

Sprague-Grundy-Theory

- First/Second-Player Win depends on number of coins

- for a single pile

Take away 1≤i≤k coins		1, 2 coins:
1, 2, 3 coins	O 1st	O 2nd
	O 2nd	O 1st
	O 1st	O 2nd
	O 1st	O 1st
	O 1st
	O 1st	

- Nimbers

Nimbers $*i, i \geq 0$, are a 'code' used for game-positions:

- o $*i, i \neq 0 \Rightarrow$ 1st player win (the player to move)
 - o $*0 \Rightarrow$ 2nd player win (the one just moved)

A nimber code implies:

- o From a $*0$ situation no legal move leads to another $*0$ situation
 - Interpretation: If I made a winning move, my opponent can not
 - o From any $*i, i \neq 0$, situation there is a legal move to a $*0$ situation
 - Interpretation: If my opponent gives me a (for them) non-optimal situation, I can make a winning move

- nimber depends on height

O		O			O	Move
O	O	O			O	MEX Rule
O	O	O			O	
O	O	O	O		O	
*4	*3	*1	*5	-	O	

NIM Rules

MEX-rule (Minimal Excluded):

- The nimber of a position P is the smallest value which is NOT a nimber of any position which is reachable by a valid move from P .

- The MEX-rule guarantees a good code!
 - From a $*0$ situation no legal move leads to another $*0$ position
 - From any $*i, i \neq 0$, situation there is a legal move to a $*0$ situation.

- XOR-rule:

- The nimber of a set of positions is the XOR-sum of the nimbers of the positions.

- Simplifies computation of nimbers for several piles.
 - That is, using the MEX-rule AND the XOR-rules makes computations much more efficient!

- For correctness the MEX-rule is sufficient.

	O		O		
*3					
1st	O	O	O		
player	O	O	O		
winner	O	O	O	O	
*4	*3	*1	*5		
4 ≈ 100		XOR:			
3 ≈ 011		011 ≈ *3			
1 ≈ 001					
5 ≈ 101					

Optimal Strategy

- compute each pile's nimber
- XOR to find out if a winning move exists
 - zero \Rightarrow no winning move
- compute for each pile if there is a winning move for this pile
- execute a optimal move
- repeat after opponent's move

NIM:	Height Number	Optimal moves:
07	111	A: 7 ≈ 111 → 100 ≈ *4 ✓
0	010	B: 2 ≈ 010 → 001 ≈ *1 ✓
5	101	C: 5 ≈ 101 → 110 ≈ *6 ↗
0	001	D: 3 ≈ 001 → 000 ≈ *φ ✓
0	000	
3	000	011 ≈ 3 *3
0	000	
2	000	
0	000	Goal: 000 ≈ *φ
0	000	
0	000	⇒ winning moves:
A B C D	2	A: remove 3 coins from A
	1	B: --- 1 coin --- B
*3 1st	0	D: --- 3 coins --- D
player win	*φ	

Variations of [[NIM-type Games]]

- take a limited number of coins each turn

$\begin{matrix} 5 & \circ \\ 0 & \circ \\ 0 & 3 \circ \\ 0 & 0 \circ \\ 0 & 0 \circ \\ 0 & 0 \circ \end{matrix}$	$\begin{matrix} 6 \\ \vdots \\ 6 \\ 5 \\ 4 \\ A \ B \ C \\ *1 \ *3 \ *2 \\ \swarrow \\ \Rightarrow * \phi \end{matrix}$	$\begin{matrix} \text{Take 1, 2, or 3 coins from one pile} \\ \text{HIGH Number} \\ A: *1 = 001 \\ B: *3 = 011 \\ C: *2 = 010 \\ \hline \text{XOR: } 000 = * \phi \\ 3 \ *3 \\ 2 \ *2 \\ 1 \ *1 \\ 0 \ ; * \phi \end{matrix}$	$\begin{matrix} \text{2nd player win} \\ \Rightarrow \text{NO winning moves} \end{matrix}$
--	---	---	--

- take 1 coin or split a pile into two (non-empty) piles

Take 1 coin OR split a pile into two piles

$\begin{matrix} 0 & * \phi \\ 0 & 2 \\ 0 & 1 \\ 0 & 0 \end{matrix}$	$H=2: \rightarrow H=1 \rightarrow *1$ $\rightarrow H=1, H=1: *1, *1 = * \phi$ $H=3: \rightarrow H=2 \rightarrow *2$ $\rightarrow H=2, H=1: *2, *1 = *3$ $H=4: \rightarrow H=3 \rightarrow *3$ $\rightarrow H=3, H=1: *3, *1 = *4$ $H=5: \rightarrow H=4 \rightarrow *4$ $\rightarrow H=3, H=2: *4, *2 = *5$ $\rightarrow H=4, H=1: *5, *1 = *6$ $\rightarrow H=3, H=2: *5, *2 = *7$ $\rightarrow H=2, H=1: *6, *1 = *8$ $\rightarrow H=1, H=0: *7, *0 = *9$
---	--

- Laskers NIM

- NIM or split a pile into two (non-empty) piles

$\begin{matrix} 10 & *10 \\ 9 & *9 \\ 8 & *8 \\ 7 & *7 \\ 6 & *6 \\ 5 & *5 \\ 4 & *4 \\ 3 & *3 \\ 2 & *2 \\ 1 & *1 \\ 0 & *0 \end{matrix}$	$\begin{matrix} LASKERS NIM \\ H=2: \rightarrow H=1: *1 \rightarrow H=0: *0 \\ \rightarrow H=1, H=1: *1, *1 = * \phi \\ H=3: \rightarrow H=2: *2 \\ \rightarrow H=1: *2, *1 = *3 \\ H=4: \rightarrow H=3: *3 \\ \rightarrow H=2: *3, *2 = *5 \\ \rightarrow H=1: *5, *1 = *6 \\ H=6: *5, *3, *2, *1, *0 \\ H=5, 1: *5, *1 = *4 \\ \rightarrow H=3: *4, *2 = *6 \\ H=4, 2: *4, *2 = *7 \\ H=3, 3: *4, *3 = *7 \\ H=7: *6, *5, *4, *3, *2, *1, *0 \\ H=6, 1: *6, *1 = *7 \\ H=5, 2: *5, *2 = *7 \\ H=4, 3: *5, *3 = *7 \end{matrix}$
--	--

LASKERS NIM		position/number to reach:	H : *
A	0	A: $4 \rightarrow *3 \triangleq 0011$	11 : *11
B	0	B: $12 \rightarrow *11 \triangleq 1011$	12 : *12
C	0	C: $5 \rightarrow *8 \triangleq 1001$	10 : *10
D	6	D: $6 \rightarrow *6 \triangleq 0110$	9 : *9
		XOR: $0111 \triangleq *7$	8 : *7
		1st player wins	7 : *8
		① Take 5 coins from pile D	6 : *6
		② Take 1 coin from pile A	5 : *5
		③ Take 1 coin from pile B	4 : *3
		④ Split pile D into 4 and 2	3 : *4
A B C D	*7		2 : *2
			1 : *1
			0 : *0

- Kayles and Dawson's Kayles

- no repeating nimber patterns

[[Algorithms and Games]]