Problem Set 4

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Problem 1-Find Asymptotic Moments

$$m_3(x) = \begin{bmatrix} \text{Mean} \\ \text{variance} \\ 1^{st} \text{ Autcorrelation} \end{bmatrix} = \begin{bmatrix} \mu_i \\ \sigma_i^2 \\ \varphi_{i,1} \end{bmatrix}$$

- μ_x:

$$\begin{aligned} \mu_x : \quad E\left[x_t\right] = E\left[\rho_0 x_{t-1} + \varepsilon_t\right] \\ E\left[x_t\right] &= \rho_0 E\left[x_{t-1}\right] + E\left[\varepsilon_t\right] \\ \mu_x &= \rho_0 \mu_x \quad \because \text{ ibid} \\ \left(1-\rho_0\right)\mu_x &= 0 \\ \mu_x &= 0 \quad \because \left(1-\rho_0\right) > 0 \end{aligned}$$

- σ_x^2 :

$$\begin{aligned} \operatorname{Var}\left(X_t\right) &= \operatorname{Var}\left(\rho_0 X_{t-1} + \varepsilon_t\right) \\ &= \rho_0^2 \operatorname{Var}\left(x_{t-1}\right) + \operatorname{Var}\left(\varepsilon_t\right) + 2\rho_0 \operatorname{Cov}\left(x_{t-1}, \varepsilon_t\right) \\ \sigma_x^2 &= \rho_0^2 \sigma_x^2 + \sigma_0^2 \quad \because \quad \varepsilon_t \quad \text{has iid dist} \\ \sigma_x^2 &= \frac{\sigma_0^2}{1-\rho_0^2} \Rightarrow \sigma_x^2 = \frac{1}{0,75} = 1.3\overline{3} \end{aligned}$$

- φ_1:

$$\begin{aligned} \varphi_1 &= \left\{E\left[X_t X_{t-1}\right] - E\left(X_t\right)E\left(X_{t-1}\right)\right\} \frac{1}{\sqrt{\operatorname{Var}\left(X_t\right)\operatorname{Var}\left(X_{t-1}\right)}} \\ &= \frac{1}{\sigma^2 x} E\left[\left(\rho_0 x_{t-1} + \varepsilon_t\right) x_{t-1}\right] \\ &= \frac{1}{\sigma_x^2}\left[E\left[\rho_0 x_{t-1}^2\right] + E\left[\varepsilon_t x_{t-1}\right]\right] \\ &= \frac{1}{\sigma_x^2}\rho_0 E\left[x_{t-1}^2\right] \\ \varphi_1 &= \rho_0 \quad \because \mu_x = 0 \Rightarrow \operatorname{Var}\left(X_{t-1}\right) = E\left[X_{t-1}^2\right] \\ &\Downarrow \\ \varphi_1 &= 0.5 \end{aligned}$$

- Compute $\nabla_b g(b_0)$:

$$\nabla_b g\left(b_0\right) = \begin{bmatrix} 0 & 0 \\ \frac{2\rho_0\sigma_0^2}{\left(1-\rho_0^2\right)^2} & \frac{1}{1-\rho_0^2} \\ 1 & 0 \end{bmatrix} \rightarrow \text{ Full Rank}$$

– It's clear that the mean is non-informative for any element in b. Only the second moment, the variance of y would help identify σ^2. However, both the variance and the autocorrelation would help in estimating ρ.

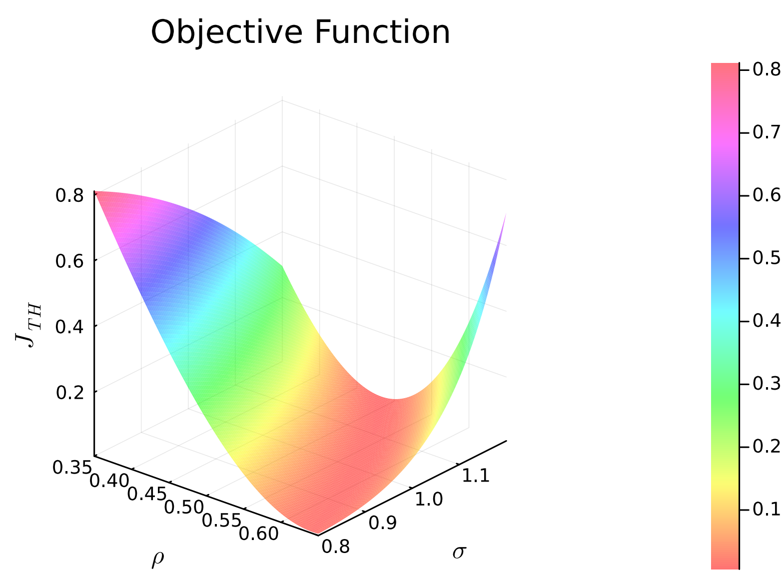
Problems 2-4: Just Identified Case with Mean and Variance

First thing we notice is that the estimates for the persistence of the AR(1) are noticeably higher than the true parameter value (0.5). In contrast, the estimates for the standard deviation of residuals are lower in both cases. These results reflect the fact that the key moment to identify persistence is autocorrelation, as shown in Part 1, and that the series variance is affected by both parameters simultaneously and in the same direction.

Parameter	$\hat{b}_1$	$\hat{b}_2$	**Standard Error**
$\hat{\rho}$	0.87978	0.879775	0.123903
$\hat{\sigma}$	0.620367	0.620379	0.260875

We can also see how the estimates change very little when we recompute the weighting matrix using Newey-West. This is what we expected because the SMM/GMM estimators should be consistent and robust to this matrix, it is only efficiency that is affected by the latter.

The sensitivity of the objective function can be shown through the following graph:



We also numerically compute the gradient of the function $g_{TH}(b)$ with respect to the two parameters we are estimating. In the table below we can see how the autocorrelation moment does not vary as we vary σ_ε^2, which makes sense since we showed in the asymptotic moments in Problem 1, that this moment does not depend directly on said parameter. As a consequence, first order autocorrelation is potentially not very useful to identify σ_ε^2, at least locally, and so the variance of the simulated series would be more helpful.

Moment	$\frac{\partial g_{TH}(b)}{\partial \rho}$	$\frac{\partial g_{TH}(b)}{\partial \sigma}$
Mean	1.370	0.246
Variance	-12.413	-5.268
Autocorrelation	-1.055	0.000

Lastly, the J-test for this scenario is 5.720e-10, which would mean that we should not reject the null hypothesis of the moments being valid and providing sufficient information for identification. However, this is not very informative as we have no degrees of freedom.

Problem 5: Just Identified with variance and autocorrelation

The estimation results for the second case show us that we can get closer to the true parameters when the moments we are using are actual functions of the parameters. As we saw in the asymptotic vector of moments, autocorrelation helps identify persistence and variance helps with both. This is consistent with the table below.

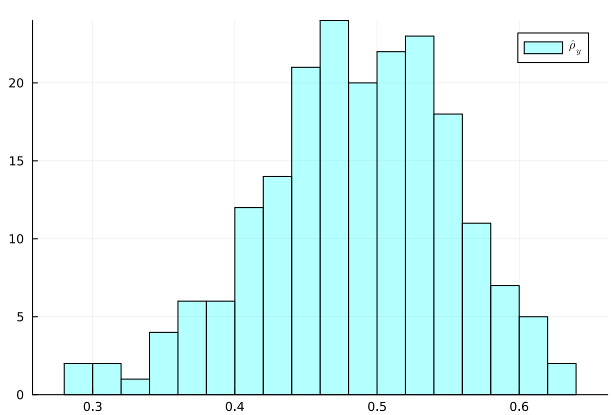
Parameter	$\hat{b}_1$	$\hat{b}_2$	**Standard Error**
$\hat{\rho}$	0.554284	0.57458	0.05896
$\hat{\sigma}$	1.06395	1.0724	0.06793

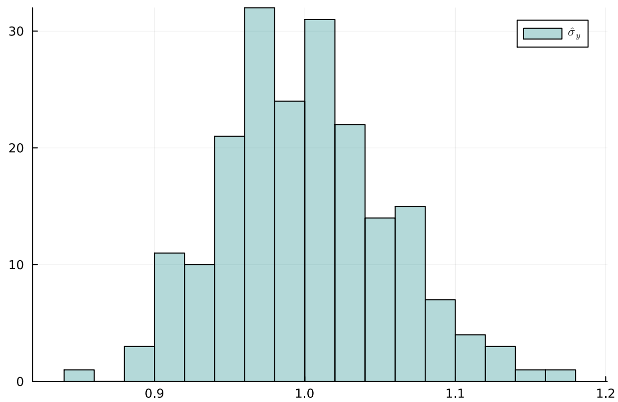
Problem 6: Over-identified Case

The estimation results for this third case show us that the point estimates improve slightly with respect to previous case, reflecting the fact that the mean is not directly an informative moment. However, since we are using more information, the standard error falls as compared to the just identified case.

Parameter	$\hat{b}_1$	$\hat{b}_2$	**Standard Error**
$\hat{\rho}$	0.59235	0.58737	0.001393
$\hat{\sigma}$	1.04759	1.03691	0.002189

We also performed the bootstrapping exercise and obtained the following histograms for $\hat{\rho}$ and $\hat{\sigma}$, respectively.





In both cases we can see that the bulk of the frequency is concentrated around the true values of the parameters, namely $\rho_o = 0.5$ and $\sigma_o = 1$. These are good news and give us confidence in the robustness of our procedure, particularly when we use as much information as we can (i.e. using the three moments).

Problem 7: Indirect Inference

We proceed by estimating MA(N) for N=1,2,3 using Maximum Likelihood on both the true data and the simulated one. Then used the Moment Matching procedure described in the question and we obtained the following estimates for $b = (\rho, \sigma)$:

Parameter	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$
$\hat{\rho}$	0.618337	0.490552	0.526151
$\hat{\sigma}$	0.909059	0.956193	0.949241

Clearly, the MA(N) is an imperfect approximation of an inverted AR(1), so we will suffer from omitted variable bias when getting the MA coefficients and the variance. Nevertheless, for not highly persistent processes this bias tends to be small and improves gradually as we increase N. For that reason, the estimates are not particularly far from the true values.