Remaining quantities and scaling laws

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Compute the rest of the 0th order quantities

For one choice of γ, E_{bg} the zeroth order solution can be calculated.

(to be exact, we choose E_* based on E_{bg} and the found relation $E_*(E_{bg})$)

this yields the normalized quantities $\lambda_*, r_p, E_0(r\to\infty), q_0(r\to\infty)$

$$\Rightarrow \quad r_* = \frac{\tilde{r}_p}{r_p}, \quad q_* = \frac{q_{bg}}{q_0(r\to\infty)}, \quad E_* = \frac{E_{bg}}{E_0(r\to\infty)}$$

Additionally, we have 4 equations that relate the quantities at r_* (all from Parks paper)

$$\lambda_* = \frac{p_*}{T_*}\Lambda_* r_*$$

$$r_*^2 \frac{p_*}{T_*} v_* = \frac{G}{4\pi}$$

$$v_* = \sqrt{\frac{\gamma T_*}{m}}$$

$$(\gamma-1)\frac{4\pi}{G}\frac{\mu}{\gamma}\lambda_*\left(q_*\frac{r_*^2}{T_*}\right) = 2$$

"known" quantities: λ_*, $\Lambda_* = \Lambda(E_*)$, r_*, γ, m, μ (model parameter), q_*

"unknown" quantities: p_*, T_*, v_*, G

solving this system of equations using sympy yields:

(disregarding complex solutions) in form (dimensionless) $\cdot$ [input parameters with dimensions]

$$T_* = 2^{-\frac{2}{3}}\frac{(\gamma-1)^{2/3}}{\gamma} m^{\frac{1}{3}}(\Lambda_*\mu q_* r_*)^{2/3}$$

$$v_* = 2^{-\frac{1}{3}}(\gamma-1)^{\frac{1}{3}}\left(\frac{\Lambda_*\mu q_* r_*}{m}\right)^{\frac{1}{3}}$$

$$p_* = 2^{-\frac{2}{3}}\lambda_*\frac{(\gamma-1)^{2/3}}{\gamma}\cdot\left(\frac{m}{\Lambda_* r_*}\right)^{\frac{1}{3}}\cdot(\mu q_*)^{2/3}$$

$$G = 2\cdot 2^{\frac{2}{3}}\pi\lambda_*(\gamma-1)^{\frac{1}{3}}\Lambda_*^{-\frac{2}{3}} m^{-\frac{1}{3}} r_*^{\frac{4}{3}}(\mu q_*)^{\frac{1}{3}}$$

$$\boxed{\begin{aligned}
&\Leftrightarrow\; T_* = \left(\frac{1}{2 q_\infty r_p}\right)^{\frac{2}{3}}\left(\frac{(\gamma-1)^2}{\gamma}\right)^{\frac{1}{3}}\cdot\left[\sqrt{m}\,\Lambda_*\mu q_{bg}\tilde{r}_p\right]^{\frac{2}{3}}\\
&\Leftrightarrow\; v_* = \left(\frac{2}{q_\infty r_p}\right)^{\frac{1}{3}}(\gamma-1)^{\frac{1}{3}}\cdot\left[\frac{\Lambda_*\mu q_{bg}\tilde{r}_p}{m}\right]^{\frac{1}{3}}\\
&\Leftrightarrow\; p_* = \left(\frac{\lambda_* r_p}{4 q_\infty^2}\right)^{\frac{1}{3}}\left(\frac{(\gamma-1)^2}{\gamma}\right)^{\frac{1}{3}}\cdot\left[\frac{m(\mu q_{bg})^2}{\Lambda_*\tilde{r}_p}\right]^{\frac{1}{3}}\\
&\Leftrightarrow\; G = 2\pi\lambda_*\left(\frac{4}{q_\infty r_p^4}\right)^{\frac{1}{3}}(\gamma-1)^{\frac{1}{3}}\cdot\left[\frac{\tilde{r}_p^4\,\mu q_{bg}}{\Lambda_*^2 m}\right]^{\frac{1}{3}}
\end{aligned}}$$

Scaling law fits

Scaling laws of the form

$$f(E_{bg}) = a_0 + a_1 \log_{10}(E_{bg})$$

are fitted to the dimensionless 0th order quantities for each γ

$$E_0(\infty),\ q_0(\infty),\ \lambda_*,\ r_p,\ T_{*,\text{prefactor}},\ V_{*,\text{prefactor}},\ P_{*,\text{prefactor}},\ G_{\text{prefactor}}$$

To the first order pressure $P_1(r_p)$ a linear function

$$P_1(E_{rel}/q_{rel}) = a\left(\frac{E_{rel}}{q_{rel}} - b\right)$$

is fitted for each combination of γ, E_{bg} in the parameter scan

Then scaling laws

$$a(E_{bg}) = a_0 + a_1 \log_{10}(E_{bg})$$

$$b(E_{bg}) = b_0 + b_1 \log_{10}(E_{bg}) + b_2 (\log_{10}(E_{bg}))^2$$

are fitted for each γ