Putting 1st order ODE in matrix form

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Set of equations

$\alpha_l = -l(l+1) \quad , \quad \beta_l = -1$

$$R_l = \left(\frac{P_l}{T_0} - \frac{P_0}{T_0^2}\tau_l\right) = \frac{1}{T_0}\left(P_l - \rho_0 \tau_l\right) \qquad (0)$$

$$\frac{\partial \rho_0}{\partial r} U_l + \rho_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U_l) + \frac{\alpha_l}{r} V_l\right] + v_0 \frac{\partial R_l}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_0) R_l = 0 \qquad (1)$$

$$\rho_0 v_0 \frac{\partial U_l}{\partial r} + \rho_0 \frac{\partial v_0}{\partial r} U_l + v_0 \frac{\partial v_0}{\partial r} R_l = -\frac{1}{\gamma}\frac{\partial P_l}{\partial r} \qquad (2)$$

$$\rho_0 v_0 \frac{\partial V_l}{\partial r} + \rho_0 \frac{v_0}{r} V_l = \frac{\beta_l}{r}\frac{1}{\gamma} P_l \qquad (3)$$

$$\left[U_l \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U_l) + \frac{\alpha_l}{r} V_l\right]\left((\gamma-1)\frac{1}{2}\rho_0 v_0^2 + P_0\right) + \left[v_0 \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_0)\right]\left((\gamma-1)\frac{1}{2} R_l v_0^2 + (\gamma-1)\rho_0 v_0 U_l + P_l\right) = \frac{2}{\lambda_*}\frac{\partial Q_l}{\partial r} \qquad (4)$$

$$\frac{\partial Q_l}{\partial r} = \lambda_*\left(R_l q_0 \Lambda(E_0) + \rho_0 Q_l \Lambda(E_0) + \rho_0 q_0 \frac{\partial \Lambda}{\partial E}\Big|_{E_0} \varepsilon_l\right) \qquad (5)$$

$$\frac{\partial \varepsilon_l}{\partial r} = 2\lambda_*\left(L(E_0) R_l + \rho_0 \frac{\partial L}{\partial E}\Big|_{E_0} \varepsilon_l\right) \qquad (6)$$

Boundary conditions

$T_1(r_p) = 0 \qquad U_1(r_p) = 0 \qquad V_1(r_p) = 0 \qquad Q_1(r_p) = 0 \qquad \varepsilon_1(r_p) = 0$?

$P_1(\infty) = 0 \qquad R_1(\infty) = 0 \qquad Q_1(\infty) = q_0(\infty) \qquad \varepsilon_1(\infty) = E_0(\infty)\frac{\varepsilon_{rel}}{q_{rel}}$

Put ODE in matrix form $\frac{\partial \vec{y}}{\partial r} = M\vec{y}$

define $\vec{y}_1 = \left(P_1, \tau_1, U_1, V_1, Q_1, \varepsilon_1\right)^T$ now use $\alpha_l = -l(l+1) \rightarrow \alpha_1 = -2$, $\beta_1 = -1$

and now abbreviate $\frac{\partial}{\partial r}$ with $()'$

$$(1): \quad \frac{\partial \rho_0}{\partial r} U_l + \rho_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U_l) + \frac{\alpha_l}{r} V_l\right] + v_0 \frac{\partial R_l}{\partial r} + \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_0)}_{\vec{\nabla}\cdot\vec{v}_0} R_l = 0$$

with $R_1' = \left(\frac{1}{T_0}\right)' P_1 - \left(\frac{\rho_0}{T_0}\right)' \tau_1 + \frac{1}{T_0} P_1' - \frac{\rho_0}{T_0}\tau_1'$

and $(r^2 U_1)' = 2r U_1 + r^2 U_1'$

$$P_1 \cdot \left[+ v_0\left(\frac{1}{T_0}\right)' + (\vec{\nabla}\cdot\vec{v}_0)\frac{1}{T_0}\right] + \tau_1 \cdot \left[-v_0\left(\frac{\rho_0}{T_0}\right)' - (\vec{\nabla}\cdot\vec{v}_0)\frac{\rho_0}{T_0}\right] + U_1 \cdot \left[+\rho_0' + \rho_0 \frac{2}{r}\right] + V_1 \cdot \left[-\frac{2\rho_0}{r}\right] = P_1' \cdot \left[-\frac{v_0}{T_0}\right] + \tau_1' \cdot \left(+\frac{v_0 \rho_0}{T_0}\right) + U_1' \cdot (-\rho_0)$$

$$(2): \quad \rho_0 v_0 \frac{\partial U_l}{\partial r} + \rho_0 \frac{\partial v_0}{\partial r} U_l + v_0 \frac{\partial v_0}{\partial r} R_l = -\frac{1}{\gamma}\frac{\partial P_l}{\partial r}$$

$$\Leftrightarrow \quad P_1 \cdot \left[v_0 v_0' \frac{1}{T_0}\right] + \tau_1 \cdot \left[-v_0 v_0' \frac{\rho_0}{T_0}\right] + U_1 \cdot \left[\rho_0 v_0'\right] = P_1' \cdot \left[-\frac{1}{\gamma}\right] + U_1' \cdot \left[-\rho_0 v_0\right]$$

$$(3): \quad \rho_0 v_0 \frac{\partial V_l}{\partial r} + \rho_0 \frac{v_0}{r} V_l = \frac{\beta_l}{r}\frac{1}{\gamma} P_l$$

$\Leftrightarrow \quad P_1 \cdot \left[\frac{1}{\gamma r}\right] + V_1 \cdot \left[\frac{v_0 \rho_0}{r}\right] = V_1' \cdot \left[-\rho_0 v_0\right] \qquad k_0 = \left(\frac{1}{2}\rho_0 v_0^2 + \frac{1}{\gamma-1} P_0\right)$

$$(3):\ \rho_0 v_0 \frac{\partial V_l}{\partial r} + \rho_0 \frac{1}{r} V_l = \frac{1}{r}\frac{1}{\gamma} P_l$$

$$\Longleftrightarrow\quad P_1\cdot\left[\frac{1}{\gamma r}\right] + V_1\cdot\left[\frac{v_0\rho_0}{r}\right] = V_1'\cdot\left[-\rho_0 v_0\right]$$

$$k_0 = \left(\tfrac{1}{2}\rho_0 v_0^2 + \tfrac{1}{\gamma-1}P_0\right)$$

$$(4):\ \left[U_l\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U_l) + \frac{\alpha_l}{r}V_l\right]\overbrace{\left((\gamma-1)\tfrac{1}{2}\rho_0 v_0^2 + P_0\right)} + \left[v_0\frac{\partial}{\partial r} + \overbrace{\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_0)}^{\vec{\nabla}\cdot\vec{v}_0}\right]\left((\gamma-1)\tfrac{1}{2}R_l v_0^2 + (\gamma-1)\rho_0 v_0 U_l + P_l\right) = \frac{2}{\lambda_*}\frac{\partial Q_l}{\partial r}$$

$$P_1\cdot\left[(\vec{\nabla}\cdot\vec{v}_0)\left(1 + (\gamma-1)\tfrac{1}{2}v_0^2\tfrac{1}{T_0}\right) + v_0(\gamma-1)\tfrac{1}{2}\left(\frac{v_0^2}{T_0}\right)'\right] + \tau_1\cdot\left[-(\vec{\nabla}\cdot\vec{v}_0)(\gamma-1)\tfrac{1}{2}\frac{\rho_0}{T_0}v_0^2 - v_0(\gamma-1)\tfrac{1}{2}\left(\frac{\rho_0 v_0^2}{T_0}\right)'\right]$$
$$+\, U_1\cdot\left[(\gamma-1)k_0' + \frac{2(\gamma-1)}{r}k_0 + v_0(\gamma-1)(\rho_0 v_0)' + (\vec{\nabla}\cdot\vec{v}_0)(\gamma-1)\rho_0 v_0\right] + V_1\cdot\left[-\frac{2(\gamma-1)}{r}k_0\right] = P_1'\cdot\left[-v_0 - v_0(\gamma-1)\tfrac{1}{2}\frac{v_0^2}{T_0}\right] + \tau_1'\cdot\left[v_0(\gamma-1)\tfrac{1}{2}\frac{v_0^2\rho_0}{T_0}\right]$$
$$+\, U_1'\cdot\left[-(\gamma-1)k_0 - (\gamma-1)\rho_0 v_0^2\right] + Q_1'\cdot\left[\frac{2}{\lambda_*}\right] \qquad \Big|\cdot\frac{1}{\gamma-1}$$

$$\Longleftrightarrow\quad P_1\cdot\left[(\vec{\nabla}\cdot\vec{v}_0)\left(\frac{1}{\gamma-1} + \frac{1}{2}\frac{v_0^2}{T_0}\right) + \frac{1}{2}v_0\left(\frac{v_0^2}{T_0}\right)'\right] + \tau_1\cdot\left[-(\vec{\nabla}\cdot\vec{v}_0)\frac{1}{2}\frac{\rho_0 v_0^2}{T_0} - \frac{1}{2}v_0\left(\frac{\rho_0 v_0^2}{T_0}\right)'\right] + U_1\cdot\left[k_0' + \frac{2}{r}k_0 + v_0(\rho_0 v_0)' + (\vec{\nabla}\vec{v}_0)\rho_0 v_0\right] + V_1\cdot\left[-\frac{2}{r}k_0\right] =$$
$$= P_1'\cdot\left[-\frac{v_0}{\gamma-1} - \frac{1}{2}\frac{v_0^3}{T_0}\right] + \tau_1'\cdot\left[\frac{1}{2}\frac{\rho_0 v_0^3}{T_0}\right] + U_1'\cdot\left[-k_0 - \rho_0 v_0^2\right] + Q_1'\cdot\left[\frac{2}{(\gamma-1)\lambda_*}\right]$$

$$(5):\ \frac{\partial Q_l}{\partial r} = \lambda_*\left(R_l q_0 \Lambda(E_0) + \rho_0 Q_l \Lambda(E_0) + \rho_0 q_0 \frac{\partial\Lambda}{\partial E}\Big|_{E_0}\varepsilon_l\right)$$

$$\Longleftrightarrow\quad P_1\cdot\left[\lambda_*\frac{q_0\Lambda(E_0)}{T_0}\right] + \tau_1\cdot\left[-\lambda_*\frac{\rho_0 q_0\Lambda(E_0)}{T_0}\right] + Q_1\cdot\left[\rho_0\Lambda(E_0)\right] + \varepsilon_1\cdot\left[\rho_0 q_0\frac{\partial\Lambda}{\partial E}\Big|_{E_0}\right] = Q_1'$$

$$(6):\ \frac{\partial\varepsilon_l}{\partial r} = 2\lambda_*\left(L(E_0)R_l + \rho_0\frac{\partial L}{\partial E}\Big|_{E_0}\varepsilon_l\right)$$

$$\Longleftrightarrow\quad P_1\cdot\left[2\lambda_*\frac{L(E_0)}{T_0}\right] + \tau_1\left[-2\lambda_*\frac{\rho_0 L(E_0)}{T_0}\right] + \varepsilon_1\cdot\left[2\lambda_*\rho_0\frac{\partial L}{\partial E}\Big|_{E_0}\right] = \varepsilon_1'$$

In total as the matrix equation $A\vec{y}' = B\vec{y}$ with

$$A = \begin{array}{c} \\ (1)\\(2)\\(3)\\(4)\\(5)\\(6)\end{array}\begin{array}{c}\begin{array}{cccccc} P_1' & \tau_1' & U_1' & V_1' & Q_1' & \varepsilon_1'\end{array}\\ \begin{pmatrix} -\frac{v_0}{T_0} & \frac{v_0\rho_0}{T_0} & -\rho_0 & 0 & 0 & 0\\ -\frac{1}{\gamma} & 0 & -\rho_0 v_0 & 0 & 0 & 0\\ 0 & 0 & 0 & -\rho_0 v_0 & 0 & 0\\ \left(-\frac{v_0}{\gamma-1} - \frac{1}{2}\frac{v_0^3}{T_0}\right) & \frac{1}{2}\frac{\rho_0 v_0^3}{T_0} & -k_0 - \rho_0 v_0^2 & 0 & \frac{2}{(\gamma-1)\lambda_*} & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}\end{array}$$

$$B = \begin{array}{c} \\ (1)\\(2)\\(3)\\(4)\\(5)\\(6)\end{array}\begin{array}{c}\begin{array}{cccccc} P_1 & \tau_1 & U_1 & V_1 & Q_1 & \varepsilon_1\end{array}\\ \begin{pmatrix} v_0\left(\frac{1}{T_0}\right)' + (\vec{\nabla}\cdot\vec{v}_0)\frac{1}{T_0} & -v_0\left(\frac{\rho_0}{T_0}\right)' - (\vec{\nabla}\cdot\vec{v}_0)\frac{\rho_0}{T_0} & \rho_0' + \frac{2}{r}\rho_0 & -\frac{2}{r}\rho_0 & 0 & 0\\ v_0 v_0'\frac{1}{T_0} & -v_0 v_0'\frac{\rho_0}{T_0} & \rho_0 v_0' & 0 & 0 & 0\\ \frac{1}{\gamma r} & 0 & 0 & \frac{1}{r}v_0\rho_0 & 0 & 0\\ (\vec{\nabla}\cdot\vec{v}_0)\left(\frac{1}{\gamma-1} + \frac{1}{2}\frac{v_0^2}{T_0}\right) + \frac{1}{2}v_0\left(\frac{v_0^2}{T_0}\right)' & -(\vec{\nabla}\cdot\vec{v}_0)\frac{1}{2}\frac{\rho_0 v_0^2}{T_0} - \frac{1}{2}v_0\left(\frac{\rho_0 v_0^2}{T_0}\right)' & k_0' + \frac{2}{r}k_0 + v_0(\rho_0 v_0)' + (\vec{\nabla}\vec{v}_0)\rho_0 v_0 & -\frac{2}{r}k_0 & 0 & 0\\ \lambda_*\frac{q_0\Lambda(E_0)}{T_0} & -\lambda_*\frac{\rho_0 q_0\Lambda(E_0)}{T_0} & 0 & 0 & \lambda_*\rho_0\Lambda(E_0) & \lambda_*\rho_0 q_0\frac{\partial\Lambda}{\partial E}\Big|_{E_0}\\ 2\lambda_*\frac{L(E_0)}{T_0} & -2\lambda_*\frac{\rho_0 L(E_0)}{T_0} & 0 & 0 & 0 & 2\lambda_*\rho_0\frac{\partial L}{\partial E}\Big|_{E_0}\end{pmatrix}\end{array}$$