

Initial set of equations

$$\vec{\nabla}(\rho \vec{v}) = 0 \quad (\text{I}) \quad (\text{mass conservation})$$

$$\underbrace{\rho \vec{v} \cdot \vec{\nabla} \vec{v}}_{\text{Jacobian}} = -\vec{\nabla} p = \rho \underbrace{(\vec{v} \cdot \vec{\nabla}) \vec{v}}_{\text{directional derivative}} \quad (\text{II}) \quad (\text{momentum conservation})$$

Parks assumption

$$\vec{\nabla} \cdot \left[\left(\frac{\partial \vec{v}}{\partial r} + \frac{\gamma p}{\gamma-1} \right) \vec{v} \right] = \vec{\nabla} \cdot \vec{q} \approx \gamma \frac{dq}{dr} \quad (\text{III}) \quad (\text{energy conservation + heat source } Q)$$

(X is fraction converted into heating the gas)
Text

$$\rho = m \frac{p}{T} \quad (0) \quad (\text{ideal gas law, } T = k_B \cdot \text{Temperature})$$

Heat flux dynamics assumptions

Like Parks has done it:

$$\frac{dE}{dr} = \cancel{2} \frac{d}{dm} L(E) \quad (\text{IV}) \quad (\text{energy absorption of electrons})$$

$L(E)$ is energy loss function

$$\frac{dq}{dr} = \frac{q}{\lambda_{\text{eff},p}(E)} = \frac{p}{m} q \Lambda(E) \quad (\text{V}) \quad (\text{effective energy flux})$$

$\Lambda(E) = \hat{\sigma}_T(E) + 2 \frac{L(E)}{E}$ = "effective energy flux crosssection"

alternatively to IV:

$$q(r) = q_\infty \frac{E(r)}{E_\infty} \exp \left[\int_{E(r)}^{E_\infty} \frac{1}{2} \frac{\partial \hat{\sigma}_T(E')}{L(E')} dE' \right] \quad (\text{V}^*)$$

$$m \cdot \frac{N_m^2 \rho}{N_\infty^2} \frac{ds}{dr}$$

Spherically symmetric model (zeroth order)

$$(\text{I}): \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 p_0(r) v_0(r) \right) = 0 \quad \xrightarrow{\text{use (0)}} \quad r^2 \frac{p_0}{T_0} v_0 = \text{const} = \frac{6}{4\pi} \quad (\text{I}_0)$$

$$(\text{II}): p_0(r) v_0(r) \frac{\partial}{\partial r} v_0(r) = - \frac{\partial}{\partial r} p_0(r) \quad \Rightarrow \quad m \frac{p_0}{T_0} v_0 \frac{\partial v_0}{\partial r} + \frac{\partial p_0}{\partial r} = 0 \quad (\text{II}_0)$$

$$(\text{III}): \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{p_0(r) v_0^2(r)}{2} + r^2 \frac{\gamma}{\gamma-1} p_0(r) v_0(r) \right] = \gamma \frac{\partial q_0}{\partial r} \quad (\text{III}_0) \quad \xrightarrow{\text{use (I)_0}} \quad \frac{G}{4\pi r^2} \frac{\partial}{\partial r} \left[\frac{m}{2} v_0^2 + \frac{\gamma}{\gamma-1} T_0 \right] = \gamma \frac{\partial q_0}{\partial r} \quad (\text{III}_0)$$

$$\text{particle ablation rate} \quad [G] = \frac{1}{s}$$

$$(\text{IV}): \frac{\partial E_0}{\partial r} = \cancel{2} \frac{p_0}{T_0} L(E_0) \quad (\text{IV}_0)$$

$$\left(p_0 = m \frac{p_0}{T_0} \right)$$

$$(\text{V}): \frac{\partial q_0}{\partial r} = \frac{p_0}{T_0} q_0 \Lambda(E_0) \quad (\text{V}_0)$$

First order angular asymmetry correctionPerturbations (small)

$$p(\vec{r}) = p_0(r) + p_1(r, \theta)$$

$$p(\vec{r}) = p_0(r) + p_1(r, \theta)$$

$$T(\vec{r}) = T_0(r) + T_1(r, \theta)$$

$$\vec{v}(\vec{r}) = v_0(r) \hat{r} + \vec{v}_1(r, \theta) = v_0 \hat{r} + u_1 \hat{r} + v_1 \hat{\theta}$$

Expansion in general basis $\{X_i\}, \{Y_i\}$

$$p_1(r, \theta) = \sum_l R_l X_l(\theta)$$

$$p_1(r, \theta) = \sum_l T_l(r) X_l(\theta)$$

$$T_1(r, \theta) = \sum_l T_l(r) X_l(\theta)$$

$$\vec{v}_1(r, \theta) = \left(\sum_l U_l(r) X_l(\theta) \right) \hat{r} + \left(\sum_l V_l(r) Y_l(\theta) \right) \hat{\theta}$$

$$\vec{q}(\vec{r}) = q_0(r)\hat{r} + q_1(r,\theta)\hat{\theta}$$

$$E(\vec{r}) = E_0(r) + E_1(r,\theta)$$

$$q_n(r,\theta) = \sum_l Q_l(r) X_l(\theta)$$

$$E_n(r,\theta) = \sum_l E_l(r) X_l(\theta)$$

Derivatives in spherical coordinates:

- gradient: $\vec{\nabla}\phi = \hat{r}\frac{\partial\phi}{\partial r} + \hat{\theta}\frac{\partial\phi}{\partial\theta} + \hat{\varphi}\frac{\partial\phi}{\partial\varphi}$

- divergence: $\vec{\nabla} \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r \sin\theta} \frac{\partial A_\varphi}{\partial\varphi}$

- vector-gradient: $\vec{\nabla}\vec{A} = \hat{r}\hat{r}\frac{\partial A_r}{\partial r} + \hat{r}\hat{\theta}\frac{\partial A_r}{\partial\theta} - \frac{A_\theta}{r} + \hat{r}\hat{\varphi}\frac{\partial A_r}{\partial\varphi} + \hat{\theta}\hat{r}\frac{\partial A_\theta}{\partial r} + \hat{\theta}\hat{\theta}\left(\frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{A_r}{r}\right) + \hat{\theta}\hat{\varphi}\left(\frac{1}{r \sin\theta}\frac{\partial A_\theta}{\partial\varphi} - \cot\theta \frac{A_\varphi}{r}\right) + \hat{\varphi}\hat{r}\frac{\partial A_\varphi}{\partial r} + \hat{\varphi}\hat{\theta}\frac{1}{r}\frac{\partial A_\varphi}{\partial\theta} + \hat{\varphi}\hat{\varphi}\left(\frac{1}{r \sin\theta}\frac{\partial A_\varphi}{\partial\varphi} + \cot\theta \frac{A_\theta}{r} + \frac{A_r}{r}\right)$

- directional derivative: $\vec{A} \cdot (\vec{\nabla}\vec{B}) = (\vec{A} \cdot \vec{\nabla})\vec{B} = \vec{\nabla}_A \vec{B} = (\vec{\nabla}\vec{B}) \cdot \vec{A}$

useful examples:

$$\vec{\nabla}P_n(r,\theta) = \hat{r}\sum_l \frac{\partial R_l}{\partial r} X_l + \hat{\theta}\sum_l R_l \frac{\partial X_l}{\partial\theta}$$

$$\vec{\nabla}V_n = \frac{1}{r^2} \sum_l \frac{\partial}{\partial r}(r^2 U_l) X_l + \frac{1}{r \sin\theta} \sum_l V_l \frac{\partial}{\partial\theta}(\sin\theta Y_l)$$

Derivation of 1st 3 eqs. through linearization

$$(I) \Leftrightarrow \vec{\nabla} \cdot (\rho \vec{v}) \underset{\text{0th order}}{\approx} \vec{\nabla} \cdot \left(\underbrace{\rho_0 \vec{v}_0 + \rho_1 \vec{v}_1 + \rho_2 \vec{v}_2}_{\text{0th order}} \right) = 0$$

$$\Leftrightarrow \vec{\nabla} \cdot (\rho_1 \vec{v}_0 + \rho_0 \vec{v}_1) = \vec{\nabla} \rho_0 \cdot \vec{v}_1 + \rho_0 \vec{\nabla} \vec{v}_1 + \vec{\nabla} \rho_1 \cdot \vec{v}_0 + \rho_1 \vec{\nabla} \vec{v}_0 = 0 \quad (I_{1,a})$$

$$\Leftrightarrow \frac{\partial \rho_0}{\partial r} \sum_l U_l X_l + \rho_0 \left[\frac{1}{r^2} \sum_l \frac{\partial}{\partial r}(r^2 U_l) X_l + \frac{1}{r \sin\theta} \sum_l V_l \frac{\partial}{\partial\theta}(\sin\theta Y_l) \right] + V_0 \sum_l \frac{\partial R_l}{\partial r} X_l + \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 v_0) \sum_l R_l X_l = 0$$

require an orthogonal basis $\langle X_l, X_k \rangle = \int X_l(\theta) X_k(\theta) d\theta = 0$ if $l \neq k$

but require also $\alpha_l X_l = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}(\sin\theta Y_l) \quad (1)$, where $\alpha_l = \text{const}$ can be chosen later

then we can project out each mode and get for mode l the differential equation (and divide by X_l)

$$\frac{\partial \rho_0}{\partial r} U_l + \rho_0 \left[\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 U_l) + \frac{\alpha_l}{r} V_l \right] + V_0 \frac{\partial R_l}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 v_0) R_l = 0 \quad (I_{1,b})$$

(linearize (II)): $\rho \vec{v} \cdot \vec{\nabla} \vec{v} = -\vec{\nabla} \rho$ and use (II₀)

$$\Leftrightarrow \rho_0 \vec{v}_0 \cdot \vec{\nabla} \vec{v}_1 + \rho_0 \vec{v}_1 \cdot \vec{\nabla} \vec{v}_0 + \rho_1 \vec{v}_0 \cdot \vec{\nabla} \vec{v}_0 = -\vec{\nabla} \rho_1 \quad (II_{1,a})$$

Directional derivatives: $\hat{r} \cdot \vec{\nabla} \vec{v}_0 = \hat{r} \frac{\partial v_0}{\partial r}, \hat{\theta} \cdot \vec{\nabla} \vec{v}_0 = \hat{\theta} \frac{v_0}{r}$

$$\hat{r} \cdot \vec{\nabla} \vec{v}_1 = \hat{r} \frac{\partial v_1}{\partial r} + \hat{\theta} \frac{\partial v_1}{\partial\theta}$$

$$\Leftrightarrow \rho_0 v_0 \left(\hat{r} \frac{\partial v_1}{\partial r} + \hat{\theta} \frac{\partial v_1}{\partial\theta} \right) + \rho_0 (v_1 \hat{r} \frac{\partial v_0}{\partial r} + v_1 \hat{\theta} \frac{v_0}{r}) + \rho_1 v_0 \hat{r} \frac{\partial v_0}{\partial r} = -\hat{r} \frac{\partial \rho_1}{\partial r} - \hat{\theta} \frac{1}{r} \frac{\partial \rho_1}{\partial\theta}$$

separate \hat{r} and $\hat{\theta}$ and insert expansion

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$$\textcircled{R} \quad P_0 V_0 \sum_l \frac{\partial U_l}{\partial r} X_l + P_0 \frac{\partial V_0}{\partial r} \sum_l U_l X_l + V_0 \frac{\partial V_0}{\partial r} \sum_l R_l X_l = - \sum_l \frac{\partial \Pi_l}{\partial r} X_l$$

$\rightarrow l$ -modes are separable and X_l can be canceled

$$\Rightarrow P_0 V_0 \frac{\partial U_l}{\partial r} + P_0 \frac{\partial V_0}{\partial r} U_l + V_0 \frac{\partial V_0}{\partial r} R_l = - \frac{\partial \Pi_l}{\partial r} \quad (\text{II}_1, b1)$$

$$\textcircled{\Theta} \quad P_0 V_0 \sum_l \frac{\partial V_l}{\partial r} Y_l + P_0 \frac{V_0}{r} \sum_l V_l Y_l = - \frac{1}{r} \sum_l \Pi_l \frac{\partial X_l}{\partial \theta}$$

To be able to separate modes require $\langle Y_l, Y_k \rangle = 0$ if $l \neq k$ and

$$\Rightarrow P_0 V_0 \frac{\partial V_l}{\partial r} + P_0 \frac{V_0}{r} V_l = \frac{\beta_l}{r} \Pi_l \quad (\text{II}_1, b2)$$

$$Y_l = - \frac{1}{\beta_l} \frac{\partial X_l}{\partial \theta} \quad (2) \quad \text{where } \beta_l = \text{const. can be chosen later}$$

$$\text{(III)} \Rightarrow \vec{\nabla} \cdot \left[\left(\frac{\gamma P}{2} + \frac{\gamma P}{\gamma-1} \right) \vec{V} \right] = \gamma \frac{d q}{dr}$$

$$(\text{linearize} \Rightarrow) \vec{\nabla} \cdot \left[\left(\frac{1}{2} P_0 V_0^2 + \frac{\gamma}{\gamma-1} P_0 \right) \vec{V}_1 + \left(\frac{1}{2} P_1 V_0^2 + P_0 \vec{V}_0 \cdot \vec{V}_1 + \frac{\gamma}{\gamma-1} P_1 \right) \vec{V}_0 \right] = \gamma \frac{d q_1}{dr} \quad (\text{III}_1, a)$$

$$\approx \vec{\nabla} \cdot \left[\left(\frac{1}{2} P_0 V_0^2 + \frac{\gamma}{\gamma-1} P_0 \right) \vec{V}_1 + \left(\frac{1}{2} P_0 V_0^2 + \frac{\gamma}{\gamma-1} P_0 \right) \vec{\nabla} \vec{V}_1 + \vec{\nabla} \cdot \left(\frac{1}{2} P_1 V_0^2 + P_0 \vec{V}_0 \cdot \vec{V}_1 + \frac{\gamma}{\gamma-1} P_1 \right) \vec{V}_0 + \left(\frac{1}{2} P_1 V_0^2 + P_0 \vec{V}_0 \cdot \vec{V}_1 + \frac{\gamma}{\gamma-1} P_1 \right) \vec{\nabla} \cdot \vec{V}_0 = \gamma \frac{d q_1}{dr}$$

insert expansion

$$\Rightarrow \frac{\partial}{\partial r} \left(\frac{1}{2} P_0 V_0^2 + \frac{\gamma}{\gamma-1} P_0 \right) \sum_l U_l X_l + \left(\frac{1}{2} P_0 V_0^2 + \frac{\gamma}{\gamma-1} P_0 \right) \left[\frac{1}{r^2} \sum_l \frac{\partial}{\partial r} (r^2 U_l) X_l + \frac{1}{r} \sin \theta \sum_l V_l \frac{\partial}{\partial \theta} (\sin \theta Y_l) \right] \\ + \frac{\partial}{\partial r} \left(\frac{1}{2} P_1 V_0^2 + P_0 \vec{V}_0 \cdot \vec{V}_1 + \frac{\gamma}{\gamma-1} P_1 \right) V_0 + \left(\frac{1}{2} P_1 V_0^2 + P_0 \vec{V}_0 \cdot \vec{V}_1 + \frac{\gamma}{\gamma-1} P_1 \right) \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 V_0) = \gamma \frac{d q_1}{dr}$$

$$\Rightarrow \frac{\partial}{\partial r} \left(\frac{1}{2} P_0 V_0^2 + \frac{\gamma}{\gamma-1} P_0 \right) \sum_l U_l X_l + \left(\frac{1}{2} P_0 V_0^2 + \frac{\gamma}{\gamma-1} P_0 \right) \left[\frac{1}{r^2} \sum_l \frac{\partial}{\partial r} (r^2 U_l) X_l + \frac{1}{r} \sin \theta \sum_l V_l \frac{\partial}{\partial \theta} (\sin \theta Y_l) \right] \\ + V_0 \sum_l \frac{\partial}{\partial r} \left(\frac{1}{2} R_l V_0^2 + P_0 V_0 U_l + \frac{\gamma}{\gamma-1} \Pi_l \right) X_l + \sum_l \left(\frac{1}{2} R_l V_0^2 + P_0 V_0 U_l + \frac{\gamma}{\gamma-1} \Pi_l \right) X_l \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 V_0) = \gamma \sum_l \frac{\partial Q_l}{\partial r} X_l$$

again the modes can be separated if (1) is required and X_l can be canceled (don't forget α_l factor)

$$\Rightarrow U_l \frac{\partial}{\partial r} \left(\frac{1}{2} P_0 V_0^2 + \frac{\gamma}{\gamma-1} P_0 \right) + \left(\frac{1}{2} P_0 V_0^2 + \frac{\gamma}{\gamma-1} P_0 \right) \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 U_l) + \frac{K_l}{r} V_l \right] \\ + \frac{\partial}{\partial r} \left(\frac{1}{2} R_l V_0^2 + P_0 V_0 U_l + \frac{\gamma}{\gamma-1} \Pi_l \right) V_0 + \left(\frac{1}{2} R_l V_0^2 + P_0 V_0 U_l + \frac{\gamma}{\gamma-1} \Pi_l \right) \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 V_0) = \gamma \frac{\partial Q_l}{\partial r} \quad (\text{eq. (D) in Per's notes})$$

$$\Rightarrow \left[U_l \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 U_l) + \frac{K_l}{r} V_l \right] \left(\frac{1}{2} P_0 V_0^2 + \frac{\gamma}{\gamma-1} P_0 \right) + \left[V_0 \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 V_0) \right] \left(\frac{1}{2} R_l V_0^2 + P_0 V_0 U_l + \frac{\gamma}{\gamma-1} \Pi_l \right) = \gamma \frac{\partial Q_l}{\partial r} \quad (\text{III}_1, b)$$

$$\text{(IV)} : \frac{dE}{dr} = \frac{1}{m} \frac{d}{dr} L(E) \xrightarrow{\text{linearize}} \frac{dE_1}{dr} = \frac{1}{m} P_1 L(E_0) + \frac{1}{2} \frac{P_0}{m} \frac{\partial L}{\partial E} \Big|_{E_0} E_1 \xrightarrow{\text{expand}} \sum_l \frac{\partial E_l}{\partial r} X_l = \frac{1}{m} L(E_0) \sum_l R_l X_l + \frac{2}{m} P_0 \frac{\partial L}{\partial E} \Big|_{E_0} \sum_l \varepsilon_l X_l$$

$$\Rightarrow \frac{\partial E_l}{\partial r} = \frac{1}{m} L(E_0) R_l + \frac{1}{m} P_0 \frac{\partial L}{\partial E} \Big|_{E_0} \varepsilon_l \quad (\text{IV}_1, b)$$

$$\text{(V)} : \frac{d\alpha}{dr} = \frac{P}{m} q \Lambda(E) \xrightarrow{\text{linearize}} m \frac{dQ_l}{dr} = P_1 q_0 \Lambda(E_0) + P_0 q_1 \Lambda(E_0) + P_0 q_0 \frac{\partial \Lambda}{\partial E} \Big|_{E_0} \cdot E_1$$

expand and separate

$$\Rightarrow m \frac{dQ_l}{dr} = R_l q_0 \Lambda(E_0) + P_0 Q_l \Lambda(E_0) + P_0 q_0 \frac{\partial \Lambda}{\partial E} \Big|_{E_0} \varepsilon_l \quad (\text{V}_1, b)$$

$$(IV) : \frac{dq}{dr} = \frac{p}{m} q \Lambda(E) \xrightarrow{\text{linearize}} m \frac{d\dot{q}_1}{dr} = p_1 q_0 \Lambda(E_0) + p_0 q_1 \Lambda(E_0) + p_0 q_0 \frac{d\Lambda}{dE} \Big|_{E_0} \cdot E,$$

expand and separate

$$\Rightarrow m \frac{d\dot{q}_1}{dr} = R_1 q_0 \Lambda(E_0) + p_0 q_1 \Lambda(E_0) + p_0 q_0 \frac{d\Lambda}{dE} \Big|_{E_0} \cdot E \quad (IV_1, b)$$

In order to replace p , i.e. R_1 in all equations:

$$(0): p = m \frac{p}{T} \xrightarrow{\text{linearize}} p_1 = m \frac{p_1}{T_0} - m \frac{p_0}{T_0} T_1 \xrightarrow{\text{expand}} \sum_i R_i X_i = \frac{m}{T_0} \sum_i T_i X_i - m \frac{p_0}{T_0} \sum_i T_i X_i$$

$\Leftrightarrow R_1 = m \left(\frac{T_1}{T_0} - \frac{p_0}{T_0} T_1 \right) \quad (0, b)$

and $p_0 = m \frac{p_0}{T_0}$

$\hat{\theta}$ -dependence

we required $\langle X_l, X_k \rangle = \delta_{lk}$, $\langle Y_l, Y_k \rangle = \delta_{lk}$ and more importantly

$$(1): \alpha_l X_l = \frac{1}{\sin \theta} \frac{d}{d\theta} (\sin \theta Y_l) \quad \text{and} \quad (2): Y_l = -\frac{1}{\beta_l} \frac{dX_l}{d\theta}$$

together: $\frac{1}{X_l} \frac{1}{\sin \theta} \frac{d}{d\theta} (\sin \theta \frac{dX_l}{d\theta}) = -\alpha_l \beta_l = -\lambda_l \quad (4)$

for this ODE to have a non-trivial, regular solution

we require $\lambda_l = l(l+1)$ (see spherical harmonics Wikipedia)

Then without loss of generality we define $X_l(\theta) = P_l^0(\cos \theta)$

and $P_l^0(\cos \theta)$ solves (4) if they are the associated Legendre polynomials (with $m=0$)

and they form automatically a orthogonal basis.

Arbitrarily choose $\alpha_l = -l(l+1)$, $\beta_l = -1$