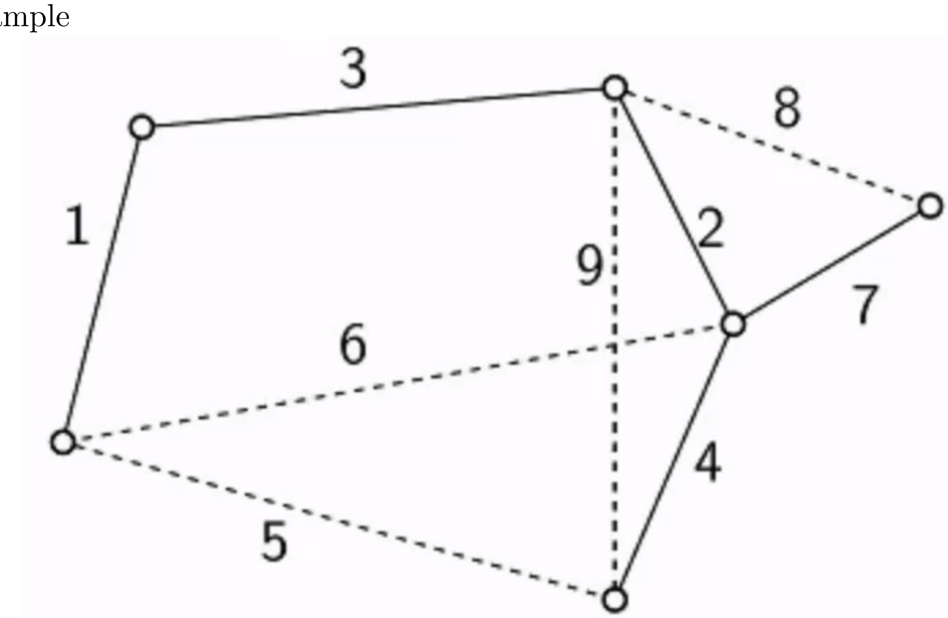
Motivation

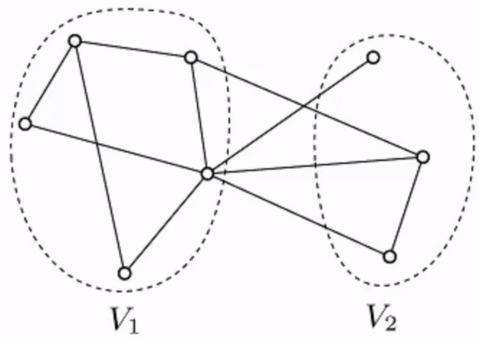
- many algorithms can be applied to unweighted trees
  - [[Spannbaumalgorithmen]]
  - [[Breadth-First Search]]
  - [[Depth-First Search]]
  - ...
- does not work for weighted graphs
- useful if direct connections from A to B not possible
  - road networks
  - electrical circuits
  - ...

Definition

- tree which connects all points which minimizes the total length of all edges
  - A minimum spanning tree of a weighted graph $G = (V, E, w)$ is a tree $T = (V, E')$ with $E' \subseteq E$ and with minimal total edge length among all spanning trees in G:
    $$w(T) = \sum_{e \in E'} w(e)$$
    is minimized over all trees in G with vertex set V.
- crossing free (planar)
  - otherwise using different edges would lead to a fewer length
- example
  - 

Algorithm Idea

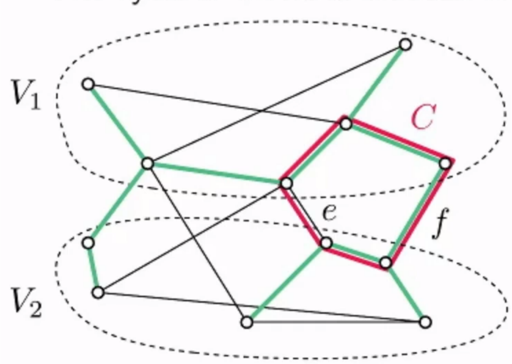
- cut
  - A **cut** of a graph $G = (V, E)$ is a partition of V into V_1, V_2.

    An edge e is called **external** for the cut (V_1, V_2) if it has one endpoint in V_1 and one in V_2; otherwise e is called **internal**.
- **Theorem:** Let E' be a subset of edges of an MST of $G = (V, E, w)$. Let (V_1, V_2) be a cut of G for which all edges of E' are internal. Then the external edge of the cut with **minimum weight** is a good edge for E'.
  - proof
    * **Proof:**
      Assume there is an MST T with E' and without e.

      $\Rightarrow$ e closes a cycle C in T.

      The cycle C contains at least one edge f of the tree T that is external for the cut.

      By definition of e, the weight $w(f) \geq w(e)$.

      $\Rightarrow T \backslash \{f\} \cup \{e\}$ is an MST of G. □

      If $w(f) > w(e)$ then T is not an MST of G.

Prim's Algorithm

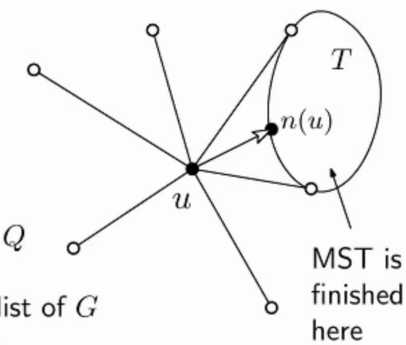
- greedy
- also works with negative weights

- Start with an arbitrary vertex s of G and iteratively 'grow' an MST T from s.
- **Iterative step:**
  Choose the 'cheapest' edge with exactly one node in T.
- Cut: V_1 = vertices of T, V_2 = vertices not yet in T.
- For each vertex $v \notin T$ we maintain:
  - Priority $p(v)$: weight of the shortest edge from v to a vertex in T (initially: ∞).
  - Nearest $n(v)$: vertex in T realizing $p(v)$: $w(v, n(v))$ is min. among neighbors of v in T (initially no vertex).
- A **queue** Q contains all vertices not yet in T, organized by priorities (e.g., in a min-heap; initially all vertices).

- pseudo code

PRIM-MST (G,s)

```
for all v ∈ V do p(v) = ∞ od
p(s) = 0,  n(s) = nil
Q = V          // build up Q
while Q ≠ 0 do
   u = MIN(Q)
   remove u from Q        // reorganize Q
   write u, n(u)
   for all v ∈ A(u) do        // A: adj. list of G
      if v ∈ Q and p(v) > w(u,v) then
         p(v) = w(u,v)        // reorganize Q
         n(v) = u
      fi
   od
od
```



Run-time-analysis:

n ... number of vertices of G

m ... number of edges of G

$d(v)$ ... Degree of vertex v in G

- Initialization, construction of the heap Q: $\Theta(n)$
- n times removing the minimum from Q: $O(n \log n)$
- report MST edges: $\Theta(n)$
- Update priorities for all neighbors of v: $O(d(v) \cdot \log n)$

$\Rightarrow$ Alltogether:

$$O(n + n \log n + \sum_{v \in V} d(v) \log n) = O(m \log n)$$

Memory requirements: $\Theta(m + n) = \Theta(m)$

Graph + queue + priorities + nearests + constant additional

Remarks:

- MST always begins at the start vertex s and grows from there as a connected tree.
- Shrinking $p(v)$ causes v to move up in the heap. $\Rightarrow O(\log n)$ time.
- Test for $v \in Q$ in $O(1)$ time when bit vector is used to store which vertices are already in T.
- Runtime can be changed to $O(n^2)$. This is useful for dense graphs (see notes on Dijkstra's algorithm in the next chapter).

Kruskal's Algorithm

- greedy
- also works with negative weights
- uses [[Union Find]] data structure
  - Label the vertex set of G as $v_1, v_2, ...v_n$ (arbitrary)
  - Initially there are n disjoint sets $M_1, M_2, ...M_n$ (each with one vertex)
  - FIND(v): returns index i if vertex v is in M_i
  - UNION(i,j): join sets M_i and M_j: $M_i = M_i \cup M_j$ (index of resulting set: minimum of i and j)
  - End of the algorithm: one component M_1 with all vertices of G.
  - Creating a 1-element set needs $\Theta(1)$ time.
  - f FIND and u UNION operations need $O(f + u \log u)$ time in total.
  - The total memory requirement of the data structure is linear in the number of initial 1-element sets.

- Start with empty edge set E'.
- Sort edge set E of $G = (V, E, w)$ in increasing order of their weights (edges will be considered in this order): $e_1, e_2, ...e_m$ with $w(e_1) < w(e_2) < ... < w(e_m)$.
- **Iterative step**:
  - E' forms a forest F (= set of disjoint subtrees, acyclic) in G and in the MST to be constructed.
  - Edge e that is added to E' is the shortest edge in $E \setminus E'$ that does not form a cycle with edges from E'.

- pseudo code

KRUSKAL-MST(G)

sort edges by weight: $\{e_1, e_2, \dots e_m\}$

for $i = 1$ **to** n **do** $M_i = \{v_i\}$ **od**

for $k = 1$ **to** m **do**

  $(u, v) = e_k$

  $i =$FIND(u)

  $j =$FIND(v)

  if $i \neq j$ **then**

    write e_k

    UNION(i, j)

  fi

od

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Runtime analysis:

- Sorting of the edges: $O(m \log m)$
- Initialize UNION-FIND data structure for vertices: $\Theta(n)$
- In total $2m$ FIND operations and $n - 1$ UNION operations: $O(m + n \log n)$
- Extract edges + write MST edges: $\Theta(m)$

$\Rightarrow$ Altogether $O(m \log m)$ time.

$\Rightarrow$ Sorting of the edges dominates the runtime.

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Memory requirements:

$\Theta(n+m) = \Theta(m)$ in total.

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