NIM-Theory

Algorithms and Games

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- $n \times m$ chess board
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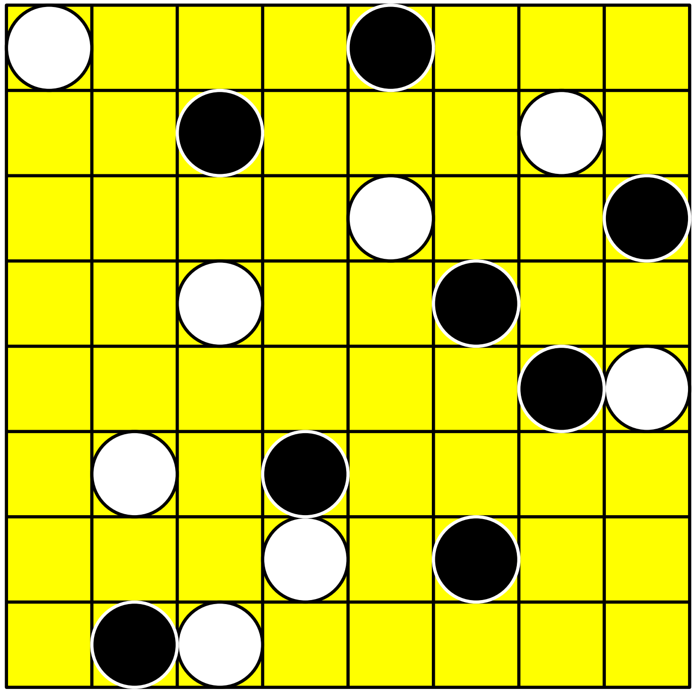

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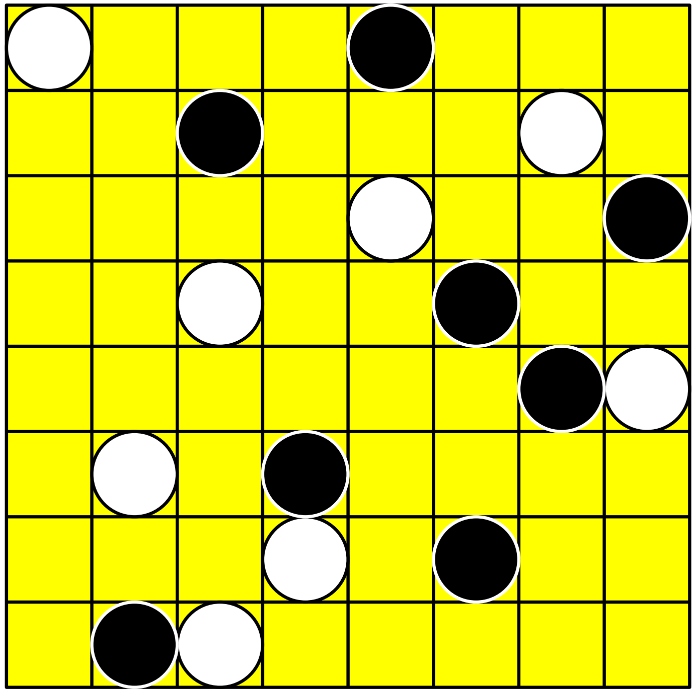

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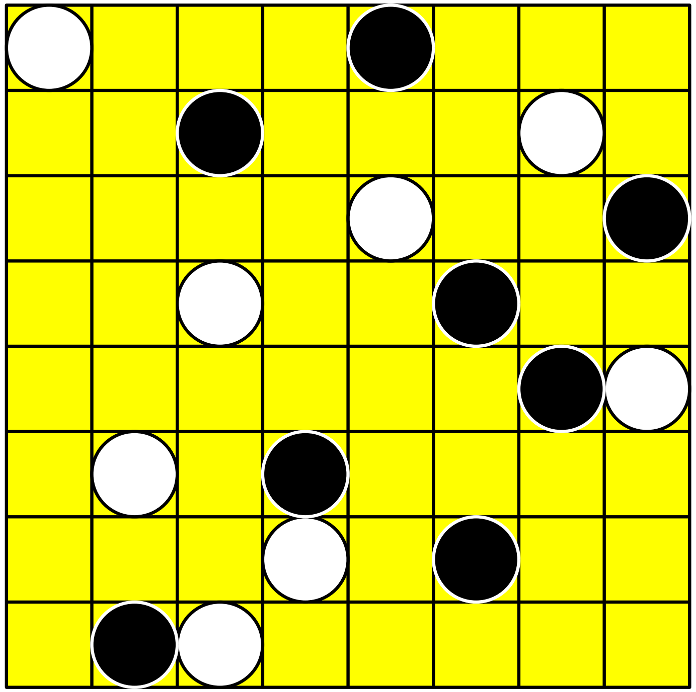

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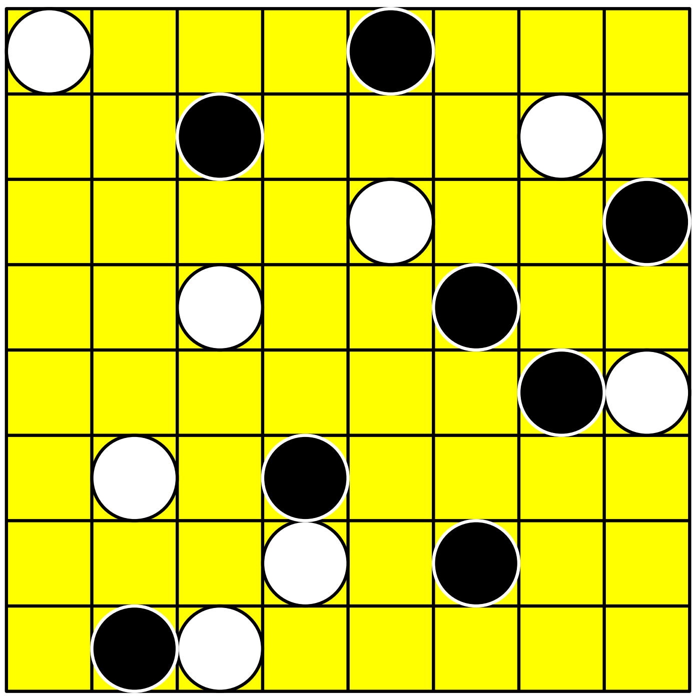

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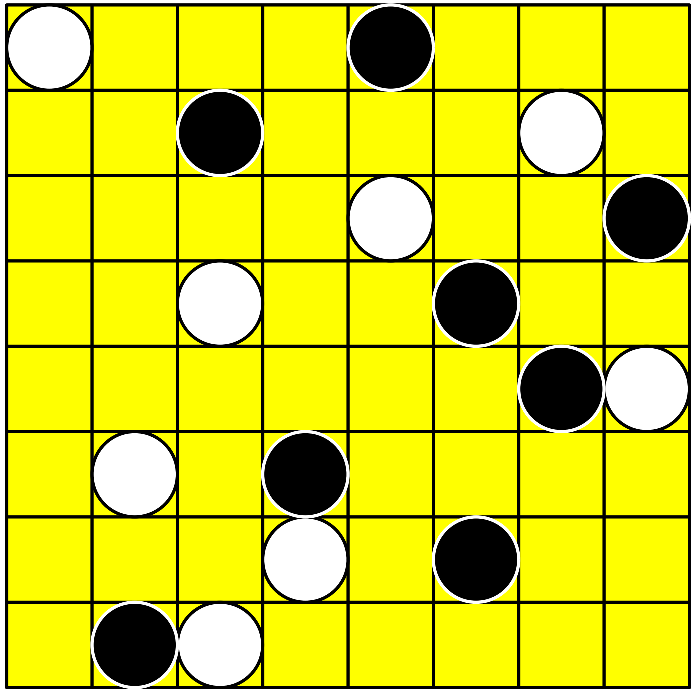

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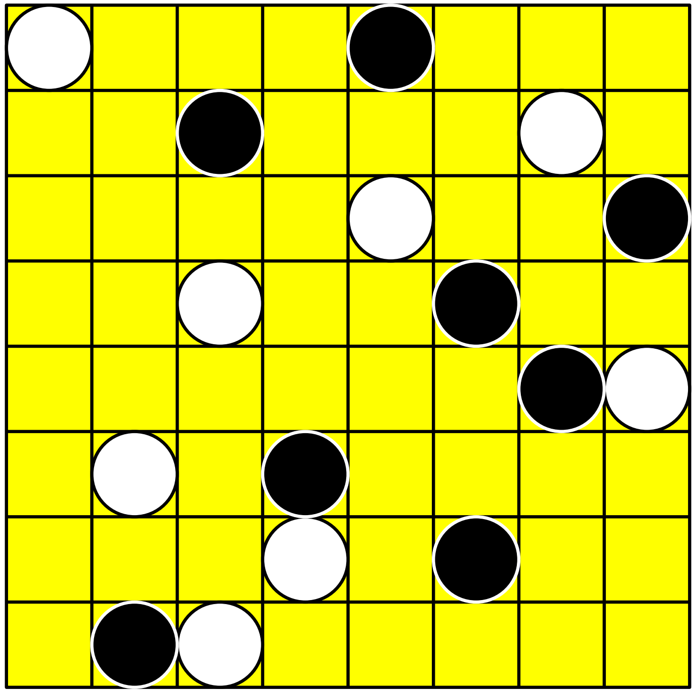

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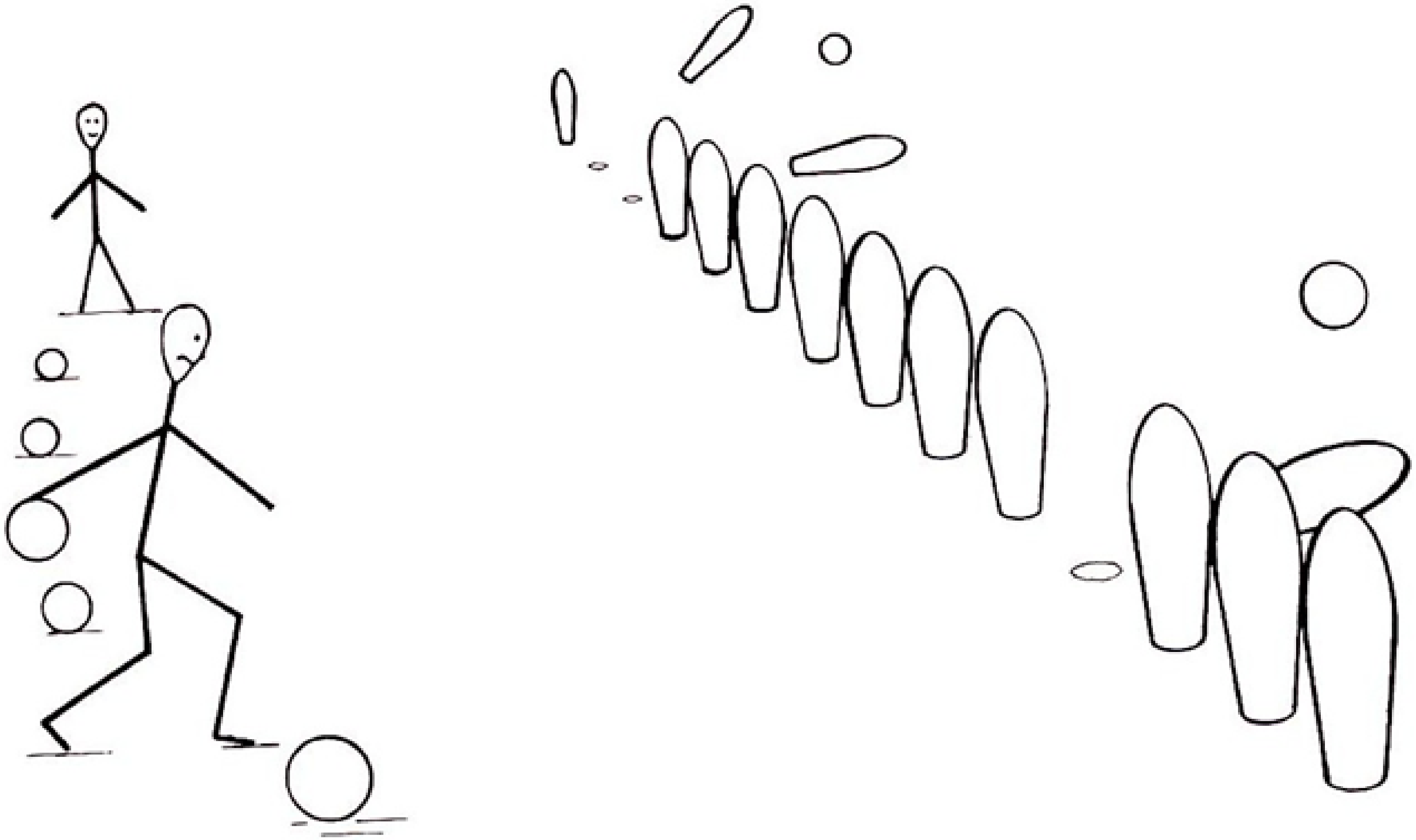
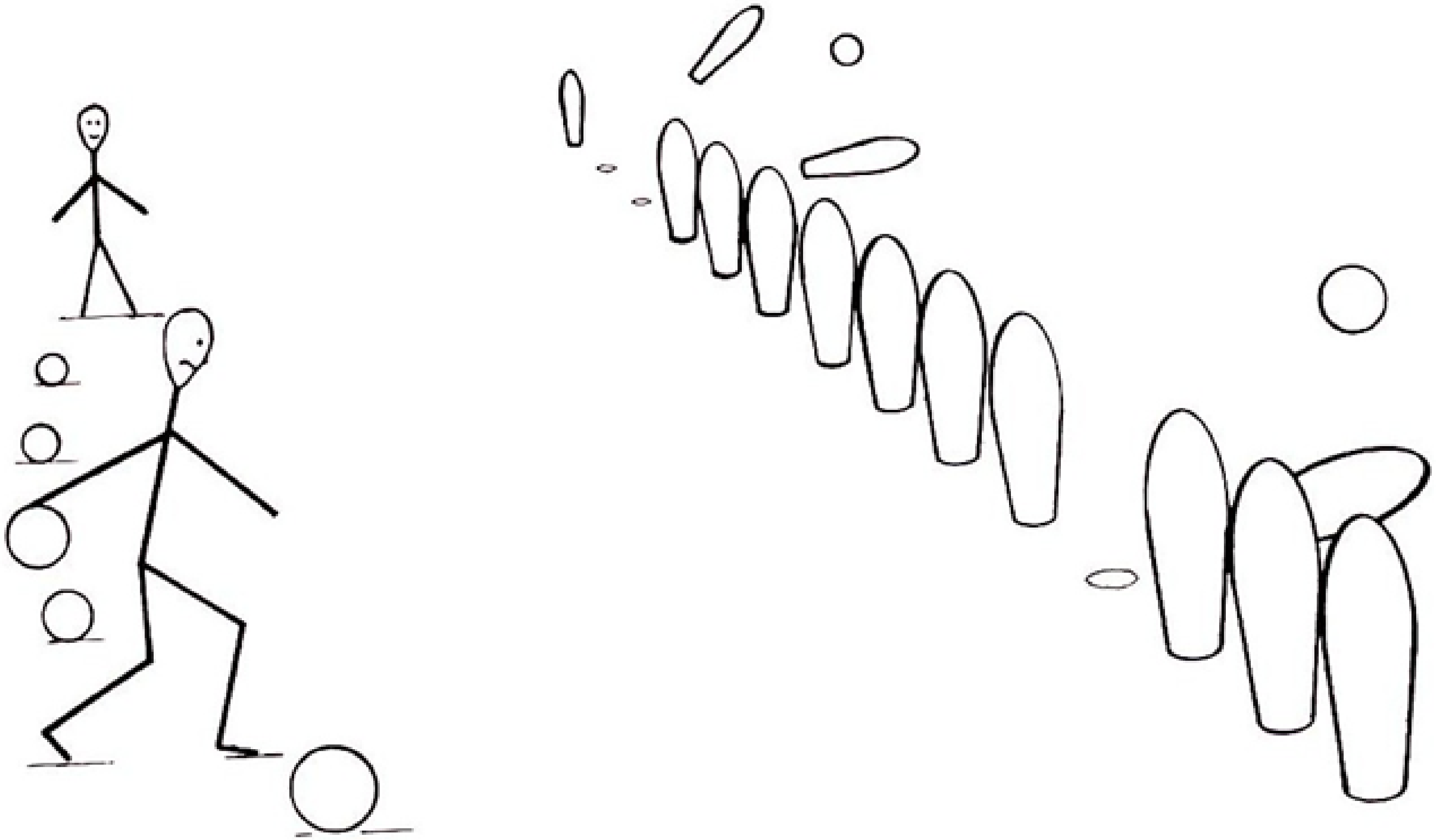
Kayles (aka Rip van Winkle's Game)



- Bowling: Row of n pins.
  In a move hit one or two neighbored pins.

Dawson's Kayles



- Bowling: Row of n pins. In a move always hit two neighbored pins. Single pins can be removed.

Kayles II

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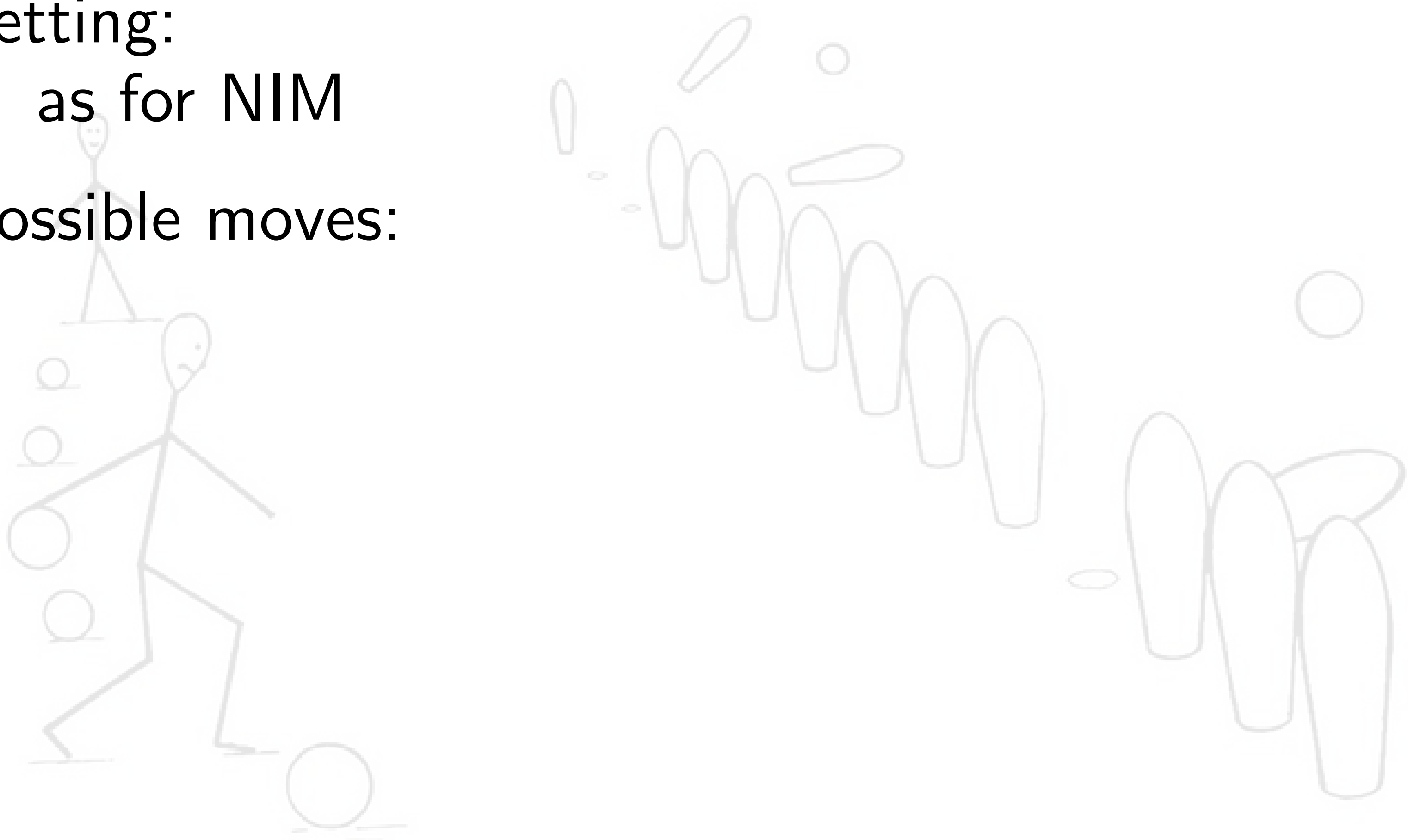

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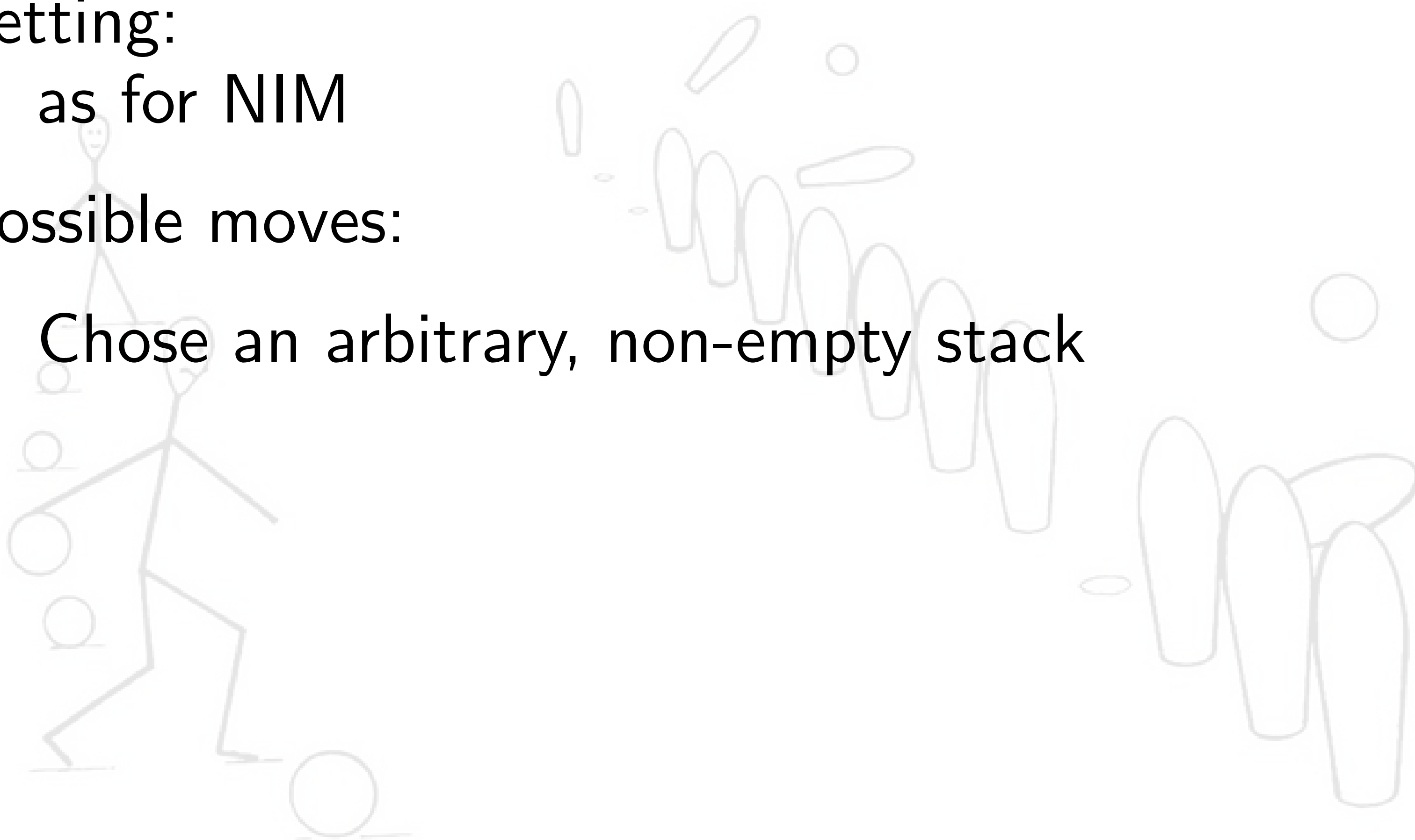

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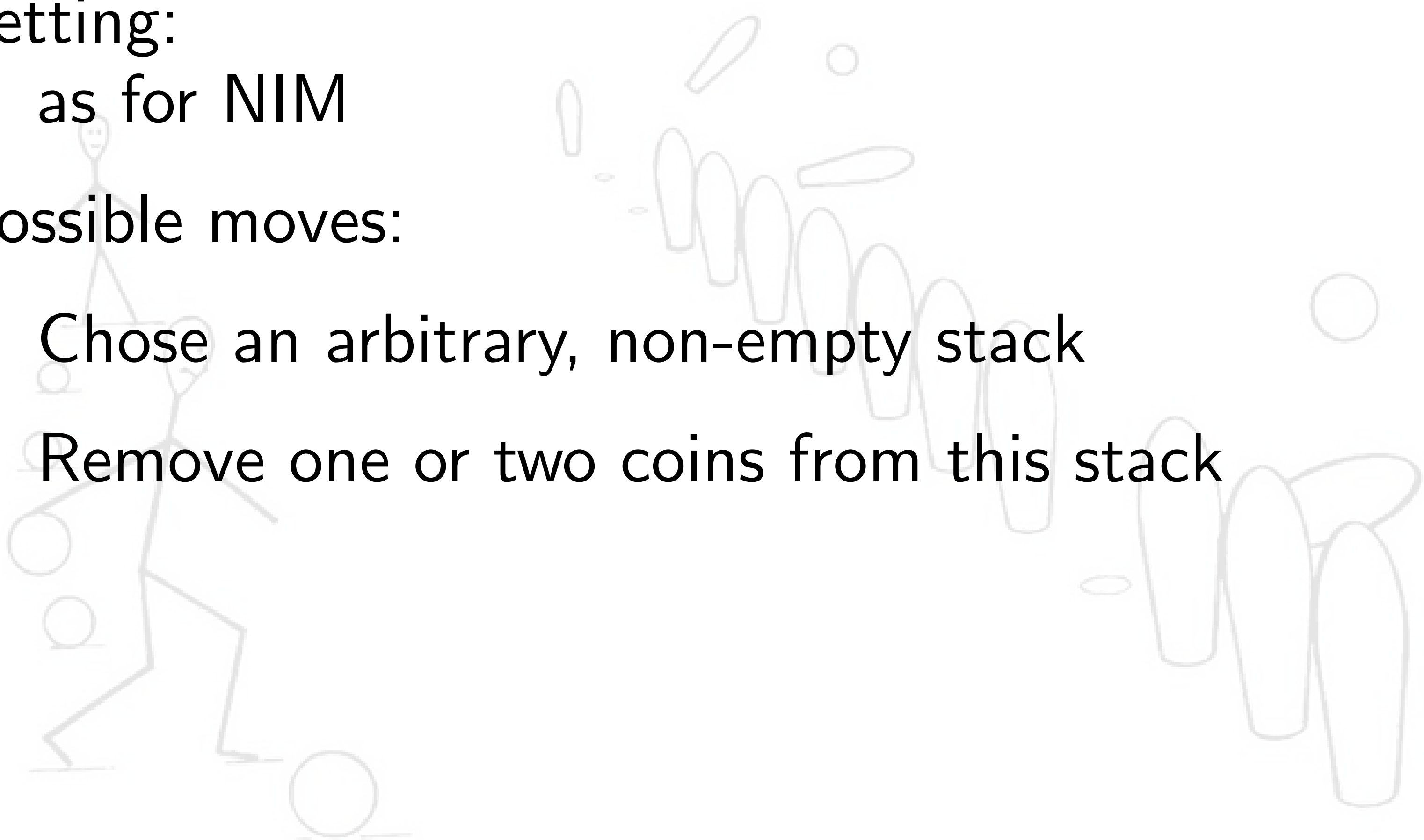

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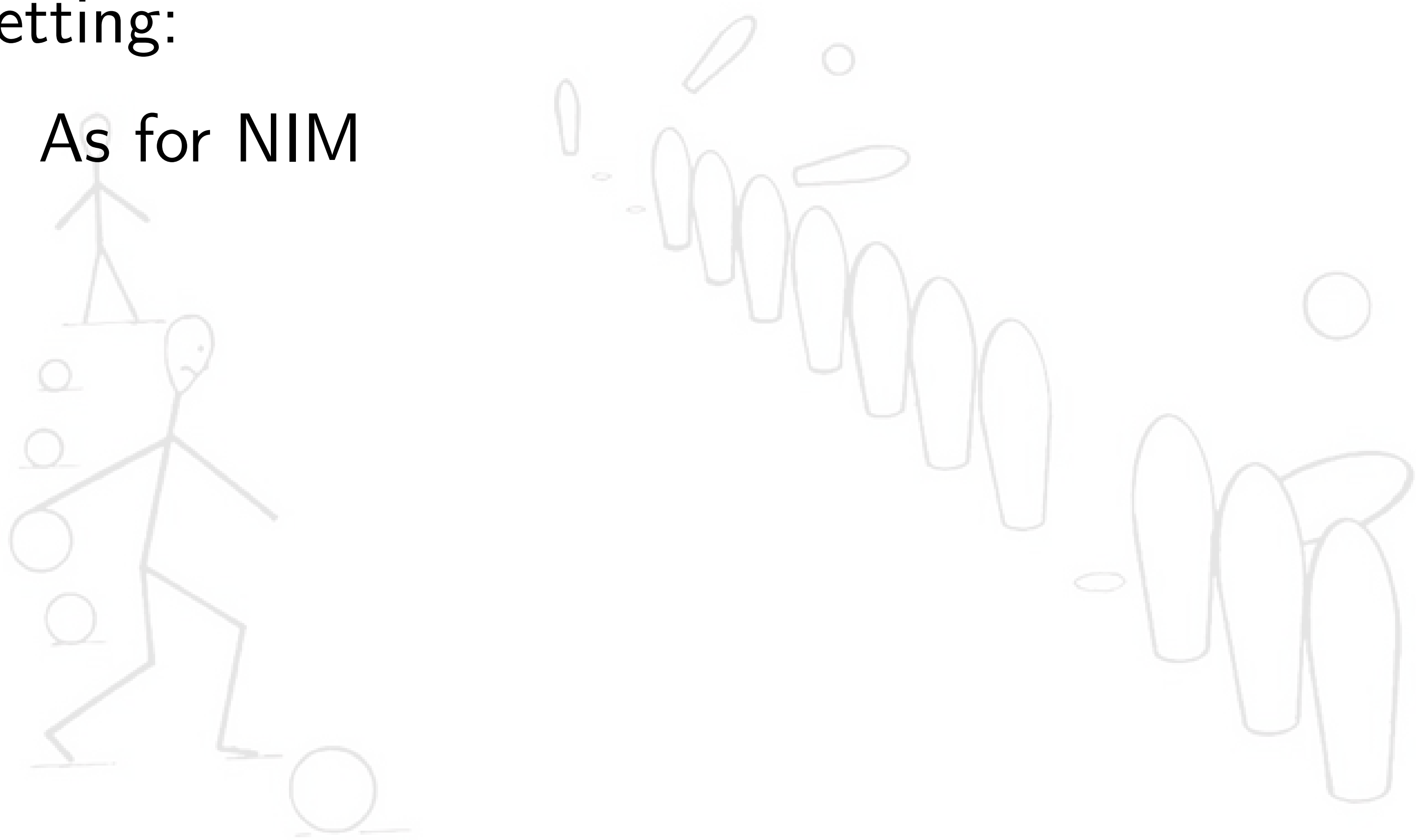

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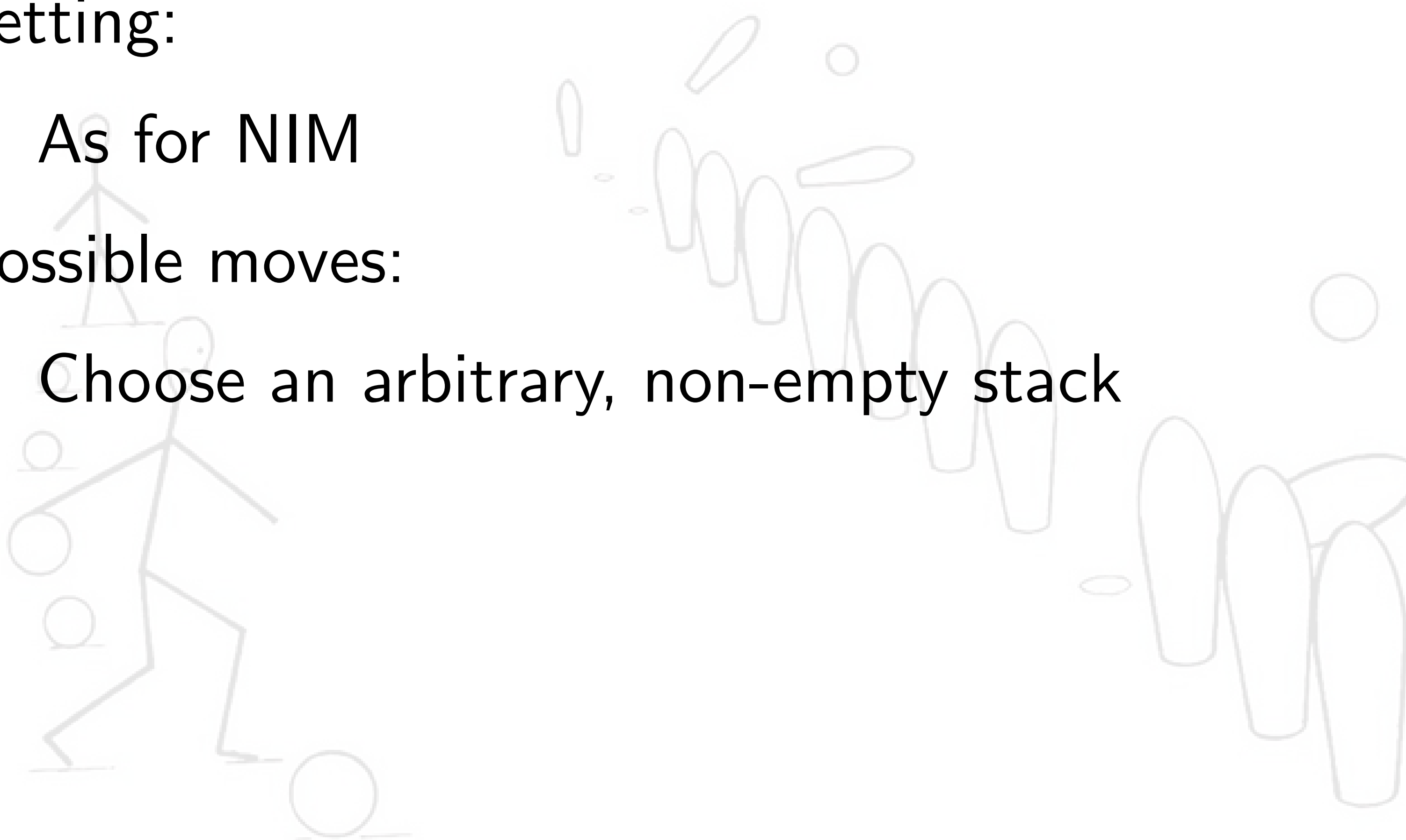

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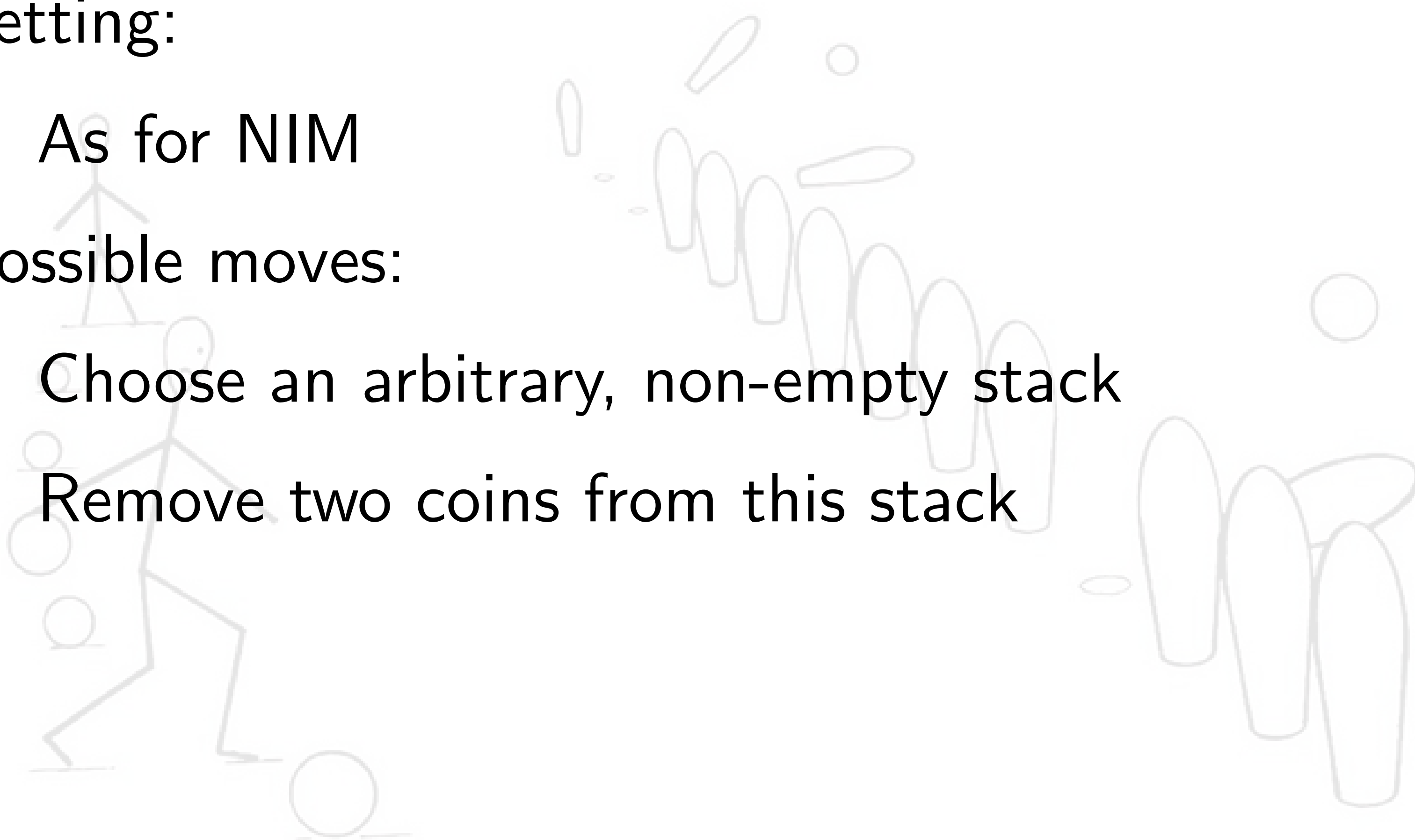

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Sprague-Grundy-Theory
(1935/39; aka NIM-theory)

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- How about several piles?

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- Terminal positions have nimber $*0$ (normal play)

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    - Interpretation: If my opponent gives me a (for them) non-optimal situation, I can make a winning move

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  - For correctness the MEX-rule is sufficient.

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- A stack of size i has nimber $*i$

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- The nimber of a group of stacks is the XOR-sum of their heights

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- $\Rightarrow$ Always try to obtain a position to get $k_1 \otimes k_2 \otimes k_3 \otimes \cdots \otimes k_n = 0$

Games, Triangulations, Theory

- Literature:
  - **Winning Ways for Your Mathematical Plays**
    E.R. Berlekamp, J.H. Conway and R.K. Guy:
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  - **Games on triangulations**
    O. Aichholzer, D. Bremner, E.D. Demaine, F. Hurtado, E. Kranakis,H. Krasser, S. Ramaswami, S. Sethia, and J. Urrutia: Theoretical Computer Science, 343(1-2):42-71,2005.

Thanks for your attention. . .