Application

- Idea: "guess" an asymptotic bound ($\mathcal{O}$, Ω) and prove it by mathematical induction

  Example: $T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + n$

  Guess: $T(n) = \mathcal{O}(n \log n)$

  We have to show: $\exists c > 0, n_0 \in \mathbb{N}$ such that
  $T(n) \leq c{\cdot}n \log n$ for all $n \geq n_0$

  Induction hypothesis: $T(n) \leq c{\cdot}n \log_2 n$ for some const. $c > 0$

  Induction base: $T(k) = \Theta(1) \leq d$ for small constant k
  and some constant $d > 0$ (by def.)

  Induction step: $T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + n$

  $$\leq 2 \cdot (c{\cdot}\lfloor \tfrac{n}{2} \rfloor \log_2 \lfloor \tfrac{n}{2} \rfloor) + n \overset{?}{\leq} c \cdot n \log_2 n$$

- Example: $T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + n$, $\quad T(n) \overset{?}{=} \mathcal{O}(n \log n)$

  $$\begin{aligned} T(n) &= 2T(\lfloor \tfrac{n}{2} \rfloor) + n \\ &\leq 2 \cdot \big(c\lfloor \tfrac{n}{2} \rfloor \log(\lfloor \tfrac{n}{2} \rfloor)\big) + n \\ &\leq c \cdot n \log_2(\tfrac{n}{2}) + n \\ &= c \cdot n \log_2 n - c \cdot n \log_2 2 + n \\ &= c \cdot n \log_2 n - c \cdot n + n \\ &= c \cdot n \log_2 n + (1 - c) \cdot n \\ &\leq c \cdot n \log_2 n \qquad\qquad \text{for } c \geq 1 \end{aligned}$$

  Choose $k \geq 3$ for induction base, $c \geq \max\{1, d\}$, $n_0 \geq 2$

  from induction base

Pitfalls

- Don't use asymptotic notation in the induction step!

  Example again: $T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + n$
  Guess: $T(n) = \mathcal{O}(n)$

  $\Rightarrow$ Show: $\exists c > 0, n_0 \in \mathbb{N} : T(n) \leq c{\cdot}n$ for $n \geq n_0$

  Induction hypothesis: $T(n) \leq c{\cdot}n$ for some const. $c > 0$

  Induction step:

  $$\begin{aligned} T(n) &= 2T(\lfloor \tfrac{n}{2} \rfloor) + n \\ &\leq 2(c \cdot \lfloor \tfrac{n}{2} \rfloor) + n \\ &\leq c \cdot n + n = (c+1) \cdot n \cancel{= \mathcal{O}(n)} \quad \textbf{WRONG !!} \end{aligned}$$

- Sometimes the "obvious" induction hypothesis doesn't work:

  Example: $T(n) = T(\lfloor \frac{n}{2} \rfloor) + T(\lceil \frac{n}{2} \rceil) + 1$
  Guess: $T(n) = \mathcal{O}(n)$

  $\Rightarrow$ Show: $\exists c > 0, n_0 \in \mathbb{N} : T(n) \leq c{\cdot}n$ for $n \geq n_0$

  Induction hypothesis: $T(n) \leq c{\cdot}n$ for some const. $c > 0$

  Induction step:

  $$\begin{aligned} T(n) &= T(\lfloor \tfrac{n}{2} \rfloor) \;+\; T(\lceil \tfrac{n}{2} \rceil) \;+1 \\ &\leq c \cdot \lfloor \tfrac{n}{2} \rfloor \;+\; c \cdot \lceil \tfrac{n}{2} \rceil \;+1 \\ &= c \cdot n \;+1 \;>\; c \cdot n \qquad \Rightarrow \text{Induction fails} \end{aligned}$$

  But: weaker hypothesis $T(n) \leq c \cdot n - d$ with $d \in \mathbb{R}$ works

- Induction hypothesis: ~~$T(n) \leq c{\cdot}n$~~ for some const. $c > 0, d \in \mathbb{R}$
  $T(n) \leq c \cdot n - d$

  Induction step:

  $$\begin{aligned} T(n) &= T(\lfloor \tfrac{n}{2} \rfloor) \;+\; T(\lceil \tfrac{n}{2} \rceil) \;+1 \quad \boxed{-d} \\ &\leq c \cdot \lfloor \tfrac{n}{2} \rfloor \;+\; c \cdot \lceil \tfrac{n}{2} \rceil \;+1 \quad \boxed{-2 \cdot d} \\ &= c \cdot n \;+1 \;\cancel{> c \cdot n \qquad \Rightarrow \text{Induction fails}} \\ &= c \cdot n - d + (1 - d) \;\leq\; c \cdot n - d \quad \text{for } d \geq 1 \end{aligned}$$

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Properties

- advantage: more powerful than the other two methods

  disadvantage: two proofs needed for Θ ($\mathcal{O}$ and Ω)

  How to make the right guess?
  - similarity to known recurrence relations
  - recursion tree

  How to get the right approach?
  - look at additional function
  - in case of doubt, try a second time