Rocket effect derivation, part 1

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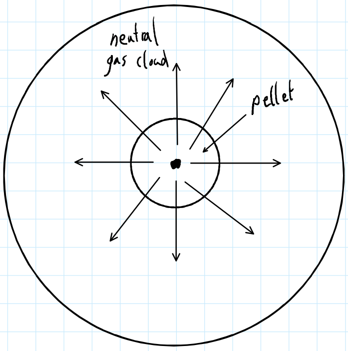
Consider the neutral gas around the pellet through fluid equations.

The pellet acts as a source of ablatant particles at the origin through boundary conditions.



To first order the characteristics of the cloud are spherically symmetric and described by the Neutral Gas Shielding model (Parks 1978).

We want to model the effect of an anisotropic heat source as a perturbation. But it should still be rotationally symmetric around z-axis.

The gas is described by the steady state equations for an ideal gas i.e. Euler equations (Navier Stokes without viscocity and thermal conductivity, Wikipedia) can be derived from kinetic theory (Vlasov equation), see book Helander & Sigmar eq. 2.15, 2.16, 2.17 (steady state $\rightarrow \frac{\partial}{\partial t} = 0$, neutral particles $\rightarrow e_a = 0$, no friction force $R_a = 0$, no viscosity $\pi_a = 0$)

$$\vec{\nabla}(\rho \vec{v}) = 0 \qquad \text{(I)} \quad \text{(mass conservation)}$$

$$\rho\, \vec{v} \cdot (\vec{\nabla} \cdot \vec{v}) = -\vec{\nabla} p \qquad \text{(II)} \quad \text{(momentum conservation)}$$

$$\vec{\nabla}\left[\left(\frac{\rho v^2}{2} + \frac{\gamma p}{\gamma - 1}\right)\vec{v}\right] = Q \qquad \text{(III)} \quad \text{(energy conservation + heat source } Q)$$

where $\rho(\vec{r})$ is mass density, $\vec{v}(\vec{r})$ is flow velocity, $p(\vec{r})$ is fluid pressure

(III) in usual form: $\vec{v} \cdot \vec{\nabla} e + \frac{p}{\rho} \vec{\nabla} \cdot \vec{v} = Q$

equation of state ideal gas: $p = \rho e (\gamma - 1) \iff e = \frac{p}{\rho} \frac{1}{\gamma - 1}$

$$\vec{v} \cdot \vec{\nabla}\left(\frac{p}{\rho} \frac{1}{\gamma - 1}\right) + \frac{p}{\rho} \vec{\nabla} \cdot \vec{v} = Q$$

$$\frac{1}{\gamma - 1}\left(\vec{v}\, \vec{\nabla}\left(\frac{p}{\rho}\right) + (\gamma - 1)\frac{p}{\rho} \vec{\nabla} \cdot \vec{v}\right) = Q$$

Origin of (III) unclear

Treat anisotropic dynamics (index 1) as a perturbation to sherically symmetric dynamics (index 0)

$\rho(\vec{r}) = \rho_0(r) + \rho_1(r, \theta)$, $\vec{v}(\vec{r}) = \underbrace{v_0(r)\hat{r}}_{\vec{v}_0} + \vec{v}_1(r, \theta)$, $p(\vec{r}) = p_0(r) + p_1(r, \theta)$, $Q(\vec{r}) = Q_0(r) + Q_1(r, \theta)$

with $\rho_0 \gg \rho_1$, $v_0 \gg v_1$, $p_0 \gg p_1$, $Q_0 \gg Q_1$

System of equations for isotropic part (0th order): with $\vec{\nabla}(f(r)\hat{r}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f(r))$ and $\vec{\nabla} f(r) = \frac{\partial}{\partial r} f(r)\, \hat{r}$

$$\cancel{\frac{1}{r^2}} \frac{\partial}{\partial r}\left(r^2 \rho_0(r) v_0(r)\right) = 0 \qquad (\mathrm{I}_0)$$

$$\rho_0(r) v_0(r) \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 v_0(r)\right) = -\frac{\partial}{\partial r} p_0(r) \qquad (\mathrm{II}_0)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{\rho_0(r) v_0^3(r)}{2} + \frac{\gamma}{\gamma - 1} p_0(r) v_0(r)\right] = Q_0(r) \qquad (\mathrm{III}_0)$$

This can be solved through the Neutral Gas Shielding model.

System of equations for perturbation (linearization) (1st order):

$$(\mathrm{I}):\quad \vec\nabla\left((\rho_0+\rho_1)(\vec v_0+\vec v_1)\right) = \vec\nabla\Big(\underbrace{\rho_0\vec v_0}_{\substack{=0\\ \text{0th order}}} + \rho_1\vec v_0 + \rho_0\vec v_1 + \underbrace{\rho_1\vec v_1}_{\substack{\text{2nd order}\\ \to\text{neglect}}}\Big) = 0$$

$$\Leftrightarrow\ \vec\nabla\left(\rho_1\vec v_0 + \rho_0\vec v_1\right) = 0 \qquad (\mathrm{I}_1, a)$$

$$(\mathrm{II}):\quad (\rho_0+\rho_1)(\vec v_0+\vec v_1)\vec\nabla(\vec v_0+\vec v_1) = -\vec\nabla p_0 - \vec\nabla p_1 \qquad \text{with} \quad \rho_0 v_0\hat r\,\vec\nabla(v_0\hat r) = -\vec\nabla p_0$$

and linearization

$$\Leftrightarrow\ \rho_0\,\vec v_0(\vec\nabla\vec v_1) + \rho_0\,\vec v_1(\vec\nabla\vec v_0) + \rho_1\vec v_0(\vec\nabla\vec v_0) = -\vec\nabla p_1 \qquad (\mathrm{II}_1, a)$$

$$(\mathrm{III}):\quad \vec\nabla\left[\left(\tfrac12(\rho_0+\rho_1)(\vec v_0+\vec v_1)^2 + \tfrac{\gamma}{\gamma-1}(p_0+p_1)\right)(\vec v_0+\vec v_1)\right] = Q_0+Q_1 \qquad \text{with} \quad \vec\nabla\left[\left(\tfrac12\rho_0 v_0^2 + \tfrac{\gamma}{\gamma-1}p_0\right)\vec v_0\right] = Q_0$$

and linearization

$$\Leftrightarrow\ \vec\nabla\left[\left(\tfrac12\rho_0 v_0^2 + \tfrac{\gamma}{\gamma-1}p_0\right)\vec v_1 + \left(\tfrac12\rho_1 v_0^2 + \rho_0\vec v_0\cdot\vec v_1 + \tfrac{\gamma}{\gamma-1}p_1\right)\vec v_0\right] = Q_1 \qquad (\mathrm{III}_1, a)$$

Assume the perturbations take the form

$$\rho_1(r,\theta) = \rho_r(r)\,\rho_\theta(\theta), \quad p_1(r,\theta) = p_r(r)\,p_\theta(\theta), \quad \vec v_1(r,\theta) = U_r(r)U_\theta(\theta)\,\hat r + V_r(r)V_\theta(\theta)\,\hat\theta$$

In spherical coordinates: (Jackson)

$$\vec\nabla\psi = \hat r\frac{\partial\psi}{\partial r} + \hat\theta\frac{\partial\psi}{\partial\theta} + \hat\varphi\frac{\partial\psi}{\partial\varphi}, \qquad \vec\nabla\cdot\vec A = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

$$\Rightarrow\ \vec\nabla\cdot\vec v_1 = \frac{1}{r^2}U_\theta\frac{\partial}{\partial r}(r^2U_r) + \frac{1}{r\sin\theta}V_r\frac{\partial}{\partial\theta}(\sin\theta\cdot V_\theta)$$

$$\vec\nabla\rho_1 = \hat r\,\rho_\theta\frac{\partial}{\partial r}\rho_r + \hat\theta\,\rho_r\frac{\partial}{\partial\theta}\rho_\theta$$

$$(\mathrm{I}_1, a) \Rightarrow \frac{1}{r^2}\rho_\theta\frac{\partial}{\partial r}(r^2\rho_r v_0) + \frac{1}{r^2}U_\theta\frac{\partial}{\partial r}(r^2\rho_0U_r) + \frac{1}{r\sin\theta}\rho_0V_r\frac{\partial}{\partial\theta}(\sin\theta\,V_\theta) = 0$$

Assume the perturbations take the form

$$\rho_1(r,\theta) = \rho_r(r)\,\rho_\theta(\theta), \quad p_1(r,\theta) = p_r(r)\,p_\theta(\theta), \quad \vec v_1(r,\theta) = U_r(r)U_\theta(\theta)\,\hat r + V_r(r)V_\theta(\theta)\,\hat\theta$$

Use fourier expansion

$$\rho_\theta(\theta) = \sum_{k=0}^{\infty}\alpha_{\rho,k}\cos(k\theta), \quad p_\theta(\theta) = \sum_{k=0}^{\infty}\alpha_{p,k}\cos(k\theta), \quad U_\theta(\theta) = \sum_{k=0}^{\infty}\alpha_{v,k}\cos(k\theta), \quad V_\theta(\theta) = \sum_{k=1}^{\infty}\beta_{v,k}\sin(k\theta)$$

and rename $\rho_r(r) = R(r)$, $p_r(r) = P(r)$, $U_r(r) = U(r)$, $V_r(r) = V(r)$

$$\Rightarrow\ \rho_1(r,\theta) = R(r)\cdot\sum_{k=0}^{\infty}\alpha_{\rho,k}\cos(k\theta), \quad p_1(r,\theta) = P(r)\sum_{k=0}^{\infty}\alpha_{p,k}\cos(k\theta)$$

$$v_1(r,\theta) = \hat r\,U(r)\sum_{k=0}^{\infty}\alpha_{v,k}\cos(k\theta) + \hat\theta\,V(r)\sum_{k=1}^{\infty}\beta_{v,k}\sin(k\theta)$$

$$\Rightarrow\ \vec\nabla\rho_1 = \hat r\frac{\partial\rho_1}{\partial r} + \hat\theta\frac{\partial\rho_1}{\partial\theta} = \hat r\frac{\partial R}{\partial r}\sum_k\alpha_{\rho,k}\cos(k\theta) - \hat\theta R\sum_k\alpha_{\rho,k}\,k\cos(k\theta)$$

$$\vec\nabla\vec v_1 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2U)\sum_k\alpha_{v,k}\cos(k\theta) + \frac{1}{r\sin\theta}V\frac{\partial}{\partial\theta}\left(\sin\theta\sum_k\beta_{v,k}\sin(k\theta)\right)$$

$$= \left(\frac{2}{r}U + \frac{\partial U}{\partial r}\right)\sum_k\alpha_{v,k}\cos(k\theta) + \frac{\cos\theta}{r\sin\theta}V\sum_k\beta_{v,k}\sin(k\theta) + \frac{1}{r}V\sum_k\beta_{v,k}\,k\cos(k\theta)$$

Use strong assumption (Per's notes): $Q_1(r,\theta) = q(r)\cos\theta$ (this is an approximation/assumption for the external heat source)

$$\left.\begin{aligned}&\rho_1(r,\theta) = R(r)\cos\theta, \quad p_1(r,\theta) = P(r)\cos\theta,\\ &\vec v_1(r,\theta) = \hat r\,U(r)\cos\theta + \hat\theta\,V(r)\sin\theta\end{aligned}\right\}\ \text{are those guesses or assumptions?}$$

$$\Rightarrow\ \vec\nabla\rho_1 = \hat r\frac{\partial R}{\partial r}\cos\theta - \hat\theta R\sin\theta$$

$$\vec\nabla\cdot\vec v_1 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2U)\cos\theta + \frac{1}{r\sin\theta}V\frac{\partial}{\partial\theta}(\sin\theta\sin\theta) \qquad \text{with} \quad \frac{\partial}{\partial\theta}\sin^2\theta = 2\sin\theta\cos\theta$$

$$\Rightarrow \vec{\nabla}\rho_1 = \hat{r}\,\frac{\partial R}{\partial r}\cos\theta - \hat{\theta}\,R\sin\theta$$

$$\vec{\nabla}\cdot\vec{v}_1 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 U\right)\cos\theta + \frac{1}{r\sin\theta}\,V\frac{\partial}{\partial\theta}\left(\sin\theta\,\sin\theta\right) \quad \text{with} \quad \frac{\partial}{\partial\theta}\sin^2\theta = 2\sin\theta\cos\theta$$

$$= \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 U\right) + \frac{2}{r}V\right]\cos\theta$$

$$(I_1, a) \Rightarrow \vec{\nabla}\left(\rho_0\vec{v}_1 + \rho_1\vec{v}_0\right) = \vec{\nabla}\rho_0\cdot\vec{v}_1 + \rho_0\vec{\nabla}\vec{v}_1 + \vec{\nabla}\rho_1\cdot\vec{v}_0 + \rho_1\cdot\vec{\nabla}\vec{v}_0$$

$$= \frac{\partial\rho_0}{\partial r}U\cos\theta + \rho_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 U\right) + \frac{2}{r}V\right]\cos\theta + \frac{\partial R}{\partial r}\cos\theta\, v_0 + R\cos\theta\,\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_0\right)$$

$$\Leftrightarrow \frac{\partial\rho_0}{\partial r}U + \rho_0\frac{1}{r^2}\frac{\partial}{\partial r}$$

Expand using general orthogonal basis $\{X_l(\theta)\}$ and $\{Y_l(\theta)\}$

$$Q_1(r,\theta) = \sum q_l(r)\,X_l(\theta)$$

$$\rho_1(r,\theta) = \sum_l R_l(r)\,X_l(\theta) \quad , \quad p_1(r,\theta) = \sum_l \Pi_l(r)\,X_l(\theta)$$

$$\vec{v}_1(r,\theta) = \hat{r}\,\sum U_l(r)\,X_l(\theta) + \hat{\theta}\sum_l V_l(r)\,Y_l(\theta)$$

$$\vec{\nabla}\rho_1(r,\theta) = \hat{r}\sum_l\frac{\partial R_l}{\partial r}X_l + \hat{\theta}\sum_l R_l\frac{\partial X_l}{\partial\theta}$$

$$\vec{\nabla}\cdot\vec{v}_1 = \frac{1}{r^2}\sum_l\frac{\partial}{\partial r}\left(r^2 U_l\right)X_l + \frac{1}{r\sin\theta}\sum_l V_l\frac{\partial}{\partial\theta}\left(\sin\theta\,Y_l\right)$$

$$(I_1, a) \Rightarrow \vec{\nabla}\left(\rho_0\vec{v}_1 + \rho_1\vec{v}_0\right) = \vec{\nabla}\rho_0\cdot\vec{v}_1 + \rho_0\vec{\nabla}\vec{v}_1 + \vec{\nabla}\rho_1\cdot\vec{v}_0 + \rho_1\cdot\vec{\nabla}\vec{v}_0$$

$$= \frac{\partial\rho_0}{\partial r}\sum_l U_l X_l + \rho_0\left[\frac{1}{r^2}\sum_l\frac{\partial}{\partial r}\left(r^2 U_l\right)X_l + \frac{1}{r\sin\theta}\sum_l V_l\frac{\partial}{\partial\theta}\left(\sin\theta\,Y_l\right)\right] + v_0\sum_l\frac{\partial R_l}{\partial r}X_l + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_0\right)\sum_l R_l X_l = 0$$

assume that each mode l is independent ?

$$\Rightarrow \left[\frac{\partial\rho_0}{\partial r}U_l + \rho_0\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 U_l\right) + v_0\frac{\partial R_l}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_0\right)R_l\right]X_l + \rho_0\left[\frac{1}{r}V_l\right]\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,Y_l\right) = 0$$

$\Rightarrow$ to be able to factor out the theta dependence require ?

$$\alpha_l X_l = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,Y_l\right) \qquad (2) \qquad (\alpha_l \text{ some constant } \neq 0)$$

$$\Rightarrow \frac{\partial\rho_0}{\partial r}U_l + \rho_0\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 U_l\right) + v_0\frac{\partial R_l}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_0\right)R_l + \alpha_l\frac{\rho_0}{r}V_l = 0 \qquad (1)$$

$$(II_1, a): \quad \rho_0\,\vec{v}_0\left(\vec{\nabla}\cdot\vec{v}_1\right) + \rho_0\,\vec{v}_1\left(\vec{\nabla}\vec{v}_0\right) + \rho_1\,\vec{v}_0\left(\vec{\nabla}\vec{v}_0\right) = -\vec{\nabla}p_1$$

$$\Rightarrow \rho_0 v_0\hat{r}\left[\frac{1}{r^2}\sum_l\frac{\partial}{\partial r}\left(r^2 U_l\right)X_l + \frac{1}{r\sin\theta}\sum_l V_l\frac{\partial}{\partial\theta}\left(\sin\theta\,Y_l\right)\right] + \rho_0\left[\hat{r}\sum_l U_l X_l + \hat{\theta}\sum_l V_l Y_l\right]\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_0\right) + \left(\sum_l R_l X_l\right)v_0\hat{r}\,\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_0\right) = -\hat{r}\sum_l\frac{\partial\Pi_l}{\partial r}X_l - \hat{\theta}\sum_l\Pi_l\frac{\partial X_l}{\partial\theta}$$

again: each mode is independent, and separate $\hat{r}$ and $\hat{\theta}$ terms

$$\hat{r}: \quad \rho_0 v_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 U_l\right)X_l + \frac{1}{r\sin\theta}V_l\frac{\partial}{\partial\theta}\left(\sin\theta\,Y_l\right)\right] + \rho_0 U_l X_l\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_0\right) + R_l X_l v_0\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_0\right) = -\frac{\partial\Pi_l}{\partial r}X_l \qquad (3)$$

$\Rightarrow$ same requirement to factor out θ-dependence

$$\hat{\theta}: \quad \rho_0 V_l Y_l\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_0\right) = -\Pi_l\frac{\partial X_l}{\partial\theta} \qquad (\text{separate } r \text{ and } \theta \text{ dependence})$$

$$\Leftrightarrow \rho_0 V_l\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_0\right)\frac{1}{\Pi_l} = -\frac{1}{Y_l}\frac{\partial X_l}{\partial\theta} = \text{const} = \beta_l$$

$\rightarrow$ 2 separate equations:

$$\frac{1}{\Pi_l}\rho_0 V_l\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_0\right) = \beta_l \qquad (4)$$

$$Y_l = -\frac{1}{\beta_l}\frac{\partial X_l}{\partial\theta} \qquad (5)$$

$$(2) \text{ and } (5) \Rightarrow \alpha_l X_l = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left(-\frac{1}{\beta_l}\frac{\partial X_l}{\partial\theta}\right)\right)$$

(2) and (5) $\Rightarrow \alpha_l X_l = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \left(-\frac{1}{\beta_l} \frac{\partial X_l}{\partial\theta} \right) \right)$

$\Leftrightarrow \quad \frac{1}{X_l} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial X_l}{\partial\theta} \right) = -\alpha_l \beta_l = -\lambda_l \quad (6)$

according to Wikipedia on spherical harmonics

the Sturm-Liouville theorem (imposing X_l to be regular at the poles)

requires $\lambda_l = l(l+1) = \alpha_l \beta_l$

without loss of generality let $X_l(\theta) = P_l(\cos\theta)$

and $P_l(\cos\theta)$ are the solutions to (6) if they are the ($m=0$) associated Legendre polynomials

$\rightarrow$ θ dependence of mode l is fixed (spherical harmonics with $m=0$)

r-dependence (after factoring out theta dependence and requiring $X_l \stackrel{!}{=} 0$)

(1) $\Rightarrow \frac{\partial \rho_0}{\partial r} U_l + \rho_0 \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 U_l) + v_0 \frac{\partial R_l}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 v_0)) R_l + \alpha_l \frac{\rho_0}{r} V_l = 0$ $\quad (I_1, b) \quad$ (A in Per's notes)

(3) $\Rightarrow \rho_0 v_0 \left[\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 U_l) + \frac{1}{r} V_l \right] + \rho_0 U_l \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 v_0) + R_l v_0 \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 v_0) = -\frac{\partial \Pi_l}{\partial r}$ $\quad (II_1, b_1) \quad$ (corresponds to his B, but wrong)

(4) $\Rightarrow \frac{1}{\Pi_l} \rho_0 V_l \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 v_0) = \frac{l(l+1)}{\alpha_l}$ $\quad$ $\alpha_l \neq 0$ is arbitrary $\quad (II_1, b_2) \quad$ (" " C, but wrong)

(Per's notes correspond to $l=1$, $\alpha_l = 2$)

(III_1, a) to (III_1, b) on next page

$$(\mathrm{III}_1, a): \quad \vec{\nabla}\left[\left(\tfrac{1}{2}\rho_0 v_0^2 + \tfrac{\gamma}{\gamma-1}p_0\right)\vec{v}_1 + \left(\tfrac{1}{2}\rho_1 v_0^2 + \rho_0\vec{v}_0\cdot\vec{v}_1 + \tfrac{\gamma}{\gamma-1}p_1\right)\vec{v}_0\right] = Q_1$$

$$Q_1(r,\theta) = \sum_l q_l(r) X_l(\theta)$$

$$\rho_1(r,\theta) = \sum_l R_l(r) X_l(\theta), \quad p_1(r,\theta) = \sum_l \Pi_l(r) X_l(\theta)$$

$$\vec{v}_1(r,\theta) = \hat{r}\sum_l U_l(r) X_l(\theta) + \hat{\theta}\sum_l V_l(r) Y_l(\theta)$$

$$\vec{\nabla}\rho_1(r,\theta) = \hat{r}\sum_l \frac{\partial R_l}{\partial r} X_l + \hat{\theta}\sum_l R_l \frac{\partial X_l}{\partial\theta}$$

$$\vec{\nabla}\cdot\vec{v}_1 = \frac{1}{r^2}\sum_l \frac{\partial}{\partial r}(r^2 U_l) X_l + \frac{1}{r\sin\theta}\sum_l V_l \frac{\partial}{\partial\theta}(\sin\theta\, Y_l)$$

$$(\mathrm{III}_1, a) \Leftrightarrow \vec{\nabla}\left(\tfrac{1}{2}\rho_0 v_0^2 + \tfrac{\gamma}{\gamma-1}p_0\right)\vec{v}_1 + \left(\tfrac{1}{2}\rho_0 v_0^2 + \tfrac{\gamma}{\gamma-1}p_0\right)\vec{\nabla}\vec{v}_1 + \vec{\nabla}\left(\tfrac{1}{2}\rho_1 v_0^2 + \rho_0\vec{v}_0\cdot\vec{v}_1 + \tfrac{\gamma}{\gamma-1}p_1\right)\vec{v}_0 + \left(\tfrac{1}{2}\rho_1 v_0^2 + \rho_0\vec{v}_0\cdot\vec{v}_1 + \tfrac{\gamma}{\gamma-1}p_1\right)\vec{\nabla}\vec{v}_0 = Q_1$$

$$\Leftrightarrow \frac{\partial}{\partial r}\left(\tfrac{1}{2}\rho_0 v_0^2 + \tfrac{\gamma}{\gamma-1}p_0\right)\sum_l U_l X_l + \left(\tfrac{1}{2}\rho_0 v_0^2 + \tfrac{\gamma}{\gamma-1}p_0\right)\left[\frac{1}{r^2}\sum_l \frac{\partial}{\partial r}(r^2 U_l) X_l + \frac{1}{r\sin\theta}\sum_l V_l \frac{\partial}{\partial\theta}(\sin\theta\, Y_l)\right]$$
$$+ \frac{\partial}{\partial r}\left(\tfrac{1}{2}\rho_1 v_0^2 + \rho_0\vec{v}_0\cdot\vec{v}_1 + \tfrac{\gamma}{\gamma-1}p_1\right)v_0 + \left(\tfrac{1}{2}\rho_1 v_0^2 + \rho_0\vec{v}_0\cdot\vec{v}_1 + \tfrac{\gamma}{\gamma-1}p_1\right)\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_0) = Q_1$$

$$\Leftrightarrow \frac{\partial}{\partial r}\left(\tfrac{1}{2}\rho_0 v_0^2 + \tfrac{\gamma}{\gamma-1}p_0\right)\sum_l U_l X_l + \left(\tfrac{1}{2}\rho_0 v_0^2 + \tfrac{\gamma}{\gamma-1}p_0\right)\left[\frac{1}{r^2}\sum_l \frac{\partial}{\partial r}(r^2 U_l) X_l + \frac{1}{r\sin\theta}\sum_l V_l \frac{\partial}{\partial\theta}(\sin\theta\, Y_l)\right]$$
$$+ v_0\sum_l \frac{\partial}{\partial r}\left(\tfrac{1}{2}R_l v_0^2 + \rho_0 v_0 U_l + \tfrac{\gamma}{\gamma-1}\Pi_l\right)X_l + \sum_l\left(\tfrac{1}{2}R_l v_0^2 + \rho_0 v_0 U_l + \tfrac{\gamma}{\gamma-1}\Pi_l\right)X_l \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_0) = \sum_l q_l X_l$$

again each mode independent and same condition to factor out θ-dependence (don't forget α_l factor here)

$$\Leftrightarrow U_l\frac{\partial}{\partial r}\left(\tfrac{1}{2}\rho_0 v_0^2 + \tfrac{\gamma}{\gamma-1}p_0\right) + \left(\tfrac{1}{2}\rho_0 v_0^2 + \tfrac{\gamma}{\gamma-1}p_0\right)\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U_l) + \frac{\alpha_l}{r}V_l\right]$$
$$+ \frac{\partial}{\partial r}\left(\tfrac{1}{2}R_l v_0^2 + \rho_0 v_0 U_l + \tfrac{\gamma}{\gamma-1}\Pi_l\right)v_0 + \left(\tfrac{1}{2}R_l v_0^2 + \rho_0 v_0 U_l + \tfrac{\gamma}{\gamma-1}\Pi_l\right)\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_0) = q_l$$

(eq. (D) in Per's notes)

$$\Leftrightarrow \left[U_l\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U_l) + \frac{\alpha_l}{r}V_l\right]\left(\tfrac{1}{2}\rho_0 v_0^2 + \tfrac{\gamma}{\gamma-1}p_0\right) + \left[v_0\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_0)\right]\left(\tfrac{1}{2}R_l v_0^2 + \rho_0 v_0 U_l + \tfrac{\gamma}{\gamma-1}\Pi_l\right) = q_l \qquad (\mathrm{III}_1, b)$$

Full system of equations

relations to the physical perturbations

$$Q_1(r,\theta) = \sum_l q_l(r) X_l(\theta), \quad \rho_1(r,\theta) = \sum_l R_l(r) X_l(\theta), \quad p_1(r,\theta) = \sum_l \Pi_l(r) X_l(\theta)$$

$$\vec{v}_1(r,\theta) = \hat{r}\sum_l U_l(r) X_l(\theta) + \hat{\theta}\sum_l V_l(r) Y_l(\theta)$$

θ-dependence

$$Y_l(\theta) = -\frac{\alpha_l}{l(l+1)}\frac{\partial X_l}{\partial\theta}, \quad X_l(\theta) = P_l(\cos\theta) = \text{associated Legendre polynomial with } m=0$$

$\alpha_l \neq 0$ and arbitrary (in Per's notes $l=1$, $\alpha_l = 2$) (either $\alpha_l = -2$ or there is some sign error somewhere)

radial part of the mode l

$$\frac{\partial\rho_0}{\partial r}U_l + \rho_0\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U_l) + v_0\frac{\partial R_l}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_0)R_l + \alpha_l\frac{\rho_0}{r}V_l = 0 \qquad (\mathrm{I}_1, b) \qquad \text{(A in Per's notes)}$$

$$\rho_0 v_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U_l) + \frac{2}{r}V_l\right] + \rho_0 U_l\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_0) + R_l v_0\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_0) = -\frac{\partial\Pi_l}{\partial r} \qquad (\mathrm{II}_1, b_1) \qquad \text{(corresponds to his B, but wrong)}$$

$$\frac{1}{\Pi_l}\rho_0 V_l\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_0) = \frac{l(l+1)}{\alpha_l} \qquad (\mathrm{II}_1, b_2) \qquad \text{(" " C, but wrong)}$$

$$\left[U_l\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U_l) + \frac{\alpha_l}{r}V_l\right]\left(\tfrac{1}{2}\rho_0 v_0^2 + \tfrac{\gamma}{\gamma-1}p_0\right) + \left[v_0\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_0)\right]\left(\tfrac{1}{2}R_l v_0^2 + \rho_0 v_0 U_l + \tfrac{\gamma}{\gamma-1}\Pi_l\right) = q_l \qquad (\mathrm{III}_1, b) \qquad \text{(D in Per's notes)}$$