Derivation of the rocket force resulting from a spherical pellet ablation with a perturbative asymmetry

Oskar Vallhagen

January 29, 2024

In this document we derive the rocket force resulting from a spherical pellet ablation with a perturbative asymmetry. We restrict ourselves (to begin with, at least) to a simplified case where the heat flux from the background plasma is assumed to flow only in the radial direction and only has a pertubatively small angular dependence. We also restrict ourselves to hydrogen pellets, for which the sublimation energy per particle is very small ($\sim 0.01\,\mathrm{eV}$). This will cause the ablation to such that both the temperature and the heat flux vanishes at the pellet surface, and the ablation rate is thus determined solely by the dynamics of the flow of the ablated material away from the pellet.

1 Neutral flow dynamics for a given heat flux asymmetry

Here we treat only the flow in the form of a neutral gas in the close proximity to the pellet, which takes the form of a nearly spherically symmetric expansion; the heat flux and electron energy reaching the neutral zone through the ionized plasmoid is treaded as a boundary condition. The dynamics of interest are then governed by the conservation equations of mass, momentum and energy, as well as the equation of state for an ideal gas:

$$\nabla \cdot (\rho \vec{v}) = 0 \tag{1}$$

$$\rho \vec{v} \cdot \nabla \vec{v} = -\nabla p \tag{2}$$

$$\nabla \cdot \left[\left(\frac{\rho v^2}{2} + \frac{\gamma p}{\gamma - 1} \right) \vec{v} \right] = \nabla \cdot \vec{q} \tag{3}$$

$$p = \frac{\rho T}{m}, \tag{4}$$

where ρ is the mass density, $\vec{v}$ is the flow velocity, p is the pressure, m is the mass per ablated particle, q is the heat flux of electrons from the background plasma and r is the radial coordinate. (For some reason, Parks [1] seems to set $\nabla \cdot \vec{q} = dq/dr$. Maybe that has something to do with that the heat flux actually only flows along the field lines and not purely along the radial direction, so that the term describing the increased heat source per unit volume due to the decrease in the spherical shell size should be excluded?).

Several different models for the heat flux absorbtion exist in the literature, adapted for various situations and with varying degree of sophistication, but a rather simple model is the one used by [1]. In this model the incoming electrons are reduced to a monoenergetic beam with initial energy $E_{\mathrm{bg}} = 2T_{\mathrm{bg}}$ equal to the ratio of the unidirectional heat and particle fluxes for a Maxwellian distribution with initial temperature T_{bg}. The attenuation of the heat flux along the direction x in which the heat flux is directed is described by

$$\frac{dq}{dx} = -\frac{q}{\lambda_{\mathrm{mfp}}(E)} = \frac{\rho}{m} q \Lambda(E) \tag{5}$$

$$\frac{dE}{dx} = -\frac{\rho}{2m} L(E), \tag{6}$$

where λ_{mfp} is the mean free path. When utilising the approximation that the flow is purely in the $-r$-direction, we can replace $-d/dx$ with d/dr. The effective heat flux cross section Λ and electron energy loss function L can be approximated by semi-empirical expressions according to

$$\Lambda(E) = \hat{\sigma}_T + \frac{2L(E)}{E} \tag{7}$$

$$L(E) = 8.62 \cdot 10^{-15} \cdot \left[\left(\frac{E}{100}\right)^{0.823} + \left(\frac{E}{60}\right)^{-0.125} + \left(\frac{E}{48}\right)^{-1.94} \right]^{-1} \mathrm{eV} \cdot \mathrm{cm}^2 \tag{8}$$

$$\hat{\sigma}_T(E > 100\,\mathrm{eV}) = \frac{8.8 \cdot 10^{-13}}{E^1.71} - \frac{1.62 \cdot 10^{-12}}{E^{1.932}}\,\mathrm{cm}^2 \tag{9}$$

$$\hat{\sigma}_T(E < 100\,\mathrm{eV}) = \frac{1.1 \cdot 10^{-14}}{E}\,\mathrm{cm}^2, \tag{10}$$

where $\hat{\sigma}_T$ is the total elastic scattering cross section.

As the pressure inside the neutral cloud is several orders of magnitude larger than in the background plasma, the latter can be considered negligible, and thus the pressure must vanish at large radii. Moreover, the heat flux and effective incoming electron energy must approach those of the background plasma at large radii. At the pellet surface, the ablated material has not yet gained any heat or flow velocity, so both the temperature and flow velocity must go to zero at the pellet surface. Finally, the very small sublimation energy means that the heat flux reaching the pellet surface will be negligible compared to the heat flux being absorbed or scattered in the neutral cloud, and the heat flux at the pellet surface can thus be taken as zero. In summary, the boundary conditions are

$$p(r \to \infty) = 0, \quad q(r \to \infty) = q_{\mathrm{bg}}, \quad E(r \to \infty) = E_{\mathrm{bg}} \tag{11}$$

$$q(r_p) = 0, \quad T(r_p) = 0, \tag{12}$$

where q_{bg} is the initial incoming heat flux and r_p is the pellet radius.

1.1 Spherically symmetric solution

We now turn to the spherically symmetric solution. As the boundary conditions are known for the pressure and temperature but not the density, we use equation (4) to eliminate the density from the equation system and express it in terms of the pressure and temperature alone. Assuming spherical symmetry, the equation system can be written as

$$\frac{d}{dr}\left(r^2 v_0 \frac{p_0}{T_0}\right) = 0 \Rightarrow r^2 v_0 \frac{p_0}{T_0} = \mathrm{const} = \frac{G}{4\pi} \tag{13}$$

$$\frac{p_0}{T_0} v \frac{dv_0}{dr} + \frac{dp_0}{dr} = 0 \tag{14}$$

$$\frac{G}{4\pi r^2} \frac{d}{dr}\left(\frac{\gamma T_0}{\gamma - 1} + \frac{v_0^2}{2}\right) = \frac{dq_0}{dr} \tag{15}$$

$$\frac{dq_0}{dr} = \frac{p_0}{T_0} q_0 \Lambda(E_0) \tag{16}$$

$$\frac{dE_0}{dr} = \frac{p_0}{2T_0} L(E_0), \tag{17}$$

where G is the particle ablation rate $[\mathrm{m}^{-3}]$ (note that [1] considers the mass ablation rate instead, which introduces some factors m, denoting the mass per neutral particle, but here we consider the particle ablation rate for ease of notation). The term $2q/r$ appearing when taking the divergence of q is disregarded as the heat flux in reality is parallell to the field lines, so that the focusing of a radially flowing heat flux described by this term does actually not occur (again, is this the actual reason [1] disregards this term?).

The solution to this equation system is outlined in [1]. It turns out to be useful to normalize the quantities of interest in terms of their values at the sonic radius (i.e. the radius where the flow becomes supersonic),

$$\bar{p}_0 = p_0/p_*, \quad \bar{T}_0 = T_0/T_*, \quad \bar{v}_0 = v_0/v*, \quad \bar{q}_0 = q_0/q_*, \quad \bar{E}_0 = E_0/E_*$$
$$\bar{r} = r/r_*, \quad \bar{\Lambda} = \Lambda(E_0)/\Lambda(E_*) \quad \bar{L} = L(E_0)/[E_*\Lambda(E_*)], \tag{18}$$

and introduce the new variable $\bar{w}_0 = \bar{v}_0^2$. After some algebra, utilising the relations between the quantities of interest at the sonic radius, [1] derives the following equation system (dropping the bar for brewity)

$$\frac{dw}{dr} = \frac{4wT}{(T-w)r}\left(\frac{q\Lambda r}{T\sqrt{w}} - 1\right) \tag{19}$$
$$\frac{dT}{dr} = \frac{2\Lambda q}{\sqrt{w}} - \frac{1}{2}(\gamma-1)\frac{dw}{dr} \tag{20}$$
$$\frac{dE}{dr} = 2\lambda_* \frac{L}{r^2\sqrt{w}} \tag{21}$$
$$\frac{dq}{dr} = \lambda_* \frac{q\Lambda}{\sqrt{w}r^2}, \tag{22}$$

where $\lambda_* = r_*\Lambda_*p_*/T_*$ is a dimensionless eigenvalue that uniquely determines the solution to the normalised equation system. The equation for dw/dr has an apparent singularity at the sonic radius, but dw/dr can still be determined by L'Hopitals rule, yielding

$$\left.\frac{dw}{dr}\right|_{r=1} = \frac{\frac{1}{2}(3-\gamma) + \{[\frac{1}{2}(3-\gamma)]^2 - \frac{1}{2}(\gamma+1)(\lambda_* + \psi_* - 1)\}^{1/2}}{(\gamma+1)/4}, \tag{23}$$

' where

$$\psi_* = \left.\frac{d\Lambda}{dr}\right|_{r=1} = \frac{2\lambda_* L}{\Lambda^2}\left.\frac{d\Lambda}{dE_0}\right|_{E_0=E_*}. \tag{24}$$

It is then possible to start integrating the normalized equation system numerically (suggestively using some standard ODE-solver from scipy.integrate.ode) from the sonic radius (where, with the present normalisation, all quantities of interest are equal to 1) towards smaller radii, for some assumed value of λ_*. The value of λ_* can then be varied until one finds a value such that the boundary conditions for q and T are simultaneously satisfied. The radius at which this happens is then interpreted as the normalised pellet radius (which is not known a-priori). When the appropriate value of λ_* is found, one can integrate from the sonic radius towards larger radii, and find what values of the normalised heat flux $q_{\rm bg}$ and effective beam energy $E_{\rm bg}$ the found solution corresponds to (these values are also not known a-priori).

As the normalised effective beam energy and heat flux attenuation depends on the dimension-full effective beam energy, λ_* will formally also be dependent on this quantity (or, rather, on E_*, which is the value one has to choose before solving the equation system). It is however found by [1] that λ_* is a very weak function of E_*, and it might therefore be sufficient to find the spherically symmetric solution only for a representative value of E_* of a few tens of keV (where $\lambda_* \approx 0.93$ according to [1]).

1.2 First order angular asymmetry correction

1.2.1 Linearized equations

We write the quantities of interest as a spherically symmetric part plus a perturbatively small angularly dependent correction,

$$p = p_0(r) + p_1(r,\theta), \tag{25}$$
$$\vec{v} = v_0(r)\hat{r} + u_1(r,\theta)\hat{r} + v_1(r,\theta)\hat{\theta}, \tag{26}$$
$$T = T_0(r) + T_1(r,\theta), \tag{27}$$
$$\vec{q} = q_0(r)\hat{r} + q_1(r,\theta)\hat{r}, \tag{28}$$
$$E = E_0(r) + E_1(r,\theta), \tag{29}$$

keeping the approximation that the heat flux is completely radial. When inserted into equations (1)-(4), the equation system for the angularly dependent part, keeping only first order terms, becomes

$$\nabla \cdot \left[\frac{p_0}{T_0}\vec{v}_1 + \left(\frac{p_1}{T_0} - \frac{p_0}{T_0^2}T_1 \right) \vec{v}_0 \right] = 0 \tag{30}$$

$$\rho_0(\vec{v}_1 \cdot \nabla \vec{v}_0 + \vec{v}_0 \cdot \nabla \vec{v}_1) + \left(\frac{p_1}{T_0} - \frac{p_0}{T_0^2}T_1 \right) \vec{v}_0 \cdot \nabla \vec{v}_0 = -\nabla p_1 \tag{31}$$

$$\nabla \cdot \left\{ \left(\frac{p_0 v_0^2}{2T_0} + \frac{\gamma p_0}{\gamma - 1} \right) \vec{v}_1 + \left[\left(\frac{p_1}{T_0} - \frac{p_0}{T_0^2}T_1 \right) \frac{v_0^2}{2} + \rho_0 \vec{v}_0 \cdot \vec{v}_1 + \frac{\gamma p_1}{\gamma - 1} \right] \vec{v}_0 \right\} = \frac{\partial q_1}{\partial r} \tag{32}$$

$$\frac{\partial q_1}{\partial r} = \left(\frac{p_1}{T_0} - \frac{p_0}{T_0^2}T_1 \right) q_0 \Lambda(E_0) + \frac{p_0}{T_0} q_0 \Lambda'(E_0) E_1 + \frac{p_0}{T_0} q_1 \Lambda(E_0) \tag{33}$$

$$\frac{\partial E_1}{\partial r} = \left(\frac{p_1}{T_0} - \frac{p_0}{T_0^2}T_1 \right) \frac{L(E_0)}{2} + \frac{p_0}{2T_0} L'(E_0) E_1. \tag{34}$$

Let us now expand the angular dependence of all quantities except for v_1 in terms of basis functions $X_l(\theta)$, and expand the angular dependence of v_1 in terms of some other basis functions $Y_l(\theta)$. By inspecting the equation system and noting that

$$\nabla \cdot v_1 \hat{\theta} = \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}(\sin\theta v_1), \tag{35}$$

$$\nabla p \cdot \hat{\theta} = \frac{1}{r} \frac{\partial p}{\partial \theta}, \tag{36}$$

$$\hat{r} \cdot \nabla \vec{v}_1 = \frac{\partial v_1}{\partial r} \hat{\theta} \tag{37}$$

one can see that the angular dependence can be factored out in each equation if these basis functions satisfy

$$\frac{1}{\sin\theta} \frac{\partial}{\theta}(\sin\theta Y_l) \propto X_l \tag{38}$$

$$Y_l \propto \frac{\partial X_l}{\partial \theta}. \tag{39}$$

By inserting (39) into (38), we obtain the defining equation for the zonal harmonics, so that $X_l = \mathcal{P}_l^0(\cos\theta)$, where $\mathcal{P}_l^0$ is the zeroth Legendre polynomial of order l. We therefore make the following expansions:

$$p_1 = \sum_{l=1}^{\infty} P_l(r) \mathcal{P}_l^0(\cos\theta), \tag{40}$$

$$T_1 = \sum_{l=1}^{\infty} \mathcal{T}_l(r) \mathcal{P}_l^0(\cos\theta), \tag{41}$$

$$u_1 = \sum_{l=1}^{\infty} U_l(r) \mathcal{P}_l^0(\cos\theta), \tag{42}$$

$$v_1 = \sum_{l=1}^{\infty} V_l(r) \frac{d\mathcal{P}_l^0(\cos\theta}{d\theta}), \tag{43}$$

$$q_1 = \sum_{l=1}^{\infty} Q_l(r) \mathcal{P}_l^0(\cos\theta), \tag{44}$$

$$E_1 = \sum_{l=1}^{\infty} \mathcal{E}_l(r) \mathcal{P}_l^0(\cos\theta) \tag{45}$$

(the zeroth mode is constant with respect to angle and is therefore included in the zeroth order solution). Inserting the above, factoring out the angular dependence of each equation and noting that the modes are linearly independend, we obtain the following equation sytem for a given mode l:

$$\frac{\partial}{\partial r}\left(\frac{p_0}{T_0}\right)U_l + \frac{p_0}{T_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 U_l\right) + v_0\frac{\partial}{\partial r}\left(\frac{P_l}{T_0} - \frac{p_0}{T_0^2}\mathcal{T}_l\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_0\right)\left(\frac{P_l}{T_0} - \frac{p_0}{T_0^2}\mathcal{T}_l\right) - l(l+1)\frac{p_0}{rT_0}V_l = 0 \tag{46}$$

$$\frac{p_0}{T_0}U_l v_0' + \frac{p_0}{T_0}v_0 U_l' + \left(\frac{P_l}{T_0} - \frac{p_0}{T_0^2}\mathcal{T}_l\right)v_0 v_0' + P_l' = 0 \tag{47}$$

$$\frac{p_0}{T_0}\frac{v_0}{r}V_l + \frac{p_0}{T_0}v_0 V_l' - \frac{P}{r} = 0 \tag{48}$$

$$\left(\frac{p_0 v_0^2}{2T_0} + \frac{\gamma p_0}{\gamma - 1}\right)\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 U_l\right) - l(l+1)\frac{V_l}{r}\right] + U_l\frac{\partial}{\partial r}\left(\frac{p_0 v_0^2}{2T_0} + \frac{\gamma p_0}{\gamma - 1}\right) + \left[\left(\frac{P_l}{T_0} - \frac{p_0}{T_0^2}\mathcal{T}_l\right)\frac{v_0^2}{2} + \frac{p_0}{T_0}v_0 U_l + \frac{\gamma P_l}{\gamma - 1}\right]\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_0\right) + v_0\frac{\partial}{\partial r}\left[\left(\frac{P_l}{T_0} - \frac{p_0}{T_0^2}\mathcal{T}_l\right)\frac{v_0^2}{2} + \frac{p_0}{T_0}v_0 U_l + \frac{\gamma P_l}{\gamma - 1}\right] = \frac{\partial Q_l}{\partial r} \tag{49}$$

$$\frac{\partial Q_l}{\partial r} = \left(\frac{P_l}{T_0} - \frac{p_0}{T_0^2}\mathcal{T}_l\right)q_0\Lambda(E_0) + \frac{p_0}{T_0}q_0\left.\frac{d\Lambda}{dE}\right|_{E=E_0}\mathcal{E}_l + \frac{p_0}{T_0}\Lambda(E_0)Q_l \tag{50}$$

$$\frac{\partial \mathcal{E}_l}{\partial r} = \left(\frac{P_l}{T_0} - \frac{p_0}{T_0^2}\mathcal{T}_l\right)\frac{L(E_0)}{2} + \frac{p_0}{2T_0}\left.\frac{dL}{dE}\right|_{E=E_0}\mathcal{E}_l, \tag{51}$$

with boundary conditions

$$... \tag{52}$$

1.2.2 Resulting rocket force

1.2.3 Outline of numerical solution

It turns otu to be convenient to introduce the following normalisations:

$$... \tag{53}$$

With these normalisations, the equation system to be solved becomes (dropping the bar for brewity and considering now only the first zonal harmonic)

$$... \tag{54}$$

and the boundary conditions become

$$... \tag{55}$$

The only parameters which can alter the solution to this normalised equation system are the eigenvalue λ_* for the zeroth order solution (which, according to [1], also determines the normalized zeroth order solution) and the normalized energy boundary condition $\mathcal{E}_{1,\mathrm{bg}}/q_{\mathrm{rel}}$ (the normalized boundary condition for the heat flux is fixed to the zeroth order solution and the other boundary conditions are zero). We can then express the rocket force as

$$... \tag{56}$$

where the dimensionless function $f(E_0, \mathcal{E}_{1,\mathrm{bg}}/q_{\mathrm{rel}})$ is left to be determined numerically. The dependence on E_0 for λ_* and the zeroth order solution is however found in [1] to be very weak, and it might therefore be sufficiently accurate to calculate them only once for a representative E_0 and then disregard the dependence of E_0 in f.

The dependence of f on $\mathcal{E}_{1,\mathrm{bg}}/q_{\mathrm{rel}}$ can be determined by integrating the equation system from $r = r_p$ for some assumed values of $P_1(r_p)$ and $\mathcal{E}_1(r_p)$ (note that they are not determined by the boundary conditions at $r = r_p$), and then vary $P_1(r_p)$ and $\mathcal{E}_1(r_p)$ to find pairs of them such that the boundary condition $Q_1(\infty) = q_{0,\mathrm{bg}}$ is satisfied. These pairs will result in different values of $\mathcal{E}_{1,\mathrm{bg}}/q_{\mathrm{rel}}$ (which are not known a-priori), and one can then map correponding values of $\mathcal{E}_{1,\mathrm{bg}}/q_{\mathrm{rel}}$, $P_1(r_p)$, $\mathcal{T}_1(r_p)/T_0(r_p)$ and, in turn, f.

2 Determining the asymmetry

2.1 Variation in shielding length due to plasmoid drift

2.2 Resulting heat flux asymmetry

References

[1] P. B. Parks & R. J. Turnbull, The Physics of Fluids **21**, 1735 (1978); doi: 10.1063/1.862088