Idea

- - The permutation test is a resampling method for testing whether two distributions are the same
  - This test is exact: it is not based on large sample approximations
  - Let $X_1, X_2, \dots, X_n$ be a random sample of size $n$, $X_1 \sim F_X$
  - Let $Y_1, Y_2, \dots, Y_m$ be a random sample of size $m$, $Y_1 \sim F_Y$
  - With permutation test we are testing:
  $$H_0 : F_X = F_Y \text{ against } H_1 : F_X \neq F_Y$$
- - Let $T_N = t(X_1, \dots, X_n, Y_1, \dots, Y_m)$ be some test statistic, where $N = n + m$
  - E.g. $T_n = |\overline{X}_n - \overline{Y}_m|$
  - We consider all $N!$ permutations of the data $X_1, \dots, X_n, Y_1, \dots, Y_m$
  - For each permutation we compute the test statistics $T$
  - We denote these values with $T_1^*, \dots, T_{N!}^*$
- - How likely are each of the $T_1^*, \dots, T_{N!}^*$ under the $H_0$?
  - Equally likely!
  - The distribution $P_0$ that puts $1/N!$ mass on each $T_j^*$ is called the **permutation distribution** of $T$
  - Let $t_n$ be the observed value of the test statistic
  - Assuming we reject when $T$ is large, the p-value of the permutation test:
  $$p - value = P_0(T* > t_N) = \frac{1}{N!} \sum_{j=1}^{N!} I(T_j^* \geq t_N)$$
- Usually, it is not practical to evaluate all $N!$ permutations

  We can approximate the p-value by simulating random permutations

  The fraction of times $T_j^* > t_N$ among these samples approximate the p-value

Example

- toy
  - Suppose the data are: $(X_1, X_2, Y_1) = (3, 9, 1)$. Let $T(X_1, X_2, Y_1) = |\overline{X}_n - \overline{Y}_m|$, i.e. $t_N = 5$. Compute the p-value of the test statistic $T$.
  - Suppose the data are: $(X_1, X_2, Y_1) = (3, 9, 1)$. Let $T(X_1, X_2, Y_1) = |\overline{X}_n - \overline{Y}_m|$, i.e. $t_N = 5$. Compute the p-value of the test statistic $T$.

| permutation | $T^*$ | $P_0(T^*)$ |
|---|---|---|
| (3,9,1) | 5 | 1/6 |
| (1,3,9) | 7 | 1/6 |
| (1,9,3) | 2 | 1/6 |
| (3,1,9) | 7 | 1/6 |
| (9,1,3) | 2 | 1/6 |
| (9,3,1) | 5 | 1/6 |

– p-value: $P(T^* \geq 5) = 4/6$

- driving behavior
  - Suppose we have a list of cities with the car driver velocities on weekdays and weekends. Test if the driving behavior in each city is different on weekdays as compared to weekends.
  - We take $H_0$: no difference in driving behavior and test it against $H_1$: driving behavior is different at weekdays and weekends for a given city. We use permutation test for the difference in means.