Definition

- given n line segments in the plane, find all intersections

Intersection Check

- other segment's endpoints must be on different sides
- for both segments
- check 4 triple-orientations
  - $\chi(a,b,c) = \mathrm{sign}\left|\begin{pmatrix} 1 & 1 & 1 \\ a_x & b_x & c_x \\ a_y & b_y & c_y \end{pmatrix}\right| = \begin{cases} +1 & \text{ccw} \\ 0 & \text{coll.} \\ -1 & \text{cw} \end{cases}$
  - $counterclockwise = left$

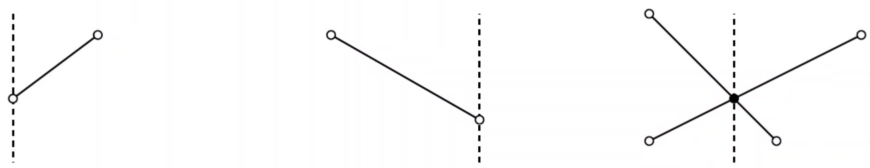
Observations

- intersection check takes constant time
- up to $\Theta(n^2)$ intersections
  - worst case takes $\Omega(n^2)$
  - output-sensitive algorithm needed

Plane Sweep Idea

- if two segments intersect $\rightarrow$ x-intervals overlap
  - inverse not always true
- scan from left to right through all x-values with vertical L
  - at every point
    * consider segments hit by L
    * check for intersection
  - intersections must be neighboured on L
    * at some point
- **Algorithm:**
  - Maintain y-order on L.
  - Check y-neighbored segments for intersections

  The plane-sweep is *event-based*, where an event is a change in the y-order.
- events



Left endpoint: insert segment

Right endpoint: remove segment

Intersection: switch y-neighbored segments

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Implementation

- used data structures
  - X
    - contains x-coordintates of known, future events
      - incoming start and endpoints
    - operations
      - insert
      - remove x-minimum
        - [[Queue]], [[Heap]]
  - search tree Y
    - contains y-ordered set of segments intersecting L
    - operations
      - insert (startpoints)
      - remove (endpoints)
      - switch neighbours (intersection)
      - dictionary, [[(2-4)-Bäume]]
- pseudocode

  $X = \varnothing, Y = \varnothing$

  Insert x-coordinates of the start- and endpoints of all segments into X.

  while $X \neq \varnothing$:

  1. Get minimum m of X and remove it from X.
  2. IF m left endpoint THEN insert its segment into Y
     ELSE IF m right endpoint THEN remove its segment from Y
     ELSE (m intersection) switch the order of the intersecting segments in Y
  3. FOR all new neighboring pairs in Y (at most two):
     IF neighboring pair intersects in p AND p is to the right of L
     THEN report p and insert x-coordinate of p into X

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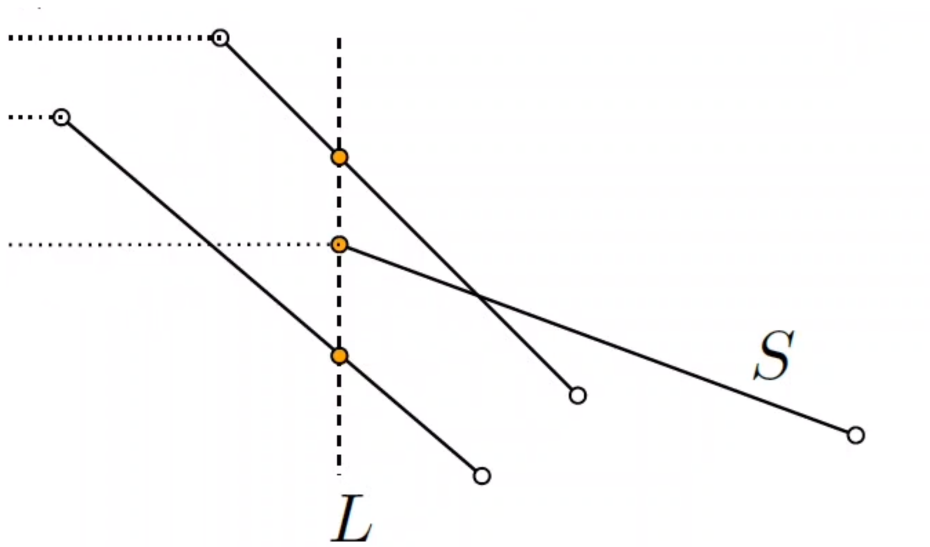
- y-order depends on actual y-coordinates at this x-coordinate



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- at most two new neighbours
  - above and below new minimum x
  - easier to find if linked
    - maybe link leaves of [[(2-4)-Bäume]] with pointers

Analysis

n segments, k intersections, $0 \leq k \leq \binom{n}{2} = \Theta(n^2)$

- **In X**: Per segment we insert two events, per intersection one. We later remove all of these events.

  $\Rightarrow O(n+k)$ space, and
  $O((n+k)\log(n+k)) = O((n+k)\log n)$ time

- **In Y**: We insert and remove every segment exactly once. For every intersection we switch a pair of segments. $O(1)$ per switch, if we link intersections to their segments (linked leaves in the 2-4-tree).

  $\Rightarrow O(n)$ space and $O(n\log n + k)$ time

In total: $O((n+k)\log n)$ time and $O(n+k)$ space.

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- detecting same intersection twice possible
  - must be prevented with check
  - does not affect time complexity

If we insert for every segment only the first not yet reached intersection into X, the algorithm uses only $O(n)$ space with the same running time.

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- can be further reduced to

- $O(n\ logn + k)$

- stop at first intersection possible
  - The algorithm works as intersection-**detector** in time $O(n \log n)$ and optimal space $O(n)$ (set $k = 0$ or $k = 1$).