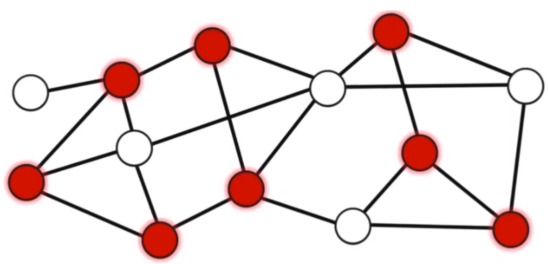
Overview

- close to optimal solutions
- provable guarantees on the solution quality
- often a simple greedy algorithm
- **Approximation ratio $\boldsymbol{\rho(n)}$** of an algorithm A: if for any n-sized input the algorithm A produces a solution with value C such that
  - $\frac{C}{OPT} \leq \rho(n)$ for a **minimization problem**, and
  - $\frac{C}{OPT} \geq \rho(n)$ for a **maximization problem**.
  - **Minimization problem:** $\rho(n) \geq 1$. **Maximization problem:** $\rho(n) \leq 1$.

Minimum Vertex Cover

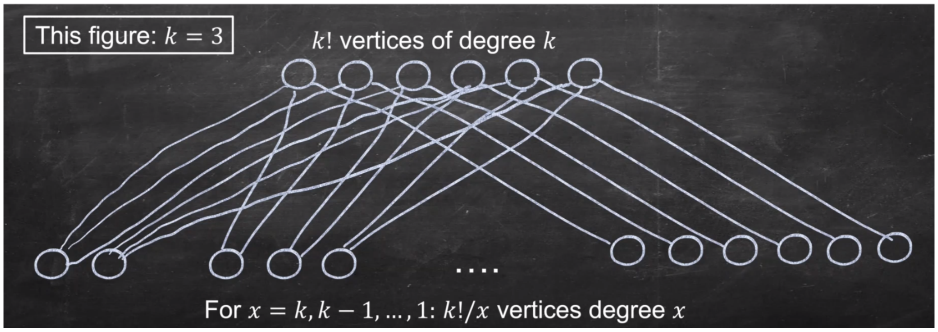


A vertex cover of an undirected graph GG is a set S of vertices such that every edge in E(G) is incident to at least one vertex in S. In other words, for every edge (u,v) in GG, at least one of the vertices u or v is in the set S.

- **Goal:** Compute a vertex cover of **minimum size**.
- NP-complete
- "vertex greedy" algorithm

```
Input: Graph G = (V, E)
Output: Vertex cover C
C = empty set
F = E(G)
while F is not empty:
   choose largest degree vertex v in G=(V-C,F)
   add v to C
   remove all edges incident on v from F
return C
```

Theorem: Greedily adding the largest degree vertex (in the graph induced by uncovered edges) is not a constant factor approximation for MVC.

- 

- "edge greedy" algorithm
  - ApproximateVertexCover

```
Input: Graph G = (V, E)
Output: Vertex cover C
C = empty set
F = E(G)
while F is not empty:
    choose any edge (u, v) from F
    add u and v to C
    remove all edges incident on u and v from F
return C
```

  - better approximation guarantee than "greedy vertex"
  - must not be always better
  - 2-approximation
    - F: the set of edges that our greedy algorithm picked. **F is a matching in $\boldsymbol{G}$.** Why?
      $C = V(F)$ be the computed vertex cover.
      - Clearly C is a vertex cover, as we continue adding vertices until all edges are covered.
      - We have $|C| = 2|F|$ (no overlap, as F is a matching in G)

      Let C_{OPT} be any optimal solution to MVC.
      $|\boldsymbol{C_{OPT}}| \geq \boldsymbol{F}$, because C_{OPT} has to cover every edge, including all edges in F.
      But no vertex of G can cover more than a single edge of F, as F is a matching.

      $$\Rightarrow |C| = 2\,|F| \leq 2|C_{OPT}|.$$

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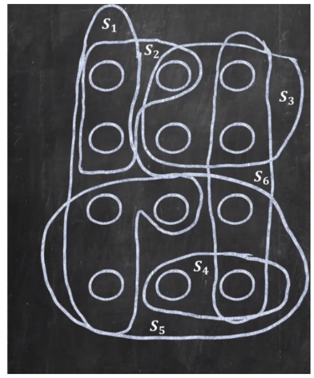
Set Cover

- **Input:**
  - **Universe** $X = \{x_1, \dots, x_n\}$ of n elements
  - **Collection of Sets** $S = \{S_1, \dots, S_k\}$, each $S_i \subseteq X$

  Set Cover: a collection of sets (indices) $I \subseteq \{1, \dots k\}$ s.t. all elements are covered, i.e.,

  $$X \subseteq \bigcup_{i \in I} S_i$$

  Goal: Select a minimum size set cover (minimize $|I|$).



- NP-complete
- greedy algorithm

```
GreedySetCover(Universe, Sets):
    I = {}
while X is not empty:
    MaxSet = argmax(Set in Sets, |Set ∩ X|)
    I = I∪{MaxSet}
    X = X - MaxSet
return I
```

  - add the set with a maximum number of not yet covered elements
  - repeat until all elements are covered => correct
  - approximation factor

    Let C_{OPT} be an optimal cover, let $t = |C_{OPT}|$
    Let X_k be the elements in iteration k. $X_0 = X$

    Claim: For all $0 \leq k$, the set X_k can be covered with t sets.
    Proof: The original set X can be covered with t sets, so the same is true for $X_k \subseteq X$

    In step k, there exists a set that covers at least $|X_k|/t$ elements (pigeonhole principle)
    ⇒ in step k the greedy algorithm is going to pick a set of size at least $|X_k|/t$
    For all k, we have $|X_{k+1}| \leq \left(1 - \frac{1}{t}\right)|X_k|$

    By induction for all $k \geq 0$: $|X_k| \leq \left(1 - \frac{1}{t}\right)^k |X_0| = \left(1 - \frac{1}{t}\right)^k \cdot |X|$

    * **When do we stop? How many sets do we choose?**
      We stop when $X_k = \emptyset$ ($|X_k| < 1$), chosen at most k sets
      For $k^* = t \cdot (\lceil \log|X| \rceil + 1)$, *we obtain*

      $$|X_{k^*}| \leq \left(1 - \frac{1}{t}\right)^{k^*} \cdot |X| \leq e^{-\frac{k^*}{t}} \cdot |X| = e^{-\frac{k^*}{t} + \log|X|} \leq e^{-1} < 1$$

    * ***At most $k^* = t \cdot (\lceil \log|X| \rceil + 1)$ sets, so we have a $(\lceil \log|X| \rceil + 1)$-approximation.***
      - $1 - x \leq e^{-x}$

Partition Problem

- **Input:**
  n positive integers $s_1, \ldots s_n$
  Goal:
  Partition the set of integers (integer indices) into two sets $A, B \subseteq \{1, \ldots n\}$ to minimize
  $$\max\left\{\sum_{i\in A} s_i, \sum_{i\in B} s_i\right\}$$
  - balance both partitions
- NP-complete
- • Dynamic programming: $O(n \cdot \sum s_i)$ (does not contradict NP-hardness)
  • Bruteforce by trying all combinations: $O(2^n)$
- Polynomial time approximation scheme
  - **Polynomial time approximation scheme (PTAS):** For each $\epsilon > 0$:
    - Computes a $(1 \pm \epsilon)$-approximation
    - Runtime is polynomial in input for fixed ϵ.
  - Fix: $m = \left\lceil \frac{1}{\epsilon} \right\rceil - 1$

    ```
    Order from largest to smallest s_1 ≥ s_2 ≥ ⋯ ≥ s_n.
    Compute an optimal solution (A,B) for the first m elements.

    Greedy for the rest:
      For i=m+1,…,n
        If weight(A) ≤ weight (B) :
          add i to A,
        else
          add i to B.
    ```
  - **Runtime:** $O(n \cdot \log n + 2^{\frac{1}{\epsilon}+1} + n)$
  - approximation factor

    Proof: Wlog assume at the end we have $weight(A) \geq weight\ (B)$
    Let s_k be the last element added to A.

    Look at the snapshot after adding k:
    - We only need to prove $\mathrm{w(A)} = w(A_k) \leq (1+\epsilon)OPT$
      but we don't know OPT, how can we compare to it?
    - perfect balancing as lower bound
      - $$L = \frac{1}{2}\sum s_i \leq OPT$$

Case 1 (k was added in the first phase) ⇒ $\boldsymbol{k \leq m}$

After the first phase, A was optimal for the smaller problem of adding the first m elements. Later, we never add anything to A. We obtain:

$$w(A) = w(A_k) \leq \; OPT(s_1, \dots, s_m) \leq OPT(s_1, \dots, s_n)$$

➔ Approximation ratio in this case is 1

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Case 2 (k was added in the second phase) ⇒ $\boldsymbol{k > m}$

Claim: $s_k \leq 2L/m$

Proof: As $s_1 \geq s_2 \geq \dots \geq s_k$ we get:
$2L \geq \sum_{i=1}^{k} s_i \geq \sum_{i=1}^{k} s_k \geq m \cdot s_k$, claim follows by dividing by m.

$W(A) \leq \; w(B_{k-1}) + s_k$ (we added k to A because of w$(A_{k-1}) \leq w(B_{k-1})$, and never changed A afterwards)

$$W(A) \leq \; w(B_{k-1}) + s_k \leq \; w(B) + s_k = 2L - w(A) + s_k$$

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$$W(A) \leq L + s_k \leq (1+\epsilon)L \leq (1+\epsilon)OPT$$

- $s_k \leq \frac{2L}{m}$ since all integers are positive
- last line of approximation works because of
  - $m = \lceil \frac{1}{\epsilon} \rceil - 1$

Theorem: For any $\epsilon > 0$ there exists an algorithm that computes a $(1+\epsilon)$-approximation of the partition problem in time $O(n \cdot \log n + 2^{\frac{1}{\epsilon}+1} + n)$.

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