Aufgabe 6

a) $f(x) = N \cdot \exp(-x/\tau) \qquad x \in [0, \infty)$

Normieren:

$$\int_0^\infty f(x)\,dx \overset{!}{=} 1$$

$$= \int_0^\infty N \exp(-x/\tau)\,dx$$

$$= N \left[(-\tau)\exp(-x/\tau)\right]_0^\infty$$

$$= N(-\tau)\left[0 - 1\right]$$

$$= N\tau \overset{!}{=} 1$$

$$\Leftrightarrow \quad N = \frac{1}{\tau}$$

Verteilungsfunktion:

$$F(x) = \int_0^x f(x)\,dx$$

$$= \int_0^x \frac{1}{\tau} \exp\left(-\frac{x}{\tau}\right) dx$$

$$= \frac{1}{\tau}\left[(-\tau)\exp\left(-\frac{x}{\tau}\right)\right]_0^x$$

$$= (-1)\left[\exp\left(-\frac{x}{\tau}\right) - 1\right]$$

$$= 1 - \exp(-\frac{x}{\tau})$$

Invertieren:

$$F(x) = 1 - \exp(-\frac{x}{\tau}) = y(x)$$

$$\Leftrightarrow \exp(-\frac{x}{\tau}) = 1 - y$$

$$\Leftrightarrow -\frac{x}{\tau} = \ln(1-y)$$

$$\Leftrightarrow x = (-\tau)\ln(1-y)$$

$$\Rightarrow F^{-1}(y) = (-\tau)\ln(1-y) \quad y \in [0,1)$$

b)

$$f(x) = N \cdot x^{-n} \quad x \in [x_{min}, x_{max}] \quad n \geq 2$$

Normierung:

$$\int_{x_{min}}^{x_{max}} f(x)dx \overset{!}{=} 1$$

$$= \int_{x_{min}}^{x_{max}} N \cdot x^{-n} dx$$

$$= N \cdot \left[\frac{1}{(-n+1)} x^{-n+1}\right]_{x_{min}}^{x_{max}}$$

$$= N \frac{1}{(1-n)} \cdot \left[x_{max}^{1-n} - x_{min}^{1-n}\right] \overset{!}{=} 1$$

$$\Leftrightarrow N = (1-n)(x_{max}^{1-n} - x_{min}^{1-n})^{-1}$$

Verteilungsfunktion:

$$F(x) = \int_{x_{min}}^{x} f(x)\,dx$$

$$= \int_{x_{min}}^{x} N x^{-n}\,dx$$

$$= N \frac{1}{(-n+1)} \left[x^{-n+1} \right]_{x_{min}}^{x}$$

$$= N \frac{1}{(1-n)} \left[x^{-n+1} - x_{min}^{-n+1} \right]$$

Invertierung:

$$F(x) = N \frac{1}{(1-n)} \left(x^{1-n} - x_{min}^{1-n} \right) = y(x)$$

$$\Leftrightarrow \quad x^{1-n} - x_{min}^{1-n} = N^{-1} (1-n)\; y$$

$$\Leftrightarrow \quad x^{1-n} = N^{-1} (1-n)\; y + x_{min}^{1-n}$$

$$\Leftrightarrow \quad x = \left(N^{-1} (1-n)\; y + x_{min}^{1-n} \right)^{\frac{1}{1-n}}$$

$$\Rightarrow \quad F^{-1}(y) = \left(N^{-1} (1-n)\; y + x_{min}^{1-n} \right)^{\frac{1}{1-n}} \qquad y \in [0,1)$$

c)

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2} \qquad x \in \mathbb{R}$$

schon normiert

Verteilungsfunktion:

$$F(x) = \int_{-\infty}^{x} \frac{1}{\pi} \frac{1}{1+x^2} \, dx$$

$$= \frac{1}{\pi} \int_{-\infty}^{x} \frac{1}{1+x^2} \, dx$$

$$= \frac{1}{\pi} \left[\arctan(x) \right]_{-\infty}^{x}$$

$$= \frac{1}{\pi} \left[\arctan(x) + \frac{\pi}{2} \right]$$

$$= \frac{1}{\pi} \arctan(x) + \frac{1}{2}$$

Invertieren:

$$F(x) = \frac{1}{\pi} \arctan(x) + \frac{1}{2} = y(x)$$

$$\Leftrightarrow \quad \arctan(x) = \pi \left(y - \frac{1}{2} \right)$$

$$x = \tan\left(\pi \left(y - \frac{1}{2} \right) \right)$$

$$\Rightarrow F^{-1}(y) = \tan\left(\pi \left(y - \frac{1}{2} \right) \right) \qquad y \in [0, 1)$$