1. $f(x,y) = x^2 + y^2,\ xy = 1$

   - $f : \mathbb{R}^2 \to \mathbb{R}\checkmark$
   - $f \in C^1\checkmark$
   - $g : \mathbb{R}^2 \to \mathbb{R}\checkmark$
   - $g \in C^1\checkmark$
   - $\boldsymbol{\nabla}(x,y) \neq (0,0)\checkmark$
   - $\boldsymbol{\nabla} f(x,y) = (2x, 2y)$
   - $\boldsymbol{\nabla} g(x,y) = (y, x)$

   $\overset{Mult\ Lagrange}{\Rightarrow}$

   $(2x, 2y) = \lambda(y, x)$

   $\begin{cases} 2x = \lambda y \\ 2y = \lambda x \\ xy = 1 \end{cases} \Rightarrow$

   $\begin{cases} x = \frac{\lambda y}{2} \\ 4y = \lambda^2 y \\ xy = 1 \end{cases} \leftrightarrow \lambda \in \{-2, 2\}$

   $\Leftrightarrow (x,y) \in \{(1,1), (-1,-1)\}$

   $f(1,1) = 2\ f(-1,-1) = 2$

2. $f(x,y) = 3x + 1,\ x^2 + y^2 = 10$

   - $f : \mathbb{R}^2 \to \mathbb{R}\checkmark$
   - $f \in C^1\checkmark$
   - $g : \mathbb{R}^2 \to \mathbb{R}\checkmark$
   - $g \in C^1\checkmark$
   - $\boldsymbol{\nabla}(x,y) \neq (0,0)\checkmark$
   - $\boldsymbol{\nabla} f(x,y) = (3, 0)$
   - $\boldsymbol{\nabla} g(x,y) = (2x, 2y)$

   $\begin{cases} 3 = \lambda 2x \\ 0 = \lambda 2y \\ x^2 + y^2 = 10 \end{cases} \leftrightarrow$

   $\begin{cases} 3 = \lambda 2x \\ 0 = \lambda 2y \\ x^2 + y^2 = 10 \end{cases} \leftrightarrow x = \pm\sqrt{10}, \lambda = \frac{3}{2\sqrt{10}}$

   $f(-\sqrt{10}, 0) = 3\sqrt{10} + 1$ max

   $f(\sqrt{10}, 0) = -3\sqrt{10} + 1$ min

3. $f(x,y) = e^{xy},\ x^3 + y^3 = 16$

   - $\boldsymbol{\nabla} f(x,y) = (ye^{xy}, xe^{xy})$
   - $\boldsymbol{\nabla} g(x,y) = (3x^2, 3y^2)$

   $\begin{cases} ye^{xy} = \lambda 3x^2 \\ xe^{xy} = \lambda 3y^2 \\ x^3 + y^3 = 16 \end{cases}$

   $y = \frac{\lambda 3x^2}{e^{xy}}$

   $xe^{xy} = \lambda 3(\frac{\lambda 3x^2}{e^{xy}})^2$

   $1 = \frac{\lambda^3 3^3 x^3}{e^{xy}}$

   $e^{xy} = \lambda^3 3^3 x^3$

4. $f(x,y) = 2x + 2y + z,\ x^2 + y^2 + z^2 = 9$

$$\begin{cases} 2 = \lambda 2x \\ 2 = \lambda 2y \\ 1 = \lambda 2z \\ x^2 + y^2 + z^2 = 9 \end{cases} \Leftrightarrow$$

$$\begin{cases} 1 = \lambda x \\ 1 = \lambda y \\ 1 = \lambda 2z \\ x^2 + y^2 + z^2 = 9 \end{cases} \Leftrightarrow$$

$\frac{2}{\lambda^2} + \frac{1}{4\lambda^2} = 9$

$\frac{9}{4\lambda^2} = 9$

$\frac{9}{36} = \lambda^2$

$\pm\frac{1}{2} = \lambda$

$(x,y,z) \in \{(2,2,1), (-2,-2,1)\}$

$f(2,2,1) = 9$ max

$f(-2,-2,-1) = -9$ min

5. $f(x,y) = x^2 + y^2 + z^2,\ x + y + z = 12$

$$\begin{cases} 2x = \lambda 1 \\ 2y = \lambda 1 \\ 2z = \lambda 1 \\ x + y + z = 12 \end{cases} \Leftrightarrow$$

$\frac{3\lambda}{2} = 12$

$\lambda = 8$

$(x,y,z) = (4,4,4)$

$f(x,y,z) = 16 \cdot 3$