Rocket effect derivation, part 3

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Assume $\rho, p, \vec{v}$ are calculated. What is the force on the pellet?

Calculate the net force that the gas exerts on a sphere at the origin.

Each point in the gas has a momentum flux tensor $\Pi_{jk} = \langle m n v_j v_k \rangle_f = p\,\delta_{jk} + \rho\, v_j v_k$.

Then the momentum flux through area element dA with normal vector $\hat{n}$ is $\overleftrightarrow{\Pi} \cdot \hat{n}\, dA$.

The net force on the pellet of radius r is then the (negative) surface integral through momentum conservation.

$$\vec{F} = -\int_S \overleftrightarrow{\Pi} \cdot d\vec{S} \overset{\overleftrightarrow{\Pi} = \overleftrightarrow{\Pi}(r,\theta)}{=} -2\pi r^2 \int_0^{\pi} \overleftrightarrow{\Pi} \cdot \hat{r} \sin\theta \, d\theta$$

The φ-symmetry leads to a net force only along the z-axis

$$F_z = \hat{z} \cdot \vec{F} = -2\pi r^2 \int_0^{\pi} \hat{z} \cdot \overleftrightarrow{\Pi} \cdot \hat{r} \sin\theta \, d\theta$$

use $\hat{z} \cdot \overleftrightarrow{\Pi} \cdot \hat{r} = \hat{z} \cdot \left(\rho (\vec{v} \otimes \vec{v}) + p\, \overleftrightarrow{\mathbb{1}}\right) \cdot \hat{r} = \rho\, v_z v_r + p \cos\theta$

$$= -2\pi r^2 \int_0^{\pi} (\rho v_z v_r + p \cos\theta) \sin\theta \, d\theta$$

now linearize with the anisotropic perturbations

$$= -2\pi r^2 \int_0^{\pi} \left[\cancel{\rho_0 v_{0z} v_{0r}} + \rho_1 v_{0z} v_{0r} + \rho_0 v_{1z} v_{0r} + \rho_0 v_{0z} v_{1r} + \cancel{p_0 \cos\theta} + p_1 \cos\theta \right] \sin\theta \, d\theta$$

The zeroth order is independent of θ and $v_{0z} = v_0 \cos\theta$ and $\int_0^{\pi} \cos\theta \sin\theta \, d\theta = 0$

$$= -2\pi r^2 \int_0^{\pi} \left[\rho_1 v_0 \cos\theta\, v_0 + \rho_0 (\hat{z} \cdot \vec{v}_1) v_0 + \rho_0 v_0 \cos\theta\, v_{1r} + p_1 \cos\theta \right] \sin\theta \, d\theta$$

with $\hat{z} \cdot \vec{v}_1 = \hat{z} \cdot \left(\hat{r} \sum_l U_l P_l(\cos\theta) + \hat{\theta} \sum_l V_l \left(-\frac{\alpha_l}{l(l+1)} \frac{\partial}{\partial\theta} P_l(\cos\theta) \right) \right)$ with $\hat{z} \cdot \hat{r} = \cos\theta$ and $\hat{z} \cdot \hat{\theta} = -\sin\theta$

$$= \sum_l \left(U_l P_l(\cos\theta) \cos\theta + V_l \frac{\alpha_l}{l(l+1)} \frac{\partial}{\partial\theta} P_l(\cos\theta) \cdot \sin\theta \right)$$

$$= -2\pi r^2 \sum_l \int_0^{\pi} \left[v_0^2 R_l P_l(\cos\theta) \cos\theta + \rho_0 v_0 \left(U_l P_l(\cos\theta) \cos\theta + V_l \frac{\alpha_l}{l(l+1)} \frac{\partial}{\partial\theta} P_l(\cos\theta) \cdot \sin\theta \right) + \rho_0 v_0 U_l P_l(\cos\theta) \cos\theta + \Pi_l P_l(\cos\theta) \cos\theta \right] \sin\theta \, d\theta$$

$$= -2\pi r^2 \sum_l \left[\left(v_0^2 R_l + 2 \rho_0 v_0 U_l + \Pi_l \right) \underbrace{\int_0^{\pi} P_l(\cos\theta) \cos\theta \sin\theta \, d\theta}_{①} + \rho_0 v_0 V_l \frac{\alpha_l}{l(l+1)} \underbrace{\int_0^{\pi} \frac{\partial}{\partial\theta} P_l(\cos\theta) \sin^2\theta \, d\theta}_{②} \right]$$

② : $\int_0^{\pi} \frac{\partial}{\partial\theta} P_l(\cos\theta) \sin^2\theta \, d\theta = \{\text{integrate by parts}\} = \underbrace{\left[P_l(\cos\theta) \sin^2\theta \right]_0^{\pi}}_{=0} - \underbrace{\int_0^{\pi} P_l(\cos\theta)\, 2 \sin\theta \cos\theta \, d\theta}_{2 \cdot ①}$

① : $\int_0^{\pi} P_l(\cos\theta) \sin\theta \cos\theta \, d\theta = \{\text{substitute } x = \cos\theta \Leftrightarrow \frac{dx}{d\theta} = -\sin\theta \Leftrightarrow d\theta = \frac{1}{-\sin\theta} dx \}$

$$= \int_{-1}^{1} P_l(x)\, x \, dx = \{\text{Mathematica}\} = -\frac{2 \sin(l\pi)}{(l + l^2 - 2)\pi}$$

has poles for $l + l^2 - 2 = 0 \Leftrightarrow l = 1$ (and $l = -2$ but unphysical)

$$P_{l=1}(x) = x \Rightarrow \int_{-1}^{1} P_{l=1}(x)\, x \, dx = \int_{-1}^{1} x^2 dx = \left[\frac{1}{3} x^3 \right]_{-1}^{1} = \frac{2}{3}$$

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$$\Rightarrow \int_0^{\pi} P_l(\cos\theta)\, \sin\theta \cos\theta\, d\theta = \frac{2}{3}\delta_{l1} = \begin{cases} \frac{2}{3} & \text{for } l = 1 \\ 0 & \text{for } l \in \mathbb{N}_0,\ l \neq 1 \end{cases}$$

$$\Rightarrow F_z = -2\pi r^2 \sum_l \left(v_0^2 R_l + 2\rho_0 v_0 U_l + \Pi_l - 2\rho_0 v_0 V_l \frac{\alpha_l}{l(l+1)} \right) \frac{2}{3}\delta_{l1}$$

$$= -\frac{4}{3}\pi r^2 \left(v_0^2 R_{l=1} + 2\rho_0 v_0 U_{l=1} + \Pi_{l=1} - \rho_0 v_0 V_{l=1} \alpha_{l=1} \right)$$

$$= -\frac{4}{3}\pi r^2 \left(v_0^2 R + \rho_0 v_0 (2U - \alpha_1 V) + P \right) \qquad (\text{Per's notes} + \rho_1 \text{ dependence})$$

$\Rightarrow$ The rocket force only depends on mode $l=1$, i.e. on the $\cos\theta$ dependence.