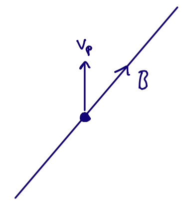
Rocket force on pellets

Consider a pellet travelling at velocity v_p through a plasma. The ablation rate is related to the penetration depth L_p by



$$\frac{\dot{m}}{m} \sim \frac{v_p}{L_p}$$

If, for some reason, the ablation is asymmetric, there will be a rocket force on the pellet

$$F \sim \varepsilon \dot{m} v_a$$

where v_a is the velocity of the ablated material, and ε is the degree of asymmetry. The resulting acceleration is

$$a \sim \frac{F}{m} \sim \varepsilon \frac{\dot{m}}{m} v_a \sim \frac{\varepsilon v_a v_p}{L_p}$$

and changes the pellet velocity by the total amount

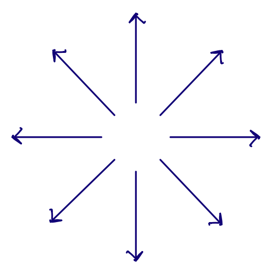
$$\Delta v_p \sim \frac{a L_p}{v_p} \sim \varepsilon v_a$$

which is significant if $\Delta v_p \sim v_p$, i.e. if

$$\varepsilon \sim \frac{v_p}{v_a} \sim 3 \cdot 10^{-3}$$

if $v_p \sim 10^3$ m/s and $v_p \sim \sqrt{T/m_i} \sim \sqrt{1{,}6 \cdot 10^{-16} / 1{,}6 \cdot 10^{-27}} \sim 3 \cdot 10^5$ m/s.

Model calculation: The pellet itself is very small and is surrounded by a much larger ablation cloud consisting of spherically outflowing gas. This cloud will not be isotropically heated by the surrounding plasma. We begin our enquiry by exploring the effect of this anisotropy. We thus suppose that there is a particle source at $r = 0$ producing a "solar wind" that is heated anisotropically.



For simplicity, we consider the case of a weak heating anisotropy and linearise the equations in the corresponding small parameter.

How does the heating change the flow pattern? More importantly, is there a non-zero "reaction force" on the source?

As we shall see, the pressure is not spherically symmetric due to the heating and produces a net force on the pellet. In addition, the temperature varies over the surface of the pellet and will presumably cause asymmetric ablation, which in turn gives rise to a rocket force.

Anisotropic pellet cloud expansion: To be more quantitative, we endeavour to calculate the effect of anisotropic heating on the otherwise spherically symmetric expansion of an ideal gas from a point source. We thus consider the steady-state equations for an ideal gas

$$\nabla \cdot (\rho \vec{v}) = 0$$

$$\rho \, \vec{v} \cdot \nabla \vec{v} = - \nabla p$$

$$\nabla \cdot \left[\left(\frac{\rho v^2}{2} + \frac{\gamma p}{\gamma - 1} \right) \vec{v} \right] = Q = \text{heating source}$$

and write $\rho = \rho_0(r) + \rho_1(r,\theta)$, $\vec{v} = v_0(r)\,\hat{r} + \vec{v}_1(r,\theta)$, $p = p_0(r) + p_1(r,\theta)$, $Q = Q_0(r) + Q_1(r,\theta)$, with $\rho_1/\rho_0 \ll 1$ etc. The zeroth-order system describes spherically symmetric outflow

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \rho_0 v_0 \right) = 0 \quad \Rightarrow \quad r^2 \rho_0 v_0 = \text{const.} = \frac{\Gamma}{4\pi} \quad \Rightarrow \quad v_0 = \frac{\Gamma}{4\pi \rho_0 r^2} \qquad (0)$$

$$\rho_0 v_0 \frac{dv_0}{dr} = - \frac{dp_0}{dr}$$

$$\frac{1}{r^2}\frac{d}{dr} r^2 \left(\frac{\rho_0 v_0^2}{2} + \frac{\gamma p_0}{\gamma - 1} \right) v_0 = Q_0$$

in spherical coordinates. Linearisation gives

$$\nabla \cdot \left(\rho_0 \vec{v}_1 + \rho_1 \vec{v}_0 \right) = 0 \qquad (1)$$

$$\rho_0 \left(\vec{v}_1 \cdot \nabla \vec{v}_0 + \vec{v}_0 \cdot \nabla \vec{v}_1 \right) + \rho_1 \, \vec{v}_0 \cdot \nabla \vec{v}_0 = - \nabla p_1 \qquad (2)$$

$$\nabla \cdot \left[\left(\frac{\rho_0 v_0^2}{2} + \frac{\gamma p_0}{\gamma - 1} \right) \vec{v}_1 + \left(\frac{\rho_1 v_0^2}{2} + \rho_0 \vec{v}_0 \cdot \vec{v}_1 + \frac{\gamma p_1}{\gamma - 1} \right) \vec{v}_0 \right] = Q_1 \qquad (3)$$

For simplicity we take $Q_1(r,\theta) = q(r) \cos\theta$ to be up-down antisymmetric and write

$$\vec{v}_1(r,\theta) = u_r(r,\theta)\,\hat{r} + u_\theta(r,\theta)\,\hat{\theta} = U(r) \cos\theta \; \hat{r} + V(r) \sin\theta \; \hat{\theta}$$

so that

$$\nabla\cdot\vec{v}_1 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(u_\theta\sin\theta\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_\theta}{r\tan\theta}$$

$$= \left[\frac{1}{r^2}\frac{d}{dr}\left(r^2 U\right) + \frac{2V}{r}\right]\cos\theta$$

Furthermore, we write

$$\rho_1(r,\theta) = R(r)\cos\theta \qquad p_1(r,\theta) = P(r)\cos\theta$$

so that the 1st-order continuity eq. (1) becomes

$$\nabla\cdot\left(\rho_0\vec{v}_1 + \rho_1\vec{v}_0\right) = \nabla\rho_0\cdot\vec{v}_1 + \rho_0\nabla\cdot\vec{v}_1 + \rho_1\nabla\cdot\vec{v}_0 + \nabla\rho_1\cdot\vec{v}_0$$

$$= \left[\rho_0' U + \frac{\rho_0}{r^2}\frac{d}{dr}\left(r^2 U\right) + \frac{2\rho_0 V}{r} + \frac{R}{r^2}\frac{d}{dr}\left(r^2 v_0\right) + v_0\frac{dR}{dr}\right]\cos\theta = 0$$

$$\Rightarrow \rho_0' U + \frac{\rho_0}{r^2}\frac{d}{dr}\left(r^2 U\right) + \frac{2\rho_0 V}{r} + \frac{R}{r^2}\frac{d}{dr}\left(r^2 v_0\right) + v_0\frac{dR}{dr} = 0 \qquad (A)$$

The 1st-order eq. of motion (2) becomes

$$\rho_0\left(\vec{v}_1\cdot\nabla\vec{v}_0 + \vec{v}_0\cdot\nabla\vec{v}_1\right) + \rho_1\vec{v}_0\cdot\nabla\vec{v}_0 = \hat{r}\left(\rho_0 U v_0' + R v_0 v_0' + \rho_0 v_0 U'\right)\cos\theta$$

$$+ \hat{\theta}\rho_0 v_0 V'\sin\theta = -\nabla p' = -\hat{r}P'\cos\theta + \frac{\hat{\theta}}{r}P\sin\theta$$

and implies

$$\rho_0 U v_0' + R v_0 v_0' + \rho_0 v_0 U' + P' = 0 \qquad (B)$$

$$\rho_0 v_0 V' - \frac{P}{r} = 0 \qquad (C)$$

Finally, we consider the energy equation (3),

$$\nabla \cdot \left[\left(\frac{\rho_0 v_0^2}{2} + \frac{\gamma p_0}{\gamma - 1} \right) \vec{v}_1 + \left(\frac{\rho_1 v_0^2}{2} + \rho_0 \vec{v}_0 \cdot \vec{v}_1 + \frac{\gamma p_1}{\gamma - 1} \right) \vec{v}_0 \right]$$

$$= \left(\frac{\rho_0 v_0^2}{2} + \frac{\gamma p_0}{\gamma - 1} \right) \nabla \cdot \vec{v}_1 + \vec{v}_1 \cdot \nabla \left(\frac{\rho_0 v_0^2}{2} + \frac{\gamma p_0}{\gamma - 1} \right)$$

$$+ \left(\frac{\rho_1 v_0^2}{2} + \rho_0 \vec{v}_0 \cdot \vec{v}_1 + \frac{\gamma p_1}{\gamma - 1} \right) \nabla \cdot \vec{v}_0 + \vec{v}_0 \cdot \nabla \left(\frac{\rho_1 v_0^2}{2} + \rho_0 \vec{v}_0 \cdot \vec{v}_1 + \frac{\gamma p_1}{\gamma - 1} \right)$$

$$= \left(\frac{\rho_0 v_0^2}{2} + \frac{\gamma p_0}{\gamma - 1} \right) \left[\frac{1}{r^2} \frac{d}{dr} \left(r^2 U \right) + \frac{2V}{r} \right] \cos\theta$$

$$+ U \frac{d}{dr} \left(\frac{\rho_0 v_0^2}{2} + \frac{\gamma p_0}{\gamma - 1} \right) \cos\theta$$

$$+ \left(\frac{R v_0^2}{2} + \rho_0 v_0 U + \frac{\gamma P}{\gamma - 1} \right) \frac{1}{r^2} \frac{d}{dr} \left(r^2 v_0 \right) \cos\theta$$

$$+ v_0 \frac{d}{dr} \left(\frac{R v_0^2}{2} + \rho_0 v_0 U + \frac{\gamma P}{\gamma - 1} \right) \cos\theta = q \cos\theta$$

so that

$$\left(\frac{\rho_0 v_0^2}{2} + \frac{\gamma p_0}{\gamma - 1} \right) \left[\frac{1}{r^2} \frac{d}{dr} \left(r^2 U \right) + \frac{2V}{r} \right] + U \frac{d}{dr} \left(\frac{\rho_0 v_0^2}{2} + \frac{\gamma p_0}{\gamma - 1} \right)$$

$$+ \left(\frac{R v_0^2}{2} + \rho_0 v_0 U + \frac{\gamma P}{\gamma - 1} \right) \frac{1}{r^2} \frac{d}{dr} \left(r^2 v_0 \right) + v_0 \frac{d}{dr} \left(\frac{R v_0^2}{2} + \rho_0 v_0 U + \frac{\gamma P}{\gamma - 1} \right) = 0 \qquad \text{(D)}$$

We thus have 4 equations, (A) - (D), for 4 unknowns (R, U, V, P).

Once these equations have been solved (numerically ?), the rocket force can be calculated from momentum conservation by integrating the momentum flux $\rho \vec{v}\vec{v} + p \overleftrightarrow{I}$ over a sphere S surrounding the origin :

$$\vec{F} = - \int_S \left(\rho \vec{v}\vec{v} + p \overleftrightarrow{I} \right) \cdot \hat{r} \, dS$$

Since this force must point in the z-direction and $\hat{z} \cdot \overleftrightarrow{I} \cdot \hat{r} = \cos\theta$

$$F_z = - \int_0^\pi \left(\rho v_z v_r + p \cos\theta \right) 2\pi r^2 \sin\theta \, d\theta$$

$$= - \int_0^\pi \left(\rho_0 v_{0z} v_{1r} + \rho_0 v_{1z} v_{0r} + p_1 \cos\theta \right) 2\pi r^2 \sin\theta \, d\theta$$

$$= \left\{ v_{1z} = \hat{z} \cdot \left(U \cos\theta \, \hat{r} + V \sin\theta \, \hat{\theta} \right) = U \cos^2\theta - V \sin^2\theta \; , \; v_{0z} v_{1r} = v_0 U \cos^2\theta \right\}$$

$$= - \int_0^\pi \left[\rho_0 v_0 \left(2U \cos^2\theta - V \sin^2\theta \right) + P \cos^2\theta \right] 2\pi r^2 \sin\theta \, d\theta$$

$$= \begin{pmatrix} \int_0^\pi \cos^2\theta \sin\theta \, d\theta = (x = \cos\theta) = \int_{-1}^1 x^2 dx = \frac{2}{3} \\ \int_0^\pi \sin^2\theta \sin\theta \, d\theta = \int_{-1}^1 (1 - x^2) \, dx = \frac{4}{3} \end{pmatrix}$$

$$= \frac{4\pi r^2}{3} \left[2 \rho_0 v_0 (U - V) + P \right] = \frac{4\pi r^2}{3} P + \frac{2\Gamma}{3} (U - V)$$

where $\Gamma = \rho_0 v_0 r^2$ is independent of r according to (1).

The result should be independent of r and can, for instance, be evaluated for $r \to r_p$, the pellet radius, in which case the first term represents the pressure force from the cloud on the pellet since $P \neq 0$.

In addition, since the temperature $\propto p/\rho$ varies over the surface of the pellet, the ablation rate will also vary, causing $U \neq 0$, which adds to the reaction force.