Notes for "The Quantum Theory of Fields 1, Foundations" - Weinberg

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I. INTRODUCTION

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II. RELATIVISTIC QUANTUM MECHANICS

A. Quantum Mechanics

B. Symmetries

1. *"For this to be unitary and linear, t must be Hermitian and linear"* ($P.51$)

Linearity is trivial and hermiticity follow from the following observation:

$$\begin{aligned}\langle U\Psi|U\Phi\rangle &= \langle(1+i\varepsilon t)\Psi|(1+i\varepsilon t)\Phi\rangle \\ &= \langle\Psi|\Phi\rangle + \varepsilon i\left(\langle\Psi|t\Phi\rangle - \langle t\Psi|\Phi\rangle\right) + \mathcal{O}\left(\varepsilon^2\right) \\ &\overset{Eq.\,(2.2.2)}{\Leftrightarrow} \langle\Psi|t\Phi\rangle = \langle t\Psi|\Phi\rangle \\ &\overset{Eq.\,(2.1.5)}{\Leftrightarrow} t^\dagger = t\end{aligned}$$

2. *Eq.* (2.2.19) ($P.54$)

f^a_{bc} and f^a have to be real as θ^a are real.

3. *Eq.* (2.2.21) ($P.54$)

From Eq. (2.2.20) we have up to $\mathcal{O}\left(\theta^2,\bar\theta^2\right)$

$$\begin{aligned}&1 + i\left(\theta^a + \bar\theta^a + f^a{}_{bc}\bar\theta^b\theta^c\right)t_a + \frac{1}{2}\left(\theta^b+\bar\theta^b\right)\left(\theta^c+\bar\theta^c\right)t_{bc} \\ &= \left[1 + i\bar\theta^a t_a + \frac{1}{2}\bar\theta^b\bar\theta^c t_{bc}\right]\cdot\left[1 + i\theta^a t_a + \frac{1}{2}\theta^b\theta^c t_{bc}\right] \\ &= 1 + i\theta^a t_a + \frac{1}{2}\theta^b\theta^c t_{bc} + i\bar\theta^a t_a - \bar\theta^b t_b\theta^c t_c + \frac{1}{2}\bar\theta^b\bar\theta^c t_{bc} \\ \Leftrightarrow\, & i f^a{}_{bc}\bar\theta^b\theta^c t_a + \frac{1}{2}\left(\theta^b\bar\theta^c + \bar\theta^b\theta^c\right)t_{bc} = -\bar\theta^b\theta^c t_b t_c \\ \Leftrightarrow\, & \bar\theta^b\theta^c\left[t_{bc} + i f^a{}_{bc}t_a + t_b t_c\right] = 0 \\ \Leftrightarrow\, & t_{bc} = -i f^a{}_{bc}t_a - t_b t_c\end{aligned}$$

4. *Eq.* (2.2.22) ($P.54$)

$$\begin{aligned}-i f^a{}_{bc}t_a - t_b t_c \overset{(2.2.21)}{=} t_{bc} = t_{cb} \overset{(2.2.21)}{=} -i f^a{}_{cb}t_a - t_c t_b \\ \Leftrightarrow [t_b, t_c] = i\left(f^a{}_{cb} - f^a{}_{bc}\right)t_a\end{aligned}$$

C. Quantum Lorentz Transformations

1. *"$\Lambda^\mu{}_\nu$ has an **inverse**"* ($P.57$)

This is true because

$$\det(\Lambda) = \pm 1 \neq 0$$

2. *"$\left(\bar\Lambda\Lambda\right)^0{}_0 \geq \bar\Lambda^0{}_0\Lambda^0{}_0 - \sqrt{\left(\Lambda^0{}_0\right)^2 - 1}\sqrt{\left(\bar\Lambda^0{}_0\right)^2 - 1} \geq 1$"* ($P.58$)

The first inequality is trivial from the preceding considerations. For the second inequality assume the contrary holds, then:

$$\begin{aligned}&0 \leq \bar\Lambda^0{}_0\Lambda^0{}_0 - 1 < \sqrt{\left(\Lambda^0{}_0\right)^2 - 1}\sqrt{\left(\bar\Lambda^0{}_0\right)^2 - 1} \\ &\Rightarrow \left(\bar\Lambda^0{}_0\right)^2\left(\Lambda^0{}_0\right)^2 - 2\bar\Lambda^0{}_0\Lambda^0{}_0 + 1 \\ &\quad < \left(\bar\Lambda^0{}_0\right)^2\left(\Lambda^0{}_0\right)^2 - \left(\bar\Lambda^0{}_0\right)^2 - \left(\Lambda^0{}_0\right)^2 + 1 \\ &\Rightarrow \left(\Lambda^0{}_0 + \bar\Lambda^0{}_0\right)^2 = \left(\bar\Lambda^0{}_0\right)^2 - 2\bar\Lambda^0{}_0\Lambda^0{}_0 + \left(\Lambda^0{}_0\right)^2 < 0\end{aligned}$$

Which is a contradiction as $\Lambda^0{}_0 + \bar\Lambda^0{}_0 \geq 1 + 1 = 2$ and therefore completes the proof.

D. The Poincaré Algebra

1. *"In order fo $U(1+\omega,\varepsilon)$ to be unitary, the operators $J^{\rho\sigma}$ and P^ρ must be **Hermitian**"* ($P.59$)

Analog to II B 1.

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2. *Eqs.* (2.4.8/9) (*P.*60)

$$
\begin{aligned}
&\frac{1}{2}\omega_{\rho\sigma}UJ^{\rho\sigma}U^{-1}-\varepsilon_\rho UP^\rho U^{-1}\\
&\overset{(2.4.7)}{=}\frac{1}{2}\left(\Lambda\omega\Lambda^{-1}\right)_{\mu\nu}J^{\mu\nu}-\left(\Lambda\varepsilon-\Lambda\omega\Lambda^{-1}a\right)_\mu P^\mu\\
&=\frac{1}{2}\Lambda_\mu{}^\rho\omega_{\rho\sigma}\left(\Lambda^{-1}\right)^\sigma{}_\nu J^{\mu\nu}\\
&-\left(\Lambda_\mu{}^\rho\varepsilon_\rho-\Lambda_\mu{}^\rho\omega_{\rho\sigma}\left(\Lambda^{-1}\right)^\sigma{}_\nu a^\nu\right)P^\mu\\
&\overset{(2.3.10)}{=}\frac{1}{2}\Lambda_\mu{}^\rho\omega_{\rho\sigma}\Lambda_\nu{}^\sigma J^{\mu\nu}\\
&-\left(\Lambda_\mu{}^\rho\varepsilon_\rho-\Lambda_\mu{}^\rho\omega_{\rho\sigma}\Lambda_\nu{}^\sigma a^\nu\right)P^\mu\\
&=\frac{1}{2}\omega_{\rho\sigma}\left(\Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma J^{\mu\nu}+\Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma a^\nu P^\mu\right)\\
&-\varepsilon_\rho\Lambda_\mu{}^\rho P^\mu
\end{aligned}
$$

In order to be able to compare coefficients in this, the coefficient of $\omega_{\rho\sigma}$ has to be anti symmatrized. With this Eqs. (2.4.8/9) follow immediately.

3. *Eqs.* (2.4.10/11) (*P.*60)

Up to $\mathcal{O}(\omega,\varepsilon)$ one can identify

$$U^{-1}(1+\omega,\varepsilon)=U(1-\omega,-\varepsilon)$$

,since

$$U(1+\omega,\varepsilon)U(1-\omega,-\varepsilon)=U(1-\omega+\omega,-\varepsilon+\varepsilon)=U(1,0).$$

With this we have up to $\mathcal{O}(\omega,\varepsilon)$

$$
\begin{aligned}
&i\left[\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}-\varepsilon_\mu P^\mu,J^{\rho\sigma}\right]\\
&=\left(1+\frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu}-i\varepsilon_\mu P^\mu\right)J^{\rho\sigma}\\
&\cdot\left(1-\frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu}+i\varepsilon_\mu P^\mu\right)-J^{\rho\sigma}\\
&=UJ^{\rho\sigma}U^{-1}-J^{\rho\sigma}\\
&\overset{(2.4.8)}{=}(1+\omega)_\mu{}^\rho(1+\omega)_\nu{}^\sigma\\
&\cdot\left(J^{\mu\nu}-\varepsilon^\mu P^\nu+\varepsilon^\nu P^\mu\right)-J^{\rho\sigma}\\
&=-\varepsilon^\rho P^\sigma+\varepsilon^\sigma P^\rho+\omega_\nu{}^\sigma J^{\rho\nu}+\omega_\mu{}^\rho J^{\mu\sigma}
\end{aligned}
$$

and also

$$
\begin{aligned}
&i\left[\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}-\varepsilon_\mu P^\mu,P^\rho\right]\\
&=\left(1+\frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu}-i\varepsilon_\mu P^\mu\right)P^\rho\\
&\cdot\left(1-\frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu}+i\varepsilon_\mu P^\mu\right)-P^\rho\\
&=UP^\rho U^{-1}-P^\rho\\
&\overset{(2.4.9)}{=}\omega_\mu{}^\rho P^\mu
\end{aligned}
$$

4. *Eqs.* (2.4.12/13/14) (*P.*60)

The only difficulty lies in the derivation of Eq. (2.4.12) because for this one has to anti symmatrize the coefficient of $\omega_{\mu\nu}$ in Eq. (2.4.10) in order to be able to compare coefficients, similar to II D 2. (Similarly if one derives Eq. (2.4.13) from Eq. (2.4.11))

5. *Eqs.* (2.4.18 − 24) (*P.*61)

First observe

$$J^{ij}=\varepsilon_{ijk}J_k$$

and

$$J_i=\frac{1}{2}\varepsilon_{lmi}J^{lm}$$

$$
\begin{aligned}
\left[J_i,J_j\right]&=\frac{\varepsilon_{lmi}\varepsilon_{kpj}}{4}\left[J^{lm},J^{kp}\right]\\
&\overset{(2.4.12)}{=}-i\frac{\varepsilon_{lmi}\varepsilon_{kpj}}{4}\left[\eta^{mk}J^{lp}-\eta^{lk}J^{mp}-\eta^{pl}J^{km}+\eta^{pm}J^{kl}\right]\\
&=-\frac{i}{2}\left[\varepsilon_{kil}\varepsilon_{kmj}J^{lm}+\varepsilon_{kim}\varepsilon_{kjl}J^{lm}\right]\\
&=-\frac{i}{2}\left[J^{ji}-J^{ij}\right]\\
&=iJ^{ij}=i\varepsilon_{ijk}J_k\\
\left[J_i,K_j\right]&=\left[J^{lm},J^{0j}\right]\frac{\varepsilon_{lmi}}{2}\\
&\overset{(2.4.12)}{=}-i\frac{\varepsilon_{lmi}}{2}\left[\eta^{m0}J^{lj}-\eta^{l0}J^{mj}-\eta^{jl}J^{0m}+\eta^{jm}J^{0l}\right]\\
&=-i\frac{\varepsilon_{lmi}}{2}\left[\delta_{jm}K_l-\delta_{jl}K_m\right]=i\varepsilon_{ijl}K_l\\
\left[K_i,K_j\right]&=\left[J^{0i},J^{0j}\right]\\
&\overset{(2.4.12)}{=}-i\left[\eta^{i0}J^{0j}-\eta^{00}J^{ij}-\eta^{j0}J^{0i}+\eta^{ij}J^{00}\right]\\
&=-iJ^{ij}=-i\varepsilon_{ijk}J_k\\
\left[J_i,P_j\right]&=\frac{\varepsilon_{lmi}}{2}\left[J^{lm},P^j\right]\\
&\overset{(2.4.13)}{=}\frac{i\varepsilon_{lmi}}{2}\left[\eta^{jl}P^m-\eta^{jm}P^l\right]\\
&=\frac{i}{2}\left[\varepsilon_{jmi}P^m-\varepsilon_{mji}P^m\right]=i\varepsilon_{ijm}P_m\\
\left[K_i,P_j\right]&=\left[J^{0i},P^j\right]\\
&\overset{(2.4.13)}{=}i\left[\eta^{j0}P^i-\eta^{ji}P^0\right]\\
&=-i\delta_{ji}P^0=-i\delta_{ij}H\\
\left[J_i,H\right]&\overset{(2.4.13)}{=}\frac{i\varepsilon_{lmi}}{2}\left[\eta^{0l}P^m-\eta^{0m}P^l\right]=0\\
\left[P_i,H\right]&=\left[P^i,P^0\right]\overset{(2.4.14)}{=}0\\
\left[H,H\right]&=\left[P^0,P^0\right]\overset{(2.4.14)}{=}0\\
\left[K_i,H\right]&=\left[J^{0i},P^0\right]\\
&\overset{(2.4.13)}{=}i\left[\eta^{00}P^i-\eta^{0i}P^0\right]=-iP_i
\end{aligned}
$$

6. Eq. (2.4.27) (*P.*61)

When trying to check this for e.g. the standard representation in 4 dimensions, in terms of infinitesimal rotations, attention with the sign in front of sin for the different rotation axis.

7. "Inspection of Eqs. (2.4.18 − 24) *shows that these commutation relations have a limit for* $v \ll 1$ *of the form ...*" (*P.*62)

Always equate same orders in v for this.

8. "$\exp(-i\mathbf{K}\cdot\mathbf{v})\exp(-i\mathbf{P}\cdot\mathbf{a}) = \exp(iM\mathbf{a}\cdot\mathbf{v}/2)\exp(-i(\mathbf{K}\cdot\mathbf{v}+\mathbf{P}\cdot\mathbf{a}))$" (*P.*62)

Use BCH Formula

$$\begin{aligned}&\exp(-iK_iv_i)\exp(-iP_ja_j)\\&=\exp\left(-i(K_iv_i+P_ja_j)+\frac{1}{2}(-i)^2[K_i,P_j]v_ia_j+0\right)\\&=\exp\left(-i(K_iv_i+P_ja_j)+\frac{1}{2}iMv_ia_i\right)\end{aligned}$$

E. One-Particle States

1. Eq. (2.5.2) (*P.*63)

Λ^{-1} shows up here since in Eq. (2.4.9) UPU^{-1} is given but here $U^{-1}PU$ is being used.

2. "with σ *within any one block by themselves furnish a representation of the inhomogeneous Lorentz group"* (*P.*63)

In this case the Blocks do not mix with other blocks.

3. "and for $p^2 \leq 0$*, also the sign of* p^0*"* (*P.*64)

For $p^2 \leq 0$ we have

$$\begin{aligned}p^2 &= -\left(p^0\right)^2+\vec{p}^2 \leq 0\\ \Rightarrow |\vec{p}| &\leq \left|p^0\right|\end{aligned}$$

and from Eq. (2.3.13) we know

$$\left|\Lambda^0{}_0\right| \geq \left|\vec{\Lambda^0}_{.}\right|.$$

First suppose $p^0 \geq 0$:

$$\begin{aligned}p'^0 &= \Lambda^0{}_0p^0+\Lambda^0{}_ip^i\\&\geq \Lambda^0{}_0p^0-\left|\vec{\Lambda^0}_{.}\right||\vec{p}|\\&=\left|\Lambda^0{}_0\right|\left|p^0\right|-\left|\vec{\Lambda^0}_{.}\right||\vec{p}|\\&\geq\left|\vec{\Lambda^0}_{.}\right|\left(\left|p^0\right|-|\vec{p}|\right)\geq 0\end{aligned}$$

Now suppose $p^0 \leq 0$:

$$\begin{aligned}p'^0 &= \Lambda^0{}_0p^0+\Lambda^0{}_ip^i\\&\leq \Lambda^0{}_0p^0+\left|\vec{\Lambda^0}_{.}\right||\vec{p}|\\&=-\left|\Lambda^0{}_0\right|\left|p^0\right|+\left|\vec{\Lambda^0}_{.}\right||\vec{p}|\\&\leq\left|\Lambda^0{}_0\right|\left(-\left|p^0\right|+|\vec{p}|\right)\leq 0\end{aligned}$$

4. Eq. (2.5.12) *"The delta function appears here because* $\Psi_{k,\sigma}$ *and* $\Psi_{k',\sigma'}$ *are eigenstates of a Hermitian operator with eigenvalues* $\mathbf{k}$ *and* $\mathbf{k}'$*, respectively."* (*P.*66)

> Note: From this point forward in this section of the book the states considered have standard momentum k^μ! (see Text in the Book preceding Eq. (2.5.12))

Suppose that for given values of $\mathbf{k}$ and $\mathbf{k}'$

$$\langle\Psi_{k',\sigma'}|\Psi_{k,\sigma}\rangle \neq 0$$

then from

$$\begin{aligned}&k^i\langle\Psi_{k',\sigma'}|\Psi_{k,\sigma}\rangle\\&=\left\langle\Psi_{k',\sigma'}\middle|P^i\Psi_{k,\sigma}\right\rangle\\&=\left\langle P^i\Psi_{k',\sigma'}\middle|\Psi_{k,\sigma}\right\rangle\\&=k'^i\langle\Psi_{k',\sigma'}|\Psi_{k,\sigma}\rangle\end{aligned}$$

follows

$$k^i = k'^i.$$

On why there is only a 3 dimensional delta function showing up:

This is because both states are assumed to have standard momentum k^μ and therefore their 0-th component is completely fixed from their spatial components:

$$k'^0 = \pm\sqrt{\mathbf{k}'^2 - k^2}$$

Choice between + and − comes from sign of k^0, i.e. + for cases (a) and (c) of Table 2.1.

5. Eq. (2.5.13) (*P*.67)

$$\begin{aligned}
&\langle U(W)\Psi_{k',\sigma'}|U(W)\Psi_{k,\sigma}\rangle \\
&\overset{(2.5.8)}{=} \sum_{\sigma''\sigma'''} \langle D_{\sigma''\sigma'}\Psi_{k',\sigma''}|D_{\sigma'''\sigma}\Psi_{k,\sigma'''}\rangle \\
&= \sum_{\sigma''\sigma'''} \left(D_{\sigma''\sigma'}\right)^\star D_{\sigma'''\sigma} \langle \Psi_{k',\sigma''}|\Psi_{k,\sigma'''}\rangle \\
&\overset{(2.5.12)}{=} \sum_{\sigma''} \left(D_{\sigma''\sigma'}\right)^\star D_{\sigma''\sigma} \delta^{(3)}(\mathbf{k'}-\mathbf{k}) \\
&\overset{(2.2.2)}{=} \langle \Psi_{k',\sigma'}|\Psi_{k,\sigma}\rangle \\
&\overset{(2.5.12)}{=} \delta_{\sigma'\sigma}\delta^{(3)}(\mathbf{k'}-\mathbf{k})
\end{aligned}$$

From this it follows

$$D^\dagger(W) = D^{-1}(W)$$

6. "$\langle\Psi_{p',\sigma'}|\Psi_{p,\sigma}\rangle = \dots$" (*P*.67)

Note: What is meant by "arbitrary momenta" is that these momenta still have the standard momentum k^μ, but now none of the states has exactly k^μ as its momentum.

First define

$$k' := L^{-1}(p)p'$$

with this we get:

$$\begin{aligned}
&\langle\Psi_{p',\sigma'}|\Psi_{p,\sigma}\rangle \\
&\overset{(2.5.5)}{=} \langle\Psi_{p',\sigma'}|N(p)U(L(p))\Psi_{k,\sigma}\rangle \\
&= N(p)\left\langle U(L^{-1}(p))\Psi_{p',\sigma'}\middle|\Psi_{k,\sigma}\right\rangle \\
&\overset{(2.5.11)}{=} N(p)\frac{N^\star(p')}{N^\star(k')} \\
&\cdot \sum_{\sigma''} D^\star_{\sigma''\sigma'}\left(W\left(L^{-1}(p),p'\right)\right)\langle\Psi_{k',\sigma''}|\Psi_{k,\sigma}\rangle \\
&\overset{(2.5.12)}{=} N(p)N^\star(p')\, D^\star_{\sigma\sigma'}\left(W\left(L^{-1}(p),p'\right)\right)\delta^{(3)}(\mathbf{k'}-\mathbf{k})
\end{aligned}$$

Where in the last step we used

$$\frac{\delta^{(3)}(\mathbf{k'}-\mathbf{k})}{N^\star(k')} = \frac{\delta^{(3)}(\mathbf{k'}-\mathbf{k})}{N^\star(k)} \overset{(2.5.5)}{=} \delta^{(3)}(\mathbf{k'}-\mathbf{k})$$

This is valid as from Eq. (2.5.5) we know that $N(p)$ is implicitly dependent on k^μ. This therefor fixes $p^2 = k^2$ and the sign of p^0 (see end of II E 4), s.t.

$$N(p) = N(\mathbf{p}).$$

Further we are allowed to use Eq. (2.5.12) in the last step since k' also has k as its standard momentum:

$$k' = L^{-1}(p)p' = \underbrace{L^{-1}(p)L(p')}_{=:L(k')}k$$

7. "$W\left(L^{-1}(p),p\right)=1$" (*P*.67)

First note

$$L(k) = 1 = L^{-1}(k)$$

with this we get:

$$\begin{aligned}
W\left(L^{-1}(p),p\right) &\overset{(2.5.10)}{=} L^{-1}\left(L^{-1}(p)p\right)L^{-1}(p)L(p) \\
&= L^{-1}(k) = 1
\end{aligned}$$

*8. "So we see that the **invariant delta function** is …"* (*P*.67)

Invariant in this case means w.r.t. proper orthochronous Lorentz transformations, which can be interpreted as change of variables under the $\int \mathrm{d}^4p$ integral.

9. Eq. (2.5.24) (*P*.68)

$$\begin{aligned}
p^0 &= L^0{}_0k^0 + L^0{}_ik^i \\
&= \frac{\sqrt{\mathbf{p}^2+M^2}}{M}M = \sqrt{\mathbf{p}^2+M^2} \\
p^i &= L^i{}_0k^0 + L^i{}_jk^j \\
&= \frac{p_i}{|\mathbf{p}|}\sqrt{\frac{\mathbf{p}^2+M^2}{M^2}-1}M \\
&= \frac{p_i}{|\mathbf{p}|}\sqrt{\frac{\mathbf{p}^2}{M^2}}M = p_i
\end{aligned}$$

10. "To see this, note that the boost Eq. (2.5.24) *may be expressed as $L(p) = R(\hat{\mathbf{p}})B(|\mathbf{p}|)R^{-1}(\hat{\mathbf{p}})$"* (*P*.68)

First note that the columns and rows of the matrix $B(|\mathbf{p}|)$ are counted in the order 1,2,3,0.

Let

$$\hat{\mathbf{p}} = \begin{pmatrix}\sin(\theta)\cos(\phi)\\ \sin(\theta)\sin(\phi)\\ \cos(\theta)\end{pmatrix}$$

then we have

$$R(-\hat{\mathbf{p}}) = R_3(-\phi)R_2(\theta) = \begin{pmatrix} \cos(\phi)\cos(\theta) & -\sin(\phi) & \cos(\phi)\sin(\theta) \\ \sin(\phi)\cos(\theta) & \cos(\phi) & \sin(\phi)\sin(\theta) \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

(see II E 20) and

$$\begin{aligned} R^{-1}(\hat{\mathbf{p}}) = R_2(\theta)R_3(\phi) &= \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta)\cos(\phi) & \cos(\theta)\sin(\phi) & -\sin(\theta) \\ -\sin(\phi) & \cos(\phi) & 0 \\ \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \end{pmatrix} = R^\top(\hat{\mathbf{p}}) \end{aligned}$$

From Eq. (2.5.24) we know

$$L(p) = \begin{pmatrix} 1+(\gamma-1)s_\theta^2 c_\phi^2 & (\gamma-1)s_\theta c_\phi s_\theta s_\phi & (\gamma-1)s_\theta c_\phi c_\theta & s_\theta c_\phi\sqrt{\gamma^2-1} \\ (\gamma-1)s_\theta c_\phi s_\theta s_\phi & 1+(\gamma-1)s_\theta^2 s_\phi^2 & (\gamma-1)s_\theta s_\phi c_\theta & s_\theta s_\phi\sqrt{\gamma^2-1} \\ (\gamma-1)s_\theta c_\phi c_\theta & (\gamma-1)s_\theta s_\phi c_\theta & 1+(\gamma-1)c_\theta^2 & c_\theta\sqrt{\gamma^2-1} \\ s_\theta c_\phi\sqrt{\gamma^2-1} & s_\theta s_\phi\sqrt{\gamma^2-1} & c_\theta\sqrt{\gamma^2-1} & \gamma \end{pmatrix}$$

where s and c are shorthand for sin and cos. With this we can now check:

$$\begin{aligned} R(\hat{\mathbf{p}})B(|\mathbf{p}|)R^{-1}(\hat{\mathbf{p}}) &= \begin{pmatrix} c_\phi c_\theta & -s_\phi & c_\phi s_\theta & 0 \\ s_\phi c_\theta & c_\phi & s_\phi s_\theta & 0 \\ -s_\theta & 0 & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & \sqrt{\gamma^2-1} \\ 0 & 0 & \sqrt{\gamma^2-1} & \gamma \end{pmatrix} \begin{pmatrix} c_\phi c_\theta & s_\phi c_\theta & -s_\theta & 0 \\ -s_\phi & c_\phi & 0 & 0 \\ c_\phi s_\theta & s_\phi s_\theta & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_\phi c_\theta & -s_\phi & c_\phi s_\theta & 0 \\ s_\phi c_\theta & c_\phi & s_\phi s_\theta & 0 \\ -s_\theta & 0 & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_\phi c_\theta & s_\phi c_\theta & -s_\theta & 0 \\ -s_\phi & c_\phi & 0 & 0 \\ \gamma c_\phi s_\theta & \gamma s_\phi s_\theta & \gamma c_\theta & \sqrt{\gamma^2-1} \\ \sqrt{\gamma^2-1}c_\phi s_\theta & \sqrt{\gamma^2-1}s_\phi s_\theta & \sqrt{\gamma^2-1}c_\theta & \gamma \end{pmatrix} = L(p) \end{aligned}$$

11. "$W(\mathbf{R},p)=\mathbf{R}$" (P.69)

To see this just substitute $R(\theta)$ back into the previous result.

12. Eq. (2.5.25) (P.70)

$$\begin{aligned} -1 = (Wt)^\mu (Wt)_\mu &= \alpha^2+\beta^2+\zeta^2-(1+\zeta)^2 \\ \Leftrightarrow \alpha^2+\beta^2 &= 2\zeta \end{aligned}$$

13. Eq. (2.5.26) (P.70)

This is a Lorentz transformation, since

$$\begin{aligned} S^\top \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} S &= \begin{pmatrix} 1 & 0 & \alpha & \alpha \\ 0 & 1 & \beta & \beta \\ -\alpha & -\beta & 1-\zeta & -\zeta \\ \alpha & \beta & \zeta & 1+\zeta \end{pmatrix} \\ &\cdot \begin{pmatrix} 1 & 0 & -\alpha & \alpha \\ 0 & 1 & -\beta & \beta \\ \alpha & \beta & 1-\zeta & \zeta \\ -\alpha & -\beta & \zeta & -1-\zeta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

14. Eqs. (2.5.29/30) (*P.*70)

Eq. (2.5.30) is trivial since the rotations around the three axis satisfy this group multiplication and

$$S(0,0) = \mathbf{1}.$$

For Eq. (2.5.29) explicit calculation shows it (see TODO) together with

$$R(0) = \mathbf{1}.$$

15. Eq. (2.5.31) (*P.*70)

Explicit calculation shows Eq. (2.5.31) (see TODO) and the invariance follows then immediately, since:

$$W'SW'^{-1} = S'\underbrace{R'SR'^{-1}}_{=S''}S'^{-1} = S'''$$

16. "$W(\theta,\alpha,\beta) = 1 + \omega$" (P.71)

From Eq. (2.5.28) we immediately get for infinitesimal θ, α, β :

$$(W(\theta,\alpha,\beta))^\mu{}_\nu = \delta^\mu_\nu + \begin{pmatrix} 0 & \theta & -\alpha & \alpha \\ -\theta & 0 & -\beta & \beta \\ \alpha & \beta & 0 & 0 \\ \alpha & \beta & 0 & 0 \end{pmatrix}^\mu{}_\nu$$

$$\Rightarrow \omega_{\mu\nu} = \begin{pmatrix} 0 & \theta & -\alpha & \alpha \\ -\theta & 0 & -\beta & \beta \\ \alpha & \beta & 0 & 0 \\ -\alpha & -\beta & 0 & 0 \end{pmatrix}_{\mu\nu}$$

17. Eq. (2.5.32) (*P.*71)

$$\begin{aligned} U(1+\omega) &\overset{(2.4.3)}{=} 1 + \frac{1}{2} i\omega_{\rho\sigma}J^{\rho\sigma} \\ &= 1 + i\theta J^{12} + i\alpha(-J^{13} + J^{10}) + i\beta(-J^{23} + J^{20}) \\ &= 1 + i\theta J_3 + i\alpha(J_2 - K_1) + i\beta(-J_1 - K_2) \end{aligned}$$

18. Eqs. (2.5.35/36/37) (*P.*71)

From Eqs. (2.4.18/19/20) we get:

$$\begin{aligned} [J_3, A] &= [J_3, J_2 - K_1] \\ &= [J_3, J_2] - [J_3, K_1] \\ &= -iJ_1 - iK_2 = iB \\ [J_3, B] &= -[J_3, J_1] - [J_3, K_2] \\ &= -iJ_2 - i(-K_1) = -iA \\ [A, B] &= -[J_2, J_1] - [J_2, K_2] + [K_1, J_1] + [K_1, K_2] \\ &= iJ_3 - iJ_3 = 0 \end{aligned}$$

19. "Since A and B are commuting Hermitian operators they can be simultaneously diagonalized by states $\Psi_{k,a,b}$" (P.71)

This is valid, including the label k, since

$$\begin{aligned} [A, P_1]\Psi_{k,a,b} &= ([J_2, P_1] - [K_1, P_1])\Psi_{k,a,b} \\ &= (-iP_3 + iP^0)\Psi_{k,a,b} = 0 \\ [A, P_2]\Psi_{k,a,b} &= ([J_2, P_2] - [K_1, P_2])\Psi_{k,a,b} \\ &= 0 \\ [A, P_3]\Psi_{k,a,b} &= ([J_2, P_3] - [K_1, P_3])\Psi_{k,a,b} \\ &= (iP_1)\Psi_{k,a,b} = 0 \\ \left[A, P^0\right]\Psi_{k,a,b} &= (\left[J_2, P^0\right] - \left[K_1, P^0\right])\Psi_{k,a,b} \\ &= (-iP_1)\Psi_{k,a,b} = 0 \\ [B, P_1]\Psi_{k,a,b} &= (-[J_1, P_1] - [K_2, P_1])\Psi_{k,a,b} \\ &= 0 \\ [B, P_2]\Psi_{k,a,b} &= ([J_1, P_2] - [K_2, P_2])\Psi_{k,a,b} \\ &= (iP_3 - iP^0)\Psi_{k,a,b} = 0 \\ [B, P_3]\Psi_{k,a,b} &= ([J_1, P_3] - [K_2, P_3])\Psi_{k,a,b} \\ &= (-iP_2)\Psi_{k,a,b} = 0 \\ \left[B, P^0\right]\Psi_{k,a,b} &= (\left[J_1, P^0\right] - \left[K_2, P^0\right])\Psi_{k,a,b} \\ &= (-iP_2)\Psi_{k,a,b} = 0 \end{aligned}$$

20. "take $R(\hat{\mathbf{p}})$ as a rotation by angle θ around the two-axis followed by a rotation by angle ϕ around the three-axis" (P.73)

Let

$$\hat{\mathbf{p}} = \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix}$$

then

$$\begin{aligned} R(\hat{\mathbf{p}}) &= R_3(-\phi)R_2(-\theta) \\ &= \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\phi)\cos(\theta) & -\sin(\phi) & \cos(\phi)\sin(\theta) \\ \sin(\phi)\cos(\theta) & \cos(\phi) & \sin(\phi)\sin(\theta) \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}, \end{aligned}$$

(With this sign convention everything is consistent, see definition of $R(\theta)$ after Eq. (2.5.27) and compare Eq. (2.5.47) to Eq. (2.4.27)) with this we have

$$\hat{\mathbf{p}} = R(\hat{\mathbf{p}})\hat{e}_3.$$

F. Space Inversion and Time-Reversal

G. Projective Representations

H. The Symmetry Representation Theorem

1. *"But $\langle \Psi'_k | \Psi'_k \rangle$ is automatically **real and positive**" (P.91)*

This follows immediately from Eq. (2.1.1).

2. *"From Eq. (2.A.1) we have $|c_{kk}| = |c_{k1}| = \frac{1}{\sqrt{2}}$ and for $l \neq k$ and $l \neq 1$: $c_{kl} = 0$" (P.91)*

$$
\begin{aligned}
|c_{kl'}|^2 &\overset{(2.A.3)}{=} \left| \sum_l c^\star_{kl} \langle \Psi'_l | \Psi'_{l'} \rangle \right|^2 = |\langle \Upsilon'_k | \Psi'_{l'} \rangle|^2 \\
&\overset{(2.A.1)}{=} |\langle \Upsilon_k | \Psi_{l'} \rangle|^2 = \begin{cases} \frac{1}{2} & \text{for } l' = 1, k \\ 0 & \text{for } l' \neq 1, k \end{cases}
\end{aligned}
$$

3. *Eq. (2.A.10) (P.92)*

$$
\begin{aligned}
|C_k|^2 + |C_1|^2 + 2\,\mathrm{Re}(C_k C_1^\star) &= |C_k + C_1|^2 \\
\overset{(2.A.9)}{=} |C'_k + C'_1|^2 &= |C'_k|^2 + |C'_1|^2 + 2\,\mathrm{Re}(C'_k C_1'^\star) \\
\overset{Eq.\,(2.A.8)}{\Rightarrow} \mathrm{Re}(C_k C_1^\star) &= \mathrm{Re}(C'_k C_1'^\star) \\
\overset{Eq.\,(2.A.8)}{\Rightarrow} \mathrm{Re}\left(\frac{C_k}{C_1}\right) &= \mathrm{Re}\left(\frac{C'_k}{C'_1}\right)
\end{aligned}
$$

4. *Eq. (2.A.11) (P.92)*

$$
\begin{aligned}
\left\{\mathrm{Re}\left(\frac{C_k}{C_1}\right)\right\}^2 + \left\{\mathrm{Im}\left(\frac{C_k}{C_1}\right)\right\}^2 &= \left|\frac{C_k}{C_1}\right|^2 \\
\overset{Eq.\,(2.A.8)}{=} \left|\frac{C'_k}{C'_1}\right|^2 &= \left\{\mathrm{Re}\left(\frac{C'_k}{C'_1}\right)\right\}^2 + \left\{\mathrm{Im}\left(\frac{C'_k}{C'_1}\right)\right\}^2 \\
\overset{Eq.\,(2.A.10)}{\Rightarrow} \mathrm{Im}\left(\frac{C_k}{C_1}\right) &= \pm\,\mathrm{Im}\left(\frac{C'_k}{C'_1}\right)
\end{aligned}
$$

5. *"This is only possible if $\mathrm{Re}\left(\frac{C_k}{C_1}\frac{C_l^\star}{C_1^\star}\right) = \mathrm{Re}\left(\frac{C_k}{C_1}\frac{C_l}{C_1}\right)$ or, in other words, if $\mathrm{Im}\left(\frac{C_k}{C_1}\right)\mathrm{Im}\left(\frac{C_l}{C_1}\right) = 0$" (P.93)*

Define

$$
\begin{aligned}
a &:= \frac{C_k}{C_1} \\
b &:= \frac{C_l}{C_1}
\end{aligned}
$$

With this we have

$$
\begin{aligned}
|1 + a + b^\star|^2 &= |1 + a + b|^2 && (1) \\
\Leftrightarrow 1 + a^\star + b + a + |a|^2 + ab + b^\star + b^\star a^\star + |b|^2 & && (2) \\
= 1 + a^\star + b^\star + a + |a|^2 + ab^\star + b + ba^\star + |b|^2 & && (3) \\
\Leftrightarrow ab + b^\star a^\star &= ab^\star + ba^\star && (4)
\end{aligned}
$$

And further rewriting yields

$$
\begin{aligned}
\mathrm{Re}\left(\frac{C_k}{C_1}\frac{C_l}{C_1}\right) &= \mathrm{Re}(ab) \overset{4}{=} \mathrm{Re}(ab^\star) = \mathrm{Re}\left(\frac{C_k}{C_1}\frac{C_l^\star}{C_1^\star}\right) \\
\mathrm{Im}\left(\frac{C_k}{C_1}\right)\mathrm{Im}\left(\frac{C_l}{C_1}\right) &= \mathrm{Im}(a)\,\mathrm{Im}(b) \\
&= -\frac{1}{4}(ab - ab^\star - a^\star b + a^\star b^\star) \overset{4}{=} 0
\end{aligned}
$$

6. *"Then the invariance of transition probabilities requires that $\left|\sum_k B_k^\star A_k\right|^2 = \left|\sum_k B_k A_k\right|^2$" (P.93)*

$$
\begin{aligned}
\left|\sum_k B_k^\star A_k\right|^2 &\overset{Eq.\,(2.A.2)}{=} \left|\sum_{kl} B_k^\star A_l \langle \Psi_k | \Psi_l \rangle\right|^2 \\
&= \left|\left\langle \sum_k B_k \Psi_k \middle| \sum_l A_l \Psi_l \right\rangle\right|^2 \\
&\overset{Eq.\,(2.A.1)}{=} \left|\left\langle U\left(\sum_k B_k \Psi_k\right) \middle| U\left(\sum_l A_l \Psi_l\right) \right\rangle\right|^2 \\
&= \left|\left\langle \sum_k B_k^\star U \Psi_k \middle| \sum_l A_l U \Psi_l \right\rangle\right|^2 \\
&= \left|\sum_{kl} B_k A_l \langle U\Psi_k | U\Psi_l \rangle\right|^2 \\
&\overset{Eq.\,(2.A.3)}{=} \left|\sum_k B_k A_k\right|^2
\end{aligned}
$$

7. Eq. (2.*A*.16)(*P*.94)

$$
\begin{aligned}
&\sum_{kl} \operatorname{Im}(B_k^\star B_l)\operatorname{Im}(A_k^\star A_l)\\
&= \operatorname{Im}\left(\sum_{kl}\operatorname{Im}(B_k^\star B_l)A_k^\star A_l\right)\\
&= \frac{1}{2i}\left[\sum_{kl}\operatorname{Im}(B_k^\star B_l)A_k^\star A_l - \text{c.c.}\right]\\
&= \frac{1}{2i}\left[\sum_{kl}\frac{1}{2i}(B_k^\star B_l - B_k B_l^\star)A_k^\star A_l - \text{c.c.}\right]\\
&= \frac{1}{2i}\left[\frac{1}{2i}\left(\sum_{kl}B_k^\star B_l A_k^\star A_l - \sum_{kl}B_k B_l^\star A_k^\star A_l\right) - \text{c.c.}\right]\\
&= \frac{1}{2i}\left[\frac{1}{2i}\left(\left|\sum_k B_k A_k\right|^2 - \left|\sum_k B_k^\star A_k\right|^2\right) - \text{c.c.}\right]\\
&= \frac{1}{2i}\left[\frac{1}{i}\left(\left|\sum_k B_k A_k\right|^2 - \left|\sum_k B_k^\star A_k\right|^2\right)\right]\\
&\overset{II\,H\,6}{=} 0
\end{aligned}
$$

8. "However, for any pair of such state-vectors, with neither A_k nor B_k ***all of the same phase****"* (*P*.94)

If they were all of the same phase then

$$\forall k,l : \operatorname{Im}\{A_k^\star A_l\} = 0$$

or

$$\forall k,l : \operatorname{Im}\{B_k^\star B_l\} = 0$$

See Footnote j, for why this is relevant.

9. "We have thus shown that for a given symmetry transformation T either all state-vectors satisfy Eq. (2.*A*.14) *or else they all satisfy Eq.* (2.*A*.15)*"* (*P*.94)

Preceding this a contradiction was derived from the assumption that Eq. (2.*A*.14) applies for a state-vector

$$\sum_k A_k \Psi_k$$

while Eq. (2.*A*.15) applies for a state vector

$$\sum_k B_k \Psi_k.$$

Such that the statement is obvious.

I. Group Operators and Homotopy Classes

J. Inversions and Degenerate Multiplets

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