Polinomio de taylor - $\mathbb{R}^2$

$P_2(x,y)$ de $f=f(x,y)$ en $p=(a,b)$ con $f\in C^2$ en D (disco centrado en p)

Hessiano de f en p : $H_f(p)=\begin{vmatrix} f_{xx}(p) & f_{xy}(p) \\ f_{yx}(p) & f_{yy}(p)\end{vmatrix}$

$\Rightarrow P_2(x,y)=f(p)+\nabla f(p)\cdot(x-a,y-b)+\frac{1}{2}\begin{vmatrix} x-a & y-b\end{vmatrix}\cdot H_f(p)\cdot\begin{vmatrix} x-a \\ y-b\end{vmatrix}$

Polinomio de taylor - $\mathbb{R}^2$

$P_2(x,y,z)$ de $f=f(x,y,z)$ en $p=(a,b,c)$ con $f\in C^2$ en D (disco centrado en p)

Hessiano de f en p : $H_f(p)=\begin{vmatrix} f_{xx}(p) & f_{xy}(p) & f_{xz}(p) \\ f_{yx}(p) & f_{yy}(p) & f_{yz}(p) \\ f_{zx}(p) & f_{zy}(p) & f_{zz}(p)\end{vmatrix}$

$\Rightarrow P_2(x,y)=f(p)+\nabla f(p)\cdot(x-a,y-b)+\frac{1}{2}\begin{vmatrix} x-a & y-b & z-c\end{vmatrix}\cdot H_f(p)\cdot\begin{vmatrix} x-a \\ y-b \\ z-c\end{vmatrix}$

1. - $f(x,y)=(x+y)^2$
   - $p=(0,0)$

   - $f_x(x,y)=2x+2y$
   - $f_y(x,y)=2x+2y$
   - $f_{xx}(x,y)=2$
   - $f_{xy}(x,y)=2$
   - $f_{yx}(x,y)=2$
   - $f_{yy}(x,y)=2$

   $P_2(x,y)=\nabla f(p)\cdot(x,y)+\frac{1}{2}\begin{vmatrix} x & y\end{vmatrix} H_f(p)\begin{vmatrix} y \\ x\end{vmatrix}=$

   $\frac{1}{2}\begin{vmatrix} x & y\end{vmatrix}\begin{vmatrix} 2 & 2 \\ 2 & 2\end{vmatrix}\begin{vmatrix} y \\ x\end{vmatrix}=\frac{1}{2}\begin{vmatrix} 2x+2y & 2x+2y\end{vmatrix}\begin{vmatrix} y \\ x\end{vmatrix}=\frac{1}{2}(2xy+2y^2+2x^2+2yx)=$

   $y^2+x^2+2xy\Rightarrow$

   $P_1(x,y)=0$

   $P_1(x,y)=x^2+y^2+2xy$

2. - $f(x,y)=e^{x+y}$
   - $p=(0,0)$

   - $f_x(x,y)=e^{x+y}$
   - $f_y(x,y)=e^{x+y}$
   - $f_{xx}(x,y)=e^{x+y}$
   - $f_{xy}(x,y)=e^{x+y}$
   - $f_{yx}(x,y)=e^{x+y}$
   - $f_{yy}(x,y)=e^{x+y}$

   $P_2(x,y)=f(0,0)+\nabla f(0,0)\cdot(x,y)+\frac{1}{2}\begin{vmatrix} x & y\end{vmatrix} H_f(0,0)\begin{vmatrix} y \\ x\end{vmatrix}=$

   $1+(1,1)\cdot(x,y)+\frac{1}{2}\begin{vmatrix} x & y\end{vmatrix}\begin{vmatrix} 1 & 1 \\ 1 & 1\end{vmatrix}\begin{vmatrix} y \\ x\end{vmatrix}=$

   $1+x+y+\frac{1}{2}\begin{vmatrix} x+y & x+y\end{vmatrix}\begin{vmatrix} y \\ x\end{vmatrix}=$

   $1+x+y+\frac{1}{2}(y(x+y)+x(x+y))=$

$1 + x + y + \frac{1}{2}(yx + y^2 + x^2 + yx) =$
$1 + x + y + \frac{1}{2}(2yx + y^2 + x^2) =$
$1 + x + y + yx + \frac{y^2}{2} + \frac{x^2}{2} \Rightarrow$
$P_1(x, y) = 1 + x + y$
$P_2(x, y) = 1 + x + y + xy + \frac{x^2}{2} + \frac{y^2}{2}$

3.
- $f(x, y) = \frac{1}{x^2+y^2+1} = (x^2 + y^2 + 1)^{-1}$
- $p = (0, 0)$

- $f_x(x, y) = \frac{-2}{(x^2+y^2+1)^2}$
- $f_y(x, y) = \frac{-2}{(x^2+y^2+1)^2}$
- $f_{xx}(x, y) = \frac{6x^2-2(y^2+1)}{(x^2+y^2+1)^3}$
- $f_{xy}(x, y) = \frac{8xy}{(x^2+y^2+1)^3}$
- $f_{yx}(x, y) = \frac{8xy}{(x^2+y^2+1)^3}$
- $f_{yy}(x, y) = \frac{2(x^2-3y^2+1)}{(x^2+y^2+1)^3}$

- $f_x(0, 0) = -2$
- $f_y(0, 0) = -2$
- $f_{xx}(0, 0) = -2$
- $f_{xy}(0, 0) = 0$
- $f_{yx}(0, 0) = 0$
- $f_{yy}(0, 0) = 2$

$P_2(x, y) = 1 - 2x - 2y + \frac{1}{2}(-2x^2 + 2x^2) =$
$1 - 2x - 2y - x^2 + x^2$
$P_1(x, y) = 1 - 2x - 2y$

4.
- $f(x, y) = (x + xy + 2y)$
- $p = (1, 1)$

- $f_x(x, y) = 1 + y$
- $f_y(x, y) = x + 2$
- $f_{xx}(x, y) = 0$
- $f_{xy}(x, y) = 1$
- $f_{yx}(x, y) = 1$
- $f_{yy}(x, y) = 0$

- $f_x(1, 1) = 2$
- $f_y(1, 1) = 3$
- $f_{xx}(1, 1) = 0$
- $f_{xy}(1, 1) = 1$
- $f_{yx}(1, 1) = 1$
- $f_{yy}(1, 1) = 0$

$P_2(x, y) = f(1, 1) \cdot \boldsymbol{\nabla} f(1, 1) \cdot (x - 1, y - 1) + \frac{1}{2} \begin{vmatrix} x - 1 & y - 1 \end{vmatrix} \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} x - 1 \\ y - 1 \end{vmatrix} =$
$4 + 2(x - 1) + 3(y - 1) + \frac{1}{2} \cdot \begin{vmatrix} y - 1 & x - 1 \end{vmatrix} \cdot \begin{vmatrix} x - 1 \\ y - 1 \end{vmatrix} =$
$4 + 2(x - 1) + 3(y - 1) + \frac{1}{2} \cdot (y - 1)(x - 1) + (x - 1)(y - 1) =$
$4 + 2(x - 1) + 3(y - 1) + \frac{1}{\cancel{2}} \cdot \cancel{2}(y - 1)(x - 1) =$
$4 + 2(x - 1) + 3(y - 1) + (y - 1)(x - 1) \Rightarrow$
$P_1(x, y) = 4 + 2(x - 1) + 3(y - 1)$
$P_2(x, y) = 4 + 2(x - 1) + 3(y - 1) + (x - 1)(y - 1)$

5.
- $f(x,y) = e^{(x-1)^2}\cos(y)$
- $p = (1,0)$

- $f_x(x,y) = 2e^{(x-1)^2}(x-1)\cos(y)$
- $f_y(x,y) = -e^{(x-1)^2}\sin(y)$
- $f_{xx}(x,y) = 2e^{(x-1)^2}(2x^2-4x+3)\cos(y)$
- $f_{xy}(x,y) = -2e^{(x-1)^2}(x-1)\sin(y)$
- $f_{yx}(x,y) = -2e^{(x-1)^2}(x-1)\sin(y)$
- $f_{yy}(x,y) = -e^{(x-1)^2}\cos(y)$

- $f_x(1,0) = 0$
- $f_y(1,0) = 0$
- $f_{xx}(1,0) = 1$
- $f_{xy}(1,0) = 0$
- $f_{yx}(1,0) = 0$
- $f_{yy}(1,0) = -1$

$\Rightarrow P_2(x,y) = 1 + \frac{x^2}{2} - \frac{y^2}{2}$
$\Rightarrow P_1(x,y) = 1$

6.
- $f(x,y) = e^x \sin(xy)$
- $p = (2, \frac{\pi}{4})$

- $f_x(x,y) = e^x(\sin(xy) + \cos(xy))$
- $f_y(x,y) = e^x \cos(xy)$
- $f_{xx}(x,y) = e^x(2y\cos(xy) - (y^2-1)\sin(xy))$
- $f_{xy}(x,y) = e^x((x+1)\cos(xy) - xy\sin(xy))$
- $f_{yx}(x,y) = e^x((x+1)\cos(xy) - xy\sin(xy))$
- $f_{yy}(x,y) = -e^x\sin(x,y)$

- $f(2, \frac{\pi}{4}) = e^2$
- $f_x(2, \frac{\pi}{4}) = e^2$
- $f_y(2, \frac{\pi}{4}) = 0$
- $f_{xx}(2, \frac{\pi}{4}) = 1$
- $f_{xy}(2, \frac{\pi}{4}) = -e^2\frac{\pi}{2}$
- $f_{yx}(2, \frac{\pi}{4}) = -e^2\frac{\pi}{2}$
- $f_{yy}(2, \frac{\pi}{4}) = -e^2$

$P_1(x,y) = e^2 + e^2(x-2)$
$P_2(x,y) = e^2 + e^2(x-2) + \frac{(x-2)^2}{2} - \frac{e^2(y-\frac{\pi}{4})^2}{2} - e^2\frac{\pi}{2}(x-2)(y-\frac{\pi}{4})$

7.
- $f(x,y) = ln(1+xy)$
- $p = (2,3)$

- $f_x(x,y) = \frac{y}{1+xy}$
- $f_y(x,y) = \frac{x}{1+xy}$
- $f_{xx}(x,y) = \frac{y^2}{(1+xy)^2}$
- $f_{xy}(x,y) = \frac{1}{(1+xy)^2}$
- $f_{yx}(x,y) = \frac{1}{(1+xy)^2}$
- $f_{yy}(x,y) = \frac{x^2}{(1+xy)^2}$

- $f(2,3) = \ln(7)$
- $f_x(2,3) = \frac{2}{7}$
- $f_y(2,3) = \frac{3}{7}$
- $f_{xx}(2,3) = -\frac{9}{49}$
- $f_{xy}(2,3) = \frac{1}{49}$
- $f_{yx}(2,3) = \frac{1}{49}$
- $f_{yy}(2,3) = -\frac{4}{49}$

$P_1(x,y) = \ln(7) + \frac{2(x-2)}{7} + \frac{3(y-2)}{7}$

$P_2(x,y) = \ln(7) + \frac{2(x-2)}{7} + \frac{3(y-2)}{7} - \frac{9(x-2)^2}{2\cdot 49} - \frac{4(y-3)^2}{2\cdot 49} + \frac{(x-2)(y-3)}{49}$

8. - $f(x,y) = x + \sqrt{y} + \sqrt[3]{z}$
   - $p = (2,3,4)$

   - $f_x(x,y,z) = 1$
   - $f_y(x,y,z) = \frac{1}{2\sqrt{y}}$
   - $f_z(x,y,z) = \frac{1}{3z^{\frac{2}{3}}}$
   - $f_{xx}(x,y,z) = 0$
   - $f_{xy}(x,y,z) = 0$
   - $f_{xz}(x,y,z) = 0$
   - $f_{yx}(x,y,z) = 0$
   - $f_{yy}(x,y,z) = -\frac{1}{4y^{\frac{3}{2}}}$
   - $f_{yz}(x,y,z) = 0$
   - $f_{zx}(x,y,z) = 0$
   - $f_{zy}(x,y,z) = 0$
   - $f_{zz}(x,y,z) = -\frac{2}{9z^{\frac{5}{3}}}$

   - $f(2,3,4) = 2 + \sqrt{3} + \sqrt[3]{4}$
   - $f_x(2,3,4) = 1$
   - $f_y(2,3,4) = \frac{1}{2\sqrt{3}}$
   - $f_z(2,3,4) = \frac{1}{3\cdot 4^{\frac{2}{3}}}$
   - $f_{xx}(2,3,4) = 0$
   - $f_{xy}(2,3,4) = 0$
   - $f_{xz}(2,3,4) = 0$
   - $f_{yx}(2,3,4) = 0$
   - $f_{yy}(2,3,4) = -\frac{1}{4\cdot 3^{\frac{3}{2}}}$
   - $f_{yz}(2,3,4) = 0$
   - $f_{zx}(2,3,4) = 0$
   - $f_{zy}(2,3,4) = 0$
   - $f_{zz}(2,3,4) = -\frac{2}{9\cdot 4^{\frac{5}{3}}}$

$P_1(x,y,z) = 2 + \sqrt{3} + \sqrt[3]{4} + (x-2) + \frac{1}{2\sqrt{3}}(y-3) + \frac{1}{3\cdot 4^{\frac{2}{3}}}(z-4)$

$P_2(x,y,z) = 2 + \sqrt{3} + \sqrt[3]{4} + (x-2) + \frac{1}{2\sqrt{3}}(y-3) + \frac{1}{3\cdot 4^{\frac{2}{3}}}(z-4) - \frac{(y-3)^2}{2\cdot 4\cdot 3^{\frac{3}{2}}} - \frac{2\cdot(z-4)^2}{2\cdot 9\cdot 4^{\frac{5}{3}}}$