Definition

- $$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

  Die Lösung der Rekursion ergibt sich je nach $f(n)$:

  1. Wenn $f(n) \in O\left(n^{log_b(a)-\varepsilon}\right)$ für ein ε>0, dann gilt
  $$T(n) \in \Theta\left(n^{log_b(a)}\right)$$
  2. Wenn $f(n) \in \Theta\left(n^{log_b(a)}\right)$, dann gilt
  $$T(n) \in \Theta\left(n^{log_b(a)}\log(n)\right)$$
  3. Wenn $f(n) \in \Omega\left(n^{log_b(a)+\varepsilon}\right)$ für ein ε>0, und gilt für alle hinreichend großen n die Abschätzung $af\left(\frac{n}{b}\right) \leq cf(n)$ für $0 < c < 1$, dann gilt
  $$T(n) \in \Theta(f(n))$$

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- same in english
  - "cooking recipe" to solve recurrences of the form
  $$T(n) = aT(\frac{n}{b}) + f(n)$$
  with $a \geq 1$ and $b > 1$.

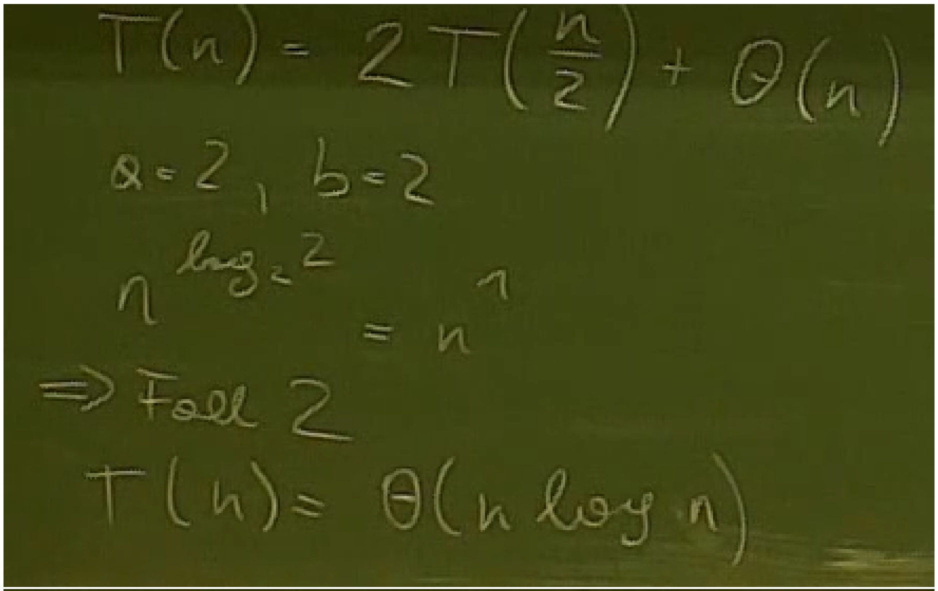
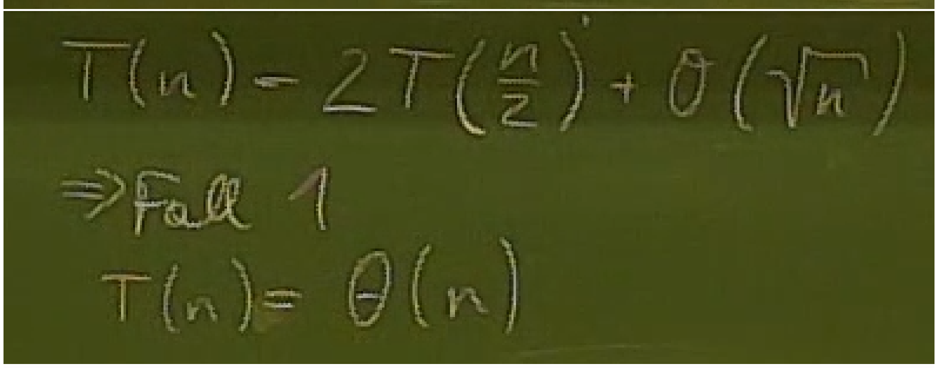
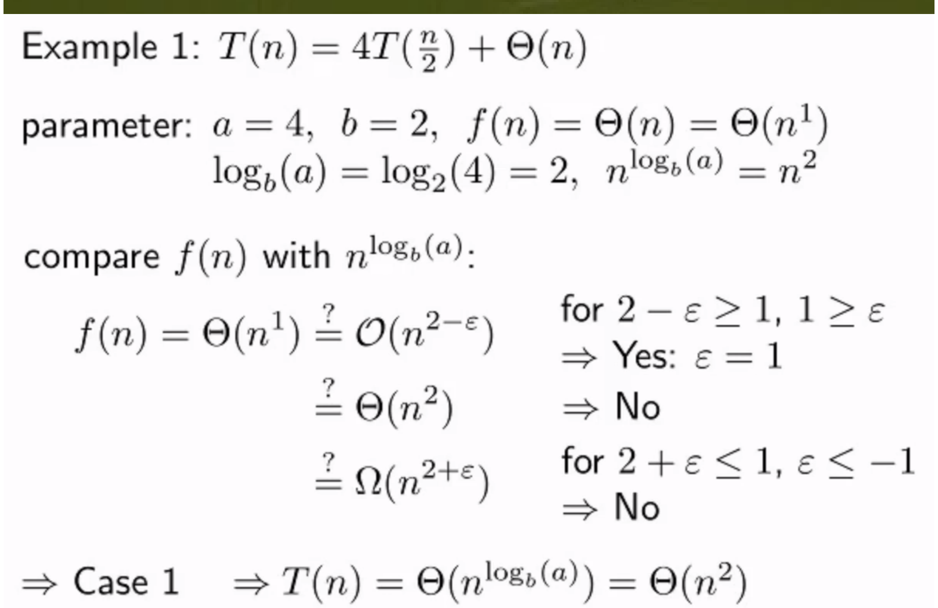
    In the following cases we directly get the solution ...

    Case 1: $f(n) = \mathcal{O}\left(n^{\log_b a - \varepsilon}\right)$ for some $\varepsilon > 0$
    $$\Rightarrow T(n) = \Theta(n^{\log_b a})$$

    Case 2: $f(n) = \Theta(n^{\log_b a})$ $\qquad \Rightarrow T(n) = \Theta(n^{\log_b a} \log n)$

    Case 3: $f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right)$ for some $\varepsilon > 0$
    and $\exists c < 1$ such that $a \cdot f(\frac{n}{b}) \leq c \cdot f(n), n \geq n_0$

Beispiele

- 
$T(n) = 2T(\frac{n}{2}) + \Theta(n)$
$a = 2, b = 2$
$n^{\log_2 2} = n^1$
$\Rightarrow$ Fall 2
$T(n) = \Theta(n \log n)$
- 
$T(n) = 2T(\frac{n}{2}) + \Theta(\sqrt{n})$
$\Rightarrow$ Fall 1
$T(n) = \Theta(n)$
- 
Example 1: $T(n) = 4T(\frac{n}{2}) + \Theta(n)$
parameter: $a = 4, \ b = 2, \ f(n) = \Theta(n) = \Theta(n^1)$
$\log_b(a) = \log_2(4) = 2, \ n^{\log_b(a)} = n^2$
compare $f(n)$ with $n^{\log_b(a)}$:
$f(n) = \Theta(n^1) \stackrel{?}{=} \mathcal{O}(n^{2-\varepsilon})$
for $2 - \varepsilon \geq 1$, $1 \geq \varepsilon$
$\Rightarrow$ Yes: $\varepsilon = 1$
$\stackrel{?}{=} \Theta(n^2)$
$\Rightarrow$ No
$\stackrel{?}{=} \Omega(n^{2+\varepsilon})$
for $2 + \varepsilon \leq 1$, $\varepsilon \leq -1$
$\Rightarrow$ No
$\Rightarrow$ Case 1 $\Rightarrow T(n) = \Theta(n^{\log_b(a)}) = \Theta(n^2)$

- Example 2: $T(n) = 4T(\frac{n}{2}) + \Theta(n^2)$

  parameter: $a = 4,\ \ b = 2,\ \ f(n) = \Theta(n^2)$
  $\log_b(a) = \log_2(4) = 2,\ \ n^{\log_b(a)} = n^2$

  compare $f(n)$ with $n^{\log_b(a)}$:

  $f(n) = \Theta(n^2) \stackrel{?}{=} \mathcal{O}(n^{2-\varepsilon})$ for $2 - \varepsilon \geq 2$, $0 \geq \varepsilon$ $\Rightarrow$ No

  $\stackrel{?}{=} \Theta(n^2)$ $\Rightarrow$ Yes

  $\stackrel{?}{=} \Omega(n^{2+\varepsilon})$ for $2 + \varepsilon \leq 2$, $\varepsilon \leq 0$ $\Rightarrow$ No

  $\Rightarrow$ Case 2 $\quad \Rightarrow T(n) = \Theta(n^{\log_b(a)} \log(n)) = \Theta(n^2 \log(n))$

- Example 3: $T(n) = 3T(\frac{n}{2}) + \Theta(n^2)$

  parameter: $a = 3,\ \ b = 2,\ \ f(n) = \Theta(n^2)$
  $\log_b(a) = \log_2(3),\ \ 1 < \log_2(3) < 2$
  $n^{\log_b(a)} = n^{\log_2(3)}$

  compare $f(n)$ with $n^{\log_b(a)}$:

  $f(n) = \Theta(n^2) \stackrel{?}{=} \mathcal{O}(n^{log_2(3)-\varepsilon})$ for $\log_2(3) - \varepsilon \geq 1$, $\log_2(3) - 2 \geq \varepsilon$ $\Rightarrow$ No

  $\stackrel{?}{=} \Theta(n^{log_2(3)})$ $\Rightarrow$ No

  $\stackrel{?}{=} \Omega(n^{log_2(3)+\varepsilon})$ for $\log_2(3) + \varepsilon \leq 2$, $\varepsilon \leq 2 - \log_2(3)$ $\Rightarrow$ Yes $\Rightarrow$ Case 3?

- parameter: $a = 3,\ \ b = 2,\ \ f(n) = \Theta(n^2)$
  $\log_b(a) = \log_2(3),\ \ 1 < \log_2(3) < 2$
  $n^{\log_b(a)} = n^{\log_2(3)}$

  $f(n) = \Theta(n^2) = \Omega(n^{\log_2(3)+\varepsilon})$

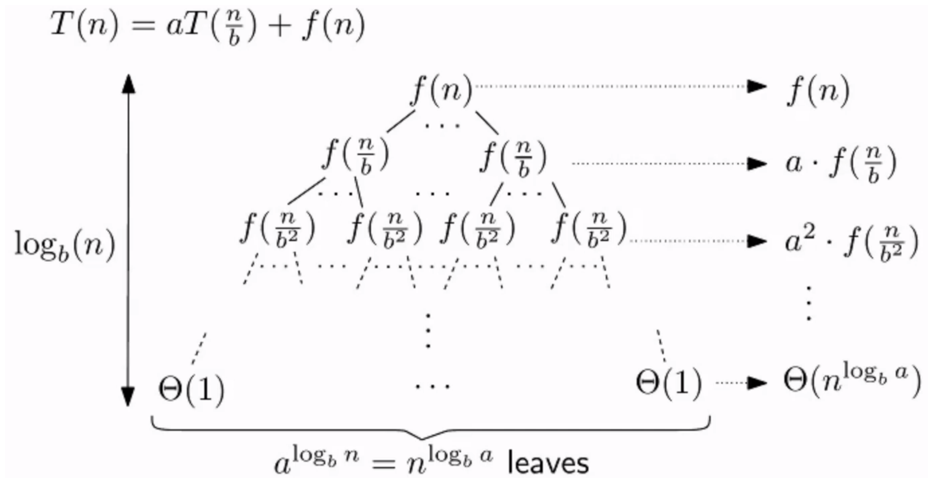
  Check additional condition:
  $\exists c < 1$ such that $a \cdot f(\frac{n}{b}) \leq c \cdot f(n)$ for $n \geq n_0$?

  $a \cdot f(\frac{n}{b}) = 3 \cdot (\frac{n}{2})^2 = \frac{3}{4} \cdot n^2 \leq c \cdot n^2$

  $\Rightarrow$ Yes for $n \geq 1$ and $\frac{3}{4} \leq c < 1$

  $\Rightarrow$ Case 3 $\quad \Rightarrow T(n) = \Theta(f(n)) = \Theta(n^2)$

Recursion Tree

- 

- Total: $\Theta(n^{\log_b a}) + \sum_{i=0}^{\log_b(n)-1} a^i \cdot f(\frac{n}{b^i})$