Motivation

- Polygons represent 2-dimensional parts of the plane
- Complexity can be rather arbitrary
- Convex sets have many important properties
- Convex hull "simplifies" a set.
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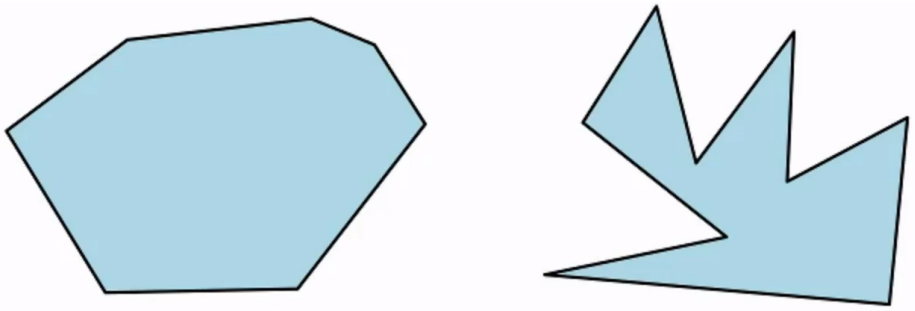

Convex Set

- **Definition:** A set $P \subseteq \mathbb{R}^d$ is *convex* if the segment between two points of P is also contained in P.
- Equivalent: intersection of P with a line is connected.
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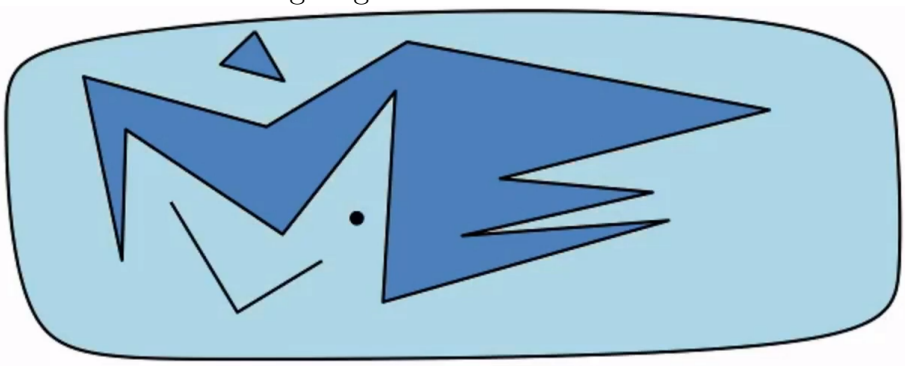
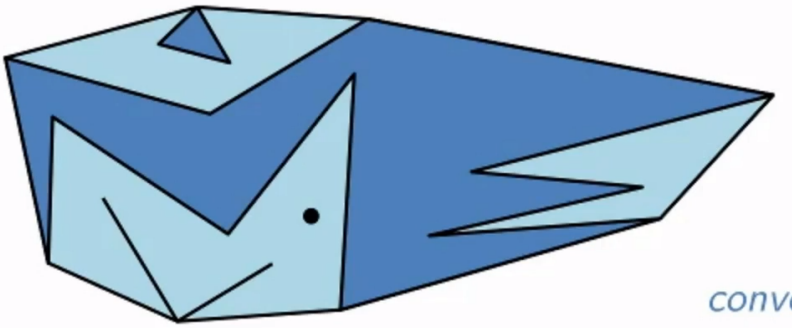
- intersection of convex sets also convex
- sets can be approximated with convex set
  - small convex set containing original set

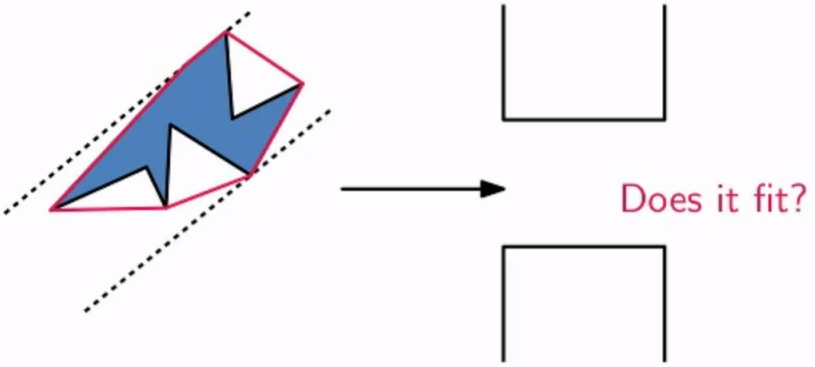


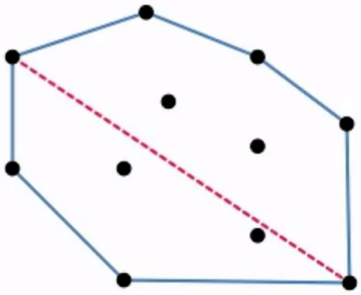
    *
- convex hull
  - smallest convex set

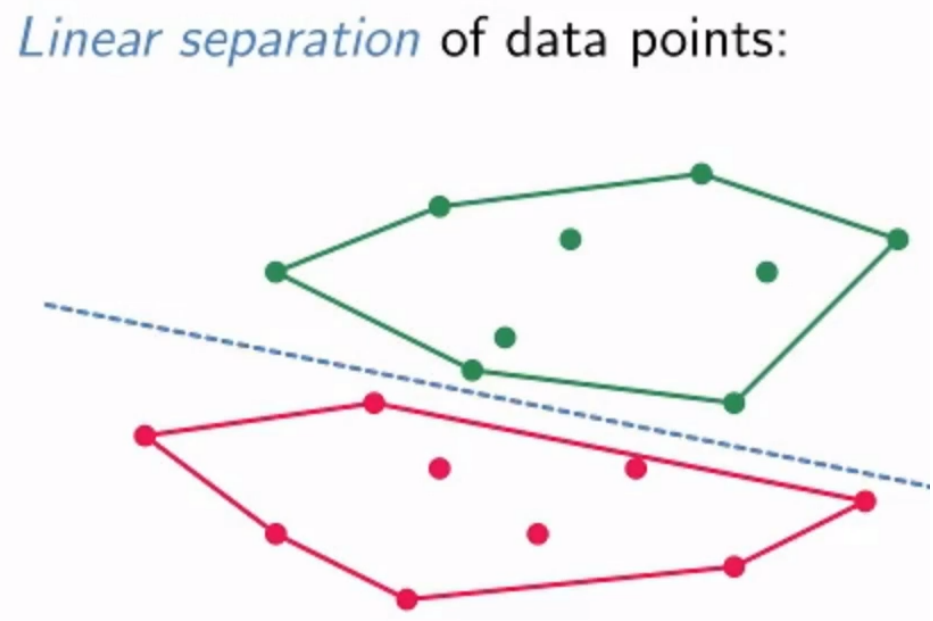
* 

  - **Definition:** The *convex hull* conv(P) of a set $P \subseteq \mathbb{R}^d$ is the intersection of all convex supersets of P.
  - contains boundary and interior

Motivation

- important geometric [[Datenstrukturen]]
- applications
  - *Width* of a polygon
    

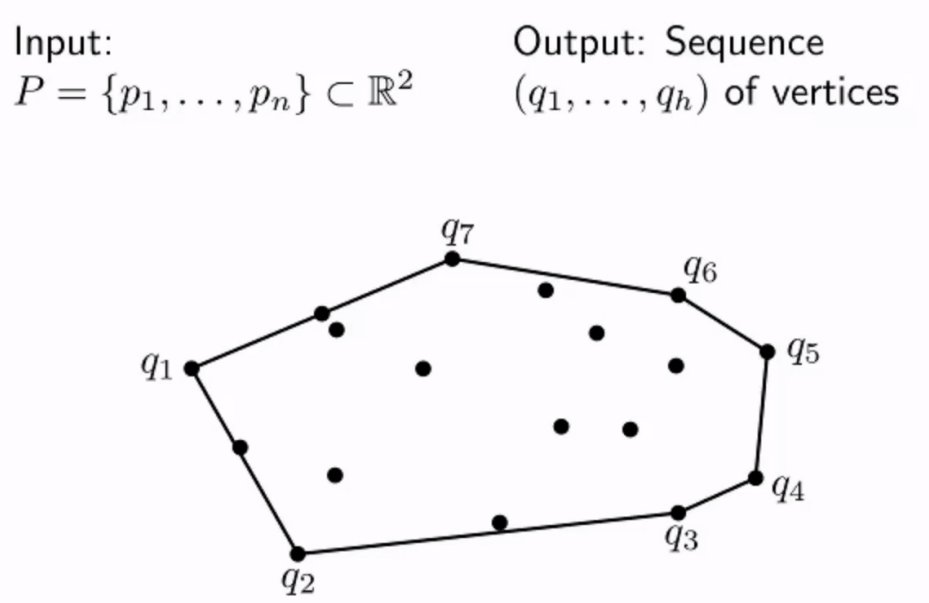
  - *Diameter* of a point set S: maximum distance between two points of S.
    



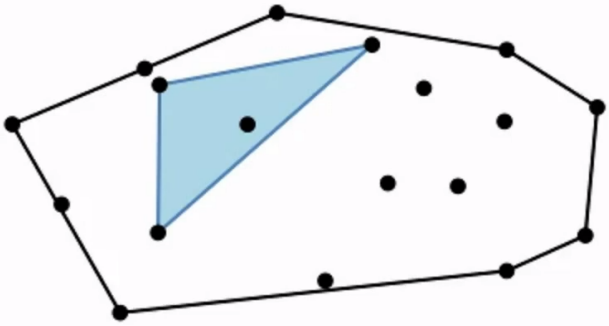
  - etc

Planar Convex Hull

- convex polygon
  - convex hull of point set
- convex hull of vertex set
  - convex hull of polygon
- basic creation algorithms



A point is on the convex hull boundary iff it is not contained in a triangle spanned by three points.

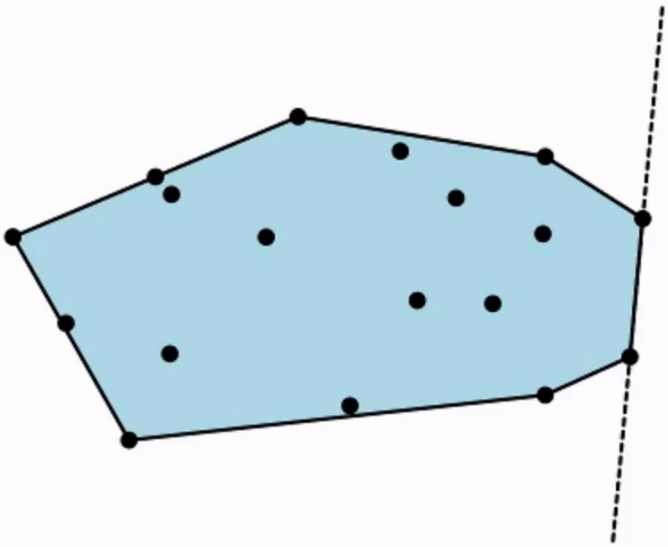


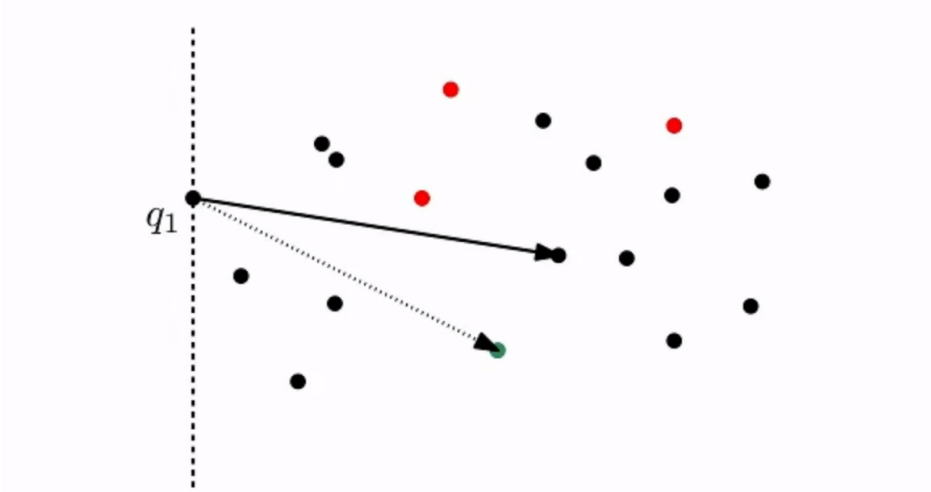
Trivial algorithm:
Check all points with all triangles in $O(n^4)$ time.

Jarvis' Wrap

- edge of convex hull if all other points on one side

Identify edges of the convex hull in $O(n^3)$ time.





- Start with extreme point
- Find edges that have all other points to the left

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– Repeat until q_1 is reached again

Input:

- Array `p[1..N]` of points (`N` ≥ 3)

Output:

- Array `q[1..h]` with convex hull vertices in order

Preparation:

- Find the point with smallest x-coordinate (`q_now`).
- `q_now` is the first known convex hull point.
- Choose a different point (`q_next`) as the first candidate for the next convex hull point.
- The array `q` is still empty.

•

Every round:

- add `q_now` to array `q`
- find next convex hull point `q_next`:
  point such that no other point is right of the directed line from `q_now` to `q_next`
- replace `q_now` by `q_next`
- replace `q_next` by new candidate different from `q_now`

End:

- When the next found convex hull point is equal to `q[1]` then the convex hull is completed.
- This is the case when `q_now` equals `q[1]` after a round.

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- pseudo code
  - preparation

```
for (i = 2 to N)
    if (p[i].x < p[1].x)
        swap(p[1], p[i])
q_now = p[1]
q_next = p[2]
h = 0
```

```
do
  h = h+1
  q[h] = q_now
  for (i = 2 to N)
    if (rightturn(q_now, q_next, p[i]))
      q_next = p[i]
  q_now = q_next
  q_next = p[1]
while (q_now != q[1])
```

- time complexity
  - Preparation: $\Theta(n)$ time
  - Outer loop is processed $h \leq n$ times
  - Inner loop is processed $\Theta(n)$ times each round.
  - $\Theta(nh) = O(n^2)$ rightturn tests
  - Worst case: $h = \Theta(n)$
  - Output sensitive (good for small hulls)
  - Asymptotically not optimal
- space complexity
  - $O(n)$ in addition to input: q and constantly many extra variables.
- correctness

- Before round k,
  - q contains $k-1$ vertices of the convex hull in order,
  - q_now is the next vertex on the convex hull.
- At the end of round k, q_now is the next convex hull vertex after q[h]: edge from q[h] to q_now has no other point to the right.
- When q_now equals q[1] the convex hull is complete.

Degeneracies:

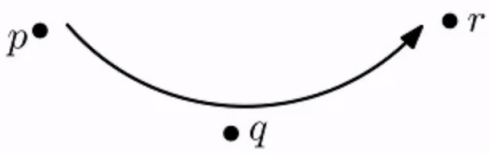
- Several points with smallest x-coordinate
- More than two points on a line

Orientation Test

Do three points make a left turn?

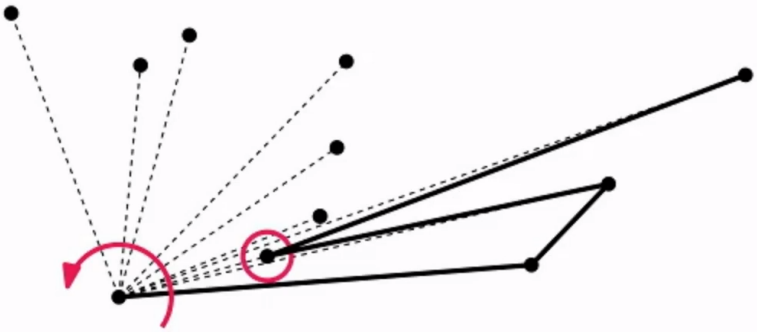
- Needed by many algorithms
- Evaluate degree 2 polynomial: *algebraic degree* 2
- Practical relevance

$$\text{sign}\left(\begin{vmatrix} 1 & p_x & p_y \\ 1 & q_x & q_y \\ 1 & r_x & r_y \end{vmatrix}\right)$$



Graham Scan

- Create hull by "Successive Local Repair"
- sequence sorted around extreme point: remove points not making a left turn



- Input:
  - Array `p[1..N]` of points ($N \geq 3$)

  Output:
  - Array `q[1..h]` with convex hull vertices (in order)

  Preparation:
  - Place the point with smallest y-coordinate into `p[1]`
  - Sort all other points counterclockwise around `p[1]`: `p[i]` is larger than `p[j]` if `p[i]` is left of the directed line from `p[1]` to `p[j]`
  - `p[1]` and `p[2]` are the first two convex hull vertices: Add them in this order to `q`.
-

```
for (i = 2 to N)
  if (p[i].y < p[1].y)
    swap(p[1], p[i])
sort p[2..N] counterclockwise around p[1]
q[1] = p[1]
q[2] = p[2]
h = 2
```

Process the remaining points from `p[3]` to `p[n]`.

Processing point `p[i]`:

- While from the last edge of the convex hull there is no left turn to `p[i]`, remove the last point from `q`.
- Add `p[i]` to `q`.

End:

- After `p[n]` has been processed, the convex hull vertices are stored in order in `q`.

```
for (i = 3 to N)
  while (h>1 and not leftturn(q[h-1],q[h],p[i]))
    h = h - 1
  h = h + 1
  q[h] = p[i]
```

- time complexity
  - Preparation in $O(n \log n)$ time due to sorting.
  - Buliding the convex hull: $\Theta(n)$ time. Why?

```
for (i = 3 to N)
  while (h>1 and not leftturn(q[h-1],q[h],p[i]))
    h = h - 1
  h = h + 1
  q[h] = p[i]
```

→ in total $O(n \log n)$ time.

- space complexity
  - $O(n)$ in addition to input

- correctness
  - Before the round for `p[i]`, `q` contains the vertices of the convex hull of `p[1..i-1]` in order. ⇐ initially true
  - After the round for `p[i]`, `q` contains the vertices of the convex hull of `p[1..i]` in order:
    - Point `p[i]` is the "last" vertex of the convex hull.
    - The last point `q[h]` is removed iff it lies in the triangle Δ `q[1] q[h-1] p[i]`.
      If it does not lie in this triangle, it is on the convex hull of `p[1..i]`.

Lower Bound

- The convex hull of n points in $\mathbb{R}^2$ can be computed in
  - $O(nh)$ time by Jarvis' Wrap:
    output sensitive, but quadratic in the worst case.
  - $O(n \log n)$ time by Graham's scan:
    worst-case optimal, but not output-sensitive.
- Lower bound of $\Omega(n \log n)$ time in the worst case follows from $\Omega(n \log n)$ worst case time for sorting.

Remark: The convex hull of n points in $\mathbb{R}^2$ can be computed in $O(n \log h)$ time by *Chan's algorithm* (a clever combination of Graham's scan and Jarvis' wrap).