

Initial set of equations

$$\vec{\nabla} \cdot (\rho \vec{v}) = 0 \quad (\text{I}) \quad (\text{mass conservation})$$

$$\rho \vec{v} \cdot \vec{\nabla} \vec{v} = -\vec{\nabla} p = \rho (\underbrace{\vec{v} \cdot \vec{\nabla}}_{\text{directional derivative}}) \vec{v} \quad (\text{II}) \quad (\text{momentum conservation})$$

Parks assumptions

$$\vec{\nabla} \cdot \left[\left(\frac{\rho v^2}{2} + \frac{\gamma p}{\gamma-1} \right) \vec{v} \right] = \vec{\nabla} \cdot \vec{q} \approx \gamma \frac{dq}{dr} \quad (\text{III}) \quad (\text{energy conservation + heat source } Q)$$

(X is fraction converted into heating the gas)
text

$$p = m \frac{p}{T} \quad (0) \quad (\text{ideal gas law, } T = k_B \cdot \text{Temperature})$$

use to replace p
because we have boundary conditions for T

Heat flux dynamics assumptions

Like Parks has done it:

$$\frac{dE}{dr} = \frac{1}{m} \frac{p}{L(E)} \quad (\text{IV}) \quad (\text{energy absorption of electrons})$$

$L(E)$ is energy loss function

$$\frac{dq}{dr} = \frac{q}{\lambda_{\text{eff}}(E)} = \frac{p}{m} q \Lambda(E) \quad (\text{V}) \quad (\text{effective energy flux})$$

$\Lambda(E) = \hat{\sigma}_T(E) + 2 \frac{L(E)}{E} = \text{"effective energy flux crosssection"}$

alternatively to V:

$\hat{\sigma}_T(E) = \text{effective backscattering crosssection}$

$$q(r) = q_\infty \frac{E(r)}{E_0} \exp \left[\int_{E(r)}^{E_0} \frac{\hat{\sigma}_T(E')}{L(E')} dE' \right] \quad (\text{V}^*)$$

Spherically symmetric model (zeroth order)

$$\frac{M \dot{m}^2}{\rho^2} \frac{1}{s}$$

$$(\text{I}): \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho(r) v_0(r)) = 0 \quad \Leftrightarrow \quad \text{use (0)}$$

$$r^2 \frac{\rho_0}{T_0} v_0 = \text{const} = \frac{6}{4\pi}$$

$$(\text{I}_0) \quad \frac{[G]}{[G]} = \frac{1}{5}$$

$$(\text{II}): \rho_0(r) v_0(r) \frac{\partial}{\partial r} v_0(r) = - \frac{\partial}{\partial r} p_0(r) \quad \Leftrightarrow$$

$$m \frac{\rho_0}{T_0} v_0 \frac{\partial v_0}{\partial r} + \frac{\partial p_0}{\partial r} = 0$$

$$(\text{II}_0)$$

$$(\text{III}): \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\rho_0(r) v_0(r)}{2} + r^2 \frac{\gamma}{\gamma-1} p_0(r) v_0(r) \right] = \gamma \frac{\partial q_0}{\partial r} \quad (\text{use (I}_0), (\text{II}_0))$$

$$\frac{6}{4\pi r^2} \frac{\partial}{\partial r} \left[\frac{m}{2} v_0^2 + \frac{\gamma}{\gamma-1} T_0 \right] = \gamma \frac{\partial q_0}{\partial r}$$

$$(\text{III}_0)$$

$$(\text{VI}): \frac{\partial E_0}{\partial r} = \frac{1}{m} \frac{p_0}{L(E_0)} \quad (\text{IV}_0)$$

$$\left(\rho_0 = m \frac{\rho_0}{T_0} \right)$$

$$(\text{V}): \frac{\partial q_0}{\partial r} = \frac{p_0}{T_0} q_0 \Lambda(E_0) \quad (\text{V}_0)$$

First order angular asymmetry correction

Perturbations (small)

$$p(\vec{r}) = p_0(r) + p_1(r, \theta)$$

$$p(\vec{r}) = p_0(r) + p_1(r, \theta)$$

$$T(\vec{r}) = T_0(r) + T_1(r, \theta)$$

$$\vec{v}(\vec{r}) = v_0(r) \hat{r} + \vec{v}_1(r, \theta) = v_0 \hat{r} + u_1 \hat{r} + v_1 \hat{\theta}$$

Expansion in general basis $\{X_L\}, \{Y_L\}$

$$p_1(r, \theta) = \sum_L R_L(r) X_L(\theta)$$

$$p_1(r, \theta) = \sum_L \pi_L(r) X_L(\theta)$$

$$T_1(r, \theta) = \sum_L \tau_L(r) X_L(\theta)$$

$$\vec{v}_1(r, \theta) = \left(\sum_L u_L(r) X_L(\theta) \right) \hat{r} + \left(\sum_L v_L(r) Y_L(\theta) \right) \hat{\theta}$$

$$\vec{q}(\vec{r}) = q_0(r) \hat{r} + q_1(r, \theta) \hat{\theta}$$

$$q_1(r, \theta) = \sum_l Q_l(r) X_l(\theta)$$

$$E(\vec{r}) = E_0(r) + E_1(r, \theta)$$

$$E_1(r, \theta) = \sum_l E_l(r) X_l(\theta)$$

Derivatives in spherical coordinates:

• gradient: $\vec{\nabla} \phi = \hat{r} \frac{\partial \phi}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial \phi}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi}$

• divergence: $\vec{\nabla} \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial A_\varphi}{\partial \varphi}$

• vector-gradient: $\vec{\nabla} \vec{A} = \hat{r} \hat{r} \frac{\partial A_r}{\partial r} + \hat{r} \hat{\theta} \left(\frac{1}{r} \frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{r} \right) + \hat{r} \hat{\phi} \left(\frac{1}{r \sin \theta} \frac{\partial A_r}{\partial \varphi} - \frac{A_\varphi}{r} \right) + \hat{\theta} \hat{r} \frac{\partial A_\theta}{\partial r} + \hat{\theta} \hat{\theta} \left(\frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{A_r}{r} \right) + \hat{\theta} \hat{\phi} \left(\frac{1}{r \sin \theta} \frac{\partial A_\theta}{\partial \varphi} - \cot \theta \frac{A_\varphi}{r} \right) + \hat{\phi} \hat{r} \frac{\partial A_\varphi}{\partial r} + \hat{\phi} \hat{\theta} \frac{1}{r} \frac{\partial A_\varphi}{\partial \theta} + \hat{\phi} \hat{\phi} \left(\frac{1}{r \sin \theta} \frac{\partial A_\varphi}{\partial \varphi} + \cot \theta \frac{A_\theta}{r} + \frac{A_r}{r} \right)$

• directional derivative: $\vec{A} \cdot (\vec{\nabla} \vec{B}) = (\vec{A} \cdot \vec{\nabla}) \vec{B} = \vec{\nabla}_A \vec{B} = (\vec{\nabla} \vec{B}) \cdot \vec{A}$

useful examples:

$$\vec{\nabla} p_1(r, \theta) = \hat{r} \sum_l \frac{\partial R_l}{\partial r} X_l + \hat{\theta} \sum_l R_l \frac{\partial X_l}{\partial \theta}$$

$$\vec{\nabla} u_1 = \frac{1}{r^2} \sum_l \frac{\partial}{\partial r} (r^2 u_l) X_l + \frac{1}{r \sin \theta} \sum_l V_l \frac{\partial}{\partial \theta} (\sin \theta Y_l)$$

Derivation of 1st 3 eqs. through linearization

$$(I) \Rightarrow \vec{\nabla} \cdot (\rho \vec{v}) \approx \underbrace{\vec{\nabla} \cdot (\rho_0 \vec{v}_0)}_{=0} + \rho_1 \vec{\nabla} \cdot \vec{v}_0 + \rho_0 \vec{\nabla} \cdot \vec{v}_1 = 0$$

$$\Rightarrow \vec{\nabla} \cdot (\rho_1 \vec{v}_0 + \rho_0 \vec{v}_1) = \vec{\nabla} \rho_0 \cdot \vec{v}_1 + \rho_0 \vec{\nabla} \cdot \vec{v}_1 + \vec{\nabla} \rho_1 \cdot \vec{v}_0 + \rho_1 \vec{\nabla} \cdot \vec{v}_0 = 0 \quad (I_1, a)$$

$$\Rightarrow \frac{\partial \rho_0}{\partial r} \sum_l u_l X_l + \rho_0 \left[\frac{1}{r^2} \sum_l \frac{\partial}{\partial r} (r^2 u_l) X_l + \frac{1}{r \sin \theta} \sum_l V_l \frac{\partial}{\partial \theta} (\sin \theta Y_l) \right] + \rho_1 \sum_l \frac{\partial R_l}{\partial r} X_l + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_0) \sum_l R_l X_l = 0$$

require an orthogonal basis $\langle X_l, X_k \rangle = \int X_l(\theta) X_k(\theta) d\theta = 0$ if $l \neq k$

but require also $\alpha_l X_l = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta Y_l) \quad (1)$, where $\alpha_l = \text{const}$ can be chosen later

then we can project out each mode and get for mode l the differential equation (and divide by X_l)

$$\frac{\partial \rho_0}{\partial r} u_l + \rho_0 \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_l) + \frac{\alpha_l}{r} V_l \right] + \rho_1 \frac{\partial R_l}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_0) R_l = 0 \quad (I_1, b)$$

linearize (II): $\rho \vec{v} \cdot \vec{\nabla} \vec{v} = -\vec{\nabla} p$ and use (II₀)

$$\Rightarrow \rho_0 \vec{v}_0 \cdot \vec{\nabla} \vec{v}_1 + \rho_0 \vec{v}_1 \cdot \vec{\nabla} \vec{v}_0 + \rho_1 \vec{v}_0 \cdot \vec{\nabla} \vec{v}_0 = -\vec{\nabla} p_1 \quad (II_1, a)$$

Directional derivatives: $\hat{r} \cdot \vec{\nabla} v_0 = \hat{r} \frac{\partial v_0}{\partial r}, \quad \hat{\theta} \cdot \vec{\nabla} v_0 = \hat{\theta} \frac{v_0}{r}$

$$\hat{r} \cdot \vec{\nabla} u_1 = \hat{r} \frac{\partial u_1}{\partial r} + \hat{\theta} \frac{\partial u_1}{\partial \theta}$$

$$\Rightarrow \rho_0 v_0 \left(\hat{r} \frac{\partial u_1}{\partial r} + \hat{\theta} \frac{\partial u_1}{\partial \theta} \right) + \rho_0 \left(u_1 \hat{r} \frac{\partial v_0}{\partial r} + v_1 \hat{\theta} \frac{v_0}{r} \right) + \rho_1 v_0 \hat{r} \frac{\partial v_0}{\partial r} = -\hat{r} \frac{\partial p_1}{\partial r} - \hat{\theta} \frac{1}{r} \frac{\partial p_1}{\partial \theta}$$

separate \hat{r} and $\hat{\theta}$ and insert expansion

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$$\textcircled{\text{I}} \quad \rho_0 v_0 \sum_L \frac{\partial U_L}{\partial r} X_L + \rho_0 \frac{\partial v_0}{\partial r} \sum_L U_L X_L + v_0 \frac{\partial v_0}{\partial r} \sum_L R_L X_L = - \sum_L \frac{\partial \pi_L}{\partial r} X_L$$

\rightarrow L-modes are separable and X_L can be canceled

$$\Rightarrow \rho_0 v_0 \frac{\partial U_L}{\partial r} + \rho_0 \frac{\partial v_0}{\partial r} U_L + v_0 \frac{\partial v_0}{\partial r} R_L = - \frac{\partial \pi_L}{\partial r} \quad (\text{II}_1, b1)$$

$$\textcircled{\text{II}} \quad \rho_0 v_0 \sum_L \frac{\partial V_L}{\partial r} Y_L + \rho_0 \frac{v_0}{r} \sum_L V_L Y_L = - \frac{1}{r} \sum_L \pi_L \frac{\partial X_L}{\partial \theta}$$

To be able to separate modes require $\langle Y_L, Y_k \rangle = 0$ if $L \neq k$ and $Y_L = -\frac{1}{\beta_L} \frac{\partial X_L}{\partial \theta} \quad (2)$ where $\beta_L = \text{const}$ can be chosen later

$$\Rightarrow \rho_0 v_0 \frac{\partial V_L}{\partial r} + \rho_0 \frac{v_0}{r} V_L = \frac{\beta_L}{r} \pi_L \quad (\text{II}_1, b2)$$

$$\textcircled{\text{III}} \Rightarrow \vec{\nabla} \cdot \left[\left(\frac{\rho_0 v^2}{2} + \frac{\gamma p}{\gamma-1} \right) \vec{v} \right] = \gamma \frac{\partial q}{\partial r}$$

$$\text{(linearize)} \Rightarrow \vec{\nabla} \cdot \left[\left(\frac{1}{2} \rho_0 v_0^2 + \frac{\gamma}{\gamma-1} p_0 \right) \vec{v}_1 + \left(\frac{1}{2} \rho_1 v_0^2 + \rho_0 \vec{v}_0 \cdot \vec{v}_1 + \frac{\gamma}{\gamma-1} p_1 \right) \vec{v}_0 \right] = \gamma \frac{\partial q_1}{\partial r} \quad (\text{III}_1, a)$$

$$\text{product rule} \Rightarrow \vec{\nabla} \cdot \left(\frac{1}{2} \rho_0 v_0^2 + \frac{\gamma}{\gamma-1} p_0 \right) \vec{v}_1 + \left(\frac{1}{2} \rho_0 v_0^2 + \frac{\gamma}{\gamma-1} p_0 \right) \vec{\nabla} \cdot \vec{v}_1 + \vec{\nabla} \cdot \left(\frac{1}{2} \rho_1 v_0^2 + \rho_0 \vec{v}_0 \cdot \vec{v}_1 + \frac{\gamma}{\gamma-1} p_1 \right) \vec{v}_0 + \left(\frac{1}{2} \rho_1 v_0^2 + \rho_0 \vec{v}_0 \cdot \vec{v}_1 + \frac{\gamma}{\gamma-1} p_1 \right) \vec{\nabla} \cdot \vec{v}_0 = \gamma \frac{\partial q_1}{\partial r}$$

insert expansion

$$\Rightarrow \frac{\partial}{\partial r} \left(\frac{1}{2} \rho_0 v_0^2 + \frac{\gamma}{\gamma-1} p_0 \right) \sum_L U_L X_L + \left(\frac{1}{2} \rho_0 v_0^2 + \frac{\gamma}{\gamma-1} p_0 \right) \left[\frac{1}{r^2} \sum_L \frac{\partial}{\partial r} (r^2 U_L) X_L + \frac{1}{r \sin \theta} \sum_L V_L \frac{\partial}{\partial \theta} (\sin \theta V_L) \right] + \frac{\partial}{\partial r} \left(\frac{1}{2} \rho_1 v_0^2 + \rho_0 \vec{v}_0 \cdot \vec{v}_1 + \frac{\gamma}{\gamma-1} p_1 \right) v_0 + \left(\frac{1}{2} \rho_1 v_0^2 + \rho_0 \vec{v}_0 \cdot \vec{v}_1 + \frac{\gamma}{\gamma-1} p_1 \right) \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_0) = \gamma \frac{\partial q_1}{\partial r}$$

$$\Rightarrow \frac{\partial}{\partial r} \left(\frac{1}{2} \rho_0 v_0^2 + \frac{\gamma}{\gamma-1} p_0 \right) \sum_L U_L X_L + \left(\frac{1}{2} \rho_0 v_0^2 + \frac{\gamma}{\gamma-1} p_0 \right) \left[\frac{1}{r^2} \sum_L \frac{\partial}{\partial r} (r^2 U_L) X_L + \frac{1}{r \sin \theta} \sum_L V_L \frac{\partial}{\partial \theta} (\sin \theta V_L) \right] + v_0 \sum_L \frac{\partial}{\partial r} \left(\frac{1}{2} R_L v_0^2 + \rho_0 v_0 U_L + \frac{\gamma}{\gamma-1} \pi_L \right) X_L + \sum_L \left(\frac{1}{2} R_L v_0^2 + \rho_0 v_0 U_L + \frac{\gamma}{\gamma-1} \pi_L \right) X_L \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_0) = \gamma \sum_L \frac{\partial Q_L}{\partial r} X_L$$

again the modes can be separated if (1) is required and X_L can be cancelled (don't forget \propto_L factor)

$$\Rightarrow U_L \frac{\partial}{\partial r} \left(\frac{1}{2} \rho_0 v_0^2 + \frac{\gamma}{\gamma-1} p_0 \right) + \left(\frac{1}{2} \rho_0 v_0^2 + \frac{\gamma}{\gamma-1} p_0 \right) \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 U_L) + \frac{\kappa_L}{r} V_L \right] + \frac{\partial}{\partial r} \left(\frac{1}{2} R_L v_0^2 + \rho_0 v_0 U_L + \frac{\gamma}{\gamma-1} \pi_L \right) v_0 + \left(\frac{1}{2} R_L v_0^2 + \rho_0 v_0 U_L + \frac{\gamma}{\gamma-1} \pi_L \right) \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_0) = \gamma \frac{\partial Q_L}{\partial r} \quad (\text{eq. (D) in Per's notes})$$

$$\Rightarrow \left[U_L \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 U_L) + \frac{\kappa_L}{r} V_L \right] \left(\frac{1}{2} \rho_0 v_0^2 + \frac{\gamma}{\gamma-1} p_0 \right) + \left[v_0 \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_0) \right] \left(\frac{1}{2} R_L v_0^2 + \rho_0 v_0 U_L + \frac{\gamma}{\gamma-1} \pi_L \right) = \gamma \frac{\partial Q_L}{\partial r} \quad (\text{III}_1, b)$$

$$\textcircled{\text{IV}} : \frac{dE}{dr} = \frac{2}{m} \frac{d}{dr} L(\theta) \xRightarrow{\text{linearize}} \frac{\partial E_1}{\partial r} = \frac{2}{m} \frac{d}{dr} L(E_0) + \frac{1}{m} \frac{d}{dr} \frac{\partial L}{\partial E} \Big|_{E_0} E_1 \xRightarrow{\text{expand}} \sum_L \frac{\partial E_L}{\partial r} X_L = \frac{2}{m} L(E_0) \sum_L R_L X_L + \frac{2}{m} \rho_0 \frac{\partial L}{\partial E} \Big|_{E_0} \sum_L E_L X_L$$

$$\xRightarrow{\text{separate}} \frac{\partial E_L}{\partial r} = \frac{2}{m} L(E_0) R_L + \frac{2}{m} \rho_0 \frac{\partial L}{\partial E} \Big|_{E_0} E_L \quad (\text{IV}_1, b)$$

$$\textcircled{\text{V}} : \frac{dq}{dr} = \frac{\rho}{m} q \Lambda(E) \xRightarrow{\text{linearize}} m \frac{\partial q_1}{\partial r} = \rho_1 q_0 \Lambda(E_0) + \rho_0 q_1 \Lambda(E_0) + \rho_0 q_0 \frac{\partial \Lambda}{\partial E} \Big|_{E_0} E_1$$

$$\xRightarrow{\text{expand and separate}} m \frac{\partial Q_L}{\partial r} = R_L q_0 \Lambda(E_0) + \rho_0 Q_L \Lambda(E_0) + \rho_0 q_0 \frac{\partial \Lambda}{\partial E} \Big|_{E_0} E_L \quad (\text{V}_1, b)$$

$$(V) : \frac{d\psi}{dr} = \frac{p}{m} \psi \Lambda(E) \xRightarrow{\text{linearize}} m \frac{d\psi}{dr} = p_1 \psi_0 \Lambda(E_0) + p_0 \psi_1 \Lambda(E_0) + p_0 \psi_0 \left. \frac{\partial \Lambda}{\partial E} \right|_{E_0} E_1$$

expand and separate
 \Rightarrow

$$m \frac{dQ_L}{dr} = R_L \psi_0 \Lambda(E_0) + p_0 Q_L \Lambda(E_0) + p_0 \psi_0 \left. \frac{\partial \Lambda}{\partial E} \right|_{E_0} E_L \quad (V, b)$$

In order to replace p , i.e. R_L in all equations:

$$(0): p = m \frac{p}{\tau} \xRightarrow{\text{linearize}} p_1 = m \frac{p_1}{\tau_0} - m \frac{p_0}{\tau_0} \tau_1 \xRightarrow{\text{expand}} \sum R_L X_L = \frac{m}{\tau_0} \sum \tau_L X_L - m \frac{p_0}{\tau_0} \sum \tau_L X_L$$

separate
 \Rightarrow

$$R_L = m \left(\frac{\tau_L}{\tau_0} - \frac{p_0}{\tau_0} \tau_L \right) \quad (0, b)$$

and $p_0 = m \frac{p_0}{\tau_0}$

$\hat{\theta}$ -dependence

we required $\langle X_L, X_k \rangle = \delta_{Lk}$, $\langle Y_L, Y_k \rangle = \delta_{Lk}$ and more importantly

$$(1): \alpha_L X_L = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta Y_L) \quad \text{and} \quad (2): Y_L = -\frac{1}{\beta_L} \frac{\partial X_L}{\partial \theta}$$

together: $\frac{1}{X_L} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial X_L}{\partial \theta}) = -\alpha_L \beta_L = -\lambda_L \quad (4)$

for this ODE to have a non-trivial, regular solution

we require $\lambda_L = L(L+1)$ (see spherical harmonics Wikipedia)

Then without loss of generality we define $X_L(\theta) = P_L^0(\cos \theta)$

and $P_L^0(\cos \theta)$ solves (4) if they are the associated Legendre polynomials (with $m=0$)

and they form automatically a orthogonal basis.

Arbitrarily choose $\alpha_L = -L(L+1)$, $\beta_L = -1$