Notes for "The Quantum Theory of Fields 1, Foundations" - Weinberg

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I. INTRODUCTION

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II. RELATIVISTIC QUANTUM MECHANICS

A. Quantum Mechanics

B. Symmetries

1. *"For this to be unitary and linear, t must be Hermitian and linear"* ($P.51$)

Linearity is trivial and hermiticity follow from the following observation:

$$\begin{aligned}\langle U\Psi|U\Phi\rangle &= \langle(1+i\varepsilon t)\Psi|(1+i\varepsilon t)\Phi\rangle\\ &= \langle\Psi|\Phi\rangle + \varepsilon i\left(\langle\Psi|t\Phi\rangle - \langle t\Psi|\Phi\rangle\right) + \mathcal{O}\left(\varepsilon^2\right)\\ &\overset{Eq.\,(2.2.2)}{\Leftrightarrow} \langle\Psi|t\Phi\rangle = \langle t\Psi|\Phi\rangle\\ &\overset{Eq.\,(2.1.5)}{\Leftrightarrow} t^\dagger = t\end{aligned}$$

2. *Eq.* (2.2.19) ($P.54$)

f^a_{bc} and f^a have to be real as θ^a are real.

3. *Eq.* (2.2.21) ($P.54$)

From Eq. (2.2.20) we have up to $\mathcal{O}\left(\theta^2, \bar{\theta}^2\right)$

$$\begin{aligned}&1 + i\left(\theta^a + \bar{\theta}^a + f^a{}_{bc}\bar{\theta}^b\theta^c\right)t_a + \frac{1}{2}\left(\theta^b + \bar{\theta}^b\right)\left(\theta^c + \bar{\theta}^c\right)t_{bc}\\ &= \left[1 + i\bar{\theta}^a t_a + \frac{1}{2}\bar{\theta}^b\bar{\theta}^c t_{bc}\right]\cdot\left[1 + i\theta^a t_a + \frac{1}{2}\theta^b\theta^c t_{bc}\right]\\ &= 1 + i\theta^a t_a + \frac{1}{2}\theta^b\theta^c t_{bc} + i\bar{\theta}^a t_a - \bar{\theta}^b t_b\theta^c t_c + \frac{1}{2}\bar{\theta}^b\bar{\theta}^c t_{bc}\\ \Leftrightarrow & i f^a{}_{bc}\bar{\theta}^b\theta^c t_a + \frac{1}{2}\left(\theta^b\bar{\theta}^c + \bar{\theta}^b\theta^c\right)t_{bc} = -\bar{\theta}^b\theta^c t_b t_c\\ \Leftrightarrow & \bar{\theta}^b\theta^c\left[t_{bc} + i f^a{}_{bc}t_a + t_b t_c\right] = 0\\ \Leftrightarrow & t_{bc} = -i f^a{}_{bc}t_a - t_b t_c\end{aligned}$$

4. *Eq.* (2.2.22) ($P.54$)

$$\begin{aligned}-i f^a{}_{bc}t_a - t_b t_c \overset{(2.2.21)}{=} t_{bc} = t_{cb} \overset{(2.2.21)}{=} -i f^a{}_{cb}t_a - t_c t_b\\ \Leftrightarrow [t_b, t_c] = i\left(f^a{}_{cb} - f^a{}_{bc}\right)t_a\end{aligned}$$

C. Quantum Lorentz Transformations

1. *"$\Lambda^\mu{}_\nu$ has an **inverse**"* ($P.57$)

This is true because

$$\det(\Lambda) = \pm 1 \neq 0$$

2. *"$\left(\bar{\Lambda}\Lambda\right)^0{}_0 \geq \bar{\Lambda}^0{}_0\Lambda^0{}_0 - \sqrt{\left(\Lambda^0{}_0\right)^2 - 1}\sqrt{\left(\bar{\Lambda}^0{}_0\right)^2 - 1} \geq 1$"* ($P.58$)

The first inequality is trivial from the preceding considerations. For the second inequality assume the contrary holds, then:

$$\begin{aligned}&0 \leq \bar{\Lambda}^0{}_0\Lambda^0{}_0 - 1 < \sqrt{\left(\Lambda^0{}_0\right)^2 - 1}\sqrt{\left(\bar{\Lambda}^0{}_0\right)^2 - 1}\\ \Rightarrow & \left(\bar{\Lambda}^0{}_0\right)^2\left(\Lambda^0{}_0\right)^2 - 2\bar{\Lambda}^0{}_0\Lambda^0{}_0 + 1\\ &\quad < \left(\bar{\Lambda}^0{}_0\right)^2\left(\Lambda^0{}_0\right)^2 - \left(\bar{\Lambda}^0{}_0\right)^2 - \left(\Lambda^0{}_0\right)^2 + 1\\ \Rightarrow & \left(\Lambda^0{}_0 + \bar{\Lambda}^0{}_0\right)^2 = \left(\bar{\Lambda}^0{}_0\right)^2 - 2\bar{\Lambda}^0{}_0\Lambda^0{}_0 + \left(\Lambda^0{}_0\right)^2 < 0\end{aligned}$$

Which is a contradiction as $\Lambda^0{}_0 + \bar{\Lambda}^0{}_0 \geq 1 + 1 = 2$ and therefore completes the proof.

D. The Poincaré Algebra

1. *"In order fo $U(1+\omega,\varepsilon)$ to be unitary, the operators $J^{\rho\sigma}$ and P^ρ must be **Hermitian**"* ($P.59$)

Analog to II B 1.

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2. *Eqs.* (2.4.8/9) (*P.*60)

$$\frac{1}{2}\omega_{\rho\sigma}UJ^{\rho\sigma}U^{-1}-\varepsilon_\rho UP^\rho U^{-1}$$
$$\overset{(2.4.7)}{=}\frac{1}{2}\left(\Lambda\omega\Lambda^{-1}\right)_{\mu\nu}J^{\mu\nu}-\left(\Lambda\varepsilon-\Lambda\omega\Lambda^{-1}a\right)_\mu P^\mu$$
$$=\frac{1}{2}\Lambda_\mu{}^\rho\omega_{\rho\sigma}\left(\Lambda^{-1}\right)^\sigma{}_\nu J^{\mu\nu}$$
$$-\left(\Lambda_\mu{}^\rho\varepsilon_\rho-\Lambda_\mu{}^\rho\omega_{\rho\sigma}\left(\Lambda^{-1}\right)^\sigma{}_\nu a^\nu\right)P^\mu$$
$$\overset{(2.3.10)}{=}\frac{1}{2}\Lambda_\mu{}^\rho\omega_{\rho\sigma}\Lambda_\nu{}^\sigma J^{\mu\nu}$$
$$-\left(\Lambda_\mu{}^\rho\varepsilon_\rho-\Lambda_\mu{}^\rho\omega_{\rho\sigma}\Lambda_\nu{}^\sigma a^\nu\right)P^\mu$$
$$=\frac{1}{2}\omega_{\rho\sigma}\left(\Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma J^{\mu\nu}+\Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma a^\nu P^\mu\right)$$
$$-\varepsilon_\rho\Lambda_\mu{}^\rho P^\mu$$

In order to be able to compare coefficients in this, the coefficient of $\omega_{\rho\sigma}$ has to be anti symmatrized. With this Eqs. (2.4.8/9) follow immediately.

3. *Eqs.* (2.4.10/11) (*P.*60)

Up to $\mathcal{O}(\omega,\varepsilon)$ one can identify
$$U^{-1}(1+\omega,\varepsilon)=U(1-\omega,-\varepsilon)$$
,since
$$U(1+\omega,\varepsilon)U(1-\omega,-\varepsilon)=U(1-\omega+\omega,-\varepsilon+\varepsilon)=U(1,0).$$
With this we have up to $\mathcal{O}(\omega,\varepsilon)$
$$i\left[\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}-\varepsilon_\mu P^\mu,J^{\rho\sigma}\right]$$
$$=\left(1+\frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu}-i\varepsilon_\mu P^\mu\right)J^{\rho\sigma}$$
$$\cdot\left(1-\frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu}+i\varepsilon_\mu P^\mu\right)-J^{\rho\sigma}$$
$$=UJ^{\rho\sigma}U^{-1}-J^{\rho\sigma}$$
$$\overset{(2.4.8)}{=}(1+\omega)_\mu{}^\rho(1+\omega)_\nu{}^\sigma$$
$$\cdot\left(J^{\mu\nu}-\varepsilon^\mu P^\nu+\varepsilon^\nu P^\mu\right)-J^{\rho\sigma}$$
$$=-\varepsilon^\rho P^\sigma+\varepsilon^\sigma P^\rho+\omega_\nu{}^\sigma J^{\rho\nu}+\omega_\mu{}^\rho J^{\mu\sigma}$$
and also
$$i\left[\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}-\varepsilon_\mu P^\mu,P^\rho\right]$$
$$=\left(1+\frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu}-i\varepsilon_\mu P^\mu\right)P^\rho$$
$$\cdot\left(1-\frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu}+i\varepsilon_\mu P^\mu\right)-P^\rho$$
$$=UP^\rho U^{-1}-P^\rho$$
$$\overset{(2.4.9)}{=}\omega_\mu{}^\rho P^\mu$$

4. *Eqs.* (2.4.12/13/14) (*P.*60)

The only difficulty lies in the derivation of Eq. (2.4.12) because for this one has to anti symmatrize the coefficient of $\omega_{\mu\nu}$ in Eq. (2.4.10) in order to be able to compare coefficients, similar to II D 2. (Similarly if one derives Eq. (2.4.13) from Eq. (2.4.11))

5. *Eqs.* (2.4.18 − 24) (*P.*61)

First observe
$$J^{ij}=\varepsilon_{ijk}J_k$$
and
$$J_i=\frac{1}{2}\varepsilon_{lmi}J^{lm}$$

$$[J_i,J_j]=\frac{\varepsilon_{lmi}\varepsilon_{kpj}}{4}\left[J^{lm},J^{kp}\right]$$
$$\overset{(2.4.12)}{=}-i\frac{\varepsilon_{lmi}\varepsilon_{kpj}}{4}\left[\eta^{mk}J^{lp}-\eta^{lk}J^{mp}-\eta^{pl}J^{km}+\eta^{pm}J^{kl}\right]$$
$$=-\frac{i}{2}\left[\varepsilon_{kil}\varepsilon_{kmj}J^{lm}+\varepsilon_{kim}\varepsilon_{kjl}J^{lm}\right]$$
$$=-\frac{i}{2}\left[J^{ji}-J^{ij}\right]$$
$$=iJ^{ij}=i\varepsilon_{ijk}J_k$$
$$[J_i,K_j]=\left[J^{lm},J^{0j}\right]\frac{\varepsilon_{lmi}}{2}$$
$$\overset{(2.4.12)}{=}-i\frac{\varepsilon_{lmi}}{2}\left[\eta^{m0}J^{lj}-\eta^{l0}J^{mj}-\eta^{jl}J^{0m}+\eta^{jm}J^{0l}\right]$$
$$=-i\frac{\varepsilon_{lmi}}{2}\left[\delta_{jm}K_l-\delta_{jl}K_m\right]=i\varepsilon_{ijl}K_l$$
$$[K_i,K_j]=\left[J^{0i},J^{0j}\right]$$
$$\overset{(2.4.12)}{=}-i\left[\eta^{i0}J^{0j}-\eta^{00}J^{ij}-\eta^{j0}J^{0i}+\eta^{ij}J^{00}\right]$$
$$=-iJ^{ij}=-i\varepsilon_{ijk}J_k$$
$$[J_i,P_j]=\frac{\varepsilon_{lmi}}{2}\left[J^{lm},P^j\right]$$
$$\overset{(2.4.13)}{=}\frac{i\varepsilon_{lmi}}{2}\left[\eta^{jl}P^m-\eta^{jm}P^l\right]$$
$$=\frac{i}{2}\left[\varepsilon_{jmi}P^m-\varepsilon_{mji}P^m\right]=i\varepsilon_{ijm}P_m$$
$$[K_i,P_j]=\left[J^{0i},P^j\right]$$
$$\overset{(2.4.13)}{=}i\left[\eta^{j0}P^i-\eta^{ji}P^0\right]$$
$$=-i\delta_{ji}P^0=-i\delta_{ij}H$$
$$[J_i,H]\overset{(2.4.13)}{=}\frac{i\varepsilon_{lmi}}{2}\left[\eta^{0l}P^m-\eta^{0m}P^l\right]=0$$
$$[P_i,H]=\left[P^i,P^0\right]\overset{(2.4.14)}{=}0$$
$$[H,H]=\left[P^0,P^0\right]\overset{(2.4.14)}{=}0$$
$$[K_i,H]=\left[J^{0i},P^0\right]$$
$$\overset{(2.4.13)}{=}i\left[\eta^{00}P^i-\eta^{0i}P^0\right]=-iP_i$$

6. Eq. (2.4.27) (*P.*61)

When trying to check this for e.g. the standard representation in 4 dimensions, in terms of infinitesimal rotations, attention with the sign in front of sin for the different rotation axis (see remark at the end ofII E 23).

7. "Inspection of Eqs. (2.4.18 − 24) *shows that these commutation relations have a limit for $v \ll 1$ of the form ..."* (*P.*62)

Always equate same orders in v for this.

8. "$\exp(-i\mathbf{K}\cdot\mathbf{v})\exp(-i\mathbf{P}\cdot\mathbf{a}) = \exp(iM\mathbf{a}\cdot\mathbf{v}/2)\exp(-i(\mathbf{K}\cdot\mathbf{v}+\mathbf{P}\cdot\mathbf{a}))$*"* (*P.*62)

Use BCH Formula

$$\begin{aligned}
&\exp(-iK_iv_i)\exp(-iP_ja_j) \\
&= \exp\left(-i(K_iv_i + P_ja_j) + \frac{1}{2}(-i)^2[K_i, P_j]v_ia_j + 0\right) \\
&= \exp\left(-i(K_iv_i + P_ja_j) + \frac{1}{2}iMv_ia_i\right)
\end{aligned}$$

E. One-Particle States

1. Eq. (2.5.2) (*P.*63)

Λ^{-1} shows up here since in Eq. (2.4.9) UPU^{-1} is given but here $U^{-1}PU$ is being used.

2. "with σ within any one block by themselves furnish a representation of the inhomogeneous Lorentz group" (*P.*63)

In this case the Blocks do not mix with other blocks.

3. "and for $p^2 \le 0$, also the sign of p^0" (*P.*64)

For $p^2 \le 0$ we have

$$\begin{aligned}
p^2 &= -\left(p^0\right)^2 + \vec{p}^2 \le 0 \\
\Rightarrow |\vec{p}| &\le \left|p^0\right|
\end{aligned}$$

and from Eq. (2.3.13) we know

$$\left|\Lambda^0{}_0\right| \ge \left|\vec{\Lambda^0}_\cdot\right|.$$

First suppose $p^0 \ge 0$:

$$\begin{aligned}
p'^0 &= \Lambda^0{}_0 p^0 + \Lambda^0{}_i p^i \\
&\ge \Lambda^0{}_0 p^0 - \left|\vec{\Lambda^0}_\cdot\right| |\vec{p}| \\
&= \left|\Lambda^0{}_0\right| \left|p^0\right| - \left|\vec{\Lambda^0}_\cdot\right| |\vec{p}| \\
&\ge \left|\vec{\Lambda^0}_\cdot\right| \left(\left|p^0\right| - |\vec{p}|\right) \ge 0
\end{aligned}$$

Now suppose $p^0 \le 0$:

$$\begin{aligned}
p'^0 &= \Lambda^0{}_0 p^0 + \Lambda^0{}_i p^i \\
&\le \Lambda^0{}_0 p^0 + \left|\vec{\Lambda^0}_\cdot\right| |\vec{p}| \\
&= -\left|\Lambda^0{}_0\right| \left|p^0\right| + \left|\vec{\Lambda^0}_\cdot\right| |\vec{p}| \\
&\le \left|\Lambda^0{}_0\right| \left(-\left|p^0\right| + |\vec{p}|\right) \le 0
\end{aligned}$$

4. Eq. (2.5.12) *"The delta function appears here because $\Psi_{k,\sigma}$ and $\Psi_{k',\sigma'}$ are eigenstates of a Hermitian operator with eigenvalues $\mathbf{k}$ and $\mathbf{k}'$, respectively."* (*P.*66)

Note: From this point forward in this section of the book the states considered have standard momentum k^μ! (see Text in the Book preceding Eq. (2.5.12))

Suppose that for given values of $\mathbf{k}$ and $\mathbf{k}'$

$$\left\langle \Psi_{k',\sigma'} \middle| \Psi_{k,\sigma} \right\rangle \ne 0$$

then from

$$\begin{aligned}
&k^i \left\langle \Psi_{k',\sigma'} \middle| \Psi_{k,\sigma} \right\rangle \\
&= \left\langle \Psi_{k',\sigma'} \middle| P^i \Psi_{k,\sigma} \right\rangle \\
&= \left\langle P^i \Psi_{k',\sigma'} \middle| \Psi_{k,\sigma} \right\rangle \\
&= k'^i \left\langle \Psi_{k',\sigma'} \middle| \Psi_{k,\sigma} \right\rangle
\end{aligned}$$

follows

$$k^i = k'^i.$$

On why there is only a 3 dimensional delta function showing up:

This is because both states are assumed to have standard momentum k^μ and therefore their 0-th component is completely fixed from their spatial components:

$$k'^0 = \pm\sqrt{\mathbf{k}'^2 - k^2}$$

Choice between + and − comes from sign of k^0, i.e. + for cases (a) and (c) of Table 2.1.

5. Eq. (2.5.13) (*P*.67)

$$
\begin{aligned}
&\langle U(W)\Psi_{k',\sigma'}|U(W)\Psi_{k,\sigma}\rangle \\
&\overset{(2.5.8)}{=} \sum_{\sigma''\sigma'''} \langle D_{\sigma''\sigma'}\Psi_{k',\sigma''}|D_{\sigma'''\sigma}\Psi_{k,\sigma'''}\rangle \\
&= \sum_{\sigma''\sigma'''} \left(D_{\sigma''\sigma'}\right)^\star D_{\sigma'''\sigma} \langle \Psi_{k',\sigma''}|\Psi_{k,\sigma'''}\rangle \\
&\overset{(2.5.12)}{=} \sum_{\sigma''} \left(D_{\sigma''\sigma'}\right)^\star D_{\sigma''\sigma} \delta^{(3)}(\mathbf{k}'-\mathbf{k}) \\
&\overset{(2.2.2)}{=} \langle \Psi_{k',\sigma'}|\Psi_{k,\sigma}\rangle \\
&\overset{(2.5.12)}{=} \delta_{\sigma'\sigma}\delta^{(3)}(\mathbf{k}'-\mathbf{k})
\end{aligned}
$$

From this it follows

$$D^\dagger(W) = D^{-1}(W)$$

6. "$\langle\Psi_{p',\sigma'}|\Psi_{p,\sigma}\rangle = \ldots$*"* (*P*.67)

Note: What is meant by "arbitrary momenta" is that these momenta still have the standard momentum k^μ, but now none of the states has exactly k^μ as its momentum.

First define

$$k' := L^{-1}(p)p'$$

with this we get:

$$
\begin{aligned}
&\langle \Psi_{p',\sigma'}|\Psi_{p,\sigma}\rangle \\
&\overset{(2.5.5)}{=} \langle \Psi_{p',\sigma'}|N(p)U(L(p))\Psi_{k,\sigma}\rangle \\
&= N(p) \left\langle U(L^{-1}(p))\Psi_{p',\sigma'}\middle|\Psi_{k,\sigma}\right\rangle \\
&\overset{(2.5.11)}{=} N(p)\frac{N^\star(p')}{N^\star(k')} \\
&\cdot \sum_{\sigma''} D^\star_{\sigma''\sigma'}\left(W\left(L^{-1}(p),p'\right)\right) \langle \Psi_{k',\sigma''}|\Psi_{k,\sigma}\rangle \\
&\overset{(2.5.12)}{=} N(p)N^\star(p')\, D^\star_{\sigma\sigma'}\left(W\left(L^{-1}(p),p'\right)\right)\delta^{(3)}(\mathbf{k}'-\mathbf{k})
\end{aligned}
$$

Where in the last step we used

$$\frac{\delta^{(3)}(\mathbf{k}'-\mathbf{k})}{N^\star(k')} = \frac{\delta^{(3)}(\mathbf{k}'-\mathbf{k})}{N^\star(k)} \overset{(2.5.5)}{=} \delta^{(3)}(\mathbf{k}'-\mathbf{k})$$

This is valid as from Eq. (2.5.5) we know that $N(p)$ is implicitly dependent on k^μ. This therefor fixes $p^2 = k^2$ and the sign of p^0 (see end of II E 4), s.t.

$$N(p) = N(\mathbf{p}).$$

Further we are allowed to use Eq. (2.5.12) in the last step since k' also has k as its standard momentum:

$$k' = L^{-1}(p)p' = \underbrace{L^{-1}(p)L(p')}_{=:L(k')} k$$

7. "W $\left(L^{-1}(p),p\right) = 1$*"* (*P*.67)

First note

$$L(k) = 1 = L^{-1}(k)$$

with this we get:

$$
\begin{aligned}
W\left(L^{-1}(p),p\right) &\overset{(2.5.10)}{=} L^{-1}\left(L^{-1}(p)p\right)L^{-1}(p)L(p) \\
&= L^{-1}(k) = 1
\end{aligned}
$$

*8. "So we see that the **invariant delta function** is ..."* (*P*.67)

Invariant in this case means w.r.t. proper orthochronous Lorentz transformations, which can be interpreted as change of variables under the $\int \mathrm{d}^4p$ integral.

9. Eq. (2.5.24) (*P*.68)

$$
\begin{aligned}
p^0 &= L^0{}_0k^0 + L^0{}_ik^i \\
&= \frac{\sqrt{\mathbf{p}^2+M^2}}{M}M = \sqrt{\mathbf{p}^2+M^2} \\
p^i &= L^i{}_0k^0 + L^i{}_jk^j \\
&= \frac{p_i}{|\mathbf{p}|}\sqrt{\frac{\mathbf{p}^2+M^2}{M^2}-1}M \\
&= \frac{p_i}{|\mathbf{p}|}\sqrt{\frac{\mathbf{p}^2}{M^2}}M = p_i
\end{aligned}
$$

10. "To see this, note that the boost Eq. (2.5.24) *may be expressed as* $L(p) = R(\hat{\mathbf{p}})B(|\mathbf{p}|)R^{-1}(\hat{\mathbf{p}})$*"* (*P*.68)

First note that the columns and rows of the matrix $B(|\mathbf{p}|)$ are counted in the order 1,2,3,0.

Let

$$\hat{\mathbf{p}} = \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix}$$

then we have

$$R(-\hat{\mathbf{p}}) = R_3(-\phi)R_2(\theta) = \begin{pmatrix} \cos(\phi)\cos(\theta) & -\sin(\phi) & \cos(\phi)\sin(\theta) \\ \sin(\phi)\cos(\theta) & \cos(\phi) & \sin(\phi)\sin(\theta) \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

(see II E 23) and

$$\begin{aligned} R^{-1}(\hat{\mathbf{p}}) = R_2(\theta)R_3(\phi) &= \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta)\cos(\phi) & \cos(\theta)\sin(\phi) & -\sin(\theta) \\ -\sin(\phi) & \cos(\phi) & 0 \\ \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \end{pmatrix} = R^\top(\hat{\mathbf{p}}) \end{aligned}$$

From Eq. (2.5.24) we know

$$L(p) = \begin{pmatrix} 1+(\gamma-1)s_\theta^2c_\phi^2 & (\gamma-1)s_\theta c_\phi s_\theta s_\phi & (\gamma-1)s_\theta c_\phi c_\theta & s_\theta c_\phi\sqrt{\gamma^2-1} \\ (\gamma-1)s_\theta c_\phi s_\theta s_\phi & 1+(\gamma-1)s_\theta^2 s_\phi^2 & (\gamma-1)s_\theta s_\phi c_\theta & s_\theta s_\phi\sqrt{\gamma^2-1} \\ (\gamma-1)s_\theta c_\phi c_\theta & (\gamma-1)s_\theta s_\phi c_\theta & 1+(\gamma-1)c_\theta^2 & c_\theta\sqrt{\gamma^2-1} \\ s_\theta c_\phi\sqrt{\gamma^2-1} & s_\theta s_\phi\sqrt{\gamma^2-1} & c_\theta\sqrt{\gamma^2-1} & \gamma \end{pmatrix}$$

where s and c are shorthand for sin and cos. With this we can now check:

$$\begin{aligned} R(\hat{\mathbf{p}})B(|\mathbf{p}|)R^{-1}(\hat{\mathbf{p}}) &= \begin{pmatrix} c_\phi c_\theta & -s_\phi & c_\phi s_\theta & 0 \\ s_\phi c_\theta & c_\phi & s_\phi s_\theta & 0 \\ -s_\theta & 0 & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & \sqrt{\gamma^2-1} \\ 0 & 0 & \sqrt{\gamma^2-1} & \gamma \end{pmatrix} \begin{pmatrix} c_\phi c_\theta & s_\phi c_\theta & -s_\theta & 0 \\ -s_\phi & c_\phi & 0 & 0 \\ c_\phi s_\theta & s_\phi s_\theta & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_\phi c_\theta & -s_\phi & c_\phi s_\theta & 0 \\ s_\phi c_\theta & c_\phi & s_\phi s_\theta & 0 \\ -s_\theta & 0 & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_\phi c_\theta & s_\phi c_\theta & -s_\theta & 0 \\ -s_\phi & c_\phi & 0 & 0 \\ \gamma c_\phi s_\theta & \gamma s_\phi s_\theta & \gamma c_\theta & \sqrt{\gamma^2-1} \\ \sqrt{\gamma^2-1}c_\phi s_\theta & \sqrt{\gamma^2-1}s_\phi s_\theta & \sqrt{\gamma^2-1}c_\theta & \gamma \end{pmatrix} = L(p) \end{aligned}$$

11. "$W(\mathbf{R}, p) = \mathbf{R}$" (P.69)

To see this just substitute $R(\theta)$ back into the previous result.

12. Eq. (2.5.25) (P.70)

$$\begin{aligned} -1 = (Wt)^\mu\,(Wt)_\mu &= \alpha^2+\beta^2+\zeta^2-(1+\zeta)^2 \\ &\Leftrightarrow \alpha^2+\beta^2 = 2\zeta \end{aligned}$$

13. Eq. (2.5.26) (P.70)

This is a Lorentz transformation, since

$$\begin{aligned} S^\top \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} S &= \begin{pmatrix} 1 & 0 & \alpha & \alpha \\ 0 & 1 & \beta & \beta \\ -\alpha & -\beta & 1-\zeta & -\zeta \\ \alpha & \beta & \zeta & 1+\zeta \end{pmatrix} \\ &\cdot \begin{pmatrix} 1 & 0 & -\alpha & \alpha \\ 0 & 1 & -\beta & \beta \\ \alpha & \beta & 1-\zeta & \zeta \\ -\alpha & -\beta & \zeta & -1-\zeta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

14. *Eqs.* (2.5.29/30) (*P.*70)

Eq. (2.5.30) is trivial since the rotations around the three axis satisfy this group multiplication and

$$S(0,0) = \mathbf{1}.$$

For Eq. (2.5.29) explicit calculation shows it (see TODO) together with

$$R(0) = \mathbf{1}.$$

15. *Eq.* (2.5.31) (*P.*70)

Explicit calculation shows Eq. (2.5.31) (see TODO) and the invariance follows then immediately, since:

$$W'SW'^{-1} = S' \underbrace{R'SR'^{-1}}_{=S''} S'^{-1} = S'''$$

16. *"$W(\theta,\alpha,\beta) = 1 + \omega$"* (*P.*71)

From Eq. (2.5.28) we immediately get for infinitesimal θ, α, β :

$$(W(\theta,\alpha,\beta))^\mu{}_\nu = \delta^\mu_\nu + \begin{pmatrix} 0 & \theta & -\alpha & \alpha \\ -\theta & 0 & -\beta & \beta \\ \alpha & \beta & 0 & 0 \\ \alpha & \beta & 0 & 0 \end{pmatrix}^{\mu}{}_{\nu}$$

$$\Rightarrow \omega_{\mu\nu} = \begin{pmatrix} 0 & \theta & -\alpha & \alpha \\ -\theta & 0 & -\beta & \beta \\ \alpha & \beta & 0 & 0 \\ -\alpha & -\beta & 0 & 0 \end{pmatrix}_{\mu\nu}$$

17. *Eq.* (2.5.32) (*P.*71)

$$\begin{aligned} U(1+\omega) &\overset{(2.4.3)}{=} 1 + \frac{1}{2} i\omega_{\rho\sigma} J^{\rho\sigma} \\ &= 1 + i\theta J^{12} + i\alpha(-J^{13} + J^{10}) + i\beta(-J^{23} + J^{20}) \\ &= 1 + i\theta J_3 + i\alpha(J_2 - K_1) + i\beta(-J_1 - K_2) \end{aligned}$$

18. *Eqs.* (2.5.35/36/37) (*P.*71)

From Eqs. (2.4.18/19/20) we get:

$$\begin{aligned} [J_3, A] &= [J_3, J_2 - K_1] \\ &= [J_3, J_2] - [J_3, K_1] \\ &= -iJ_1 - iK_2 = iB \\ [J_3, B] &= -[J_3, J_1] - [J_3, K_2] \\ &= -iJ_2 - i(-K_1) = -iA \\ [A, B] &= -[J_2, J_1] - [J_2, K_2] + [K_1, J_1] + [K_1, K_2] \\ &= iJ_3 - iJ_3 = 0 \end{aligned}$$

19. *"Since A and B are commuting Hermitian operators they can be simultaneously diagonalized by states $\Psi_{k,a,b}$"* (*P.*71)

This is valid, including the label k, since

$$\begin{aligned} [A, P_1]\Psi_{k,a,b} &= ([J_2, P_1] - [K_1, P_1])\Psi_{k,a,b} \\ &= (-iP_3 + iP^0)\Psi_{k,a,b} = 0 \\ [A, P_2]\Psi_{k,a,b} &= ([J_2, P_2] - [K_1, P_2])\Psi_{k,a,b} \\ &= 0 \\ [A, P_3]\Psi_{k,a,b} &= ([J_2, P_3] - [K_1, P_3])\Psi_{k,a,b} \\ &= (iP_1)\Psi_{k,a,b} = 0 \\ \left[A, P^0\right]\Psi_{k,a,b} &= (\left[J_2, P^0\right] - \left[K_1, P^0\right])\Psi_{k,a,b} \\ &= (-iP_1)\Psi_{k,a,b} = 0 \\ [B, P_1]\Psi_{k,a,b} &= (-[J_1, P_1] - [K_2, P_1])\Psi_{k,a,b} \\ &= 0 \\ [B, P_2]\Psi_{k,a,b} &= ([J_1, P_2] - [K_2, P_2])\Psi_{k,a,b} \\ &= (iP_3 - iP^0)\Psi_{k,a,b} = 0 \\ [B, P_3]\Psi_{k,a,b} &= ([J_1, P_3] - [K_2, P_3])\Psi_{k,a,b} \\ &= (-iP_2)\Psi_{k,a,b} = 0 \\ \left[B, P^0\right]\Psi_{k,a,b} &= (\left[J_1, P^0\right] - \left[K_2, P^0\right])\Psi_{k,a,b} \\ &= (-iP_2)\Psi_{k,a,b} = 0 \end{aligned}$$

20. *"σ gives the component of angular momentum in the direction of motion, or* helicity*"* (*P.*72)

The derivation of Eq. (2.5.26) starts among other conditions from the explicit form of k. Which results in

$$\textit{Eqs.}\,(2.5.27/28) \to \textit{Eq.}\,(2.5.32) \to 2.5.39,$$

s.t. this is really connected to the direction of motion.

21. *"$U(W)\Psi_{k,\sigma} = \exp(i\theta\sigma)\Psi_{k,\sigma}$"* (*P.*72)

Use Eqs. (2.5.38/39).

22. Eq. (2.5.44) (*P.*73)

With $B(u)$ from Eq. (2.5.45) we get:

$$
\begin{aligned}
B\left(\frac{|\mathbf{p}|}{\kappa}\right)k &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\left(\frac{|\mathbf{p}|}{\kappa}\right)^2+1}{2\frac{|\mathbf{p}|}{\kappa}} & \frac{\left(\frac{|\mathbf{p}|}{\kappa}\right)^2-1}{2\frac{|\mathbf{p}|}{\kappa}} \\ 0 & 0 & \frac{\left(\frac{|\mathbf{p}|}{\kappa}\right)^2-1}{2\frac{|\mathbf{p}|}{\kappa}} & \frac{\left(\frac{|\mathbf{p}|}{\kappa}\right)^2+1}{2\frac{|\mathbf{p}|}{\kappa}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \kappa \\ \kappa \end{pmatrix} \\
&= |\mathbf{p}| \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}
\end{aligned}
$$

23. "take $R(\hat{\mathbf{p}})$ as a rotation by angle θ around the two-axis followed by a rotation by angle ϕ around the three-axis" (*P.*73)

Let

$$
\hat{\mathbf{p}} = \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix}
$$

then

$$
\begin{aligned}
R(\hat{\mathbf{p}}) &= R_3(-\phi)R_2(-\theta) \\
&= \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\phi)\cos(\theta) & -\sin(\phi) & \cos(\phi)\sin(\theta) \\ \sin(\phi)\cos(\theta) & \cos(\phi) & \sin(\phi)\sin(\theta) \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix},
\end{aligned}
$$

(With this sign convention everything is consistent, see definition of $R(\theta)$ after Eq. (2.5.27) and compare Eq. (2.5.47) to Eq. (2.4.27)) with this we have

$$
\hat{\mathbf{p}} = R(\hat{\mathbf{p}})\hat{e}_3.
$$

F. Space Inversion and Time-Reversal

1. Eqs. (2.6.7 − 12) (*P.*76)

$$
\begin{aligned}
\mathsf{P}J_i\mathsf{P}^{-1} &= \frac{1}{2}\varepsilon_{ijk}\mathsf{P}J^{jk}\mathsf{P}^{-1} \\
&= \frac{1}{2}\varepsilon_{ijk}\left(-\delta^j_l\right)\left(-\delta^k_m\right)J^{lm} \\
&= \frac{1}{2}\varepsilon_{ijk}J^{jk} = J_i \\
\mathsf{P}K_i\mathsf{P}^{-1} &= \mathsf{P}J^{0i}\mathsf{P}^{-1} \\
&= \delta^0_\mu\left(-\delta^j_\nu\right)J^{\mu\nu} \\
&= -J^{0i} = -K_i \\
\mathsf{P}P_i\mathsf{P}^{-1} &= \left(-\delta^i_\nu\right)P^\mu = -P_i \\
\mathsf{T}J_i\mathsf{T}^{-1} &= \frac{1}{2}\varepsilon_{ijk}\mathsf{T}J^{jk}\mathsf{T}^{-1} \\
&= -\frac{1}{2}\varepsilon_{ijk}\delta^j_l\delta^k_mJ^{lm} \\
&= -\frac{1}{2}\varepsilon_{ijk}J^{jk} = -J_i \\
\mathsf{T}K_i\mathsf{T}^{-1} &= \mathsf{T}J^{0i}\mathsf{T}^{-1} \\
&= -\left(-\delta^0_\mu\right)\delta^j_\nu J^{\mu\nu} \\
&= J^{0i} = K_i \\
\mathsf{T}P_i\mathsf{T}^{-1} &= -\delta^i_\nu P^\mu = -P_i
\end{aligned}
$$

2. "$\mathcal{P}L(p)\mathcal{P}^{-1} = L(\mathcal{P}p)$" (*P.*77)

From Eq. (2.5.24) we immediately see:

$$
\begin{aligned}
\left(\mathcal{P}L(p)\mathcal{P}^{-1}\right)^i{}_k &= \left(L(p)\right)^i{}_k = \left(L(\mathcal{P}p)\right)^i{}_k \\
\left(\mathcal{P}L(p)\mathcal{P}^{-1}\right)^i{}_0 &= -\left(L(p)\right)^i{}_0 = \left(L(\mathcal{P}p)\right)^i{}_0 \\
\left(\mathcal{P}L(p)\mathcal{P}^{-1}\right)^0{}_0 &= \left(L(p)\right)^0{}_0 = \left(L(\mathcal{P}p)\right)^0{}_0
\end{aligned}
$$

3. "Using Eq. (2.6.14) *again on the left, we see that the square-root factors cancel,..."* (*P.*77)

$$
\begin{aligned}
&\left(-J_1 \pm iJ_2\right)\zeta_\sigma\Psi_{k,-\sigma} = -\left(J_1 \mp iJ_2\right)\zeta_\sigma\Psi_{k,-\sigma} \\
&\overset{(2.6.14)}{=} -\sqrt{(j \pm (-\sigma))(j \mp (-\sigma)+1)}\zeta_\sigma\Psi_{k,-\sigma\mp1} \\
&= -\sqrt{(j \mp \sigma)(j \pm \sigma + 1)}\zeta_\sigma\Psi_{k,-\sigma\mp1}
\end{aligned}
$$

4. "The time-reversal phase ζ has no physical significance." (*P.*78)

This redefinition only works because $\mathsf{T}$is *anti-linear*.

5. "$\mathcal{T}L(p)\mathcal{T}^{-1} = L(\mathcal{P}p)$" (P.78)

From Eq. (2.5.24) we immediately see:

$$\begin{aligned}\left(\mathcal{T}L(p)\mathcal{T}^{-1}\right)^i{}_k &= (L(p))^i{}_k = (L(\mathcal{P}p))^i{}_k \\ \left(\mathcal{T}L(p)\mathcal{T}^{-1}\right)^i{}_0 &= -(L(p))^i{}_0 = (L(\mathcal{P}p))^i{}_0 \\ \left(\mathcal{T}L(p)\mathcal{T}^{-1}\right)^0{}_0 &= (L(p))^0{}_0 = (L(\mathcal{P}p))^0{}_0\end{aligned}$$

6. "P yields a state with four-momentum ..." (P.78)

What is meant by this is:

$$\begin{aligned}P^i\mathsf{P}\Psi_{k,\sigma} &\overset{(2.6.9)}{=} \mathsf{P}\left(-P^i\right)\Psi_{k,\sigma} = \left(-\delta^i_3\kappa\right)\mathsf{P}\Psi_{k,\sigma} \\ H\mathsf{P}\Psi_{k,\sigma} &\overset{(2.6.13)}{=} \mathsf{P}H\Psi_{k,\sigma} = \kappa\mathsf{P}\Psi_{k,\sigma} \\ J_3\mathsf{P}\Psi_{k,\sigma} &\overset{(2.6.7)}{=} \mathsf{P}J_3\Psi_{k,\sigma} = \sigma\mathsf{P}\Psi_{k,\sigma}\end{aligned}$$

7. Eq. (2.6.20) (P.79)

We have

$$R_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = R_2^{-1}$$

from which we get (Is $U(R_2^{-1}) = U^{-1}(R_2)$? TODO)

$$\begin{aligned}U^{-1}(R_2)J_3U(R_2) &= U^{-1}(R_2)J^{12}U(R_2) \\ &\overset{(2.4.8)}{=} \left(R_2^{-1}\right)^1{}_\mu \left(R_2^{-1}\right)^2{}_\nu J^{\mu\nu} \\ &= -J^{12} = -J_3 \\ U^{-1}(R_2)P^\nu U(R_2) &\overset{(2.4.9)}{=} \left(R_2^{-1}\right)^\nu{}_\mu P^\mu\end{aligned}$$

such that

$$\begin{aligned}J_3U(R_2^{-1})\mathsf{P}\Psi_{k,\sigma} &= -U(R_2^{-1})J_3\mathsf{P}\Psi_{k,\sigma} \\ &\overset{II\,F\,6}{=} -\sigma U(R_2^{-1})\mathsf{P}\Psi_{k,\sigma} \\ P^iU(R_2^{-1})\mathsf{P}\Psi_{k,\sigma} &\overset{II\,F\,6}{=} U(R_2^{-1})\left(-\left(-\delta^i_3\kappa\right)\right)\mathsf{P}\Psi_{k,\sigma} \\ &= k^iU(R_2^{-1})\mathsf{P}\Psi_{k,\sigma} \\ HU(R_2^{-1})\mathsf{P}\Psi_{k,\sigma} &\overset{II\,F\,6}{=} \kappa U(R_2^{-1})\mathsf{P}\Psi_{k,\sigma} \\ &= k^0U(R_2^{-1})\mathsf{P}\Psi_{k,\sigma}.\end{aligned}$$

Further we get

$$R_2^{-1}\mathcal{P} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

8. "$\mathcal{P}$ commutes with the rotation $R(\hat{\mathbf{p}})$" (P.79)

This is true because $R(\hat{\mathbf{p}})$ only acts on space components non trivially, which all just get a "-" sign from $\mathcal{P}$.

9. "$\mathsf{P}\Psi_{p,\sigma} = \sqrt{\frac{\kappa}{p^0}}\eta_\sigma U\left(R(\hat{\mathbf{p}})R_2B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,-\sigma}$" (P.79)

$$\begin{aligned}\mathsf{P}\Psi_{p,\sigma} &\overset{(2.5.5)}{=} N(p)\mathsf{P}U(L(p))\Psi_{k,\sigma} \\ &\overset{(2.5.44)}{=} \sqrt{\frac{\kappa}{p^0}}\mathsf{P}U\left(R(\hat{\mathbf{p}})B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma} \\ &= \sqrt{\frac{\kappa}{p^0}}U\left(\mathcal{P}R(\hat{\mathbf{p}})B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma} \\ &= \sqrt{\frac{\kappa}{p^0}}U\left(R(\hat{\mathbf{p}})\mathcal{P}B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma} \\ &= \sqrt{\frac{\kappa}{p^0}}U\left(R(\hat{\mathbf{p}})R_2R_2^{-1}\mathcal{P}B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma} \\ &= \sqrt{\frac{\kappa}{p^0}}U\left(R(\hat{\mathbf{p}})R_2B\left(\frac{|\mathbf{p}|}{\kappa}\right)R_2^{-1}\mathcal{P}\right)\Psi_{k,\sigma} \\ &= \sqrt{\frac{\kappa}{p^0}}U\left(R(\hat{\mathbf{p}})R_2B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)U\left(R_2^{-1}\right)\mathsf{P}\Psi_{k,\sigma} \\ &\overset{(2.6.20)}{=} \sqrt{\frac{\kappa}{p^0}}\eta_\sigma U\left(R(\hat{\mathbf{p}})R_2B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,-\sigma}\end{aligned}$$

10. "But a rotation of $\pm 180°$ around the three-axis reverses the sign of $J_2, \ldots$" (P.79)

Analogously to II F 6 ($2 \leftrightarrow 3$).

11. Eq. (2.6.22) (P.79)

First note that

$$B\left(\frac{|\mathbf{p}|}{\kappa}\right)$$

and rotations around the three-axis commute (see Eq. (2.5.45)).

$$\begin{aligned}
\mathsf{P}\Psi_{p,\sigma} &\overset{II\,F\,9}{=} \sqrt{\frac{\kappa}{p^0}}\eta_\sigma U\left(R\left(\hat{\mathbf{p}}\right)R_2 B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,-\sigma} \\
&\overset{Eq.\,(2.6.21)}{=} \sqrt{\frac{\kappa}{p^0}}\eta_\sigma U\left(R\left(-\hat{\mathbf{p}}\right)\right) \\
&\cdot \exp(\pm i\pi J_3) U\left(B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,-\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}}\eta_\sigma U\left(R\left(-\hat{\mathbf{p}}\right)\right) \\
&\cdot U\left(B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\exp(\pm i\pi J_3)\Psi_{k,-\sigma} \\
&= \eta_\sigma \exp(\pm i\pi(-\sigma))\sqrt{\frac{\kappa}{p^0}} U\left(L(\mathcal{P}p)\right)\Psi_{k,-\sigma} \\
&= \eta_\sigma \exp(\mp i\pi\sigma)\Psi_{\mathcal{P}p,-\sigma}
\end{aligned}$$

12. *"This peculiar change of sign in the operation of parity for mass-less particles of half-integer spin is due to the convention adopted in Eq.* (2.5.47) *for the rotation used to define mass-less particles states of arbitrary momentum."* (P.79)

This is based on the fact that the choice of $R\left(\hat{\mathbf{p}}\right)$, which transforms the three axis into the unit vector $\hat{\mathbf{p}}$, is *not* unique. As mentioned in the text right after Eq. (2.5.47), one could always add an initial rotation around the three axis. This is also why the factor of

$$\exp(\pm i\pi J_3)$$

shows up in Eq. (2.6.21), despite $R\left(\hat{\mathbf{p}}\right)R_2$ and $R\left(-\hat{\mathbf{p}}\right)$ both transforming the three axis into the unit vector $-\hat{\mathbf{p}}$.

13. "T *yields a state which has values ..."* (P.79)

What is meant by this is:

$$\begin{aligned}
P^i\mathsf{T}\Psi_{k,\sigma} &\overset{(2.6.12)}{=} \mathsf{T}\left(-P^i\right)\Psi_{k,\sigma} = \left(-\delta_3^i\kappa\right)\mathsf{T}\Psi_{k,\sigma} \\
H\mathsf{T}\Psi_{k,\sigma} &\overset{(2.6.13)}{=} \mathsf{T}H\Psi_{k,\sigma} = \kappa\mathsf{T}\Psi_{k,\sigma} \\
J_3\mathsf{T}\Psi_{k,\sigma} &\overset{(2.6.10)}{=} \mathsf{T}\left(-J_3\right)\Psi_{k,\sigma} = -\sigma\mathsf{T}\Psi_{k,\sigma}
\end{aligned}$$

14. *Eq.* (2.6.23) (P.80)

This is completely analog to Eq. (2.6.20) when using II F 13 (see II F 7). Further we get

$$R_2^{-1}\mathsf{T} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

15. *Eq.* (2.6.24) (P.80)

$$\begin{aligned}
\mathsf{T}\Psi_{p,\sigma} &\overset{(2.5.5)}{=} N(p)\mathsf{T}U(L(p))\Psi_{k,\sigma} \\
&\overset{(2.5.44)}{=} \sqrt{\frac{\kappa}{p^0}}\mathsf{T}U\left(R\left(\hat{\mathbf{p}}\right)B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}}U\left(\mathcal{T}R\left(\hat{\mathbf{p}}\right)B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}}U\left(R\left(\hat{\mathbf{p}}\right)\mathcal{T}B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}}U\left(R\left(\hat{\mathbf{p}}\right)R_2R_2^{-1}\mathcal{T}B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}}U\left(R\left(\hat{\mathbf{p}}\right)R_2B\left(\frac{|\mathbf{p}|}{\kappa}\right)R_2^{-1}\mathcal{T}\right)\Psi_{k,\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}}U\left(R\left(\hat{\mathbf{p}}\right)R_2B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)U\left(R_2^{-1}\right)\mathsf{T}\Psi_{k,\sigma} \\
&\overset{(2.6.23)}{=} \sqrt{\frac{\kappa}{p^0}}\zeta_\sigma U\left(R\left(\hat{\mathbf{p}}\right)R_2B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma}
\end{aligned}$$

16. *Eq.* (2.6.25) (P.80)

First note that

$$B\left(\frac{|\mathbf{p}|}{\kappa}\right)$$

and rotations around the three-axis commute (see Eq. (2.5.45)).

$$\begin{aligned}
\mathsf{T}\Psi_{p,\sigma} &\overset{II\,F\,15}{=} \sqrt{\frac{\kappa}{p^0}}\zeta_\sigma U\left(R\left(\hat{\mathbf{p}}\right)R_2B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma} \\
&\overset{Eq.\,(2.6.21)}{=} \sqrt{\frac{\kappa}{p^0}}\zeta_\sigma U\left(R\left(-\hat{\mathbf{p}}\right)\right) \\
&\cdot \exp(\pm i\pi J_3)U\left(B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}}\zeta_\sigma U\left(R\left(-\hat{\mathbf{p}}\right)\right) \\
&\cdot U\left(B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\exp(\pm i\pi J_3)\Psi_{k,\sigma} \\
&= \zeta_\sigma\exp(\pm i\pi\sigma)\sqrt{\frac{\kappa}{p^0}}U\left(L(\mathcal{P}p)\right)\Psi_{k,\sigma} \\
&= \zeta_\sigma\exp(\pm i\pi\sigma)\Psi_{\mathcal{P}p,\sigma}
\end{aligned}$$

17. *"... the total angular momentum j of any state of this system would have to be a half-integer,..."* (P.81)

This is true as all spins and helicities except for one half integer spin/helicity would couple to a an integer angular momentum. And this would then couple with the remaining half integer spin/helicity to a half-integer total angular momentum j.

G. Projective Representations

1. Eq. (2.7.6) (P.83)

From Eq. (2.2.20) together with the modification of Eq. (2.7.1) we have up to $\mathcal{O}(\theta^2, \bar\theta^2)$

$$\begin{aligned}
&\left(1 + i f_{ab}\bar\theta^a\theta^b\right) \\
&\cdot\left[1 + i\left(\theta^a + \bar\theta^a + f^a{}_{bc}\bar\theta^b\theta^c\right)t_a + \frac{1}{2}\left(\theta^b + \bar\theta^b\right)\left(\theta^c + \bar\theta^c\right)t_{bc}\right] \\
&= \left[1 + i\bar\theta^a t_a + \frac{1}{2}\bar\theta^b\bar\theta^c t_{bc}\right]\cdot\left[1 + i\theta^a t_a + \frac{1}{2}\theta^b\theta^c t_{bc}\right] \\
&= 1 + i\theta^a t_a + \frac{1}{2}\theta^b\theta^c t_{bc} + i\bar\theta^a t_a - \bar\theta^b t_b\theta^c t_c + \frac{1}{2}\bar\theta^b\bar\theta^c t_{bc} \\
\Leftrightarrow & i f^a{}_{bc}\bar\theta^b\theta^c t_a + \frac{1}{2}\left(\theta^b\bar\theta^c + \bar\theta^b\theta^c\right)t_{bc} + i f_{ab}\bar\theta^a\theta^b = -\bar\theta^b\theta^c t_b t_c \\
\Leftrightarrow & \bar\theta^b\theta^c\left[t_{bc} + i f^a{}_{bc}t_a + t_b t_c + i f_{bc}\right] = 0 \\
\Leftrightarrow & t_{bc} = -i f^a{}_{bc}t_a - t_b t_c - i f_{bc}
\end{aligned}$$

Note that here the order of multiplication is adapted from Eq. (2.2.18) and not from Eq. (2.7.1), which does not matter for the result since θ and $\bar\theta$ are dropped in the end. With this we now get the analog of Eq. (2.2.22):

$$\begin{aligned}
&- i f^a{}_{bc}t_a - t_b t_c - i f_{bc} \\
&= t_{bc} = t_{cb} = -i f^a{}_{cb}t_a - t_c t_b - i f_{cb} \\
\Leftrightarrow & \left[t_b, t_c\right] = i\left(f^a{}_{cb} - f^a{}_{bc}\right)t_a + i\left(f_{cb} - f_{bc}\right)\mathbf{1}
\end{aligned}$$

2. Eq. (2.7.12) (P.83)

$$\left[\tilde t_b, \tilde t_c\right] = \left[t_b, t_c\right] = iC^a{}_{bc}t_a + iC^a{}_{bc}\phi_a\mathbf{1} = iC^a{}_{bc}\tilde t_a$$

3. Eqs. (2.7.23/24/25) (P.85)

Inserting Eqs. (2.7.14 – 16) into Eq. (2.7.20) we get:

$$\begin{aligned}
0 &\overset{(2.7.20)}{=} \left[J^{\mu\nu}, C^{\rho,\mu}\right] \\
&+ \left[P^\sigma, \eta^{\nu\rho}P^\mu - \eta^{\mu\rho}P^\nu + C^{\rho,\mu\nu}\right] \\
&+ \left[P^\rho, \eta^{\sigma\mu}P^\nu - \eta^{\sigma\nu}P^\mu + C^{\mu\nu,\sigma}\right] \\
&= \eta^{\nu\rho}\left[P^\sigma, P^\mu\right] - \eta^{\mu\rho}\left[P^\sigma, P^\nu\right] \\
&+ \eta^{\sigma\mu}\left[P^\rho, P^\nu\right] - \eta^{\sigma\nu}\left[P^\rho, P^\mu\right] \\
\overset{(2.7.16)}{\Rightarrow} 0 &= \eta^{\nu\rho}C^{\mu,\sigma} - \eta^{\mu\rho}C^{\nu,\sigma} \\
&+ \eta^{\sigma\mu}C^{\nu,\rho} - \eta^{\sigma\nu}C^{\mu,\rho}
\end{aligned}$$

Inserting Eqs. (2.7.13 – 15) into Eq. (2.7.21) we get:

$$\begin{aligned}
0 &\overset{(2.7.21)}{=} \left[J^{\lambda\eta}, \eta^{\nu\rho}P^\mu - \eta^{\mu\rho}P^\nu + C^{\rho,\mu\nu}\right] \\
&+ \left[P^\rho, -\eta^{\nu\lambda}J^{\mu\eta} + \eta^{\mu\lambda}J^{\nu\eta} + \eta^{\eta\mu}J^{\lambda\nu} - \eta^{\eta\nu}J^{\lambda\mu} - C^{\lambda\eta,\mu\nu}\right] \\
&+ \left[J^{\mu\nu}, \eta^{\rho\lambda}P^\eta - \eta^{\rho\eta}P^\lambda + C^{\lambda\eta,\rho}\right] \\
&= \eta^{\nu\rho}\left[J^{\lambda\eta}, P^\mu\right] - \eta^{\mu\rho}\left[J^{\lambda\eta}, P^\nu\right] \\
&- \eta^{\nu\lambda}\left[P^\rho, J^{\mu\eta}\right] + \eta^{\mu\lambda}\left[P^\rho, J^{\nu\eta}\right] \\
&+ \eta^{\eta\mu}\left[P^\rho, J^{\lambda\nu}\right] - \eta^{\eta\nu}\left[P^\rho, J^{\lambda\mu}\right] \\
&+ \eta^{\rho\lambda}\left[J^{\mu\nu}, P^\eta\right] - \eta^{\rho\eta}\left[J^{\mu\nu}, P^\lambda\right]
\end{aligned}$$

Using Eqs. (2.7.14/15) we finally get: TODO

Rest analog TODO

4. Eq. (2.7.26) (P.85)

$$\begin{aligned}
0 &= 4C^{\mu,\sigma} - \delta^\mu_\nu C^{\nu,\sigma} \\
&- \delta^\sigma_\rho C^{\mu,\rho} + \eta^{\sigma\mu}C^{\nu,\rho}\eta_{\nu\rho} \\
&= 2C^{\mu,\sigma} + 0 = 2C^{\mu,\sigma}
\end{aligned}$$

5. Eqs. (2.7.27/28) (P.85)

$$\begin{aligned}
0 &= 4C^{\mu,\lambda\eta} - \delta^\mu_\nu C^{\nu,\lambda\eta} - \eta^{\mu\eta}C^{\rho,\lambda\nu}\eta_{\nu\rho} \\
&+ \eta^{\lambda\mu}C^{\rho,\eta\nu}\eta_{\nu\rho} + \delta^\lambda_\rho C^{\rho,\mu\eta} - \delta^\eta_\rho C^{\rho,\mu\lambda} \\
&+ \delta^\lambda_\nu C^{\eta,\mu\nu} - \delta^\eta_\nu C^{\lambda,\mu\nu} \\
&= 3C^{\mu,\lambda\eta} - \eta^{\mu\eta}C^{\rho,\lambda\nu}\eta_{\nu\rho} + \eta^{\lambda\mu}C^{\rho,\eta\nu}\eta_{\nu\rho} \\
&= 3\left(C^{\mu,\lambda\eta} - \eta^{\mu\eta}C^\lambda + \eta^{\lambda\mu}C^\eta\right)
\end{aligned}$$

6. Eqs. (2.7.29/30) (P.85)

First note

$$C^{\rho\nu,\lambda\eta}\eta_{\nu\rho} = 0$$

from the antisymmetry of $J^{\rho\sigma}$ in Eq. (2.7.13).

$$\begin{aligned}
0 &= 4C^{\mu\sigma,\lambda\eta} - \delta^\mu_\nu C^{\nu\sigma,\lambda\eta} - \eta^{\sigma\mu}C^{\rho\nu,\lambda\eta}\eta_{\nu\rho} + \delta^\sigma_\rho C^{\rho\mu,\lambda\eta} \\
&+ \eta^{\eta\mu}C^{\lambda\nu,\rho\sigma}\eta_{\nu\rho} - \eta^{\lambda\mu}C^{\eta\nu,\rho\sigma}\eta_{\nu\rho} - \delta^\lambda_\rho C^{\mu\eta,\rho\sigma} + \delta^\eta_\rho C^{\mu\lambda,\rho\sigma} \\
&+ \eta^{\sigma\lambda}C^{\rho\eta,\mu\nu}\eta_{\nu\rho} - \delta^\lambda_\nu C^{\sigma\eta,\mu\nu} - \delta^\eta_\nu C^{\lambda\sigma,\mu\nu} + \eta^{\eta\sigma}C^{\lambda\rho,\mu\nu}\eta_{\nu\rho} \\
&= 2C^{\mu\sigma,\lambda\eta} - 2\eta^{\eta\mu}C^{\lambda\sigma} + 2\eta^{\lambda\mu}C^{\eta\sigma} \\
&- 2\eta^{\sigma\lambda}C^{\eta\mu} + 2\eta^{\eta\sigma}C^{\lambda\mu} \\
&= 2\left(C^{\mu\sigma,\lambda\eta} - \eta^{\eta\mu}C^{\lambda\sigma} + \eta^{\lambda\mu}C^{\eta\sigma} - \eta^{\sigma\lambda}C^{\eta\mu} + \eta^{\eta\sigma}C^{\lambda\mu}\right)
\end{aligned}$$

7. Eqs. (2.7.33/34/35) (*P*.86)

$$
\begin{aligned}
& i\left[\tilde{J}^{\mu\nu}, \tilde{J}^{\rho\sigma}\right] = i[J^{\mu\nu}, J^{\rho\sigma}] \\
&= \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\sigma\mu}J^{\rho\nu} + \eta^{\sigma\nu}J^{\rho\mu} + C^{\rho\sigma,\mu\nu} \\
&\overset{(2.7.29)}{=} \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\sigma\mu}J^{\rho\nu} + \eta^{\sigma\nu}J^{\rho\mu} \\
&+ \eta^{\nu\rho}C^{\mu\sigma} - \eta^{\mu\rho}C^{\nu\sigma} + \eta^{\sigma\mu}C^{\nu\rho} - \eta^{\nu\sigma}C^{\mu\rho} \\
&= \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\sigma\mu}J^{\rho\nu} + \eta^{\sigma\nu}J^{\rho\mu} \\
&+ \eta^{\nu\rho}C^{\mu\sigma} - \eta^{\mu\rho}C^{\nu\sigma} - \eta^{\sigma\mu}C^{\rho\nu} + \eta^{\sigma\nu}C^{\rho\mu} \\
&= \eta^{\nu\rho}\tilde{J}^{\mu\sigma} - \eta^{\mu\rho}\tilde{J}^{\nu\sigma} - \eta^{\sigma\mu}\tilde{J}^{\rho\nu} + \eta^{\sigma\nu}\tilde{J}^{\rho\mu}
\end{aligned}
$$

$$
\begin{aligned}
& i\left[\tilde{J}^{\mu\nu}, \tilde{P}^{\rho}\right] = i[J^{\mu\nu}, P^{\rho}] \\
&= \eta^{\nu\rho}P^{\mu} - \eta^{\mu\rho}P^{\nu} + C^{\rho,\mu\nu} \\
&\overset{(2.7.27)}{=} \eta^{\nu\rho}P^{\mu} - \eta^{\mu\rho}P^{\nu} + \eta^{\rho\nu}C^{\mu} - \eta^{\rho\mu}C^{\nu} \\
&= \eta^{\nu\rho}\tilde{P}^{\mu} - \eta^{\mu\rho}\tilde{P}^{\nu}
\end{aligned}
$$

$$
i\left[\tilde{P}^{\mu}, \tilde{P}^{\rho}\right] = [P^{\mu}, P^{\rho}] = 0
$$

8. Eq. (2.7.42) (*P*.87)

This does not set the overall phase completely because

$$
1 = \det(\exp(i\theta)\lambda) = \exp(i2\theta)\det(\lambda) = \exp(i2\theta)
$$

...so for $\theta = \pi$, $\exp(i\theta)\lambda = -\lambda$ satisfies the condition if λ satisfies it.

9. "The group elements depend on $4 - 1 = 3$ complex parameters,..." (*P*.87)

This can easily been seen by calculating the determinant of a general 2×2 complex matrix:

$$
\begin{aligned}
& \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = 1 \\
& \overset{d \neq 0}{\Rightarrow} a = \frac{1 + bc}{d} \\
& \overset{d = 0}{\Rightarrow} b = -\frac{1}{c}
\end{aligned}
$$

10. "... produces a Lorentz transformation $\Lambda(\lambda(\theta))$ which is just a rotation by an angle θ around the three-axis,..." (*P*.87)

$$
\begin{aligned}
& \lambda(\theta) v \lambda^{\dagger}(\theta) \\
&= \begin{pmatrix} \exp\left(i\frac{\theta}{2}\right) & 0 \\ 0 & \exp\left(-i\frac{\theta}{2}\right) \end{pmatrix} \begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix} \begin{pmatrix} \exp\left(-i\frac{\theta}{2}\right) & 0 \\ 0 & \exp\left(i\frac{\theta}{2}\right) \end{pmatrix} \\
&= \begin{pmatrix} \exp\left(i\frac{\theta}{2}\right) & 0 \\ 0 & \exp\left(-i\frac{\theta}{2}\right) \end{pmatrix} \begin{pmatrix} \exp\left(-i\frac{\theta}{2}\right)(V^0 + V^3) & \exp\left(i\frac{\theta}{2}\right)(V^1 - iV^2) \\ \exp\left(-i\frac{\theta}{2}\right)(V^1 + iV^2) & \exp\left(i\frac{\theta}{2}\right)(V^0 - V^3) \end{pmatrix} \\
&= \begin{pmatrix} V^0 + V^3 & \exp(i\theta)(V^1 - iV^2) \\ \exp(-i\theta)(V^1 + iV^2) & V^0 - V^3 \end{pmatrix}
\end{aligned}
$$

From this we can read of:

$$
\begin{aligned}
V'^0 &= V^0 \\
V'^1 &= \mathrm{Re}\{\exp(i\theta)(V^1 - iV^2)\} = \mathrm{Re}\{(\cos(\theta) + i\sin(\theta))(V^1 - iV^2)\} \\
&= \cos(\theta)V^1 + \sin(\theta)V^2 \\
V'^2 &= \mathrm{Im}\{\exp(-i\theta)(V^1 + iV^2)\} = \mathrm{Im}\{(\cos(\theta) - i\sin(\theta))(V^1 + iV^2)\} \\
&= -\sin(\theta)V^1 + \cos(\theta)V^2
\end{aligned}
$$

11. "... $\det(\exp(h)) = \exp(\mathrm{tr}(h))$ is real and positive" (*P*.88)

This is true because eigenvalues of an hermitian matrix are real, s.t.:

$$
\exp(tr(h)) = \exp\left(\sum_i e_i\right) > 0
$$

12. "…with d, e, f, g subject to the single non-linear constraint $d^2+e^2+f^2+g^2=1$,…" ($P.88$)

$$u^\dagger u = 1$$

and

$$\det(u) = 1$$

yield the same constraint.

13. "because $\exp(h)$ is always positive" ($P.89$)

The eigenvalues of $\exp(h)$ are positive, since

$$\exp(h) = \exp\big(u\mathrm{diag}\,(e_i)\,u^\dagger\big) = u\mathrm{diag}\,(\exp(e_i))\,u^\dagger$$

and $e_i \in \mathbb{R}$.

14. "$\big[U(\Lambda)U(\bar\Lambda)U^{-1}(\Lambda\bar\Lambda)\big]^2 = \mathbf{1}$" ($P.89$)

This follows from the discussion in Appendix B, more precisely see III6, because a contraction of the double loop to a point is possible.

15. "These two cases correspond to the two irreducible representations of the first homotopy group Z_2" ($P.89$)

These are the trivial representation

$$1 \to 1 \qquad -1 \to 1$$

and the faithful representation

$$1 \to 1 \qquad -1 \to -1.$$

16. "We must not mix states of integer and half-integer spin." ($P.89$)

Because they are different representations of Z_2 or because some of their loops are contractable to a point compare Superselection rule in Section 2.2. TODO

*17. "…, so the factor $\exp(4\pi i\sigma)$ must be **unity**, and hence σ must be an integer or half-integer." ($P.90$)*

This can be seen from the transformation behavior of massless one particle states Eq. (2.5.42) and

$$\begin{aligned}\mathbf{1}\Psi_{p,\sigma} &= [U(R(2\pi))]^2\,\Psi_{p,\sigma}\\ &= (\exp(i\sigma 2\pi))^2\,\Psi_{p,\sigma}\\ &= \exp(i\sigma 4\pi)\Psi_{p,\sigma}\end{aligned}$$

TODO

H. The Symmetry Representation Theorem

*1. "But $\langle\Psi'_k|\Psi'_k\rangle$ is automatically **real and positive**" ($P.91$)*

This follows immediately from Eq. (2.1.1).

2. "From Eq. (2.A.1) we have $|c_{kk}| = |c_{k1}| = \frac{1}{\sqrt{2}}$ and for $l \neq k$ and $l \neq 1$: $c_{kl} = 0$" ($P.91$)

$$\begin{aligned}|c_{kl'}|^2 &\overset{(2.A.3)}{=} \left|\sum_l c^\star_{kl}\,\langle\Psi'_l|\Psi'_{l'}\rangle\right|^2 = |\langle\Upsilon'_k|\Psi'_{l'}\rangle|^2\\ &\overset{(2.A.1)}{=} |\langle\Upsilon_k|\Psi_{l'}\rangle|^2 = \begin{cases}\frac{1}{2} & \text{for } l' = 1, k\\ 0 & \text{for } l' \neq 1, k\end{cases}\end{aligned}$$

3. Eq. (2.A.10) ($P.92$)

$$\begin{aligned}|C_k|^2 + |C_1|^2 + 2\,\mathrm{Re}(C_kC_1^\star) &= |C_k + C_1|^2\\ \overset{(2.A.9)}{=} |C'_k + C'_1|^2 &= |C'_k|^2 + |C'_1|^2 + 2\,\mathrm{Re}(C'_kC_1'^\star)\\ &\overset{Eq.\,(2.A.8)}{\Rightarrow} \mathrm{Re}(C_kC_1^\star) = \mathrm{Re}(C'_kC_1'^\star)\\ &\overset{Eq.\,(2.A.8)}{\Rightarrow} \mathrm{Re}\left(\frac{C_k}{C_1}\right) = \mathrm{Re}\left(\frac{C'_k}{C'_1}\right)\end{aligned}$$

4. Eq. (2.A.11) ($P.92$)

$$\begin{aligned}\left\{\mathrm{Re}\left(\frac{C_k}{C_1}\right)\right\}^2 + \left\{\mathrm{Im}\left(\frac{C_k}{C_1}\right)\right\}^2 &= \left|\frac{C_k}{C_1}\right|^2\\ \overset{Eq.\,(2.A.8)}{=} \left|\frac{C'_k}{C'_1}\right|^2 &= \left\{\mathrm{Re}\left(\frac{C'_k}{C'_1}\right)\right\}^2 + \left\{\mathrm{Im}\left(\frac{C'_k}{C'_1}\right)\right\}^2\\ &\overset{Eq.\,(2.A.10)}{\Rightarrow} \mathrm{Im}\left(\frac{C_k}{C_1}\right) = \pm\,\mathrm{Im}\left(\frac{C'_k}{C'_1}\right)\end{aligned}$$

5. "This is only possible if $\mathrm{Re}\left(\frac{C_k}{C_1}\frac{C_l^\star}{C_1^\star}\right) = \mathrm{Re}\left(\frac{C_k}{C_1}\frac{C_l}{C_1}\right)$ or, in other words, if $\mathrm{Im}\left(\frac{C_k}{C_1}\right)\mathrm{Im}\left(\frac{C_l}{C_1}\right) = 0$" ($P.93$)

Define

$$a := \frac{C_k}{C_1}$$

$$b := \frac{C_l}{C_1}$$

With this we have

$$
\begin{aligned}
|1+a+b^\star|^2 &= |1+a+b|^2 && (1)\\
\Leftrightarrow 1+a^\star+b+a+|a|^2+ab+b^\star+b^\star a^\star+|b|^2 & && (2)\\
=1+a^\star+b^\star+a+|a|^2+ab^\star+b+ba^\star+|b|^2 & && (3)\\
\Leftrightarrow ab+b^\star a^\star &= ab^\star+ba^\star && (4)
\end{aligned}
$$

And further rewriting yields

$$
\begin{aligned}
\mathrm{Re}\left(\frac{C_k}{C_1}\frac{C_l}{C_1}\right) &= \mathrm{Re}(ab) \stackrel{4}{=} \mathrm{Re}(ab^\star) = \mathrm{Re}\left(\frac{C_k}{C_1}\frac{C_l^\star}{C_1^\star}\right)\\
\mathrm{Im}\left(\frac{C_k}{C_1}\right)\mathrm{Im}\left(\frac{C_l}{C_1}\right) &= \mathrm{Im}(a)\,\mathrm{Im}(b)\\
&= -\frac{1}{4}(ab-ab^\star-a^\star b+a^\star b^\star) \stackrel{4}{=} 0
\end{aligned}
$$

6. "Then the invariance of transition probabilities requires that $\left|\sum_k B_k^\star A_k\right|^2 = \left|\sum_k B_k A_k\right|^2$*"* ($P.93$)

$$
\begin{aligned}
\left|\sum_k B_k^\star A_k\right|^2 &\stackrel{Eq.\,(2.A.2)}{=} \left|\sum_{kl} B_k^\star A_l \langle\Psi_k|\Psi_l\rangle\right|^2\\
&= \left|\left\langle \sum_k B_k\Psi_k \middle| \sum_l A_l\Psi_l\right\rangle\right|^2\\
&\stackrel{Eq.\,(2.A.1)}{=} \left|\left\langle U\left(\sum_k B_k\Psi_k\right) \middle| U\left(\sum_l A_l\Psi_l\right)\right\rangle\right|^2\\
&= \left|\left\langle \sum_k B_k^\star U\Psi_k \middle| \sum_l A_l U\Psi_l\right\rangle\right|^2\\
&= \left|\sum_{kl} B_k A_l \langle U\Psi_k|U\Psi_l\rangle\right|^2\\
&\stackrel{Eq.\,(2.A.3)}{=} \left|\sum_k B_k A_k\right|^2
\end{aligned}
$$

7. Eq. (2.A.16)(P.94)

$$
\begin{aligned}
&\sum_{kl}\mathrm{Im}(B_k^\star B_l)\,\mathrm{Im}(A_k^\star A_l)\\
&= \mathrm{Im}\left(\sum_{kl}\mathrm{Im}(B_k^\star B_l)A_k^\star A_l\right)\\
&= \frac{1}{2i}\left[\sum_{kl}\mathrm{Im}(B_k^\star B_l)A_k^\star A_l - \text{c.c.}\right]\\
&= \frac{1}{2i}\left[\sum_{kl}\frac{1}{2i}(B_k^\star B_l - B_k B_l^\star)A_k^\star A_l - \text{c.c.}\right]\\
&= \frac{1}{2i}\left[\frac{1}{2i}\left(\sum_{kl}B_k^\star B_l A_k^\star A_l - \sum_{kl}B_k B_l^\star A_k^\star A_l\right) - \text{c.c.}\right]\\
&= \frac{1}{2i}\left[\frac{1}{2i}\left(\left|\sum_k B_k A_k\right|^2 - \left|\sum_k B_k^\star A_k\right|^2\right) - \text{c.c.}\right]\\
&= \frac{1}{2i}\left[\frac{1}{i}\left(\left|\sum_k B_k A_k\right|^2 - \left|\sum_k B_k^\star A_k\right|^2\right)\right]\\
&\stackrel{II\,H\,6}{=} 0
\end{aligned}
$$

8. "However, for any pair of such state-vectors, with neither A_k *nor* B_k ***all of the same phase****"* ($P.94$)

If they were all of the same phase then

$$\forall k,l : \mathrm{Im}\{A_k^\star A_l\} = 0$$

or

$$\forall k,l : \mathrm{Im}\{B_k^\star B_l\} = 0$$

See Footnote j, for why this is relevant.

9. "We have thus shown that for a given symmetry transformation T either all state-vectors satisfy Eq. (2.A.14) *or else they all satisfy Eq.* (2.A.15)*"* ($P.94$)

Preceding this a contradiction was derived from the assumption that Eq. (2.A.14) applies for a state-vector

$$\sum_k A_k\Psi_k$$

while Eq. (2.A.15) applies for a state vector

$$\sum_k B_k\Psi_k.$$

Such that the statement is obvious.

I. Group Operators and Homotopy Classes

1. *Eq.* (2.*B*.7) (*P*.97)

Taylor evolving Eq. (2.*B*.6) in θ_3^c up to $\mathcal{O}\big(\theta_3^2\big)$ yields:

$$\begin{aligned}
&f^a(\theta_2,\theta_1) + \left[h^{-1}\right]^a{}_c\left(f(\theta_2,\theta_1)\right)\theta_3^c \\
&= f^a(0,f(\theta_2,\theta_1)) + \left[\frac{\partial f^a(\bar\theta, f(\theta_2,\theta_1))}{\partial\bar\theta^c}\right]_{\bar\theta=0}\theta_3^c \\
&= f^a(\theta_3, f(\theta_2,\theta_1)) \\
&\overset{(2.B.6)}{=} f^a(f(\theta_3,\theta_2),\theta_1) \\
&= f^a\left(f(0,\theta_2) + \left[\frac{\partial f^a(\bar\theta,\theta_2)}{\partial\bar\theta^c}\right]_{\bar\theta=0}\theta_3^c, \theta_1\right) \\
&= f^a(\theta_2,\theta_1) + \left[\frac{\partial f^a(\bar\theta,\theta_1)}{\partial\bar\theta^b}\right]_{\bar\theta=\theta_2}\left[\frac{\partial f^b(\bar\theta,\theta_2)}{\partial\bar\theta^c}\right]_{\bar\theta=0}\theta_3^c \\
&= f^a(\theta_2,\theta_1) + \left[\frac{\partial f^a(\bar\theta,\theta_1)}{\partial\bar\theta^b}\right]_{\bar\theta=\theta_2}\left[h^{-1}\right]^b{}_c(\theta_2)\,\theta_3^c
\end{aligned}$$

Equating coefficients of θ_3^c, we get:

$$\begin{aligned}
\left[h^{-1}\right]^a{}_c\left(f(\theta_2,\theta_1)\right) &= \left[\frac{\partial f^a(\bar\theta,\theta_1)}{\partial\bar\theta^b}\right]_{\bar\theta=\theta_2}\left[h^{-1}\right]^b{}_c(\theta_2) \\
\Leftrightarrow h^c{}_b(\theta_2) &= \left[\frac{\partial f^a(\bar\theta,\theta_1)}{\partial\bar\theta^b}\right]_{\bar\theta=\theta_2} h^c{}_a\left(f(\theta_2,\theta_1)\right)
\end{aligned}$$

2. *"Along the second segment the differential equation Eq.* (2.*B*.2) *for* $U_{\mathcal{P}}(s)$ *is thus the same as the differential equation for* $U_{\theta_2}(2s-1)$*."* (*P*.97)

In the second segment $\left(\frac{1}{2}\leq s\leq 1\right)$ the differential equation for $U_{\mathcal{P}}(s)$ reads:

$$\begin{aligned}
&\frac{\mathrm{d}U_{\mathcal{P}}(s)}{\mathrm{d}s} \\
&= it_cU_{\mathcal{P}}(s)h^c{}_a\left(\Theta_{\mathcal{P}}(s)\right)\frac{\mathrm{d}\Theta^c_{\mathcal{P}}(s)}{\mathrm{d}s} \\
&= it_cU_{\mathcal{P}}(s)h^c{}_a\left(f\left(\Theta_{\theta_2}(2s-1),\theta_1\right)\right)\frac{\mathrm{d}f^c\left(\Theta_{\theta_2}(2s-1),\theta_1\right)}{\mathrm{d}s} \\
&= it_cU_{\mathcal{P}}(s)h^c{}_a\left(f\left(\Theta_{\theta_2}(2s-1),\theta_1\right)\right)\left[\frac{\partial f^a(\bar\theta,\theta_1)}{\partial\bar\theta^b}\right]_{\bar\theta=\Theta_{\theta_2}(2s-1)} \\
&\cdot\frac{\mathrm{d}\Theta^b_{\theta_2}(2s-1)}{\mathrm{d}s} \\
&\overset{(2.B.7)}{=} it_cU_{\mathcal{P}}(s)h^c{}_b\left(\Theta_{\theta_2}(2s-1)\right)\frac{\mathrm{d}\Theta^b_{\theta_2}(2s-1)}{\mathrm{d}s}
\end{aligned}$$

Which is just the differential equation for $U_{\theta_2}(2s-1)$:

$$\begin{aligned}
\frac{\mathrm{d}U_{\theta_2}(2s-1)}{\mathrm{d}s} &= \left[\frac{\mathrm{d}U_{\theta_2}(t)}{\mathrm{d}t}\right]_{t=2s-1}2 \\
&= it_cU_{\theta_2}(2s-1)h^c{}_b\left(\Theta_{\theta_2}(2s-1)\right)\left[\frac{\mathrm{d}\Theta^b_{\theta_2}(t)}{\mathrm{d}t}\right]_{t=2s-1}2 \\
&= it_cU_{\theta_2}(2s-1)h^c{}_b\left(\Theta_{\theta_2}(2s-1)\right)\frac{\mathrm{d}\Theta^b_{\theta_2}(2s-1)}{\mathrm{d}s}
\end{aligned}$$

3. *Eq.* (2.*B*.9) (*P*.98)

First note

$$\frac{\mathrm{d}U^{-1}}{\mathrm{d}s} = \left(\frac{\mathrm{d}U}{\mathrm{d}s}\right)^\dagger \overset{(2.B.2)}{=} -i\left(t_aU\right)^\dagger h^a{}_b\frac{\mathrm{d}\Theta^b}{\mathrm{d}s} \tag{5}$$

$$= -iU^{-1}t_ah^a{}_b\frac{\mathrm{d}\Theta^b}{\mathrm{d}s}, \tag{6}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(U^{-1}t_aUh^a{}_b\right) \tag{7}$$

$$= \frac{\mathrm{d}U^{-1}}{\mathrm{d}s}t_eUh^e{}_b + U^{-1}t_e\frac{\mathrm{d}U}{\mathrm{d}s}h^e{}_b + U^{-1}t_aU\frac{\mathrm{d}h^a{}_b}{\mathrm{d}s} \tag{8}$$

$$\overset{(2.B.2)}{=} -iU^{-1}t_dh^d{}_c\frac{\mathrm{d}\Theta^c}{\mathrm{d}s}t_eUh^e{}_b \tag{9}$$

$$+ U^{-1}t_eit_dUh^d{}_c\frac{\mathrm{d}\Theta^c}{\mathrm{d}s}h^e{}_b \tag{10}$$

$$+ U^{-1}t_aUh^a{}_{b,c}\frac{\mathrm{d}\Theta^c}{\mathrm{d}s} \tag{11}$$

$$= i\frac{\mathrm{d}\Theta^c}{\mathrm{d}s}h^d{}_ch^e{}_bU^{-1}[t_e,t_d]U + U^{-1}t_aU\frac{\mathrm{d}\Theta^c}{\mathrm{d}s}h^a{}_{b,c} \tag{12}$$

$$\overset{(2.2.22)}{=} U^{-1}t_aU\frac{\mathrm{d}\Theta^c}{\mathrm{d}s}\left(ih^d{}_ch^e{}_biC^a{}_{ed} + h^a{}_{b,c}\right). \tag{13}$$

By using Eq. (2.2.22) we are making use of condition (a) of the Theorem. With this we get:

$$
\begin{aligned}
&\frac{\mathrm{d}U^{-1}\delta U}{\mathrm{d}s} \\
&= \frac{\mathrm{d}U^{-1}}{\mathrm{d}s}\delta U + U^{-1}\frac{\mathrm{d}\delta U}{\mathrm{d}s} \\
&\overset{6}{=} -iU^{-1}t_a h^a{}_b \frac{\mathrm{d}\Theta^b}{\mathrm{d}s}\delta U + U^{-1}\frac{\mathrm{d}\delta U}{\mathrm{d}s} \\
&= iU^{-1}t_a U h^a{}_{c,b}\left(\delta\Theta^b\right)\frac{\mathrm{d}\Theta^c}{\mathrm{d}s} \\
&+ iU^{-1}t_a U h^a{}_b \frac{\mathrm{d}\delta\Theta^b}{\mathrm{d}s} \\
&= iU^{-1}t_a U\left(\delta\Theta^b\right)\frac{\mathrm{d}\Theta^c}{\mathrm{d}s} h^a{}_{c,b} \\
&+ \frac{\mathrm{d}}{\mathrm{d}s}\left(iU^{-1}t_a U h^a{}_b \delta\Theta^b\right) - i\left(\delta\Theta^b\right)\frac{\mathrm{d}}{\mathrm{d}s}\left(U^{-1}t_a U h^a{}_b\right) \\
&\overset{13}{=} iU^{-1}t_a U\left(\delta\Theta^b\right)\frac{\mathrm{d}\Theta^c}{\mathrm{d}s}\left(h^a{}_{c,b} - \left(-h^d{}_c h^e{}_b C^a{}_{ed} + h^a{}_{b,c}\right)\right) \\
&+ \frac{\mathrm{d}}{\mathrm{d}s}\left(iU^{-1}t_a U h^a{}_b \delta\Theta^b\right) \\
&= iU^{-1}t_a U\left(\delta\Theta^b\right)\frac{\mathrm{d}\Theta^c}{\mathrm{d}s}\left(h^a{}_{c,b} + h^d{}_c h^e{}_b C^a{}_{ed} - h^a{}_{b,c}\right) \\
&+ \frac{\mathrm{d}}{\mathrm{d}s}\left(iU^{-1}t_a U h^a{}_b \delta\Theta^b\right)
\end{aligned}
$$

Where we inserted the expression for $\frac{\mathrm{d}\delta U}{\mathrm{d}s}$ in the third step.

4. Eq. (2.*B*.10) (*P*.98)

Taylor evolving Eq. (2.*B*.6) in θ_3, θ_2 up to $\mathcal{O}\left(\theta_3^3, \theta_2^3\right)$ yields:

$$
\begin{aligned}
&f^a(0,\theta_1) + \left[h^{-1}\right]^a{}_b(\theta_1)\left(\theta_3^b + \theta_2^b + f^b{}_{ec}\theta_3^e\theta_2^c\right) \\
&= f^a(0,\theta_1) + \left[\frac{\partial f^a(\bar\theta,\theta_1)}{\partial\bar\theta^b}\right]_{\bar\theta=0}\left(\theta_3^b + \theta_2^b + f^b{}_{ec}\theta_3^e\theta_2^c\right) \\
&= f^a\left(\theta_3 + \theta_2 + f^{\cdot}{}_{ec}\theta_3^e\theta_2^c, \theta_1\right) \\
&\overset{(2.2.19)}{=} f^a(f(\theta_3,\theta_2),\theta_1) \\
&\overset{(2.B.6)}{=} f^a(\theta_3, f(\theta_2,\theta_1)) \\
&= f^a\left(\theta_3, f(0,\theta_1) + \left[\frac{\partial f^a(\bar\theta,\theta_1)}{\partial\bar\theta^e}\right]_{\bar\theta=0}\theta_2^e\right) \\
&= f^a\left(\theta_3, \theta_1 + \left[h^{-1}\right]^{\cdot}{}_e(\theta_1)\theta_2^e\right) \\
&= f^a(0,\theta_1) + \left[\frac{\partial f^a(\bar\theta,\theta_1)}{\partial\bar\theta^b}\right]_{\bar\theta=0}\theta_3^b \\
&+ \left[\frac{\partial f^a(0,\bar\theta)}{\partial\bar\theta^b}\right]_{\bar\theta=\theta_1}\left[h^{-1}\right]^b{}_e(\theta_1)\theta_2^e \\
&+ \frac{1}{2}\left[\frac{\partial}{\partial\tilde\theta^c}\left[\frac{\partial f^a(\bar\theta,\tilde\theta)}{\partial\bar\theta^b}\right]_{\bar\theta=0}\right]_{\tilde\theta=\theta_1}\theta_3^b\left[h^{-1}\right]^c{}_e(\theta_1)\theta_2^e \\
&+ \frac{1}{2}\left[\frac{\partial}{\partial\bar\theta^b}\left[\frac{\partial f^a(\bar\theta,\tilde\theta)}{\partial\tilde\theta^c}\right]_{\tilde\theta=\theta_1}\right]_{\bar\theta=0}\theta_3^b\left[h^{-1}\right]^c{}_e(\theta_1)\theta_2^e \\
&= f^a(0,\theta_1) + \left[h^{-1}\right]^a{}_b(\theta_1)\theta_3^b + \left[h^{-1}\right]^a{}_e(\theta_1)\theta_2^e \\
&+ \frac{\partial}{\partial\theta_1^c}\left(\left[h^{-1}\right]^a{}_b(\theta_1)\right)\theta_3^b\left[h^{-1}\right]^c{}_e(\theta_1)\theta_2^e
\end{aligned}
$$

No θ_3^2, θ_2^2 terms show up and

$$
\left[\frac{\partial f^a(0,\bar\theta)}{\partial\bar\theta^b}\right]_{\bar\theta=\theta_1} = \delta_b^a,
$$

see Eq. (2.2.19). Equating coefficients of $\theta_3^b\theta_2^c$, we get:

$$
\begin{aligned}
&\left[h^{-1}\right]^a{}_e(\theta_1)f^e{}_{bc} \\
&= \frac{\partial}{\partial\theta_1^d}\left(\left[h^{-1}\right]^a{}_b(\theta_1)\right)\left[h^{-1}\right]^d{}_c(\theta_1) \\
\Leftrightarrow f^e{}_{bc} &= h^e{}_a(\theta_1)\frac{\partial}{\partial\theta_1^d}\left(\left[h^{-1}\right]^a{}_b(\theta_1)\right)\left[h^{-1}\right]^d{}_c(\theta_1) \\
&= \left(0 - h^e{}_{a,d}\left[h^{-1}\right]^a{}_b\right)\left[h^{-1}\right]^d{}_c \\
\Leftrightarrow h^e{}_{a,d} &= -f^e{}_{bc}h^b{}_a h^c{}_d
\end{aligned}
$$

5. *Eq.* (2.*B*.11) (*P*.98)

$$\begin{aligned} h^a{}_{c,b} - h^a{}_{b,c} &\overset{(2.B.10)}{=} -f^a{}_{de} h^d{}_c h^e{}_b + f^a{}_{de} h^d{}_b h^e{}_c \\ &= h^d{}_c h^e{}_b \left(-f^a{}_{de} + f^a{}_{ed}\right) \\ &\overset{(2.2.23)}{=} h^d{}_c h^e{}_b \left(-C^a{}_{ed}\right) \end{aligned}$$

6. *"It follows that $U_\theta(1)$ is stationary under any infinitesimal variation of the path that leaves the endpoints $\Theta(0) = 0$ and $\Theta(1) = \theta$ (and $U_\theta(0) = \mathbf{1}$) fixed."* (*P*.98)

We have

$$U^{-1}\delta U - iU^{-1} t_a U h^a{}_b \delta\Theta^b = C = \text{const}$$

For $s = 0$ this gives

$$0 = \delta U(0) = it_a U(0) h^a{}_b(0) \underbrace{\delta\Theta^b(0)}_{=0} + U(0)C = \mathbf{1}C$$

such that $C = 0$. For $s = 1$ we then get

$$\delta U(1) = it_a U(1) h^a{}_b(1) \underbrace{\delta\Theta^b(1)}_{=0} + U(1) \underbrace{C}_{=0} = 0.$$

7. *"$0 = \frac{\partial}{\partial\theta^b}\left\{U[\theta]^{-1}\tilde{U}[\theta]\right\} + i\phi_b(\theta) U[\theta]^{-1}\tilde{U}[\theta]$ where $\phi_b(\theta) = h^a{}_b \left[\frac{\partial\phi(\theta',\theta)}{\partial\theta'^a}\right]_{\theta'=0}$"* (*P*.99)

By taking the derivative of the previous equation w.r.t. θ'^a, we obtain:

$$\begin{aligned} 0 &= U[\theta]^{-1}(-t_a + t_a)\tilde{U}[\theta] \\ &= \left[\frac{\partial}{\partial\theta'^a} U[\theta]^{-1} U[\theta']^{-1} \tilde{U}[\theta'] \tilde{U}[\theta]\right]_{\theta'=0} \\ &= \left[\frac{\partial}{\partial\theta'^a} U[f(\theta',\theta)]^{-1} \tilde{U}[f(\theta',\theta)] \exp(i\phi(\theta',\theta))\right]_{\theta'=0} \\ &= \left[\frac{\partial}{\partial\bar{\theta}^b} U[\bar{\theta}]^{-1}\tilde{U}[\bar{\theta}]\right]_{\bar{\theta}=f(0,\theta)=\theta} \left[\frac{\partial f^b(\theta',\theta)}{\partial\theta'^a}\right]_{\theta'=0} \exp(i\phi(0,\theta)) \\ &+ U[\theta]^{-1}\tilde{U}[\theta] i \left[\frac{\partial\phi(\theta',\theta)}{\partial\theta'^a}\right]_{\theta'=0} \exp(i\phi(0,\theta)) \\ &= \frac{\partial}{\partial\theta^b}\left\{U[\theta]^{-1}\tilde{U}[\theta]\right\} \left[h^{-1}\right]^b{}_a (\theta) \\ &+ U[\theta]^{-1}\tilde{U}[\theta] i \left[\frac{\partial\phi(\theta',\theta)}{\partial\theta'^a}\right]_{\theta'=0} \end{aligned}$$

Multiplying by $h^a{}_b(\theta)$ yields finally

$$0 = \frac{\partial}{\partial\theta^b}\left\{U[\theta]^{-1}\tilde{U}[\theta]\right\} + i\phi_b(\theta) U[\theta]^{-1}\tilde{U}[\theta].$$

8. *"$0 = \frac{\partial\phi_b(\theta)}{\partial\theta^c} - \frac{\partial\phi_c(\theta)}{\partial\theta^b}$"* (*P*.99)

Differentiating the result of III7 w.r.t θ^c yields

$$\begin{aligned} 0 &= \frac{\partial^2}{\partial\theta^c\partial\theta^b}\left\{U[\theta]^{-1}\tilde{U}[\theta]\right\} + i\frac{\partial\phi_b(\theta)}{\partial\theta^c} U[\theta]^{-1}\tilde{U}[\theta] \\ &+ i\phi_b(\theta)\frac{\partial}{\partial\theta^c}\left\{U[\theta]^{-1}\tilde{U}[\theta]\right\} \\ &\overset{III7}{=} \frac{\partial^2}{\partial\theta^c\partial\theta^b}\left\{U[\theta]^{-1}\tilde{U}[\theta]\right\} + i\frac{\partial\phi_b(\theta)}{\partial\theta^c} U[\theta]^{-1}\tilde{U}[\theta] \\ &+ \phi_b(\theta)\phi_c(\theta) U[\theta]^{-1}\tilde{U}[\theta]. \end{aligned}$$

Antisymmetrizing then gives

$$0 = \frac{\partial\phi_b(\theta)}{\partial\theta^c} - \frac{\partial\phi_c(\theta)}{\partial\theta^b}.$$

9. *"Then $U^{-1}(f(\theta_2,\theta_1))U(\theta_2)U(\theta_1)$ can be a phase factor $\exp(i\phi(\theta_2,\theta_1)) \neq 1$, but ϕ will be the same for all other loops into which this can be continuously deformed."* (*P*.99)

This can be seen from the statement of III6 by setting $\theta = 0$.

J. Inversions and Degenerate Multiplets

1. *"…the corresponding proportionality factor for T^2 can only be ± 1,…"* (*P*.100)

Suppose

$$\mathsf{T}^2 = \varphi\mathbf{1}$$

then we have

$$\mathsf{T}^3 = \mathsf{T}^2\mathsf{T} = \varphi\mathsf{T}$$

but also

$$\mathsf{T}^3 = \mathsf{T}\mathsf{T}^2 = \mathsf{T}\varphi = \varphi^\star\mathsf{T}$$

such that

$$\varphi^\star = \varphi = \pm 1.$$

2. *"…because T is anti unitary, $\mathcal{T}$ must be unitary"* (*P*.101)

Using basic orthonormality properties we see

$$
\begin{aligned}
&\delta_{\sigma'\sigma}\delta^{(3)}(\mathbf{p}'-\mathbf{p})\delta_{ni} \\
&\overset{(2.5.19)}{=} \langle \Psi_{\mathbf{p}',\sigma',n} | \Psi_{\mathbf{p},\sigma,i}\rangle^\star \\
&= \langle \mathsf{T}\Psi_{\mathbf{p}',\sigma',n} | \mathsf{T}\Psi_{\mathbf{p},\sigma,i}\rangle \\
&\overset{(2.C.1)}{=} \left\langle (-1)^{j'-\sigma'}\sum_m \mathcal{T}_{mn}\Psi_{-\mathbf{p}',-\sigma',m} \middle| (-1)^{j-\sigma}\sum_l \mathcal{T}_{li}\Psi_{-\mathbf{p},-\sigma,l}\right\rangle \\
&= (-1)^{j'+j-\sigma'-\sigma}\sum_{m,l}\mathcal{T}^\star_{mn}\mathcal{T}_{li}\left\langle \Psi_{-\mathbf{p}',-\sigma',m} | \Psi_{-\mathbf{p},-\sigma,l}\right\rangle \\
&\overset{(2.5.19)}{=} (-1)^{2(j-\sigma)}\sum_m \mathcal{T}^\star_{mn}\mathcal{T}_{mi}\delta_{\sigma'\sigma}\delta^{(3)}(\mathbf{p}'-\mathbf{p}) \\
&= \delta_{\sigma'\sigma}\delta^{(3)}(\mathbf{p}'-\mathbf{p})\sum_m \left(\mathcal{T}^\dagger\right)_{nm}\mathcal{T}_{mi}
\end{aligned}
$$

such that

$$\delta_{ni} = \sum_m \left(\mathcal{T}^\dagger\right)_{nm}\mathcal{T}_{mi}.$$

3. Eq. (2.C.2) (P.101)

$$
\begin{aligned}
&\mathsf{T}\Psi'_{\mathbf{p},\sigma,n} \\
&= \sum_m \mathsf{T}\mathcal{U}_{mn}\Psi_{\mathbf{p},\sigma,m} \\
&= \sum_m \mathcal{U}^\star_{mn}\mathsf{T}\Psi_{\mathbf{p},\sigma,m} \\
&\overset{(2.C.1)}{=} \sum_{m,k}\mathcal{U}^\star_{mn}(-1)^{j-\sigma}\mathcal{T}_{km}\Psi_{-\mathbf{p},-\sigma,k} \\
&= \sum_{m,k,l}\mathcal{U}^\star_{mn}(-1)^{j-\sigma}\mathcal{T}_{km}\left(\mathcal{U}^{-1}\right)_{lk}\Psi'_{-\mathbf{p},-\sigma,l} \\
&= (-1)^{j-\sigma}\sum_l \left(\mathcal{U}^{-1}\mathcal{T}\mathcal{U}^\star\right)_{ln}\Psi'_{-\mathbf{p},-\sigma,l}
\end{aligned}
$$

*4. "… **unitary** matrix $\mathcal{T}\mathcal{T}^\star$." (P.101)*

$$
\begin{aligned}
\left(\mathcal{T}\mathcal{T}^\star\right)^\dagger\mathcal{T}\mathcal{T}^\star &= \left(\mathcal{T}^\dagger\right)^\star\mathcal{T}^\dagger\mathcal{T}\mathcal{T}^\star \\
&= \left(\mathcal{T}^\dagger\right)^\star\mathcal{T}^\star \\
&= \left(\mathcal{T}^\dagger\mathcal{T}\right)^\star \\
&= \mathbf{1}^\star = \mathbf{1}
\end{aligned}
$$

5. Eq. (2.C.4) (P.101)

We have

$$\mathcal{T}\mathcal{T}^\star = D$$

multiplying by $\mathcal{T}^\top = \left(\mathcal{T}^\dagger\right)^\star$ from the right we get

$$\mathcal{T} = \mathcal{T}\left(\mathcal{T}\mathcal{T}^\dagger\right)^\star = \mathcal{T}\mathcal{T}^\star\mathcal{T}^\top = D\mathcal{T}^\top.$$

6. "… the diagonal element $\mathcal{T}_{nn}$ vanishes unless $\exp(i\phi_n) = 1$. Furthermore, if $\exp(i\phi_n) = 1$ but $\exp(i\phi_m) \neq 1$, then Eq. (2.C4) tells us that $\mathcal{T}_{nm} = \mathcal{T}_{mn} = 0$." (P.101)

From Eq. (2.C.4) we have

$$\mathcal{T}_{nm} = \exp(i\phi_n)\mathcal{T}_{mn}.$$

From this we get

$$
\begin{aligned}
&\mathcal{T}_{nn} = \exp(i\phi_n)\mathcal{T}_{nn} \\
&\Rightarrow 0 = \left(1-\exp(i\phi_n)\right)\mathcal{T}_{nn}
\end{aligned}
$$

such that for $\exp(i\phi_n) \neq 1$ we have

$$\mathcal{T}_{nn} = 0.$$

Furthermore, if $\exp(i\phi_n) = 1$ but $\exp(i\phi_m) \neq 1$, then we have

$$
\begin{aligned}
\mathcal{T}_{mn} &= \exp(i\phi_m)\mathcal{T}_{nm} \\
&= \exp(i\phi_m)\exp(i\phi_n)\mathcal{T}_{mn} \\
&= \underbrace{\exp(i\phi_m)}_{\neq 1}\mathcal{T}_{mn} \\
\Rightarrow \mathcal{T}_{mn} &= \mathcal{T}_{nm} = 0
\end{aligned}
$$

7. "… $\mathcal{A}$ is symmetric as well as unitary, … " (P.101)

Unitarity follows directly from the unitarity of $\mathcal{T}$and symmetry can be seen from

$$\mathcal{T}_{nm} = \exp(i\phi_n)\mathcal{T}_{mn}$$

because $\mathcal{A}$ only contains rows and columns for which $\exp(i\phi_n) = 1$.

8. "Because $\mathcal{A}$ is symmetric, it can be expressed as the exponential of a symmetric anti-Hermitian matrix, so it can be diagonalized by a transformation Eq. (2.C.2) acting on $\mathcal{A}$, with the corresponding submatrix of $\mathcal{U}$ real and hence orthogonal." (P.101)

We know that $\mathcal{A}$ is unitary and symmetric, i.e.

$$\mathcal{A}^\dagger = \mathcal{A}^{-1} \qquad \mathcal{A}^\top = \mathcal{A}.$$

Suppose $\mathcal{A}$ can be written as

$$\mathcal{A} = \exp(a)$$

where a is a symmetric anti-Hermitian matrix, i.e.

$$a^\dagger = -a \qquad a^\top = a.$$

We can easily check that this satisfies all properties imposed on $\mathcal{A}$:

$$\begin{aligned}\mathcal{A}^\dagger\mathcal{A} &= \exp\left(a^\dagger\right)\exp(a)\\ &= \exp(-a)\exp(a)\\ &= \exp(-a+a) = \mathbf{1}\\ \mathcal{A}^\top &= \exp\left(a^\top\right) = \exp(a) = \mathcal{A}\end{aligned}$$

Any symmetric anti-Hermitian matrix a can be written in terms of a symmetric Hermitian matrix h as

$$a = ih,$$

such that

$$\mathcal{A} = \exp(a) = \exp(ih).$$

Observe that h is real, since

$$h^\star = \left(h^\top\right)^\dagger = h^\dagger = h.$$

Since h is real and by definition symmetric, it can be diagonalized by an orthogonal matrix:

$$h = O^{-1}DO$$

Inserting this into the expression for $\mathcal{A}$ we see that O also diagonalizes $\mathcal{A}$:

$$\begin{aligned}\mathcal{A} &= \exp(ih) = \exp\left(iO^{-1}DO\right)\\ &= \exp\left(O^{-1}iDO\right) = O^{-1}\exp(iD)O\end{aligned}$$

Since O is orthogonal and in particular real it can be set as a submatrix of $\mathcal{U}$ in the transformation Eq. (2.C.2).

9. *Eq.* (2.C.7) (P.102)

For components of $\mathcal{T}$ in the same block $\mathcal{B}_i$ we have

$$\exp(i\phi_m) = \exp(-i\phi_n).$$

Such that pulling a factor of $\exp\left(-i\frac{\phi_n}{2}\right)$ out of $\mathcal{T}_{mn}$ we can write

$$\begin{aligned}\mathcal{T}_{mn} &= \exp\left(-i\frac{\phi_n}{2}\right)z\\ \Rightarrow \mathcal{T}_{nm} &= \exp(i\phi_n)\exp\left(-i\frac{\phi_n}{2}\right)z\\ &= \exp\left(i\frac{\phi_n}{2}\right)z\end{aligned}$$

with z some complex number specific to the combination of indices m, n.

10. *"…$\mathcal{C}_i\mathcal{C}_i^\dagger = \mathcal{C}_i^\dagger\mathcal{C}_i = \mathbf{1}$, and hence $\mathcal{C}_i$ is square and unitary"* (P.102)

The Unitarity of $\mathcal{T}$ implies the Unitarity of $\mathcal{B}$ which in turn implies the Unitarity of each $\mathcal{B}_i$. This imposes the following condition:

$$\begin{aligned}\mathcal{B}_i^\dagger\mathcal{B}_i &= \begin{pmatrix} 0 & \exp\left(i\frac{\phi_n}{2}\right)\mathcal{C}_i^\star \\ \exp\left(-i\frac{\phi_n}{2}\right)\mathcal{C}_i^\dagger & 0\end{pmatrix}\begin{pmatrix} 0 & \exp\left(i\frac{\phi_n}{2}\right)\mathcal{C}_i \\ \exp\left(-i\frac{\phi_n}{2}\right)\mathcal{C}_i^\top & 0\end{pmatrix}\\ &= \begin{pmatrix}\mathcal{C}_i^\star\mathcal{C}_i^\top & 0\\ 0 & \mathcal{C}_i^\dagger\mathcal{C}_i\end{pmatrix} = \begin{pmatrix}\left(\mathcal{C}_i\mathcal{C}_i^\dagger\right)^\top & 0\\ 0 & \mathcal{C}_i^\dagger\mathcal{C}_i\end{pmatrix} \stackrel{!}{=} \mathbf{1}\\ \Rightarrow \mathcal{C}_i^\dagger\mathcal{C}_i &= \mathbf{1} = \mathbf{1}^\top = \mathcal{C}_i\mathcal{C}_i^\dagger\end{aligned}$$

11. *"…$\exp\left(\pm i\frac{\phi}{2}\right)c_\pm^\star = |\lambda|^2 c_\pm^\star\exp\left(\mp i\frac{\phi}{2}\right)$, which is impossible unless either $c_+ = c_- = 0$ or $\exp(i\phi)$ is unity,…"* (P.103)

We have

$$\begin{aligned}\exp\left(i\frac{\phi}{2}\right)c_+^\star = \lambda c_- &\Rightarrow c_+ = \lambda^\star c_-^\star\exp\left(i\frac{\phi}{2}\right)\\ \exp\left(-i\frac{\phi}{2}\right)c_-^\star = \lambda c_+ &\Rightarrow c_- = \lambda^\star c_+^\star\exp\left(-i\frac{\phi}{2}\right)\end{aligned}$$

such that

$$\begin{aligned}\exp\left(\pm i\frac{\phi}{2}\right)c_\pm^\star &= \lambda c_\mp\\ &= |\lambda|^2 c_\pm^\star\exp\left(\mp i\frac{\phi}{2}\right)\end{aligned}$$

which is equivalent to

$$\exp(\pm i\phi)c_\pm^\star = |\lambda|^2 c_\pm^\star.$$

This is only possible if either $c_+ = c_- = 0$ or $\exp(i\phi)$ is unity.

12. Eq. (2.C.16) (P.104)

From Eq. (2.C.8) we have

$$\begin{aligned}\Psi_{\mathbf{p},\sigma,\pm} &= \mathsf{T}^{-1}\mathsf{T}\Psi_{\mathbf{p},\sigma,\pm} \\ &= \exp\left(\mp i\frac{\phi}{2}\right)(-1)^{j-\sigma}\mathsf{T}^{-1}\Psi_{-\mathbf{p},-\sigma,\mp} \\ \Rightarrow \mathsf{T}^{-1}\Psi_{-\mathbf{p},-\sigma,\mp} &= \exp\left(\pm i\frac{\phi}{2}\right)(-1)^{-j+\sigma}\Psi_{\mathbf{p},\sigma,\pm} \\ \Rightarrow \mathsf{T}^{-1}\Psi_{\mathbf{p},\sigma,\pm} &= \exp\left(\mp i\frac{\phi}{2}\right)(-1)^{-j-\sigma}\Psi_{-\mathbf{p},-\sigma,\mp}\end{aligned}$$

with this we get:

$$\begin{aligned}\mathsf{CP}\Psi_{\mathbf{p},\sigma,\pm} &= (\mathsf{CPT})\,\mathsf{T}^{-1}\Psi_{\mathbf{p},\sigma,\pm} \\ &= (\mathsf{CPT})\exp\left(\mp i\frac{\phi}{2}\right)(-1)^{-j-\sigma}\Psi_{-\mathbf{p},-\sigma,\mp} \\ &= \exp\left(\pm i\frac{\phi}{2}\right)(-1)^{-j-\sigma}\,(\mathsf{CPT})\,\Psi_{-\mathbf{p},-\sigma,\mp} \\ &= \exp\left(\pm i\frac{\phi}{2}\right)(-1)^{-j-\sigma}(-1)^{j+\sigma}\Psi_{-\mathbf{p},\sigma,\mp^C} \\ &= \exp\left(\pm i\frac{\phi}{2}\right)\Psi_{-\mathbf{p},\sigma,\mp^C}\end{aligned}$$

K. Problems

1. Problem

Observer $\mathcal{O}$ sees a W-Boson with:

$$\begin{aligned}j &= 1 \\ \text{mass} &= m \\ \mathbf{p} &= k\hat{e}_y \\ j_z &= \sigma\end{aligned}$$

Observer $\mathcal{O}'$ moves relative to $\mathcal{O}$ with

$$\mathbf{v} = w\hat{e}_z.$$

We work in case a) of Table 2.1, s.t.:

$$\begin{aligned}p^0 &> 0 \\ -m^2 &= p^2 \\ &= -\left(p^0\right)^2 + \mathbf{p}^2 \\ &= -\left(p^0\right)^2 + k^2 \\ \Rightarrow p^0 &= \sqrt{m^2+k^2} \\ \Rightarrow p &= \begin{pmatrix} 0 \\ k \\ 0 \\ \sqrt{m^2+k^2}\end{pmatrix}\end{aligned}$$

The Lorentz transformation from $\mathcal{O}$ to $\mathcal{O}'$ is given by

$$\Lambda(\mathbf{v}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1+(\gamma_w-1)\frac{w^2}{\mathbf{v}^2} & -\gamma_w w \\ 0 & 0 & -\gamma_w w & \gamma_w \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_w & -\gamma_w w \\ 0 & 0 & -\gamma_w w & \gamma_w \end{pmatrix} \qquad \gamma_w = \frac{1}{\sqrt{1-w^2}}.$$

From this we get

$$\Lambda p = \begin{pmatrix} 0 \\ k \\ -\gamma_w w\sqrt{m^2+k^2} \\ \gamma_w\sqrt{m^2+k^2}\end{pmatrix}.$$

In order to be able to apply Eq. (2.5.23) we first need to calculate $W(\Lambda, p)$ and for this we need $L(p)$ and $L(\Lambda p)$:

$$
\begin{aligned}
\gamma_k &= \frac{\sqrt{\mathbf{p}^2+m^2}}{m} = \sqrt{1+\frac{k^2}{m^2}} \\
\sqrt{\gamma_k^2-1} &= \sqrt{1+\frac{k^2}{m^2}-1} = \frac{k}{m} \\
\hat{\mathbf{p}} &= \hat{e}_y \\
L(p) &\overset{(2.5.24)}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+(\gamma_k-1) & 0 & \sqrt{\gamma_k^2-1} \\ 0 & 0 & 1 & 0 \\ 0 & \sqrt{\gamma_k^2-1} & 0 & \gamma_k \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_k & 0 & \sqrt{\gamma_k^2-1} \\ 0 & 0 & 1 & 0 \\ 0 & \sqrt{\gamma_k^2-1} & 0 & \gamma_k \end{pmatrix} \\
\gamma_{wk} &= \frac{\sqrt{\mathbf{\Lambda p}^2+m^2}}{m} = \sqrt{1+\frac{k^2+\gamma_w^2 w^2(m^2+k^2)}{m^2}} \\
&= \sqrt{1+\frac{k^2}{m^2}+\gamma_w^2 w^2\left(1+\frac{k^2}{m^2}\right)} = \sqrt{(1+\gamma_w^2 w^2)\left(1+\frac{k^2}{m^2}\right)} \\
&= \sqrt{\gamma_w^2\,(1-w^2+w^2)}\gamma_k = \gamma_w\gamma_k \\
\sqrt{\gamma_{wk}^2-1} &= \sqrt{\gamma_w^2\gamma_k^2-1} \\
|\mathbf{\Lambda p}| &= \sqrt{\gamma_w^2\gamma_k^2 m^2-m^2} = m\sqrt{\gamma_w^2\gamma_k^2-1} \\
\widehat{\mathbf{\Lambda p}} &= \frac{1}{m\sqrt{\gamma_w^2\gamma_k^2-1}} \begin{pmatrix} 0 \\ k \\ -\gamma_w w\sqrt{m^2+k^2} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{\gamma_w^2\gamma_k^2-1}} \begin{pmatrix} 0 \\ \sqrt{\gamma_k^2-1} \\ -\gamma_w w\gamma_k \\ 0 \end{pmatrix} \\
L(\Lambda p) &\overset{(2.5.24)}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+(\gamma_w\gamma_k-1)\frac{\gamma_k^2-1}{\gamma_w^2\gamma_k^2-1} & (\gamma_w\gamma_k-1)\frac{-w\gamma_w\gamma_k\sqrt{\gamma_k^2-1}}{\gamma_w^2\gamma_k^2-1} & \frac{\sqrt{\gamma_k^2-1}}{\sqrt{\gamma_w^2\gamma_k^2-1}}\sqrt{\gamma_w^2\gamma_k^2-1} \\ 0 & (\gamma_w\gamma_k-1)\frac{-w\gamma_w\gamma_k\sqrt{\gamma_k^2-1}}{\gamma_w^2\gamma_k^2-1} & 1+(\gamma_w\gamma_k-1)\frac{w^2\gamma_w^2\gamma_k^2}{\gamma_w^2\gamma_k^2-1} & \frac{-\gamma_w w\gamma_k}{\sqrt{\gamma_w^2\gamma_k^2-1}}\sqrt{\gamma_w^2\gamma_k^2-1} \\ 0 & \frac{\sqrt{\gamma_k^2-1}}{\sqrt{\gamma_w^2\gamma_k^2-1}}\sqrt{\gamma_w^2\gamma_k^2-1} & \frac{-\gamma_w w\gamma_k}{\sqrt{\gamma_w^2\gamma_k^2-1}}\sqrt{\gamma_w^2\gamma_k^2-1} & \gamma_w\gamma_k \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\frac{\gamma_k^2-1}{\gamma_w\gamma_k+1} & -\frac{w\gamma_w\gamma_k\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1} & \sqrt{\gamma_k^2-1} \\ 0 & -\frac{w\gamma_w\gamma_k\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1} & 1+\frac{w^2\gamma_w^2\gamma_k^2}{\gamma_w\gamma_k+1} & -\gamma_w w\gamma_k \\ 0 & \sqrt{\gamma_k^2-1} & -\gamma_w w\gamma_k & \gamma_w\gamma_k \end{pmatrix}
\end{aligned}
$$

Using Eq. (2.3.10) we get:

$$
\begin{aligned}
\left(L^{-1}\right)^0{}_0 &= (-1)(-1)L^0{}_0 = L^0{}_0 \\
\left(L^{-1}\right)^i{}_k &= (+1)(+1)L^k{}_i = L^k{}_i = L^i{}_k \\
\left(L^{-1}\right)^i{}_0 &= (-1)(+1)L^0{}_i = -L^0{}_i = \left(L^{-1}\right)^0{}_i \\
\Rightarrow L^{-1}(\Lambda p) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\frac{\gamma_k^2-1}{\gamma_w\gamma_k+1} & -\frac{w\gamma_w\gamma_k\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1} & -\sqrt{\gamma_k^2-1} \\ 0 & -\frac{w\gamma_w\gamma_k\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1} & 1+\frac{w^2\gamma_w^2\gamma_k^2}{\gamma_w\gamma_k+1} & \gamma_w w\gamma_k \\ 0 & -\sqrt{\gamma_k^2-1} & \gamma_w w\gamma_k & \gamma_w\gamma_k \end{pmatrix}
\end{aligned}
$$

Putting everything together we obtain:

$$W(\Lambda,p) \overset{(2.5.10)}{=} L^{-1}(\Lambda p)\Lambda L(p) = L^{-1}(\Lambda p)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_w & -\gamma_w w \\ 0 & 0 & -\gamma_w w & \gamma_w \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_k & 0 & \sqrt{\gamma_k^2-1} \\ 0 & 0 & 1 & 0 \\ 0 & \sqrt{\gamma_k^2-1} & 0 & \gamma_k \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\frac{\gamma_k^2-1}{\gamma_w\gamma_k+1} & -\frac{w\gamma_w\gamma_k\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1} & -\sqrt{\gamma_k^2-1} \\ 0 & -\frac{w\gamma_w\gamma_k\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1} & 1+\frac{w^2\gamma_w^2\gamma_k^2}{\gamma_w\gamma_k+1} & \gamma_w w\gamma_k \\ 0 & -\sqrt{\gamma_k^2-1} & \gamma_w w\gamma_k & \gamma_w\gamma_k \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_k & 0 & \sqrt{\gamma_k^2-1} \\ 0 & -w\gamma_w\sqrt{\gamma_k^2-1} & \gamma_w & -w\gamma_w\gamma_k \\ 0 & \gamma_w\sqrt{\gamma_k^2-1} & -w\gamma_w & \gamma_w\gamma_k \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_k+\frac{\gamma_k^2-1}{\gamma_w\gamma_k+1}\left(\gamma_k+w^2\gamma_w^2\gamma_k-\gamma_w(\gamma_w\gamma_k+1)\right) & w\gamma_w\sqrt{\gamma_k^2-1}\left(-\frac{\gamma_w\gamma_k}{\gamma_w\gamma_k+1}+1\right) & 0 \\ 0 & \frac{-w\gamma_w\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1}(\gamma_k^2+\gamma_w\gamma_k+1+w^2\gamma_w^2\gamma_k^2-\gamma_w\gamma_k(1+\gamma_w\gamma_k)) & \frac{\gamma_w(\gamma_w\gamma_k+1)+w^2\gamma_w^2\gamma_k^2\gamma_w-w^2\gamma_w^2\gamma_k(1+\gamma_w\gamma_k)}{\gamma_w\gamma_k+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_k+\frac{\gamma_k^2-1}{\gamma_w\gamma_k+1}(-\gamma_w) & w\gamma_w\sqrt{\gamma_k^2-1}\left(\frac{1}{\gamma_w\gamma_k+1}\right) & 0 \\ 0 & \frac{-w\gamma_w\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1}(\gamma_k^2+1-\gamma_k^2) & \frac{\gamma_k+\gamma_w}{\gamma_w\gamma_k+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\gamma_w\gamma_k+1}\left(\gamma_w\gamma_k^2+\gamma_k-\gamma_w\gamma_k^2+\gamma_w\right) & \frac{w\gamma_w\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1} & 0 \\ 0 & \frac{-w\gamma_w\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1} & \frac{\gamma_k+\gamma_w}{\gamma_w\gamma_k+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\gamma_k+\gamma_w}{\gamma_w\gamma_k+1} & \frac{w\gamma_w\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1} & 0 \\ 0 & \frac{-w\gamma_w\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1} & \frac{\gamma_k+\gamma_w}{\gamma_w\gamma_k+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This explicit calculation can also be checked (see TODO):

$$\frac{w\gamma_w\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1} = \frac{w\gamma_w\frac{k}{m}(\gamma_w\gamma_k-1)}{\gamma_w^2\gamma_k^2-1} = \frac{w\gamma_w\frac{k}{m}(\frac{\gamma_k}{\gamma_w}-\frac{1}{\gamma_w^2})}{\gamma_k^2-\frac{1}{\gamma_w^2}} = \frac{w\frac{k}{m}(\gamma_k-\frac{1}{\gamma_w})}{1+\frac{k^2}{m^2}-1+w^2} = \frac{wkm(\gamma_k-\frac{1}{\gamma_w})}{k^2+m^2w^2}$$
$$= \frac{wk(\sqrt{k^2+m^2}-m\sqrt{1-w^2})}{k^2+m^2w^2}$$

$$\frac{\gamma_k+\gamma_w}{\gamma_w\gamma_k+1} = \frac{(\gamma_w+\gamma_k)(\gamma_w\gamma_k-1)}{\gamma_w^2\gamma_k^2-1} = \frac{\gamma_k-\frac{1}{\gamma_w}+\frac{\gamma_k^2}{\gamma_w}-\frac{\gamma_k}{\gamma_w^2}}{\gamma_k^2-\frac{1}{\gamma_w^2}} = \frac{\gamma_k(1-(1-w^2))+\sqrt{1-w^2}(1+\frac{k^2}{m^2}-1)}{1+\frac{k^2}{m^2}-1+w^2}$$
$$= \frac{w^2m\sqrt{m^2+k^2}+k^2\sqrt{1-w^2}}{k^2+m^2w^2}$$

In order to apply Eq. (2.5.23), we need to identify $W(\Lambda,p)$ with a rotation, in this case a simple rotation around the x-axis (For the sign convention see the discussion in II E 23):

$$W(\Lambda,p) \overset{!}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\Rightarrow \cos(\theta) = \frac{\gamma_k+\gamma_w}{\gamma_w\gamma_k+1}$$
$$\Rightarrow \sin(\theta) = \frac{w\gamma_w\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1}$$

With this we see that Observer $\mathcal{O}'$ observes the state

$$
\begin{aligned}
U(\Lambda)\Psi_{\mathbf{p},\sigma} &\overset{(2.5.23)}{=} \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\sigma'=-1}^{1} D^{(1)}_{\sigma'\sigma}(W(\Lambda,p))\Psi_{\mathbf{\Lambda p},\sigma'} \\
&= \gamma_w \sum_{\sigma'=-1}^{1} D^{(1)}_{\sigma'\sigma}(W(\Lambda,p))\Psi_{\mathbf{\Lambda p},\sigma'}
\end{aligned}
$$

where

$$
\begin{aligned}
D^{(1)}(W(\Lambda,p)) &\overset{(2.5.20)}{=} \exp\Big(i\theta J_1^{(1)}\Big) \\
J_1^{(1)} &\overset{(2.5.21)}{=} \frac{1}{2}\left(J_+ + J_-\right) = \frac{1}{2}\left(\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\Rightarrow \left(J_1^{(1)}\right)^2 &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad \left(J_1^{(1)}\right)^3 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} = J_1^{(1)} \qquad \left(J_1^{(1)}\right)^4 = \left(J_1^{(1)}\right)^2 \qquad \dots \\
D^{(1)}(W(\Lambda,p)) &= \sum_{n=0}^{\infty}\frac{1}{n!}(i\theta)^n\left(J_1^{(1)}\right)^n = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}(i\theta)^{2n+1}\left(J_1^{(1)}\right)^{2n+1} + \mathbf{1} + \sum_{n=1}^{\infty}\frac{1}{(2n)!}(i\theta)^{2n}\left(J_1^{(1)}\right)^{2n} \\
&= J_1^{(1)} i \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\theta^{2n+1} + \mathbf{1} + \left(J_1^{(1)}\right)^2 \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}(\theta)^{2n} \\
&= J_1^{(1)} i\sin(\theta) + \mathbf{1} + \left(J_1^{(1)}\right)^2(\cos(\theta)-1) = i\sin(\theta)J_1^{(1)} + \cos(\theta)\left(J_1^{(1)}\right)^2 + \mathbf{1} - \left(J_1^{(1)}\right)^2 \\
&= i\frac{w\gamma_w\sqrt{\gamma_k^2-1}}{\gamma_w\gamma_k+1}J_1^{(1)} + \frac{\gamma_k+\gamma_w}{\gamma_w\gamma_k+1}\left(J_1^{(1)}\right)^2 + \mathbf{1} - \left(J_1^{(1)}\right)^2
\end{aligned}
$$

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