Normalize asymmetric part

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System of equations for 1st order (radial part) (non-normalized)

$$\frac{\partial \rho_0}{\partial r} U_l + \rho_0 \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U_l) + \frac{\alpha_l}{r} V_l\right] + v_0 \frac{\partial R_l}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_0) R_l = 0 \qquad (I_1, b)$$

$$\rho_0 v_0 \frac{\partial U_l}{\partial r} + \rho_0 \frac{\partial v_0}{\partial r} U_l + v_0 \frac{\partial v_0}{\partial r} R_l = -\frac{\partial P_l}{\partial r} \qquad (II_1, b1)$$

$$\rho_0 v_0 \frac{\partial V_l}{\partial r} + \rho_0 \frac{v_0}{r} V_l = \frac{\beta_l}{r} P_l \qquad (II_1, b2)$$

$$\left[U_l \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U_l) + \frac{\alpha_l}{r} V_l\right]\left(\frac{1}{2}\rho_0 v_0^2 + \frac{\gamma}{\gamma-1}P_0\right) + \left[v_0 \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_0)\right]\left(\frac{1}{2} R_l v_0^2 + \rho_0 v_0 U_l + \frac{\gamma}{\gamma-1} P_l\right) = \mu \frac{\partial Q_l}{\partial r} \qquad (III_1, b)$$

$$\frac{\partial \varepsilon_l}{\partial r} = \frac{2}{m} L(E_0) R_l + \frac{2}{m}\rho_0 \left.\frac{\partial L}{\partial E}\right|_{E_0} \varepsilon_l \qquad (IV_1, b)$$

$$m \frac{\partial Q_l}{\partial r} = R_l q_0 \Lambda(E_0) + \rho_0 Q_l \Lambda(E_0) + \rho_0 q_0 \left.\frac{\partial \Lambda}{\partial E}\right|_{E_0} \varepsilon_l \qquad (V_1, b)$$

$$R_l = m\left(\frac{P_l}{T_0} - \frac{P_0}{T_0^2}\tau_l\right) \qquad (0, b) \qquad \left(\rho_0 = m\frac{P_0}{T_0}\right)$$

we choose $\boxed{\alpha_l = -l(l+1) \quad , \quad \beta_l = -1}$

Boundary conditions (non-normalized)

$\tau_l(r_p) = 0 \qquad U_l(r_p) = 0 \qquad V_l(r_p) = 0 \qquad Q_l(r_p) = 0 \qquad$ (maybe $\varepsilon_l(r_p) = 0$)

$P_l(\infty) = 0 \qquad R_l(\infty) = 0$? $\qquad Q_l(\infty) = \int_0^{\pi} q_1 X_l \, d\theta \qquad \varepsilon_l(\infty) = \int_0^{\pi} E_1 X_l \, d\theta$

Choice of normalization

The 0th order is normalized the way Parks does it. (to sonic radius)

For the rocket force, we only need to consider $l = 1$.

Suppose we know $Q_1(\infty)$, $q_0(\infty) = q_{bg}$, $E_1(\infty)$, $E_0(\infty) = E_{bg}$

Then we define the relative heat contribution associated to the asymmetry

as $q_r = \frac{Q_1(\infty)}{q_{bg}}$, $E_r = \frac{E_1(\infty)}{E_{bg}}$.

To get dimensionless quantities of order 1 we choose:

(quantities with a bar are non-normalized, with bar are normalized)

$P_1 = \tilde{P}_1 / p_* q_r \qquad \tau_1 = \tilde{\tau}_1 / T_* q_r \quad \rightarrow \quad R_1 = \tilde{R}_1 / \rho_* q_r$

$U_1 = \tilde{U}_1 / v_* q_r \qquad V_1 = \tilde{V}_1 / v_* q_r \qquad Q_1 = \tilde{Q}_1 / q_* q_r \qquad \varepsilon_1 = \tilde{\varepsilon}_1 / \varepsilon_* q_r$

$P_1 = \tilde{P}_1 / p_* q_r \qquad \tau_1 = \tilde{\tau}_1 / T_* q_r \quad \rightarrow \quad R_1 = \tilde{R}_1 / \rho_* q_r$

$U_1 = \tilde{U}_1 / v_* q_r \qquad V_1 = \tilde{V}_1 / v_* q_r \qquad Q_1 = \tilde{Q}_1 / q_* q_r \qquad \mathcal{E}_1 = \tilde{\mathcal{E}}_1 / \mathcal{E}_* q_r$

Also, don't forget $r = \tilde{r}/r_*$ and 0th order normalizations

Relations at the sonic radius

$$\lambda_* = \frac{p_*}{T_*} \Lambda_* r_* \qquad (\mathrm{I}_*)$$

$$r_*^2 \frac{p_*}{T_*} v_* = \frac{G}{4\pi} \qquad (\mathrm{II}_*)$$

$$v_* = \sqrt{\frac{\gamma T_*}{m}} \qquad (\mathrm{III}_*)$$

$$(\gamma - 1) \frac{4\pi}{G} \frac{\mu}{\gamma} \lambda_* \left(q_* \frac{r_*^2}{T_*} \right) = 2 \qquad (\mathrm{IV}_*)$$

$$\rho_* = m \frac{p_*}{T_*} \qquad (\mathrm{V}_*)$$

Derive normalized equations

$$(0,b): \quad \tilde{R}_l = m\left(\frac{\tilde{P}_l}{\tilde{T}_0} - \frac{\tilde{p}_0}{\tilde{T}_0^2} \tilde{\tau}_l \right) \iff q_r \rho_* R_l = m\left(q_r \frac{p_*}{T_*} \frac{P_l}{T_0} - \frac{p_*}{T_*^2} \frac{p_0}{T_0^2} q_r T_* \tau_l \right)$$

with $\rho_* = m \frac{p_*}{T_*} \Rightarrow$

$$\boxed{R_l = \left(\frac{P_l}{T_0} - \frac{p_0}{T_0^2} \tau_l \right) \qquad (0)}$$

$\rho_0 = \frac{p_0}{T_0}$

(From now on I drop all ~, the first eq. is always non-normalized)

$$(\mathrm{I}, b): \quad \frac{\partial \rho_0}{\partial r} U_l + \rho_0 \left[\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 U_l) + \frac{\alpha_l}{r} V_l \right] + v_0 \frac{\partial R_l}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 v_0) R_l = 0$$

each term gets a factor $\frac{\rho_*}{r_*} q_r v_*$ so we can just divide by it and the normalized equation looks the same

$$\iff \boxed{\frac{\partial \rho_0}{\partial r} U_l + \rho_0 \left[\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 U_l) + \frac{\alpha_l}{r} V_l \right] + v_0 \frac{\partial R_l}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 v_0) R_l = 0 \qquad (1)}$$

$$(\mathrm{II}, b1): \quad \rho_0 v_0 \frac{\partial U_l}{\partial r} + \rho_0 \frac{\partial v_0}{\partial r} U_l + v_0 \frac{\partial v_0}{\partial r} R_l = - \frac{\partial P_l}{\partial r}$$

left side gets a factor $\frac{\rho_* v_*^2 q_r}{r_*}$ right side gets $q_r \frac{p_*}{r_*}$

$$\frac{\rho_* v_*^2}{r_*} \overset{(\mathrm{III}_*)}{\underset{(\mathrm{V}_*)}{=}} m \frac{p_*}{T_*} \frac{\gamma T_*}{m} \frac{1}{r_*} = \gamma \frac{p_*}{r_*} \Rightarrow \text{cancels with right side except for } \gamma$$

$$\iff \boxed{\rho_0 v_0 \frac{\partial U_l}{\partial r} + \rho_0 \frac{\partial v_0}{\partial r} U_l + v_0 \frac{\partial v_0}{\partial r} R_l = - \frac{1}{\gamma} \frac{\partial P_l}{\partial r} \qquad (2)}$$

$$(\mathrm{II}, b2): \quad \rho_0 v_0 \frac{\partial V_l}{\partial r} + \rho_0 \frac{v_0}{r} V_l = \frac{\beta_l}{r} P_l$$

left side gets factor $\frac{\rho_* V_*^2 q_r}{r_*}$, right side: $\frac{p_* q_r}{r_*}$ $\Rightarrow$ cancels up to a γ factor again

$$\Leftrightarrow \boxed{\rho_0 V_0 \frac{\partial V_L}{\partial r} + \rho_0 \frac{V_0}{r} V_L = \frac{\beta_L}{r}\frac{1}{\gamma} P_L \qquad (3)}$$

$$(\mathrm{III}, b): \left[U_L \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U_L) + \frac{\alpha_L}{r} V_L\right]\left(\frac{1}{2}\rho_0 V_0^2 + \frac{\gamma}{\gamma-1}p_0\right) + \left[V_0 \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_0)\right]\left(\frac{1}{2} R_L V_0^2 + \rho_0 V_0 U_L + \frac{\gamma}{\gamma-1}P_L\right) = \mu \frac{\partial Q_L}{\partial r}$$

1st $[\dots]$ gets factor $\frac{V_* q_r}{r_*}$ 1st $(\dots)$ gets factor $\rho_* V_*^2$ and p_* 2nd $[\dots]$ gets factor $\frac{V_*}{r_*}$ 2nd $(\dots)$ gets factor $V_*^2 \rho_* q_r$ and $p_* q_r$ right side gets $\frac{q_* q_r}{r_*}$

with again $\rho_* V_*^2 = m \frac{p_*}{T_*}\frac{\gamma}{m} T_* = \gamma p_*$

so left side $\frac{V_* p_*}{r_*} q_r$ right side $\frac{q_*}{r_*} q_r$ $\big((\dots)$ gets γ in front of non-p terms$\big)$

$\rightarrow$ put everything on the right $\rightarrow$ factor $\frac{q_*}{p_* V_*}$

(I_*) into (IV_*) $\Rightarrow$ $\frac{\gamma-1}{\gamma}\mu \lambda_* q_* \left(\frac{1}{p_* V_*}\right) = 2 \Leftrightarrow \frac{q_*}{p_* V_*} = \frac{\gamma}{\gamma-1}\frac{2}{\mu\lambda_*}$

$$\Rightarrow \left[U_L \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U_L) + \frac{\alpha_L}{r} V_L\right]\left(\frac{\gamma}{2}\rho_0 V_0^2 + \frac{\gamma}{\gamma-1}p_0\right) + \left[V_0 \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_0)\right]\left(\frac{\gamma}{2} R_L V_0^2 + \gamma\rho_0 V_0 U_L + \frac{\gamma}{\gamma-1}P_L\right) = \frac{\gamma}{\gamma-1}\frac{2}{\mu\lambda_*}\mu\frac{\partial Q_L}{\partial r} \Big| \cdot \frac{\gamma-1}{\gamma}$$

$$\Leftrightarrow \boxed{\left[U_L \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U_L) + \frac{\alpha_L}{r} V_L\right]\left((\gamma-1)\frac{1}{2}\rho_0 V_0^2 + p_0\right) + \left[V_0 \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_0)\right]\left((\gamma-1)\frac{1}{2} R_L V_0^2 + (\gamma-1)\rho_0 V_0 U_L + P_L\right) = \frac{2}{\lambda_*}\frac{\partial Q_L}{\partial r} \qquad (4)}$$

$$(\mathrm{IV}, b): \frac{\partial \varepsilon_L}{\partial r} = \frac{2}{m} L(\varepsilon_0) R_L + \frac{2}{m}\rho_0 \frac{\partial L}{\partial E}\Big|_{\varepsilon_0} \varepsilon_L$$

factor on the left: $\frac{E_* q_r}{r_*}$ 1st on right $E_* \Lambda_* \rho_* q_r$ 2nd on right $\rho_* \frac{E_* \Lambda_*}{E_*} E_* q_r$

in total on the right: $r_* \rho_* \Lambda_* = m\lambda_*$

$$\Leftrightarrow \boxed{\frac{\partial \varepsilon_L}{\partial r} = 2\lambda_*\left(L(\varepsilon_0) R_L + \rho_0 \frac{\partial L}{\partial E}\Big|_{\varepsilon_0} \varepsilon_L\right) \qquad (6)}$$

$$(\mathrm{V}, b): m\frac{\partial Q_L}{\partial r} = R_L q_0 \Lambda(\varepsilon_0) + \rho_0 Q_L \Lambda(\varepsilon_0) + \rho_0 q_0 \frac{\partial \Lambda}{\partial E}\Big|_{\varepsilon_0} \varepsilon_L$$

left factor $\frac{q_* q_r}{r_*}$ right factor $\rho_* q_r q_* \Lambda_*$

in total on the right: $\rho_* r_* \Lambda_* = m\lambda_*$

$$\Rightarrow \boxed{\frac{\partial Q_L}{\partial r} = \lambda_*\left(R_L q_0 \Lambda(\varepsilon_0) + \rho_0 Q_L \Lambda(\varepsilon_0) + \rho_0 q_0 \frac{\partial \Lambda}{\partial E}\Big|_{\varepsilon_0} \varepsilon_L\right) \qquad (5)}$$

All equations are the same as in Oskars document.

Summary

Set of equations

$\alpha_l = -l(l+1)$, $\beta_l = -1$

$$R_l = \left(\frac{P_l}{T_0} - \frac{P_0}{T_0^2}\tau_l\right) \qquad (0)$$

$$\frac{\partial \rho_0}{\partial r} U_l + \rho_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U_l) + \frac{\alpha_l}{r} V_l\right)\Big] + v_0 \frac{\partial R_l}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_0) R_l = 0 \qquad (1)$$

$$\rho_0 v_0 \frac{\partial U_l}{\partial r} + \rho_0 \frac{\partial v_0}{\partial r} U_l + v_0 \frac{\partial v_0}{\partial r} R_l = -\frac{1}{\gamma}\frac{\partial P_l}{\partial r} \qquad (2)$$

$$\rho_0 v_0 \frac{\partial V_l}{\partial r} + \rho_0 \frac{v_0}{r} V_l = \frac{\beta_l}{r}\frac{1}{\gamma} P_l \qquad (3)$$

$$\left[U_l \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U_l) + \frac{\alpha_l}{r} V_l\right]\left((\gamma-1)\frac{1}{2}\rho_0 v_0^2 + P_0\right) + \left[v_0 \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_0)\right]\left((\gamma-1)\frac{1}{2} R_l v_0^2 + (\gamma-1)\rho_0 v_0 U_l + P_l\right) = \frac{2}{\lambda_*}\frac{\partial Q_l}{\partial r} \qquad (4)$$

$$\frac{\partial Q_l}{\partial r} = \lambda_*\left(R_l q_0 \Lambda(\mathcal{E}_0) + \rho_0 Q_l \Lambda(\mathcal{E}_0) + \rho_0 q_0 \frac{\partial \Lambda}{\partial \mathcal{E}}\Big|_{\mathcal{E}_0} \mathcal{E}_l\right) \qquad (5)$$

$$\frac{\partial \mathcal{E}_l}{\partial r} = 2\lambda_*\left(L(\mathcal{E}_0) R_l + \rho_0 \frac{\partial L}{\partial \mathcal{E}}\Big|_{\mathcal{E}_0} \mathcal{E}_l\right) \qquad (6)$$

Boundary conditions

$T_1(r_p) = 0$ $\quad U_1(r_p) = 0$ $\quad V_1(r_p) = 0$ $\quad Q_1(r_p) = 0$ $\quad \left(\mathcal{E}_1(r_p) = 0 \ ?\right)$

$P_1(\infty) = 0$ $\quad R_1(\infty) = 0$ $\quad Q_1(\infty) = q_0(\infty)$ $\quad \mathcal{E}_1(\infty) = \mathcal{E}_0(\infty)\frac{\mathcal{E}_{rel}}{q_{rel}}$