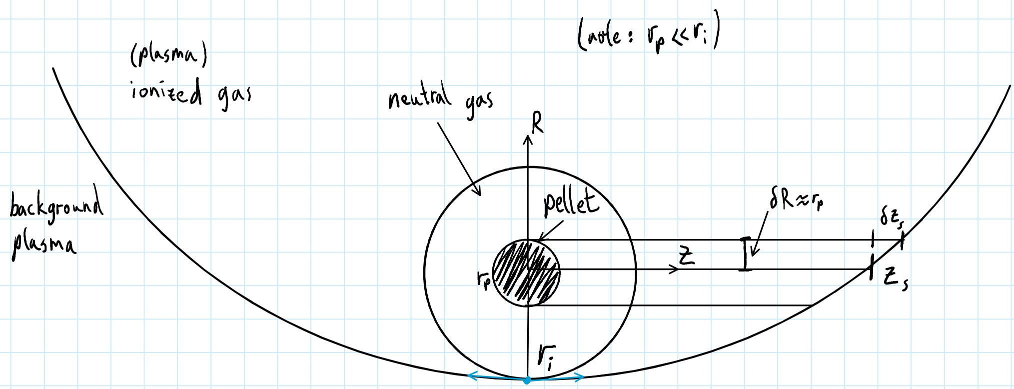
To be able to use the perturbation model of the neutral ablation cloud,
it is important to model the heat flux which reaches the neutral gas
together with the average energy of the incoming electrons.
Specifically the previously defined asymmetry factors q_{rel}, E_{rel} have to be determined.

The first step is to determine the shape of the ionized part of the
ablation cloud and the shielding length which the electrons have to pass.

Shielding geometry



Assume the neutral gas cloud has a spherical boundary
and follows the pellet with the pellet velocity v_p.
At the ionization radius r_i there is a sharp boundary
to an ionized gas of inonization degree α.
The expansion velocity is at the speed of sound c_{s0} but is
stopped perpendicular to the magnetic field line direction z.
In Oskar's drift paper it is shown that the charged particles
have a constant acceleration $\dot{v}_0$ along the major radius direction R.
To determine the plasma cloud boundary, calculate the
trajectory of particles stating at $(R=-r_i, z=0)$

$$R = -r_i + \underbrace{v_p t}_{\text{follows pellet}} + \underbrace{\tfrac{1}{2}\dot{v}_0 t^2}_{\text{drift}} \qquad (1) \qquad (\text{initial velocity } v_0 = v_p)$$

$$z = c_{s0} t \qquad (2)$$

$$(1) \Leftrightarrow \quad t^2 + t\, 2\frac{v_p}{\dot{v}_0} - \frac{2}{\dot{v}_0}(R+r_i) = 0$$

$$\Rightarrow \quad t = -\frac{v_p}{\dot{v}_0} \pm \sqrt{\left(\frac{v_p}{\dot{v}_0}\right)^2 + \frac{2}{\dot{v}_0}(R+r_i)} \qquad \text{(only + is important)}$$

insert into (2) $\Rightarrow$

$$\boxed{z = c_{s0}\left(-\frac{v_p}{\dot{v}_0} + \sqrt{\left(\frac{v_p}{\dot{v}_0}\right)^2 + \frac{2}{\dot{v}_0}(R+r_i)}\right) \qquad (3)}$$

$\Rightarrow$ zentral shielding length

$$\boxed{z_s = z(R=0) - r_i = c_{s0}\left(-\frac{v_p}{\dot{v}_0} + \sqrt{\left(\frac{v_p}{\dot{v}_0}\right)^2 + \frac{2}{\dot{v}_0} r_i}\right) - r_i \qquad (4)}$$

In the limit of a negligible background pressure (ideal gas $n k_B T = p$)

and $\left(\frac{v_p}{c_{s0}}\right)^2 \ll \frac{R+r_i}{R_m}$ by inserting $\dot{v}_0$

$$\Rightarrow z \approx \sqrt{\frac{\gamma_e \langle Z \rangle + \gamma_i}{1+\langle Z \rangle}} \sqrt{R_m (R+r_i)}$$ which is similar to the expression used by Senichenkov

To determine the heatflux asymmetry calculate how much the shielding length

varies over a small distance δR along the major radius

$$\delta z_s \approx \frac{dz}{dR}\Big|_{R=0} \delta R \qquad \text{(assuming linearity over } \delta R)$$

with $\frac{\partial z}{\partial R} = c_{s0} \frac{2}{\dot{v}_0} \frac{1}{2} \left[\left(\frac{v_p}{\dot{v}_0}\right)^2 + \frac{2}{\dot{v}_0}(R+r_i)\right]^{-\frac{1}{2}}$

$$\Rightarrow \quad \delta z_s \approx \frac{c_{s0}}{\dot{v}_0}\left[\left(\frac{v_p}{\dot{v}_0}\right)^2 + \frac{2}{\dot{v}_0} r_i\right]^{-\frac{1}{2}} \delta R$$

$$\Leftrightarrow \quad \delta z_s \approx \frac{c_{s0}\, \delta R}{\sqrt{v_p^2 + 2\dot{v}_0 r_i}} \qquad (5)$$

Resulting heat flux asymmetry

Due to the complex dynamics of how the incoming electrons traverse the plasma cloud, a full describtion of the shielding mechanisms is outside the scope of this project.

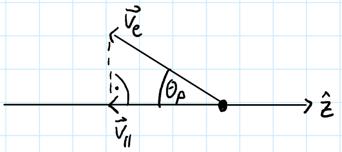
As a first estimation the incoming heatflux is modelled by assuming that electrons with a mean free path λ_{mfp} shorter than the distance traveled through the plasma cloud d are fully shilded and the other electrons reach the neutral gas cloud unaffected.

The distance travelled is not exactly the same as the shielding length z because the electrons gyrate around the magnetic field lines.

because the electrons gyrate around the magnetic field lines.

This helical motion means the electrons vectorial velocity is neither parallel nor perpendicular to $\vec{B}$.

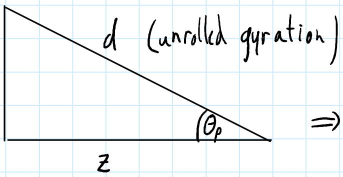
The angle between $\vec{B}$ and $\vec{v}_e$ is called the pitch angle θ_p



$$\Rightarrow \xi := \cos\theta_p = \frac{v_{\parallel}}{v_e} \qquad (6)$$

This pitch angle stays constant during the gyration.

Therefore the total distance travelled becomes clear when imagining to unroll the gyration



$$\Rightarrow \quad z = d\cos\theta_p \Leftrightarrow d = \frac{z}{\xi} \qquad (7)$$

Therefore electrons reach the neutral cloud if

$$\boxed{\frac{z}{\xi} \leq \lambda_{mfp} \qquad (8)}$$

The mean free path for the slowing down part of electron-electron collisions is (from Per's book)

$$\Rightarrow \lambda_{mfp} = \frac{v_e}{\nu_{ei}} = \frac{4\,\varepsilon_0^2 m_e^2 v_e^4}{n_e e^4 \ln\Lambda} \qquad (9) \qquad \lambda_{mfp} = \left(\frac{v_e}{v_{th}}\right)^4 \lambda_T$$

($\lambda_T \hat{=}$ mean free path at thermal speed v_{th})

$$v_{th} = \sqrt{2\frac{k_B T}{m_e}}$$

as mentioned, electrons contribute to the heatflux into the neutral cloud if

$\frac{z}{\xi} \leq \lambda_{mfp}$ or expressed in terms of a critical velocity

$$\Leftrightarrow v_e \geq v_c(\xi, z)$$

$$\Rightarrow \boxed{v_c(\xi, z) = \left[\frac{z}{\xi\,\lambda_T}\right]^{1/4} v_{th} \qquad (10)}$$

Then the total heat flux along one specific field line reaching the neutral cloud is (from one side)

$$q_p = \iiint q(\vec{v}_e)\, f(\vec{v}_e)\, d^3 v_e \quad \text{over allowed velocities} \qquad \text{(normalized to 1)}$$

with the Maxwellian distribution $f(\vec{v}) = \left[\frac{m_e}{2\pi k_B T}\right]^{3/2} \exp\left(-\frac{m v^2}{2 k_B T}\right) = \left(\frac{1}{\sqrt{\pi}\, v_{th}}\right)^3 \exp\left(-\left(\frac{v}{v_{th}}\right)^2\right)$

and the heat flux $q(\vec{v}) = \xi v_e \frac{m v_e^2}{2} n_e$ (remember $v_{\parallel} = \xi v_e$)

integrate in "spherical coordinates" $d^3 v_e = v_e^2 \sin\theta_p\, dv_e\, d\theta_p\, d\varphi$ $\quad$ $\varphi \hat{=}$ gyration angle, $\theta_p \hat{=}$ pitch angle integrate in "spherical coordinates" $d^3 v_e = v_e^2 \sin\theta_p \, dv_e \, d\theta_p \, d\varphi$ $\qquad$ $\varphi \hat{=}$ gyration angle, $\theta_p \hat{=}$ pitch angle

substitute $\xi = \cos\theta_p \Leftrightarrow \frac{d\xi}{d\theta_p} = -\sin\theta_p \Leftrightarrow d\theta_p = -\frac{1}{\sin\theta_p} d\xi \Rightarrow d^3 v_e = -v_e^2 \, dv_e \, d\xi \, d\varphi$

the interval in θ_p would be $\theta_p \in [0, \frac{\pi}{2}] \Leftrightarrow \xi \in [1, 0] \rightarrow$ switching the bounds gets rid of the minus sign

$$\Rightarrow \boxed{q_p = \int_0^{2\pi} d\varphi \int_0^1 d\xi \int_{v_c}^{\infty} dv_e \, v_e^2 \left(\xi v_e \frac{m v_e^2}{2} n_e\right) \left(\frac{1}{\sqrt{\pi} v_{th}}\right)^3 \exp\left(-\left(\frac{v_e}{v_{th}}\right)^2\right) \qquad (11)}$$

no φ-dependence $\Rightarrow \int_0^{2\pi} d\varphi = 2\pi$

$$q_p = 2\pi \frac{m_e}{2} n_e \, \pi^{-\frac{3}{2}} v_{th}^2 \int_0^1 d\xi \, \xi \cdot \int_{v_c}^{\infty} dv_e \left(\frac{v_e}{v_{th}}\right)^5 \exp\left(-\left(\frac{v_e}{v_{th}}\right)^2\right)$$

substitute $u = \frac{v_e}{v_{th}} \Rightarrow \frac{du}{dv_e} = \frac{1}{v_{th}} \Leftrightarrow dv_e = v_{th} \, du$ $\qquad u(v_e = \infty) = \infty$, $u(v_e = v_c) = \left[\frac{z}{\xi} \lambda_T\right]^{1/4} = (\xi x)^{-\frac{1}{4}}$ with $x := \frac{\lambda_T}{z}$

$$q_p = \pi m_e n_e \left(\frac{v_{th}}{\sqrt{\pi}}\right)^3 \int_0^1 d\xi \, \xi \int_{u(v_c)}^{\infty} du \, u^5 \exp(-u^2) \quad \text{use Wolfram Alpha}$$

$$= \pi m_e n_e \left(\frac{v_{th}}{\sqrt{\pi}}\right)^3 \int_0^1 d\xi \, \xi \left[-\frac{1}{2} e^{-u^2}(u^4 + 2u^2 + 2)\right]_{u(v_c)}^{\infty}$$

$$= \pi m_e n_e \left(\frac{v_{th}}{\sqrt{\pi}}\right)^3 \int_0^1 d\xi \, \xi \left[\frac{1}{2} e^{-(\xi x)^{-\frac{1}{2}}} \left((\xi x)^{-1} + 2(\xi x)^{-\frac{1}{2}} + 2\right)\right] \quad \text{use Wolfram Alpha}$$

$$= \pi m_e n_e \left(\frac{v_{th}}{\sqrt{\pi}}\right)^3 \frac{1}{2} \cdot \frac{1}{2x^2} \left[\exp\left(-(\xi x)^{-\frac{1}{2}}\right)\left(2(\xi x)^{3/2} + 2(\xi x)^2 + (\xi x) - \sqrt{\xi x}\right) - E_i\left(-(\xi x)^{-\frac{1}{2}}\right)\right]_0^1$$

with $E_i(z) := -\int_{-z}^{\infty} \frac{e^{-t}}{t} dt$ $\qquad$ with $E_i(-\infty) = 0$

$$q_p = \pi m_e n_e \left(\frac{v_{th}}{\sqrt{\pi}}\right)^3 \frac{1}{4} \frac{1}{x^2} \left[\exp\left(-x^{-\frac{1}{2}}\right)\left(2x^{3/2} + 2x^2 + x - \sqrt{x}\right) - E_i\left(-x^{-\frac{1}{2}}\right)\right] \qquad (12^*)$$

with $v_{th} = \sqrt{2 \frac{k_B T}{m_e}}$ $\qquad x = \frac{\lambda_T}{z} \Rightarrow \frac{dx}{dz} = -\frac{\lambda_T}{z^2} = -\frac{x}{z}$

$$\boxed{q_p = \sqrt{\frac{(k_B T)^3}{2\pi m_e}} \, n_e \frac{1}{x^2} \left[e^{-\frac{1}{\sqrt{x}}}\left(2x^{3/2} + 2x^2 + x - \sqrt{x}\right) - E_i\left(-\frac{1}{\sqrt{x}}\right)\right] \qquad (12)}$$

in Parks paper $q_{bg} = 2\sqrt{\frac{(k_B T)^3}{2\pi m_e}} \, n_e$

$$\boxed{\frac{dq_p}{dz} = \frac{\partial q_p}{\partial x} \frac{dx}{dz} = -\frac{x}{z} \sqrt{\frac{(k_B T)^3}{2\pi m_e}} \, n_e \frac{2}{x^3}\left(E_i\left(-\frac{1}{\sqrt{x}}\right) + \exp\left(-\frac{1}{\sqrt{x}}\right)\sqrt{x}\right) \qquad (13)}$$

The particle flux reaching the neutral cloud along one field line is

$$\Phi_p = \iiint \phi(\vec{v}_e) f(\vec{v}_e) d^3 v_e \qquad \text{with} \quad \phi(\vec{v}_e) = \xi v_e n_e \qquad (\text{remember } v_\parallel = \xi v_e)$$

$$\Phi_p = \int_0^{2\pi} d\varphi \int_0^1 d\xi \int^{\infty} dv_e \, v_e^2 \, \xi v_e n_e \left(\frac{1}{\sqrt{\pi} v_{th}}\right)^3 \exp\left(-\left(\frac{v_e}{v_{th}}\right)^2\right)$$

$$\phi_p = \int_0^{2\pi} d\varphi \int_0^1 d\xi \int_{v_c}^{\infty} dv_e\, v_e^2\, \xi\, v_e\, n_e \left(\frac{1}{\sqrt{\pi} v_{th}}\right)^3 \exp\left(-\left(\frac{v_e}{v_{th}}\right)^2\right)$$

$$= 2\pi n_e\, \pi^{-\frac{3}{2}} \int_0^1 d\xi\, \xi \cdot \int_{v_c}^{\infty} dv_e \left(\frac{v_e}{v_{th}}\right)^3 \exp\left(-\left(\frac{v_e}{v_{th}}\right)^2\right)$$

subtitute $u = \frac{v_e}{v_{th}} \Rightarrow \frac{du}{dv_e} = \frac{1}{v_{th}} \Leftrightarrow dv_e = v_{th}\, du$

$$= \frac{2}{\sqrt{\pi}} n_e v_{th} \int_0^1 d\xi\, \xi \cdot \int_{u(v_c)}^{\infty} du\, u^3 \exp(-u^2)$$ use Wolfram Alpha

$$= \frac{2}{\sqrt{\pi}} n_e v_{th} \int_0^1 d\xi\, \xi \cdot \left[-\frac{1}{2}\exp(-u^2)(u^2+1)\right]_{u(v_c)}^{\infty}$$ with $u(v_c) = (\xi x)^{-\frac{1}{4}}$ with $x = \frac{\lambda_T}{z_s}$

$$= \frac{1}{\sqrt{\pi}} n_e v_{th} \int_0^1 d\xi\, \xi \cdot \exp\left(-(\xi x)^{-\frac{1}{2}}\right)\left((\xi x)^{-\frac{1}{2}}+1\right)$$ use Wolfram Alpha

$$= \frac{1}{\sqrt{\pi}} n_e v_{th} \frac{1}{4x^2}\left[\exp\left(-(\xi x)^{-\frac{1}{2}}\right)\left(2(\xi x)^{3/2} + 2(\xi x)^2 + \sqrt{\xi x} - (\xi x)\right) + Ei\left(-(\xi x)^{-\frac{1}{2}}\right)\right]_0^1$$

$$\boxed{\phi_p = \frac{1}{\sqrt{\pi}} n_e v_{th} \frac{1}{4x^2}\left[e^{-\frac{1}{\sqrt{x}}}\left(2x^{3/2} + 2x^2 + \sqrt{x} - x\right) + E_i\left(-\frac{1}{\sqrt{x}}\right)\right] \qquad (14)}$$

Similar to Parks paper the effective energy is $E_p = \frac{q_p}{\phi_p}$

(12*): $q_p = \frac{1}{\sqrt{\pi}} m_e n_e v_{th}^3 \frac{1}{4} \frac{1}{x^2}\left[\exp\left(-x^{-\frac{1}{2}}\right)\left(2x^{3/2} + 2x^2 + x - \sqrt{x}\right) - E_i\left(-x^{-\frac{1}{2}}\right)\right]$

$$\Rightarrow \boxed{E_p = \frac{q_p}{\phi_p} = m_e v_{th}^2 \frac{e^{-\frac{1}{\sqrt{x}}}\left(2x^{3/2} + 2x^2 + x - \sqrt{x}\right) - E_i\left(-\frac{1}{\sqrt{x}}\right)}{e^{-\frac{1}{\sqrt{x}}}\left(2x^{3/2} + 2x^2 - x + \sqrt{x}\right) + E_i\left(-\frac{1}{\sqrt{x}}\right)} \qquad (15)}$$

in Parks paper $E_{bg} = 2k_BT = m_e v_{th}$

$\frac{dE_p}{dz} = \frac{dx}{dz}\frac{\partial E_p}{\partial x} = -\frac{1}{z^2}\frac{\partial E_p}{\partial x}$ ← very large expression in Wolfram Alpha

Relation to the first order perturbation

perturbation: $q = q_0(r) + q_1(r,\theta)$

$E = E_0(r) + E_1(r,\theta)$

here we have $q_p(z)$, $E_p(z)$ where z is the shielding length at the specific field line

but $z = z(R)$ (eq. (3)) and $R = R(\theta) = -\delta R \cos\theta$ where δR is the radius we consider to be r_p

Taylor around $R=0$

$q_p \approx q_p(R=0) + \left.\frac{dq_p}{dR}\right|_{R=0} \cdot R + \mathcal{O}(R^2)$ higher orders would be different Legendre polynomial modes? (it would be $(\cos\theta)^n$ modes)

$$\Rightarrow \boxed{q_0(r\to\infty) = q_p(R=0) \qquad q_1(r\to\infty,\theta) = \frac{dq_p}{dR}\Big|_{R=0} (-\delta R\cos\theta) + \mathcal{O}(R^2)}$$

$$\boxed{\frac{dq_p}{dR}\Big|_{R=0} = \frac{dq_p}{dz}\Big|_{z=z_s} \frac{dz}{dR}\Big|_{R=0}} \qquad \text{with } \frac{\partial z}{\partial R} = \frac{c_{s0}}{\dot{v}_0}\left[\left(\frac{v_p}{\dot{v}_0}\right)^2 + \frac{2}{\dot{v}_0}(R+r_i)\right]^{-\frac{1}{2}} \qquad \text{from the beginning}$$

similarly

$$\boxed{E_0(r\to\infty) = E_p(R=0) \qquad E_1(r\to\infty,\theta) = \frac{dE_p}{dR}\Big|_{R=0}(-\delta R\cos\theta) + \mathcal{O}(R^2)}$$

with the definition $q_{rel} = \frac{Q_1(\infty)}{q_{bg}}$ ← $\cos\theta$ mode ← spherical mode

$$\boxed{\begin{aligned} q_{rel} &= -\frac{1}{q_p(z=z_s)} \frac{dq_p}{dx}\Big|_{x=x_0} \frac{dx}{dz}\Big|_{z=z_s} \frac{dz}{dR}\Big|_{R=0} \delta R \\ E_{rel} &= -\frac{1}{E_p(z=z_s)} \frac{dE_p}{dx}\Big|_{x=x_0} \frac{dx}{dz}\Big|_{z=z_s} \frac{dz}{dR}\Big|_{R=0} \delta R \end{aligned}} \qquad x = \frac{\lambda_T}{z} \qquad \frac{dx}{dz} = -\frac{\lambda_T}{z^2} = -\frac{x}{z}$$

Summary

shielding length $z(R) = c_{s0}\left(-\frac{v_p}{\dot{v}_0} + \sqrt{\left(\frac{v_p}{\dot{v}_0}\right)^2 + \frac{2}{\dot{v}_0}(R+r_i)}\right)$ $\qquad z_s = z(R=0)$

with $\quad c_{s0} \mathrel{\hat{=}}$ sound speed $\quad v_p \mathrel{\hat{=}}$ pellet speed $\quad r_i \mathrel{\hat{=}}$ ionization radius

Drift acceleration $\dot{v}_0$ given in Oskars paper

defining the mean free path at thermal velocity $\lambda_T = \frac{4\,\varepsilon_0^2 m_e^2 v_{th}^4}{n_e e^4 \ln\Lambda}$

with Coulomb logarithm $\ln\Lambda$, $\quad v_{th} = \sqrt{2\frac{k_B T}{m_e}}$ $\quad$ where $T = T_{e,bg}$

and $x = \frac{\lambda_T}{z}$, $\quad x_s = x(z=z_s)$

Parks paper: $q_{bg} = 2\sqrt{\frac{(k_B T)^3}{2\pi m_e}}\, n_e$, $\quad E_{bg} = 2 k_B T$

while solving the integrals leads to

$$q_0(\infty) = q_{bg}\, f_q(x_s) \qquad \text{with } f_q(x) = 2x^{\frac{1}{2}}\left[e^{-\frac{1}{\sqrt{x}}}\left(2x^{3/2} + 2x^2 + x - \sqrt{x}\right) - E_i\left(-\frac{1}{\sqrt{x}}\right)\right]$$

$$E_0(\infty) = E_{bg}\, f_E(x_s) \qquad \text{with } f_E(x) = \frac{e^{-\frac{1}{\sqrt{x}}}\left(2x^{3/2} + 2x^2 + x - \sqrt{x}\right) - E_i\left(-\frac{1}{\sqrt{x}}\right)}{e^{-\frac{1}{\sqrt{x}}}\left(2x^{3/2} + 2x^2 - x + \sqrt{x}\right) + E_i\left(-\frac{1}{\sqrt{x}}\right)}$$

$$q_{rel} = \frac{f'(x_s)}{f(x_s)} \frac{x_s}{z_s} \frac{dz}{dR}\Big|_{R=0} \delta R$$

$$E_{rel} = \frac{f_E'(x_s)}{f_E(x_s)} \frac{x_s}{z_s} \frac{dz}{dR}\Big|_{R=0} \delta R$$

where $\delta R = r_p$ (assumption)