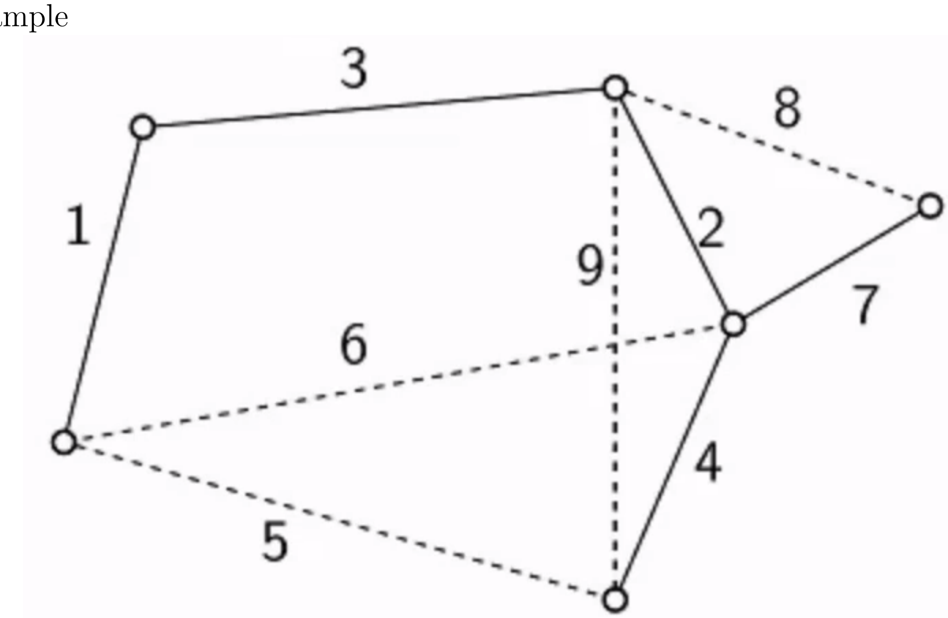
Motivation

- many algorithms can be applied to unweighted trees
  - [[Spannbaumalgorithmen]]
  - [[Breadth-First Search]]
  - [[Depth-First Search]]
  - ...
- does not work for weighted graphs
- useful if direct connections from A to B not possible
  - road networks
  - electrical circuits
  - ...

Definition

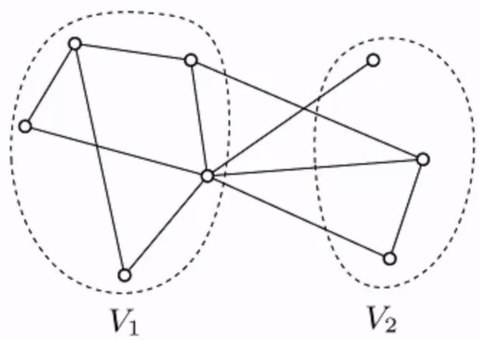
- tree which connects all points which minimizes the total length of all edges
  - A minimum spanning tree of a weighted graph $G = (V, E, w)$ is a tree $T = (V, E')$ with $E' \subseteq E$ and with minimal total edge length among all spanning trees in G:

    $$w(T) = \sum_{e \in E'} w(e)$$

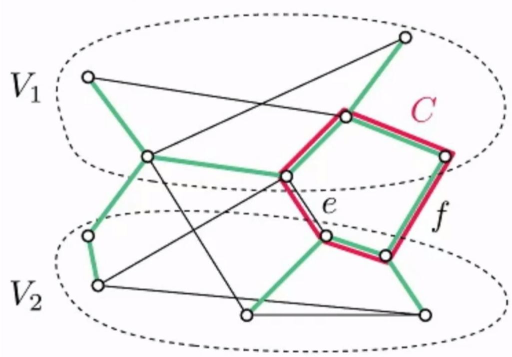
    is minimized over all trees in G with vertex set V.
- crossing free (planar)
  - otherwise using different edges would lead to a fewer length
- example
  - 

Algorithm Idea

- cut
  - A **cut** of a graph $G = (V, E)$ is a partition of V into V_1, V_2.

    An edge e is called **external** for the cut (V_1, V_2) if it has one endpoint in V_1 and one in V_2; otherwise e is called **internal**.
- **Theorem:** Let E' be a subset of edges of an MST of $G = (V, E, w)$. Let (V_1, V_2) be a cut of G for which all edges of E' are internal. Then the external edge of the cut with **minimum weight** is a good edge for E'.
  - proof
    * **Proof:**
      Assume there is an MST T with E' and without e.
      $\Rightarrow$ e closes a cycle C in T.
      The cycle C contains at least one edge f of the tree T that is external for the cut.

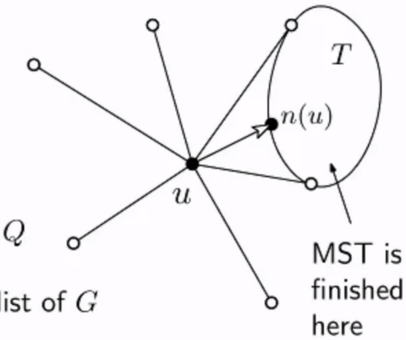
      By definition of e, the weight $w(f) \geq w(e)$.
      $\Rightarrow T \backslash \{f\} \cup \{e\}$ is an MST of G. $\square$
      If $w(f) > w(e)$ then T is not an MST of G.

Prim's Algorithm

- greedy
- also works with negative weights

- - Start with an arbitrary vertex s of G and iteratively 'grow' an MST T from s.
  - **Iterative step:**
    Choose the 'cheapest' edge with exactly one node in T.
  - Cut: V_1 = vertices of T, V_2 = vertices not yet in T.
  - For each vertex $v \notin T$ we maintain:
    - Priority $p(v)$: weight of the shortest edge from v to a vertex in T (initially: ∞).
    - Nearest $n(v)$: vertex in T realizing $p(v)$: $w(v, n(v))$ is min. among neighbors of v in T (initially no vertex).
  - A **queue** Q contains all vertices not yet in T, organized by priorities (e.g., in a min-heap; initially all vertices).
- pseudo code

```
PRIM-MST (G,s)
for all v ∈ V do p(v) = ∞ od
p(s) = 0,  n(s) = nil
Q = V          // build up Q
while Q ≠ 0 do
  u = MIN(Q)
  remove u from Q        // reorganize Q
  write u, n(u)
  for all v ∈ A(u) do        // A: adj. list of G
    if v ∈ Q and p(v) > w(u,v) then
      p(v) = w(u,v)        // reorganize Q
      n(v) = u
    fi
  od
od
```



Run-time-analysis:

n ... number of vertices of G
m ... number of edges of G
$d(v)$ ... Degree of vertex v in G

- Initialization, construction of the heap Q: $\Theta(n)$
- n times removing the minimum from Q: $O(n \log n)$
- report MST edges: $\Theta(n)$
- Update priorities for all neighbors of v: $O(d(v) \cdot \log n)$

$\Rightarrow$ Alltogether:

$$O(n + n \log n + \sum_{v \in V} d(v) \log n) = O(m \log n)$$

Memory requirements: $\Theta(m + n) = \Theta(m)$
Graph + queue + priorities + nearests + constant additional

- **Remarks:**
  - MST always begins at the start vertex s and grows from there as a connected tree.
  - Shrinking $p(v)$ causes v to move up in the heap. $\Rightarrow O(\log n)$ time.
  - Test for $v \in Q$ in $O(1)$ time when bit vector is used to store which vertices are already in T.
  - Runtime can be changed to $O(n^2)$. This is useful for dense graphs (see notes on Dijkstra's algorithm in the next chapter).

Kruskal's Algorithm

- greedy
- also works with negative weights
- uses [[Union Find]] data structure
  - - Label the vertex set of G as $v_1, v_2, ...v_n$ (arbitrary)
    - Initially there are n disjoint sets $M_1, M_2, ...M_n$ (each with one vertex)
    - FIND(v): returns index i if vertex v is in M_i
    - UNION(i,j): join sets M_i and M_j: $M_i = M_i \cup M_j$ (index of resulting set: minimum of i and j)
    - End of the algorithm: one component M_1 with all vertices of G.
  - - Creating a 1-element set needs $\Theta(1)$ time.
    - f FIND and u UNION operations need $O(f + u \log u)$ time in total.
    - The total memory requirement of the data structure is linear in the number of initial 1-element sets.

- - Start with empty edge set E'.
  - Sort edge set E of $G = (V, E, w)$ in increasing order of their weights (edges will be considered in this order): $e_1, e_2, ...e_m$ with $w(e_1) < w(e_2) < ... < w(e_m)$.
  - **Iterative step**:
    - E' forms a forest F (= set of disjoint subtrees, acyclic) in G and in the MST to be constructed.
    - Edge e that is added to E' is the shortest edge in $E \setminus E'$ that does not form a cycle with edges from E'.
- pseudo code

  – **KRUSKAL-MST(G)**

```
sort edges by weight: {e_1, e_2, ... e_m}
for i = 1 to n do M_i = {v_i} od
for k = 1 to m do
    (u, v) = e_k
    i =FIND(u)
    j =FIND(v)
    if i ≠ j then
        write e_k
        UNION(i, j)
    fi
od
```

  Runtime analysis:

  - Sorting of the edges: $O(m \log m)$
  - Initialize UNION-FIND data structure for vertices: $\Theta(n)$
  - In total $2m$ FIND operations and $n - 1$ UNION operations: $O(m + n \log n)$
  - Extract edges + write MST edges: $\Theta(m)$

  $\Rightarrow$ Altogether $O(m \log m)$ time.

  $\Rightarrow$ Sorting of the edges dominates the runtime.
-

Memory requirements:

$\Theta(n+m) = \Theta(m)$ in total.

-
-