Spherical part summary for numerical calculation

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Normalization relations (barred quantities are non-normalized, starred are value at sonic radius)

$p_0 = \tilde{p}_0/p_*$ $T_0 = \tilde{T}_0/T_*$ $v_0 = \tilde{v}_0/v_*$ $q_0 = \tilde{q}_0/q_*$ $\rho_0 = \tilde{\rho}_0/\rho_*$

$E_0 = \tilde{E}_0/E_*$ $r = \tilde{r}/r_*$ $\Lambda(\tilde{E}_0) = \tilde{\Lambda}(\tilde{E}_0)/\Lambda(E_*)$ $L(\tilde{E}_0) = \tilde{L}(\tilde{E}_0)/[E_* \tilde{\Lambda}(E_*)]$

Relations at the sonic radius

$$\rho_* = m\frac{p_*}{T_*}$$

$$r_*^2 \frac{p_*}{T_*} v_* = \frac{G}{4\pi}$$

$$v_* = \sqrt{\frac{\gamma T_*}{m}}$$

$$\lambda_* = \frac{p_*}{T_*}\Lambda_* r_*$$

$$(\gamma-1)\frac{4\pi}{G}\frac{\mu}{\gamma}\lambda_*\left(q_*\frac{r_*^2}{T_*}\right) = 2$$

relations to get pressure and density (normalized)

$$\rho_0 = \frac{p_0}{T_0}$$

$$r^2\frac{p_0}{T_0}v_0 = 1$$

Normalized system of differential equations

$$\frac{\partial v_0^2}{\partial r} = \frac{4v_0^2}{(v_0^2 - T_0)}\left[\frac{T_0}{r} - \frac{q_0\Lambda(E_0)}{v_0}\right] \qquad (I_0)$$

$$\frac{\partial T_0}{\partial r} = \frac{2q_0\Lambda(E_0)}{v_0} - \frac{(\gamma-1)}{2}\frac{\partial v_0^2}{\partial r} \qquad (II_0)$$

$$\frac{\partial q_0}{\partial r} = \lambda_*\frac{q_0\Lambda(E_0)}{r^2 v_0} \qquad (III_0)$$

$$\frac{\partial E_0}{\partial r} = 2\lambda_* \frac{L(E_0)}{r^2 v_0} \qquad (\underline{IV}_0)$$

singularity for $\left.\frac{\partial v_0^2}{\partial r}\right|_{r=1}$:

$$\left.\frac{\partial v_0^2}{\partial r}\right|_{r=1} = \frac{2}{\gamma+1}\left[-(\gamma-3) + \sqrt{(\gamma-3)^2 - 2(\lambda_* + \Psi_* - 1)(\gamma+1)}\right]$$

with $\Psi_* = 2\lambda_* L(E_*) \left.\frac{\partial \Lambda}{\partial E}\right|_{E=E_*}$

Empirical energy attenuation cross sections

$$\tilde{\Lambda}(E) = \hat{\sigma}_T(E) + \frac{2\tilde{L}(E)}{E}$$

$$\tilde{L}(E) = 8.62 \cdot 10^{-15} \left[\left(\frac{E}{100}\right)^{0.823} + \left(\frac{E}{60}\right)^{-0.125} + \left(\frac{E}{48}\right)^{-1.94}\right]^{-1} eV \cdot cm^2$$

$$\hat{\sigma}_T(E > 100\,eV) = \frac{8.8 \cdot 10^{-13}}{E^{1.71}} - \frac{1.62 \cdot 10^{-12}}{E^{1.932}} \; cm^2$$

$$\hat{\sigma}_T(E < 100\,eV) = \frac{1.1 \cdot 10^{-14}}{E} \; cm^2$$

boundary conditions (normalized quantities)

at sonic radius:

$$v_0^2(1) = 1 \qquad T_0(1) = 1 \qquad q_0(1) = 1 \qquad E_0(1) = 1$$

$$\Lambda(E_0(1)) = 1 \qquad L(\tilde{E}_0(1)) = \tilde{L}(E_*)/(E_* \tilde{\Lambda}(E_*))$$

at pellet radius:

$$v_0^2(r_p) = ? \qquad T_0(r_p) = 0 \qquad q_0(r_p) = 0 \qquad E_0(r_p) = ?$$

$$r_p = \tilde{r}_p / r_*$$

at cloud boundary $(r \to \infty)$:

$$\rho(\infty) = 0$$

$$v_0^2(\infty) = ? \qquad T_0(\infty) = ? \qquad q(\infty) = q_{bg} \qquad E(\infty) = E_{bg}$$

make boundary value problem for $0 \leq \hat{r} \leq 1$

we want to solve eq. system (I_0) to (IV_0)

for a given pellet radius r_p, plasma temperature T_{bg}, plasma electron density $n_{e,bg}$

this eq. system is in terms of the variable $r = \frac{\tilde{r}}{r_*}$

which is $r=1$ at the sonic radius but $r=$ unknown at pellet radius

transform the eq. system into one in terms of the

variable $\hat{r}$ so that $\hat{r}=1$ at sonic radius and $\hat{r}=0$ at pellet radius

$$\Rightarrow \hat{r}(\tilde{r}=r_*)=1 \text{ and } \hat{r}(\tilde{r}=r_p)=0 \Rightarrow \quad \hat{r} = \frac{\tilde{r}-r_p}{r_*-r_p}$$

but we need $r = r(\hat{r})$ to transform the system $\quad \left(\hat{r}\in[0,1], \; r \in [\frac{r_p}{r_*}, 1], \; \tilde{r}\in[r_p, r_*]\right)$

$$r = \frac{\tilde{r}}{r_*} \Leftrightarrow \tilde{r} = r\, r_*$$

$$\Rightarrow \hat{r} = \frac{r\, r_* - r_p}{r_* - r_p} \Leftrightarrow \frac{1}{r_*}\left[(r_* - r_p)\hat{r} + r_p\right] = r$$

$$\Leftrightarrow r = \left(1 - \frac{r_p}{r_*}\right)\hat{r} + \frac{r_p}{r_*} \qquad \text{define} \quad \eta_* = \frac{r_p}{r_*} \quad \text{(dimensionless)}$$

$$r = (1-\eta_*)\hat{r} + \eta_*$$

$$\Rightarrow \frac{dr}{d\hat{r}} = 1 - \eta_* \Leftrightarrow dr = (1-\eta_*)\, d\hat{r}$$

Transform eq. sys.

$$(\mathrm{I}_0) \Leftrightarrow \frac{dv_0^2}{d\hat{r}} = (1-\eta_*)\frac{4 v_0^2}{(v_0^2 - T_0)}\left[\frac{T_0}{(1-\eta_*)\hat{r} + \eta_*} - \frac{q_0 \Lambda}{v_0}\right]$$

$$(\mathrm{II}_0) \Leftrightarrow \frac{dT_0}{d\hat{r}} = (1-\eta_*)\frac{2 q_0 \Lambda}{v_0} - \frac{\gamma - 1}{2}\frac{dv_0^2}{d\hat{r}}$$

$$(\mathrm{III}_0) \Leftrightarrow \frac{dq_0}{d\hat{r}} = (1-\eta_*)\,\lambda_*\,\frac{q_0 \Lambda}{v_0}\left((1-\eta_*)\hat{r} - \eta_*\right)^{-2}$$

$$\left(\boxed{\text{IV}}_0\right) \Leftrightarrow \frac{dE_0}{d\hat{r}} = (1-\eta_*)2\lambda_* \frac{L}{v_0}\left((1-\eta_*)\hat{r}-\eta_*\right)^{-2}$$

$$\left.\frac{dv_0^2}{d\hat{r}}\right|_{\hat{r}=1} = (1-\eta_*)\left.\frac{dv_0^2}{dr}\right|_{r=1}$$

now there are 4 eqs with 2 unknown parameters η_*, λ_*

boundary conditions:

$v_0^2(\hat{r}=1)=1$, $T_0(\hat{r}=1)=1$, $q_0(\hat{r}=1)=1$, $E_0(\hat{r}=1)=1$

$T_0(\hat{r}=0)=0$, $q_0(\hat{r}=0)=0$

to be able to calculate Λ and L we need E_*

so instead of setting $\tilde{E}(\tilde{r}\to\infty)=E_{bg}$ we set E_*

and in the second step find out which E_{bg} this corresponds to

now we have enough to solve the boundary value problem $\hat{r}\in[0,1]$

numerical solution via scipy.integrate.solve_bvp yields

values for η_*, λ_*

and the values of v_0^2, T_0, q_0, E_0 on a chosen grid on $\hat{r}\in[0,1]$

To find E_{bg} and q_{bg}, we then solve the system with known η_*, λ_*, E_*

as an initial value problem $\hat{r}\in[1,\infty)$

where we stop at some $\hat{r}$ where all parameters have converged to be constant

Then we know $v_0^2(\hat{r}\to\infty)$, $T_0(\hat{r}\to\infty)$, $E_0(\hat{r}\to\infty)$, $q_0(\hat{r}\to\infty)$

Then we also know $E_{bg}=E_0(\hat{r}\to\infty)E_*$ and through $q_{bg}=\frac{n_{e,bg}\,v_{e,bg}}{4}E_{bg}$ (Park eq. 1)

we can find q_*

<u>unknown still: v_*, T_*, P_*, G</u>

$$\lambda_* = \frac{P_*}{T_*}\Lambda_* r_* \qquad (I_*)$$

$$\lambda_* = \frac{\rho_*}{T_*} \Lambda_* r_* \qquad (\mathrm{I}_*)$$

$$r_*^2 \frac{\rho_*}{T_*} v_* = \frac{G}{4\pi} \qquad (\mathrm{II}_*)$$

$$v_* = \sqrt{\frac{\gamma T_*}{m}} \qquad (\mathrm{III}_*)$$

$$(\gamma-1) \frac{4\pi}{G} \frac{\mu}{\gamma} \lambda_* \left(q_* \frac{r_*^2}{T_*} \right) = 2 \qquad (\mathrm{IV}_*)$$

solve (I_*) to (IV_*) numerically for v_*, ρ_*, T_*, G