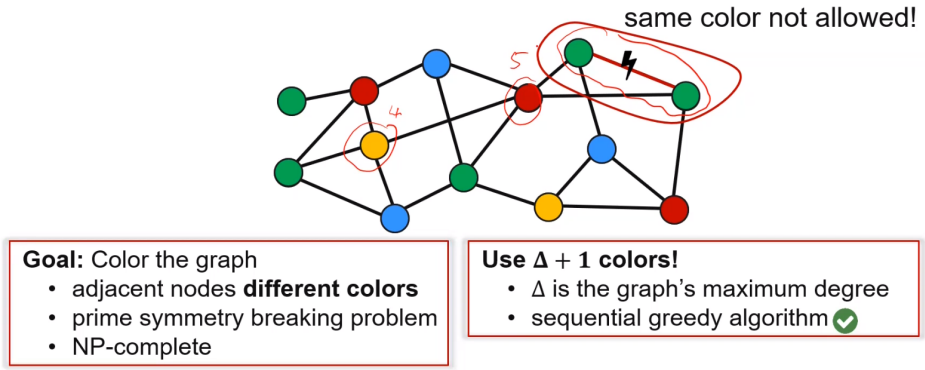
Overview

- solve a problem using multiple distributed devices
  - data might not fit onto a single device
    - disk cluster
  - maybe All-to-all communication
    - no locality restrictions
    - bandwidth restrictions
- output may be just a part of the solution
- time complexity wrt communication rounds instead of computation time
  - efficient = $polylog(n) << n$
  - log^*n
    - $$\log^* n = \min\{i | \log^{(i)} n \leq 2\}$$
      $$\log^{(1)} n = \log n$$
      $$\log^{(i+1)} n = \log(\log^{(i)} n)$$

Vertex Coloring

- symmetry breaking
  - mutual exclusive access to shared resource
  - colors = time slots => resource scheduling



- color local minima vertices first
  - minima if no neighbor has a higher IP
  - extremely slow
  - linear time
- random coloring

- **Repeat** until colored
  - pick a random **free color** c
  - keep it, if no *conflict*

- **Randomized:** $O(\log n)$ rounds for 2Δ-coloring, whp.
- **Randomized:** $O(\log n)$ rounds for $(\Delta+1)$-coloring, whp.

Simple Color Reduction

- reduce one color per round
  - cannot reduce below $\delta+1$ colors

Input: Graph $G=(V,E)$ with C-vertex input coloring ϕ
Output: Coloring with $\max\{C-1, \Delta+1\}$ colors

Each node v executes the following code in parallel

If $\phi(v) \neq C$ **then**
  output $\phi(v)$
else
  output $min\,\{1,\dots,\Delta+1\} \setminus \{\phi(u) \mid u \in N(v)\}$

Messages (2 rounds):
when node v is processed, ask its neighbors $u \in N(v)$ for their color $\phi(u)$.
Messages (1 round, alternative implementation):
every node sends its current color

- from $O(\delta$š$)$ to $\delta+1$ colors in $O(\delta$š$)$ rounds
  - **Theorem:** $(\Delta+1)$**-coloring can be done in** $O(\Delta^2+\log^* n)$ rounds.

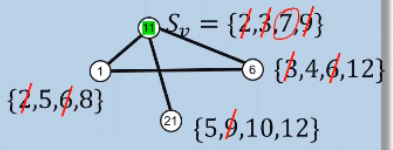
Linial's Algorithm

- theorems
  - **Linial's LB:** Coloring rings with $O(1)$ **colors** requires $\Omega(\log^* n)$ rounds
  - **Linial's algorithm:** $O(\log^* n)$ rounds for $O(\Delta^2)$**-coloring.**
    - Δ = highest degree
  - Linial's color reduction
    - Given an m-coloring, we can compute a $(100\Delta^2 \cdot \log m)$**-coloring** in 1 round.
    - Given a Δ^3-coloring, we can compute a $(100\Delta^2)$**-coloring** in 1 round.

Algorithm (1 round):

- node v selects a **candidate color set** $S_v \subseteq [100\Delta^2]$
- v **picks** color in

$$S_v \setminus \bigcup_{w \in N(v)} S_w \neq \emptyset$$



- Fix prime $q \in [10 \cdot \Delta, 20\Delta]$. Polynomials $p_0, p_1, p_2, \ldots$ of degree 3 over $\mathbb{F}_q$

(we assume just $q = 10\Delta$)

$$p\colon \mathbb{F}_q \to \mathbb{F}_q, \qquad x \mapsto p(x)$$

Node v with **input color** $i \in \Delta^3$ picks polynomial p_i (#polynomials=$q^{d+1} \geq \Delta^3$)

Low intersecting sets

$$S_v = S_i = \{(x, p_i(x)) \mid x \in \mathbb{F}_q\} \subseteq \mathbb{F}_q \times \mathbb{F}_q$$

- **Example:**
  prime $q = 7$.
  $p_v(x) = 6x^3 + 4x^2 + 3 \in \mathbb{F}_q$

Low intersecting sets

$$S_v = S_i = \{(x, p_i(x)) \mid x \in \mathbb{F}_q\} \subseteq \mathbb{F}_q \times \mathbb{F}_q$$

$$S_v = \{(0, p_v(0)), (1, p_v(1)), (2, p_v(2)), (3, p_v(3)), (4, p_v(4)), (5, p_v(5)), (6, p_v(6))\}$$
$$S_v = \{(0,3), (1,13), (2,67), (3,201), (4,451), (5,853), (6,1443)\}$$
$$S_v = \{(0,3), (1,6), (2,4), (3,5), (4,3), (5,6), (6,1)\}$$

- Interpret each tuple (x, y) as an output color
- $|S_v| = q = 7$

- small intersection set
  - Two polynomials of degree d over a prime field can intersect in at most d points.

Input coloring provides enough symmetry breaking to assign distinct polynomials to neighboring nodes

Input color $i \mapsto$ polynomial $p_i \mapsto$ set S_i

(adjacent nodes do not receive the same set)

No communication!

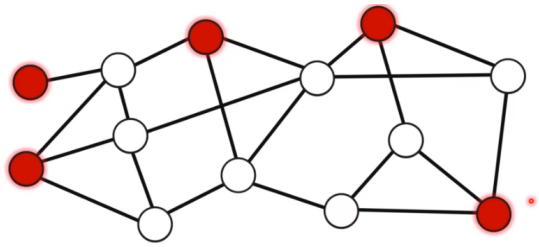
ID-assignment is a coloring |ID-SPACE|=N colors

N-coloring $\to O(\Delta^2 \log N) \to O(\Delta^2(\log \Delta + \log\log N) \to \cdots \to O(\Delta^2 \log \Delta) \to^* O(\Delta^2)$

$\Delta^3 \to^* O(\Delta^2)$

$O(\log^* N)$ iterations/rounds

Maximal Independent Set



*A **maximal independent set (MIS)** of a graph $G = (V, E)$ is a subset $S \subseteq V$ of the nodes such that*

- *S is an **independent set** in G (no two vertices in S are adjacent)*
- *S is **maximal**, that is, for any $v \in V \setminus S$, the set $S \cup \{v\}$ is **not** an independent set*

- at the end each node knows whether is in the independent set
- different from the maximum independent set problem
  - independent set need not be of maximum size
- naive centralized greedy algorithm

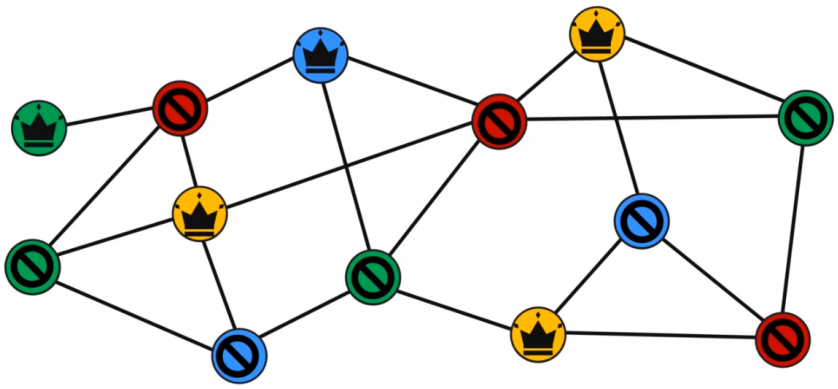
```
S = ∅.
iterate through the vertices in arbitrary order
    add a node to S, if none of its neighbors is already in S.
```

- distributed version
  - **Input: Graph $G = (V, E)$ with C-vertex coloring ϕ**
  - executed by each node v in parallel/synchronous

```
S=∅
For i=1 to C
    If φ(v) = i then
        if N(v) ∩ S = ∅, join S and notify neighbors
Output whether v ∈ S.
```

  - coloring acts as scheduler
  - notifying/asking neighbors
    - implementation dependent
    - may take 2 additional rounds
    - does not change time complexity

```
For i=1 to C
    If φ(v) = i then
        if N(v) ∩ S = ∅, join S and notify neighbors
```



- There is a deterministic distributed algorithm to compute an MIS in $O(C)$ rounds, given a C-vertex coloring.

Proof (correctness, independent set):
Notation: $S_i, i = 0, \dots, C$, the set S after iteration i
Induction: S_i is an independent set.
Induction start: Initially $S_0 = \emptyset$ is an IS
Induction Step ($S_i \to S_{i+1}$):
Assume for contradiction that there exists adjacent nodes $u, v \in S_{i+1}$:
As S_i is an IS, u or v needs to have input color $i+1$. Wlog $\phi(v) = i+1$.
Case 1 ($\phi(u) = i+1$): Contradiction, as u, v, are adjacent
Case 2 ($\phi(u) < i+1$): Contradiction, as then $u \in S_i$, and v would not have joined S_i, as the test $N(v) \cap S = \emptyset$ would have been negative.

Proof (correctness, maximality):
Assume for contradiction there exists some node $v \in V \setminus S$ such that $S \cup \{v\}$ is an IS.
As S only keeps growing, this implies that v had no neighbor in $S_{\phi(v)-1}$ when v was processed, and v would have joined $S_{\phi(v)} \subseteq S$, a contradiction.

Proof (runtime):
The previous algorithm requires one round of communication per color class to inform neighbors whether one has joined the MIS or not

Corollary:
There is a deterministic distributed algorithm to compute an MIS in $O(\Delta^2 + \log^* n)$ rounds.

Proof:
Use Linial's algorithm to compute an $O(\Delta^2)$-coloring in $O(\log^* n)$ rounds. Then use the theorem above.