

Sprague-Grundy-Theory

- First/Second-Player Win depends on number of coins

- for a single pile

<p>Take away 1≤i≤k coins 1, 2, 3 coins</p>	<p>1, 2 coins :</p> <table style="margin-left: 20px; border-collapse: collapse;"> <tr><td>○ 1st</td><td>○ 2nd</td></tr> <tr><td>○ 2nd</td><td>○ 1st</td></tr> <tr><td>○ 1st</td><td>○ 1st</td></tr> <tr><td>.....</td><td>.....</td></tr> <tr><td>○ 1st</td><td>○ 1st</td></tr> <tr><td>○ 1st</td><td>○ 1st</td></tr> </table>	○ 1st	○ 2nd	○ 2nd	○ 1st	○ 1st	○ 1st	○ 1st	○ 1st	○ 1st	○ 1st
○ 1st	○ 2nd												
○ 2nd	○ 1st												
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○ 1st	○ 1st												
○ 1st	○ 1st												

- Nimbers

Nimbers $*i, i \geq 0$, are a 'code' used for game-positions:

- $*i, i \neq 0 \Rightarrow 1^{\text{st}}$ player win (the player to move)
- $*0 \Rightarrow 2^{\text{nd}}$ player win (the one just moved)

A nimber code implies:

- From a $*0$ situation no legal move leads to another $*0$ situation
 - Interpretation: If I made a winning move, my opponent can not
- From any $*i, i \neq 0$, situation there is a legal move to a $*0$ situation
 - Interpretation: If my opponent gives me a (for them) non-optimal situation, I can make a winning move

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- nimber depends on height

				MOVE	MEX
				Highest i	RULE
	O	O		O	*
	O	O	O	O	*
	O	O	O	O	*
	O	O	O	O	*
*	*	*	*		

NIM Rules

MEX-rule (Minimal Excluded):

- The nimber of a position P is the smallest value which is NOT a nimber of any position which is reachable by a valid move from P .

- The MEX-rule guarantees a good code!
 - From a $*0$ situation no legal move leads to another $*0$ position
 - From any $*i, i \neq 0$, situation there is a legal move to a $*0$ situation.

- XOR-rule:

- The nimber of a set of positions is the XOR-sum of the nimbers of the positions.

- Simplifies computation of nimbers for several piles.
 - That is, using the MEX-rule AND the XOR-rules makes computations much more efficient!
 - For correctness the MEX-rule is sufficient.

	O		O		
*3					
1. st	O	O	O		
player	O	O	O		
winner	O	O	O	O	
*4	*3	*1	*5		
4 ≈ 100		XOR:			
3 ≈ 011		011 ≈ *3			
1 ≈ 001					
5 ≈ 101					

Optimal Strategy

- compute each pile's nimber
- XOR to find out if a winning move exists
 - zero \Rightarrow no winning move
- compute for each pile if there is a winning move for this pile
- execute a optimal move
- repeat after opponent's move

NIM:	High N. Number	A: 7 ≈ 111 → 100 ≈ *4 ✓	optimal moves:
07	7	B: 2 ≈ 010 → 001 ≈ *1 ✓	
0 5	9 *9	C: 5 ≈ 101 → 110 ≈ *6 ↗	
0 0	8 *8	D: 3 ≈ 011 → 000 ≈ *∅ ✓	
0 0 3	7 *7		
0 2 0 0	6 *6	011 ≈ 3 *3	
0 0 0 0	5 *5	Goal: 000 ≈ *∅	
0 0 0 0	4 *4	⇒ winning moves:	
A B C D	3 *3	A: remove 3 coins from A	
*3 1st	2 *2	B: → 1 coin → B	
player win	1 *1	D: → 3 coins → D	

Variations of [[NIM-type Games]]

- take a limited number of coins each turn

$\begin{array}{r} 5 \\ 0 \\ 0 \end{array}$	$\begin{array}{r} 6 \\ 0 \\ 0 \end{array}$	Take 1, 2, or 3 coins from one pile <small>HIGH Number</small>	
$\begin{array}{r} 0 \\ 3 \\ 0 \end{array}$	\vdots	$A: *1 = 001$	
$\begin{array}{r} 0 \\ 0 \\ 0 \end{array}$	6	$B: *3 = 011$	
$\begin{array}{r} 0 \\ 0 \\ 0 \end{array}$	5	$C: *2 = 010$	
$\begin{array}{r} 0 \\ 0 \\ 0 \end{array}$	4	$*\phi$	
$\begin{array}{r} A \\ B \\ C \end{array}$	3	$*3$	$XOR: 000 \triangleq *\phi$
$\begin{array}{r} *1 \\ *3 \\ *2 \end{array}$	2	$*2$	
$\Rightarrow *\phi$	1	$*1$	<small>2nd player win</small>
$\Rightarrow *\phi$	0	$;\ *\phi$	$\Rightarrow NO$ winning moves

- take 1 coin or split a pile into two (non-empty) piles

Take 1 coin OR split a pile into two piles

$\begin{array}{r} 9 \\ 8 \\ 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ \phi \end{array}$	$\begin{array}{r} *\phi \\ *2 \\ *1 \\ *2 \\ *\phi \\ *2 \\ *1 \\ *2 \\ *1 \\ *\phi \end{array}$	$H=2: \rightarrow H=1 \rightarrow *1$ $\rightarrow H=1, H=1: *1, *1 \triangleq *\phi$ $H=3: \rightarrow H=2 \rightarrow *2$ $\rightarrow H=2, H=1: *2, *1 \triangleq *3$ $H=4: \rightarrow H=3 \rightarrow *\phi$ $\rightarrow H=3, H=1: *\phi, *1 \triangleq *1$ $H=5: \rightarrow H=4 \rightarrow *3$ $\rightarrow H=3: *\phi$ $H=6: *\phi, *2 \triangleq *2$ $H=7: *\phi, *1 \triangleq *6$ $H=8: *\phi$ $H=9: *\phi$ $H=10: *\phi$
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- Laskers NIM

- NIM or split a pile into two (non-empty) piles

LASKERS NIM		$H=5: \rightarrow H=4: *3$ $\rightarrow H=3: *\phi$	$H=4, 1: *3, *1$ $\rightarrow H=3: *\phi$
$\begin{array}{r} 10 \\ 9 \\ 8 \\ 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ \phi \end{array}$	$\begin{array}{r} *\phi \\ *1 \\ *2 \\ *1 \\ *\phi \\ *2 \\ *1 \\ *2 \\ *1 \\ *\phi \end{array}$	$H=2: \rightarrow H=1: *1 \rightarrow H=\phi: *\phi$ $\rightarrow H=1, 1: *1, *1 \triangleq *\phi$	$H=2: *\phi$ $H=3, 2: *4, *2$ $H=1: *1$ $H=\phi: *\phi$
$\begin{array}{r} 10 \\ 9 \\ 8 \\ 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ \phi \end{array}$	$\begin{array}{r} *\phi \\ *1 \\ *2 \\ *1 \\ *\phi \\ *2 \\ *1 \\ *2 \\ *1 \\ *\phi \end{array}$	$H=3: \rightarrow H=2: *2$ $\rightarrow H=1: *1$	$H=6: *5, *3, *4, *2, *1, *\phi$ $H=5, 1: *5, *1 \triangleq *4$ $H=4, 2: *3, *2 \triangleq *1$ $H=3, 3: *4, *4 \triangleq *\phi$
$\begin{array}{r} 10 \\ 9 \\ 8 \\ 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ \phi \end{array}$	$\begin{array}{r} *\phi \\ *1 \\ *2 \\ *1 \\ *\phi \\ *2 \\ *1 \\ *2 \\ *1 \\ *\phi \end{array}$	$H=4: \rightarrow H=3: *4$ $H=2: *\phi$	$H=7: *\phi, ... *\phi$ $H=6, 1: *6, *1 \triangleq *7$ $H=5, 2: *5, *2 \triangleq *7$ $H=4, 3: *3, *4 \triangleq *7$
$\begin{array}{r} 10 \\ 9 \\ 8 \\ 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ \phi \end{array}$	$\begin{array}{r} *\phi \\ *1 \\ *2 \\ *1 \\ *\phi \\ *2 \\ *1 \\ *2 \\ *1 \\ *\phi \end{array}$	$H=3, 1: *4, *1 \triangleq *5$	

LASKERS NIM		position/number to reach:	H : *
A	0	A: $4 \rightarrow *3 \triangleq 0011$	11 : *11
B	0	B: $12 \rightarrow *11 \triangleq 1011$	12 : *12
C	9	C: $5 \rightarrow *8 \triangleq 1001$	10 : *10
D	0	D: $6 \rightarrow *6 \triangleq 0110$	9 : *9
	6		8 : *7
	0		7 : *8
	0		6 : *6
	0	XOR: $0111 \triangleq *7$	5 : *5
	0	1st player wins	4 : *3
	0	① Take 5 coins from pile D	3 : *4
	0	② Take 1 coins from pile A	2 : *2
A	B	③ Take 1 coin from pile B	1 : *1
C	D	④ Split pile D into 4 and 2	0 : *0
*	7		

- Kayles and Dawson's Kayles
 - no repeating nimber patterns

[[Algorithms and Games]]