Rocket effect derivation, full set, new

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Initial set of equations

$$\vec{\nabla}\cdot(\rho\vec{v}) = 0 \qquad (\mathrm{I}) \qquad \text{(mass conservation)}$$

$$\rho\,\underbrace{\vec{v}\cdot\vec{\nabla}}_{\text{Jacobian}}\vec{v} = -\vec{\nabla}p = \rho\underbrace{(\vec{v}\cdot\vec{\nabla})}_{\text{directional derivative}}\vec{v} \qquad (\mathrm{II}) \qquad \text{(momentum conservation)}$$

$$\vec{\nabla}\cdot\left[\left(\frac{\rho v^2}{2} + \frac{\gamma p}{\gamma-1}\right)\vec{v}\right] = \vec{\nabla}\cdot\vec{q} \overset{\text{Parks assumptions}}{\approx} \gamma\frac{dq}{dr} \qquad (\mathrm{III}) \qquad \text{(energy conservation + heat source } Q)$$

(α is fraction converted into heating the gas)

$$\rho = m\frac{p}{T} \qquad (0) \qquad (\text{ideal gas law}, T = k_B \cdot \text{Temperature})$$

use to replace ρ because we have boundary conditions for T

Heat flux dynamics assumptions

Like Parks has done it:

$$\frac{dE}{dr} = \frac{1}{2}\frac{\rho}{m}L(E) \qquad (\mathrm{IV}) \qquad \text{(energy absorption of electrons)}$$

$L(E)$ is energy loss function

$$\frac{dq}{dr} = \frac{q}{\lambda_{mfp}(E)} = \frac{\rho}{m}q\,\Lambda(E) \qquad (\mathrm{V}) \qquad \text{(effective energy flux)}$$

$\Lambda(E) = \hat{\sigma}_T(E) + 2\frac{L(E)}{E}$ = "effective energy flux crossection"

$\hat{\sigma}_T(E)$ = effective backscattering crossection

alternatively to V:

$$q(r) = q_\infty\frac{E(r)}{E_{bg}}\exp\left[\int_{E(r)}^{E_{bg}}\frac{1}{2}\frac{\hat{\sigma}_T(E')}{L(E')}\,dE'\right] \qquad (\mathrm{V}^*)$$

$\cancel{m^x}\cdot\frac{\cancel{N}\cancel{m^2}}{\cancel{N}\cancel{x}}\frac{\cancel{m}}{s}$

Spherically symmetric model (zeroth order)

$$(\mathrm{I}):\ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\rho_0(r)v_0(r)\right) = 0 \quad \overset{\text{use }(0)}{\Longleftrightarrow} \quad r^2\frac{p_0}{T_0}v_0 = \text{const} = \frac{G}{4\pi} \qquad (\mathrm{I}_0)$$

particle ablation rate $[G] = \frac{1}{s}$

$$(\mathrm{II}):\ \rho_0(r)v_0(r)\frac{\partial}{\partial r}v_0(r) = -\frac{\partial}{\partial r}p_0(r) \quad \Longleftrightarrow \quad m\frac{p_0}{T_0}v_0\frac{\partial v_0}{\partial r} + \frac{\partial p_0}{\partial r} = 0 \qquad (\mathrm{II}_0)$$

$$(\mathrm{III}):\ \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\rho_0(r)v_0^3(r)}{2} + r^2\frac{\gamma}{\gamma-1}p_0(r)v_0(r)\right] = \gamma\frac{\partial q_0}{\partial r} \quad \overset{\text{use }(0),(\mathrm{I}_0)}{\Longleftrightarrow} \quad \frac{G}{4\pi r^2}\frac{\partial}{\partial r}\left[\frac{m}{2}v_0^2 + \frac{\gamma}{\gamma-1}T_0\right] = \gamma\frac{\partial q_0}{\partial r} \qquad (\mathrm{III}_0)$$

$$(\mathrm{VI}):\ \frac{\partial E_0}{\partial r} = \frac{1}{2}\frac{p_0}{T_0}L(E_0) \qquad (\mathrm{IV}_0)$$

$$\left(\rho_0 = m\frac{p_0}{T_0}\right)$$

$$(\mathrm{V}):\ \frac{\partial q_0}{\partial r} = \frac{p_0}{T_0}q_0\,\Lambda(E_0) \qquad (\mathrm{V}_0)$$

First order angular asymmetry correction

Perturbations (small)

$$\rho(\vec{r}) = \rho_0(r) + \rho(r,\theta)$$

$$p(\vec{r}) = p_0(r) + p_1(r,\theta)$$

$$T(\vec{r}) = T_0(r) + T_1(r,\theta)$$

$$\vec{v}(\vec{r}) = v_0(r)\hat{r} + \vec{v}_1(r,\theta) = v_0\hat{r} + u_1\hat{r} + v_1\hat{\theta}$$

Expansion in general basis $\{X_l\}, \{Y_l\}$

$$\rho_1(r,\theta) = \sum_l R_l X_l(\theta)$$

$$p_1(r,\theta) = \sum_l \Pi_l(r)X_l(\theta)$$

$$T_1(r,\theta) = \sum_l \tau_l(r)X_l(\theta)$$

$$\vec{v}_1(r,\theta) = \left(\sum_l U_l(r)X_l(\theta)\right)\hat{r} + \left(\sum_l V_l(r)Y_l(\theta)\right)\hat{\theta}$$

$\vec q(\vec r) = q_0(r)\hat r + q_1(r,\theta)\hat r \qquad q_1(r,\theta) = \sum_l Q_l(r) X_l(\theta)$

$E(\vec r) = E_0(r) + E_1(r,\theta) \qquad E_1(r,\theta) = \sum_l \varepsilon_l(r) X_l(\theta)$

Derivatives in spherical coordinates:

- gradient: $\vec\nabla\phi = \hat r \frac{\partial \phi}{\partial r} + \hat\theta \frac{\partial \phi}{\partial \theta} + \hat\varphi \frac{\partial \phi}{\partial \varphi}$
- divergence: $\vec\nabla\cdot\vec A = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
- vector-gradient:

$$\vec\nabla\vec A = \hat r\otimes\hat r\, \frac{\partial A_r}{\partial r} + \hat r\otimes\hat\theta\left(\frac{1}{r}\frac{\partial A_r}{\partial\theta} - \frac{A_\theta}{r}\right) + \hat r\otimes\hat\varphi\left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{A_\varphi}{r}\right)$$
$$+ \hat\theta\otimes\hat r\, \frac{\partial A_\theta}{r} + \hat\theta\otimes\hat\theta\left(\frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{A_r}{r}\right) + \hat\theta\otimes\hat\varphi\left(\frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\varphi} - \cot\theta\,\frac{A_\varphi}{r}\right)$$
$$+ \hat\varphi\otimes\hat r\, \frac{\partial A_\varphi}{\partial r} + \hat\varphi\otimes\hat\theta\, \frac{1}{r}\frac{\partial A_\varphi}{\partial\theta} + \hat\varphi\otimes\hat\varphi\left(\frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi} + \cot\theta\,\frac{A_\theta}{r} + \frac{A_r}{r}\right)$$

- directional derivative: $\vec A\cdot(\vec\nabla\vec B) = (\vec A\cdot\vec\nabla)\vec B = \vec\nabla_{\vec A}\vec B = (\vec\nabla\vec B)\cdot A$

useful examples:

$$\vec\nabla\rho_1(r,\theta) = \hat r\sum_l \frac{\partial R_l}{\partial r} X_l + \hat\theta\sum_l R_l \frac{\partial X_l}{\partial\theta}$$

$$\vec\nabla\cdot\vec v_1 = \frac{1}{r^2}\sum_l \frac{\partial}{\partial r}(r^2 U_l) X_l + \frac{1}{r\sin\theta}\sum_l V_l \frac{\partial}{\partial\theta}(\sin\theta\, Y_l)$$

<u>Derivation of 1st 3 eqs. through linearization</u>

$$(I) \Leftrightarrow \vec\nabla\cdot(\rho\vec v) \approx \vec\nabla\cdot(\underbrace{\rho_0\vec v_0}_{\substack{=0\\ \text{0th order}}} + \rho_1\vec v_0 + \rho_0\vec v_1) = 0$$

$$\Leftrightarrow \vec\nabla\cdot(\rho_1\vec v_0 + \rho_0\vec v_1) = \vec\nabla\rho_0\cdot\vec v_1 + \rho_0\vec\nabla\cdot\vec v_1 + \vec\nabla\rho_1\cdot\vec v_0 + \rho_1\vec\nabla\cdot\vec v_0 = 0 \qquad (I_1, a)$$

$$\Leftrightarrow \frac{\partial\rho_0}{\partial r}\sum_l U_l X_l + \rho_0\left[\frac{1}{r^2}\sum_l\frac{\partial}{\partial r}(r^2U_l)X_l + \frac{1}{r\sin\theta}\sum_l V_l\frac{\partial}{\partial\theta}(\sin\theta\, Y_l)\right] + v_0\sum_l\frac{\partial R_l}{\partial r}X_l + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2v_0)\sum_l R_lX_l = 0$$

require an orthogonal basis $\langle X_l, X_k\rangle = \int X_l(\theta)X_k(\theta)\,d\theta = 0$ if $l\neq k$

but require also $\boxed{\alpha_l X_l = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, Y_l) \quad (1)}$, where $\alpha_l = \text{const}$ can be chosen later

then we can project out each mode and get for mode l the differential equation (and divide by X_l)

$$\boxed{\frac{\partial\rho_0}{\partial r}U_l + \rho_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2U_l) + \frac{\alpha_l}{r}V_l\right] + v_0\frac{\partial R_l}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2v_0)R_l = 0 \qquad (I_1, b)}$$

linearize (II): $\rho\vec v\cdot\vec\nabla\vec v = -\vec\nabla p$ and use (II_0)

$$\Leftrightarrow \rho_0\,\vec v_0\cdot\vec\nabla\vec v_1 + \rho_0\,\vec v_1\cdot\vec\nabla\vec v_0 + \rho_1\vec v_0\cdot\vec\nabla\vec v_0 = -\vec\nabla p_1 \qquad (II_1, a)$$

Directional derivatives: $\hat r\cdot\vec\nabla\vec v_0 = \hat r\frac{\partial v_0}{\partial r}$, $\hat\theta\cdot\vec\nabla\vec v_0 = \hat\theta\frac{v_0}{r}$

$\hat r\cdot\vec\nabla\vec v_1 = \hat r\frac{\partial u_1}{\partial r} + \hat\theta\frac{\partial v_1}{\partial r}$

$$\Leftrightarrow \rho_0\, v_0\left(\hat r\frac{\partial u_1}{\partial r} + \hat\theta\frac{\partial v_1}{\partial r}\right) + \rho_0\left(u_1\hat r\frac{\partial v_0}{\partial r} + v_1\hat\theta\frac{v_0}{r}\right) + \rho_1 v_0\hat r\frac{\partial v_0}{\partial r} = -\hat r\frac{\partial p_1}{\partial r} - \hat\theta\frac{1}{r}\frac{\partial p_1}{\partial\theta}$$

separate $\hat r$ and $\hat\theta$ and insert expansion separate $\hat{r}$ and $\hat{\theta}$ and insert expansion

$(\hat{r})$ $\rho_0 v_0 \sum_l \frac{\partial U_l}{\partial r} X_l + \rho_0 \frac{\partial v_0}{\partial r} \sum_l U_l X_l + v_0 \frac{\partial v_0}{\partial r} \sum_l R_l X_l = -\sum_l \frac{\partial \Pi_l}{\partial r} X_l$

→ l-modes are separable and X_l can be canceled

$$\Leftrightarrow \boxed{\rho_0 v_0 \frac{\partial U_l}{\partial r} + \rho_0 \frac{\partial v_0}{\partial r} U_l + v_0 \frac{\partial v_0}{\partial r} R_l = -\frac{\partial \Pi_l}{\partial r} \qquad (\mathrm{II}_1, b1)}$$

$(\hat{\theta})$ $\rho_0 v_0 \sum_l \frac{\partial V_l}{\partial r} Y_l + \rho_0 \frac{v_0}{r} \sum_l V_l Y_l = -\frac{1}{r} \sum_l \Pi_l \frac{\partial X_l}{\partial \theta}$

To be able to separate modes require $\langle Y_l, Y_k \rangle = 0$ if $l \neq k$ and $\boxed{Y_l = -\frac{1}{\beta_l} \frac{\partial X_l}{\partial \theta} \quad (2)}$ where $\beta_l = \text{const}$ can be chosen later

$$\Leftrightarrow \boxed{\rho_0 v_0 \frac{\partial V_l}{\partial r} + \rho_0 \frac{v_0}{r} V_l = \frac{\beta_l}{r} \Pi_l \qquad (\mathrm{II}_1, b2)}$$

$(\mathrm{III}) \Leftrightarrow \vec{\nabla} \cdot \left[\left(\frac{\rho v^2}{2} + \frac{\gamma p}{\gamma - 1} \right) \vec{v} \right] = \gamma \frac{dq}{dr}$

linearize $\Rightarrow$ $\vec{\nabla} \left[\left(\frac{1}{2} \rho_0 v_0^2 + \frac{\gamma}{\gamma-1} p_0 \right) \vec{v}_1 + \left(\frac{1}{2} \rho_1 v_0^2 + \rho_0 \vec{v}_0 \cdot \vec{v}_1 + \frac{\gamma}{\gamma-1} p_1 \right) \vec{v}_0 \right] = \gamma \frac{\partial q_1}{\partial r}$ $\qquad (\mathrm{III}_1, a)$

product rule

$$\Leftrightarrow \vec{\nabla}\left(\tfrac{1}{2}\rho_0 v_0^2 + \tfrac{\gamma}{\gamma-1} p_0\right) \vec{v}_1 + \left(\tfrac{1}{2}\rho_0 v_0^2 + \tfrac{\gamma}{\gamma-1} p_0\right) \vec{\nabla} \vec{v}_1 + \vec{\nabla}\left(\tfrac{1}{2}\rho_1 v_0^2 + \rho_0 \vec{v}_0 \cdot \vec{v}_1 + \tfrac{\gamma}{\gamma-1} p_1\right) \vec{v}_0 + \left(\tfrac{1}{2}\rho_1 v_0^2 + \rho_0 \vec{v}_0 \cdot \vec{v}_1 + \tfrac{\gamma}{\gamma-1} p_1\right) \vec{\nabla} \cdot \vec{v}_0 = \gamma \frac{\partial q_1}{\partial r}$$

insert expansion

$$\Leftrightarrow \frac{\partial}{\partial r}\left(\tfrac{1}{2}\rho_0 v_0^2 + \tfrac{\gamma}{\gamma-1} p_0\right) \sum_l U_l X_l + \left(\tfrac{1}{2}\rho_0 v_0^2 + \tfrac{\gamma}{\gamma-1} p_0\right) \left[\frac{1}{r^2} \sum_l \frac{\partial}{\partial r}(r^2 U_l) X_l + \frac{1}{r \sin\theta} \sum_l V_l \frac{\partial}{\partial \theta}(\sin\theta\, Y_l) \right]$$
$$+ \frac{\partial}{\partial r}\left(\tfrac{1}{2}\rho_1 v_0^2 + \rho_0 \vec{v}_0 \cdot \vec{v}_1 + \tfrac{\gamma}{\gamma-1} p_1\right) v_0 + \left(\tfrac{1}{2}\rho_1 v_0^2 + \rho_0 \vec{v}_0 \cdot \vec{v}_1 + \tfrac{\gamma}{\gamma-1} p_1\right) \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 v_0) = \gamma \frac{\partial q_1}{\partial r}$$

$$\Leftrightarrow \frac{\partial}{\partial r}\left(\tfrac{1}{2}\rho_0 v_0^2 + \tfrac{\gamma}{\gamma-1} p_0\right) \sum_l U_l X_l + \left(\tfrac{1}{2}\rho_0 v_0^2 + \tfrac{\gamma}{\gamma-1} p_0\right) \left[\frac{1}{r^2} \sum_l \frac{\partial}{\partial r}(r^2 U_l) X_l + \frac{1}{r \sin\theta} \sum_l V_l \frac{\partial}{\partial \theta}(\sin\theta\, Y_l) \right]$$
$$+ v_0 \sum_l \frac{\partial}{\partial r}\left(\tfrac{1}{2} R_l v_0^2 + \rho_0 v_0 U_l + \tfrac{\gamma}{\gamma-1} \Pi_l\right) X_l + \sum_l \left(\tfrac{1}{2} R_l v_0^2 + \rho_0 v_0 U_l + \tfrac{\gamma}{\gamma-1} \Pi_l\right) X_l \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 v_0) = \gamma \sum_l \frac{\partial Q_l}{\partial r} X_l$$

again the modes can be separated if (1) is required and X_l can be cancelled (don't forget α_l factor)

$$\Leftrightarrow U_l \frac{\partial}{\partial r}\left(\tfrac{1}{2}\rho_0 v_0^2 + \tfrac{\gamma}{\gamma-1} p_0\right) + \left(\tfrac{1}{2}\rho_0 v_0^2 + \tfrac{\gamma}{\gamma-1} p_0\right) \left[\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 U_l) + \frac{\alpha_l}{r} V_l \right]$$
$$+ \frac{\partial}{\partial r}\left(\tfrac{1}{2} R_l v_0^2 + \rho_0 v_0 U_l + \tfrac{\gamma}{\gamma-1} \Pi_l\right) v_0 + \left(\tfrac{1}{2} R_l v_0^2 + \rho_0 v_0 U_l + \tfrac{\gamma}{\gamma-1} \Pi_l\right) \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 v_0) = \gamma \frac{\partial Q_l}{\partial r}$$

(eq. (D) in Per's notes)

$$\Leftrightarrow \boxed{\left[U_l \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 U_l) + \frac{\alpha_l}{r} V_l \right] \left(\tfrac{1}{2}\rho_0 v_0^2 + \tfrac{\gamma}{\gamma-1} p_0\right) + \left[v_0 \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 v_0) \right] \left(\tfrac{1}{2} R_l v_0^2 + \rho_0 v_0 U_l + \tfrac{\gamma}{\gamma-1} \Pi_l\right) = \gamma \frac{\partial Q_l}{\partial r} \qquad (\mathrm{III}_1, b)}$$

(IV) : $\frac{dE}{dr} = \frac{1}{2} \frac{\rho}{m} L(E) \overset{\text{linearize}}{\Longrightarrow} \frac{\partial E_1}{\partial r} = \frac{1}{2} \frac{\rho_1}{m} L(E_0) + \frac{1}{2} \frac{\rho_0}{m} \frac{\partial L}{\partial E}\Big|_{E_0} E_1 \overset{\text{expand}}{\Longleftrightarrow} \sum_l \frac{\partial \mathcal{E}_l}{\partial r} X_l = \frac{1}{2m} L(E_0) \sum_l R_l X_l + \frac{1}{2m} \rho_0 \frac{\partial L}{\partial E}\Big|_{E_0} \sum_l \mathcal{E}_l X_l$

$$\overset{\text{separate}}{\Longleftrightarrow} \boxed{\frac{\partial \mathcal{E}_l}{\partial r} = \frac{1}{2m} L(E_0) R_l + \frac{1}{2m} \rho_0 \frac{\partial L}{\partial E}\Big|_{E_0} \mathcal{E}_l \qquad (\mathrm{IV}_1, b)}$$

(V) : $\frac{dq}{dr} = \frac{\rho}{m} q \Lambda(E) \overset{\text{linearize}}{\Longrightarrow} m \frac{\partial q_1}{\partial r} = \rho_1 q_0 \Lambda(E_0) + \rho_0 q_1 \Lambda(E_0) + \rho_0 q_0 \frac{\partial \Lambda}{\partial E}\Big|_{E_0} \cdot E_1$

$$\overset{\text{expand and separate}}{\Longleftrightarrow} \boxed{m \frac{\partial Q_l}{\partial r} = R_l q_0 \Lambda(E_0) + \rho_0 Q_l \Lambda(E_0) + \rho_0 q_0 \frac{\partial \Lambda}{\partial E}\Big|_{E_0} \mathcal{E}_l \qquad (\mathrm{V}_1, b)}$$

(V) : $\frac{dq}{dr} = \frac{\rho}{m} q \Lambda(\epsilon) \overset{\text{linearize}}{\Longrightarrow} m\frac{\partial q_1}{\partial r} = \rho_1 q_0 \Lambda(\epsilon_0) + \rho_0 q_1 \Lambda(\epsilon_0) + \rho_0 q_0 \frac{\partial \Lambda}{\partial E}\Big|_{E_0} \cdot \epsilon_1$

expand and separate $\Longleftrightarrow$

$$\boxed{m \frac{\partial Q_l}{\partial r} = R_l q_0 \Lambda(\epsilon_0) + \rho_0 Q_l \Lambda(\epsilon_0) + \rho_0 q_0 \frac{\partial \Lambda}{\partial E}\Big|_{E_0} \varepsilon_l \qquad (\mathrm{V}_1, b)}$$

In order to replace ρ, i.e. R_l in all equations:

(0): $\rho = m\frac{p}{T} \overset{\text{linearize}}{\Longrightarrow} \rho_1 = m\frac{p_1}{T_0} - m\frac{p_0}{T_0^2} T_1 \overset{\text{expand}}{\Longleftrightarrow} \sum_l R_l X_l = \frac{m}{T_0} \sum_l \Pi_l X_l - m\frac{p_0}{T_0^2} \sum_l \tau_l X_l$

separate $\Longleftrightarrow$ $\boxed{R_l = m\left(\frac{\Pi_l}{T_0} - \frac{p_0}{T_0^2} \tau_l\right) \qquad (0, b)}$

and $\boxed{\rho_0 = m\frac{p_0}{T_0}}$

$\hat{\theta}$-dependence

we required $\langle X_l, X_k \rangle = \delta_{lk}$, $\langle Y_l, Y_k \rangle = \delta_{lk}$ and more importantly

(1): $\alpha_l X_l = \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}(\sin\theta \, Y_l)$ and (2): $Y_l = -\frac{1}{\beta_l}\frac{\partial X_l}{\partial \theta}$

together: $\boxed{\frac{1}{X_l} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial X_l}{\partial\theta}\right) = -\alpha_l \beta_l = -\lambda_l \qquad (4)}$

for this ODE to have a non-trivial, regular solution

we require $\boxed{\lambda_l = l(l+1)}$ (see spherical harmonics Wikipedia)

Then without loss o generality we define $\boxed{X_l(\theta) = P_l^0(\cos\theta)}$

and $P_l^0(\cos\theta)$ solves (4) if they are the associated Legendre polynomials (with $m=0$)

and they form automatically a orthogonal basis.

Arbitrarily choose $\alpha_l = -l(l+1)$, $\beta_l = -1$