Notes for "The Quantum Theory of Fields 1, Foundations" - Weinberg

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I. INTRODUCTION

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II. RELATIVISTIC QUANTUM MECHANICS

A. Quantum Mechanics

B. Symmetries

1. *"For this to be unitary and linear, t must be Hermitian and linear"* ($P.51$)

Linearity is trivial and hermiticity follow from the following observation:

$$\begin{aligned}\langle U\Psi|U\Phi\rangle &= \langle(1+i\varepsilon t)\Psi|(1+i\varepsilon t)\Phi\rangle\\ &= \langle\Psi|\Phi\rangle + \varepsilon i\left(\langle\Psi|t\Phi\rangle - \langle t\Psi|\Phi\rangle\right) + \mathcal{O}\left(\varepsilon^2\right)\\ &\overset{Eq.\,(2.2.2)}{\Leftrightarrow} \langle\Psi|t\Phi\rangle = \langle t\Psi|\Phi\rangle\\ &\overset{Eq.\,(2.1.5)}{\Leftrightarrow} t^\dagger = t\end{aligned}$$

2. *Eq.* (2.2.19) ($P.54$)

f^a_{bc} and f^a have to be real as θ^a are real.

3. *Eq.* (2.2.21) ($P.54$)

From Eq. (2.2.20) we have up to $\mathcal{O}\left(\theta^2, \bar\theta^2\right)$

$$\begin{aligned}&1 + i\left(\theta^a + \bar\theta^a + f^a{}_{bc}\bar\theta^b\theta^c\right)t_a + \frac{1}{2}\left(\theta^b + \bar\theta^b\right)\left(\theta^c + \bar\theta^c\right)t_{bc}\\ &= \left[1 + i\bar\theta^a t_a + \frac{1}{2}\bar\theta^b\bar\theta^c t_{bc}\right]\cdot\left[1 + i\theta^a t_a + \frac{1}{2}\theta^b\theta^c t_{bc}\right]\\ &= 1 + i\theta^a t_a + \frac{1}{2}\theta^b\theta^c t_{bc} + i\bar\theta^a t_a - \bar\theta^b t_b\theta^c t_c + \frac{1}{2}\bar\theta^b\bar\theta^c t_{bc}\\ \Leftrightarrow &if^a{}_{bc}\bar\theta^b\theta^c t_a + \frac{1}{2}\left(\theta^b\bar\theta^c + \bar\theta^b\theta^c\right)t_{bc} = -\bar\theta^b\theta^c t_b t_c\\ \Leftrightarrow &\bar\theta^b\theta^c\left[t_{bc} + if^a{}_{bc}t_a + t_b t_c\right] = 0\\ \Leftrightarrow &t_{bc} = -if^a{}_{bc}t_a - t_b t_c\end{aligned}$$

4. *Eq.* (2.2.22) ($P.54$)

$$\begin{aligned}-if^a{}_{bc}t_a - t_b t_c \overset{(2.2.21)}{=} t_{bc} = t_{cb} \overset{(2.2.21)}{=} -if^a{}_{cb}t_a - t_c t_b\\ \Leftrightarrow [t_b, t_c] = i\left(f^a{}_{cb} - f^a{}_{bc}\right)t_a\end{aligned}$$

C. Quantum Lorentz Transformations

1. *"$\Lambda^\mu{}_\nu$ has an **inverse**"* ($P.57$)

This is true because

$$\det(\Lambda) = \pm 1 \neq 0$$

2. *"$\left(\bar\Lambda\Lambda\right)^0{}_0 \geq \bar\Lambda^0{}_0\Lambda^0{}_0 - \sqrt{\left(\Lambda^0{}_0\right)^2 - 1}\sqrt{\left(\bar\Lambda^0{}_0\right)^2 - 1} \geq 1$"* ($P.58$)

The first inequality is trivial from the preceding considerations. For the second inequality assume the contrary holds, then:

$$\begin{aligned}&0 \leq \bar\Lambda^0{}_0\Lambda^0{}_0 - 1 < \sqrt{\left(\Lambda^0{}_0\right)^2 - 1}\sqrt{\left(\bar\Lambda^0{}_0\right)^2 - 1}\\ \Rightarrow &\left(\bar\Lambda^0{}_0\right)^2\left(\Lambda^0{}_0\right)^2 - 2\bar\Lambda^0{}_0\Lambda^0{}_0 + 1\\ &\quad < \left(\bar\Lambda^0{}_0\right)^2\left(\Lambda^0{}_0\right)^2 - \left(\bar\Lambda^0{}_0\right)^2 - \left(\Lambda^0{}_0\right)^2 + 1\\ \Rightarrow &\left(\Lambda^0{}_0 + \bar\Lambda^0{}_0\right)^2 = \left(\bar\Lambda^0{}_0\right)^2 - 2\bar\Lambda^0{}_0\Lambda^0{}_0 + \left(\Lambda^0{}_0\right)^2 < 0\end{aligned}$$

Which is a contradiction as $\Lambda^0{}_0 + \bar\Lambda^0{}_0 \geq 1 + 1 = 2$ and therefore completes the proof.

D. The Poincaré Algebra

1. *"In order fo $U(1+\omega, \varepsilon)$ to be unitary, the operators $J^{\rho\sigma}$ and P^ρ must be **Hermitian**"* ($P.59$)

Analog to II B 1.

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2. *Eqs.* (2.4.8/9) (*P.*60)

$$
\begin{aligned}
&\frac{1}{2}\omega_{\rho\sigma} U J^{\rho\sigma} U^{-1} - \varepsilon_\rho U P^\rho U^{-1} \\
&\overset{(2.4.7)}{=} \frac{1}{2}\left(\Lambda\omega\Lambda^{-1}\right)_{\mu\nu} J^{\mu\nu} - \left(\Lambda\varepsilon - \Lambda\omega\Lambda^{-1}a\right)_\mu P^\mu \\
&= \frac{1}{2}\Lambda_\mu{}^\rho \omega_{\rho\sigma}\left(\Lambda^{-1}\right)^\sigma{}_\nu J^{\mu\nu} \\
&- \left(\Lambda_\mu{}^\rho \varepsilon_\rho - \Lambda_\mu{}^\rho \omega_{\rho\sigma}\left(\Lambda^{-1}\right)^\sigma{}_\nu a^\nu\right) P^\mu \\
&\overset{(2.3.10)}{=} \frac{1}{2}\Lambda_\mu{}^\rho \omega_{\rho\sigma}\Lambda_\nu{}^\sigma J^{\mu\nu} \\
&- \left(\Lambda_\mu{}^\rho \varepsilon_\rho - \Lambda_\mu{}^\rho \omega_{\rho\sigma}\Lambda_\nu{}^\sigma a^\nu\right) P^\mu \\
&= \frac{1}{2}\omega_{\rho\sigma}\left(\Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma J^{\mu\nu} + \Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma a^\nu P^\mu\right) \\
&- \varepsilon_\rho \Lambda_\mu{}^\rho P^\mu
\end{aligned}
$$

In order to be able to compare coefficients in this, the coefficient of $\omega_{\rho\sigma}$ has to be anti symmatrized. With this Eqs. (2.4.8/9) follow immediately.

3. *Eqs.* (2.4.10/11) (*P.*60)

Up to $\mathcal{O}(\omega, \varepsilon)$ one can identify

$$U^{-1}(1+\omega, \varepsilon) = U(1-\omega, -\varepsilon)$$

,since

$$U(1+\omega,\varepsilon)U(1-\omega,-\varepsilon) = U(1-\omega+\omega, -\varepsilon+\varepsilon) = U(1,0).$$

With this we have up to $\mathcal{O}(\omega, \varepsilon)$

$$
\begin{aligned}
& i\left[\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - \varepsilon_\mu P^\mu, J^{\rho\sigma}\right] \\
&= \left(1 + \frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu} - i\varepsilon_\mu P^\mu\right) J^{\rho\sigma} \\
&\cdot \left(1 - \frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu} + i\varepsilon_\mu P^\mu\right) - J^{\rho\sigma} \\
&= U J^{\rho\sigma} U^{-1} - J^{\rho\sigma} \\
&\overset{(2.4.8)}{=} (1+\omega)_\mu{}^\rho (1+\omega)_\nu{}^\sigma \\
&\cdot \left(J^{\mu\nu} - \varepsilon^\mu P^\nu + \varepsilon^\nu P^\mu\right) - J^{\rho\sigma} \\
&= -\varepsilon^\rho P^\sigma + \varepsilon^\sigma P^\rho + \omega_\nu{}^\sigma J^{\rho\nu} + \omega_\mu{}^\rho J^{\mu\sigma}
\end{aligned}
$$

and also

$$
\begin{aligned}
& i\left[\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - \varepsilon_\mu P^\mu, P^\rho\right] \\
&= \left(1 + \frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu} - i\varepsilon_\mu P^\mu\right) P^\rho \\
&\cdot \left(1 - \frac{1}{2}i\omega_{\mu\nu}J^{\mu\nu} + i\varepsilon_\mu P^\mu\right) - P^\rho \\
&= U P^\rho U^{-1} - P^\rho \\
&\overset{(2.4.9)}{=} \omega_\mu{}^\rho P^\mu
\end{aligned}
$$

4. *Eqs.* (2.4.12/13/14) (*P.*60)

The only difficulty lies in the derivation of Eq. (2.4.12) because for this one has to anti symmatrize the coefficient of $\omega_{\mu\nu}$ in Eq. (2.4.10) in order to be able to compare coefficients, similar to II D 2. (Similarly if one derives Eq. (2.4.13) from Eq. (2.4.11))

5. *Eqs.* (2.4.18 − 24) (*P.*61)

First observe

$$J^{ij} = \varepsilon_{ijk} J_k$$

and

$$J_i = \frac{1}{2}\varepsilon_{lmi} J^{lm}$$

$$
\begin{aligned}
[J_i, J_j] &= \frac{\varepsilon_{lmi}\varepsilon_{kpj}}{4}\left[J^{lm}, J^{kp}\right] \\
&\overset{(2.4.12)}{=} -i\frac{\varepsilon_{lmi}\varepsilon_{kpj}}{4}\left[\eta^{mk}J^{lp} - \eta^{lk}J^{mp} - \eta^{pl}J^{km} + \eta^{pm}J^{kl}\right] \\
&= -\frac{i}{2}\left[\varepsilon_{kil}\varepsilon_{kmj}J^{lm} + \varepsilon_{kim}\varepsilon_{kjl}J^{lm}\right] \\
&= -\frac{i}{2}\left[J^{ji} - J^{ij}\right] \\
&= iJ^{ij} = i\varepsilon_{ijk}J_k \\
[J_i, K_j] &= \left[J^{lm}, J^{0j}\right]\frac{\varepsilon_{lmi}}{2} \\
&\overset{(2.4.12)}{=} -i\frac{\varepsilon_{lmi}}{2}\left[\eta^{m0}J^{lj} - \eta^{l0}J^{mj} - \eta^{jl}J^{0m} + \eta^{jm}J^{0l}\right] \\
&= -i\frac{\varepsilon_{lmi}}{2}\left[\delta_{jm}K_l - \delta_{jl}K_m\right] = i\varepsilon_{ijl}K_l \\
[K_i, K_j] &= \left[J^{0i}, J^{0j}\right] \\
&\overset{(2.4.12)}{=} -i\left[\eta^{i0}J^{0j} - \eta^{00}J^{ij} - \eta^{j0}J^{0i} + \eta^{ij}J^{00}\right] \\
&= -iJ^{ij} = -i\varepsilon_{ijk}J_k \\
[J_i, P_j] &= \frac{\varepsilon_{lmi}}{2}\left[J^{lm}, P^j\right] \\
&\overset{(2.4.13)}{=} \frac{i\varepsilon_{lmi}}{2}\left[\eta^{jl}P^m - \eta^{jm}P^l\right] \\
&= \frac{i}{2}\left[\varepsilon_{jmi}P^m - \varepsilon_{mji}P^m\right] = i\varepsilon_{ijm}P_m \\
[K_i, P_j] &= \left[J^{0i}, P^j\right] \\
&\overset{(2.4.13)}{=} i\left[\eta^{j0}P^i - \eta^{ji}P^0\right] \\
&= -i\delta_{ji}P^0 = -i\delta_{ij}H \\
[J_i, H] &\overset{(2.4.13)}{=} \frac{i\varepsilon_{lmi}}{2}\left[\eta^{0l}P^m - \eta^{0m}P^l\right] = 0 \\
[P_i, H] &= \left[P^i, P^0\right] \overset{(2.4.14)}{=} 0 \\
[H, H] &= \left[P^0, P^0\right] \overset{(2.4.14)}{=} 0 \\
[K_i, H] &= \left[J^{0i}, P^0\right] \\
&\overset{(2.4.13)}{=} i\left[\eta^{00}P^i - \eta^{0i}P^0\right] = -iP_i
\end{aligned}
$$

6. Eq. (2.4.27) (*P.*61)

When trying to check this for e.g. the standard representation in 4 dimensions, in terms of infinitesimal rotations, attention with the sign in front of sin for the different rotation axis (see remark at the end ofII E 23).

7. "Inspection of Eqs. (2.4.18 − 24) *shows that these commutation relations have a limit for $v \ll 1$ of the form ...*" (*P.*62)

Always equate same orders in v for this.

8. "$\exp(-i\mathbf{K}\cdot\mathbf{v})\exp(-i\mathbf{P}\cdot\mathbf{a}) = \exp(iM\mathbf{a}\cdot\mathbf{v}/2)\exp(-i(\mathbf{K}\cdot\mathbf{v}+\mathbf{P}\cdot\mathbf{a}))$" (*P.*62)

Use BCH Formula

$$
\begin{aligned}
&\exp(-iK_iv_i)\exp(-iP_ja_j)\\
&=\exp\left(-i(K_iv_i+P_ja_j)+\frac{1}{2}(-i)^2[K_i,P_j]v_ia_j+0\right)\\
&=\exp\left(-i(K_iv_i+P_ja_j)+\frac{1}{2}iMv_ia_i\right)
\end{aligned}
$$

E. One-Particle States

1. Eq. (2.5.2) (*P.*63)

Λ^{-1} shows up here since in Eq. (2.4.9) UPU^{-1} is given but here $U^{-1}PU$ is being used.

2. "with σ within any one block by themselves furnish a representation of the inhomogeneous Lorentz group" (*P.*63)

In this case the Blocks do not mix with other blocks.

3. "and for $p^2 \le 0$, also the sign of p^0" (*P.*64)

For $p^2 \le 0$ we have

$$
\begin{aligned}
p^2 &= -\left(p^0\right)^2+\vec{p}^2 \le 0\\
\Rightarrow |\vec{p}| &\le \left|p^0\right|
\end{aligned}
$$

and from Eq. (2.3.13) we know

$$
\left|\Lambda^0{}_0\right| \ge \left|\vec{\Lambda^0}_{.}\right|.
$$

First suppose $p^0 \ge 0$:

$$
\begin{aligned}
p'^0 &= \Lambda^0{}_0p^0+\Lambda^0{}_ip^i\\
&\ge \Lambda^0{}_0p^0-\left|\vec{\Lambda^0}_{.}\right||\vec{p}|\\
&= \left|\Lambda^0{}_0\right|\left|p^0\right|-\left|\vec{\Lambda^0}_{.}\right||\vec{p}|\\
&\ge \left|\vec{\Lambda^0}_{.}\right|\left(\left|p^0\right|-|\vec{p}|\right)\ge 0
\end{aligned}
$$

Now suppose $p^0 \le 0$:

$$
\begin{aligned}
p'^0 &= \Lambda^0{}_0p^0+\Lambda^0{}_ip^i\\
&\le \Lambda^0{}_0p^0+\left|\vec{\Lambda^0}_{.}\right||\vec{p}|\\
&= -\left|\Lambda^0{}_0\right|\left|p^0\right|+\left|\vec{\Lambda^0}_{.}\right||\vec{p}|\\
&\le \left|\Lambda^0{}_0\right|\left(-\left|p^0\right|+|\vec{p}|\right)\le 0
\end{aligned}
$$

4. Eq. (2.5.12) *"The delta function appears here because $\Psi_{k,\sigma}$ and $\Psi_{k',\sigma'}$ are eigenstates of a Hermitian operator with eigenvalues $\mathbf{k}$ and $\mathbf{k}'$, respectively."* (*P.*66)

> Note: From this point forward in this section of the book the states considered have standard momentum k^μ! (see Text in the Book preceding Eq. (2.5.12))

Suppose that for given values of $\mathbf{k}$ and $\mathbf{k}'$

$$
\langle\Psi_{k',\sigma'}|\Psi_{k,\sigma}\rangle \ne 0
$$

then from

$$
\begin{aligned}
&k^i\langle\Psi_{k',\sigma'}|\Psi_{k,\sigma}\rangle\\
&=\left\langle\Psi_{k',\sigma'}\middle|P^i\Psi_{k,\sigma}\right\rangle\\
&=\left\langle P^i\Psi_{k',\sigma'}\middle|\Psi_{k,\sigma}\right\rangle\\
&=k'^i\langle\Psi_{k',\sigma'}|\Psi_{k,\sigma}\rangle
\end{aligned}
$$

follows

$$
k^i = k'^i.
$$

On why there is only a 3 dimensional delta function showing up:

This is because both states are assumed to have standard momentum k^μ and therefore their 0-th component is completely fixed from their spatial components:

$$
k'^0 = \pm\sqrt{\mathbf{k}'^2-k^2}
$$

Choice between + and − comes from sign of k^0, i.e. + for cases (a) and (c) of Table 2.1.

5. *Eq.* (2.5.13) (*P.*67)

$$
\begin{aligned}
&\langle U(W)\Psi_{k',\sigma'}|U(W)\Psi_{k,\sigma}\rangle \\
&\overset{(2.5.8)}{=} \sum_{\sigma''\sigma'''} \langle D_{\sigma''\sigma'}\Psi_{k',\sigma''}|D_{\sigma'''\sigma}\Psi_{k,\sigma'''}\rangle \\
&= \sum_{\sigma''\sigma'''} \left(D_{\sigma''\sigma'}\right)^{\star} D_{\sigma'''\sigma} \langle \Psi_{k',\sigma''}|\Psi_{k,\sigma'''}\rangle \\
&\overset{(2.5.12)}{=} \sum_{\sigma''} \left(D_{\sigma''\sigma'}\right)^{\star} D_{\sigma''\sigma}\delta^{(3)}(\mathbf{k}'-\mathbf{k}) \\
&\overset{(2.2.2)}{=} \langle \Psi_{k',\sigma'}|\Psi_{k,\sigma}\rangle \\
&\overset{(2.5.12)}{=} \delta_{\sigma'\sigma}\delta^{(3)}(\mathbf{k}'-\mathbf{k})
\end{aligned}
$$

From this it follows

$$D^{\dagger}(W) = D^{-1}(W)$$

6. "$\langle\Psi_{p',\sigma'}|\Psi_{p,\sigma}\rangle = \dots$" (*P.*67)

Note: What is meant by "arbitrary momenta" is that these momenta still have the standard momentum k^{μ}, but now none of the states has exactly k^{μ} as its momentum.

First define

$$k' := L^{-1}(p)p'$$

with this we get:

$$
\begin{aligned}
&\langle\Psi_{p',\sigma'}|\Psi_{p,\sigma}\rangle \\
&\overset{(2.5.5)}{=} \langle\Psi_{p',\sigma'}|N(p)U(L(p))\Psi_{k,\sigma}\rangle \\
&= N(p)\left\langle U(L^{-1}(p))\Psi_{p',\sigma'}\middle|\Psi_{k,\sigma}\right\rangle \\
&\overset{(2.5.11)}{=} N(p)\frac{N^{\star}(p')}{N^{\star}(k')} \\
&\cdot \sum_{\sigma''} D^{\star}_{\sigma''\sigma'}\left(W\left(L^{-1}(p),p'\right)\right)\langle\Psi_{k',\sigma''}|\Psi_{k,\sigma}\rangle \\
&\overset{(2.5.12)}{=} N(p)N^{\star}(p')\,D^{\star}_{\sigma\sigma'}\left(W\left(L^{-1}(p),p'\right)\right)\delta^{(3)}(\mathbf{k}'-\mathbf{k})
\end{aligned}
$$

Where in the last step we used

$$\frac{\delta^{(3)}(\mathbf{k}'-\mathbf{k})}{N^{\star}(k')} = \frac{\delta^{(3)}(\mathbf{k}'-\mathbf{k})}{N^{\star}(k)} \overset{(2.5.5)}{=} \delta^{(3)}(\mathbf{k}'-\mathbf{k})$$

This is valid as from Eq. (2.5.5) we know that $N(p)$ is implicitly dependent on k^{μ}. This therefor fixes $p^2 = k^2$ and the sign of p^0 (see end of II E 4), s.t.

$$N(p) = N(\mathbf{p}).$$

Further we are allowed to use Eq. (2.5.12) in the last step since k' also has k as its standard momentum:

$$k' = L^{-1}(p)p' = \underbrace{L^{-1}(p)L(p')}_{=:L(k')}k$$

7. "$W\left(L^{-1}(p),p\right) = 1$" (*P.*67)

First note

$$L(k) = 1 = L^{-1}(k)$$

with this we get:

$$
\begin{aligned}
W\left(L^{-1}(p),p\right) &\overset{(2.5.10)}{=} L^{-1}\left(L^{-1}(p)p\right)L^{-1}(p)L(p) \\
&= L^{-1}(k) = 1
\end{aligned}
$$

8. "*So we see that the* ***invariant delta function*** *is* ..." (*P.*67)

Invariant in this case means w.r.t. proper orthochronous Lorentz transformations, which can be interpreted as change of variables under the $\int \mathrm{d}^4p$ integral.

9. *Eq.* (2.5.24) (*P.*68)

$$
\begin{aligned}
p^0 &= L^0{}_0k^0 + L^0{}_ik^i \\
&= \frac{\sqrt{\mathbf{p}^2+M^2}}{M}M = \sqrt{\mathbf{p}^2+M^2} \\
p^i &= L^i{}_0k^0 + L^i{}_jk^j \\
&= \frac{p_i}{|\mathbf{p}|}\sqrt{\frac{\mathbf{p}^2+M^2}{M^2}-1}M \\
&= \frac{p_i}{|\mathbf{p}|}\sqrt{\frac{\mathbf{p}^2}{M^2}}M = p_i
\end{aligned}
$$

10. "*To see this, note that the boost Eq.* (2.5.24) *may be expressed as* $L(p) = R(\hat{\mathbf{p}})B(|\mathbf{p}|)R^{-1}(\hat{\mathbf{p}})$" (*P.*68)

First note that the columns and rows of the matrix $B(|\mathbf{p}|)$ are counted in the order 1,2,3,0.

Let

$$\hat{\mathbf{p}} = \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix}$$

then we have

$$R(-\hat{\mathbf{p}}) = R_3(-\phi)R_2(\theta) = \begin{pmatrix} \cos(\phi)\cos(\theta) & -\sin(\phi) & \cos(\phi)\sin(\theta) \\ \sin(\phi)\cos(\theta) & \cos(\phi) & \sin(\phi)\sin(\theta) \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

(see II E 23) and

$$\begin{aligned} R^{-1}(\hat{\mathbf{p}}) = R_2(\theta)R_3(\phi) &= \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta)\cos(\phi) & \cos(\theta)\sin(\phi) & -\sin(\theta) \\ -\sin(\phi) & \cos(\phi) & 0 \\ \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \end{pmatrix} = R^\top(\hat{\mathbf{p}}) \end{aligned}$$

From Eq. (2.5.24) we know

$$L(p) = \begin{pmatrix} 1+(\gamma-1)s_\theta^2 c_\phi^2 & (\gamma-1)s_\theta c_\phi s_\theta s_\phi & (\gamma-1)s_\theta c_\phi c_\theta & s_\theta c_\phi\sqrt{\gamma^2-1} \\ (\gamma-1)s_\theta c_\phi s_\theta s_\phi & 1+(\gamma-1)s_\theta^2 s_\phi^2 & (\gamma-1)s_\theta s_\phi c_\theta & s_\theta s_\phi\sqrt{\gamma^2-1} \\ (\gamma-1)s_\theta c_\phi c_\theta & (\gamma-1)s_\theta s_\phi c_\theta & 1+(\gamma-1)c_\theta^2 & c_\theta\sqrt{\gamma^2-1} \\ s_\theta c_\phi\sqrt{\gamma^2-1} & s_\theta s_\phi\sqrt{\gamma^2-1} & c_\theta\sqrt{\gamma^2-1} & \gamma \end{pmatrix}$$

where s and c are shorthand for sin and cos. With this we can now check:

$$\begin{aligned} R(\hat{\mathbf{p}})B(|\mathbf{p}|)R^{-1}(\hat{\mathbf{p}}) &= \begin{pmatrix} c_\phi c_\theta & -s_\phi & c_\phi s_\theta & 0 \\ s_\phi c_\theta & c_\phi & s_\phi s_\theta & 0 \\ -s_\theta & 0 & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & \sqrt{\gamma^2-1} \\ 0 & 0 & \sqrt{\gamma^2-1} & \gamma \end{pmatrix} \begin{pmatrix} c_\phi c_\theta & s_\phi c_\theta & -s_\theta & 0 \\ -s_\phi & c_\phi & 0 & 0 \\ c_\phi s_\theta & s_\phi s_\theta & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_\phi c_\theta & -s_\phi & c_\phi s_\theta & 0 \\ s_\phi c_\theta & c_\phi & s_\phi s_\theta & 0 \\ -s_\theta & 0 & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_\phi c_\theta & s_\phi c_\theta & -s_\theta & 0 \\ -s_\phi & c_\phi & 0 & 0 \\ \gamma c_\phi s_\theta & \gamma s_\phi s_\theta & \gamma c_\theta & \sqrt{\gamma^2-1} \\ \sqrt{\gamma^2-1}c_\phi s_\theta & \sqrt{\gamma^2-1}s_\phi s_\theta & \sqrt{\gamma^2-1}c_\theta & \gamma \end{pmatrix} = L(p) \end{aligned}$$

11. *"$W(\mathbf{R}, p) = \mathbf{R}$"* (*P.*69)

To see this just substitute $R(\theta)$ back into the previous result.

12. *Eq.* (2.5.25) (*P.*70)

$$\begin{aligned} -1 = (Wt)^\mu\,(Wt)_\mu &= \alpha^2+\beta^2+\zeta^2-(1+\zeta)^2 \\ \Leftrightarrow \alpha^2+\beta^2 &= 2\zeta \end{aligned}$$

13. *Eq.* (2.5.26) (*P.*70)

This is a Lorentz transformation, since

$$\begin{aligned} S^\top \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} S &= \begin{pmatrix} 1 & 0 & \alpha & \alpha \\ 0 & 1 & \beta & \beta \\ -\alpha & -\beta & 1-\zeta & -\zeta \\ \alpha & \beta & \zeta & 1+\zeta \end{pmatrix} \\ &\cdot \begin{pmatrix} 1 & 0 & -\alpha & \alpha \\ 0 & 1 & -\beta & \beta \\ \alpha & \beta & 1-\zeta & \zeta \\ -\alpha & -\beta & \zeta & -1-\zeta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

14. Eqs. (2.5.29/30) (*P.*70)

Eq. (2.5.30) is trivial since the rotations around the three axis satisfy this group multiplication and

$$S(0,0) = \mathbf{1}.$$

For Eq. (2.5.29) explicit calculation shows it (see TODO) together with

$$R(0) = \mathbf{1}.$$

15. Eq. (2.5.31) (*P.*70)

Explicit calculation shows Eq. (2.5.31) (see TODO) and the invariance follows then immediately, since:

$$W'SW'^{-1} = S' \underbrace{R'SR'^{-1}}_{=S''} S'^{-1} = S'''$$

16. "$W(\theta,\alpha,\beta) = 1 + \omega$" (P.71)

From Eq. (2.5.28) we immediately get for infinitesimal θ, α, β :

$$(W(\theta,\alpha,\beta))^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \begin{pmatrix} 0 & \theta & -\alpha & \alpha \\ -\theta & 0 & -\beta & \beta \\ \alpha & \beta & 0 & 0 \\ \alpha & \beta & 0 & 0 \end{pmatrix}^{\mu}{}_{\nu}$$

$$\Rightarrow \omega_{\mu\nu} = \begin{pmatrix} 0 & \theta & -\alpha & \alpha \\ -\theta & 0 & -\beta & \beta \\ \alpha & \beta & 0 & 0 \\ -\alpha & -\beta & 0 & 0 \end{pmatrix}_{\mu\nu}$$

17. Eq. (2.5.32) (*P.*71)

$$\begin{aligned} U(1+\omega) &\overset{(2.4.3)}{=} 1 + \frac{1}{2} i\omega_{\rho\sigma} J^{\rho\sigma} \\ &= 1 + i\theta J^{12} + i\alpha(-J^{13} + J^{10}) + i\beta(-J^{23} + J^{20}) \\ &= 1 + i\theta J_3 + i\alpha(J_2 - K_1) + i\beta(-J_1 - K_2) \end{aligned}$$

18. Eqs. (2.5.35/36/37) (*P.*71)

From Eqs. (2.4.18/19/20) we get:

$$\begin{aligned} [J_3, A] &= [J_3, J_2 - K_1] \\ &= [J_3, J_2] - [J_3, K_1] \\ &= -iJ_1 - iK_2 = iB \\ [J_3, B] &= -[J_3, J_1] - [J_3, K_2] \\ &= -iJ_2 - i(-K_1) = -iA \\ [A, B] &= -[J_2, J_1] - [J_2, K_2] + [K_1, J_1] + [K_1, K_2] \\ &= iJ_3 - iJ_3 = 0 \end{aligned}$$

19. "Since A and B are commuting Hermitian operators they can be simultaneously diagonalized by states $\Psi_{k,a,b}$" (P.71)

This is valid, including the label k, since

$$\begin{aligned} [A, P_1]\Psi_{k,a,b} &= ([J_2, P_1] - [K_1, P_1])\Psi_{k,a,b} \\ &= (-iP_3 + iP^0)\Psi_{k,a,b} = 0 \\ [A, P_2]\Psi_{k,a,b} &= ([J_2, P_2] - [K_1, P_2])\Psi_{k,a,b} \\ &= 0 \\ [A, P_3]\Psi_{k,a,b} &= ([J_2, P_3] - [K_1, P_3])\Psi_{k,a,b} \\ &= (iP_1)\Psi_{k,a,b} = 0 \\ \left[A, P^0\right]\Psi_{k,a,b} &= (\left[J_2, P^0\right] - \left[K_1, P^0\right])\Psi_{k,a,b} \\ &= (-iP_1)\Psi_{k,a,b} = 0 \\ [B, P_1]\Psi_{k,a,b} &= (-[J_1, P_1] - [K_2, P_1])\Psi_{k,a,b} \\ &= 0 \\ [B, P_2]\Psi_{k,a,b} &= ([J_1, P_2] - [K_2, P_2])\Psi_{k,a,b} \\ &= (iP_3 - iP^0)\Psi_{k,a,b} = 0 \\ [B, P_3]\Psi_{k,a,b} &= ([J_1, P_3] - [K_2, P_3])\Psi_{k,a,b} \\ &= (-iP_2)\Psi_{k,a,b} = 0 \\ \left[B, P^0\right]\Psi_{k,a,b} &= (\left[J_1, P^0\right] - \left[K_2, P^0\right])\Psi_{k,a,b} \\ &= (-iP_2)\Psi_{k,a,b} = 0 \end{aligned}$$

20. "σ gives the component of angular momentum in the direction of motion, or helicity*" (P.72)*

The derivation of Eq. (2.5.26) starts among other conditions from the explicit form of k. Which results in

$$\textit{Eqs.}\,(2.5.27/28) \rightarrow \textit{Eq.}\,(2.5.32) \rightarrow 2.5.39,$$

s.t. this is really connected to the direction of motion.

21. "$U(W)\Psi_{k,\sigma} = \exp(i\theta\sigma)\Psi_{k,\sigma}$" (P.72)

Use Eqs. (2.5.38/39).

22. Eq. (2.5.44) (*P*.73)

With $B(u)$ from Eq. (2.5.45) we get:

$$B\left(\frac{|\mathbf{p}|}{\kappa}\right)k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\left(\frac{|\mathbf{p}|}{\kappa}\right)^2+1}{2\frac{|\mathbf{p}|}{\kappa}} & \frac{\left(\frac{|\mathbf{p}|}{\kappa}\right)^2-1}{2\frac{|\mathbf{p}|}{\kappa}} \\ 0 & 0 & \frac{\left(\frac{|\mathbf{p}|}{\kappa}\right)^2-1}{2\frac{|\mathbf{p}|}{\kappa}} & \frac{\left(\frac{|\mathbf{p}|}{\kappa}\right)^2+1}{2\frac{|\mathbf{p}|}{\kappa}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \kappa \\ \kappa \end{pmatrix}$$
$$= |\mathbf{p}| \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

23. "take $R(\hat{\mathbf{p}})$ as a rotation by angle θ around the two-axis followed by a rotation by angle ϕ around the three-axis" (*P*.73)

Let

$$\hat{\mathbf{p}} = \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix}$$

then

$$\begin{aligned} R(\hat{\mathbf{p}}) &= R_3(-\phi)R_2(-\theta) \\ &= \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\phi)\cos(\theta) & -\sin(\phi) & \cos(\phi)\sin(\theta) \\ \sin(\phi)\cos(\theta) & \cos(\phi) & \sin(\phi)\sin(\theta) \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}, \end{aligned}$$

(With this sign convention everything is consistent, see definition of $R(\theta)$ after Eq. (2.5.27) and compare Eq. (2.5.47) to Eq. (2.4.27)) with this we have

$$\hat{\mathbf{p}} = R(\hat{\mathbf{p}})\hat{e}_3.$$

F. Space Inversion and Time-Reversal

1. Eqs. (2.6.7 − 12) (*P*.76)

$$\begin{aligned} \mathsf{P}J_i\mathsf{P}^{-1} &= \frac{1}{2}\varepsilon_{ijk}\mathsf{P}J^{jk}\mathsf{P}^{-1} \\ &= \frac{1}{2}\varepsilon_{ijk}\left(-\delta^j_l\right)\left(-\delta^k_m\right)J^{lm} \\ &= \frac{1}{2}\varepsilon_{ijk}J^{jk} = J_i \\ \mathsf{P}K_i\mathsf{P}^{-1} &= \mathsf{P}J^{0i}\mathsf{P}^{-1} \\ &= \delta^0_\mu\left(-\delta^j_\nu\right)J^{\mu\nu} \\ &= -J^{0i} = -K_i \\ \mathsf{P}P_i\mathsf{P}^{-1} &= \left(-\delta^i_\nu\right)P^\mu = -P_i \\ \mathsf{T}J_i\mathsf{T}^{-1} &= \frac{1}{2}\varepsilon_{ijk}\mathsf{T}J^{jk}\mathsf{T}^{-1} \\ &= -\frac{1}{2}\varepsilon_{ijk}\delta^j_l\delta^k_m J^{lm} \\ &= -\frac{1}{2}\varepsilon_{ijk}J^{jk} = -J_i \\ \mathsf{T}K_i\mathsf{T}^{-1} &= \mathsf{T}J^{0i}\mathsf{T}^{-1} \\ &= -\left(-\delta^0_\mu\right)\delta^j_\nu J^{\mu\nu} \\ &= J^{0i} = K_i \\ \mathsf{T}P_i\mathsf{T}^{-1} &= -\delta^i_\nu P^\mu = -P_i \end{aligned}$$

2. "$\mathcal{P}L(p)\mathcal{P}^{-1} = L(\mathcal{P}p)$" (*P*.77)

From Eq. (2.5.24) we immediately see:

$$\begin{aligned} \left(\mathcal{P}L(p)\mathcal{P}^{-1}\right)^i{}_k &= (L(p))^i{}_k = (L(\mathcal{P}p))^i{}_k \\ \left(\mathcal{P}L(p)\mathcal{P}^{-1}\right)^i{}_0 &= -(L(p))^i{}_0 = (L(\mathcal{P}p))^i{}_0 \\ \left(\mathcal{P}L(p)\mathcal{P}^{-1}\right)^0{}_0 &= (L(p))^0{}_0 = (L(\mathcal{P}p))^0{}_0 \end{aligned}$$

3. "Using Eq. (2.6.14) *again on the left, we see that the square-root factors cancel,..."* (*P*.77)

$$\begin{aligned} &(-J_1 \pm iJ_2)\,\zeta_\sigma\Psi_{k,-\sigma} = -(J_1 \mp iJ_2)\,\zeta_\sigma\Psi_{k,-\sigma} \\ &\overset{(2.6.14)}{=} -\sqrt{(j \pm (-\sigma))(j \mp (-\sigma)+1)}\zeta_\sigma\Psi_{k,-\sigma\mp1} \\ &= -\sqrt{(j \mp \sigma)(j \pm \sigma + 1)}\zeta_\sigma\Psi_{k,-\sigma\mp1} \end{aligned}$$

4. "The time-reversal phase ζ has no physical significance." (*P*.78)

This redefinition only works because $\mathsf{T}$is *anti-linear*.

5. "$\mathcal{T}L(p)\mathcal{T}^{-1} = L(\mathcal{P}p)$" (P.78)

From Eq. (2.5.24) we immediately see:

$$\begin{aligned}
\left(\mathcal{T}L(p)\mathcal{T}^{-1}\right)^i{}_k &= (L(p))^i{}_k = (L(\mathcal{P}p))^i{}_k \\
\left(\mathcal{T}L(p)\mathcal{T}^{-1}\right)^i{}_0 &= -(L(p))^i{}_0 = (L(\mathcal{P}p))^i{}_0 \\
\left(\mathcal{T}L(p)\mathcal{T}^{-1}\right)^0{}_0 &= (L(p))^0{}_0 = (L(\mathcal{P}p))^0{}_0
\end{aligned}$$

6. "P yields a state with four-momentum ..." (P.78)

What is meant by this is:

$$\begin{aligned}
P^i \mathsf{P}\Psi_{k,\sigma} &\overset{(2.6.9)}{=} \mathsf{P}\left(-P^i\right)\Psi_{k,\sigma} = \left(-\delta^i_3\kappa\right)\mathsf{P}\Psi_{k,\sigma} \\
H\mathsf{P}\Psi_{k,\sigma} &\overset{(2.6.13)}{=} \mathsf{P}H\Psi_{k,\sigma} = \kappa\mathsf{P}\Psi_{k,\sigma} \\
J_3\mathsf{P}\Psi_{k,\sigma} &\overset{(2.6.7)}{=} \mathsf{P}J_3\Psi_{k,\sigma} = \sigma\mathsf{P}\Psi_{k,\sigma}
\end{aligned}$$

7. Eq. (2.6.20) (P.79)

We have

$$R_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = R_2^{-1}$$

from which we get

$$\begin{aligned}
U^{-1}(R_2)J_3U(R_2) &= U^{-1}(R_2)J^{12}U(R_2) \\
&\overset{(2.4.8)}{=} \left(R_2^{-1}\right)^1{}_\mu \left(R_2^{-1}\right)^2{}_\nu J^{\mu\nu} \\
&= -J^{12} = -J_3 \\
U^{-1}(R_2)P^\nu U(R_2) &\overset{(2.4.9)}{=} \left(R_2^{-1}\right)^\nu{}_\mu P^\mu
\end{aligned}$$

such that

$$\begin{aligned}
J_3U(R_2^{-1})\mathsf{P}\Psi_{k,\sigma} &= -U(R_2^{-1})J_3\mathsf{P}\Psi_{k,\sigma} \\
&\overset{II\,F\,6}{=} -\sigma U(R_2^{-1})\mathsf{P}\Psi_{k,\sigma} \\
P^iU(R_2^{-1})\mathsf{P}\Psi_{k,\sigma} &\overset{II\,F\,6}{=} U(R_2^{-1})\left(-\left(-\delta^i_3\kappa\right)\right)\mathsf{P}\Psi_{k,\sigma} \\
&= k^iU(R_2^{-1})\mathsf{P}\Psi_{k,\sigma} \\
HU(R_2^{-1})\mathsf{P}\Psi_{k,\sigma} &\overset{II\,F\,6}{=} \kappa U(R_2^{-1})\mathsf{P}\Psi_{k,\sigma} \\
&= k^0U(R_2^{-1})\mathsf{P}\Psi_{k,\sigma}.
\end{aligned}$$

Further we get

$$R_2^{-1}\mathcal{P} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

8. "$\mathcal{P}$ commutes with the rotation $R(\hat{\mathbf{p}})$" (P.79)

This is true because $R(\hat{\mathbf{p}})$ only acts on space components non trivially, which all just get a "-" sign from $\mathcal{P}$.

9. "$\mathsf{P}\Psi_{p,\sigma} = \sqrt{\frac{\kappa}{p^0}}\eta_\sigma U\left(R(\hat{\mathbf{p}})R_2B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,-\sigma}$" (P.79)

$$\begin{aligned}
\mathsf{P}\Psi_{p,\sigma} &\overset{(2.5.5)}{=} N(p)\mathsf{P}U(L(p))\Psi_{k,\sigma} \\
&\overset{(2.5.44)}{=} \sqrt{\frac{\kappa}{p^0}}\mathsf{P}U\left(R(\hat{\mathbf{p}})B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}}U\left(\mathcal{P}R(\hat{\mathbf{p}})B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}}U\left(R(\hat{\mathbf{p}})\mathcal{P}B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}}U\left(R(\hat{\mathbf{p}})R_2R_2^{-1}\mathcal{P}B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}}U\left(R(\hat{\mathbf{p}})R_2B\left(\frac{|\mathbf{p}|}{\kappa}\right)R_2^{-1}\mathcal{P}\right)\Psi_{k,\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}}U\left(R(\hat{\mathbf{p}})R_2B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)U\left(R_2^{-1}\right)\mathsf{P}\Psi_{k,\sigma} \\
&\overset{(2.6.20)}{=} \sqrt{\frac{\kappa}{p^0}}\eta_\sigma U\left(R(\hat{\mathbf{p}})R_2B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,-\sigma}
\end{aligned}$$

10. "But a rotation of $\pm 180^\circ$ around the three-axis reverses the sign of J_2, ..." (P.79)

Analogously to II F 6 ($2 \leftrightarrow 3$).

11. Eq. (2.6.22) (P.79)

First note that

$$B\left(\frac{|\mathbf{p}|}{\kappa}\right)$$

and rotations around the three-axis commute (see Eq. (2.5.45)).

$$\begin{aligned}
\mathsf{P}\Psi_{p,\sigma} &\overset{II\,F\,9}{=} \sqrt{\frac{\kappa}{p^0}}\eta_\sigma U\left(R\left(\hat{\mathbf{p}}\right)R_2 B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,-\sigma} \\
&\overset{Eq.\,(2.6.21)}{=} \sqrt{\frac{\kappa}{p^0}}\eta_\sigma U\left(R\left(-\hat{\mathbf{p}}\right)\right) \\
&\cdot \exp(\pm i\pi J_3) U\left(B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\Psi_{k,-\sigma} \\
&= \sqrt{\frac{\kappa}{p^0}}\eta_\sigma U\left(R\left(-\hat{\mathbf{p}}\right)\right) \\
&\cdot U\left(B\left(\frac{|\mathbf{p}|}{\kappa}\right)\right)\exp(\pm i\pi J_3)\Psi_{k,-\sigma} \\
&= \eta_\sigma \exp(\pm i\pi(-\sigma))\sqrt{\frac{\kappa}{p^0}}U\left(L(\mathcal{P}p)\right)\Psi_{k,-\sigma} \\
&= \eta_\sigma \exp(\mp i\pi\sigma)\Psi_{\mathcal{P}p,-\sigma}
\end{aligned}$$

G. Projective Representations

H. The Symmetry Representation Theorem

1. *“But $\langle\Psi'_k|\Psi'_k\rangle$ is automatically* ***real and positive****”* ($P.91$)

This follows immediately from Eq. (2.1.1).

2. *“From Eq. (2.A.1) we have $|c_{kk}| = |c_{k1}| = \frac{1}{\sqrt{2}}$ and for $l \neq k$ and $l \neq 1$: $c_{kl} = 0$”* ($P.91$)

$$\begin{aligned}
|c_{kl'}|^2 &\overset{(2.A.3)}{=} \left|\sum_l c^\star_{kl}\langle\Psi'_l|\Psi'_{l'}\rangle\right|^2 = |\langle\Upsilon'_k|\Psi'_{l'}\rangle|^2 \\
&\overset{(2.A.1)}{=} |\langle\Upsilon_k|\Psi_{l'}\rangle|^2 = \begin{cases}\frac{1}{2} & \text{for } l' = 1,k \\ 0 & \text{for } l' \neq 1,k\end{cases}
\end{aligned}$$

3. *Eq.* (2.A.10) ($P.92$)

$$\begin{aligned}
|C_k|^2 + |C_1|^2 + 2\,\mathrm{Re}(C_k C_1^\star) &= |C_k + C_1|^2 \\
\overset{(2.A.9)}{=} |C'_k + C'_1|^2 &= |C'_k|^2 + |C'_1|^2 + 2\,\mathrm{Re}(C'_k C_1'^\star) \\
\overset{Eq.\,(2.A.8)}{\Rightarrow} \mathrm{Re}(C_k C_1^\star) &= \mathrm{Re}(C'_k C_1'^\star) \\
\overset{Eq.\,(2.A.8)}{\Rightarrow} \mathrm{Re}\left(\frac{C_k}{C_1}\right) &= \mathrm{Re}\left(\frac{C'_k}{C'_1}\right)
\end{aligned}$$

4. *Eq.* (2.A.11) ($P.92$)

$$\begin{aligned}
\left\{\mathrm{Re}\left(\frac{C_k}{C_1}\right)\right\}^2 + \left\{\mathrm{Im}\left(\frac{C_k}{C_1}\right)\right\}^2 &= \left|\frac{C_k}{C_1}\right|^2 \\
\overset{Eq.\,(2.A.8)}{=} \left|\frac{C'_k}{C'_1}\right|^2 &= \left\{\mathrm{Re}\left(\frac{C'_k}{C'_1}\right)\right\}^2 + \left\{\mathrm{Im}\left(\frac{C'_k}{C'_1}\right)\right\}^2 \\
\overset{Eq.\,(2.A.10)}{\Rightarrow} \mathrm{Im}\left(\frac{C_k}{C_1}\right) &= \pm\,\mathrm{Im}\left(\frac{C'_k}{C'_1}\right)
\end{aligned}$$

5. *“This is only possible if $\mathrm{Re}\left(\frac{C_k}{C_1}\frac{C_l^\star}{C_1^\star}\right) = \mathrm{Re}\left(\frac{C_k}{C_1}\frac{C_l}{C_1}\right)$ or, in other words, if $\mathrm{Im}\left(\frac{C_k}{C_1}\right)\mathrm{Im}\left(\frac{C_l}{C_1}\right) = 0$”* ($P.93$)

Define

$$\begin{aligned}
a &:= \frac{C_k}{C_1} \\
b &:= \frac{C_l}{C_1}
\end{aligned}$$

With this we have

$$\begin{aligned}
|1 + a + b^\star|^2 &= |1 + a + b|^2 && (1) \\
\Leftrightarrow 1 + a^\star + b + a + |a|^2 + ab + b^\star + b^\star a^\star + |b|^2 & && (2) \\
= 1 + a^\star + b^\star + a + |a|^2 + ab^\star + b + ba^\star + |b|^2 & && (3) \\
\Leftrightarrow ab + b^\star a^\star &= ab^\star + ba^\star && (4)
\end{aligned}$$

And further rewriting yields

$$\begin{aligned}
\mathrm{Re}\left(\frac{C_k}{C_1}\frac{C_l}{C_1}\right) &= \mathrm{Re}(ab) \overset{4}{=} \mathrm{Re}(ab^\star) = \mathrm{Re}\left(\frac{C_k}{C_1}\frac{C_l^\star}{C_1^\star}\right) \\
\mathrm{Im}\left(\frac{C_k}{C_1}\right)\mathrm{Im}\left(\frac{C_l}{C_1}\right) &= \mathrm{Im}(a)\,\mathrm{Im}(b) \\
&= -\frac{1}{4}(ab - ab^\star - a^\star b + a^\star b^\star) \overset{4}{=} 0
\end{aligned}$$

6. "Then the invariance of transition probabilities requires that $\left|\sum_k B_k^\star A_k\right|^2 = \left|\sum_k B_k A_k\right|^2$*"* ($P.93$)

$$\left|\sum_k B_k^\star A_k\right|^2 \overset{Eq.\,(2.A.2)}{=} \left|\sum_{kl} B_k^\star A_l \left\langle \Psi_k | \Psi_l \right\rangle\right|^2$$
$$= \left|\left\langle \sum_k B_k \Psi_k \middle| \sum_l A_l \Psi_l \right\rangle\right|^2$$
$$\overset{Eq.\,(2.A.1)}{=} \left|\left\langle U\left(\sum_k B_k \Psi_k\right) \middle| U\left(\sum_l A_l \Psi_l\right) \right\rangle\right|^2$$
$$= \left|\left\langle \sum_k B_k^\star U\Psi_k \middle| \sum_l A_l U\Psi_l \right\rangle\right|^2$$
$$= \left|\sum_{kl} B_k A_l \left\langle U\Psi_k | U\Psi_l \right\rangle\right|^2$$
$$\overset{Eq.\,(2.A.3)}{=} \left|\sum_k B_k A_k\right|^2$$

7. Eq. (2.A.16)(P.94)

$$\sum_{kl} \mathrm{Im}(B_k^\star B_l)\,\mathrm{Im}(A_k^\star A_l)$$
$$= \mathrm{Im}\left(\sum_{kl} \mathrm{Im}(B_k^\star B_l) A_k^\star A_l\right)$$
$$= \frac{1}{2i}\left[\sum_{kl} \mathrm{Im}(B_k^\star B_l) A_k^\star A_l - \text{c.c.}\right]$$
$$= \frac{1}{2i}\left[\sum_{kl} \frac{1}{2i}(B_k^\star B_l - B_k B_l^\star) A_k^\star A_l - \text{c.c.}\right]$$
$$= \frac{1}{2i}\left[\frac{1}{2i}\left(\sum_{kl} B_k^\star B_l A_k^\star A_l - \sum_{kl} B_k B_l^\star A_k^\star A_l\right) - \text{c.c.}\right]$$
$$= \frac{1}{2i}\left[\frac{1}{2i}\left(\left|\sum_k B_k A_k\right|^2 - \left|\sum_k B_k^\star A_k\right|^2\right) - \text{c.c.}\right]$$
$$= \frac{1}{2i}\left[\frac{1}{i}\left(\left|\sum_k B_k A_k\right|^2 - \left|\sum_k B_k^\star A_k\right|^2\right)\right]$$
$$\overset{II\,H\,6}{=} 0$$

8. "However, for any pair of such state-vectors, with neither A_k *nor* B_k ***all of the same phase****"* ($P.94$)

If they were all of the same phase then

$$\forall k, l : \mathrm{Im}\{A_k^\star A_l\} = 0$$

or

$$\forall k, l : \mathrm{Im}\{B_k^\star B_l\} = 0$$

See Footnote j, for why this is relevant.

9. "We have thus shown that for a given symmetry transformation T *either all state-vectors satisfy Eq.* (2.A.14) *or else they all satisfy Eq.* (2.A.15)*"* ($P.94$)

Preceding this a contradiction was derived from the assumption that Eq. (2.A.14) applies for a state-vector

$$\sum_k A_k \Psi_k$$

while Eq. (2.A.15) applies for a state vector

$$\sum_k B_k \Psi_k.$$

Such that the statement is obvious.

I. Group Operators and Homotopy Classes

J. Inversions and Degenerate Multiplets

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