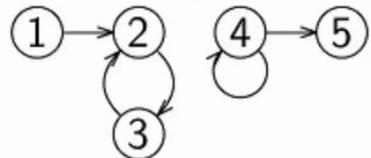


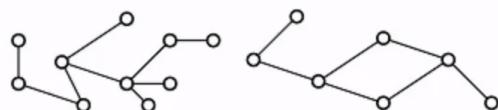
Terminology

- **Directed graph** $G = (V, E)$:
 V is the set of vertices and
 E is the set of edges: $E \subseteq V \times V$
 edge (u, v) : ordered pair of vertices,
 $u = \text{startpoint}$, $v = \text{endpoint}$

• Example: $V = \{1, 2, 3, 4, 5\}$,
 $E = \{(1, 2), (2, 3), (3, 2), (4, 5), (4, 4)\}$

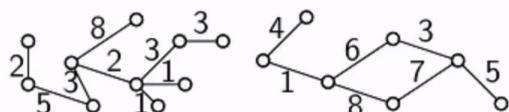


- A **loop** at a vertex $v \in V$ is an edge (v, v)
- A directed Graph is called **simple** if it has no loops
- **Undirected graph**: like a simple directed graph where E is symmetric, that is, $(v, w) \in E \Leftrightarrow (w, v) \in E$



$\Leftrightarrow E$ is a set of unordered pairs (v, w) with $v \neq w \in V$

- A (directed or undirected) **weighted graph** is a triple $G = (V, E, g)$ where g assigns a weight to each edge



- A **path** in $G = (V, E)$ from v to w is a sequence of vertices $v = v_0, v_1, \dots, v_k = w \in V$ with edges $(v_i, v_{i+1}) \in E$ for all $0 \leq i < k$



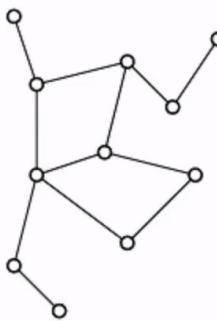
- **Simple path:** contains every vertex in V at most once
- **Length** of a path: number of its edges (v_i, v_{i+1})
length in *weighted graphs*: sum of the edge weights
- **Distance** $d_G(v, w)$ from v to w : length of the *shortest path* from v to w in G (or ∞ if no such path exists)
- A **cycle** in $G = (V, E)$ is a path v_0, v_1, \dots, v_k with $v_0 = v_k$



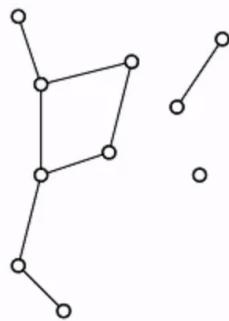
- **Simple cycle:** contains every vertex in V at most once, except for $v_0 = v_k$
- **Length** of a cycle: number of its edges (v_i, v_{i+1})
length in *weighted graphs*: sum of the edge weights
- **Trivial cycle:** contains only one vertex:  or two for undirected graphs: 
- A **subgraph** $G' = (E', V')$ of $G = (E, V)$ is a graph with $V' \subseteq V$ and $E' \subseteq E$. Note that for every edge $(u', v') \in E'$ we must have $u', v' \in V'$.

Example :

Graph G

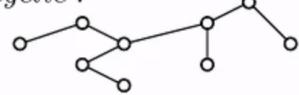


Subgraph G'



- A graph is **acyclic** if it does not contain any subgraph that is a non-trivial simple cycle

acyclic :



not acyclic :

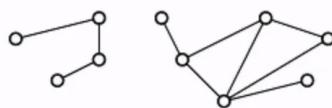


- A graph is **connected** if it contains a path from v to w for every pair $v, w \in V$ (not necessarily an edge (v, w))

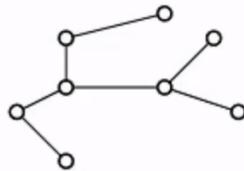
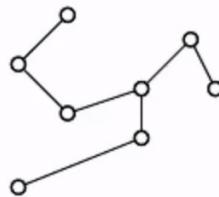
connected :



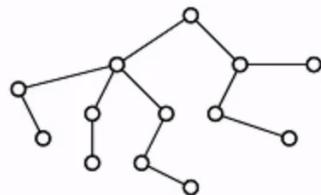
disconnected :



- An undirected graph is a **forest** if it is acyclic



- An undirected graph is a **tree** if it is acyclic and connected



- The **in-degree (out-degree)** of a vertex $v \in V$ is the number of edges ending (starting) in v :

$$\text{in-degree } (v) := |\{w \mid (w, v) \in E\}|$$

$$\text{out-degree } (v) := |\{w \mid (v, w) \in E\}|$$

- For undirected graphs we have

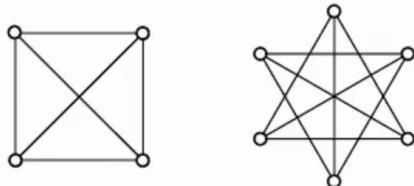
$$\text{in-degree}(v) = \text{out-degree}(v) \quad \forall v \in V.$$

In this case it is called the **degree** of the vertex v .

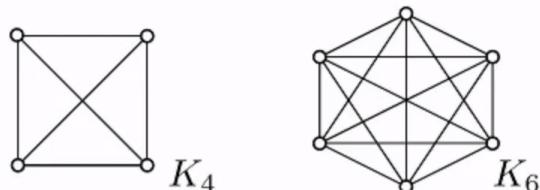
Degrees of vertices in an undirected graph $G = (V, E)$:

- degree $(v) = 0$: v is called an **isolated vertex** of G
- degree $(v) = 1$: v is called a **leaf** (end vertex) of G
- degree $(v) = r \quad \forall v \in V$: The graph G is called a **regular graph** of degree r

Example: regular graph(s) of degree 3



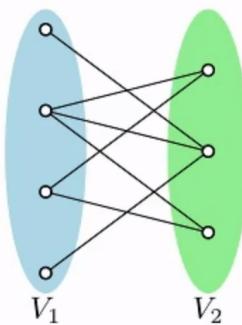
- An undirected **complete graph** on n vertices contains all $\binom{n}{2}$ possible edges (regular of degree $n - 1$)



- An **empty graph** has no edges (regular of degree 0)



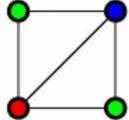
A graph $G(V, E)$ is called **bipartite**, if there exists a partition of V into V_1, V_2 such that all $(u, v) \in E$ have one endpoint in V_1 and the other one in V_2 .



In other words:
 G contains no edges within V_1 or V_2

k -Colorability of a Graph

A graph $G(V, E)$ is called **k -colorable** if its nodes can be colored with $\leq k$ colors, so that no nodes of the same color share an edge.

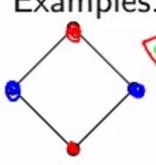


Example: How many colors are needed?

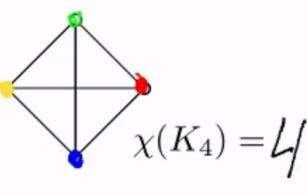
$k = 3$ colors are enough

The **chromatic number** $\chi(G)$ (spoken 'Chi of G') of a graph G is the minimum k such that G is k -colorable.

Examples:



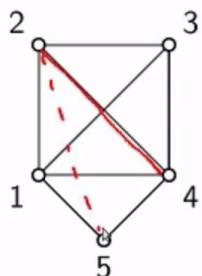
$$\chi(\text{cycle}) = \begin{cases} 2 & \text{even#rings} \\ 3 & \text{odd#rings} \end{cases}$$



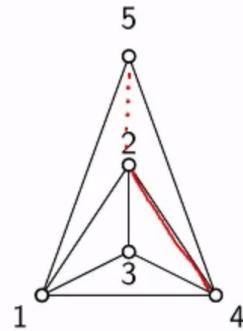
$$\chi(K_4) = 4$$

- Two graphs $G = (V, E)$ and $G'(V', E')$ are called **isomorphic** if there exists a bijection $\phi : V \rightarrow V'$, so that $(v, w) \in E \Leftrightarrow (\phi(v), \phi(w)) \in E'$

Example:



\Leftrightarrow



Definition 1:

A graph $G = (V, E)$ is called **planar** if there exists an injective mapping of V to $n = |V|$ distinct points in the Euclidian plane and a mapping of E to pairwise disjoint open curves in the Euclidian plane, so that:

1. $e = (x_i, x_j) \Rightarrow$ curve (e) connects point (x_i) with point (x_j)
2. $e = (x_i, x_j) \Rightarrow$ curve (e) does not contain any other point (x_k) , $k \neq i, j$

Definition 2:

A graph is called **planar** if it can be drawn in the plane

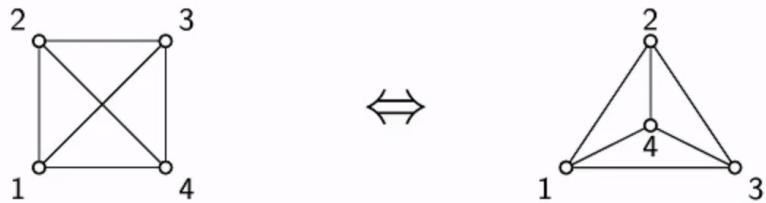
- without intersections between its edges (except in endpoints).

A drawing of a graph is called **plane** if it is free of intersections.

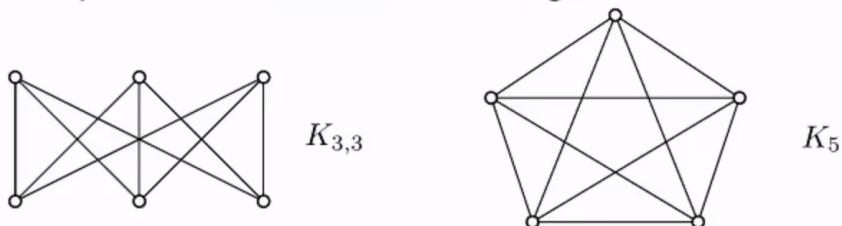
A non-plane drawing of a graph is called **embeddable** if it is isomorphic to a planar graph.

A **plane graph** is a plane drawing of a planar graph.

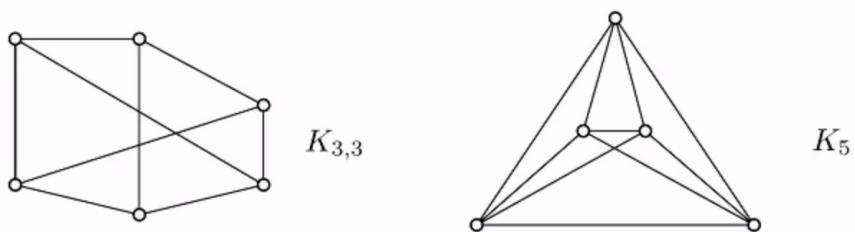
Example of an embeddable / plane drawing:



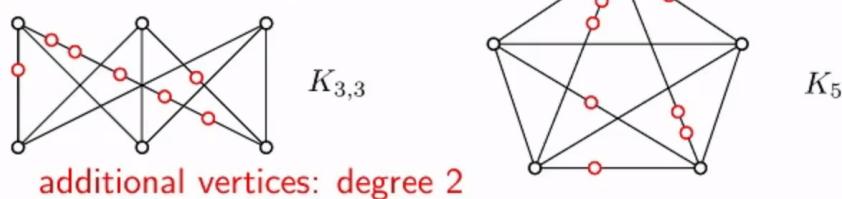
- Examples of non-embeddable drawings:



Non-embeddable drawing \Leftrightarrow drawing of a non-planar graph



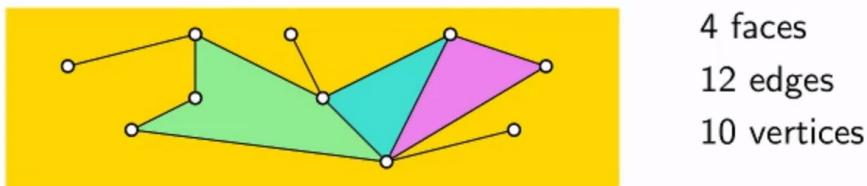
embeddability is invariant under subdivision of edges



Kuratowski's Theorem: A graph G is planar if and only if it does not contain a subgraph G' that is isomorphic to a subdivision of $K_{3,3}$ or K_5 , respectively. [without proof]

Subdivision of a graph G : some vertices are added "on edges" of G (some edges of G replaced by paths)

A plane graph (= plane drawing of a planar graph) divides the plane into regions called the *faces* of the drawing



Theorem: Euler's Formula (on plane graphs)

Given a connected, plane graph with exactly f faces, $n = |V|, m = |E|$. Then it holds that:

$$n + f = m + 2$$

- In other words: $\text{vertices} - \text{edges} + \text{faces} = 2$

- also applies for polyhedrons

$$\underline{n+f} = \underline{m+2}$$

Induction on Vtcs & edges

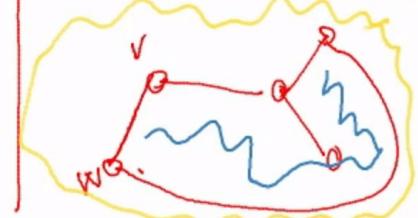
$$G = (V, E)$$

Smallest connected graph: 1 vertex, no edge

$$\begin{array}{ll} n=1 & 1+1=0+2 \cdot 1 \\ m=0 & \\ f=1 & \end{array}$$

$$\begin{array}{l} +1 \text{ edge} \\ +1 \text{ vertex} \\ +0 \text{ faces} \end{array} \quad \boxed{\checkmark}$$

$$\begin{array}{l} n-1 \text{ edges} \\ n \text{ vertices} \\ 1 \text{ face} \end{array} \quad \left. \begin{array}{l} \text{add edge:} \\ +1 \text{ edge} \\ +1 \text{ face} \\ +0 \text{ vertices} \end{array} \right.$$



- Planarity

Theorem: For a planar connected graph $G = (V, E)$ with $|V| = n \geq 3$ and $|E| = m$ it holds that $m \leq 3 \cdot n - 6$

Proof: Consider a plane drawing of G with f faces. Let $e_i, i = 1, \dots, f$ be the number of edges which bound face i .

- $\sum_{i=1}^f e_i \geq \sum_{i=1}^f 3 = 3 \cdot f$ (since $e_i \geq 3$)

- $\sum_{i=1}^f e_i = 2 \cdot m$

(since each edge is counted twice – once from each side)

\Rightarrow Combine the two results: $3 \cdot f \leq 2 \cdot m$

\Rightarrow Combine with Euler's Formula (multiplied by 3):

$$3 \cdot m + 6 = 3 \cdot n + 3 \cdot f \leq 3 \cdot n + 2 \cdot m$$

$$\Leftrightarrow m \leq 3 \cdot n - 6$$

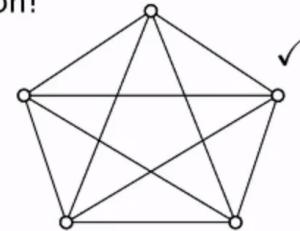
✓

*

Claim: K_5 is not planar.

Proof:

- K_5 has $n = 5$ vertices and $m = \binom{5}{2} = 10$ edges
- Assume K_5 is planar
 \Rightarrow We can apply the previous theorem
 $m \leq 3 \cdot n - 6$, with $n = 5, m = 10$
 $10 \leq 3 \cdot 5 - 6 = 9$ contradiction!
 $\Rightarrow K_5$ cannot be planar.



Theorem: For a bipartite planar connected graph $G = (V, E)$ with $|V| = n \geq 3$ and $|E| = m$ it holds that $m \leq 2 \cdot n - 4$

Proof: Consider a plane drawing of G with f faces. Let $e_i, i = 1, \dots, f$ be the number of edges which bound face i .

- G bipartite \Rightarrow every face is 2-colorable $\Rightarrow e_i \geq 4$
- $\sum_{i=1}^f e_i \geq \sum_{i=1}^f 4 = 4 \cdot f, \quad \sum_{i=1}^f e_i = 2 \cdot m$
- \Rightarrow Combine the two results: $4 \cdot f \leq 2 \cdot m \Leftrightarrow 2 \cdot f \leq m$
- \Rightarrow Combine with Euler's Formula (multiplied by 2):

$$2 \cdot m + 4 = 2 \cdot n + 2 \cdot f \leq 2 \cdot n + m$$

$$\Leftrightarrow m \leq 2 \cdot n - 4$$

✓

*

Claim: K_5 is not planar.

Proof:

- K_5 has $n = 5$ vertices and $m = \binom{5}{2} = 10$ edges
- Assume K_5 is planar
 \Rightarrow We can apply the previous theorem
 $m \leq 3 \cdot n - 6$, with $n = 5, m = 10$
 $10 \leq 3 \cdot 5 - 6 = 9$ contradiction!
 $\Rightarrow K_5$ cannot be planar.

by this,
 $K_{3,3}$ could
be planar!

Question:

Can we show in the same way that $K_{3,3}$ is not planar?

$n=6$ v.tcs, $m=3 \cdot 3 = 9$ edges
 $9 \leq 3 \cdot 6 - 6 = 12$ ✓

◆

Claim: $K_{3,3}$ is not planar.

Proof:

- $K_{3,3}$ has $n = 6$ vertices and $m = 3 \cdot 3 = 9$ edges
- Assume $K_{3,3}$ is planar

⇒ We can apply the previous theorem

$$m \leq 2 \cdot n - 4, \quad \text{with } \underline{n = 6}, m = 9$$

$$9 \leq 2 \cdot 6 - 4 = 8 \quad \text{contradiction!} \quad \blacksquare$$

⇒ $K_{3,3}$ cannot be planar.

◆

