ECON880, Computational Economics
Fall 2024, University of Wisconsin
Instructor: Dean Corbae

Final Project **- Due Date 11/4/24**

1. **Baseline:** Often it is good to have a simple baseline around which you will build your problem. While the final project is to compute the Krusell-Smith model, it is the aggregate shock version of the Aiyagari model with idiosyncratic uncertainty which itself is a version of a simple growth model. In this part we start with solving the representative agent nonstochastic growth model by hand which can at least give us a "ballpark" idea level of aggregate capital. Preferences are given by
$$\sum_{\tau=0}^{\infty} \beta^t \ln(C_t)$$
where $\beta = 0.99$.The production technology is given by
$$Y_t = K_t^{\alpha} L_t^{1-\alpha}$$
where $\alpha = 0.36$ and capital depreciates at rate $\delta = 0.025$. Assume there is a unit measure of identical people so $L_t = \overline{e} * \pi$, where $\overline{e} = 0.3271$ denotes labor efficiency per unit of time worked and $\pi = .93$ is the fraction of agents employed in the corresponding Aiyagari model. Solve the planner's problem for the steady state level of capital. What is it? We will use that as an initial guess.

2. **Aiyagari Steady State:** In the attachd Julia file named Aiyagari.jl we provide you with code to solve for a steady state of the Aiyagari (1994) model. There agents have 1 unit of time and face idiosyncratic employment opportunities $\varepsilon_t \in \{0, 1\}$ where $\varepsilon_t = 1$ means the agent is employed an receives wage $w_t\overline{e}$ and $\varepsilon_t = 0$ means he is unemployed. The Markov transition matrix between employment and unemployment is
$$\Pi = \begin{bmatrix} 0.9624 & 0.0376 \\ 0.5 & 0.5 \end{bmatrix}$$
where the first row is $\varepsilon = 1$ and the second row is $\varepsilon = 0$. This corresponds to a steady state unemployment rate of $u = .07$ and an average length of unemployment of 2 periods. The code we provide uses Julia's linear interpolation function to compute the value function over individual capital holdings k_t.[1] You are to plot individual decision rules $k_{t+1} = g(k_t, \varepsilon_t)$ across k_t for people who are employed and unemployed in a steady state of that paper. What is the steady state level of capital without aggregate uncertainty? How different is it than the representative agent version? What is the economics behind that?

3. **Krusell-Smith:** You are to compute an approximate equilibrium of the Aiyagari (1994) model with aggregate uncertainty using the techniques in Krusell and Smith (1998). Since we gave you the code to do Aiyagari without aggregate uncertainty, you will have

[1] The lines of code with interpolation begin with:
V1_interp = interpolate(V[:, 1], BSpline(Linear()))

to add 2 more state variables to the problem. One is the aggregate technology shock $z_t \in \{z_g = 1.01, z_b = 0.99\}$ which is drawn from a markov process. The other state variable is the aggregate capital stock K_t in any given period. That means the state space is now $(k_t, \varepsilon_t; K_t, z_t)$. Now you will need to append the program we gave you to compute the Aiyagari model to use the bilinear interpolation function over both k_t and K_t. In the attached Julia file named KS_skeleton.jl we provide skeleton code to guide you through the algorithm to implement Krusell-Smith. Specifically, the skeleton Julia code is based on the attached file named "Skeleton Writeup for solving Krusell-Smith".

The probability of transition from state (z, ε) to (z', ε'), denoted $\pi_{zz'\varepsilon\varepsilon'}$ must statisfy certain conditions:

$$\pi_{zz'00} + \pi_{zz'01} = \pi_{zz'10} + \pi_{zz'11} = \pi_{zz'}$$

and

$$u_z \frac{\pi_{zz'00}}{\pi_{zz'}} + (1 - u_z)\frac{\pi_{zz'10}}{\pi_{zz'}} = u_{z'}$$

where u_z denotes the fraction of those unemployed in state z with $u_g = 4\%$ and $u_b = 10\%$. The other restrictions on $\pi_{zz'\varepsilon\varepsilon'}$ necessary to pin down the transition matrix are that: the average duration of good and bad times is 8 quarters; the average duration of unemployment spells is 1.5 quarters in good times and 2.5 quarters in bad times; and

$$\frac{\pi_{gb00}}{\pi_{gb}} = 1.25 \cdot \frac{\pi_{bb00}}{\pi_{bb}} \text{ and } \frac{\pi_{bg00}}{\pi_{bg}} 0.75 \cdot \frac{\pi_{gg00}}{\pi_{gg}}.$$

See my website for the file transmatrix.m which actually computes the transition matrix for you. Capital is the only asset to self insure fluctuations; households rent their capital $k_t \in [0, \infty)$ to firms and receive rate of return r_t. Without loss of generality, we can consider one firm which hires L_t units of labor efficiency units (so that $L_t = \bar{e}(1 - u_t)$) and rents capital K so that wages and rental rates are given by their marginal products:

$$\begin{aligned} w_t &\equiv w(K_t, L_t, z_t) = (1-\alpha) z_t \left(\frac{K_t}{L_t}\right)^{\alpha} \qquad (1) \\ r_t &\equiv r(K_t, L_t, z_t) = \alpha z_t \left(\frac{K_t}{L_t}\right)^{\alpha - 1} \end{aligned}$$

As in Krusell and Smith, approximate the true distribution Γ_t over (k_t, ε_t) in state z_t by I moments and let the law of motion for the moment be $m' = h_I(m, z, z')$.

To start the Krusell-Smith algorithm, we need initial conditions. There's only 2 possibilities for (z_t, ε_t) so choose the ones that are most likely (i.e. z_g and use $L_g = 1 - u_g = 0.96$ to generate $\varepsilon_{t=0}$). Use the steady state level of capital you found in part 2 as an initial condition for K_0 as well as k_0. As you will see below we set the lower bound of the interval of possible K to 11 but the upper bound all the way to 15.

Algorithm

a. Let $I = 1$ (which means only average capital holdings matter).

b. Generate a $T = 11,000$ sequence of z_t using the $\pi_{zz'}$ markov matrix starting state z_g and for each generate z_t generate the $N = 5000$ of ε_t shocks using $\pi_{zz'\varepsilon\varepsilon'}$. Save these $Tx1$ vector of z_t (call it Z) and NxT matrix (where the row is a person's employment status in a given aggregate state calling it $\mathcal{E}$).

c. Conjecture a log linear functional form for h_1;Specifically let

$$\log K' = \begin{cases} a_0 + a_1 \log K & \text{if } z = z_g \\ b_0 + b_1 \log K & \text{if } z = z_b \end{cases} \tag{2}$$

As an initial guess, one could simply start with $a_0 = b_0 = 0$ and $a_1 = b_1 = 1$.To speed things along, choose $a_0 = 0.095, b_0 = 0.085$ and $a_1 = b_1 = 0.999$.

d. Given h_I, solve the conumers problem. For the above example

$$v(k, \varepsilon; K, z) = \max_{c,k'} u(c) + \beta E_t \left[v(k', \varepsilon'; K', z') \right]$$

s.t.

$$c + k' = r(K, L, z)k + w(K, L, z)\varepsilon + (1 - \delta)k$$

as well as (1) and (2). Let k lie in $[0, 15]$ and K lie in $[11, 15]$ and use bilinear interpolation over these two dimensions of the state vector (the other 4 states are discrete so simply index 4 different value functions by $(g, 0), (b, 0), (g, 1), (b, 1)$). On my website is code written by Phil Coyle to do bilinear interpolation (this can be found in Numerical Recipes, Section 3.6).

e. Use the decision rules generated in step 4 and the $\mathcal{E}$ matrix in step 3 to simulate the savings behavior of N households starting from an initial condition $K^{ss} = 11.55$, discarding the first 1000 periods (to deal with initial condition dependence).In This generates a huge $N \times \widetilde{T}$ matrix where each row is a different agent's k_{t+1} choice in state (ε_t, z_t). Call it V.

f. Use the simulated data in step 5 to (re-)estimate a set of parameters for the functional form conjectured in step 3. That is, average over all agents in each period (i.e. down a column of V). The resulting $\widetilde{T} \times 1$ vector of aggregate capital holdings is K. Run the (auto)regression in (2) using the information in Z to know which branch to run. Obtain a measure of "goodness of fit" (e.g. R^2 of regression in (2)).

g. If the new parameter vector (a'_0, a'_1, b'_0, b'_1) is sufficiently close to the original parameter vector (a_0, a_1, b_0, b_1) and the "goodness of fit" (e.g. R^2 of regression in (2)), is sufficiently high, stop. If the new parameter vector (a'_0, a'_1, b'_0, b'_1) is not sufficiently close to the original parameter vector (a_0, a_1, b_0, b_1), go back to step 3 with the new parameter vector. If the parameter values have converged, but the goodness of fit is not high enough, increase I in step 1 or try a different functional form in step 3.