blatt02_Guth_Venker_Jaekel

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1 Abgabe SMD Blatt 01

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1.1 Aufgabe 1: Würfel

a)

$$
\begin{aligned}
P(W_r + W_b = 9) &= P(W_r = 3 \wedge W_b = 6) + P(W_r = 4 \wedge W_b = 5) \\
&+ P(W_r = 6 \wedge W_b = 3) + P(W_r = 5 \wedge W_b = 4) \\
&= \frac{4}{36}
\end{aligned}
$$

b)

$$
\begin{aligned}
P(W_r + W_b \geq 9) &= P(W_r \geq 3 \wedge W_b = 6) + P(W_r \geq 4 \wedge W_b = 5) + ... + P(W_r = 6 \wedge W_b \geq 3) + ... \\
&= \frac{10}{36}
\end{aligned}
$$

c)

$$
\begin{aligned}
P((W_r = 4 \wedge W_b = 5) \vee (W_r = 5 \wedge W_b = 4)) &= P(W_r = 4 \wedge W_b = 5) + P(W_r = 5 \wedge W_b = 4) \\
&= \frac{2}{36}
\end{aligned}
$$

d)

$$
\begin{aligned}
P(W_r = 4 \wedge W_b = 5) &= P(W_r = 4) \cdot P(W_b = 5) \\
&= \frac{1}{36}
\end{aligned}
$$

e)

$$
\begin{aligned}
P(W_r + W_b = 9 | W_r = 4) &= P(W_b = 5 | W_r = 4) \\
&= P(W_b = 5) \\
&= \frac{1}{6}
\end{aligned}
$$

f)

$$
\begin{aligned}
P(W_r + W_b \geq 9 | W_r = 4) &= P(W_b = 5) + P(W_b = 6) \\
&= \frac{1}{3}
\end{aligned}
$$

g)

$$P(W_b = 5|W_r = 4) = P(W_b = 5)$$
$$= \frac{1}{6}$$

1.2 Aufgabe 4 *Maxwellsche Geschwindigkeitsverteilung* (Code Teil)

$f(v) = N \cdot \exp(\frac{-m\,v^2}{2\,k_\mathrm{B}\,T})\;4 \cdot \pi\,v^2$

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
import scipy as sc
from scipy import special
from scipy.optimize import root_scalar
```

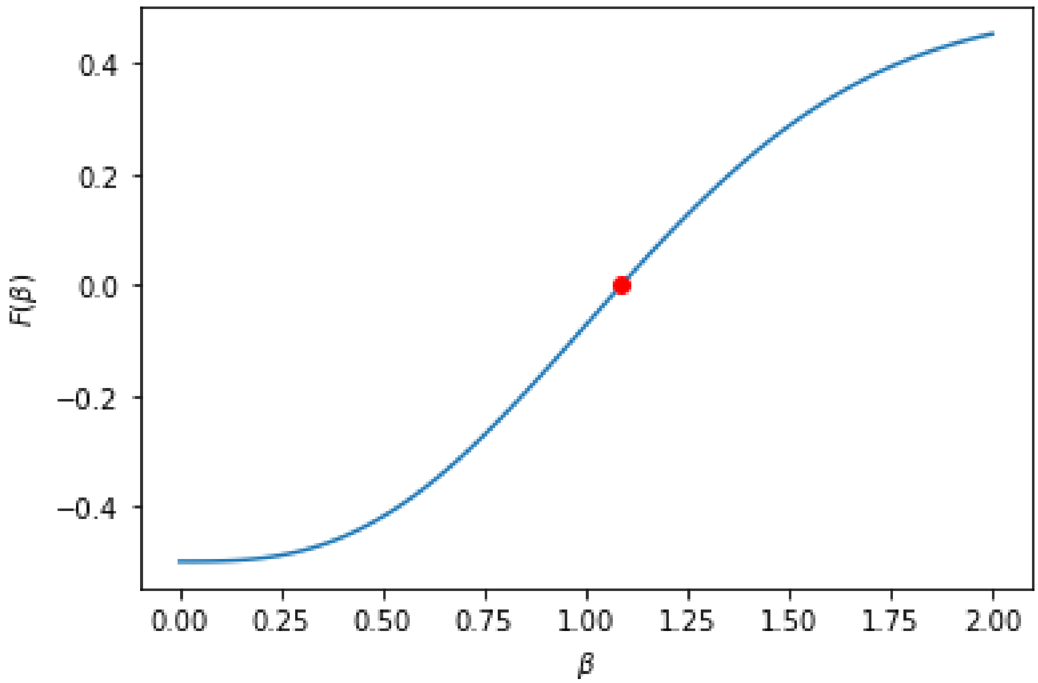
1.3 c) (Code Teil)

$F(\beta) := \mathrm{erf}(\beta) - \frac{2}{\sqrt{\pi}}\beta\exp(-\beta^2) - \frac{1}{2} \stackrel{!}{=} 0$
mit $\beta := \frac{v_{0,5}}{v_m}$

[2]:
```
def F(b):
    return( special.erf(b) - 2/np.sqrt(np.pi) * b * np.exp(- (b)**2) - 1/2  )
```

[3]:
```
x = np.linspace(0, 2, 100)
plt.plot(x,F(x))
v_0 = sc.optimize.root_scalar(F, x0=0,x1=1 ).root
plt.plot(v_0,F(v_0), "ro")
plt.xlabel(r'$\beta$')
plt.ylabel(r'$F(\beta)$')
print("Nullstelle:", v_0)
```

```
Nullstelle: 1.087652031758168
```

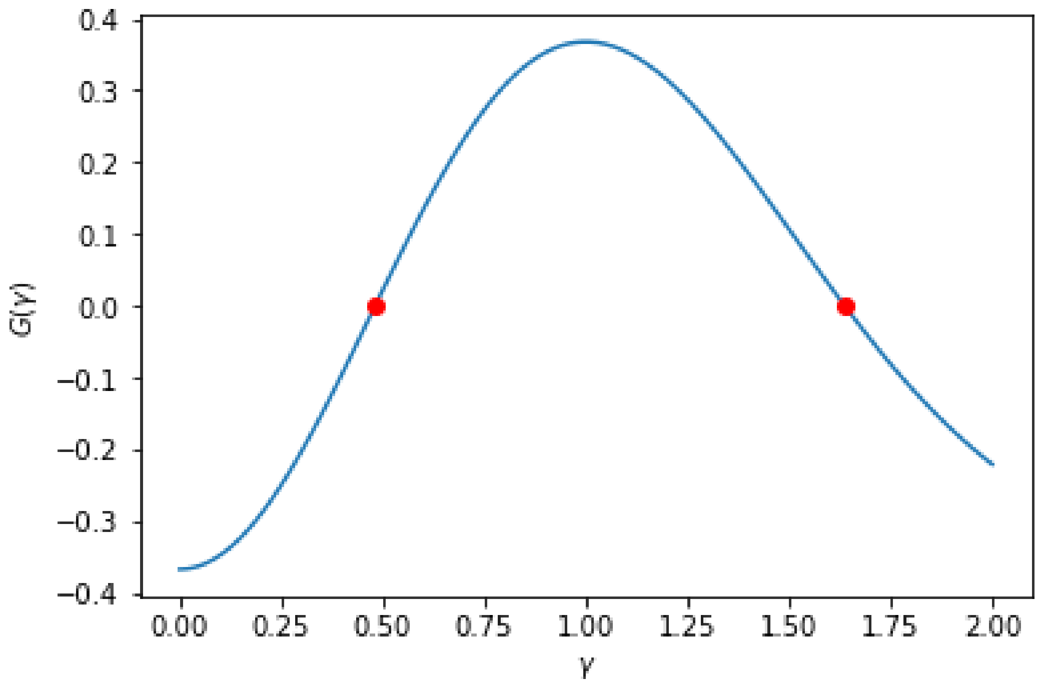


1.4 d) (Code Teil)

$G(\gamma) := 2\gamma^2 \exp(-\gamma^2) - \exp(-1) \stackrel{!}{=} 0$
mit $\gamma := \frac{v_w}{v_m}$

[4]:
```
def G(g):
    return ( 2 * g**2 * np.exp(- (g)**2)  - np.exp(-1))
```

[5]:
```
y = np.linspace(0,2,100)
plt.plot(y,G(y))
v_1 = sc.optimize.root_scalar(G, x0=0,x1=1 ).root
v_2 = sc.optimize.root_scalar(G, x0=1,x1=2 ).root
plt.plot(v_1,G(v_1), "ro")
plt.plot(v_2,G(v_2), "ro")
plt.xlabel(r'$\gamma$')
plt.ylabel(r'$G(\gamma)$');
```



[6]:
```
print("Linke Nullstelle:", v_1)
print("Rechte Nullstelle:", v_2)
print("Breite:",v_2 - v_1)
```

```
Linke Nullstelle: 0.4816232479713991
Rechte Nullstelle: 1.6365656082224935
Breite: 1.1549423602510944
```

[ ]:
```
```

Maxwellsche Geschwindigkeitsverteilung

Montag, 10. Mai 2021 15:36

(4)

$$f(v) = N \cdot \exp\left(\frac{-mv^2}{2k_BT}\right) \cdot 4\pi v^2$$

$$1 = \int_0^\infty N \cdot \exp\left(\frac{-mv^2}{2k_BT}\right) 4\pi v^2 \, dv$$

$$1 = 4\pi N \cdot \int_0^\infty \exp\left(\frac{-m}{2k_BT} \cdot v^2\right) v^2 \, dv$$

$$= 4\pi N \cdot \int_0^\infty -\frac{2k_BT}{\cancel{v^2}} \cdot \frac{d}{dm} \exp\left(-\frac{m}{2k_BT} \cdot v^2\right) \cancel{v^2} \, dv$$

$$= -8\pi N k_BT \cdot \frac{d}{dm} \int_0^\infty \exp\left(-\frac{m}{2k_BT} \cdot v^2\right) dv$$

$$= -4\pi N k_BT \frac{d}{dm} \cdot \sqrt{\pi \cdot \frac{2k_BT}{m}}$$

$$= -4\pi N k_BT \cdot \sqrt{2\pi k_BT} \cdot \frac{d}{dm} m^{-\frac{1}{2}}$$

$$1 = 2\pi N k_BT \sqrt{2\pi k_BT} \cdot m^{-\frac{3}{2}}$$

$$\Leftrightarrow N = \frac{m^{\frac{3}{2}}}{2\pi k_BT \sqrt{2\pi k_BT}}$$

$$= \left(\frac{m}{2\pi k_BT}\right)^{3/2}$$

$$f(v) = \left(\frac{m}{2\pi k_BT}\right)^{3/2} \cdot \exp\left(\frac{-mv^2}{2k_BT}\right) \cdot 4\pi v^2$$

a) Gesucht Maximum von $f(v)$

$$f'(v) = \underbrace{4\pi \left(\frac{m}{2\pi k_BT}\right)^{3/2}}_{c} \frac{d}{dv} \exp\left(\frac{-mv^2}{2k_BT}\right) v^2$$

$$0 \overset{!}{=} c \cdot \left(\exp\left(\frac{-mv^2}{2k_BT}\right) \cdot 2v - \frac{mv}{k_BT} \cdot \exp\left(\frac{-mv^2}{2k_BT}\right) \cdot v^2\right)$$

$$= c \cdot \exp(\sim) \left(2v - \frac{mv}{k_BT} \cdot v^2\right)$$

$$0 \overset{!}{=} -v^3 \frac{m}{k_BT} + 2v$$

$$= v \cdot \left(-v^2 \frac{m}{k_BT} + 2\right) \qquad 0 = -v^2 \frac{m}{k_BT} + 2$$

$\hookrightarrow v_1 = 0$

$$\frac{2k_BT}{m} = v^2$$

$$\boxed{v_{2/3} = \pm\sqrt{\frac{2k_BT}{m}} = v_m} \quad \text{Nur } v_2 = +\sqrt{\frac{2k_BT}{m}}$$

$$f(v_m) = \left(\frac{m}{2\pi k_BT}\right)^{3/2} \cdot \exp\left(\frac{-\cancel{m}}{\cancel{2k_BT}} \cdot \frac{\cancel{2k_BT}}{\cancel{m}}\right) \cdot 4\pi \frac{2k_BT}{m}$$

$$= \left(\frac{m}{2\pi k_BT}\right)^{1/2} \cdot \exp(-1) \, 4$$

$$= \frac{1}{v_m} \frac{4}{\sqrt{\pi e}}$$

$$f(v_1) = 0$$

b) $$\langle v \rangle = \int_0^\infty v \cdot f(v) \, dv = \int_0^\infty v \cdot \left(\frac{m}{2\pi k_BT}\right)^{3/2} \cdot \exp\left(\frac{-mv^2}{2k_BT}\right) \cdot 4\pi v^2 \, dv$$

$$= 4\pi N \frac{d}{dm} \int_0^\infty v^3 \frac{-2k_BT}{v^2} \exp\left(-\frac{m}{2k_BT} \cdot v^2\right) dv$$

Substitution: $u = -\frac{m}{2k_BT} v^2$

$$\frac{du}{dv} = -\frac{m}{k_BT} v$$

$$\Leftrightarrow dv = -\frac{k_BT}{mv} \cdot du$$

$$= 4\pi N (-2k_BT) \frac{d}{dm} \int_0^\infty v \exp\left(-\frac{m}{2k_BT} \cdot v^2\right) dv$$

$$= 4\pi N (2k_BT) \frac{d}{dm} \int_{u(0)}^{u(\infty)} \frac{k_BT}{m} \exp(u) \, du$$

$$= 8\pi N (k_BT)^2 \frac{d}{dm} \frac{1}{m} \Big[\exp(u)\Big]_{u(0) = 0}^{u(\infty) = -\infty}$$

$$= 8\pi N (k_BT)^2 m^{-2}$$

$$= 8\pi \left(\frac{m}{2\pi k_BT}\right)^{3/2} \cdot \left(\frac{k_BT}{m}\right)^2 = \left(\frac{k_BT}{m}\right)^{1/2} \frac{(2^{1/2})^4 \cdot 2\pi}{(2\pi)^{3/2}} = \frac{(2^{1/2})^3}{(2\pi)^{1/2}} \cdot v_m \quad \boxed{= \frac{2}{\sqrt{\pi}} \cdot v_m = \langle v \rangle}$$

c) $$F(v) = \int^v f(v) \, dv, \quad f(v) = N \cdot \exp\left(\frac{-mv^2}{2k_BT}\right) \cdot 4\pi v^2$$

c) $F(v) = \int_0^v f(v)\,dv$, $f(v) = N \cdot \exp\left(\frac{-mv^2}{2k_BT}\right)\cdot 4\pi v^2$

$$F(v) = 4\pi N \cdot \int_0^V \exp\left(\frac{-mv^2}{2k_BT}\right) v^2\, dV$$

$$= 4\pi N \cdot \int_0^V \frac{d}{dm}\left(-\frac{2k_BT}{\cancel{v^2}}\right)\exp\left(-\frac{mv^2}{2k_BT}\right)\cancel{v^2}\, dV$$

$$= 4\pi N \frac{d}{dm}\int_0^V (-2k_BT)\exp\left(-\frac{mv^2}{2k_BT}\right) dV$$

$\frac{m}{2k_BT}v^2 = u^2$,

$\sqrt{\frac{m}{2k_BT}}\, v = u$, $\frac{du}{dv} = \sqrt{\frac{m}{2k_BT}}$

$$= 4\pi N \frac{d}{dm}(-2k_BT)\cdot\sqrt{\frac{2k_BT}{m}}\int_{u(0)}^{u(v)} \exp(-u^2)\,du$$

$\left[\operatorname{erf} z = \frac{2}{\sqrt\pi}\int_0^z \exp(-t^2)\,dt = \frac{2}{\sqrt\pi}\int \exp(-z^2)\,dz\right]$

$$= 4\pi N \frac{d}{dm}(-2k_BT)\sqrt{\frac{2k_BT}{m}}\cdot\left[\frac{\sqrt\pi}{2}\cdot \operatorname{erf}(u)\right]_{u(0)}^{u(v)}$$

$u(v) = \sqrt{\frac{m}{2k_BT}}\cdot v$, $u(0) = 0$

$$= -4\pi N (2k_BT)^{3/2}\frac{\sqrt\pi}{2}\frac{d}{dm}\left[\frac{1}{\sqrt m}\cdot \operatorname{erf}\left(\sqrt{\frac{m}{2k_BT}}\cdot v\right)\right]$$

$\left[\text{mit } \frac{d}{dx}\operatorname{erf}(x) = \frac{2}{\sqrt\pi}\exp(-x^2)\right]$

$$F(v) = -4\pi N (2k_BT)^{3/2}\frac{\sqrt\pi}{2}\left\{-\frac12 (m)^{-\frac32}\cdot \operatorname{erf}\left(\frac{1}{v_m}\cdot v\right) + (m)^{-\frac12}\cdot\frac{2}{\sqrt\pi}\cdot\exp\left(-\frac{1}{v_m^2}\cdot v^2\right)\cdot\frac12 (2k_BTm)^{-\frac12}\cdot v\right\}$$

$$N = \left(\frac{m}{2\pi k_BT}\right)^{3/2} = \frac{1}{(\pi)^{3/2}}\cdot\frac{1}{v_m^3}$$

$$F(v) = -4\pi\frac{1}{v_m^3}\cdot(2k_BT)^{3/2}\frac{1}{(\pi)^{3/2}}\frac{\sqrt\pi}{2}\left(-\frac12\frac{1}{m^{3/2}}\cdot \operatorname{erf}\left(\frac{v}{v_m}\right) + \frac{1}{m^{\frac12}}\cdot\frac{2}{\sqrt\pi}\cdot\exp\left(\frac{v^2}{v_m^2}\right)\cdot\frac12\,\frac{1}{(2k_BT)^{1/2}}\,\frac{1}{m^{1/2}}\, v\right)$$

$$= \operatorname{erf}\left(\frac{v}{v_m}\right) - \cancel{4}^{2}\pi\,\frac{1}{(\pi)^{3/2}}\,\frac{1}{v_m^3}\,\frac{\sqrt\pi}{2}\,\frac{\cancel{2}}{\sqrt\pi}\cdot\cancel{\frac12}\cdot(2k_BT)^{3/2}\,\frac{1}{(2k_BT)^{1/2}}\,\frac{1}{m^{\frac12}}\,\frac{1}{m^{1/2}}\, v\,\exp\left(-\frac{v^2}{v_m^2}\right)$$

$$= \operatorname{erf}\left(\frac{v}{v_m}\right) - 2\cdot v\,\frac{1}{v_m^3}\cdot\frac{1}{\pi^{1/2}}\,v_m^2\,\exp\left(-\frac{v^2}{v_m^2}\right)$$

$$F(v) = \operatorname{erf}\left(\frac{v}{v_m}\right) - 2v\cdot v_m^{-1}\,\pi^{-\frac12}\exp\left(-\frac{v^2}{v_m^2}\right)$$

$F(v_{0,5}) = 0{,}5$

$$0{,}5 = \operatorname{erf}\left(\frac{v_{0,5}}{v_m}\right) - \frac{2\,v_{0,5}\cdot\exp\left(-\frac{v_{0,5}^2}{v_m^2}\right)}{v_m\,\sqrt\pi}$$

$\beta = \frac{v_{0,5}}{v_m}$

$$\beta = \frac{v_{0,5}}{v_m}$$

$$0 = \operatorname{erf}(\beta) - \frac{2}{\sqrt{\pi}} \beta \cdot \exp(-\beta^2) - \frac{1}{2}$$

Muss numerisch gelöst werden: $\beta = 1,0877$

$$\boxed{\Rightarrow v_{0,5} = \beta \cdot v_m = 1,0877 \cdot v_m}$$

d) Maximum bei $v = v_m$, $v_{FWHM} = v_w = \frac{1}{2} v_m$

$$2 \cdot f(v_w) = f(v_m)$$

$$f(v) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \cdot \exp\left(\frac{-mv^2}{2k_B T}\right) \cdot 4\pi v^2$$

$$= \frac{1}{\pi^{3/2}} \cdot \frac{1}{v_m^3} \cdot \exp\left(-\frac{v^2}{v_m^2}\right) \cdot 4\pi v^2$$

$$= \frac{4}{\sqrt{\pi}} \frac{v^2}{v_m^3} \cdot \exp\left(-\frac{v^2}{v_m^2}\right)$$

$$2 \cdot f(v_w) = 2 \cdot \frac{\cancel{4}}{\sqrt{\pi}} \frac{v_w^2}{v_m^3} \cdot \exp\left(-\frac{v_w^2}{v_m^2}\right) = \frac{\cancel{4}}{\sqrt{\pi}} \cdot \frac{1}{v_m} \cdot \exp(-1)$$

$$2 \frac{v_w^2}{v_m^2} \exp\left(-\frac{v_w^2}{v_m^2}\right) = \exp(-1)$$

$$\Rightarrow 0 = 2 \frac{v_w^2}{v_m^2} \exp\left(-\frac{v_w^2}{v_m^2}\right) - \exp(-1)$$

$$\frac{v_w}{v_m} = \gamma, \quad 0 = 2\gamma^2 \exp(-\gamma^2) - \exp(-1)$$

Numerische Lösung: Linke Nullstelle: $v_1 = 0,4816$

Rechte Nullstelle: $v_2 = 1,6366$

Breite: $v_{FWHM} = v_2 - v_1 = 1,1549$

e) $Var[X] = \langle X^2 \rangle - \langle x \rangle^2$

$$\langle v^2 \rangle = \int_{-\infty}^{\infty} v^2 \cdot f(v)\, dv = \int_0^{\infty} v^2 \cdot \left(\frac{m}{2\pi k_B T}\right)^{3/2} \cdot \exp\left(\frac{-mv^2}{2k_B T}\right) \cdot 4\pi v^2 \, dv^2$$

$$= \left(\frac{m}{2\pi k_B T}\right)^{3/2} \cdot 4\pi \int_0^{\infty} v^4 \cdot \exp\left(\frac{-mv^2}{2k_B T}\right) dv = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \cdot 4\pi \frac{d}{dm} \int_0^{\infty} v^4 \cdot \frac{-2k_B T}{v^2} \cdot \exp\left(-\frac{mv^2}{2k_B T}\right) dv$$

$$= \left(\frac{m}{2\pi k_B T}\right)^{3/2} \cdot 4\pi \frac{d^2}{dm^2} \int_0^{\infty} \cancel{v^4} \cdot \left(-\frac{2k_B T}{\cancel{v^2}}\right)^2 \cdot \exp\left(-\frac{mv^2}{2k_B T}\right) dv = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \cdot 4\pi \frac{d^2}{dm^2} \cdot \left(-2k_B T\right)^2 \int_0^{\infty} \exp\left(-\frac{mv^2}{2k_B T}\right) dv$$

$$= \left(\frac{m}{2\pi k_B T}\right)^{3/2} \cdot 4\pi \frac{d^2}{dm^2} \cdot \left(-2k_B T\right)^2 \cdot \frac{1}{2}\sqrt{\pi \cdot \frac{2k_B T}{m}} = \left(\frac{m}{2\pi k_B T}\right)^{3/2} 8\pi \cdot (k_B T)^2 \cdot (\pi \cdot 2k_B T)^{1/2} \cdot \frac{3}{4} \cdot m^{-\frac{5}{2}}$$

$$= \left(\frac{m}{2\pi k_B T}\right)^{3/2} 6\pi \cdot (2\pi)^{1/2} \left(\frac{k_B T}{m}\right)^{5/2} = \left(\frac{m}{\cancel{2\pi} k_B T}\right)^{3/2} 3 \cdot \cancel{(2\pi)^{3/2}} \left(\frac{k_B T}{m}\right)^{5/2}$$

$$= \frac{3k_B T}{} = \langle v^2 \rangle = \frac{3}{2} \frac{2k_B T}{m}$$

$$= \frac{3k_BT}{m} = \langle v^2 \rangle = \frac{3}{2}\,\frac{2k_BT}{m}$$

$$= \frac{3}{2} \cdot v_m^2$$

$$\mathrm{Var}[v] = \langle v^2 \rangle - \langle v \rangle^2$$

$$= \frac{3}{2} v_m^2 - \frac{4}{\pi} \cdot v_m^2$$

$$= \left(\frac{3}{2} - \frac{4}{\pi}\right) v_m^2$$

$$\boxed{\sigma_v = \sqrt{\frac{3}{2} - \frac{4}{\pi}}\; v_m}$$