

Maxwell'sche Geschwindigkeitsverteilung

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(4)

$$f(v) = N \cdot \exp\left(-\frac{mv^2}{2k_B T}\right) \cdot 4\pi v^2$$

$$1 = \int_0^\infty N \cdot \exp\left(-\frac{mv^2}{2k_B T}\right) 4\pi v^2 dv$$

$$1 = 4\pi N \cdot \int_0^\infty \exp\left(-\frac{m}{2k_B T} \cdot v^2\right) v^2 dv$$

$$= 4\pi N \cdot \int_0^\infty \frac{-2k_B T}{v^2} \cdot \frac{d}{dm} \exp\left(-\frac{m}{2k_B T} \cdot v^2\right) v^2 dv$$

$$= -8\pi N k_B T \cdot \frac{d}{dm} \int_0^\infty \exp\left(-\frac{m}{2k_B T} \cdot v^2\right) dv$$

$$= -4\pi N k_B T \cdot \frac{d}{dm} \sqrt{\pi \cdot \frac{m}{2k_B T}}$$

$$= -4\pi N k_B T \cdot \sqrt{\pi \cdot \frac{m}{2k_B T}} \cdot \frac{d}{dm} m^{-\frac{1}{2}}$$

$$1 = 2\pi N k_B T \sqrt{\pi \cdot \frac{m}{2k_B T}} \cdot m^{-\frac{1}{2}}$$

$$\Leftrightarrow N = \frac{m^{\frac{1}{2}}}{2\pi k_B T \sqrt{\pi \cdot \frac{m}{2k_B T}}}$$

$$= \left(\frac{m}{2\pi k_B T}\right)^{\frac{1}{2}}$$

$$f(v) = \left(\frac{m}{2\pi k_B T}\right)^{\frac{1}{2}} \cdot \exp\left(-\frac{mv^2}{2k_B T}\right) \cdot 4\pi v^2$$

a) Gesucht Maximum von $f(v)$

$$f'(v) = \underbrace{4\pi \left(\frac{m}{2\pi k_B T}\right)^{\frac{1}{2}}}_{C} \frac{d}{dv} \exp\left(-\frac{mv^2}{2k_B T}\right) v^2$$

$$0 = C \cdot \left(\exp\left(-\frac{mv^2}{2k_B T}\right) \cdot 2v - \frac{mv}{k_B T} \cdot \exp\left(-\frac{mv^2}{2k_B T}\right) \cdot v^2 \right)$$

$$= C \cdot \exp(-) \cdot \left(2v - \frac{mv}{k_B T} \cdot v^2 \right)$$

$$0 = -v^3 \frac{m}{k_B T} + 2v$$

$$= v \cdot \left(-v^2 \frac{m}{k_B T} + 2 \right)$$

$$\hookrightarrow v_{n=0}$$

$$0 = -v^2 \frac{m}{k_B T} + 2$$

$$\frac{2k_B T}{m} = v^2$$

$$v_{n=3} = \pm \sqrt{\frac{2k_B T}{m}} = v_m \quad \text{Nur } v_2 = +\sqrt{\frac{2k_B T}{m}}$$

$$\begin{aligned} f(v_m) &= \left(\frac{m}{2\pi k_B T}\right)^{\frac{1}{2}} \cdot \exp\left(-\frac{m}{2k_B T} - \frac{2k_B T}{m}\right) \cdot 4\pi \frac{2k_B T}{m} \\ &= \left(\frac{m}{2\pi k_B T}\right)^{\frac{1}{2}} \cdot \exp(-1) \cdot 4 \\ &= \frac{1}{v_m} \frac{4}{\sqrt{\pi m}} \end{aligned}$$

$$f(v_n) = 0$$

$$b) \langle v \rangle = \int_0^\infty v \cdot f(v) dv = \int_0^\infty v \cdot \left(\frac{m}{2\pi k_B T}\right)^{\frac{1}{2}} \cdot \exp\left(-\frac{mv^2}{2k_B T}\right) \cdot 4\pi v^2 dv$$

$$= 4\pi N \frac{d}{dm} \int_0^\infty v^3 \cdot \frac{-2k_B T}{v^2} \exp\left(-\frac{m}{2k_B T} \cdot v^2\right) dv \quad \boxed{\text{Substitution: } u = -\frac{m}{2k_B T} v^2}$$

$$= 4\pi N \left(\frac{d}{dm}\right) \int_0^\infty v \exp\left(-\frac{m}{2k_B T} \cdot v^2\right) dv$$

$$= 4\pi N \left(2k_B T\right) \frac{d}{dm} \int_{u(0)}^{\frac{m}{2k_B T}} \exp(u) du$$

$$\frac{du}{dv} = -\frac{m}{2k_B T} v$$

$$\Rightarrow dv = -\frac{2k_B T}{m v} du$$

$$= 8\pi N \left(\frac{1}{2k_B T}\right)^2 \frac{d}{dm} \frac{1}{m} \left[\exp(u) \right]_{u(0)}^{u(\infty)} = -\infty$$

$$= 8\pi N \left(\frac{1}{2k_B T}\right)^2 m^{-2}$$

$$= 8\pi \left(\frac{m}{2\pi k_B T}\right)^{\frac{1}{2}} \cdot \left(\frac{k_B T}{m}\right)^2 = \left(\frac{k_B T}{m}\right)^{\frac{1}{2}} \cdot \frac{(2\pi)^{\frac{1}{2}} \cdot 2\pi}{(2\pi)^{\frac{1}{2}}} = \frac{(2\pi)^{\frac{3}{2}}}{(2\pi)^{\frac{1}{2}}} \cdot V_m = \frac{2}{\sqrt{\pi}} \cdot V_m = \langle v \rangle$$

$$c) F(v) = \int_v^\infty f(v) dv, \quad f(v) = N \cdot \exp\left(-\frac{mv^2}{2k_B T}\right) \cdot 4\pi v^2$$

$$\text{d}) \quad F(v) = \int_0^v f(v) dv, \quad f(v) = N \cdot \exp\left(\frac{-mv^2}{2k_B T}\right) \cdot 4\pi v^2$$

$$F(v) = 4\pi N \int_0^v \exp\left(\frac{-mv^2}{2k_B T}\right) v^2 dv$$

$$= 4\pi N \int_0^v \frac{d}{dm} \left(-\frac{mv^2}{2k_B T}\right) \exp\left(-\frac{mv^2}{2k_B T}\right) v^2 dv$$

$$= 4\pi N \frac{d}{dm} \int_0^v \left(-\frac{mv^2}{2k_B T}\right) \exp\left(-\frac{mv^2}{2k_B T}\right) dv$$

$$\frac{m}{2k_B T} v^2 = u^2,$$

$$\sqrt{\frac{m}{2k_B T}} v = u, \quad \frac{du}{dv} = \sqrt{\frac{m}{2k_B T}}$$

$$= 4\pi N \frac{d}{dm} \left(-\frac{m}{2k_B T}\right) \cdot \sqrt{\frac{m}{2k_B T}} \int_{u(0)}^{u(v)} \exp(-u^2) du$$

$$\left[\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2) dt = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-z^2) dz \right]$$

$$= 4\pi N \frac{d}{dm} \left(-\frac{m}{2k_B T}\right) \sqrt{\frac{m}{2k_B T}} \cdot \left[\frac{\sqrt{\pi}}{2} \cdot \operatorname{erf}(u) \right]_{u(0)}^{u(v)}$$

$$u(v) = \sqrt{\frac{m}{2k_B T}} \cdot v, \quad u(0) = 0$$

$$= 4\pi N \frac{3k_B T}{2} \frac{d}{dm} \left[\frac{1}{\sqrt{m}} \cdot \operatorname{erf}\left(\sqrt{\frac{m}{2k_B T}} \cdot v\right) \right]$$

$$\left[\text{mit } \frac{d}{dx} \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \exp(-x^2) \right]$$

$$F(v) = -4\pi N (2k_B T)^{3/2} \frac{\sqrt{\pi}}{2} \left\{ -\frac{1}{2} (m)^{-3/2} \cdot \operatorname{erf}\left(\frac{1}{\sqrt{m}} \cdot v\right) + (m)^{-1/2} \cdot \frac{2}{\sqrt{\pi}} \cdot \exp\left(-\frac{1}{m} \cdot v^2\right) \cdot \frac{1}{2} (2k_B T m)^{-1/2} \cdot v \right\}$$

$$N = \left(\frac{m}{2\pi k_B T}\right)^{3/2} = \frac{1}{(\pi)^{3/2}} \cdot \frac{1}{v_m^3}$$

$$F(v) = -4\pi \frac{1}{v_m^3} \cdot (2k_B T)^{3/2} \frac{1}{(\pi)^{3/2}} \frac{\sqrt{\pi}}{2} \left(-\frac{1}{2} \frac{1}{m^{3/2}} \cdot \operatorname{erf}\left(\frac{v}{\sqrt{m}}\right) + \frac{1}{m^{1/2}} \cdot \frac{2}{\sqrt{\pi}} \cdot \exp\left(-\frac{v^2}{m}\right) \cdot \frac{1}{2} \frac{1}{(2k_B T)^{1/2}} \frac{1}{m^{1/2}} \cdot v \right)$$

$$= \operatorname{erf}\left(\frac{v}{\sqrt{m}}\right) - \frac{1}{4\pi} \frac{1}{m^{3/2}} \frac{1}{v_m^3} \frac{\sqrt{\pi}}{2} \cdot \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2} \cdot (2k_B T)^{3/2} \frac{1}{(2k_B T)^{1/2}} \frac{1}{m^{1/2}} \frac{1}{m^{1/2}} \cdot v \exp\left(-\frac{v^2}{m}\right)$$

$$= \operatorname{erf}\left(\frac{v}{\sqrt{m}}\right) - 2 \cdot v \cdot \frac{1}{v_m^3} \cdot \frac{1}{m^{1/2}} \cdot v_m^{-2} \exp\left(-\frac{v^2}{m}\right)$$

$$F(v) = \operatorname{erf}\left(\frac{v}{\sqrt{m}}\right) - 2v \cdot \frac{1}{v_m^3} \cdot \frac{1}{m^{1/2}} \exp\left(-\frac{v^2}{m}\right)$$

$$F(v_{0.5}) = 0.5$$

$$0.5 = \operatorname{erf}\left(\frac{v_{0.5}}{\sqrt{m}}\right) - \frac{2 \cdot v_{0.5} \cdot \exp\left(-\frac{v_{0.5}^2}{m}\right)}{v_m \sqrt{\pi}}$$

$$\beta = \frac{v_{0.5}}{v_m}$$

$$n = 1/n^2 \cdot n^2 = 1/n^2 \cdot 1$$

$$\beta = \frac{v_{0,5}}{v_m}$$

$$0 = \operatorname{erf}(\beta) - \frac{2}{\sqrt{\pi}} \beta \cdot \exp(-\beta^2) - \frac{1}{2}$$

Muss numerisch gelöst werden: $\beta = 1,0877$

$$\Rightarrow v_{0,5} = \beta \cdot v_m = 1,0877 \cdot v_m$$

d) Maximum bei $v = v_m$, $v_{FWHM} = v_w = \frac{1}{2} v_m$

$$2 \cdot f(v_w) = f(v_m)$$

$$f(v) = \left(\frac{m}{2\pi K_B T} \right)^{3/2} \cdot \exp\left(-\frac{mv^2}{2K_B T}\right) \cdot 4\pi v^2$$

$$= \frac{1}{\pi^{3/2}} \cdot \frac{1}{v_m^3} \cdot \exp\left(-\frac{v^2}{v_m^2}\right) \cdot 4\pi v^2$$

$$= \frac{4}{\sqrt{\pi}} \cdot \frac{v^2}{v_m^3} \cdot \exp\left(-\frac{v^2}{v_m^2}\right)$$

$$2 \cdot f(v_w) = 2 \cdot \frac{4}{\sqrt{\pi}} \cdot \frac{v_w^2}{v_m^3} \cdot \exp\left(-\frac{v_w^2}{v_m^2}\right) = \frac{4}{\sqrt{\pi}} \cdot \frac{1}{v_m} \cdot \exp(-1)$$

$$2 \cdot \frac{v_w^2}{v_m^2} \exp\left(-\frac{v_w^2}{v_m^2}\right) = \exp(-1)$$

$$\Rightarrow 0 = 2 \cdot \frac{v_w^2}{v_m^2} \exp\left(-\frac{v_w^2}{v_m^2}\right) - \exp(-1)$$

$$\frac{v_w}{v_m} = \gamma, \quad 0 = 2 \gamma^2 \exp(-\gamma^2) - \exp(-1)$$

Numerische Lösung: Linke Nullstelle: $v_1 = 0,4876$

Rechte Nullstelle: $v_2 = 1,6366$

$$\text{Breite: } v_{FWHM} = v_2 - v_1 = 1,1569$$

$$\text{e)} \quad \operatorname{Var}[X] = \langle X^2 \rangle - \langle X \rangle^2$$

$$\begin{aligned} \langle v^2 \rangle &= \int_{-\infty}^{\infty} v^2 f(v) dv = \int_0^{\infty} v^2 \left(\frac{m}{2\pi K_B T} \right)^{3/2} \cdot \exp\left(-\frac{mv^2}{2K_B T}\right) \cdot 4\pi v^2 dv \\ &= \left(\frac{m}{2\pi K_B T} \right)^{3/2} \cdot 4\pi \int_0^{\infty} v^4 \exp\left(-\frac{mv^2}{2K_B T}\right) dv = \left(\frac{m}{2\pi K_B T} \right)^{3/2} \cdot 4\pi \frac{1}{dm} \int_0^{\infty} v^4 \cdot \frac{-2K_B T}{v^2} \cdot \exp\left(-\frac{mv^2}{2K_B T}\right) dv \\ &= \left(\frac{m}{2\pi K_B T} \right)^{3/2} \cdot 4\pi \frac{1}{dm} \int_0^{\infty} v^4 \cdot \left(\frac{-2K_B T}{v^2} \right)^2 \cdot \exp\left(-\frac{mv^2}{2K_B T}\right) dv = \left(\frac{m}{2\pi K_B T} \right)^{3/2} \cdot 4\pi \frac{1}{dm} \cdot \left(-2K_B T \right)^2 \int_0^{\infty} v^4 \cdot \exp\left(-\frac{mv^2}{2K_B T}\right) dv \\ &= \left(\frac{m}{2\pi K_B T} \right)^{3/2} \cdot 4\pi \frac{1}{dm} \cdot \left(-2K_B T \right)^2 \cdot \frac{1}{2} \sqrt{\pi} \cdot \frac{2K_B T}{m} = \left(\frac{m}{2\pi K_B T} \right)^{3/2} 8 \cdot \pi \cdot \left(K_B T \right)^2 \cdot \left(\pi \cdot 2K_B T \right)^{1/2} \cdot \frac{3}{4} \cdot m^{-5/2} \\ &= \left(\frac{m}{2\pi K_B T} \right)^{3/2} 6 \cdot \pi \cdot \left(2\pi \right)^{1/2} \left(\frac{K_B T}{m} \right)^{5/2} = \left(\frac{m}{2\pi K_B T} \right)^{3/2} 3 \cdot \left(2\pi \right)^{1/2} \left(\frac{K_B T}{m} \right)^{5/2} \\ &= \underline{3 \frac{K_B T}{m}} = \langle v^2 \rangle = \frac{3}{2} \frac{2K_B T}{m} \end{aligned}$$

$$= \frac{3k_B T}{m} = \langle v^2 \rangle = \frac{3}{2} \frac{2k_B T}{m}$$

$$= \frac{3}{2} v_m^2$$

$$\text{Var}[v] = \langle v^2 \rangle - \langle v \rangle^2$$

$$= \frac{3}{2} v_m^2 - \frac{g}{\rho} v_m^2$$

$$= \left(\frac{3}{2} - \frac{g}{\rho} \right) v_m^2$$

$$\boxed{\sigma_v = \sqrt{\frac{3}{2} - \frac{g}{\rho}} v_m}$$