Derivation of the rocket force resulting from a spherical pellet ablation with a perturbative asymmetry

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In this document we derive the rocket force resulting from a spherical pellet ablation with a perturbative asymmetry. We restrict ourselves (to begin with, at least) to a simplified case where the heat source due to the heat flux from the background plasma is approximated to be spherically symmetric and only has a pertubatively small angular dependence. We also restrict ourselves to hydrogen pellets, for which the sublimation energy per particle is very small ($\sim 0.01\,\mathrm{eV}$). This will cause the ablation to such that both the temperature and the heat flux vanishes at the pellet surface, and the ablation rate is thus determined solely by the dynamics of the flow of the ablated material away from the pellet.

1 Neutral flow dynamics for a given heat flux asymmetry

Here we treat only the flow in the form of a neutral gas in the close proximity to the pellet, which takes the form of a nearly spherically symmetric expansion; the heat flux and electron energy reaching the neutral zone through the ionized plasmoid is treaded as a boundary condition. The dynamics of interest are then governed by the conservation equations of mass, momentum and energy, as well as the equation of state for an ideal gas:

$$\nabla\cdot(\rho\vec{v}) = 0 \tag{1}$$

$$\rho\vec{v}\cdot\nabla\vec{v} = -\nabla p \tag{2}$$

$$\nabla\cdot\left[\left(\frac{\rho v^2}{2}+\frac{\gamma p}{\gamma-1}\right)\vec{v}\right] = -f_{\mathrm{heat}}\nabla\cdot\vec{q} = -f_{\mathrm{heat}}\frac{dq}{dx} \tag{3}$$

$$p = \frac{\rho T}{m}, \tag{4}$$

where ρ is the mass density, $\vec{v}$ is the flow velocity, p is the pressure, m is the mass per ablated particle, q is the heat flux of electrons from the background plasma flowing along the magnetic field lines in the x-direction, f_{heat} is the fraction of the absorbed heat flux that goes into heating the cloud ($f_{\mathrm{heat}} \approx 0.65$ according to [1]) and r is the radial coordinate.

Several different models for the heat flux absorbtion exist in the literature, adapted for various situations and with varying degree of sophistication, but a rather simple model is the one used by [1]. In this model the incoming electrons are reduced to a monoenergetic beam with initial energy $E_{\mathrm{bg}} = 2T_{\mathrm{bg}}$ equal to the ratio of the unidirectional heat and particle fluxes for a Maxwellian distribution with initial temperature T_{bg}. The attenuation of the heat flux is then

$$\frac{dq}{dx} = -\frac{q}{\lambda_{\mathrm{mfp}}(E)} = -\frac{\rho}{m}q\Lambda(E) \tag{5}$$

$$\frac{dE}{dx} = -2\frac{\rho}{m}L(E), \tag{6}$$

where λ_{mfp} is the mean free path.

A reasonable way to implement the approximation that the magnitude of the heat source is almost spherically symmetric is to approximate all paths taken by the heat flux through the cloud with the purely radial path that leads to the same point, corresponding to replacing dq/dx with $-dq/dr$ (does this agree with the reasoning in [1]?). Note that the attenuation of the heat flux, and thus the corresponding heat source, only depends on the length of the path taken through the cloud, i.e. does not depend on the direction of the heat flux directly, so that this approximation only affects the results through an order unity modification to the heat source via a modification to the path length. If the heat flux is homogeneous in the cross section perpendicular to the field lines, this approximation will lead to a spherically symmetric heat source, so that any asymmetry can only come from an asymmetry in the incoming heat flux.

The effective heat flux cross section Λ and electron energy loss function L can be approximated by semi-empirical expressions according to

$$\Lambda(E) = \hat{\sigma}_T(E) + \frac{2L(E)}{E} \tag{7}$$

$$L(E) = 8.62 \cdot 10^{-15} \cdot \left[\left(\frac{E}{100}\right)^{0.823} + \left(\frac{E}{60}\right)^{-0.125} + \left(\frac{E}{48}\right)^{-1.94} \right]^{-1} \text{eV} \cdot \text{cm}^2 \tag{8}$$

$$\hat{\sigma}_T(E > 100\,\text{eV}) = \frac{8.8 \cdot 10^{-13}}{E^{1.71}} - \frac{1.62 \cdot 10^{-12}}{E^{1.932}}\,\text{cm}^2 \tag{9}$$

$$\hat{\sigma}_T(E < 100\,\text{eV}) = \frac{1.1 \cdot 10^{-14}}{E}\,\text{cm}^2, \tag{10}$$

where $\hat{\sigma}_T$ is the total elastic scattering cross section.

As the pressure inside the neutral cloud is several orders of magnitude larger than in the background plasma, the latter can be considered negligible, and thus the pressure must vanish at large radii. Moreover, the heat flux and effective incoming electron energy must approach those of the background plasma at large radii. At the pellet surface, the ablated material has not yet gained any heat or flow velocity, so both the temperature and flow velocity must go to zero at the pellet surface. Finally, the very small sublimation energy means that the heat flux reaching the pellet surface will be negligible compared to the heat flux being absorbed or scattered in the neutral cloud, and the heat flux at the pellet surface can thus be taken as zero. In summary, the boundary conditions are

$$p(r \to \infty) = 0, \quad q(r \to \infty) = q_{\text{bg}}, \quad E(r \to \infty) = E_{\text{bg}} \tag{11}$$

$$q(r_p) = 0, \quad T(r_p) = 0, \tag{12}$$

where q_{bg} is the initial incoming heat flux and r_p is the pellet radius.

1.1 Spherically symmetric solution

We now turn to the spherically symmetric solution. As the boundary conditions are known for the pressure and temperature but not the density, we use equation (4) to eliminate the density from the equation system and express it in terms of the pressure and temperature alone. Assuming spherical symmetry, the equation system can be written as

$$\frac{d}{dr}\left(r^2 v_0 \frac{p_0}{T_0}\right) = 0 \Rightarrow r^2 v_0 \frac{p_0}{T_0} = \text{const} = \frac{G}{4\pi} \tag{13}$$

$$m\frac{p_0}{T_0} v \frac{dv_0}{dr} + \frac{dp_0}{dr} = 0 \tag{14}$$

$$\frac{G}{4\pi r^2}\frac{d}{dr}\left(\frac{\gamma T_0}{\gamma - 1} + m\frac{v_0^2}{2}\right) = f_{\text{heat}}\frac{dq_0}{dr} \tag{15}$$

$$\frac{dq_0}{dr} = \frac{p_0}{T_0} q_0 \Lambda(E_0) \tag{16}$$

$$\frac{dE_0}{dr} = 2\frac{p_0}{T_0} L(E_0), \tag{17}$$

where G is the particle ablation rate [m^{-3}] (note that [1] considers the mass ablation rate instead, which introduces some factors m, denoting the mass per neutral particle, but here we consider the particle ablation rate for ease of notation).

The solution to this equation system is outlined in [1]. It turns out to be useful to normalize the quantities of interest in terms of their values at the sonic radius (i.e. the radius where the flow becomes supersonic),

$$\begin{aligned}&\bar{p}_0 = p_0/p_*, \quad \bar{T}_0 = T_0/T_*, \quad \bar{v}_0 = v_0/v*, \quad \bar{q}_0 = q_0/q_*, \quad \bar{E}_0 = E_0/E_* \\ &\bar{r} = r/r_*, \quad \bar{\Lambda} = \Lambda(E_0)/\Lambda(E_*) \quad \bar{L} = L(E_0)/[E_*\Lambda(E_*)],\end{aligned} \tag{18}$$

and introduce the new variable $\bar{w}_0 = \bar{v}_0^2$. After some algebra, utilising the relations between the quantities of interest at the sonic radius, [1] derives the following equation system (dropping the bar for brewity)

$$\frac{dw}{dr} = \frac{4wT}{(T-w)r}\left(\frac{q\Lambda r}{T\sqrt{w}} - 1\right) \tag{19}$$

$$\frac{dT}{dr} = \frac{2\Lambda q}{\sqrt{w}} - \frac{1}{2}(\gamma-1)\frac{dw}{dr} \tag{20}$$

$$\frac{dE}{dr} = 2\lambda_* \frac{L}{r^2\sqrt{w}} \tag{21}$$

$$\frac{dq}{dr} = \lambda_* \frac{q\Lambda}{\sqrt{w}r^2}, \tag{22}$$

where $\lambda_* = r_*\Lambda_* p_*/T_*$ is a dimensionless eigenvalue that uniquely determines the solution to the normalised equation system. The equation for dw/dr has an apparent singularity at the sonic radius, but dw/dr can still be determined by L'Hopitals rule, yielding

$$\left.\frac{dw}{dr}\right|_{r=1} = \frac{\frac{1}{2}(3-\gamma) + \{[\frac{1}{2}(3-\gamma)]^2 - \frac{1}{2}(\gamma+1)(\lambda_* + \psi_* - 1)\}^{1/2}}{(\gamma+1)/4}, \tag{23}$$

' where

$$\psi_* = \left.\frac{d\Lambda}{dr}\right|_{r=1} = \frac{2\lambda_* L}{\Lambda^2}\left.\frac{d\Lambda}{dE_0}\right|_{E_0=E_*}. \tag{24}$$

It is then possible to start integrating the normalized equation system numerically (suggestively using some standard ODE-solver from scipy.integrate.ode) from the sonic radius (where, with the present normalisation, all quantities of interest are equal to 1) towards smaller radii, for some assumed value of λ_*. The value of λ_* can then be varied until one finds a value such that the boundary conditions for q and T are simultaneously satisfied. The radius at which this happens is then interpreted as the normalised pellet radius (which is not known a-priori). When the appropriate value of λ_* is found, one can integrate from the sonic radius towards larger radii, and find what values of the normalised heat flux $q_{\rm bg}$ and effective beam energy $E_{\rm bg}$ the found solution corresponds to (these values are also not known a-priori).

As the normalised effective beam energy and heat flux attenuation depends on the dimension-full effective beam energy, λ_* will formally also be dependent on this quantity (or, rather, on E_*, which is the value one has to choose before solving the equation system). It is however found by [1] that λ_* is a very weak function of E_*, and it might therefore be sufficient to find the spherically symmetric solution only for a representative value of E_* of a few tens of keV (where $\lambda_* \approx 0.93$ according to [1]).

1.2 First order angular asymmetry correction

1.2.1 Linearized equations

We write the quantities of interest as a spherically symmetric part plus a perturbatively small angularly dependent correction,

$$p = p_0(r) + p_1(r,\theta), \tag{25}$$

$$\vec{v} = v_0(r)\hat{r} + u_1(r,\theta)\hat{r} + v_1(r,\theta)\hat{\theta}, \tag{26}$$

$$T = T_0(r) + T_1(r,\theta), \tag{27}$$

$$q = q_0(r) + q_1(r,\theta), \tag{28}$$

$$E = E_0(r) + E_1(r,\theta), \tag{29}$$

keeping the approximation that the heat flux is completely radial. When inserted into equations (1)-(4), the equation system for the angularly dependent part, keeping only first order terms, becomes

$$\nabla \cdot \left[\frac{p_0}{T_0} \vec{v}_1 + \left(\frac{p_1}{T_0} - \frac{p_0}{T_0^2} T_1 \right) \vec{v}_0 \right] = 0 \tag{30}$$

$$\frac{p_0}{T_{0\,0}} (\vec{v}_1 \cdot \nabla \vec{v}_0 + \vec{v}_0 \cdot \nabla \vec{v}_1) + \left(\frac{p_1}{T_0} - \frac{p_0}{T_0^2} T_1 \right) \vec{v}_0 \cdot \nabla \vec{v}_0 = \frac{-\nabla p_1}{m} \tag{31}$$

$$\nabla \cdot \left\{ \left(m \frac{p_0 v_0^2}{2T_0} + \frac{\gamma p_0}{\gamma - 1} \right) \vec{v}_1 + \left[m \left(\frac{p_1}{T_0} - \frac{p_0}{T_0^2} T_1 \right) \frac{v_0^2}{2} + m \frac{p_0}{T_0} \vec{v}_0 \cdot \vec{v}_1 + \frac{\gamma p_1}{\gamma - 1} \right] \vec{v}_0 \right\} = f_{\text{heat}} \frac{\partial q_1}{\partial r} \tag{32}$$

$$\frac{\partial q_1}{\partial r} = \left(\frac{p_1}{T_0} - \frac{p_0}{T_0^2} T_1 \right) q_0 \Lambda(E_0) + \frac{p_0}{T_0} q_0 \Lambda'(E_0) E_1 + \frac{p_0}{T_0} q_1 \Lambda(E_0) \tag{33}$$

$$\frac{\partial E_1}{\partial r} = 2 \left(\frac{p_1}{T_0} - \frac{p_0}{T_0^2} T_1 \right) L(E_0) + 2 \frac{p_0}{T_0} L'(E_0) E_1. \tag{34}$$

Let us now expand the angular dependence of all quantities except for v_1 in terms of basis functions $X_l(\theta)$, and expand the angular dependence of v_1 in terms of some other basis functions $Y_l(\theta)$. By inspecting the equation system and noting that

$$\nabla \cdot v_1 \hat{\theta} = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta v_1), \tag{35}$$

$$\nabla p \cdot \hat{\theta} = \frac{1}{r} \frac{\partial p}{\partial \theta}, \tag{36}$$

$$\hat{r} \cdot \nabla \vec{v}_1 = \frac{\partial v_1}{\partial r} \hat{\theta} \tag{37}$$

one can see that the angular dependence can be factored out in each equation if these basis functions satisfy

$$\frac{1}{\sin\theta} \frac{\partial}{\theta} (\sin\theta Y_l) \propto X_l \tag{38}$$

$$Y_l \propto \frac{\partial X_l}{\partial \theta}. \tag{39}$$

By inserting (39) into (38), we obtain the defining equation for the zonal harmonics, so that $X_l = \mathcal{P}_l^0(\cos\theta)$, where $\mathcal{P}_l^0$ is the zeroth Legendre polynomial of order l. We therefore make the following expansions:

$$p_1 = \sum_{l=1}^{\infty} P_l(r) \mathcal{P}_l^0(\cos\theta), \tag{40}$$

$$T_1 = \sum_{l=1}^{\infty} \mathcal{T}_l(r) \mathcal{P}_l^0(\cos\theta), \tag{41}$$

$$u_1 = \sum_{l=1}^{\infty} U_l(r) \mathcal{P}_l^0(\cos\theta), \tag{42}$$

$$v_1 = \sum_{l=1}^{\infty} V_l(r) \frac{d\mathcal{P}_l^0(\cos\theta)}{d\theta}, \tag{43}$$

$$q_1 = \sum_{l=1}^{\infty} Q_l(r) \mathcal{P}_l^0(\cos\theta), \tag{44}$$

$$E_1 = \sum_{l=1}^{\infty} \mathcal{E}_l(r) \mathcal{P}_l^0(\cos\theta) \tag{45}$$

(the zeroth mode is constant with respect to angle and is therefore included in the zeroth order solution). Inserting the above, factoring out the angular dependence of each equation and noting that the modes are linearly independend, we obtain the following equation sytem for a given mode l:

$$\frac{\partial}{\partial r}\left(\frac{p_0}{T_0}\right)U_l+\frac{p_0}{T_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2U_l\right)+v_0\frac{\partial}{\partial r}\left(\frac{P_l}{T_0}-\frac{p_0}{T_0^2}\mathcal{T}_l\right)+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2v_0\right)\left(\frac{P_l}{T_0}-\frac{p_0}{T_0^2}\mathcal{T}_l\right)-$$
$$l(l+1)\frac{p_0}{rT_0}V_l=0 \tag{46}$$
$$\frac{p_0}{T_0}U_lv_0'+\frac{p_0}{T_0}v_0U_l'+\left(\frac{P_l}{T_0}-\frac{p_0}{T_0^2}\mathcal{T}_l\right)v_0v_0'+\frac{P_l'}{m}=0 \tag{47}$$
$$\frac{p_0}{T_0}\frac{v_0}{r}V_l+\frac{p_0}{T_0}v_0V_l'+\frac{P}{mr}=0 \tag{48}$$
$$\left(m\frac{p_0v_0^2}{2T_0}+\frac{\gamma p_0}{\gamma-1}\right)\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2U_l\right)-l(l+1)\frac{V_l}{r}\right]+U_l\frac{\partial}{\partial r}\left(m\frac{p_0v_0^2}{2T_0}+\frac{\gamma p_0}{\gamma-1}\right)+$$
$$\left[m\left(\frac{P_l}{T_0}-\frac{p_0}{T_0^2}\mathcal{T}_l\right)\frac{v_0^2}{2}+m\frac{p_0}{T_0}v_0U_l+\frac{\gamma P_l}{\gamma-1}\right]\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2v_0\right)+$$
$$v_0\frac{\partial}{\partial r}\left[m\left(\frac{P_l}{T_0}-\frac{p_0}{T_0^2}\mathcal{T}_l\right)\frac{v_0^2}{2}+m\frac{p_0}{T_0}v_0U_l+\frac{\gamma P_l}{\gamma-1}\right]=f_{\mathrm{heat}}\frac{\partial Q_l}{\partial r} \tag{49}$$
$$\frac{\partial Q_l}{\partial r}=\left(\frac{P_l}{T_0}-\frac{p_0}{T_0^2}\mathcal{T}_l\right)q_0\Lambda(E_0)+\frac{p_0}{T_0}q_0\frac{d\Lambda}{dE}\Big|_{E=E_0}\mathcal{E}_l+\frac{p_0}{T_0}\Lambda(E_0)Q_l \tag{50}$$
$$\frac{\partial\mathcal{E}_l}{\partial r}=2\left(\frac{P_l}{T_0}-\frac{p_0}{T_0^2}\mathcal{T}_l\right)L(E_0)+2\frac{p_0}{T_0}\frac{dL}{dE}\Big|_{E=E_0}\mathcal{E}_l, \tag{51}$$

with boundary conditions

$$\mathcal{T}_l(r_p)=0,\quad Q_l(r_p)=0,$$
$$P_l(\infty)=0,\quad Q_l(\infty)=\frac{\int_0^\pi q_1X_l\sin\theta d\theta}{\int_0^\pi X_l^2\sin\theta d\theta},\quad \mathcal{E}_l(\infty)=\frac{\int_0^\pi E_1X_l\sin\theta d\theta}{\int_0^\pi X_l^2\sin\theta d\theta}. \tag{52}$$

It might appear as if these boundary conditions are not enough to uniquely determine the solution to the equation system, as there are 6 unknown quantities and only 5 boundary conditions. However, as shown in Nicos' notes (appendix A), when writing this equation system in a matrix form (using sympy), one finds an apparent singularity at the sonic radius similarly as in the zeroth order system. The requirement that this singularity should vanish imposes a relation between the unknown quantities at the sonic radius, which provides the missing constraint that uniquely determines the solution. Based on the zeroth order solution one would expect that at least $U_l(r_p)=0$, if there appears to be any redundancy in the above conditions one could perhaps enforce that as well?

1.2.2 Resulting rocket force

1.2.3 Outline of numerical solution

Let us introduce the relative contribution q_{rel} to the incomming heat flux asymmetry from the first harmonic compared to the symmetric heat flux, and the corresponding contribution E_{rel} for the energy asymmetry, i.e. $Q_1(\infty)=q_{\mathrm{rel}}q_{\mathrm{bg}}$, $\mathcal{E}_1(\infty)=E_{\mathrm{rel}}E_{\mathrm{bg}}$. With these quantities defined, it turns out to be convenient to normalise the first order parameters in terms of the zeroth order solution values at the sonic radius and q_{rel},

$$\bar{P}_1=P_1/(p_*q_{\mathrm{rel}}),\quad \bar{\mathcal{T}}_1=\mathcal{T}_1/(T_*q_{\mathrm{rel}}),\quad \bar{U}_1=U_1/(v_*q_{\mathrm{rel}}),$$
$$\bar{V}_1=V_1/(v_*q_{\mathrm{rel}}),\quad \bar{Q}_1=Q_1/(q_*q_{\mathrm{rel}}),\quad \bar{\mathcal{E}}_1=\mathcal{E}_1/(E_*q_{\mathrm{rel}}), \tag{53}$$

while normalising the radial coordinate and the zeroth order solutions as in equation (18). Noting that $v_*=\sqrt{\gamma T_*/m}$, the equation system to be solved becomes (dropping the bar for brewity and considering now only the first zonal harmonic)

$$\frac{\partial}{\partial r}\left(\frac{p_0}{T_0}\right)U_1+\frac{p_0}{T_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2U_1\right)+v_0\frac{\partial}{\partial r}\left(\frac{P_1}{T_0}-\frac{p_0}{T_0^2}\mathcal{T}_1\right)+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2v_0\right)\left(\frac{P_1}{T_0}-\frac{p_0}{T_0^2}\mathcal{T}_1\right)-$$
$$2\frac{p_0}{rT_0}V_1=0 \tag{54}$$
$$\frac{p_0}{T_0}U_1v_0'+\frac{p_0}{T_0}v_0U_1'+\left(\frac{P_1}{T_0}-\frac{p_0}{T_0^2}\mathcal{T}_1\right)v_0v_0'+\frac{P_1'}{\gamma}=0 \tag{55}$$
$$\frac{p_0}{T_0}\frac{v_0}{r}V_1+\frac{p_0}{T_0}v_0V_1'+\frac{P}{\gamma r}=0 \tag{56}$$
$$\left(\frac{p_0v_0^2}{2T_0}+\frac{p_0}{\gamma-1}\right)\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2U_1\right)-2\frac{V_1}{r}\right]+U_1\frac{\partial}{\partial r}\left(\frac{p_0v_0^2}{2T_0}+\frac{p_0}{\gamma-1}\right)+$$
$$\left[\left(\frac{P_1}{T_0}-\frac{p_0}{T_0^2}\mathcal{T}_1\right)\frac{v_0^2}{2}+\frac{p_0}{T_0}v_0U_1+\frac{P_1}{\gamma-1}\right]\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2v_0\right)+$$
$$v_0\frac{\partial}{\partial r}\left[\left(\frac{P_1}{T_0}-\frac{p_0}{T_0^2}\mathcal{T}_1\right)\frac{v_0^2}{2}+\frac{p_0}{T_0}v_0U_1+\frac{P_1}{\gamma-1}\right]=\frac{2}{(\gamma-1)\lambda_*}\frac{\partial Q_1}{\partial r} \tag{57}$$
$$\frac{\partial Q_1}{\partial r}=\lambda_*\left[\left(\frac{P_1}{T_0}-\frac{p_0}{T_0^2}\mathcal{T}_1\right)q_0\Lambda(E_0)+\frac{p_0}{T_0}q_0\frac{d\Lambda}{dE}\Big|_{E=E_0}\mathcal{E}_1+\frac{p_0}{T_0}\Lambda(E_0)Q_1\right] \tag{58}$$
$$\frac{\partial \mathcal{E}_1}{\partial r}=2\lambda_*\left[\left(\frac{P_1}{T_0}-\frac{p_0}{T_0^2}\mathcal{T}_1\right)L(E_0)+2\frac{p_0}{T_0}\frac{dL}{dE}\Big|_{E=E_0}\mathcal{E}_1\right], \tag{59}$$

where we used equations (13), (15) and (20) in [1] to find the relation

$$\frac{f_{\text{heat}}q_*}{\gamma p_*v_*}=\frac{2}{(\gamma-1)\lambda_*}. \tag{60}$$

The boundary conditions become

$$\begin{aligned}&\mathcal{T}_1(r_p)=0,\quad Q_1(r_p)=0,\\ &P_1(\infty)=0,\quad Q_1(\infty)=q_0(\infty),\quad \mathcal{E}_1(\infty)=E_0(\infty)E_{\text{rel}}/q_{\text{rel}}.\end{aligned} \tag{61}$$

The only parameters which can alter the solution to this normalised equation system are E_* which, according to [1], determines the eigenvalue λ_* and the normalized zeroth order solution, and the ratio of the incoming relative effective beam energy and heat flux asymmetry $E_{\text{rel}}/q_{\text{rel}}$ (the normalized boundary condition for the heat flux is fixed to the zeroth order solution and the other boundary conditions are zero).

It might appear problematic here that U_1 and V_1 may not go to zero at the pellet surface, while we know that v_0 does, making the linearisation invalid close to the pellet surface. However, as the normalised U_1 and V_1 are independent on q_{rel}, the non-normalised ratio of the first and zeroth order velocities will be proportional to q_{rel}. Thus, this ratio can be made small arbitrarily close to the pellet surface by making q_{rel} smaller, so that the linearisation is valid arbitrarily close to the pellet surface in the limit where $q_{\text{rel}}\to 0$.

We can then express the rocket force as

$$F=\frac{4\pi r_p^2}{3}p_*q_{\text{rel}}\left[\gamma v_0^2\left(\frac{P_1}{T_0}-\frac{p_0}{T_0^2}\mathcal{T}_1\right)+P_1(r_p)+\frac{2\gamma}{r^2}(U_1+V_1)\right]\Big|_{r\to r_p}=\frac{4\pi r_p^2}{3}p_*q_{\text{rel}}f(E_*,E_{\text{rel}}/q_{\text{rel}}) \tag{62}$$

where p_* can be determined from the spherically symmetric solution according to equation (30) in [1], and the dimensionless function $f(E_*,E_{\text{rel}}/q_{\text{rel}})$ is left to be determined numerically. The dependence on E_* for λ_* and the zeroth order solution is however, as mentioned above in 1.1, found in [1] to be very weak, and it might therefore be sufficiently accurate to calculate them only once for a representative E_* and then disregard the dependence of E_* in f.

Perhaps the first term in f is zero? That would correspond to saying that the kinetic energy density is zero at the pellet surface, which seems to be true for the spherically symmetric solution judging from the figures in [1]; p_0 is finite and T_0 apporaches zero slower than $w_0=v_0^2$, so that $v_0^2/T_0\to 0$.

The dependence of f on $E_{\rm rel}/q_{\rm rel}$ can be determined by mapping values of $E_{\rm rel}/q_{\rm rel}$ to the corresponding relevant quantities at the pellet surface. Solving the equation system is however complicated by the fact that some of the derivatives and coefficients (based on the zeroth order solution) diverge at both the pellet surface and at infinity. These points are therefore not suitable as starting points for the integration, even if those are the only points where the quantities are known a priori (due to the boundary conditions). Moreover, one also needs to enforce the relation between the quantities at the sonic radius required for the apparent singularity there to vanish, which is difficult if the sonic radius is not used as the starting point for the integration. However, while the apparent singularity imposes a condition that eliminates one quantity at the sonic radius (see Nicos' notes (appendix A)), the values of the remaining five are not known there. These remaining starting values at the sonic radius therefore have to be treated as free parameters, to be optimised for numerically so that the boundary conditions are satisfied. One can then choose to either set an a priori value for $E_{\rm rel}/q_{\rm rel}$ to solve for, or to not impose any boundary condition on $\mathcal{E}_1$ and instead find the 1-dimensional space of starting values at the sonic rdius which satisfy the other boundary conditions, and then map the corresponding values of $E_{\rm rel}/q_{\rm rel}$ and f that results from this calculation.

A straightforward way of integrating the equation system is to write it in a matrix form as

$$
\begin{bmatrix} P_1' \\ \mathcal{T}_1' \\ U_1' \\ V_1' \\ Q_1' \\ \mathcal{E}_1' \end{bmatrix} =
$$
$$
\begin{bmatrix}
-\frac{v_0}{T_0}, & \frac{v_0 p_0}{T_0^2}, & -\frac{p_0}{T_0}, & 0, & 0, & 0 \\
-\frac{1}{\gamma}, & 0, & -\frac{p_0 v_0}{T_0}, & 0, & 0, & 0 \\
0, & 0, & 0, & -\frac{p_0 v_0}{T_0}, & 0, & 0 \\
-\frac{v_0^3}{2T_0} - \frac{v_0}{\gamma-1}, & \frac{v_0^3 p_0}{2T_0^2}, & -k - \frac{v_0^2 p_0}{T_0}, & 0, & \frac{2}{(\gamma-1)\lambda_*}, & 0 \\
0, & 0, & 0, & 0, & 1, & 0 \\
0, & 0, & 0, & 0, & 0, & 1
\end{bmatrix}^{-1}
$$
$$
\begin{bmatrix}
v_0\left(\frac{1}{T_0}\right)' + \frac{1}{T_0}\nabla\cdot\vec{v_0}, & -v_0\left(\frac{p_0}{T_0^2}\right)' - \frac{p_0}{T_0^2}\nabla\cdot\vec{v_0}, & \left(\frac{p_0}{T_0}\right)' + \frac{2p_0}{rT_0}, & -\frac{2p_0}{rT_0}, & 0, & 0 \\
\frac{v_0 v_0'}{T_0}, & -\frac{p_0 v_0 v_0'}{T_0^2}, & \frac{p_0}{T_0} v_0', & 0, & 0, & 0 \\
\frac{1}{\gamma r}, & 0, & 0, & \frac{p_0 v_0}{T_0}, & 0, & 0 \\
\frac{v_0^2}{2T_0}\nabla\cdot\vec{v_0} + \frac{v_0}{2}\left(\frac{v_0^2}{T_0}\right)', & -\frac{p_0 v_0^2}{2T_0^2}\nabla\cdot\vec{v_0} - \frac{v_0}{2}\left(\frac{v_0^2 p_0}{T_0}\right)', & \frac{2}{r}k + k' + \nabla\cdot\vec{v_0}\frac{p_0 v_0}{T_0} + \left(\frac{p_0 v_0}{T_0}\right)' v_0, & -\frac{2}{r}k, & 0, & 0 \\
\frac{\lambda_* q_0 \Lambda_0}{T_0}, & -\frac{\lambda_* q_0 \Lambda_0 p_0}{T_0^2}, & 0, & 0, & \frac{p_0}{T_0}\Lambda_0, & \frac{p_0}{T_0} q_0 \left.\frac{d\Lambda}{dE}\right|_{E=E_0} \\
2\frac{\lambda_* q_0 L_0}{T_0}, & -2\frac{\lambda_* q_0 L_0 p_0}{T_0^2}, & 0, & 0, & 0, & 2\frac{p_0}{T_0}\left.\frac{dL}{dE}\right|_{E=E_0}
\end{bmatrix}
$$
$$
\begin{bmatrix} P_1 \\ \mathcal{T}_1 \\ U_1 \\ V_1 \\ Q_1 \\ \mathcal{E}_1 \end{bmatrix}, \tag{63}
$$

where

$$
k = \left(\frac{p_0 v_0^2}{2T_0} + \frac{p_0}{\gamma - 1}\right). \tag{64}
$$

This can be given as input to some standard ODE solver similarly to the spherically symmetric solution. The inversion of the first matrix should not be a problem to perform numerically in case one does not want to bother doing it analytically. However, as shown in Nicos' notes (appendix A) it is useful to calculate it analytically using sympy (or any other symbolic calculation tool, as the expressions become very large), as this reveals the apparent singularity at the sonic radius and the relation between the quantities there required for this singularity to vanish.

2 Determining the asymmetry

2.1 Variation in shielding length due to plasmoid drift

The shielding length through the cold but ionized plasma surrounding the pellet will vary over the pellet along the major radius direction, as the ∇B-induced charge separation gives rise to an $e \times B$-drift in this direction. The drift velocity is determined by the poloidal electric field being established inside the cloud due to the charge separation, which is regulated by the currents arising in response to this electric field which counteract further charge separation. During the initial acceleration phase of the drift motion of interest here[1], the ∇B-current is balanced by the polarisation current (related to the time variation of the electric field) alone, resulting in a constant acceleration given by equation (3.1) in [2]:

$$\dot{v}_0 = \frac{2(1+\langle z\rangle)}{\langle m_i\rangle R_\mathrm{m}}\left(T_0 - \frac{n_\mathrm{bg}}{n_0}T_\mathrm{bg}\right) = \frac{2(1+\langle z\rangle)c_\mathrm{s0}^2}{(\gamma_\mathrm{e}\langle z\rangle+\gamma_\mathrm{i})R_\mathrm{m}}\left(1-\frac{n_\mathrm{bg}T_\mathrm{bg}}{n_0T_0}\right). \tag{65}$$

Here $\langle z\rangle$ denotes the average charge, $\langle m_i\rangle$ the average ion mass (the electron mass is neglected), T_0 the temperature (assumed to be the same for electrons and ions), c_s0 the sound speed, and n_0 the particle density (sum of ions and electrons) inside the cloud. The major radius is denoted R_m, and n_bg and T_bg denote the backgrund plasma density and temperature. γ_e and γ_i are the adiabatic indices for electrons and ions, respectively.

In addition to accelerating along the major radius direction, the cloud also expands at the speed of sound c_s0 inside the cloud along the magnetic field lines (z-direction). The boundary of the plasma cloud, illustrated in figure 1, can therefore be parameterised by the time t which has elapsed since a given parcel of material was ablated according to

$$R = -\Delta y + v_0 t + \dot{v}_0\frac{t^2}{2} \tag{66}$$

$$z = c_\mathrm{s0}t. \tag{67}$$

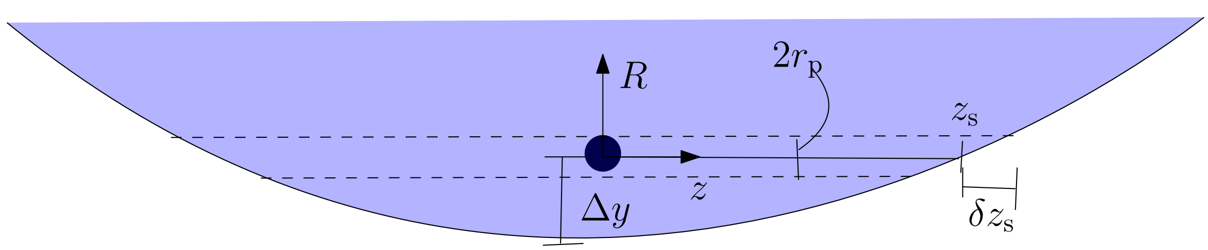

Figure 1: Illustration of the ionized cloud drifting relative to the pellet, resulting in an asymmetry in the shielding length. The magnetic field lines are stretched out to align with the z direction.

The initial velocity $v_0 = v_\mathrm{p}$ (pellet velocity) as the ablated material, after being ionized and attached to the field lines, travel with a velocity v_p in the positive R-direction relative to the pellet. We can therefore express the z-coordinate of the boundary as

$$z = -\frac{c_\mathrm{s0}v_\mathrm{p}}{\dot{v}_0} + \sqrt{\left(\frac{c_\mathrm{s0}v_\mathrm{p}}{\dot{v}_0}\right)^2 + \frac{2c_\mathrm{s0}^2}{\dot{v}_0}(\Delta y + R)}. \tag{68}$$

The central shielding length occurs where $R = 0$, i.e.

$$z_\mathrm{s} = -\frac{c_\mathrm{s0}v_\mathrm{p}}{\dot{v}_0} + \sqrt{\left(\frac{c_\mathrm{s0}v_\mathrm{p}}{\dot{v}_0}\right)^2 + \frac{2c_\mathrm{s0}^2}{\dot{v}_0}\Delta y}. \tag{69}$$

[1] the ablated material drifts past the pellet before it has reached a substantial part of its maximum speed

The change in the shielding length at some small distance δR (compared to Δy) from the pellet centern in the major radius direction is then

$$\delta z_{\mathrm{s}} = \left.\frac{dz}{dR}\right|_{R=0} \delta R = \frac{c_{\mathrm{s0}}\delta R}{\sqrt{v_{\mathrm{p}}^2 + 2\dot{v}_0 \Delta y}}. \tag{70}$$

Notably, Δz_{s} decreases with an increasing $\dot{v}_0$, which might seem counter-intuitive. This can however be understood graphically by noting that a larger $\dot{v}_0$ corresponds to increasing the slope of the boundary in figure 1, which would compress δz_{s}.

The above expressions can be simplified in some relevant limits. If the background pressure is negligible compared to the pressure inside the cloud (which is quite commonly the case, especially during the initial phase of the drift), we can express

$$z = R_{\mathrm{m}} \frac{\gamma_{\mathrm{e}}\langle z\rangle + \gamma_{\mathrm{i}}}{2(1+\langle z\rangle)} \left(-\frac{v_{\mathrm{p}}}{c_{\mathrm{s0}}} + + \sqrt{\left(\frac{v_{\mathrm{p}}}{c_{\mathrm{s0}}}\right)^2 + \frac{4(1+\langle z\rangle)}{\gamma_{\mathrm{e}}\langle z\rangle + \gamma_{\mathrm{i}}} \frac{\Delta y + R}{R_{\mathrm{m}}}} \right). \tag{71}$$

If we also consider the limit where $(v_{\mathrm{p}}/c_{\mathrm{s0}})^2 \ll (\Delta y + R)/R_{\mathrm{m}}$ (i.e. where the shielding length is limited by the drift and not the pellet motion), we get

$$z = \sqrt{\frac{\gamma_{\mathrm{e}}\langle z\rangle + \gamma_{\mathrm{i}}}{(1+\langle z\rangle)}} \sqrt{R_{\mathrm{m}}(\Delta y + R)}. \tag{72}$$

For a fully ionized deuterium plasma (I have been under the impression that the plasma cloud is fully ionized, but maybe that is not necessarily the case?), we have $\langle z\rangle = 1$, $\gamma_{\mathrm{e}} = 1$ (assuming high electron thermal conductivity, so that the electrons can be considered isothermal on the time scale of ion acoustic waves) and $\gamma_{\mathrm{i}} = 3$ (assuming a 1D ion motion due to ion acoustic waves). This reduces the expression above to simply

$$z = \sqrt{2R_{\mathrm{m}}(\Delta y + R)}. \tag{73}$$

Notably, this means that the ablated material drifts relative to the magnetic field in such a way that it expands in a straight line in the labb frame. This expression is similar to the one used by e.g. [3], except for the factor $\sqrt{2}$ (which might come from different assumptions on $\langle z\rangle$ and/or the adiabatic indices?).

Equations (72) and (73) are probably quite valid in most present day experiments, but not necessarily in future devices. The ion sound speed at the temperatures $T_0 = 1 - 2\,\mathrm{eV}$ prevailing when the plasma cloud becomes ionized are roughly $c_{\mathrm{s0}} \sim 10^4\,\mathrm{m/s}$, and $\Delta y \sim 10^{-2}\,\mathrm{m}$ essentially independently of the device. For conditions representative of most present day experiments, where $v_{\mathrm{p}} \sim 10^2\,\mathrm{m/s}$ and $R_{\mathrm{m}} \sim 1\,\mathrm{m/s}$, so that $(v_{\mathrm{p}}/c_{\mathrm{s0}})^2/[(\Delta y + R)/R_{\mathrm{m}}] \sim 10^{-2}$. However, the ITER disruption mitigation system might be able to inject pellets with speeds at $v_{\mathrm{p}} \lesssim 10^3\,\mathrm{m/s}$, and the major radius will be $R_{\mathrm{m}} \sim 6\,\mathrm{m}$, in which case $(v_{\mathrm{p}}/c_{\mathrm{s0}})^2/[(\Delta y + R)/R_{\mathrm{m}}] \sim 10^0$ or even larger. Finally, for a small pellet or pellet shard and/or when the background plasma pressure has already dropped a bit during a thermal quench, the background plasma pressure might also not be negligible in equation (65).

2.2 Resulting heat flux asymmetry

One may estimate the assymmetries q_{rel} and E_{rel} in the heat flux and energy by assuming that all electrons with a mean free path λ_{mfp} shorter than the distance they travel through the plasma cloud are completely stopped before they reach the neutral cloud, while the rest pass through the plasma cloud unaffected. As the distance d traveled through the plasma cloud depends on the cosine of the pitch angle $\xi = p_z/p$ (where p_z is the momentum component in the z-direction and p is the magnitude of the total momentum) as $d = z_{\mathrm{s}}/\xi$, this gives a condition on the velocity and pitch angle needed to pass through the plasma cloud according to

$$\frac{z_{\mathrm{s}}}{\xi} < \lambda_{\mathrm{mfp}} = \frac{4\pi\epsilon_0^2 m_{\mathrm{e}}^2 v_{\mathrm{e}}^4}{n_0 e^4 \ln\Lambda}, \tag{74}$$

where m_{e} and v_{e} denote the electron mass and speed, respectively. Here λ_{mfp} is taken to be the mean free path for slowing down against the cold electrons in the cloud (slowing down against the ions is smaller by the electron-ion mass ratio, and for simplicity we neglect pitch angle scattering altogether). From this we can determine a pitch angle-dependent critical velocity $v_c(\xi)$ for passing through the plasma cloud found when assuming equality in equation (74). In terms of this critical velocity the heat and particle flux for the distribution passing through the plasma cloud can be found by evaluating the following integrals:

$$q_p = \int_0^{2\pi} \int_0^1 \int_{v_c}^{\infty} \xi v_e \frac{m v_e^2}{2} n_e e^{-\frac{m v_e^2}{2T}} \left(\frac{m_e}{2\pi T}\right)^{3/2} v_e^2 dv_e d\xi d\phi \tag{75}$$

$$\Phi_p = \int_0^{2\pi} \int_0^1 \int_{v_c}^{\infty} \xi v_e n_e e^{-\frac{m v_e^2}{2T}} \left(\frac{m_e}{2\pi T}\right)^{3/2} v_e^2 dv_e d\xi d\phi. \tag{76}$$

The effective energy can then be expressed as $E_p = q_p/\Phi_p$, similarly to [1] chose as an effective energy in the absence of the plasma cloud, and the asymmetries may be expressed as

$$q_1 = \frac{dq_p}{dz_s}\delta z_s = \frac{dq_p}{dz_s}\frac{c_{s0}\delta R}{\sqrt{v_p^2 + 2\dot{v}_0 \Delta y}} \tag{77}$$

$$E_1 = \frac{dE_p}{dz_s}\frac{c_{s0}\delta R}{\sqrt{v_p^2 + 2\dot{v}_0 \Delta y}}. \tag{78}$$

With some help from Wolfram (I might write up some of the steps later), one may obtain (with $x = \lambda_T/z_s$, where λ_T is the mean free path at the thermal speed)

$$q_p = 2\sqrt{\frac{T^3}{2\pi m_e}} n_e \frac{1}{x^2}\left[e^{\frac{-1}{\sqrt{x}}}\left(-\frac{1}{2}\sqrt{x} + \frac{1}{2}x + x^{3/2} + x^2\right) - \frac{1}{2}\mathrm{Ei}\left(-\frac{1}{\sqrt{x}}\right)\right] \tag{79}$$

$$\frac{dq_p}{dz_s} = -2\sqrt{\frac{T^3}{2\pi m_e}}\frac{1}{x^2}\left[\mathrm{Ei}\left(-\frac{1}{\sqrt{x}}\right) + \sqrt{x}e^{-\frac{1}{\sqrt{x}}}\right]\frac{1}{z_s} \tag{80}$$

$$E_p = 2T\frac{e^{\frac{-1}{\sqrt{x}}}\left(-\frac{1}{2}\sqrt{x} + \frac{1}{2}x + x^{3/2} + x^2\right) - \frac{1}{2}\mathrm{Ei}\left(-\frac{1}{\sqrt{x}}\right)}{\frac{1}{2}\mathrm{Ei}\left(-\frac{1}{\sqrt{x}}\right) + e^{\frac{-1}{\sqrt{x}}}\left(\frac{1}{2}\sqrt{x} - \frac{1}{2}x + x^{3/2} + x^2\right)} \tag{81}$$

$$\frac{dE_p}{dz_s} = ... \tag{82}$$

The mapping between δR and the angular coordinate θ for the neutral cloud calculation is somewhat ambiguous, as the heat flux is actually not directed only along the radial direction. However, if we only account for the heat flux from particles whose trajectories actually cross the pellet, which play the most important role for the neutral cloud dynamics close to the pellet, we should do this mapping at the pellet radius. At this radius we simply have $\delta R = -\cos\theta r_p$ (with $\theta = 0$ defined to be at the least shielded side, where δR is negative), so that the projections onto $\mathcal{P}_1^0(\cos\theta$ become

$$q_{rel} = \frac{1}{q_p}\frac{\int_0^{\pi} -\frac{dq_p}{dz_s}\frac{c_{s0}\cos\theta r_p}{\sqrt{v_p^2+2\dot{v}_0\Delta y}}\cos\theta\sin\theta d\theta}{\int_0^{\pi}\cos(\theta)^2\sin\theta d\theta} = -\frac{1}{q_p}\frac{dq_p}{dz_s}\frac{c_{s0}r_p}{\sqrt{v_p^2 + 2\dot{v}_0\Delta y}} \tag{83}$$

$$E_{rel} = -\frac{1}{E_p}\frac{dE_p}{dz_s}\frac{c_{s0}r_p}{\sqrt{v_p^2 + 2\dot{v}_0\Delta y}}. \tag{84}$$

As the dimensionless quantity of interest here, λ_T/z_s, is typically of order 10^{-1} (at least for typical ITER parameters, I'll write up a specific example later), it may be usefull to simplify the above expressions by the leading order term in the taylor series, which are

$$q_{rel} = \left\{\frac{1}{2}\sqrt{\frac{z_s}{\lambda_T}} + \mathcal{O}\left[\left(\frac{\lambda_T}{z_s}\right)^0\right]\right\}\frac{1}{z_s}\frac{c_{s0}r_p}{\sqrt{v_p^2 + 2\dot{v}_0\Delta y}} \tag{85}$$

$$\frac{E_{rel}}{q_{rel}} = -\sqrt{\frac{\lambda_T}{z_s}} + \mathcal{O}\left[\left(\frac{\lambda_T}{z_s}\right)^1\right]. \tag{86}$$

2.2.1 Areas for improvement and further consideration

- It is not clear from the above analysis to what extent it is realistic that $E_{\rm rel}/q_{\rm rel}$ is negative, and to what extent this is a consequence of the monoenergetic beam approximation. On the one hand, the extra heat flux on the less shielded side is indeed carried by electrons with a lower energy than those penetrating on the more shielded side, which should indeed lower the effective energy when calculating the heat absorption (which is the way in which the energy matters), making $E_{\rm rel}/q_{\rm rel}$ negative to some extent. On the other hand, the lower energy electrons will be absorbed more quickly, while there are still in reality the same high energy tails present on both sides of the pellet, which will penetrate further. Thus, the effective energy close to the pellet might actually be more similar than the monoenergetic beam approximation suggests, motivating using a value of $E_{\rm rel}/q_{\rm rel}$ closer to zero than that calculated above. From Nicos' numerical calculations, we see that a negative enough $E_{\rm rel}/q_{\rm rel}$ will lead to a negative rocket force, which is therefore not necessarily realistic. However, the only way to conclusively answer these questions is probably to do a fully kinetic calculation, which is probably a quite major project on its own.

- In addition to the plasma shielding of th heat flux calculated above, the heat flux will also be reduced due to electrostatic shielding, which there are several models in the literature accounting for (look up some!). The effective cross section area seen by the heat flux should also be the circular cross section of the pellet, which is only half as large as the surface of the entire pellet as seen by the heat flux when making the radial flow approximation.

- It might be worth including one more term in the taylor series in equations (85) and (86), as $\sqrt{\frac{\lambda_{\rm T}}{z_{\rm s}}}$ is no that small.

- It might be possible to relax the assumption of a strict cutoff between electrons being completely stopped by the plasma cloud and passing through unaffected. As long as the plasma cloud remains static (i.e. as long as there is no feedback mechanism between the heat flux from the background plasma and the plasma cloud), and pitch angle scattering is neglected, the solution should still be possible to express in terms of scaling factors and relatively simple dimensionless integrals. Although they might not be possible to evaluate analytically, it might be possible to calculate the leading terms in the taylor series, or they may be evaluated numerically.

References

[1] P. B. Parks & R. J. Turnbull, The Physics of Fluids **21**, 1735 (1978); doi: `10.1063/1.862088`

[2] O. Vallhagen *et al* JPP 2023 (drift paper)

[3] Senichenkov *et al* EPS 2007

A Matrix form of linearised ablation flow equations (from Nico)

We have written the first order differential equation system in the matrix-vector equation

$$A\vec{y}' = B\vec{y}\,. \tag{1}$$

Then I copied A and B into sympy and calculated

$$C = A^{-1}B\,. \tag{2}$$

The expressions in this matrix are quite large, therefore I have split the matrix into 4 blocks of size 3x3.

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \tag{3}$$

Here are the results (all 0 indices are dropped):

$$C_{11} = \frac{1}{p-\rho v^2}\begin{bmatrix} \frac{\gamma v\left(-2\Lambda q r\rho + Tr\rho\frac{d}{dr}v + 2T\rho v - pr\frac{d}{dr}v\right)}{Tr} & \frac{\gamma\rho v\left(2\Lambda q\rho + p\frac{d}{dr}v\right)}{T} & \frac{\gamma\rho\left(2\gamma r\rho v^2\frac{d}{dr}v + \gamma r v^3\frac{d}{dr}\rho + 2\gamma\rho v^3 - pr\frac{d}{dr}v + 2pv - 2r\rho v^2\frac{d}{dr}v - rv^3\frac{d}{dr}\rho + rv\frac{d}{dr}p - 2\rho v^3\right)}{r} \\ \frac{2\Lambda q}{v} - \frac{2\Lambda\gamma q v}{T} - \frac{T\frac{d}{dr}v}{v} - \frac{2T}{r} + \gamma v\frac{d}{dr}v + \frac{2\gamma v^2}{r} + \frac{p\frac{d}{dr}v}{\rho v} + \frac{2p}{r\rho} - \frac{2v^2}{r} - \frac{\gamma p v\frac{d}{dr}v}{T\rho} - \frac{p\frac{d}{dr}T}{T\rho} + \frac{v^2\frac{d}{dr}T}{T} & -\frac{2\Lambda q\rho}{v} + \frac{2\Lambda\gamma q\rho v}{T} - \frac{p\frac{d}{dr}v}{v} - \frac{p\frac{d}{dr}\rho}{\rho} - \frac{2p}{r} + v^2\frac{d}{dr}\rho + \frac{2\rho v^2}{r} + \frac{\gamma p v\frac{d}{dr}v}{T} + \frac{p\frac{d}{dr}T}{T} - \frac{\rho v^2\frac{d}{dr}T}{T} & \frac{-\gamma r\rho v(\gamma-1)\left(-T\rho + \gamma\rho v^2 + p - \rho v^2\right)\frac{d}{dr}v + \rho\left(-T\gamma + T + \gamma^2 v^2 - \gamma v^2\right)\left(2p + rv(\gamma-1)\left(2\rho\frac{d}{dr}v + \frac{3v\frac{d}{dr}\rho}{2}\right) + r\frac{d}{dr}p + \rho v^2(\gamma-1) + \rho v(\gamma-1)\left(r\frac{d}{dr}v + 2v\right)\right) + (\gamma-1)\left(r\frac{d}{dr}\rho + 2\rho\right)\left(\frac{T\gamma\rho v^2}{2} + Tp - \frac{3T\rho v^2}{2} - \frac{\gamma^2\rho v^4}{2} + \frac{\gamma\rho v^4}{2}\right)}{r\rho v(\gamma-1)} \\ \frac{2\Lambda q}{T} - \frac{d}{dr}v - \frac{2v}{r} + \frac{v^2\frac{d}{dr}v}{T} & -\frac{\rho\left(2\Lambda q + v^2\frac{d}{dr}v\right)}{T} & -2\gamma\rho v\frac{d}{dr}v - \gamma v^2\frac{d}{dr}\rho - \frac{2\gamma\rho v^2}{r} - \frac{2p}{r} + 3\rho v\frac{d}{dr}v + v^2\frac{d}{dr}\rho - \frac{d}{dr}p + \frac{2\rho v^2}{r} \end{bmatrix} \tag{4}$$

$$C_{12} = \frac{1}{p-\rho v^2}\begin{bmatrix} -\frac{2\gamma p\rho v}{r} & -2\Lambda\gamma\rho^2 v & -2\gamma q\rho^2 v\frac{d}{dE}\Lambda \\ \frac{2v(T\rho-\gamma p)}{r} & \frac{2\Lambda\rho(T-\gamma v^2)}{v} & \frac{2q\rho(T-\gamma v^2)\frac{d}{dE}\Lambda}{v} \\ \frac{2p}{r} & 2\Lambda\rho & 2q\rho\frac{d}{dE}\Lambda \end{bmatrix} \tag{5}$$

$$C_{21} = \begin{bmatrix} -\frac{1}{\gamma r\rho v} & 0 & 0 \\ \frac{\Lambda\lambda q}{T} & -\frac{\Lambda\lambda q\rho}{T} & 0 \\ \frac{2L\lambda}{T} & -\frac{2L\lambda\rho}{T} & 0 \end{bmatrix} \tag{6}$$

$$C_{22} = \begin{bmatrix} -\frac{1}{r} & 0 & 0 \\ 0 & \Lambda\lambda\rho & \lambda q\rho\frac{d}{dE}\Lambda \\ 0 & 0 & 2\lambda\rho\frac{d}{dE}L \end{bmatrix} \tag{7}$$

Or after substituting $\rho = p/T$:

$$C_{11} = \frac{1}{T-v^2}\begin{bmatrix} \frac{2\gamma v(-\Lambda q r + Tv)}{Tr} & \frac{\gamma p v\left(2\Lambda q + T\frac{d}{dr}v\right)}{T^2} & \frac{\gamma\left(T\left(\gamma r v^3\left(\frac{\frac{d}{dr}p}{T} - \frac{p\frac{d}{dr}T}{T^2}\right) - pr\frac{d}{dr}v + 2pv - rv^3\left(\frac{\frac{d}{dr}p}{T} - \frac{p\frac{d}{dr}T}{T^2}\right) + rv\frac{d}{dr}p\right) + 2\gamma p r v^2\frac{d}{dr}v + 2\gamma p v^3 - 2prv^2\frac{d}{dr}v - 2pv^3\right)}{Tr} \\ \frac{-2\Lambda\gamma q r v^2 - T\left(-2\Lambda q r - 2\gamma v^3 + rv\frac{d}{dr}T + 2v^3\right) + rv^3\frac{d}{dr}T}{prv} & -\frac{2\Lambda q}{v} + \frac{2\Lambda\gamma q v}{T} - \frac{T^2\left(\frac{\frac{d}{dr}p}{T} - \frac{p\frac{d}{dr}T}{T^2}\right)}{p} - \frac{T\frac{d}{dr}v}{v} - \frac{2T}{r} + \frac{Tv^2\left(\frac{\frac{d}{dr}p}{T} - \frac{p\frac{d}{dr}T}{T^2}\right)}{p} + \gamma v\frac{d}{dr}v + \frac{d}{dr}T + \frac{2v^2}{r} - \frac{v^2\frac{d}{dr}T}{T} & \frac{-\gamma p r v^3(\gamma-1)^2\frac{d}{dr}v + (\gamma-1)\left(Tr\left(\frac{\frac{d}{dr}p}{T} - \frac{p\frac{d}{dr}T}{T^2}\right) + 2p\right)\left(T\left(T + \frac{\gamma v^2}{2} - \frac{3v^2}{2}\right) - \frac{\gamma^2 v^4}{2} + \frac{\gamma v^4}{2}\right) + \left(-T\gamma + T + \gamma^2 v^2 - \gamma v^2\right)\left(T\left(2p + r\frac{d}{dr}p\right) + pv^2(\gamma-1) + pv(\gamma-1)\left(r\frac{d}{dr}v + 2v\right) + rv(\gamma-1)\left(\frac{3Tv\left(\frac{\frac{d}{dr}p}{T} - \frac{p\frac{d}{dr}T}{T^2}\right)}{2} + 2p\frac{d}{dr}v\right)\right)}{prv(\gamma-1)} \\ \frac{2\Lambda q r - Tr\frac{d}{dr}v - 2Tv + rv^2\frac{d}{dr}v}{pr} & -\frac{2\Lambda q + v^2\frac{d}{dr}v}{T} & -\frac{T\gamma v^2\left(\frac{\frac{d}{dr}p}{T} - \frac{p\frac{d}{dr}T}{T^2}\right)}{p} - \frac{2T}{r} + \frac{Tv^2\left(\frac{\frac{d}{dr}p}{T} - \frac{p\frac{d}{dr}T}{T^2}\right)}{p} - \frac{T\frac{d}{dr}p}{p} - 2\gamma v\frac{d}{dr}v - \frac{2\gamma v^2}{r} + 3v\frac{d}{dr}v + \frac{2v^2}{r} \end{bmatrix} \tag{8}$$

$$C_{12} = \frac{1}{T-v^2}\begin{bmatrix} -\frac{2\gamma p v}{r} & -\frac{2\Lambda\gamma p v}{T} & -\frac{2\gamma p q v\frac{d}{dE}\Lambda}{T} \\ \frac{2Tv(1-\gamma)}{r} & \frac{2\Lambda(T-\gamma v^2)}{v} & \frac{2q(T-\gamma v^2)\frac{d}{dE}\Lambda}{v} \\ \frac{2T}{r} & 2\Lambda & 2q\frac{d}{dE}\Lambda \end{bmatrix} \tag{9}$$

$$C_{21} = \begin{bmatrix} -\frac{T}{\gamma p r v} & 0 & 0 \\ \frac{\Lambda\lambda q}{T} & -\frac{\Lambda\lambda p q}{T^2} & 0 \\ \frac{2L\lambda}{T} & -\frac{2L\lambda p}{T^2} & 0 \end{bmatrix} \tag{10}$$

$$C_{22} = \begin{bmatrix} -\frac{1}{r} & 0 & 0 \\ 0 & \frac{\Lambda\lambda p}{T} & \frac{\lambda p q\frac{d}{dE}\Lambda}{T} \\ 0 & 0 & \frac{2\lambda p\frac{d}{dE}L}{T} \end{bmatrix} \tag{11}$$