NIM-Type Games

Algorithms and Games

NIM-Theory

- Nimbers $*i, i \geq 0$, are a 'code' used for game-positions:
  - $*i, i \neq 0 \Rightarrow 1^{st}$ player win (the player to move)
  - $*0 \qquad\qquad \Rightarrow 2^{nd}$ player win (the one just moved)

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  - The nimber of a position P is the smallest value which is NOT a nimber of any position which is reachable by a valid move from P.

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- **MEX-rule (Minimal Excluded):**
  - The nimber of a position P is the smallest value which is NOT a nimber of any position which is reachable by a valid move from P.
- XOR-rule:
  - The nimber of a set of positions is the XOR-sum of the nimbers of the positions.

How to analyze and play a game

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- Compute the nimber of a given position using the XOR-rule
- If the position has nimber $*0$: no winning move
- If the position has nimber $*i$, $i > 0$: Compute a (all) winning moves:
  - Compute all positions with nimber $*0$ which are reachable by a valid move
  - Reconstruct the moves leading to those positions. These are winning moves

How to analyze and play a game

pre-computing (game definition)

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playing a game

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NIM-type games

Alle games are based on piles of coins. All games are normal play, that is, the last one to make a valid move wins. Taking coins is always from **one** pile.

0. NIM: Take 1,2,3,... up to all coins of a pile
1. Take 1 or 2 coins
2. Take 1 or 2 or 3 coins
3. Take 1 to k coins
4. Take 1 or 4 coins
5. Take 1 coin OR split one pile into 2 non-empty piles

NIM-type games

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6. Laskers NIM: Play NIM OR split one pile into 2 non-empty piles
7. Kayles: Take 1 or 2 coins. Then **optional** split this pile into 2 non-empty piles
8. Dawsons Kayles: Take 2 coins. Then **optional** split this pile into 2 non-empty piles
9. Take 1 OR 3 coins OR split one pile into 2 non-empty piles

Nimbers for Kayles and Dawson's Kayles

Kayles:

Hight	Nimber
12	*4
11	*6
10	*2
9	*4
8	*1
7	*2
6	*3
5	*4
4	*1
3	*3
2	*2
1	*1
0	*0

Dawson's Kayles:

Hight	Nimber
12	*2
11	*3
10	*3
9	*0
8	*1
7	*1
6	*3
5	*0
4	*2
3	*1
2	*1
1	*0
0	*0

XOR-Rule

	0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111
0000	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0001	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
0010	2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
0011	3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
0100	4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
0101	5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
0110	6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
0111	7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
1000	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
1001	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
1010	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
1011	11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
1100	12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
1101	13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
1110	14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
1111	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

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	0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111
0000	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0001	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
0010	2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
0011	3	2	1	0	7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	15	14	13	12
0100	4	5	6	7	0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	8	9	10	11
0101	5	4	7	6	1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	9	8	11	10
0110	6	7	4	5	2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	10	11	8	9
0111	7	6	5	4	3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	11	10	9	8
1000	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
1001	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
1010	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
1011	11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
1100	12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
1101	13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
1110	14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
1111	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

Induction base from $*0 + *i = *i$, and $*i + *i = *0$

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0000	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0001	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
0010	2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
0011	3	2	1	0	7										13	12
0100	4	5	6	7	0										10	11
0101	5	4	7	6	1										11	10
0110	6	7	4	5	2										8	9
0111	7	6	5	4	3										9	8
1000	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
1001	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
1010	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
1011	11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
1100	12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
1101	13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
1110	14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
1111	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

$$A \rightarrow \begin{array}{c|c} A & B \\ \hline B & A \end{array} \text{ with } B = A + (2^k)$$

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0000	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0001	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
0010	2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
0011	3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
0100	4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
0101	5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
0110	6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
0111	7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
1000	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
1001	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
1010	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
1011	11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
1100	12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
1101	13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
1110	14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
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0011	3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
0100	4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
0101	5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
0110	6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
0111	7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
1000	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
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1010	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
1011	11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
1100	12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
1101	13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
1110	14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
1111	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

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0100	4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
0101	5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
0110	6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
0111	7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
1000	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
1001	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
1010	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
1011	11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
1100	12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
1101	13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
1110	14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
1111	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0