Normalize spherical part

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System of equations

$$\rho_0 = m\frac{p_0}{T_0} \qquad (0_0)$$

$$r^2 \frac{p_0}{T_0} v_0 = \frac{G}{4\pi} \qquad (\mathrm{I}_0)$$

$$m\frac{p_0}{T_0} v_0 \frac{\partial v_0}{\partial r} = -\frac{\partial p_0}{\partial r} \qquad (\mathrm{II}_0)$$

$$\frac{G}{4\pi r^2}\frac{\partial}{\partial r}\left[\frac{m}{2}v_0^2 + \frac{\gamma}{\gamma-1}T_0\right] = \mu\frac{\partial q_0}{\partial r} \qquad (\mathrm{III}_0)$$

$$\frac{\partial E_0}{\partial r} = 2\frac{p_0}{T_0}L(E_0) \qquad (\mathrm{IV}_0)$$

$$\frac{\partial q_0}{\partial r} = \frac{p_0}{T_0} q_0 \Lambda(E_0) \qquad (\mathrm{V}_0)$$

Normalized quantities to value at sonic radius r_*

$\tilde{p}_0 = p_0/p_*$ $\quad \tilde{T}_0 = T_0/T_*$ $\quad \tilde{v}_0 = v_0/v_*$ $\quad \tilde{q}_0 = q_0/q_*$ $\quad \tilde{\rho}_0 = \rho_0/\rho_*$

$\tilde{E}_0 = E_0/E_*$ $\quad \tilde{r} = r/r_*$ $\quad \tilde{\Lambda}(E_0) = \Lambda(E_0)/\Lambda(E_*)$ $\quad \tilde{L}(E_0) = L(E_0)/[E_*\Lambda(E_*)]$

all of eq. (I_0) to (V_0) also hold true at the sonic radius

specifically also $\quad r_*^2\frac{p_*}{T_*}v_* = \frac{G}{4\pi}$ (1) and $\quad v_* = \sqrt{\frac{\gamma T_*}{m}} \Leftrightarrow \frac{v_*^2}{T_*} = \frac{\gamma}{m}$ (2)

Insert normalized variables and drop the bar notation

$$(\mathrm{I}_0) \Leftrightarrow \left(r^2\frac{p_0}{T_0}v_0\right)\cdot\left(r_*^2\frac{p_*}{T_*}v_*\right) = \frac{G}{4\pi} \overset{(1)}{\Leftrightarrow} r^2\frac{p_0}{T_0}v_0 = 1 \qquad (\mathrm{I}_*)$$

$$(\mathrm{II}_0) \Leftrightarrow \left(m\frac{p_0}{T_0}v_0\frac{\partial v_0}{\partial r}\right)\left(\frac{p_*}{T_*}v_*^2\right) = -\frac{\partial p_0}{\partial r}p_* \Leftrightarrow m\frac{p_0}{T_0}v_0\frac{\partial v_0}{\partial r}\frac{v_*^2}{T_*} = -\frac{\partial p_0}{\partial r}$$

$$\overset{(2)}{\Leftrightarrow} \gamma\frac{p_0}{T_0}v_0\frac{\partial v_0}{\partial r} = -\frac{\partial p_0}{\partial r} \qquad (\mathrm{II}_*)$$

$$(\mathrm{III}_0) \Leftrightarrow \frac{G}{4\pi r^2 r_*^2}\frac{\partial}{\partial r}\left[\frac{m}{2}v_0^2 v_*^2 + \frac{\gamma}{\gamma-1}T_0 T_*\right] = \mu\frac{\partial q_0}{\partial r}q_* \qquad \Big| \cdot \frac{r_*^2}{T_*}$$

$$\overset{(2)}{\Leftrightarrow} \frac{G}{4\pi r^2}\frac{\partial}{\partial r}\left[\frac{\gamma}{2}v_0^2 + \frac{\gamma}{\gamma-1}T_0\right] = \mu\frac{\partial q_0}{\partial r}\left(q_*\frac{r_*^2}{T_*}\right) \qquad (\mathrm{III}_*)$$

$$(\mathrm{IV}_0) \Leftrightarrow \frac{E_*}{r_*}\frac{\partial E_0}{\partial r} = 2\frac{p_0}{T_0}L(E_0)\frac{p_*}{T_*}E_*\Lambda_* \Leftrightarrow \frac{\partial E_0}{\partial r} = 2\frac{p_0}{T_0}L(E_0)\underbrace{\left(\frac{p_*}{T_*}\Lambda_* r_*\right)}_{\lambda_*}$$

$$(\mathrm{V}_0) \Leftrightarrow \frac{q_*}{r_*}\frac{\partial q_0}{\partial r} = \frac{p_0}{T_0}q_0\Lambda(E_0)\frac{p_*}{T_*}q_*\Lambda_* \Leftrightarrow \frac{\partial q_0}{\partial r} = \frac{p_0}{T_0}q_0\Lambda(E_0)\overbrace{\left(\frac{p_*}{T_*}\Lambda_* r_*\right)}^{\lambda_*}$$

$$(\mathrm{IV}_0) \Leftrightarrow \frac{\;}{r_*}\frac{\partial}{\partial r} = 2\frac{\;}{T_0}L(E_0)\frac{\;}{T_*}L_*\Lambda_* \Leftrightarrow \frac{\partial}{\partial r} = 2\frac{\;}{T_0}L(E_0)\underbrace{\left(\frac{\;}{T_*}\Lambda_* r_*\right)}_{\lambda_*}$$

$$(\mathrm{V}_0) \Leftrightarrow \frac{q_*}{r_*}\frac{\partial q_0}{\partial r} = \frac{\rho_0}{T_0} q_0 \Lambda(E_0)\frac{\rho_*}{T_*} q_* \Lambda_* \Leftrightarrow \frac{\partial q_0}{\partial r} = \frac{\rho_0}{T_0} q_0 \Lambda(E_0)\left(\frac{\rho_*}{T_*}\Lambda_* r_*\right)$$

define $\boxed{\lambda_* = \frac{\rho_*}{T_*}\Lambda_* r_* \qquad (3)}$ and use (I_*)

$$\Rightarrow \boxed{\frac{\partial E_0}{\partial r} = 2\lambda_*\frac{L(E_0)}{r^2 v_0} \qquad (\mathrm{IV}_*)} \quad \text{and} \quad \boxed{\frac{\partial q_0}{\partial r} = \lambda_* \frac{q_0 \Lambda(E_0)}{r^2 v_0} \qquad (\mathrm{V}_*)}$$

use $(\mathrm{I}_*) \Leftrightarrow \rho_0 = \frac{T_0}{r^2 v_0} \Leftrightarrow \frac{\rho_0}{T_0} = \frac{1}{r^2 v_0}$ to eliminate ρ_0

$$(\mathrm{II}_*) \Leftrightarrow \gamma \frac{1}{r^2 v_0} v_0 \frac{\partial v_0}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{T_0}{r^2 v_0}\right) = -\frac{1}{r^2 v_0}\frac{\partial T_0}{\partial r} + \frac{T_0}{r^2 v_0^2}\frac{\partial v_0}{\partial r} + 2\frac{T_0}{r^3 v_0} \quad \Big| \cdot r^2 v_0$$

$$\Leftrightarrow \gamma v_0 \frac{\partial v_0}{\partial r} = -\frac{\partial T_0}{\partial r} + \frac{T_0}{v_0}\frac{\partial v_0}{\partial r} + 2\frac{T_0}{r} \quad \text{use} \quad \frac{\partial v_0^2}{\partial r} = 2 v_0 \frac{\partial v_0}{\partial r} \Leftrightarrow \frac{\partial v_0}{\partial r} = \frac{1}{2 v_0}\frac{\partial v_0^2}{\partial r}$$

$$\Leftrightarrow \frac{\gamma}{2}\frac{\partial v_0^2}{\partial r} = -\frac{\partial T_0}{\partial r} + \frac{T_0}{2 v_0^2}\frac{\partial v_0^2}{\partial r} + 2\frac{T_0}{r}$$

$$\Leftrightarrow \frac{\partial v_0^2}{\partial r}\left(\frac{\gamma}{2} - \frac{T_0}{2 v_0^2}\right) = \frac{2 T_0}{r} - \frac{\partial T_0}{\partial r} \quad \Big| \cdot 2 v_0^2$$

$$\Leftrightarrow \frac{\partial v_0^2}{\partial r}\left(\gamma v_0^2 - T_0\right) = 2 v_0^2\left(2\frac{T_0}{r} - \frac{\partial T_0}{\partial r}\right) \qquad (4)$$

$$(\mathrm{III}_*) \Leftrightarrow \frac{G}{4\pi r^2}\left[\frac{\gamma}{2}\frac{\partial v_0^2}{\partial r} + \frac{\gamma}{\gamma-1}\frac{\partial T_0}{\partial r}\right] = \mu \frac{\partial q_0}{\partial r}\left(q_* \frac{r_*^2}{T_*}\right)$$

$$\Leftrightarrow \frac{1}{2}\frac{\partial v_0^2}{\partial r} + \frac{1}{\gamma-1}\frac{\partial T_0}{\partial r} = \frac{4\pi r^2}{G}\frac{\mu}{\gamma}\frac{\partial q_0}{\partial r}\left(q_* \frac{r_*^2}{T_*}\right)$$

$$\Leftrightarrow \frac{\partial T_0}{\partial r} = (\gamma - 1)\left(\frac{4\pi r^2}{G}\frac{\mu}{\gamma}\frac{\partial q_0}{\partial r}\left(q_* \frac{r_*^2}{T_*}\right) - \frac{1}{2}\frac{\partial v_0^2}{\partial r}\right) \qquad (5)$$

$$(5) \text{ into } (4) \Rightarrow \frac{\partial v_0^2}{\partial r}\left(\gamma v_0^2 - T_0\right) = 2 v_0^2\left[\frac{2 T_0}{r} - (\gamma - 1)\left(\frac{4\pi r^2}{G}\frac{\mu}{\gamma}\frac{\partial q_0}{\partial r}\left(q_* \frac{r_*^2}{T_*}\right) - \frac{1}{2}\frac{\partial v_0^2}{\partial r}\right)\right]$$

$$\Leftrightarrow \frac{\partial v_0^2}{\partial r}\left(\gamma v_0^2 - T_0 - v_0^2(\gamma - 1)\right) = 2 v_0^2\left[\frac{2 T_0}{r} - (\gamma - 1)\frac{r^2}{\frac{G}{4\pi}}\frac{\mu}{\gamma}\frac{\partial q_0}{\partial r}\left(q_* \frac{r_*^2}{T_*}\right)\right]$$

use (V_*): $\frac{\partial q_0}{\partial r} = \frac{1}{r^2 v_0} q_0 \Lambda(E_0)\lambda_*$

$$\Leftrightarrow \frac{\partial v_0^2}{\partial r}\left(v_0^2 - T_0\right) = 2 v_0^2\left[\frac{2 T_0}{r} - (\gamma - 1)\frac{4\pi r^2}{G}\frac{\mu}{\gamma}\frac{1}{r^2 v_0} q_0 \Lambda(E_0)\lambda_*\left(q_* \frac{r_*^2}{T_*}\right)\right] \qquad (6)$$

both $\frac{\partial v_0^2}{\partial r}$ and v_0^2 are always positive

at the sonic radius $(v_0^2 - T_0) = 0$ therefore the right side must also vanish at sonic radius

(all quantities are 1 at sonic radius)

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$$\Rightarrow (\gamma-1)\frac{4\pi}{G}\frac{\mu}{\gamma}\lambda_*\left(q_*\frac{r_*^2}{T_*}\right) \stackrel{!}{=} 2 \qquad (7)$$

(6) with (7) $\Rightarrow$ $$\boxed{\frac{\partial v_0^2}{\partial r} = \frac{4 v_0^2}{(v_0^2 - T_0)}\left[\frac{T_0}{r} - \frac{q_0 \Lambda(E_0)}{v_0}\right] \qquad (8)}$$

insert (7) into (5)

$$\Rightarrow \frac{\partial T_0}{\partial r} = \frac{2}{\lambda_*} r^2 \frac{\partial q_0}{\partial r} - \frac{(\gamma-1)}{2}\frac{\partial v_0^2}{\partial r} \quad \text{insert } (\underline{V}_*)$$

$$\Leftrightarrow \boxed{\frac{\partial T_0}{\partial r} = \frac{2 q_0 \Lambda(E_0)}{v_0} - \frac{(\gamma-1)}{2}\frac{\partial v_0^2}{\partial r} \qquad (9)}$$

Singularity in (8)

singularity at the sonic radius:

$$\left.\frac{\partial v_0^2}{\partial r}\right|_{r=1} = \left.\left(\frac{4 v_0^2}{(v_0^2 - T_0)}\left[\frac{T_0}{r} - \frac{q_0 \Lambda(E_0)}{v_0}\right]\right)\right|_{r=1} = 4\,\text{"}\frac{0}{0}\text{"}$$

use L'Hôpital's rule

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

with $\frac{\partial v_0}{\partial r} = \frac{1}{2 v_0}\frac{\partial v_0^2}{\partial r}$

$$\left.\frac{\partial}{\partial r}\left[\frac{T_0}{r} - \frac{q_0 \Lambda(E_0)}{v_0}\right]\right|_{r=1} = \left.\left[-\frac{T_0}{r^2} + \frac{1}{r}\frac{\partial T_0}{\partial r} - \frac{\partial q_0}{\partial r}\frac{\Lambda(E_0)}{v_0} - \frac{\partial \Lambda(E_0)}{\partial r}\frac{q_0}{v_0} + \frac{q_0 \Lambda(E_0)}{v_0^2}\frac{\partial v_0}{\partial r}\right]\right|_{r=1}$$

with $\left.\frac{\partial T_0}{\partial r}\right|_{r=1} = 2 - \frac{\gamma-1}{2}\left.\frac{\partial v_0^2}{\partial r}\right|_{r=1}$ and $\left.\frac{\partial q_0}{\partial r}\right|_{r=1} = \lambda_*$ and define $\Psi_* = \left.\frac{\partial \Lambda}{\partial r}\right|_{r=1}$

$$\Rightarrow \left.\frac{\partial}{\partial r}\left[\frac{T_0}{r} - \frac{q_0 \Lambda(E_0)}{v_0}\right]\right|_{r=1} = -1 + \left(2 - \frac{\gamma-1}{2}\left.\frac{\partial v_0^2}{\partial r}\right|_{r=1}\right) - \lambda_* - \Psi_* + \frac{1}{2}\left.\frac{\partial v_0^2}{\partial r}\right|_{r=1} = 1 - \lambda_* - \Psi_* + \left(\frac{1}{2} - \frac{\gamma-1}{2}\right)\left.\frac{\partial v_0^2}{\partial r}\right|_{r=1}$$

$$\left.\frac{\partial}{\partial r}\left(v_0^2 - T_0\right)\right|_{r=1} = \left.\frac{\partial v_0^2}{\partial r}\right|_{r=1} - \frac{\partial T_0}{\partial r} = \left.\frac{\partial v_0^2}{\partial r}\right|_{r=1} - \left(2 - \frac{\gamma-1}{2}\left.\frac{\partial v_0^2}{\partial r}\right|_{r=1}\right) = \left(1 + \frac{\gamma-1}{2}\right)\left.\frac{\partial v_0^2}{\partial r}\right|_{r=1} - 2$$

$$\Rightarrow \left.\frac{\partial v_0^2}{\partial r}\right|_{r=1} = 4\left[1-\lambda_* - \Psi_* + \left(\frac{1}{2}-\frac{\gamma-1}{2}\right)\left.\frac{\partial v_0^2}{\partial r}\right|_{r=1}\right] \Big/ \left[\left(1+\frac{\gamma-1}{2}\right)\left.\frac{\partial v_0^2}{\partial r}\right|_{r=1} - 2\right]$$

introduce short hand $x = \left.\frac{\partial v_0^2}{\partial r}\right|_{r=1}$

$$\Leftrightarrow \quad x \cdot \left[\left(1+\frac{\gamma-1}{2}\right)x - 2\right] = 4\left[1-\lambda_* - \Psi_*\right] + (4-2\gamma)x$$

$$\Leftrightarrow \quad \frac{1}{2}(1+\gamma)x^2 + 2(\gamma-3)x + 4\left[\lambda_* + \Psi_* - 1\right] = 0$$

$$\Leftrightarrow \quad x^2 + 4\,\frac{\gamma-3}{\gamma+1}\,x + 8\,\frac{\lambda_*+\Psi_*-1}{\gamma+1} = 0$$

$$x = -2\,\frac{\gamma-3}{\gamma+1} \pm \sqrt{4\left(\frac{\gamma-3}{\gamma+1}\right)^2 - 8\,\frac{\lambda_*+\Psi_*-1}{\gamma+1}}$$

$$\Rightarrow \quad \boxed{\left.\frac{\partial v_0^2}{\partial r}\right|_{r=1} = x = \frac{2}{\gamma+1}\left[-(\gamma-3) \pm \sqrt{(\gamma-3)^2 - 2(\lambda_*+\Psi_*-1)(\gamma+1)}\right] \qquad (10)}$$

with $\Psi_* = \left.\frac{\partial \Lambda}{\partial r}\right|_{r=1} = \left.\frac{\partial E_0}{\partial r}\right|_{r=1} \cdot \left.\frac{\partial \Lambda}{\partial E}\right|_{r=1}$

$$\Rightarrow \quad \boxed{\Psi_* = 2\lambda_* L(E_*)\left.\frac{\partial \Lambda}{\partial E}\right|_{E=E_*} \qquad (11)}$$