

Potentialgleichung

- $\Delta u = 0$
 - Lösung dieser Gleichung kommt mit zweiter Funktion v
 - * v erfüllt ebenso Gleichung
 - * v heißt konjugiert harmonische Funktion
 - * v ist mit u über CR-Gleichungen verbunden
 - ♦ bzw. $f(x + iy) = u(x, y) + iv(x, y)$
 - jede Lösung ergibt quellenfreies Gradientenfeld $\text{grad}(u)$
 - * $\text{grad}(u)$ senkrecht auf $\text{grad}(v) \iff \langle \text{grad}(u), \text{grad}(v) \rangle = 0$
 - * Äquipotentiallinien (Niveaulinien) von u und v senkrecht aufeinander
 - ♦ außer $\text{grad}(v) = \text{grad}(u) = 0$

♦ $\{(x, y) \in U \mid u(x, y) = \cos(x)\} \quad \{(x, y) \in U \mid v(x, y) = \sin(x)\}$

Bestimmen von $v(x, y)$

- Gradientenfeld gegeben $\implies v(x, y)$ als Integral

$$v(x, y) = \int_{(x_0, y_0)}^{(x, y)} \underbrace{\frac{\partial v}{\partial x}(\xi, \eta)}_P d\xi + \underbrace{\frac{\partial v}{\partial y}(\xi, \eta)}_Q d\eta = \int_{(x_0, y_0)}^{(x, y)} -\frac{\partial u}{\partial y}(\xi, \eta) d\xi + \frac{\partial u}{\partial x}(\xi, \eta) d\eta$$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} \iff \Delta u = 0$

- Integrabilitätsbedingung prüfen \implies wegunabhängig

- * $u_{xx} = u_{yy}$
- $u_x = v_y$
- $u_y = -v_x$
- Beispiel WTF?

Bsp: $u(x, y) = e^{-y}(x \cos(x) - (y+1) \sin(x))$

* $\frac{\partial u}{\partial x} = e^{-y}(\cos(x) - x \sin(x) - (y+1) \sin(x)) = e^{-y}(-y \cos(x) - x \sin(x))$
 $\frac{\partial^2 u}{\partial x^2} = e^{-y}(-y \sin(x) - \sin(x) - x \cos(x)) = e^{-y}(-(y+1) \sin(x) - x \cos(x))$
 $\frac{\partial u}{\partial y} = -e^{-y}(x \cos(x) - (y+1) \sin(x)) = e^{-y}(-(y+1) \sin(x) - x \cos(x))$
 $\frac{\partial^2 u}{\partial y^2} = -e^{-y}(y \sin(x) - x \cos(x)) + e^{-y}(\sin(x)) = e^{-y}(-y \sin(x) + x \cos(x) + \sin(x))$

* $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} = e^{-y}(-y \sin(x) - x \cos(x)) = e^{-y}(-(y+1) \sin(x) - x \cos(x))$
 $\implies \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} \implies \Delta u = 0$

* $\frac{\partial u}{\partial x} = e^{-y}(-y \cos(x) - x \sin(x))$
 $\frac{\partial u}{\partial y} = -e^{-y}(x \cos(x) - (y+1) \sin(x))$
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 $\frac{\partial^2 u}{\partial y^2} = -e^{-y}(y \sin(x) - x \cos(x)) + e^{-y}(\sin(x))$
 $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} \implies \Delta u = 0$

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 $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} \implies \Delta u = 0$

Randwertaufgabe

$\Delta u = 0$ $U = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \}$ Einheitskreisscheibe
 Randwertaufgabe: $u(\cos(t), \sin(t)) = g(t)$ gegeben $0 \leq t < 2\pi$
 Plancherel
 $u(x,y) \in \mathcal{R}_0^2(\mathbb{R}^2)$ $f(z) = \sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} b_n \bar{z}^n e^{in\varphi} = \sum_{n=0}^{\infty} a_n \bar{b}_n (\cos(n\varphi) + i \sin(n\varphi))$
 Plancherel: $x = r \cos(\varphi)$ $y = r \sin(\varphi)$ $z = re^{i\varphi}$ $\bar{z} = r e^{-i\varphi}$
 $f(z) = \sum_{n=0}^{\infty} a_n (\cos(n\varphi) + i \sin(n\varphi)) + i \sum_{n=0}^{\infty} b_n (\cos(n\varphi) - i \sin(n\varphi))$
 $u(x,y) = \sum_{n=0}^{\infty} (a_n \cos(n\varphi) + b_n \sin(n\varphi))$

Poissonsche Integralformel

- sei $g: [0, 2\pi] \rightarrow \mathbb{R}$ eine Funktion
 - $u(r \cos(\varphi), r \sin(\varphi)) = \frac{1}{2\pi} \int_0^{2\pi} g(t) \frac{1-r^2}{1-2r \cos(t-\varphi) + r^2} dt$
 - Lösung der Potentialgleichung $\Delta u = 0$ mit

$u(\cos(\varphi), \sin(\varphi)) = g(\varphi)$ für $\varphi \in [0, 2\pi]$
 $\lim_{r \rightarrow 1-} u(r \cos(\varphi), r \sin(\varphi)) = g(\varphi)$ f. – –
 in allen Stetigkeitspunkten von g .

*

- Beispiel

Bsp: $g(t) = \begin{cases} 1 & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ 0 & \text{sonst} \end{cases} \quad t \in [-\pi, \pi]$
 $\alpha_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 dt = \frac{1}{2}$
 $\alpha_n = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 \cos(nt) dt = \frac{2 \sin(n\frac{\pi}{2})}{n\pi}$
 $\beta_n = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 \sin(nt) dt = \frac{2 \cos(n\frac{\pi}{2})}{n\pi}$
 $u(r \cos(\varphi), r \sin(\varphi)) = \frac{1}{2} + \sum_{n=1}^{\infty} r^n \left(\frac{2 \sin(n\frac{\pi}{2})}{n\pi} \cos(n\varphi) + \frac{2 \cos(n\frac{\pi}{2})}{n\pi} \sin(n\varphi) \right)$

Fourier-Reihe

- sei $g: I \rightarrow \mathbb{R}$
 - $I \dots$ Intervall der Länge 2π
 - Koeffizienten
 - * $\alpha_0 = \frac{1}{2\pi} \int_I g(t) dt$
 - * $\alpha_n = \frac{1}{\pi} \int_I g(t) \cos(nt) dt$
 - * $\beta_n = \frac{1}{\pi} \int_I g(t) \sin(nt) dt$
- wenn Reihe konvergiert, dann gilt in allen Stetigkeitspunkten von g
 - $g(\varphi) = \alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \cos(n\varphi) + \beta_n \sin(n\varphi))$
- Lösung der Potentialgleichung
 - $u(r \cos(\varphi), r \sin(\varphi)) = \alpha_0 + \sum_{n=1}^{\infty} r^n (\alpha_n \cos(n\varphi) + \beta_n \sin(n\varphi))$

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- Beispiele:

$$\begin{aligned}
 \gamma &= \gamma_1 \cup \gamma_2 \quad \gamma(t) = |t| \quad \gamma_1: |t| \leq \tau \\
 \alpha_0 &= \frac{1}{2\pi} \int_{-\tau}^{\tau} |t| dt = \frac{1}{2\pi} \int_0^{\tau} t dt + \frac{1}{\pi} \int_0^{\tau} t dt = \frac{1}{\pi} \int_0^{\tau} t dt = \frac{1}{\pi} \cdot \frac{\tau^2}{2} = \frac{\tau^2}{2\pi} \\
 \alpha_1 &= \frac{1}{\pi} \int_{-\tau}^{\tau} |t| \cos(ut) dt = \frac{2}{\pi} \int_0^{\tau} t \cos(ut) dt = \frac{2}{\pi} \left[t \cdot \frac{\sin(ut)}{u} - \int_0^{\tau} \frac{\sin(ut)}{u} dt \right] \\
 &= \frac{2}{\pi} \left[\frac{\cos(ut)}{u^2} \right]_0^{\tau} = \frac{2}{\pi} \left[\frac{\cos(u\tau) - 1}{u^2} \right] = \frac{2}{\pi} \left[\frac{\cos(u\tau) - 1}{u^2} \right] \\
 \alpha_1 &= \frac{2}{\pi} \int_0^{\tau} |t| \sin(ut) dt = 0 \\
 |t| &= \frac{\tau}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} \cos((2n+1)t) \quad \gamma_1: |t| \leq \tau \\
 &= \frac{\tau}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} \cos((2n+1)\varphi) \\
 &= \frac{\tau}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} \cos((2n+1)\varphi) \\
 &= \frac{\tau}{2} - \frac{4}{\pi} \left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots \right)
 \end{aligned}$$

[[Komplexe Kurvenintegrale]]