\frac{\partial f^a(\bar\theta,\theta_1)}{\partial\bar\theta^b}\right]_{\bar\theta=0}\left(\theta_3^b + \theta_2^b + f^b{}_{ec}\theta_3^e\theta_2^c\right) \\
&= f^a\left(\theta_3 + \theta_2 + f^{\cdot}{}_{ec}\theta_3^e\theta_2^c, \theta_1\right) \\
&\overset{(2.2.19)}{=} f^a(f(\theta_3,\theta_2),\theta_1) \\
&\overset{(2.B.6)}{=} f^a(\theta_3, f(\theta_2,\theta_1)) \\
&= f^a\left(\theta_3, f(0,\theta_1) + \left[\frac{\partial f^a(\bar\theta,\theta_1)}{\partial\bar\theta^e}\right]_{\bar\theta=0}\theta_2^e\right) \\
&= f^a\left(\theta_3, \theta_1 + \left[h^{-1}\right]^{\cdot}{}_e(\theta_1)\theta_2^e\right) \\
&= f^a(0,\theta_1) + \left[\frac{\partial f^a(\bar\theta,\theta_1)}{\partial\bar\theta^b}\right]_{\bar\theta=0}\theta_3^b \\
&+ \left[\frac{\partial f^a(0,\bar\theta)}{\partial\bar\theta^b}\right]_{\bar\theta=\theta_1}\left[h^{-1}\right]^b{}_e(\theta_1)\theta_2^e \\
&+ \frac{1}{2}\left[\frac{\partial}{\partial\tilde\theta^c}\left[\frac{\partial f^a(\bar\theta,\tilde\theta)}{\partial\bar\theta^b}\right]_{\bar\theta=0}\right]_{\tilde\theta=\theta_1}\theta_3^b\left[h^{-1}\right]^c{}_e(\theta_1)\theta_2^e \\
&+ \frac{1}{2}\left[\frac{\partial}{\partial\bar\theta^b}\left[\frac{\partial f^a(\bar\theta,\tilde\theta)}{\partial\tilde\theta^c}\right]_{\tilde\theta=\theta_1}\right]_{\bar\theta=0}\theta_3^b\left[h^{-1}\right]^c{}_e(\theta_1)\theta_2^e \\
&= f^a(0,\theta_1) + \left[h^{-1}\right]^a{}_b(\theta_1)\theta_3^b + \left[h^{-1}\right]^a{}_e(\theta_1)\theta_2^e \\
&+ \frac{\partial}{\partial\theta_1^c}\left(\left[h^{-1}\right]^a{}_b(\theta_1)\right)\theta_3^b\left[h^{-1}\right]^c{}_e(\theta_1)\theta_2^e
\end{aligned}
$$

No θ_3^2, θ_2^2 terms show up and

$$
\left[\frac{\partial f^a(0,\bar\theta)}{\partial\bar\theta^b}\right]_{\bar\theta=\theta_1} = \delta^a_b,
$$

see Eq. (2.2.19). Equating coefficients of $\theta_3^b\theta_2^c$, we get:

$$
\begin{aligned}
&\left[h^{-1}\right]^a{}_e(\theta_1) f^e{}_{bc} \\
&= \frac{\partial}{\partial\theta_1^d}\left(\left[h^{-1}\right]^a{}_b(\theta_1)\right)\left[h^{-1}\right]^d{}_c(\theta_1) \\
\Leftrightarrow\ f^e{}_{bc} &= h^e{}_a(\theta_1)\frac{\partial}{\partial\theta_1^d}\left(\left[h^{-1}\right]^a{}_b(\theta_1)\right)\left[h^{-1}\right]^d{}_c(\theta_1) \\
&= \left(0 - h^e{}_{a,d}\left[h^{-1}\right]^a{}_b\right)\left[h^{-1}\right]^d{}_c \\
\Leftrightarrow\ h^e{}_{a,d} &= -f^e{}_{bc}h^b{}_a h^c{}_d
\end{aligned}
$$

5. *Eq.* (2.*B*.11) (*P*.98)

$$\begin{aligned} {h^a}_{c,b} - {h^a}_{b,c} &\overset{(2.B.10)}{=} -{f^a}_{de}{h^d}_c{h^e}_b + {f^a}_{de}{h^d}_b{h^e}_c \\ &= {h^d}_c{h^e}_b \left(-{f^a}_{de} + {f^a}_{ed}\right) \\ &\overset{(2.2.23)}{=} {h^d}_c{h^e}_b \left(-{C^a}_{ed}\right) \end{aligned}$$

6. *"It follows that $U_\theta(1)$ is stationary under any infinitesimal variation of the path that leaves the endpoints $\Theta(0) = 0$ and $\Theta(1) = \theta$ (and $U_\theta(0) = \mathbf{1}$) fixed."* (*P*.98)

We have

$$U^{-1}\delta U - iU^{-1}t_a U {h^a}_b \delta\Theta^b = C = \text{const}$$

For $s = 0$ this gives

$$0 = \delta U(0) = it_a U(0){h^a}_b(0)\underbrace{\delta\Theta^b(0)}_{=0} + U(0)C = \mathbf{1}C$$

such that $C = 0$. For $s = 1$ we then get

$$\delta U(1) = it_a U(1){h^a}_b(1)\underbrace{\delta\Theta^b(1)}_{=0} + U(1)\underbrace{C}_{=0} = 0.$$

7. *"$0 = \frac{\partial}{\partial\theta^b}\left\{U[\theta]^{-1}\tilde{U}[\theta]\right\} + i\phi_b(\theta)U[\theta]^{-1}\tilde{U}[\theta]$ where $\phi_b(\theta) = {h^a}_b\left[\frac{\partial\phi(\theta',\theta)}{\partial\theta'^a}\right]_{\theta'=0}$"* (*P*.99)

By taking the derivative of the previous equation w.r.t. θ'^a, we obtain:

$$\begin{aligned} 0 &= U[\theta]^{-1}(-t_a + t_a)\tilde{U}[\theta] \\ &= \left[\frac{\partial}{\partial\theta'^a}U[\theta]^{-1}U[\theta']^{-1}\tilde{U}[\theta']\tilde{U}[\theta]\right]_{\theta'=0} \\ &= \left[\frac{\partial}{\partial\theta'^a}U[f(\theta',\theta)]^{-1}\tilde{U}[f(\theta',\theta)]\exp(i\phi(\theta',\theta))\right]_{\theta'=0} \\ &= \left[\frac{\partial}{\partial\bar{\theta}^b}U[\bar{\theta}]^{-1}\tilde{U}[\bar{\theta}]\right]_{\bar{\theta}=f(0,\theta)=\theta}\left[\frac{\partial f^b(\theta',\theta)}{\partial\theta'^a}\right]_{\theta'=0}\exp(i\phi(0,\theta)) \\ &+ U[\theta]^{-1}\tilde{U}[\theta]i\left[\frac{\partial\phi(\theta',\theta)}{\partial\theta'^a}\right]_{\theta'=0}\exp(i\phi(0,\theta)) \\ &= \frac{\partial}{\partial\theta^b}\left\{U[\theta]^{-1}\tilde{U}[\theta]\right\}{\left[h^{-1}\right]^b}_a(\theta) \\ &+ U[\theta]^{-1}\tilde{U}[\theta]i\left[\frac{\partial\phi(\theta',\theta)}{\partial\theta'^a}\right]_{\theta'=0} \end{aligned}$$

Multiplying by ${h^a}_b(\theta)$ yields finally

$$0 = \frac{\partial}{\partial\theta^b}\left\{U[\theta]^{-1}\tilde{U}[\theta]\right\} + i\phi_b(\theta)U[\theta]^{-1}\tilde{U}[\theta].$$

8. *"$0 = \frac{\partial\phi_b(\theta)}{\partial\theta^c} - \frac{\partial\phi_c(\theta)}{\partial\theta^b}$"* (*P*.99)

Differentiating the result of III7 w.r.t θ^c yields

$$\begin{aligned} 0 &= \frac{\partial^2}{\partial\theta^c\partial\theta^b}\left\{U[\theta]^{-1}\tilde{U}[\theta]\right\} + i\frac{\partial\phi_b(\theta)}{\partial\theta^c}U[\theta]^{-1}\tilde{U}[\theta] \\ &+ i\phi_b(\theta)\frac{\partial}{\partial\theta^c}\left\{U[\theta]^{-1}\tilde{U}[\theta]\right\} \\ &\overset{III7}{=} \frac{\partial^2}{\partial\theta^c\partial\theta^b}\left\{U[\theta]^{-1}\tilde{U}[\theta]\right\} + i\frac{\partial\phi_b(\theta)}{\partial\theta^c}U[\theta]^{-1}\tilde{U}[\theta] \\ &+ \phi_b(\theta)\phi_c(\theta)U[\theta]^{-1}\tilde{U}[\theta]. \end{aligned}$$

Antisymmetrizing then gives

$$0 = \frac{\partial\phi_b(\theta)}{\partial\theta^c} - \frac{\partial\phi_c(\theta)}{\partial\theta^b}.$$

9. *"Then $U^{-1}(f(\theta_2,\theta_1))U(\theta_2)U(\theta_1)$ can be a phase factor $\exp(i\phi(\theta_2,\theta_1)) \neq 1$, but ϕ will be the same for all other loops into which this can be continuously deformed."* (*P*.99)

This can be seen from the statement of III6 by setting $\theta = 0$.

J. Inversions and Degenerate Multiplets

1. *"... the corresponding proportionality factor for T^2 can only be ±1,..."* (*P*.100)

Suppose

$$\mathsf{T}^2 = \varphi\mathbf{1}$$

then we have

$$\mathsf{T}^3 = \mathsf{T}^2\mathsf{T} = \varphi\mathsf{T}$$

but also

$$\mathsf{T}^3 = \mathsf{T}\mathsf{T}^2 = \mathsf{T}\varphi = \varphi^\star\mathsf{T}$$

such that

$$\varphi^\star = \varphi = \pm1.$$

2. *"... because T is anti unitary, $\mathcal{T}$ must be unitary"* (*P*.101)

Using basic orthonormality properties we see

$$
\begin{aligned}
&\delta_{\sigma'\sigma}\delta^{(3)}(\mathbf{p}'-\mathbf{p})\delta_{ni}\\
&\overset{(2.5.19)}{=} \langle \Psi_{\mathbf{p}',\sigma',n}|\Psi_{\mathbf{p},\sigma,i}\rangle^\star\\
&= \langle \mathsf{T}\Psi_{\mathbf{p}',\sigma',n}|\mathsf{T}\Psi_{\mathbf{p},\sigma,i}\rangle\\
&\overset{(2.C.1)}{=} \left\langle (-1)^{j'-\sigma'}\sum_m \mathcal{T}_{mn}\Psi_{-\mathbf{p}',-\sigma',m}\middle|(-1)^{j-\sigma}\sum_l \mathcal{T}_{li}\Psi_{-\mathbf{p},-\sigma,l}\right\rangle\\
&= (-1)^{j'+j-\sigma'-\sigma}\sum_{m,l}\mathcal{T}^\star_{mn}\mathcal{T}_{li}\langle \Psi_{-\mathbf{p}',-\sigma',m}|\Psi_{-\mathbf{p},-\sigma,l}\rangle\\
&\overset{(2.5.19)}{=} (-1)^{2(j-\sigma)}\sum_m \mathcal{T}^\star_{mn}\mathcal{T}_{mi}\delta_{\sigma'\sigma}\delta^{(3)}(\mathbf{p}'-\mathbf{p})\\
&= \delta_{\sigma'\sigma}\delta^{(3)}(\mathbf{p}'-\mathbf{p})\sum_m \left(\mathcal{T}^\dagger\right)_{nm}\mathcal{T}_{mi}
\end{aligned}
$$

such that

$$\delta_{ni} = \sum_m \left(\mathcal{T}^\dagger\right)_{nm}\mathcal{T}_{mi}.$$

3. Eq. (2.C.2) (P.101)

$$
\begin{aligned}
&\mathsf{T}\Psi'_{\mathbf{p},\sigma,n}\\
&= \sum_m \mathsf{T}\mathcal{U}_{mn}\Psi_{\mathbf{p},\sigma,m}\\
&= \sum_m \mathcal{U}^\star_{mn}\mathsf{T}\Psi_{\mathbf{p},\sigma,m}\\
&\overset{(2.C.1)}{=} \sum_{m,k}\mathcal{U}^\star_{mn}(-1)^{j-\sigma}\mathcal{T}_{km}\Psi_{-\mathbf{p},-\sigma,k}\\
&= \sum_{m,k,l}\mathcal{U}^\star_{mn}(-1)^{j-\sigma}\mathcal{T}_{km}\left(\mathcal{U}^{-1}\right)_{lk}\Psi'_{-\mathbf{p},-\sigma,l}\\
&= (-1)^{j-\sigma}\sum_l \left(\mathcal{U}^{-1}\mathcal{T}\mathcal{U}^\star\right)_{ln}\Psi'_{-\mathbf{p},-\sigma,l}
\end{aligned}
$$

*4. "... **unitary** matrix $\mathcal{T}\mathcal{T}^\star$." (P.101)*

$$
\begin{aligned}
\left(\mathcal{T}\mathcal{T}^\star\right)^\dagger\mathcal{T}\mathcal{T}^\star &= \left(\mathcal{T}^\dagger\right)^\star\mathcal{T}^\dagger\mathcal{T}\mathcal{T}^\star\\
&= \left(\mathcal{T}^\dagger\right)^\star\mathcal{T}^\star\\
&= \left(\mathcal{T}^\dagger\mathcal{T}\right)^\star\\
&= \mathbf{1}^\star = \mathbf{1}
\end{aligned}
$$

5. Eq. (2.C.4) (P.101)

We have

$$\mathcal{T}\mathcal{T}^\star = D$$

multiplying by $\mathcal{T}^\top = \left(\mathcal{T}^\dagger\right)^\star$ from the right we get

$$\mathcal{T} = \mathcal{T}\left(\mathcal{T}\mathcal{T}^\dagger\right)^\star = \mathcal{T}\mathcal{T}^\star\mathcal{T}^\top = D\mathcal{T}^\top.$$

6. "... the diagonal element $\mathcal{T}_{nn}$ vanishes unless $\exp(i\phi_n) = 1$. Furthermore, if $\exp(i\phi_n) = 1$ but $\exp(i\phi_m) \neq 1$, then Eq. (2.C4) tells us that $\mathcal{T}_{nm} = \mathcal{T}_{mn} = 0$." (P.101)

From Eq. (2.C.4) we have

$$\mathcal{T}_{nm} = \exp(i\phi_n)\mathcal{T}_{mn}.$$

From this we get

$$
\begin{aligned}
\mathcal{T}_{nn} &= \exp(i\phi_n)\mathcal{T}_{nn}\\
\Rightarrow 0 &= \left(1-\exp(i\phi_n)\right)\mathcal{T}_{nn}
\end{aligned}
$$

such that for $\exp(i\phi_n) \neq 1$ we have

$$\mathcal{T}_{nn} = 0.$$

Furthermore, if $\exp(i\phi_n) = 1$ but $\exp(i\phi_m) \neq 1$, then we have

$$
\begin{aligned}
\mathcal{T}_{mn} &= \exp(i\phi_m)\mathcal{T}_{nm}\\
&= \exp(i\phi_m)\exp(i\phi_n)\mathcal{T}_{mn}\\
&= \underbrace{\exp(i\phi_m)}_{\neq 1}\mathcal{T}_{mn}\\
\Rightarrow \mathcal{T}_{mn} &= \mathcal{T}_{nm} = 0
\end{aligned}
$$

7. "... $\mathcal{A}$ is symmetric as well as unitary, ..." (P.101)

Unitarity follows directly from the unitarity of $\mathcal{T}$and symmetry can be seen from

$$\mathcal{T}_{nm} = \exp(i\phi_n)\mathcal{T}_{mn}$$

because $\mathcal{A}$ only contains rows and columns for which $\exp(i\phi_n) = 1$.

8. "Because $\mathcal{A}$ is symmetric, it can be expressed as the exponential of a symmetric anti-Hermitian matrix, so it can be diagonalized by a transformation Eq. (2.C.2) acting on $\mathcal{A}$, with the corresponding submatrix of $\mathcal{U}$ real and hence orthogonal." (P.101)

We know that $\mathcal{A}$ is unitary and symmetric, i.e.

$$\mathcal{A}^\dagger = \mathcal{A}^{-1} \qquad \mathcal{A}^\top = \mathcal{A}.$$

Suppose $\mathcal{A}$ can be written as

$$\mathcal{A} = \exp(a)$$

where a is a symmetric anti-Hermitian matrix, i.e.

$$a^\dagger = -a \qquad a^\top = a.$$

We can easily check that this satisfies all properties imposed on $\mathcal{A}$:

$$\begin{aligned}\mathcal{A}^\dagger \mathcal{A} &= \exp\bigl(a^\dagger\bigr)\exp(a)\\ &= \exp(-a)\exp(a)\\ &= \exp(-a+a) = \mathbf{1}\\ \mathcal{A}^\top &= \exp\bigl(a^\top\bigr) = \exp(a) = \mathcal{A}\end{aligned}$$

Any symmetric anti-Hermitian matrix a can be written in terms of a symmetric Hermitian matrix h as

$$a = ih,$$

such that

$$\mathcal{A} = \exp(a) = \exp(ih).$$

Observe that h is real, since

$$h^\star = \bigl(h^\top\bigr)^\dagger = h^\dagger = h.$$

Since h is real and by definition symmetric, it can be diagonalized by an orthogonal matrix:

$$h = O^{-1}DO$$

Inserting this into the expression for $\mathcal{A}$ we see that O also diagonalizes $\mathcal{A}$:

$$\begin{aligned}\mathcal{A} &= \exp(ih) = \exp\bigl(iO^{-1}DO\bigr)\\ &= \exp\bigl(O^{-1}iDO\bigr) = O^{-1}\exp(iD)O\end{aligned}$$

Since O is orthogonal and in particular real it can be set as a submatrix of $\mathcal{U}$ in the transformation Eq. (2.C.2).

9. Eq. (2.C.7) (P.102)

For components of $\mathcal{T}$ in the same block $\mathcal{B}_i$ we have

$$\exp(i\phi_m) = \exp(-i\phi_n).$$

Such that pulling a factor of $\exp\left(-i\frac{\phi_n}{2}\right)$ out of $\mathcal{T}_{mn}$ we can write

$$\begin{aligned}\mathcal{T}_{mn} &= \exp\left(-i\frac{\phi_n}{2}\right)z\\ \Rightarrow \mathcal{T}_{nm} &= \exp(i\phi_n)\exp\left(-i\frac{\phi_n}{2}\right)z\\ &= \exp\left(i\frac{\phi_n}{2}\right)z\end{aligned}$$

with z some complex number specific to the combination of indices m, n.

10. "…$\mathcal{C}_i\mathcal{C}_i^\dagger = \mathcal{C}_i^\dagger\mathcal{C}_i = \mathbf{1}$, and hence $\mathcal{C}_i$ is square and unitary" (P.102)

The Unitarity of $\mathcal{T}$ implies the Unitarity of $\mathcal{B}$ which in turn implies the Unitarity of each $\mathcal{B}_i$. This imposes the following condition:

$$\begin{aligned}\mathcal{B}_i^\dagger\mathcal{B}_i &= \begin{pmatrix} 0 & \exp\left(i\frac{\phi_n}{2}\right)\mathcal{C}_i^\star \\ \exp\left(-i\frac{\phi_n}{2}\right)\mathcal{C}_i^\dagger & 0\end{pmatrix}\begin{pmatrix} 0 & \exp\left(i\frac{\phi_n}{2}\right)\mathcal{C}_i \\ \exp\left(-i\frac{\phi_n}{2}\right)\mathcal{C}_i^\top & 0\end{pmatrix}\\ &= \begin{pmatrix}\mathcal{C}_i^\star\mathcal{C}_i^\top & 0\\ 0 & \mathcal{C}_i^\dagger\mathcal{C}_i\end{pmatrix} = \begin{pmatrix}\left(\mathcal{C}_i\mathcal{C}_i^\dagger\right)^\top & 0\\ 0 & \mathcal{C}_i^\dagger\mathcal{C}_i\end{pmatrix} \stackrel{!}{=} \mathbf{1}\\ \Rightarrow \mathcal{C}_i^\dagger\mathcal{C}_i &= \mathbf{1} = \mathbf{1}^\top = \mathcal{C}_i\mathcal{C}_i^\dagger\end{aligned}$$

11. "…$\exp\left(\pm i\frac{\phi}{2}\right)c_\pm^\star = |\lambda|^2 c_\pm^\star\exp\left(\mp i\frac{\phi}{2}\right)$, which is impossible unless either $c_+ = c_- = 0$ or $\exp(i\phi)$ is unity,…" (P.103)

We have

$$\begin{aligned}\exp\left(i\frac{\phi}{2}\right)c_+^\star = \lambda c_- &\Rightarrow c_+ = \lambda^\star c_-^\star\exp\left(i\frac{\phi}{2}\right)\\ \exp\left(-i\frac{\phi}{2}\right)c_-^\star = \lambda c_+ &\Rightarrow c_- = \lambda^\star c_+^\star\exp\left(-i\frac{\phi}{2}\right)\end{aligned}$$

such that

$$\begin{aligned}\exp\left(\pm i\frac{\phi}{2}\right)c_\pm^\star &= \lambda c_\mp\\ &= |\lambda|^2 c_\pm^\star\exp\left(\mp i\frac{\phi}{2}\right)\end{aligned}$$

which is equivalent to

$$\exp(\pm i\phi)c_\pm^\star = |\lambda|^2 c_\pm^\star.$$

This is only possible if either $c_+ = c_- = 0$ or $\exp(i\phi)$ is unity.

12. Eq. (2.C.16) (P.104)

From Eq. (2.C.8) we have

$$\begin{aligned}\Psi_{\mathbf{p},\sigma,\pm} &= \mathsf{T}^{-1}\mathsf{T}\Psi_{\mathbf{p},\sigma,\pm}\\ &= \exp\left(\mp i\frac{\phi}{2}\right)(-1)^{j-\sigma}\mathsf{T}^{-1}\Psi_{-\mathbf{p},-\sigma,\mp}\\ \Rightarrow \mathsf{T}^{-1}\Psi_{-\mathbf{p},-\sigma,\mp} &= \exp\left(\pm i\frac{\phi}{2}\right)(-1)^{-j+\sigma}\Psi_{\mathbf{p},\sigma,\pm}\\ \Rightarrow \mathsf{T}^{-1}\Psi_{\mathbf{p},\sigma,\pm} &= \exp\left(\mp i\frac{\phi}{2}\right)(-1)^{-j-\sigma}\Psi_{-\mathbf{p},-\sigma,\mp}\end{aligned}$$

with this we get:

$$\begin{aligned}\mathsf{CP}\Psi_{\mathbf{p},\sigma,\pm} &= (\mathsf{CPT})\,\mathsf{T}^{-1}\Psi_{\mathbf{p},\sigma,\pm}\\ &= (\mathsf{CPT})\exp\left(\mp i\frac{\phi}{2}\right)(-1)^{-j-\sigma}\Psi_{-\mathbf{p},-\sigma,\mp}\\ &= \exp\left(\pm i\frac{\phi}{2}\right)(-1)^{-j-\sigma}\,(\mathsf{CPT})\,\Psi_{-\mathbf{p},-\sigma,\mp}\\ &= \exp\left(\pm i\frac{\phi}{2}\right)(-1)^{-j-\sigma}(-1)^{j+\sigma}\Psi_{-\mathbf{p},\sigma,\mp^C}\\ &= \exp\left(\pm i\frac{\phi}{2}\right)\Psi_{-\mathbf{p},\sigma,\mp^C}\end{aligned}$$

K. Problems

1. Problem

Observer $\mathcal{O}$ sees a W-Boson with:

$$\begin{aligned}j &= 1\\ \text{mass} &= m\\ \mathbf{p} &= k\hat{e}_y\\ j_z &= \sigma\end{aligned}$$

Observer $\mathcal{O}'$ moves relative to $\mathcal{O}$ with

$$\mathbf{v} = w\hat{e}_z.$$

We work in case a) of Table 2.1, s.t.:

$$\begin{aligned}p^0 &> 0\\ -m^2 &= p^2\\ &= -\left(p^0\right)^2 + \mathbf{p}^2\\ &= -\left(p^0\right)^2 + k^2\\ \Rightarrow p^0 &= \sqrt{m^2+k^2}\\ \Rightarrow p &= \begin{pmatrix}0\\ k\\ 0\\ \sqrt{m^2+k^2}\end{pmatrix}\end{aligned}$$

The Lorentz transformation from $\mathcal{O}$ to $\mathcal{O}'$ is given by

$$\Lambda(\mathbf{v}) = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1+(\gamma_w-1)\frac{w^2}{\mathbf{v}^2} & -\gamma_w w\\ 0 & 0 & -\gamma_w w & \gamma_w\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \gamma_w & -\gamma_w w\\ 0 & 0 & -\gamma_w w & \gamma_w\end{pmatrix} \qquad \gamma_w = \frac{1}{\sqrt{1-w^2}}.$$

From this we get

$$\Lambda p = \begin{pmatrix}0\\ k\\ -\gamma_w w\sqrt{m^2+k^2}\\ \gamma_w\sqrt{m^2+k^2}\end{pmatrix}.$$

In order to be able to apply Eq. (2.5.23) we first need to calculate $W(\Lambda, p)$ and for this we need $L(p)$ and $L(\Lambda p)$:

$$
\begin{aligned}
\gamma_k &= \frac{\sqrt{\mathbf{p}^2+m^2}}{m} = \sqrt{1+\frac{k^2}{m^2}} \\
\sqrt{\gamma_k^2-1} &= \sqrt{1+\frac{k^2}{m^2}-1} = \frac{k}{m} \\
\hat{\mathbf{p}} &= \hat{e}_y \\
L(p) &\overset{(2.5.24)}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+(\gamma_k-1) & 0 & \sqrt{\gamma_k^2-1} \\ 0 & 0 & 1 & 0 \\ 0 & \sqrt{\gamma_k^2-1} & 0 & \gamma_k \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_k & 0 & \sqrt{\gamma_k^2-1} \\ 0 & 0 & 1 & 0 \\ 0 & \sqrt{\gamma_k^2-1} & 0 & \gamma_k \end{pmatrix} \\
\gamma_{wk} &= \frac{\sqrt{\mathbf{\Lambda p}^2+m^2}}{m} = \sqrt{1+\frac{k^2+\gamma_w^2 w^2(m^2+k^2)}{m^2}} \\
&= \sqrt{1+\frac{k^2}{m^2}+\gamma_w^2 w^2\left(1+\frac{k^2}{m^2}\right)} = \sqrt{(1+\gamma_w^2 w^2)\left(1+\frac{k^2}{m^2}\right)} \\
&= \sqrt{\gamma_w^2\,(1-w^2+w^2)}\gamma_k = \gamma_w\gamma_k \\
\sqrt{\gamma_{wk}^2-1} &= \sqrt{\gamma_w^2\gamma_k^2-1} \\
|\mathbf{\Lambda p}| &= \sqrt{\gamma_w^2\gamma_k^2 m^2 - m^2} = m\sqrt{\gamma_w^2\gamma_k^2-1} \\
\widehat{\mathbf{\Lambda p}} &= \frac{1}{m\sqrt{\gamma_w^2\gamma_k^2-1}} \begin{pmatrix} 0 \\ k \\ -\gamma_w w\sqrt{m^2+k^2} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{\gamma_w^2\gamma_k^2-1}} \begin{pmatrix} 0 \\ \sqrt{\gamma_k^2-1} \\ -\gamma_w w\gamma_k \\ 0 \end{pmatrix} \\
L(\Lambda p) &\overset{(2.5.24)}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+(\gamma_w\gamma_k-1)\frac{\gamma_k^2-1}{\gamma_w^2\gamma_k^2-1} & (\gamma_w\gamma_k-1)\frac{-w\gamma_w\gamma_k\sqrt{\gamma_k^2-1}}{\gamma_w^2\gamma_k^2-1} & \frac{\sqrt{\gamma_k^2-1}}{\sqrt{\gamma_w^2\gamma_k^2-1}}\sqrt{\gamma_w^2\gamma_k^2-1} \\ 0 & (\gamma_w\gamma_k-1)\frac{-w\gamma_w\gamma_k\sqrt{\gamma_k^2-1}}{\gamma_w^2\gamma_k^2-1} & 1+(\gamma_w\gamma_k-1)\frac{w^2\gamma_w^2\gamma_k^2}{\gamma_w^2\gamma_k^2-1} & \frac{-\gamma_w w\gamma_k}{\sqrt{\gamma_w^2\gamma_k^2-1}}\sqrt{\gamma_w^2\gamma_k^2-1} \\ 0 & \frac{\sqrt{\gamma_k^2-1}}{\sqrt{\gamma_w^2\gamma_k^2-1}}\sqrt{\gamma_w^2\gamma_k^2-1} & \frac{-\gamma_w w\gamma_k}{\sqrt{\gamma_w^2\gamma_k^2-1}}\sqrt{\gamma_w^2\gamma_k^2-1} & \gamma_w\gamma_k \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\frac{\gamma_k^2-1}{\gamma_w\gamma_k+1} & -\frac{w\gamma_w\gamma_k\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1} & \sqrt{\gamma_k^2-1} \\ 0 & -\frac{w\gamma_w\gamma_k\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1} & 1+\frac{w^2\gamma_w^2\gamma_k^2}{\gamma_w\gamma_k+1} & -\gamma_w w\gamma_k \\ 0 & \sqrt{\gamma_k^2-1} & -\gamma_w w\gamma_k & \gamma_w\gamma_k \end{pmatrix}
\end{aligned}
$$

Using Eq. (2.3.10) we get:

$$
\begin{aligned}
\left(L^{-1}\right)^0{}_0 &= (-1)(-1)L^0{}_0 = L^0{}_0 \\
\left(L^{-1}\right)^i{}_k &= (+1)(+1)L^k{}_i = L^k{}_i = L^i{}_k \\
\left(L^{-1}\right)^i{}_0 &= (-1)(+1)L^0{}_i = -L^0{}_i = \left(L^{-1}\right)^0{}_i \\
\Rightarrow L^{-1}(\Lambda p) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\frac{\gamma_k^2-1}{\gamma_w\gamma_k+1} & -\frac{w\gamma_w\gamma_k\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1} & -\sqrt{\gamma_k^2-1} \\ 0 & -\frac{w\gamma_w\gamma_k\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1} & 1+\frac{w^2\gamma_w^2\gamma_k^2}{\gamma_w\gamma_k+1} & \gamma_w w\gamma_k \\ 0 & -\sqrt{\gamma_k^2-1} & \gamma_w w\gamma_k & \gamma_w\gamma_k \end{pmatrix}
\end{aligned}
$$

Putting everything together we obtain:

$$W(\Lambda,p) \overset{(2.5.10)}{=} L^{-1}(\Lambda p)\Lambda L(p) = L^{-1}(\Lambda p)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_w & -\gamma_w w \\ 0 & 0 & -\gamma_w w & \gamma_w \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_k & 0 & \sqrt{\gamma_k^2-1} \\ 0 & 0 & 1 & 0 \\ 0 & \sqrt{\gamma_k^2-1} & 0 & \gamma_k \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\frac{\gamma_k^2-1}{\gamma_w\gamma_k+1} & -\frac{w\gamma_w\gamma_k\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1} & -\sqrt{\gamma_k^2-1} \\ 0 & -\frac{w\gamma_w\gamma_k\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1} & 1+\frac{w^2\gamma_w^2\gamma_k^2}{\gamma_w\gamma_k+1} & \gamma_w w\gamma_k \\ 0 & -\sqrt{\gamma_k^2-1} & \gamma_w w\gamma_k & \gamma_w\gamma_k \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_k & 0 & \sqrt{\gamma_k^2-1} \\ 0 & -w\gamma_w\sqrt{\gamma_k^2-1} & \gamma_w & -w\gamma_w\gamma_k \\ 0 & \gamma_w\sqrt{\gamma_k^2-1} & -w\gamma_w & \gamma_w\gamma_k \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_k+\frac{\gamma_k^2-1}{\gamma_w\gamma_k+1}\left(\gamma_k+w^2\gamma_w^2\gamma_k-\gamma_w(\gamma_w\gamma_k+1)\right) & w\gamma_w\sqrt{\gamma_k^2-1}\left(-\frac{\gamma_w\gamma_k}{\gamma_w\gamma_k+1}+1\right) & 0 \\ 0 & \frac{-w\gamma_w\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1}(\gamma_k^2+\gamma_w\gamma_k+1+w^2\gamma_w^2\gamma_k^2-\gamma_w\gamma_k(1+\gamma_w\gamma_k)) & \frac{\gamma_w(\gamma_w\gamma_k+1)+w^2\gamma_w^2\gamma_k^2\gamma_w-w^2\gamma_w^2\gamma_k(1+\gamma_w\gamma_k)}{\gamma_w\gamma_k+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_k+\frac{\gamma_k^2-1}{\gamma_w\gamma_k+1}(-\gamma_w) & w\gamma_w\sqrt{\gamma_k^2-1}\left(\frac{1}{\gamma_w\gamma_k+1}\right) & 0 \\ 0 & \frac{-w\gamma_w\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1}(\gamma_k^2+1-\gamma_k^2) & \frac{\gamma_k+\gamma_w}{\gamma_w\gamma_k+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\gamma_w\gamma_k+1}\left(\gamma_w\gamma_k^2+\gamma_k-\gamma_w\gamma_k^2+\gamma_w\right) & \frac{w\gamma_w\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1} & 0 \\ 0 & \frac{-w\gamma_w\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1} & \frac{\gamma_k+\gamma_w}{\gamma_w\gamma_k+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\gamma_k+\gamma_w}{\gamma_w\gamma_k+1} & \frac{w\gamma_w\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1} & 0 \\ 0 & \frac{-w\gamma_w\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1} & \frac{\gamma_k+\gamma_w}{\gamma_w\gamma_k+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This explicit calculation can also be checked (see TODO):

$$\frac{w\gamma_w\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1} = \frac{w\gamma_w\frac{k}{m}(\gamma_w\gamma_k-1)}{\gamma_w^2\gamma_k^2-1} = \frac{w\gamma_w\frac{k}{m}(\frac{\gamma_k}{\gamma_w}-\frac{1}{\gamma_w^2})}{\gamma_k^2-\frac{1}{\gamma_w^2}} = \frac{w\frac{k}{m}(\gamma_k-\frac{1}{\gamma_w})}{1+\frac{k^2}{m^2}-1+w^2} = \frac{wkm(\gamma_k-\frac{1}{\gamma_w})}{k^2+m^2w^2}$$
$$= \frac{wk(\sqrt{k^2+m^2}-m\sqrt{1-w^2})}{k^2+m^2w^2}$$

$$\frac{\gamma_k+\gamma_w}{\gamma_w\gamma_k+1} = \frac{(\gamma_w+\gamma_k)(\gamma_w\gamma_k-1)}{\gamma_w^2\gamma_k^2-1} = \frac{\gamma_k-\frac{1}{\gamma_w}+\frac{\gamma_k^2}{\gamma_w}-\frac{\gamma_k}{\gamma_w^2}}{\gamma_k^2-\frac{1}{\gamma_w^2}} = \frac{\gamma_k(1-(1-w^2))+\sqrt{1-w^2}(1+\frac{k^2}{m^2}-1)}{1+\frac{k^2}{m^2}-1+w^2}$$
$$= \frac{w^2m\sqrt{m^2+k^2}+k^2\sqrt{1-w^2}}{k^2+m^2w^2}$$

In order to apply Eq. (2.5.23), we need to identify $W(\Lambda,p)$ with a rotation, in this case a simple rotation around the x-axis (For the sign convention see the discussion in II E 23):

$$W(\Lambda,p) \overset{!}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\Rightarrow \cos(\theta) = \frac{\gamma_k+\gamma_w}{\gamma_w\gamma_k+1}$$
$$\Rightarrow \sin(\theta) = \frac{w\gamma_w\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1}$$

With this we see that Observer $\mathcal{O}'$ observes the state

$$
\begin{aligned}
U(\Lambda)\Psi_{\mathbf{p},\sigma} &\overset{(2.5.23)}{=} \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\sigma'=-1}^{1} D^{(1)}_{\sigma'\sigma}(W(\Lambda,p))\Psi_{\mathbf{\Lambda p},\sigma'} \\
&= \gamma_w \sum_{\sigma'=-1}^{1} D^{(1)}_{\sigma'\sigma}(W(\Lambda,p))\Psi_{\mathbf{\Lambda p},\sigma'}
\end{aligned}
$$

where

$$
\begin{aligned}
D^{(1)}(W(\Lambda,p)) &\overset{(2.5.20)}{=} \exp\left(i\theta J^{(1)}_1\right) \\
J^{(1)}_1 &\overset{(2.5.21)}{=} \frac{1}{2}\left(J_+ + J_-\right) = \frac{1}{2}\left(\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\Rightarrow \left(J^{(1)}_1\right)^2 &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad \left(J^{(1)}_1\right)^3 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} = J^{(1)}_1 \qquad \left(J^{(1)}_1\right)^4 = \left(J^{(1)}_1\right)^2 \qquad \ldots \\
D^{(1)}(W(\Lambda,p)) &= \sum_{n=0}^{\infty} \frac{1}{n!}(i\theta)^n \left(J^{(1)}_1\right)^n = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(i\theta)^{2n+1}\left(J^{(1)}_1\right)^{2n+1} + \mathbf{1} + \sum_{n=1}^{\infty} \frac{1}{(2n)!}(i\theta)^{2n}\left(J^{(1)}_1\right)^{2n} \\
&= J^{(1)}_1 i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\theta^{2n+1} + \mathbf{1} + \left(J^{(1)}_1\right)^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}(\theta)^{2n} \\
&= J^{(1)}_1 i\sin(\theta) + \mathbf{1} + \left(J^{(1)}_1\right)^2(\cos(\theta)-1) = i\sin(\theta)J^{(1)}_1 + \cos(\theta)\left(J^{(1)}_1\right)^2 + \mathbf{1} - \left(J^{(1)}_1\right)^2 \\
&= i\frac{w\gamma_w\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1}J^{(1)}_1 + \frac{\gamma_k+\gamma_w}{\gamma_w\gamma_k+1}\left(J^{(1)}_1\right)^2 + \mathbf{1} - \left(J^{(1)}_1\right)^2
\end{aligned}
$$

2. *Problem*

TODO For polarization see text at end of section 2.5 massless case, rest analog to previous problem.

3. *Problem*

TODO

4. *Problem*

TODO

5. *Problem*

TODO

6. *Problem*

TODO

III. SCATTERING THEORY

A. "In" and "Out" States

1. *Eq.* (3.1.1) (*P.*107)

For the translation part of the Poincaré transformation Eq. (2.5.1) and Eq. (2.4.26) were used. And for the Lorentz transformation part Eq. (2.5.23) and Eq. (2.5.42) were used for the massive and massless case respectively.

2. *"$U(\Lambda,a) = \exp(iH\tau)$"* (*P.*109)

This follow immediately from Eq. (2.4.26) with

$$\Lambda = \mathbf{1} \qquad a = (0,0,0,\tau).$$

3. *"... state long before or long after the collision (...) is found by applying a time-translation operator $\exp(-iH\tau)$ with $\tau \to -\infty$ or $\tau \to +\infty$, respectively."* (*P.*109)

This is clear from the following consideration:
$t' = 0$ is at $t = \tau \to \mp\infty$.

4. Eq. (3.1.16) (*P.*111)

Start of from the ansatz

$$\Psi^{\pm}_{\alpha} = \Phi_{\alpha} + \tilde{\Psi}^{\pm}_{\alpha},$$

inserting we get:

$$\begin{aligned}
V\Psi^{\pm}_{\alpha} &= (E_{\alpha} - H_0)\Psi^{\pm}_{\alpha} \\
&= (E_{\alpha} - H_0)\Phi_{\alpha} + (E_{\alpha} - H_0)\tilde{\Psi}^{\pm}_{\alpha} \\
&= (E_{\alpha} - H_0)\tilde{\Psi}^{\pm}_{\alpha} \\
\Rightarrow \tilde{\Psi}^{\pm}_{\alpha} &= (E_{\alpha} - H_0 \pm i\varepsilon)^{-1} V\Psi^{\pm}_{\alpha} \\
\Rightarrow \Psi^{\pm}_{\alpha} &= \Phi_{\alpha} + (E_{\alpha} - H_0 \pm i\varepsilon)^{-1} V\Psi^{\pm}_{\alpha}
\end{aligned}$$

5. Eq. (3.1.21) (*P.*112)

$$\begin{aligned}
\Psi^{\pm}_{g}(t) &\overset{(3.1.19)}{=} \int \mathrm{d}\alpha \exp(-iE_{\alpha}t) g(\alpha) \Psi^{\pm}_{\alpha} \\
&\overset{(3.1.17)}{=} \int \mathrm{d}\alpha \exp(-iE_{\alpha}t) g(\alpha) \Phi_{\alpha} \\
&+ \int \mathrm{d}\alpha \int \mathrm{d}\beta \exp(-iE_{\alpha}t) g(\alpha) \frac{T^{\pm}_{\beta\alpha}\Phi_{\beta}}{E_{\alpha} - E_{\beta} \pm i\varepsilon} \\
&= \Phi_g(t) \\
&+ \int \mathrm{d}\beta \left(\int \mathrm{d}\alpha \, \frac{\exp(-iE_{\alpha}t) g(\alpha) T^{\pm}_{\beta\alpha}}{E_{\alpha} - E_{\beta} \pm i\varepsilon} \right) \Phi_{\beta} \\
&= \Phi_g(t) + \int \mathrm{d}\beta \, \mathcal{T}^{\pm}_{\beta} \Phi_{\beta}
\end{aligned}$$

6. "For $t \to -\infty$, we can close the contour of integration for the energy variable E_{α} in the upper half-plane…" (*P.*112)

From Eq. (3.1.4) we know

$$\int \mathrm{d}\alpha \cdots = \sum_{n_1\sigma_1 n_2\sigma_2} \int \mathrm{d}^3 p_1 \, \mathrm{d}^3 p_2 \ldots$$

and in addition the considerations between Eq. (2.5.14) and Eq. (2.5.15) yield:

$$\int \mathrm{d}^3\mathbf{p} \, \frac{f(\mathbf{p}, \sqrt{\mathbf{p}^2 + M^2})}{2\sqrt{\mathbf{p}^2 + M^2}} = \int \mathrm{d}^4 p \, \delta(p^2 + M^2)\theta(p^0) f(p)$$

Such that we can identify the E_{α} integration by the p^0_1 integration, which now covers the complete real axis and can be solved with procedure described, i.e. applying the residue theorem.

If we don't do this then the

$$E_{\alpha} \overset{(3.1.7)}{=} p^1_0 + p^2_0 + \cdots = \sqrt{M_1^2 + \mathbf{p_1}^2} + \sqrt{M_2^2 + \mathbf{p_2}^2} \ldots$$

integration would only cover the positive reals and in such a case residue integration gets a lot more complicated.

7. "Specifically, $-t$ must be much greater than both the time-uncertainty in the wave-packet $g(\alpha)$ and the duration of the collision, which respectively govern the location of the singularities of $g(\alpha)$ and $T^{\pm}_{\beta\alpha}$ in the complex E_{α} plane." (*P.*112)

TODO

8. Eqs. (3.1.22/23/24) (*P.*113)

$$\begin{aligned}
\frac{\mathcal{P}_{\varepsilon}}{E} \mp i\pi\delta_{\varepsilon}(E) &= \frac{E}{E^2 + \varepsilon^2} \mp \frac{i\pi\varepsilon}{\pi(E^2 + \varepsilon^2)} \\
&= \frac{1}{E^2 + \varepsilon^2}(E \mp i\varepsilon) \\
&= \frac{1}{(E + i\varepsilon)(E - i\varepsilon)}(E \mp i\varepsilon) \\
&= \frac{1}{E \pm i\varepsilon}
\end{aligned}$$

9. "The function Eq. (3.1.24)*…gives unity when integrated over all E, …"* (*P.*113)

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}E \, \delta_{\varepsilon}(E) &= \int_{-\infty}^{\infty} \mathrm{d}E \, \frac{\varepsilon}{\pi(E^2 + \varepsilon^2)} \\
&= \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} \mathrm{d}E \, \frac{1}{E^2 + \varepsilon^2} \\
&\overset{WA}{=} \frac{\varepsilon}{\pi} \pi \sqrt{\frac{1}{\varepsilon^2}} = 1
\end{aligned}$$

B. The S-Matrix

1. "$\int \mathrm{d}\beta \, S^{\star}_{\beta\gamma} S_{\beta\alpha} = \cdots = \langle \Psi^{+}_{\gamma} | \Psi^{+}_{\alpha} \rangle$" (*P.*114)

Note, that according to the previous discussion we can apply Eq. (3.1.5) in the following sense to Ψ^{+}_{γ}:

$$\Psi^{+}_{\gamma} = \int \mathrm{d}\beta \, \Psi^{-}_{\beta} \left\langle \Psi^{-}_{\beta} \middle| \Psi^{+}_{\gamma} \right\rangle$$

$$\begin{aligned}
\langle \Psi^{+}_{\gamma} | \Psi^{+}_{\alpha} \rangle &= \int \mathrm{d}\beta \left\langle \Psi^{-}_{\beta} \middle| \Psi^{+}_{\gamma} \right\rangle^{\star} \left\langle \Psi^{-}_{\beta} \middle| \Psi^{+}_{\alpha} \right\rangle \\
&= \int \mathrm{d}\beta \, S^{\star}_{\beta\gamma} S_{\beta\alpha}
\end{aligned}$$

In the same way we have:

$$\Psi^{-}_{\gamma} = \int \mathrm{d}\beta \, \Psi^{+}_{\beta} \left\langle \Psi^{+}_{\beta} \middle| \Psi^{-}_{\gamma} \right\rangle$$

$$\left\langle \Psi_\gamma^- \middle| \Psi_\alpha^- \right\rangle = \int \mathrm{d}\beta \left\langle \Psi_\beta^+ \middle| \Psi_\gamma^- \right\rangle^\star \left\langle \Psi_\beta^+ \middle| \Psi_\alpha^- \right\rangle$$
$$= \int \mathrm{d}\beta \left\langle \Psi_\gamma^- \middle| \Psi_\beta^+ \right\rangle \left\langle \Psi_\alpha^- \middle| \Psi_\beta^+ \right\rangle^\star$$
$$= \int \mathrm{d}\beta\, S_{\gamma\beta} S^\star_{\alpha\beta}$$

2. Eq. (3.2.5) (*P*.114)

$$S_{\beta\alpha} \overset{(3.2.1)}{=} \left\langle \Psi_\beta^- \middle| \Psi_\alpha^+ \right\rangle$$
$$\overset{(3.1.13)}{=} \langle \Omega(+\infty)\Phi_\beta | \Omega(-\infty)\Phi_\alpha \rangle$$
$$= \left\langle \Phi_\beta \middle| \Omega(+\infty)^\dagger \Omega(-\infty)\Phi_\alpha \right\rangle$$
$$\overset{(3.2.4)}{=} \langle \Phi_\beta | S\Phi_\alpha \rangle$$

3. "...for $t \to +\infty$, $\Psi_g^+(t) \to \ldots$" (*P*.115)

From III A 5 we know:

$$\Psi_g^\pm(t) = \Phi_g(t) + \int \mathrm{d}\beta\, \mathcal{T}_\beta^\pm \Phi_\beta$$
$$= \int \mathrm{d}\alpha \exp(-iE_\alpha t) g(\alpha)\Phi_\alpha + \int \mathrm{d}\beta\, \mathcal{T}_\beta^\pm \Phi_\beta$$
$$= \int \mathrm{d}\beta\, \Phi_\beta \left(\exp(-iE_\beta t) g(\beta) + \mathcal{T}_\beta^\pm \right)$$

in the limit $t \to +\infty$ we thus have

$$\Psi_g^+(t) \to \int \mathrm{d}\beta\, \Phi_\beta (\exp(-iE_\beta t) g(\beta)$$
$$- 2\pi i \exp(-iE_\beta t) \int \mathrm{d}\alpha\, \delta(E_\alpha - E_\beta) g(\alpha) T^+_{\beta\alpha})$$
$$= \int \mathrm{d}\beta\, \Phi_\beta \exp(-iE_\beta t)(g(\beta)$$
$$- 2\pi i \int \mathrm{d}\alpha\, \delta(E_\alpha - E_\beta) g(\alpha) T^+_{\beta\alpha})$$

4. "$\Psi_g^+(t) = \ldots$" (*P*.115)

From Eq. (3.1.19) we get:

$$\Psi_g^+(t) = \int \mathrm{d}\alpha \exp(-iE_\alpha t) g(\alpha) \Psi_\alpha^+$$
$$\overset{(3.1.5)}{=} \int \mathrm{d}\alpha \exp(-iE_\alpha t) g(\alpha) \int \mathrm{d}\beta\, \Psi_\beta^- \left\langle \Psi_\beta^- \middle| \Psi_\alpha^+ \right\rangle$$
$$= \int \mathrm{d}\alpha \exp(-iE_\alpha t) g(\alpha) \int \mathrm{d}\beta\, \Psi_\beta^- \underbrace{S_{\beta\alpha}}_{\propto \delta(E_\alpha - E_\beta)}$$
$$= \int \mathrm{d}\beta \exp(-iE_\beta t) \Psi_\beta^- \underbrace{\int \mathrm{d}\alpha\, g(\alpha) S_{\beta\alpha}}_{=:D(\beta)}$$
$$= \int \mathrm{d}\beta \exp(-iE_\beta t) \Psi_\beta^- D(\beta)$$

For the asymptotic behavior for $t \to +\infty$, $D(\beta)$ gets interpreted as the $g(\alpha)$ in Eq. (3.1.12), s.t.:

$$\Psi_g^+(t) \to \int \mathrm{d}\beta \exp(-iE_\beta t) \Phi_\beta D(\beta)$$
$$= \int \mathrm{d}\beta \exp(-iE_\beta t) \Phi_\beta \int \mathrm{d}\alpha\, g(\alpha) S_{\beta\alpha}$$

5. Eq. (3.2.7) (*P*.115)

This follows from the previous result because it held for arbitrary g.

6. Eq. (3.2.8) (*P*.115)

$$T^+_{\beta\alpha} \overset{(3.1.18)}{=} \left\langle \Phi_\beta \middle| V \Psi_\alpha^+ \right\rangle$$
$$\overset{(3.1.16/17)}{\approx} \langle \Phi_\beta | V \Phi_\alpha \rangle$$

Where the last approximation is for small V.

7. "...proof of the orthonormality *of these states and the* unitarity *of the S-matrix, as well as Eq.* (3.2.7)*, without having to deal with limits as $t \to \mp\infty$"* (*P*.115)

The orthonormality was previously shown in Eq. (3.1.15) by making use of results in the limit $\tau \to \mp\infty$.

The unitarity of the S-matrix was previously shown in Eqs. (3.2.2/3).

Eq. (3.2.7) was derived from results in the limit $t \to \infty$.

8. "$\left\langle \Psi_\beta^\pm \middle| V\Phi_\alpha \right\rangle + \left\langle \Psi_\beta^\pm \middle| V(E_\alpha - H_0 \pm i\varepsilon)^{-1} V \Psi_\alpha^\pm \right\rangle = \left\langle \Phi_\beta \middle| V\Psi_\alpha^\pm \right\rangle + \left\langle \Psi_\beta^\pm \middle| V(E_\beta - H_0 \mp i\varepsilon)^{-1} V \Psi_\alpha^\pm \right\rangle$" $(P.116)$

$$
\begin{aligned}
&\left\langle \Psi_\beta^\pm \middle| V\Phi_\alpha \right\rangle + \left\langle \Psi_\beta^\pm \middle| V(E_\alpha - H_0 \pm i\varepsilon)^{-1} V \Psi_\alpha^\pm \right\rangle \\
&= \left\langle \Psi_\beta^\pm \middle| V \Psi_\alpha^\pm \right\rangle \\
&= \left\langle \Phi_\beta \middle| V\Psi_\alpha^\pm \right\rangle + \left\langle (E_\beta - H_0 \pm i\varepsilon)^{-1} V \Psi_\beta^\pm \middle| V \Psi_\alpha^\pm \right\rangle \\
&= \left\langle \Phi_\beta \middle| V\Psi_\alpha^\pm \right\rangle + \left\langle \Psi_\beta^\pm \middle| V(E_\beta - H_0 \mp i\varepsilon)^{-1} V \Psi_\alpha^\pm \right\rangle
\end{aligned}
$$

9. *Eq.* (3.2.9) $(P.116)$

From the previous result we get:

$$
\begin{aligned}
&\left(T_{\alpha\beta}^\pm\right)^\star - T_{\beta\alpha}^\pm \\
&= \left\langle \Phi_\alpha \middle| V\Psi_\beta^\pm \right\rangle^\star - T_{\beta\alpha}^\pm = \left\langle V\Phi_\alpha \middle| \Psi_\beta^\pm \right\rangle^\star - T_{\beta\alpha}^\pm = \left\langle \Psi_\beta^\pm \middle| V\Phi_\alpha \right\rangle - \left\langle \Phi_\beta \middle| V\Psi_\alpha^\pm \right\rangle \\
&= \left\langle \Psi_\beta^\pm \middle| V(E_\beta - H_0 \mp i\varepsilon)^{-1} V \Psi_\alpha^\pm \right\rangle - \left\langle \Psi_\beta^\pm \middle| V(E_\alpha - H_0 \pm i\varepsilon)^{-1} V \Psi_\alpha^\pm \right\rangle \\
&= \left\langle \Psi_\beta^\pm \middle| V \left[(E_\beta - H_0 \mp i\varepsilon)^{-1} - (E_\alpha - H_0 \pm i\varepsilon)^{-1}\right] V \Psi_\alpha^\pm \right\rangle \\
&= \int \mathrm{d}\gamma \left\langle \Psi_\beta^\pm \middle| V \left[(E_\beta - H_0 \mp i\varepsilon)^{-1} - (E_\alpha - H_0 \pm i\varepsilon)^{-1}\right] \Phi_\gamma \right\rangle \left\langle \Phi_\gamma \middle| V\Psi_\alpha^\pm \right\rangle \\
&= \int \mathrm{d}\gamma \left\langle \Psi_\beta^\pm \middle| V\Phi_\gamma \right\rangle \left[(E_\beta - E_\gamma \mp i\varepsilon)^{-1} - (E_\alpha - E_\gamma \pm i\varepsilon)^{-1}\right] T_{\gamma\alpha}^\pm \\
&= \int \mathrm{d}\gamma \left(T_{\gamma\beta}^\pm\right)^\star \left[(E_\beta - E_\gamma \mp i\varepsilon)^{-1} - (E_\alpha - E_\gamma \pm i\varepsilon)^{-1}\right] T_{\gamma\alpha}^\pm
\end{aligned}
$$

10. *“To prove the orthonormality of the ”in” and ”out” states, divide Eq.* (3.2.9) *by* $E_\alpha - E_\beta \pm 2i\varepsilon$*. This gives ...”* (*P.*116)

$$
\begin{aligned}
&\left(\frac{T^{\pm}_{\alpha\beta}}{E_\beta - E_\alpha \pm 2i\varepsilon}\right)^{\star} + \frac{T^{\pm}_{\beta\alpha}}{E_\alpha - E_\beta \pm 2i\varepsilon} \\
&= \frac{\left(T^{\pm}_{\alpha\beta}\right)^{\star}}{E_\beta - E_\alpha \mp 2i\varepsilon} + \frac{T^{\pm}_{\beta\alpha}}{E_\alpha - E_\beta \pm 2i\varepsilon} \\
&= -\frac{1}{E_\alpha - E_\beta \pm 2i\varepsilon}\left\{\left(T^{\pm}_{\alpha\beta}\right)^{\star} - T^{\pm}_{\beta\alpha}\right\} \\
&\overset{(3.2.9)}{=} -\frac{1}{E_\alpha - E_\beta \pm 2i\varepsilon}\int \mathrm{d}\gamma \left(T^{\pm}_{\gamma\beta}\right)^{\star} T^{\pm}_{\gamma\alpha}\left[(E_\beta - E_\gamma \mp i\varepsilon)^{-1} - (E_\alpha - E_\gamma \pm i\varepsilon)^{-1}\right] \\
&= -\frac{1}{E_\alpha - E_\beta \pm 2i\varepsilon}\int \mathrm{d}\gamma \left(T^{\pm}_{\gamma\beta}\right)^{\star} T^{\pm}_{\gamma\alpha}\left[\frac{E_\alpha - E_\gamma \pm i\varepsilon - (E_\beta - E_\gamma \mp i\varepsilon)}{(E_\beta - E_\gamma \mp i\varepsilon)(E_\alpha - E_\gamma \pm i\varepsilon)}\right] \\
&= -\frac{1}{E_\alpha - E_\beta \pm 2i\varepsilon}\int \mathrm{d}\gamma \left(T^{\pm}_{\gamma\beta}\right)^{\star} T^{\pm}_{\gamma\alpha}\left[\frac{E_\alpha \pm 2i\varepsilon - E_\beta}{(E_\beta - E_\gamma \mp i\varepsilon)(E_\alpha - E_\gamma \pm i\varepsilon)}\right] \\
&= -\int \mathrm{d}\gamma \left(T^{\pm}_{\gamma\beta}\right)^{\star} T^{\pm}_{\gamma\alpha}\frac{1}{(E_\beta - E_\gamma \mp i\varepsilon)(E_\alpha - E_\gamma \pm i\varepsilon)} \\
&= -\int \mathrm{d}\gamma \left(\frac{T^{\pm}_{\gamma\beta}}{E_\beta - E_\gamma \pm i\varepsilon}\right)^{\star}\frac{T^{\pm}_{\gamma\alpha}}{E_\alpha - E_\gamma \pm i\varepsilon}
\end{aligned}
$$

11. *“We see the that* $\delta(\beta-\alpha) + T^{\pm}_{\beta\alpha}/(E_\alpha - E_\beta \pm i\varepsilon)$ *is* unitary*. With Eq.* (3.1.17)*, this is just the statement that the* $\Psi^{\pm}_\alpha$ *form two* orthonormal *sets of state vectors.”* (*P.*116)

First define

$$A^{\pm}_{\beta\alpha} = \delta(\beta-\alpha) + \frac{T^{\pm}_{\beta\alpha}}{E_\alpha - E_\beta \pm i\varepsilon}.$$

With this we get:

$$
\begin{aligned}
&\int \mathrm{d}\gamma \left(A^{\pm}_{\gamma\beta}\right)^{\star} A^{\pm}_{\gamma\alpha} \\
&= \int \mathrm{d}\gamma \left(\delta(\gamma-\beta) + \frac{T^{\pm}_{\gamma\beta}}{E_\beta - E_\gamma \pm i\varepsilon}\right)^{\star} \\
&\cdot \left(\delta(\gamma-\alpha) + \frac{T^{\pm}_{\gamma\alpha}}{E_\alpha - E_\gamma \pm i\varepsilon}\right) \\
&\overset{III\,B\,10}{=} \int \mathrm{d}\gamma\, \delta(\gamma-\beta)\delta(\gamma-\alpha) \\
&= \delta(\beta-\alpha)
\end{aligned}
$$

Further, Eq. (3.1.17) can be written as

$$\Psi^{\pm}_\alpha = \int \mathrm{d}\beta\, A^{\pm}_{\beta\alpha}\Phi_\beta.$$

Such that we can already show the orthonormality of the states:

$$
\begin{aligned}
\langle \Psi^{\pm}_\gamma | \Psi^{\pm}_\alpha \rangle &= \int \mathrm{d}\delta \int \mathrm{d}\beta \left(A^{\pm}_{\delta\gamma}\right)^{\star} A^{\pm}_{\beta\alpha} \underbrace{\langle \Phi_\delta | \Phi_\beta \rangle}_{=\delta(\delta-\beta)} \\
&= \int \mathrm{d}\beta \left(A^{\pm}_{\beta\gamma}\right)^{\star} A^{\pm}_{\beta\alpha} \\
&= \delta(\gamma-\alpha) \qquad (14)
\end{aligned}
$$

Showing the full statement includes proving that $A^{\pm}$ is unitary, i.e. we still have to show:

$$\int \mathrm{d}\gamma\, A^{\pm}_{\beta\gamma}\left(A^{\pm}_{\alpha\gamma}\right)^{\star} \overset{!}{=} \delta(\beta-\alpha)$$

For this consider an analog inner-product to the one considered in III B 8 and III B 9 (such that all the indices will be swapped in the end):

$$
\begin{aligned}
&\left(T^{\pm}_{\alpha\beta}\right)^{\star} + \int \mathrm{d}\gamma\, \frac{(T^{\pm}_{\alpha\gamma})^{\star}\, T^{\pm}_{\beta\gamma}}{E_\gamma - E_\beta \pm i\varepsilon} \\
&= \int \mathrm{d}\gamma \left(\langle \Phi_\beta | \Phi_\gamma \rangle + \left\langle \Phi_\beta \middle| (E_\gamma - H_0 \pm i\varepsilon)^{-1} V \Psi^{\pm}_\gamma \right\rangle\right) (T^{\pm}_{\alpha\gamma})^{\star} \\
&= \int \mathrm{d}\gamma \left\langle \Phi_\beta \middle| \Psi^{\pm}_\gamma \right\rangle \left\langle \Psi^{\pm}_\gamma \middle| V \Phi_\alpha \right\rangle \\
&= \langle \Phi_\beta | V \Phi_\alpha \rangle \\
&= \int \mathrm{d}\gamma \left\langle \Phi_\beta \middle| V \Psi^{\pm}_\gamma \right\rangle \left\langle \Psi^{\pm}_\gamma \middle| \Phi_\alpha \right\rangle \\
&= \int \mathrm{d}\gamma\, T^{\pm}_{\beta\gamma} \left(\langle \Phi_\gamma | \Phi_\alpha \rangle + \left\langle (E_\gamma - H_0 \pm i\varepsilon)^{-1} V \Psi^{\pm}_\gamma \middle| \Phi_\alpha \right\rangle\right) \\
&= T^{\pm}_{\beta\alpha} + \int \mathrm{d}\gamma\, \frac{(T^{\pm}_{\alpha\gamma})^{\star}\, T^{\pm}_{\beta\gamma}}{E_\gamma - E_\alpha \mp i\varepsilon}
\end{aligned}
$$

We are allowed to introduce

$$
\mathbf{1} = \int \mathrm{d}\gamma \left|\Psi^{\pm}_\gamma\right\rangle\!\left\langle\Psi^{\pm}_\gamma\right|
$$

because $\left|\Psi^{\pm}_\gamma\right\rangle$ form a complete set of states and we showed the orthonormality of the $\left|\Psi^{\pm}_\gamma\right\rangle$ in 14.

We can continue by rearranging the terms

$$
\begin{aligned}
&\left(T^{\pm}_{\alpha\beta}\right)^{\star} - T^{\pm}_{\beta\alpha} \\
&= \int \mathrm{d}\gamma\, (T^{\pm}_{\alpha\gamma})^{\star}\, T^{\pm}_{\beta\gamma} \left(\frac{1}{E_\gamma - E_\alpha \mp i\varepsilon} - \frac{1}{E_\gamma - E_\beta \pm i\varepsilon}\right) \\
&= -\int \mathrm{d}\gamma\, (T^{\pm}_{\alpha\gamma})^{\star}\, T^{\pm}_{\beta\gamma} \left(\frac{1}{E_\alpha - E_\gamma \pm i\varepsilon} - \frac{1}{E_\beta - E_\gamma \mp i\varepsilon}\right)
\end{aligned}
$$

and by following the same steps as in III B 10, we obtain the analog relation to Eq. (3.2.9):

$$
\begin{aligned}
&\left(\frac{T^{\pm}_{\alpha\beta}}{E_\beta - E_\alpha \pm 2i\varepsilon}\right)^{\star} + \frac{T^{\pm}_{\beta\alpha}}{E_\alpha - E_\beta \pm 2i\varepsilon} \\
&= -\int \mathrm{d}\gamma\, (T^{\pm}_{\alpha\gamma})^{\star}\, T^{\pm}_{\beta\gamma}\, \frac{1}{(E_\beta - E_\gamma \mp i\varepsilon)(E_\alpha - E_\gamma \pm i\varepsilon)} \\
&= -\int \mathrm{d}\gamma \left(-\frac{T^{\pm}_{\alpha\gamma}}{E_\gamma - E_\alpha \pm i\varepsilon}\right)^{\star} \left(-\frac{T^{\pm}_{\beta\gamma}}{E_\gamma - E_\beta \pm i\varepsilon}\right) \\
&= -\int \mathrm{d}\gamma \left(\frac{T^{\pm}_{\alpha\gamma}}{E_\gamma - E_\alpha \pm i\varepsilon}\right)^{\star} \frac{T^{\pm}_{\beta\gamma}}{E_\gamma - E_\beta \pm i\varepsilon}
\end{aligned}
\tag{15}
$$

With this we can finally show the second part of $A^{\pm}$ being unitary:

$$
\begin{aligned}
&\int \mathrm{d}\gamma\, A^{\pm}_{\beta\gamma}\, (A^{\pm}_{\alpha\gamma})^{\star} \\
&= \int \mathrm{d}\gamma \left(\delta(\beta - \gamma) + \frac{T^{\pm}_{\beta\gamma}}{E_\gamma - E_\beta \pm i\varepsilon}\right) \\
&\cdot \left(\delta(\alpha - \gamma) + \frac{T^{\pm}_{\alpha\gamma}}{E_\gamma - E_\alpha \pm i\varepsilon}\right)^{\star} \\
&\overset{15}{=} \int \mathrm{d}\gamma\, \delta(\beta - \gamma)\delta(\alpha - \gamma) \\
&= \delta(\beta - \alpha).
\end{aligned}
$$

12. "The unitarity of the S-matrix can be proved in a similar fashion by multiplying Eq. (3.2.9) *with* $\delta\left(E_\beta - E_\alpha\right)$ *. . . "* (*P.*116)

We first start of by proving Eq. (3.2.7) without using results in the $t \to \pm\infty$ limit. For this consider a similar inner-product to the one considered in III B 8:

$$
\begin{aligned}
&\left\langle \Psi^{-}_\beta \middle| V \Phi_\alpha \right\rangle + \left\langle \Psi^{-}_\beta \middle| V (E_\alpha - H_0 + i\varepsilon)^{-1} V \Psi^{+}_\alpha \right\rangle \\
&= \left\langle \Psi^{-}_\beta \middle| V \Psi^{+}_\alpha \right\rangle \\
&= \left\langle \Phi_\beta \middle| V \Psi^{+}_\alpha \right\rangle + \left\langle (E_\beta - H_0 - i\varepsilon)^{-1} V \Psi^{-}_\beta \middle| V \Psi^{+}_\alpha \right\rangle \\
&= \left\langle \Phi_\beta \middle| V \Psi^{+}_\alpha \right\rangle + \left\langle \Psi^{-}_\beta \middle| V (E_\beta - H_0 + i\varepsilon)^{-1} V \Psi^{+}_\alpha \right\rangle
\end{aligned}
$$

And follow similar steps to III B 9 afterwards:

$$\begin{aligned}
&\left(T^-_{\alpha\beta}\right)^\star - T^+_{\beta\alpha}\\
&= \left\langle \Phi_\alpha \middle| V\Psi^-_\beta \right\rangle^\star - T^\pm_{\beta+} = \left\langle V\Phi_\alpha \middle| \Psi^-_\beta \right\rangle^\star - T^+_{\beta\alpha} = \left\langle \Psi^-_\beta \middle| V\Phi_\alpha \right\rangle - \left\langle \Phi_\beta \middle| V\Psi^+_\alpha \right\rangle\\
&= \left\langle \Psi^-_\beta \middle| V(E_\beta - H_0 + i\varepsilon)^{-1} V\Psi^+_\alpha \right\rangle - \left\langle \Psi^-_\beta \middle| V(E_\alpha - H_0 + i\varepsilon)^{-1} V\Psi^+_\alpha \right\rangle\\
&= \left\langle \Psi^-_\beta \middle| V\left[(E_\beta - H_0 + i\varepsilon)^{-1} - (E_\alpha - H_0 + i\varepsilon)^{-1}\right] V\Psi^+_\alpha \right\rangle\\
&= \int \mathrm{d}\gamma \left\langle \Psi^-_\beta \middle| V\left[(E_\beta - H_0 + i\varepsilon)^{-1} - (E_\alpha - H_0 + i\varepsilon)^{-1}\right] \Phi_\gamma \right\rangle \left\langle \Phi_\gamma \middle| V\Psi^+_\alpha \right\rangle\\
&= \int \mathrm{d}\gamma \left\langle \Psi^-_\beta \middle| V\Phi_\gamma \right\rangle \left[(E_\beta - E_\gamma + i\varepsilon)^{-1} - (E_\alpha - E_\gamma + i\varepsilon)^{-1}\right] T^+_{\gamma\alpha}\\
&= \int \mathrm{d}\gamma \left(T^-_{\gamma\beta}\right)^\star \left[(E_\beta - E_\gamma + i\varepsilon)^{-1} - (E_\alpha - E_\gamma + i\varepsilon)^{-1}\right] T^+_{\gamma\alpha}
\end{aligned} \tag{16}$$

With this we can now show Eq. (3.2.7):

$$\begin{aligned}
S_{\beta\alpha} &\overset{Eq.\,(3.2.1)}{=} \left\langle \Psi^-_\beta \middle| \Psi^+_\alpha \right\rangle\\
&= \left\langle \Phi_\beta + (E_\beta - H_0 - i\varepsilon)^{-1} V\Psi^-_\beta \middle| \Phi_\alpha + (E_\alpha - H_0 + i\varepsilon)^{-1} V\Psi^+_\alpha \right\rangle\\
&= \delta(\beta - \alpha) + (E_\alpha - E_\beta + i\varepsilon)^{-1} \left\langle \Phi_\beta \middle| V\Psi^+_\alpha \right\rangle + (E_\beta - E_\alpha + i\varepsilon)^{-1} \left\langle V\Psi^-_\beta \middle| \Phi_\alpha \right\rangle\\
&+ \left\langle (E_\beta - H_0 - i\varepsilon)^{-1} V\Psi^-_\beta \middle| (E_\alpha - H_0 + i\varepsilon)^{-1} V\Psi^+_\alpha \right\rangle\\
&= \delta(\beta - \alpha) + \frac{T^+_{\beta\alpha}}{E_\alpha - E_\beta + i\varepsilon} + \frac{\left(T^-_{\alpha\beta}\right)^\star}{E_\beta - E_\alpha + i\varepsilon}\\
&+ \int \mathrm{d}\gamma \left\langle V\Psi^-_\beta \middle| \Phi_\gamma \right\rangle \frac{1}{E_\beta - E_\gamma + i\varepsilon} \frac{1}{E_\alpha - E_\gamma + i\varepsilon} \left\langle \Phi_\gamma \middle| V\Psi^+_\alpha \right\rangle\\
&\overset{16}{=} \delta(\beta - \alpha) + T^+_{\beta\alpha} \left(\frac{1}{E_\alpha - E_\beta + i\varepsilon} + \frac{1}{E_\beta - E_\alpha + i\varepsilon}\right)\\
&+ \int \mathrm{d}\gamma \left(T^-_{\gamma\beta}\right)^\star T^+_{\gamma\alpha} \left[\frac{1}{E_\beta - E_\gamma + i\varepsilon} \frac{1}{E_\alpha - E_\gamma + i\varepsilon} + \frac{1}{E_\beta - E_\alpha + i\varepsilon}\left(\frac{1}{E_\beta - E_\gamma + i\varepsilon} - \frac{1}{E_\alpha - E_\gamma + i\varepsilon}\right)\right]\\
&= \delta(\beta - \alpha) + T^+_{\beta\alpha} \frac{E_\alpha - E_\beta - i\varepsilon - (E_\alpha - E_\beta + i\varepsilon)}{(E_\alpha - E_\beta)^2 + \varepsilon^2}\\
&+ \int \mathrm{d}\gamma \left(T^-_{\gamma\beta}\right)^\star T^+_{\gamma\alpha} \left[\frac{1}{E_\beta - E_\gamma + i\varepsilon} \frac{1}{E_\alpha - E_\gamma + i\varepsilon} + \frac{1}{E_\beta - E_\alpha + i\varepsilon}\left(\frac{E_\alpha - E_\gamma + i\varepsilon - (E_\beta - E_\gamma + i\varepsilon)}{(E_\beta - E_\gamma + i\varepsilon)(E_\alpha - E_\gamma + i\varepsilon)}\right)\right]\\
&= \delta(\beta - \alpha) + T^+_{\beta\alpha} \frac{(-2i\varepsilon)}{(E_\alpha - E_\beta)^2 + \varepsilon^2}\\
&+ \int \mathrm{d}\gamma \left(T^-_{\gamma\beta}\right)^\star T^+_{\gamma\alpha} \left[\frac{1}{E_\beta - E_\gamma + i\varepsilon} \frac{1}{E_\alpha - E_\gamma + i\varepsilon} + \frac{1}{E_\beta - E_\alpha + i\varepsilon}\left(\frac{E_\alpha - E_\beta}{(E_\beta - E_\gamma + i\varepsilon)(E_\alpha - E_\gamma + i\varepsilon)}\right)\right]\\
&\overset{Eq.\,(3.1.24)}{=} \delta(\beta - \alpha) - 2\pi i \delta(E_\alpha - E_\beta) T^+_{\beta\alpha} + \int \mathrm{d}\gamma \left(T^-_{\gamma\beta}\right)^\star T^+_{\gamma\alpha} \cdot 0\\
&= \delta(\beta - \alpha) - 2\pi i \delta(E_\alpha - E_\beta) T^+_{\beta\alpha}
\end{aligned} \tag{17}$$

To prove the unitarity of the S-matrix we start of by multiplying Eq. (3.2.9) with $\delta\left(E_\beta - E_\alpha\right)$:

$$
\begin{aligned}
&\delta\left(E_\beta - E_\alpha\right)\left(\left(T^{\pm}_{\alpha\beta}\right)^{\star} - T^{\pm}_{\beta\alpha}\right)\\
&\overset{Eq.\,(3.2.9)}{=} -\int \mathrm{d}\gamma \left(T^{\pm}_{\gamma\beta}\right)^{\star}\left[\left(E_\alpha - E_\gamma \pm i\varepsilon\right)^{-1} - \left(E_\beta - E_\gamma \mp i\varepsilon\right)^{-1}\right] T^{\pm}_{\gamma\alpha}\delta\left(E_\beta - E_\alpha\right)\\
&= -\int \mathrm{d}\gamma \left(T^{\pm}_{\gamma\beta}\right)^{\star} T^{\pm}_{\gamma\alpha}\delta\left(E_\beta - E_\alpha\right)\left[\left(E_\alpha - E_\gamma \pm i\varepsilon\right)^{-1} - \left(E_\alpha - E_\gamma \mp i\varepsilon\right)^{-1}\right]\\
&= -\int \mathrm{d}\gamma \left(T^{\pm}_{\gamma\beta}\right)^{\star} T^{\pm}_{\gamma\alpha}\delta\left(E_\beta - E_\alpha\right)\left[\frac{E_\alpha - E_\gamma \mp i\varepsilon - \left(E_\alpha - E_\gamma \pm i\varepsilon\right)}{\left(E_\alpha - E_\gamma \pm i\varepsilon\right)\left(E_\alpha - E_\gamma \mp i\varepsilon\right)}\right]\\
&= -\int \mathrm{d}\gamma \left(T^{\pm}_{\gamma\beta}\right)^{\star} T^{\pm}_{\gamma\alpha}\delta\left(E_\beta - E_\alpha\right)\frac{\mp 2i\varepsilon}{\left(E_\alpha - E_\gamma\right)^2 + \varepsilon^2}\\
&\overset{Eq.\,(3.1.24)}{=} \pm 2\pi i \int \mathrm{d}\gamma \left(T^{\pm}_{\gamma\beta}\right)^{\star} T^{\pm}_{\gamma\alpha}\delta\left(E_\beta - E_\alpha\right)\delta\left(E_\alpha - E_\gamma\right)
\end{aligned}
\tag{18}
$$

With this we get:

$$
\begin{aligned}
&\int \mathrm{d}\gamma \left(S_{\gamma\beta}\right)^{\star} S_{\gamma\alpha}\\
&\overset{17}{=} \int \mathrm{d}\gamma \left(\delta(\gamma-\beta) + 2\pi i\delta(E_\beta - E_\gamma)\left(T^{+}_{\gamma\beta}\right)^{\star}\right)\left(\delta(\gamma-\alpha) - 2\pi i\delta(E_\alpha - E_\gamma)T^{+}_{\gamma\alpha}\right)\\
&= \delta(\beta-\alpha) - 2\pi i\delta(E_\alpha - E_\beta)T^{+}_{\beta\alpha} + 2\pi i\delta(E_\beta - E_\alpha)\left(T^{+}_{\alpha\beta}\right)^{\star}\\
&- (2\pi i)^2 \int \mathrm{d}\gamma \left(T^{+}_{\gamma\beta}\right)^{\star} T^{+}_{\gamma\alpha}\delta(E_\beta - E_\alpha)\delta(E_\alpha - E_\gamma)\\
&\overset{18}{=} \delta(\beta-\alpha) - 2\pi i\delta(E_\alpha - E_\beta)T^{+}_{\beta\alpha} + 2\pi i\delta(E_\beta - E_\alpha)\left(T^{+}_{\alpha\beta}\right)^{\star}\\
&- 2\pi i\delta\left(E_\beta - E_\alpha\right)\left(\left(T^{+}_{\alpha\beta}\right)^{\star} - T^{+}_{\beta\alpha}\right)\\
&= \delta(\beta-\alpha)\\
&\int \mathrm{d}\gamma\, S_{\beta\gamma}\left(S_{\alpha\gamma}\right)^{\star}
\end{aligned}
$$

C. Symmetries of the S-Matrix

1. Eq. (3.3.1) (*P*.117)

One inserts Eq. (3.1.1) for both Ψ^{-}_{β} and Ψ^{+}_{α}, keeping in mind that the coefficients of Ψ^{-}_{β} have to be complex conjugated when pulled out of the inner-product.

Note: This assumes that there exists a unitary operator that acts as in Eq. (3.1.1) on *both* "in" and "out" states. Which in turn places conditions on the possible Hamiltonians.

2. "Eq. (3.3.1) *will thus hold if this unitary operator commutes with the S-operator..."*(*P*.118)

First define a shorthand notation for the transformation behavior in Eq. (3.1.1):

$$U(\Lambda, a)\Psi_{p_i,\sigma_i,n_i} = D_{\sigma_i',\sigma_i}(\Lambda, a, p_i, n_i)\Psi_{\Lambda p_i,\sigma_i',n_i}$$

With this the action of U_0 on Φ is given by

$$U_0(\Lambda, a)\Phi_{p_i,\sigma_i,n_i} = D_{\sigma_i',\sigma_i}(\Lambda, a, p_i, n_i)\Phi_{\Lambda p_i,\sigma_i',n_i} \tag{19}$$

and Eq. (3.3.1) can be written as:

$$\begin{aligned}
S_{p_i',\sigma_i',n_i';p_i,\sigma_i,n_i} &= D_{\bar\sigma_i,\sigma_i}(\Lambda,a,p_i,n_i)D^\star_{\bar\sigma_i',\sigma_i'}(\Lambda,a,p_i',n_i')S_{\Lambda p_i',\bar\sigma_i',n_i';\Lambda p_i,\bar\sigma_i,n_i} \\
&\overset{Eq.\,(3.2.4)}{=} D_{\bar\sigma_i,\sigma_i}(\Lambda,a,p_i,n_i)D^\star_{\bar\sigma_i',\sigma_i'}(\Lambda,a,p_i',n_i')\left\langle \Phi_{\Lambda p_i',\bar\sigma_i',n_i'} \middle| S\Phi_{\Lambda p_i,\bar\sigma_i,n_i}\right\rangle \\
&= \left\langle D_{\bar\sigma_i',\sigma_i'}(\Lambda,a,p_i',n_i')\Phi_{\Lambda p_i',\bar\sigma_i',n_i'} \middle| SD_{\bar\sigma_i,\sigma_i}(\Lambda,a,p_i,n_i)\Phi_{\Lambda p_i,\bar\sigma_i,n_i}\right\rangle \\
&\overset{19}{=} \left\langle U_0(\Lambda,a)\Phi_{p_i',\sigma_i',n_i'} \middle| SU_0(\Lambda,a)\Phi_{p_i,\sigma_i,n_i}\right\rangle
\end{aligned} \tag{20}$$

From Eq. (3.2.4) we further know:

$$\begin{aligned}
S_{p_i',\sigma_i',n_i';p_i,\sigma_i,n_i} &= \left\langle \Phi_{p_i',\sigma_i',n_i'} \middle| S\Phi_{p_i,\sigma_i,n_i}\right\rangle \\
&= \left\langle U_0(\Lambda,a)\Phi_{p_i',\sigma_i',n_i'} \middle| U_0(\Lambda,a)S\Phi_{p_i,\sigma_i,n_i}\right\rangle
\end{aligned} \tag{21}$$

From this we see

$$U_0(\Lambda,a)S = SU_0(\Lambda,a) \Rightarrow Eq.\,(3.3.1).$$

And given Eq. (3.3.1) holds then 20 and 21 hold for arbitrary free particle states such that

$$Eq.\,(3.3.1) \Rightarrow U_0(\Lambda,a)S = SU_0(\Lambda,a).$$

In total:

$$Eq.\,(3.3.1) \Leftrightarrow U_0(\Lambda,a)S = SU_0(\Lambda,a)$$

3. "Eq. (3.3.1) is equivalent to the statement that... the S-operator commutes with these generators..."(P.118)

Note: The free particle states Φ_α furnish a representation of the inhomogeneous Lorentz group. Which is a group of symmetry transformation in the sense of Section 2.2. For which we can therefore always define the unitary operator U_0 (See II H). Such that it has to satisfy the considerations of Section 2.4.

The equivalence

$$Eq.\,(3.3.1) \Leftrightarrow U_0(\Lambda,a)S = SU_0(\Lambda,a)$$

was already shown in III C 2. Further applying the considerations of Section 2.4 to the operator $U_0(\Lambda,a)$ we know

$$[U_0,S]=0 \Leftrightarrow [H_0,S]=[\mathbf{P}_0,S]=[\mathbf{J}_0,S]=[\mathbf{K}_0,S]=0,$$

such that

$$Eq.\,(3.3.1) \Leftrightarrow [H_0,S]=[\mathbf{P}_0,S]=[\mathbf{J}_0,S]=[\mathbf{K}_0,S]=0.$$

4. "Because the operators $H_0, \boldsymbol{P}_0, \boldsymbol{J}_0$ and $\boldsymbol{K}_0$ generate infinitesimal inhomogeneous Lorentz transformations on the Φ_α, they automatically satisfy the commutation relations..."(P.118)

This follows from the considerations in Section 2.4 and the way we *defined* U_0 to act on the Φ_α's.

D. Rates and Cross-Sections

E. Perturbation Theory

F. Implications of Unitarity

G. Partial-Wave Expansions

H. Resonances

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