# Force derivation for normalized first order

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In "Rocket effect derivation, part 3" I derived the force on the pellet from the first order perturbation.

$$\rightarrow F_z = -\frac{4}{3}\pi r_p^2 \left( v_0^2 R_1 + \rho_0 v_0 (2U_1 - \alpha_1 V_1) + P_1 \right)\Big|_{r=r_p}$$

But here all quantities are their non-normalized versions.

We chose previously $\alpha_l = -l(l+1) \Rightarrow \alpha_1 = -2$

Now, let us insert the normalized quantities for example $\underset{\text{non-norm.}}{R_1} \rightarrow \rho_* q_{rel} \underset{\text{normalized}}{R_1}$

(while $F_z$ is still with the unit of force)

We need to keep some non-normalized quantities, which I label as $\tilde{r}_p, E_{bg}, q_{bg}, \ldots$

$$\Rightarrow F_z = -\frac{4}{3}\pi \tilde{r}_p^2 \left( \rho_* v_*^2 q_{rel} v_0^2 R_1 + 2\rho_* v_*^2 q_{rel} \rho_0 v_0 (U_1 + V_1) + p_* q_{rel} P_1 \right)\Big|_{r=r_p}$$

Use the relations at the sonic radius: $\rho_* = m\frac{p_*}{T_*}$ and $v_*^2 = \frac{1}{m}\gamma T_*$

$$\Rightarrow \rho_* v_*^2 = m\frac{p_*}{T_*} \cdot \frac{1}{m}\gamma T_* = \gamma p_*$$

$$\Rightarrow F_z = -\frac{4}{3}\pi \tilde{r}_p^2 p_* q_{rel} \left( \gamma v_0^2 R_1 + \gamma 2 \rho_0 v_0 (U_1 + V_1) + P_1 \right)\Big|_{r=r_p}$$

eliminate the density: $R_1 = \frac{P_1}{T_0} - \frac{P_0}{T_0^2}\tau_1$

$\rho_0 = \frac{P_0}{T_0}$ and $r^2 \frac{P_0}{T_0} v_0 = 1 \Rightarrow \rho_0 v_0 = \frac{1}{r^2}$

$$\Rightarrow F_z = -\frac{4}{3}\pi \tilde{r}_p^2 p_* q_{rel} \left( \gamma v_0^2 \left( \frac{P_1}{T_0} - \frac{P_0}{T_0^2}\tau_1 \right) + \frac{2\gamma}{r_p^2}(U_1 + V_1) + P_1 \right)\Big|_{r=r_p}$$

We can disregard the first order velocity quantities since our boundary conditions force $U_1(r_p) = 0$, $V_1(r_p) = 0$

Also, numerical solutions show that $v_0^2(r_p) R_1(r_p) \ll P_1(r_p)$

(How can we argue for that analytically or physically?)

$$\Rightarrow \boxed{F_z \approx -\frac{4}{3}\pi \tilde{r}_p^2 p_* q_{rel} P_1(r = r_p)}$$

$P_1(r = r_p)$ results from solving the first order ODE system and thus only depends on $\gamma, E_*(E_{bg}), \frac{E_{rel}}{q_{rel}}$

The dependence on $E_*$ seems quite low. (From looking at my first numerical results)

$\rightarrow$ our goal is to calculate $P_1(r=r_p)$ once and for all

for a representative range of $\gamma, E_*(E_{bg}), \frac{E_{rel}}{q_{rel}}$

$q_{rel}$ and $\tilde{r}_p$ are input parameters (not needed for $P_1(r=r_p)$) normalized pellet radius

$p_*$ can be calculated from Parks model and an expression is derived in the following section.

## Derivation of $p_*$ from the zeroth order solution

For one choice of $\gamma, E_{bg}$ the zeroth order solution can be calculated.

(to be exact, we choose $E_*$ based on $E_{bg}$ and the found relation $E_*(E_{bg})$)

this yields the normalized quantities $\lambda_*, r_p, E_0(r\to\infty), q_0(r\to\infty)$

$$\Rightarrow \quad r_* = \frac{\tilde{r}_p}{r_p}, \quad q_* = \frac{q_{bg}}{q_0(r\to\infty)}, \quad E_* = \frac{E_{bg}}{E_0(r\to\infty)}$$

$\tilde{r}_p$ and $E_{bg}$ are free parameters (physical inputs)

Equation 1 in Parks paper: $q_{bg} = \frac{n_{e,bg} \cdot v_{e,bg}}{4} \cdot E_{bg}$

with $v_{e,bg} = \sqrt{\frac{8 k_B T_{e,bg}}{\pi m_e}}$, $E_{bg} = 2 k_B T_{e,bg}$

together: $q_{bg} = \frac{n_{e\,bg}}{4} \cdot \sqrt{\frac{4 E_{bg}}{\pi m_e}} \, E_{bg}$

$$\Leftrightarrow \quad q_{bg} = \frac{1}{2\sqrt{\pi}} \cdot \frac{n_{e,bg}}{\sqrt{m_e}} E_{bg}^{3/2}$$

Additionally, we have 4 equations that relate the quantities at $r_*$ (all from Parks paper)

$$\lambda_* = \frac{p_*}{T_*} \Lambda_* r_*$$

$$r_*^2 \frac{p_*}{T_*} v_* = \frac{G}{4\pi}$$

$$v_* = \sqrt{\frac{\gamma T_*}{m}}$$

$(\gamma - 1)\, 4\pi \ldots \mu \ldots \left( d \ldots r_*^2 \right) = 2$

$$v_* = \sqrt{\frac{\gamma T_*}{m}}$$

$$(\gamma-1)\frac{4\pi}{G}\frac{\mu}{\gamma}\lambda_*\left(q_*\frac{r_*^2}{T_*}\right) = 2$$

"known" quantities: $\lambda_*$, $\Lambda_* = \Lambda(E_*)$, $r_*$, $\gamma$, $m$, $\mu$ (model parameter), $q_*$

"unknown" quantities: $p_*$, $T_*$, $v_*$, $G$

solving this system of equations using sympy yields:

(disregarding complex solutions)

$$\Rightarrow p_* = 2^{-\frac{2}{3}}\lambda_*\frac{(\gamma-1)^{2/3}}{\gamma}\cdot\left(\frac{m}{\Lambda_* r_*}\right)^{\frac{1}{3}}\cdot(\mu q_*)^{2/3}$$

insert $r_* = \frac{\tilde{r}_p}{r_p}$ and $q_* = \frac{q_{bg}}{q_0(\infty)}$

$$\Leftrightarrow \quad p_* = 2^{-\frac{2}{3}}\lambda_*\frac{(\gamma-1)^{2/3}}{\gamma}\cdot\left(\frac{m\, r_p}{\Lambda_*\tilde{r}_p}\right)^{1/3}\left(\frac{\mu}{q_0(\infty)}\frac{1}{2\sqrt{\pi}}\frac{n_{e,bg}}{\sqrt{m_e}}E_{bg}^{3/2}\right)^{2/3}$$

separate visually

$$\Rightarrow \boxed{p_* = \underbrace{\frac{1}{2^{5/3}\sqrt{\pi}}\lambda_*\, r_p^{1/3}\left(\frac{1}{q_0(\infty)}\right)^{\frac{2}{3}}\frac{(\gamma-1)^{2/3}}{\gamma}}_{\substack{\approx 0.04\pm 0.003 \\ \text{weakly dependent on } E_* \\ \text{dimensionless}}}\underbrace{\left(\frac{m}{m_e}\frac{1}{\tilde{r}_p}\mu^2 n_{e,bg}^2\frac{1}{\Lambda_*}\right)^{\frac{1}{3}}E_{bg}}_{\substack{\text{physical quantities} \\ \text{"model inputs"}}}}$$

<u>In total needed quantities to calculate the pellet rocket force:</u>

needed for precalculating the model:

$\gamma$, $E_{bg}$, $(E_{rel}/q_{rel})$ (or better $E_*$ than $E_{bg}$)

(main output of this project is $P_1(r=r_p)$)

needed to calculate $p_*$ from pre-calculated model:

- $m$ (mass of one ablated atom/molecule) divided by the electron mass $m_e$
- $\tilde{r}_p$ (pellet radius)
- $n_{e,bg}$ (electron density reaching the neutral cloud)
- $E_{bg}$ (average energy of incoming electrons)

$n_{e,bg}$ ([illegible])

$E_{bg}$ (average energy of incoming electrons)

$\mu$ (fraction of electron heat loss, heating the neutral gas cloud)
Park assumes $\mu \approx 0.6 - 0.7$

additionally needed for the force:

$q_{rel}$ (from $\tilde{Q}_1(\infty) = q_{rel}\, q_{bg} = \int_0^{\pi} q_1(r=\infty, \theta) \cos\theta \, d\theta$)