Non-linear Regression, Logistic Regression

Machine Learning 1 — Lecture 5

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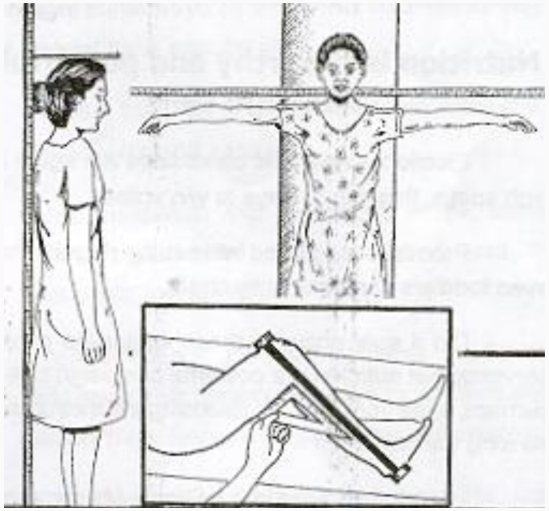


knee height [cm]	arm span [cm]	height [cm]
50	166	171
56	172	175
52	174	168
...	...	...

- Last lecture, we discussed **regression**, i.e. predicting a continuous **target** variable y from an input vector $\boldsymbol{x}$
- In particular, **linear regression** using an affine model:

$$\hat{y} = \boldsymbol{w}^T \boldsymbol{x} + b$$

- **Least-squares loss**:

$$\mathcal{L}(\boldsymbol{w}, b) = \frac{1}{N} \sum_{i=1}^{N} \left(\hat{y}^{(i)} - y^{(i)} \right)^2$$

Analytical Least-Squares Solution

- Collect all input vectors in the **design matrix**; include a "dummy feature" $x_0 \equiv 1$ to account for the bias b:

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{x}^{(1)\,T} \\ \boldsymbol{x}^{(2)\,T} \\ \vdots \\ \boldsymbol{x}^{(N)\,T} \end{pmatrix} = \begin{pmatrix} 1 & x_1^{(1)} & x_2^{(1)} & \dots & x_D^{(1)} \\ 1 & x_1^{(2)} & x_2^{(2)} & \dots & x_D^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_1^{(N)} & x_2^{(N)} & \dots & x_D^{(N)} \end{pmatrix}$$

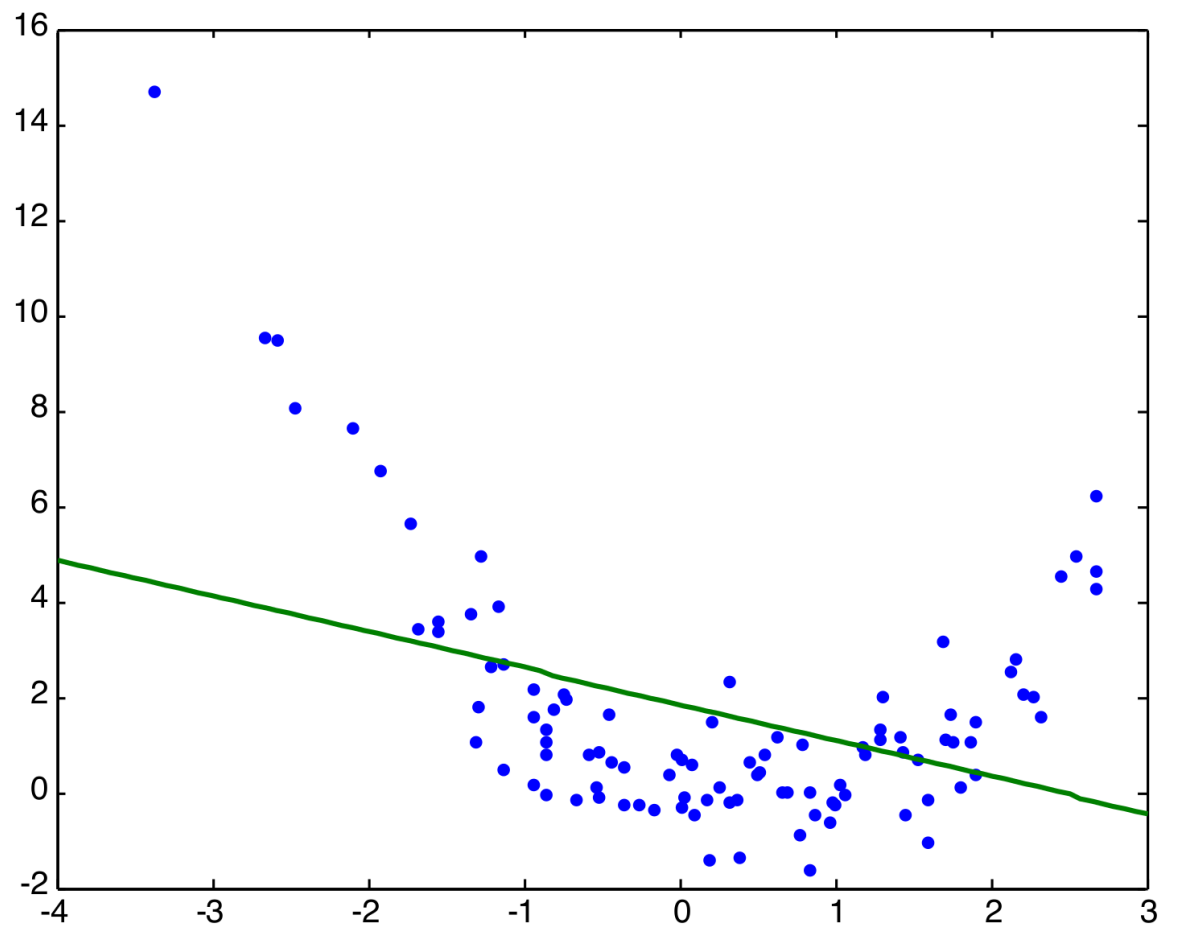
- Parameter vector $\boldsymbol{\theta} = (\theta_0, \theta_1, \dots, \theta_D)^T = (b, w_1, \dots, w_D)^T$
- **Least-squares solution** (minimizes $\mathcal{L}(\boldsymbol{\theta})$):

$$\boldsymbol{\theta}^* = \underbrace{(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T}_{\text{Moore-Penrose Inverse}}\ \boldsymbol{y}$$

Non-linear Features

Linearity Assumption

Linear dependency is often a strong assumption:



Linear fit: $f_{\theta^*}(x) = 2.1 - 0.76x$

Non-Linear Features

$$\overbrace{\begin{pmatrix} \hat{y}^{(1)} \\ \hat{y}^{(2)} \\ \vdots \\ \hat{y}^{(N)} \end{pmatrix}}^{\text{prediction } \hat{\boldsymbol{y}}} = \overbrace{\begin{pmatrix} 1 & x_1^{(1)} & \dots & x_D^{(1)} \\ 1 & x_1^{(2)} & \dots & x_D^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_1^{(N)} & \dots & x_D^{(N)} \end{pmatrix}}^{\text{design matrix } \boldsymbol{X}} \overbrace{\begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \vdots \\ \theta_D \end{pmatrix}}^{\text{parameters } \boldsymbol{\theta}}$$

The input features $x_1, \dots, x_D$ in the design matrix were provided/measured by the user.

We can apply any pre-processing (non-linear function) to them!

Non-Linear Features cont'd

Replace the original D features $x_1, x_2, \ldots x_D$ with K (non-linear) features $\phi_1(\boldsymbol{x}), \phi_2(\boldsymbol{x}), \ldots, \phi_K(\boldsymbol{x})$.

features, **basis functions**: $\phi_k : \mathbb{R}^D \mapsto \mathbb{R}$, $1 \leq k \leq K$

Parameter vector $\boldsymbol{\theta}$ is now $(K+1)$-dimensional.

$$\overbrace{\begin{pmatrix} \hat{y}^{(1)} \\ \hat{y}^{(2)} \\ \vdots \\ \hat{y}^{(N)} \end{pmatrix}}^{\text{prediction } \hat{\boldsymbol{y}}} = \overbrace{\begin{pmatrix} 1 & \phi_1(\boldsymbol{x}^{(1)}) & \ldots & \phi_K(\boldsymbol{x}^{(1)}) \\ 1 & \phi_1(\boldsymbol{x}^{(2)}) & \ldots & \phi_K(\boldsymbol{x}^{(2)}) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_1(\boldsymbol{x}^{(N)}) & \ldots & \phi_K(\boldsymbol{x}^{(N)}) \end{pmatrix}}^{\text{design matrix } \boldsymbol{\Phi}} \overbrace{\begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \vdots \\ \theta_K \end{pmatrix}}^{\text{parameters } \boldsymbol{\theta}}$$

Note that the constant "dummy" feature can also be interpreted as a basis function $\phi_0(\boldsymbol{x}) \equiv 1$.

Non-Linear Features cont'd

We can be creative and use **any** transformation as features, for example:

- **polynomials** $x_1^2,\ x_1x_2,\ x_2^2,\ \ldots$
- **sinusoidals** $\sin(\omega_i x_1),\ \cos(\omega_i x_1),\ \omega_i > 0$
- Localized features, like **radial basis functions** (Gaussian shaped)

$$\phi_i(\boldsymbol{x}) = \exp\left(-\frac{1}{\sigma^2}\|\boldsymbol{x} - \boldsymbol{c}_i\|\right)$$

- We can also include the original (raw) features $x_1, x_2, x_3, \ldots$, i.e. **identity functions**

How to Learn with Non-linear Features?

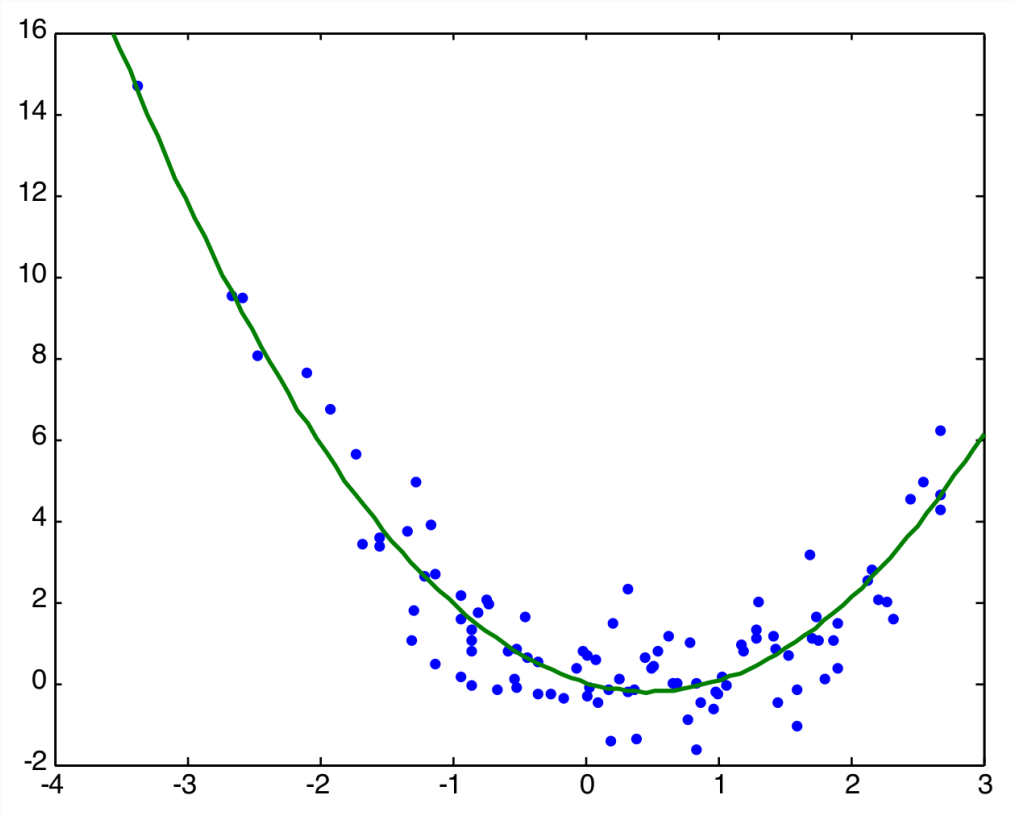
Regression with non-linear features works **exactly the same** as before. Simply replace $\boldsymbol{X}$ with $\boldsymbol{\Phi}$.

$$\boldsymbol{\theta}^* = \underbrace{(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T}_{\text{Moore-Penrose Inverse}}\ \boldsymbol{y} \quad \Rightarrow \quad \boldsymbol{\theta}^* = \underbrace{(\boldsymbol{\Phi}^T\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^T}_{\text{Moore-Penrose Inverse}}\ \boldsymbol{y}$$

If the features (columns of $\boldsymbol{\Phi}$) are linearly independent, $\boldsymbol{\Phi}^T\boldsymbol{\Phi}$ is invertible and the least-squares solution is unique.

Non-linear (Quadratic) Fit

Example

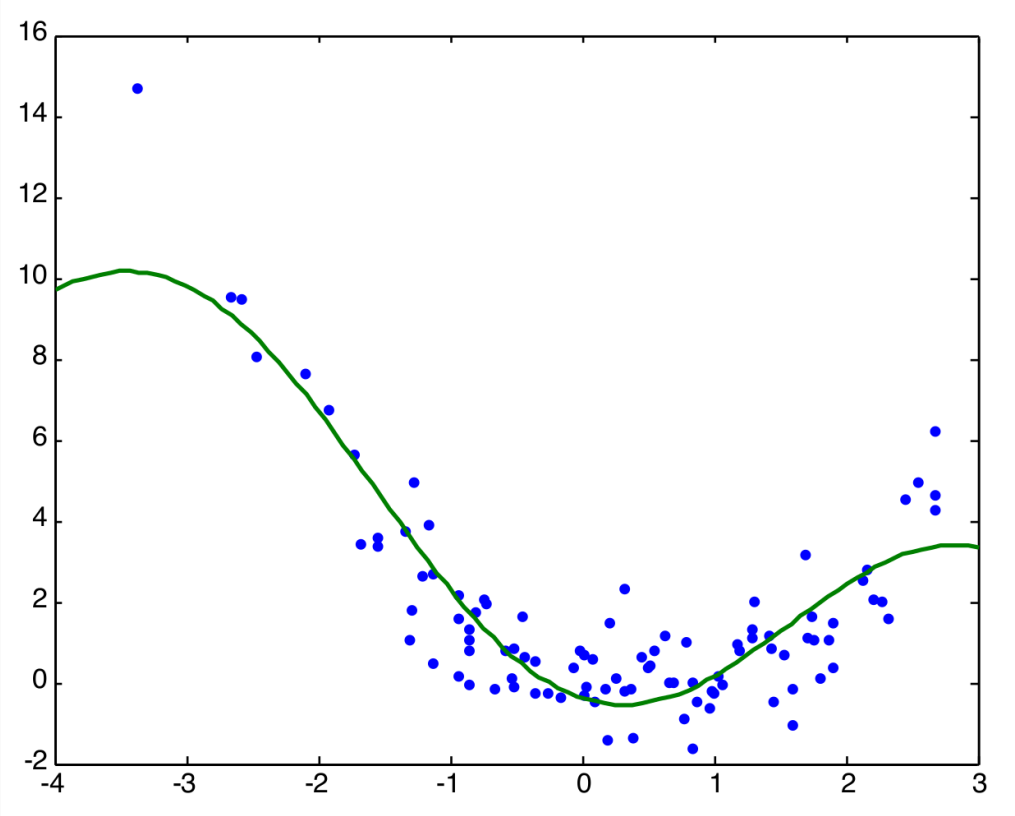


$$f_{\theta^*}(x) = 0.02 \cdot \underbrace{1}_{\phi_0} + 0.95 \cdot \underbrace{x}_{\phi_1} + 0.99 \cdot \underbrace{x^2}_{\phi_2}$$

linear model + non-linear features = non-linear model!

Non-linear (Sinusoidal) Fit

Example

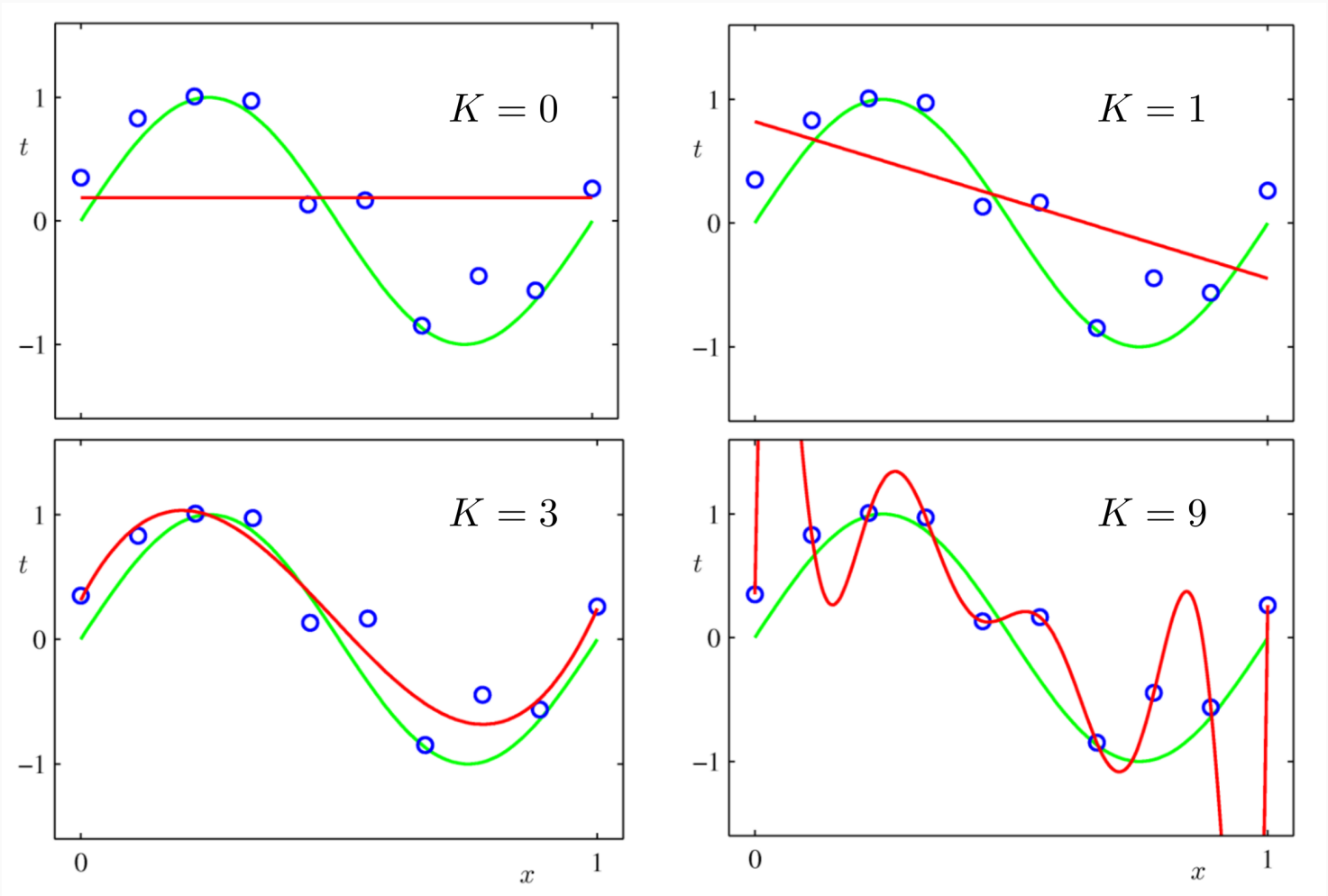


$$f_{\theta^*}(x) = 3.12 \cdot \underbrace{1}_{\phi_0} - 1.07 \cdot \underbrace{x}_{\phi_1} - 3.5 \cdot \underbrace{\cos(x)}_{\phi_2}$$

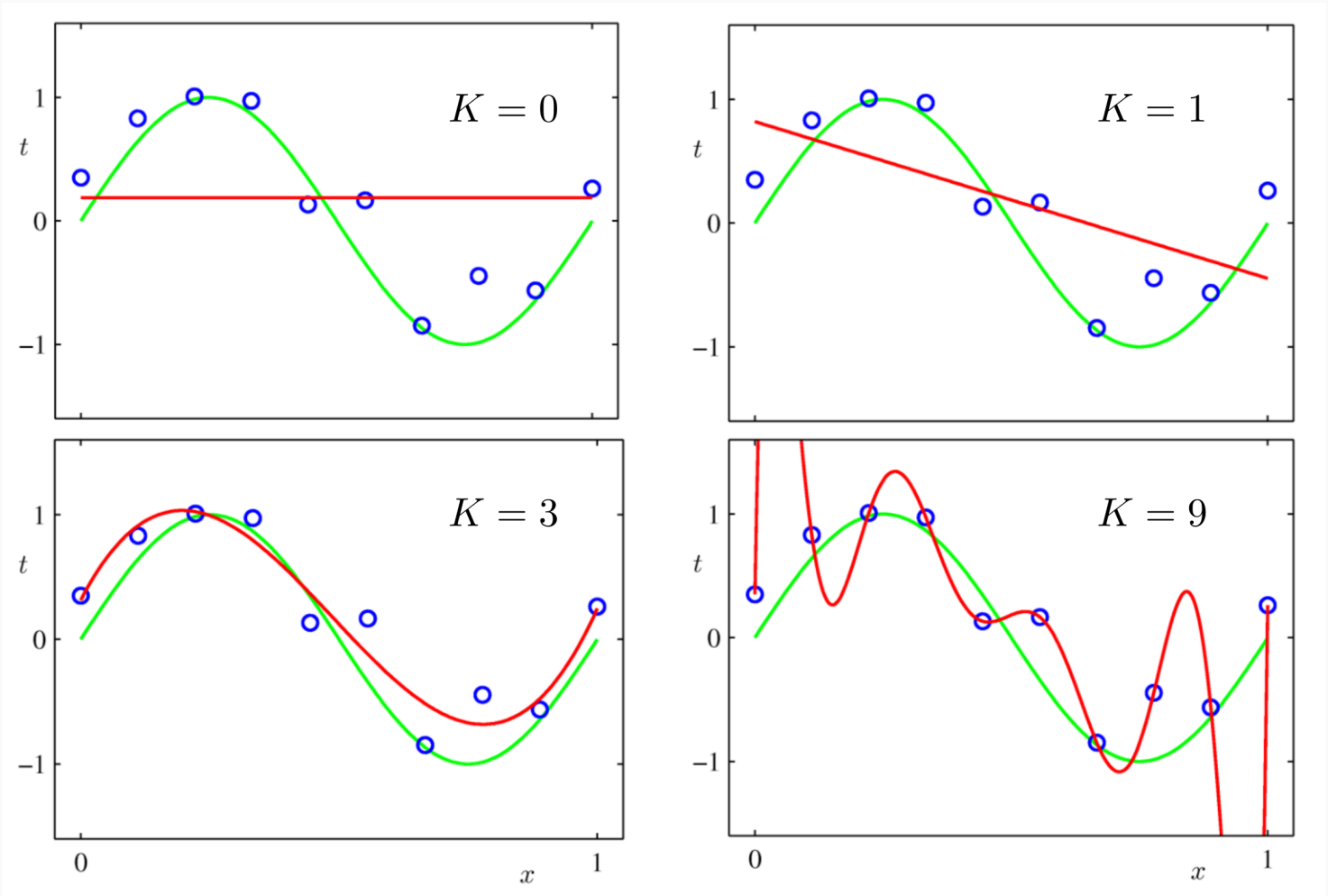
linear model + non-linear features = non-linear model!

Polynomial regression: Basis functions are all powers of up to order K. For example, for $D = 2$ and $K = 3$:

$$
\begin{aligned}
f_\theta(x_1, x_2) = \; & \theta_0 1 + && \text{constant} \\
& \theta_1 x_1 + \theta_2 x_2 + && \text{linear} \\
& \theta_3 x_1^2 + \theta_4 x_1 x_2 + \theta_5 x_2^2 && \text{quadratic} \\
& \theta_6 x_1^3 + \theta_7 x_1^2 x_2 + \theta_8 x_1 x_2^2 + \theta_9 x_2^3 && \text{cubic}
\end{aligned}
$$



What is going on with $K = 9$?



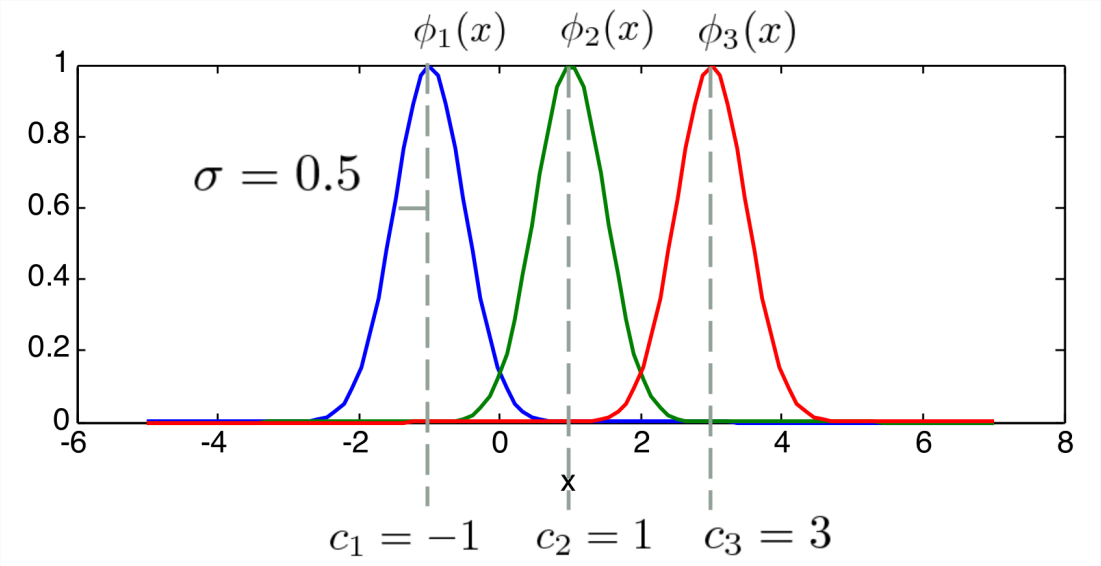
What is going on with $K = 9$?

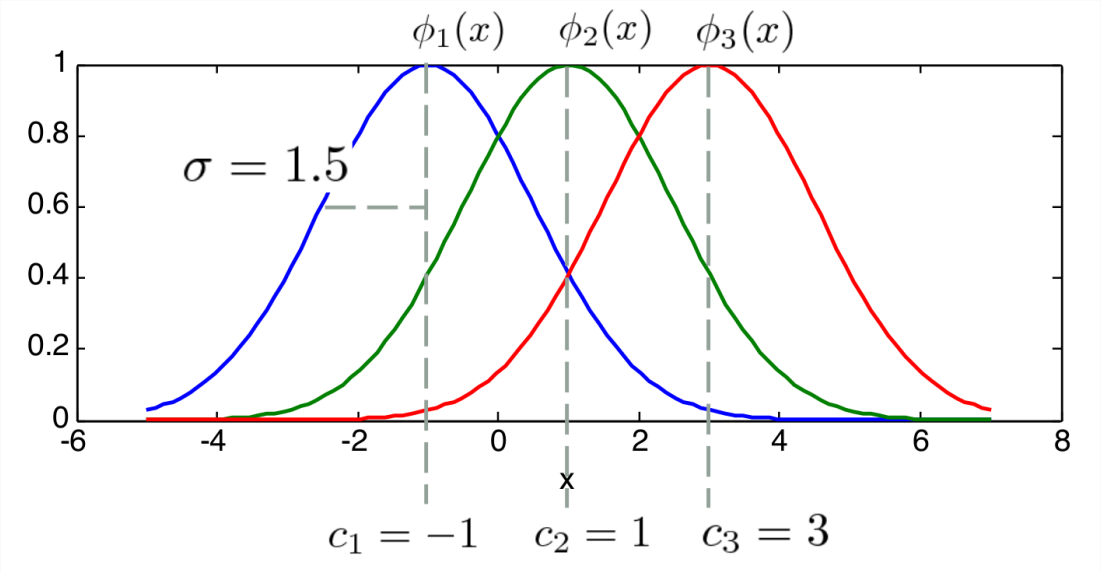
$\Rightarrow$ To be discussed in a future lecture.

Radial basis functions (RBFs) are another prominent type of non-linear basis functions. They have the shape of the Gaussian bell curve and are parametrized by a **center vector** $\boldsymbol{c}$. The width of the RBF is determined by a parameter σ.

$$\phi_i(\boldsymbol{x}) = \exp\left(-\frac{1}{\sigma^2}\|\boldsymbol{x} - \boldsymbol{c}_i\|\right)$$

RBFs quickly approach 0 when moving away from the center vector – thus, they are so-called **local basis functions**.


$\phi_1(x)$ $\phi_2(x)$ $\phi_3(x)$
$\sigma = 0.5$
1
0.8
0.6
0.4
0.2
0
-6 -4 -2 0 2 4 6 8
x
$c_1 = -1$ $c_2 = 1$ $c_3 = 3$


$\phi_1(x)$ $\phi_2(x)$ $\phi_3(x)$
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1
0.8
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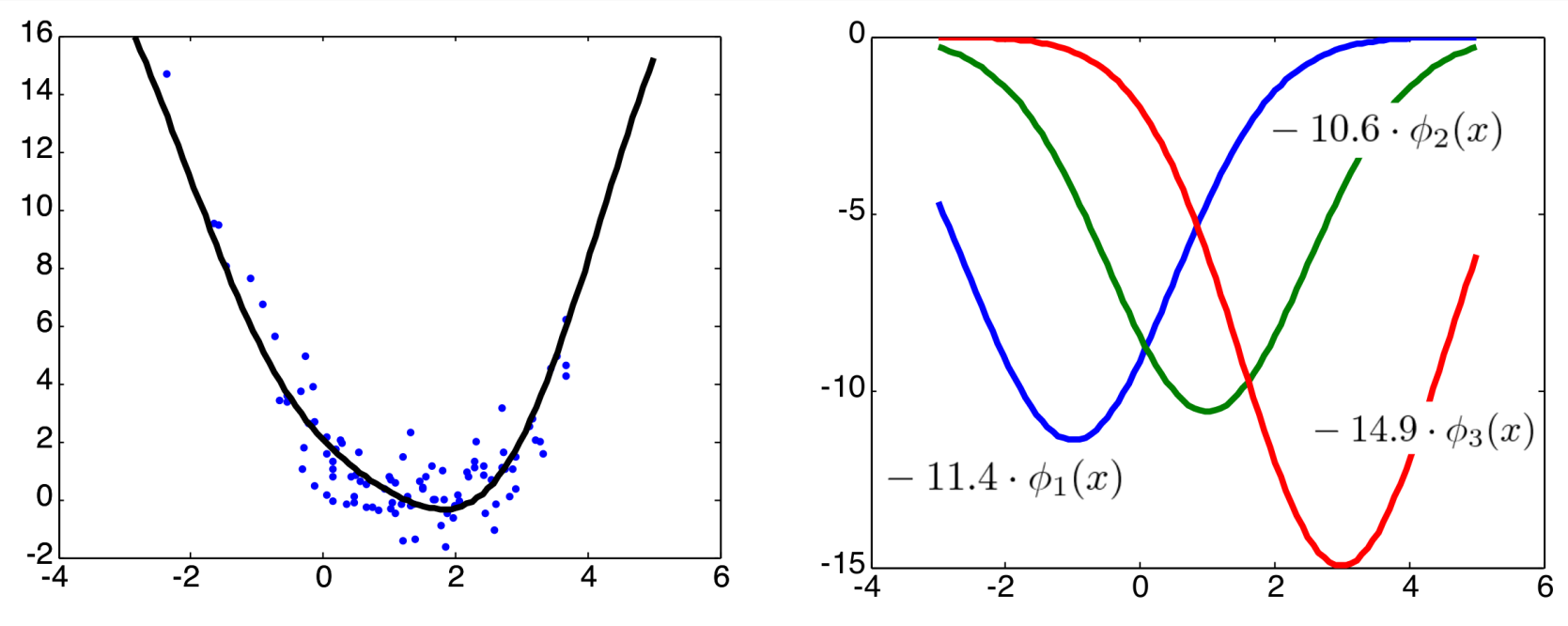
0
-6 -4 -2 0 2 4 6 8
x
$c_1 = -1$ $c_2 = 1$ $c_3 = 3$



Using RBFs with $\sigma = 1.5$:

$$f_{\boldsymbol{\theta}}(x) = \theta_0 \cdot 1 + \theta_1 \cdot \phi_1(x) + \theta_2 \cdot \phi_2(x) + \theta_3 \cdot \phi_3(x)$$
$$= 21.7 \cdot 1 - 11.4 \cdot \phi_1(x) - 10.6 \cdot \phi_2(x) - 14.9 \cdot \phi_3(x)$$

Classification, Logistic Regression

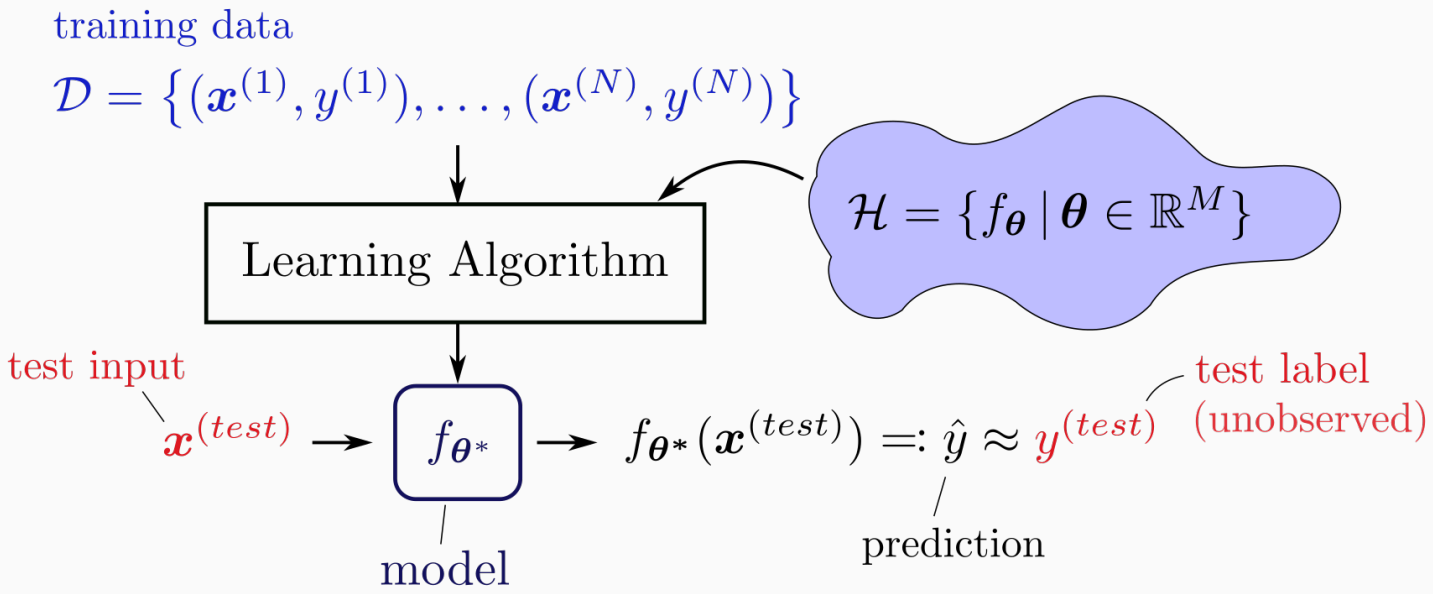
Coming Next

- **Regression**: Predict a **continuous** target y
- **Classification**: Predict a **categorical** target y (**label**, **class value**)
- Labels are **discrete** and assumed to have **no natural order**:
  - $\mathcal{C} = \{\text{'dog'}, \text{'cat'}, \text{'rabbit'}\}$
  - $\mathcal{C} = \{\text{'ice cream'}, \text{'chocolate'}, \text{'piece of cake'}, \text{'lollipop'}\}$
  - $\mathcal{C} = \{\text{'pear'}, \text{'apple'}\}$
- **Logistic regression**: a modification of linear regression for the classification problem
- *The* most widely used classification algorithm
- Confusing name: It's called logistic **regression**, even though it is a classification model 

Classification: The General Setup

- The true label y is a function $f : \mathbb{R}^D \mapsto \mathcal{C}$, where $\mathcal{C}$ is the set of possible labels (e.g. $\mathcal{C} = \{\text{'dog'}, \text{'cat'}, \text{'rabbit'}\}$
- f is sometimes called the **classification concept**
- We have a **training set** of **labelled examples** $(\boldsymbol{x}^{(1)}, y^{(1)}), \ldots, (\boldsymbol{x}^{(N)}, y^{(N)})$

Similar picture as for regression (**supervised** learning):



The Iris Dataset

Example

Iris Setosa

Iris Versicolor

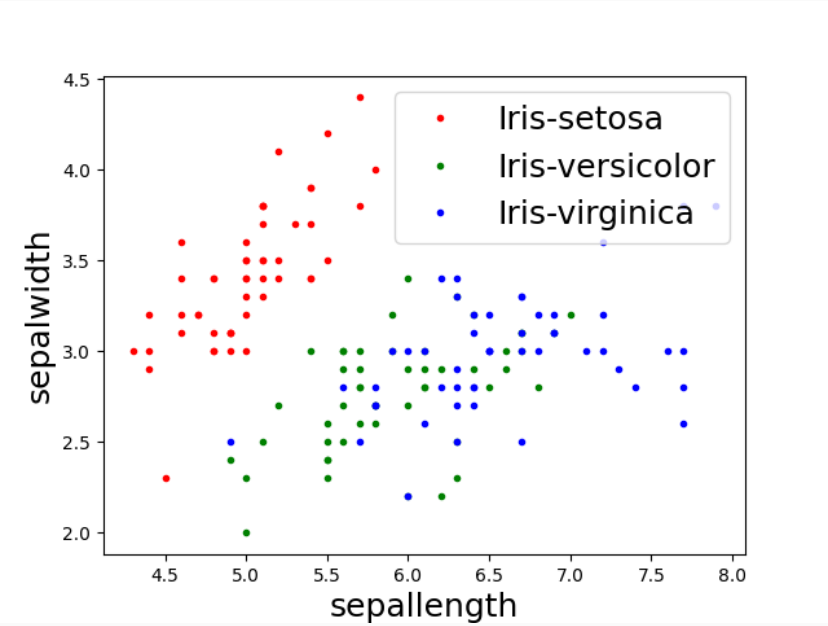
Iris Virginica

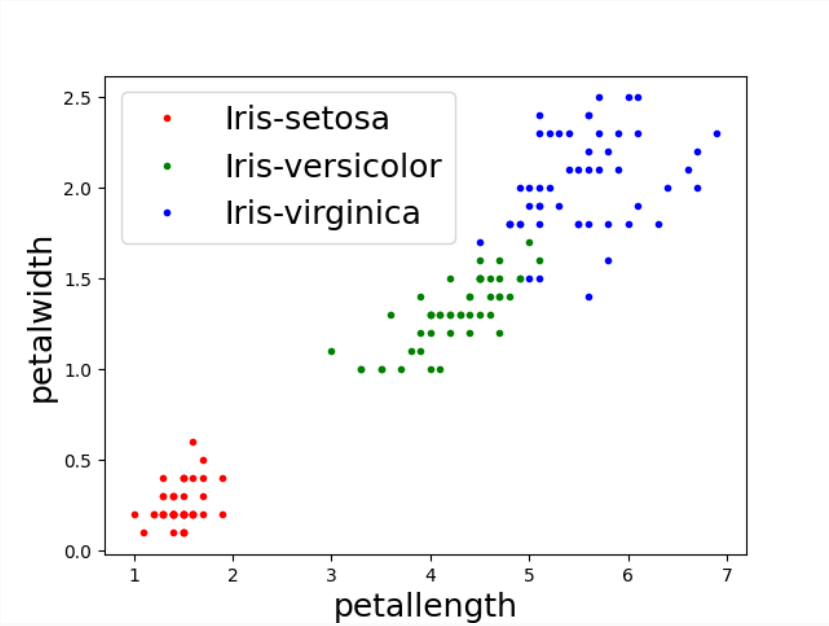
The **Iris dataset** is a classical dataset collected by R. Fisher in the 1930's. It contains $50 \times 3 = 150$ samples of three types of iris flowers.

"sepallength", "sepalwidth", "petallength", "petalwidth", "class"
5.1, 3.5, 1.4, 0.2, Iris-setosa
4.9, 3.0, 1.4, 0.2, Iris-setosa
...
7.0, 3.2, 4.7, 1.4, Iris-versicolor
6.4, 3.2, 4.5, 1.5, Iris-versicolor
...
6.3, 3.3, 6.0, 2.5, Iris-virginica
5.8, 2.7, 5.1, 1.9, Iris-virginica

The Iris Dataset

Example





How can we apply linear functions to the classification problem?

We focus first on the case of **binary classification**, i.e. $|\mathcal{C}| = 2$.
We might encode the labels as $y \in \{-1, +1\}$. For example:

$\mathcal{C} = \{\text{'dog'}, \text{'cat'}\}$

'dog':	$y = -1$	**negative class**
'cat':	$y = +1$	**positive class**

As **classifier** we use a linear function:

$$f(\phi) = \theta^T \phi$$

where $\phi = (1, \phi_1(\boldsymbol{x}), \ldots, \phi_K(\boldsymbol{x}))$ is a feature vector—this also subsumes the "standard" linear case, i.e. when $\phi_k(\boldsymbol{x}) = x_k$.

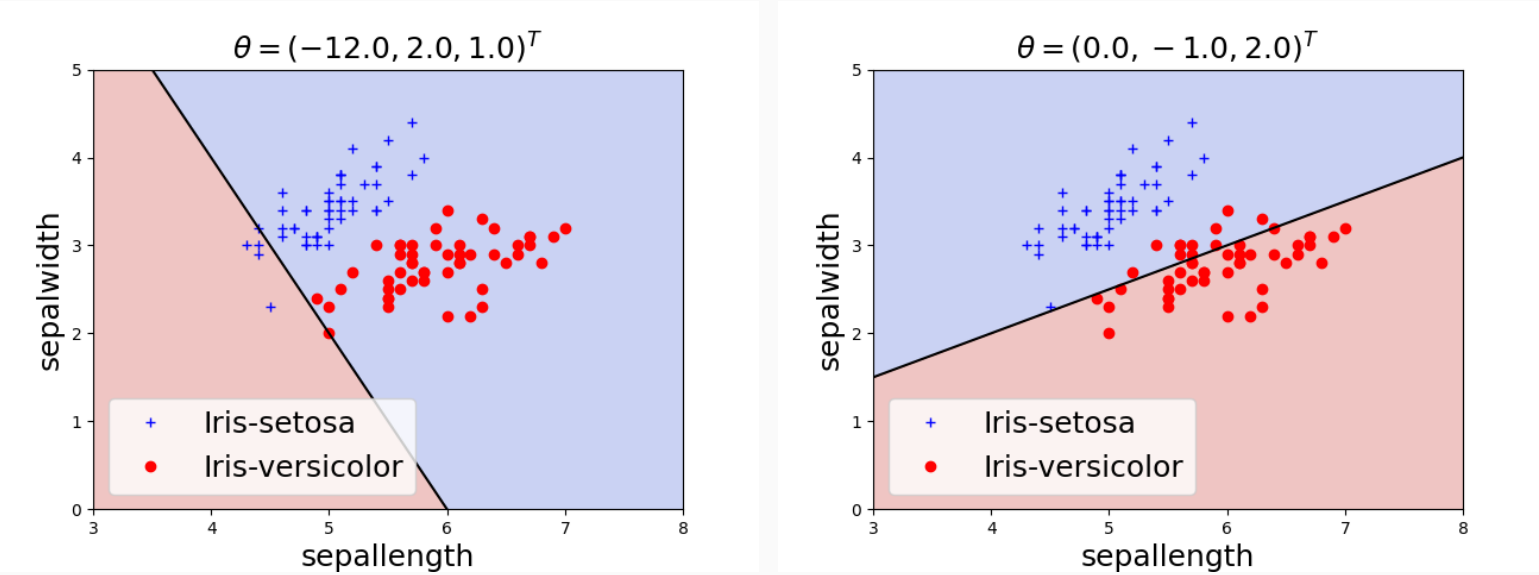
Classification rule given by **half spaces**:

$$\begin{aligned} \theta^T \phi &< 0 \quad \Rightarrow \text{predict } \hat{y} = -1 \\ \theta^T \phi &\geq 0 \quad \Rightarrow \text{predict } \hat{y} = +1 \end{aligned}$$

In short:

$$\hat{y} = sign\left(\theta^T \phi\right)$$

Two features $\phi_1(\boldsymbol{x}) = x_1$, $\phi_2(\boldsymbol{x}) = x_2$ and two classes from "Iris":



- Linear classifiers yield a **linear decision boundary** (hyper-plane) in the **feature space** ϕ
- If ϕ includes non-linear features, the decision boundary becomes non-linear in the original feature space $\boldsymbol{x}$!
- A dataset is called **linearly separable** if there exists a linear classifier (hyper-plane) with 0 classification errors

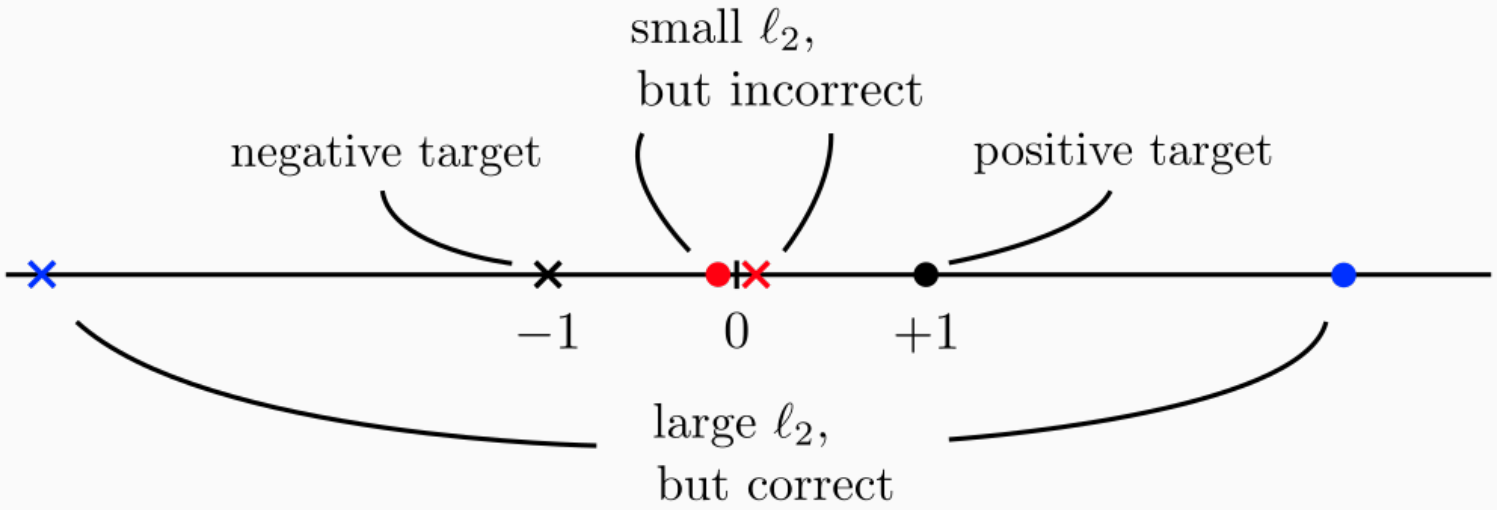
How to learn θ?

Idea: We could treat the problem as a regression problem (interpreting the labels $y = \pm 1$ as real numbers) and optimize squared loss. As we saw, squared loss is convex and optimization is easy.

But is this a good idea? 

Squared Loss vs. Correct Classification

ℓ_2-error in label space and correct classification are only loosely related:



Treating classification as a regression problem will work to a certain extend. But a loss functions reflecting the classification problem will deliver better results.

Minimizing Classification Error?

The most direct way seems to minimize the number of classification errors, or equivalently, the **classification error rate**:

$$\min_{\theta} \; CER = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1} \left[\hat{y}^{(i)} \neq y^{(i)} \right]$$

where $\mathbb{1}[\cdot]$ is the **indicator function**: $\mathbb{1}[arg] = \begin{cases} 0 & \text{if } arg = \text{False} \\ 1 & \text{if } arg = \text{True} \end{cases}$

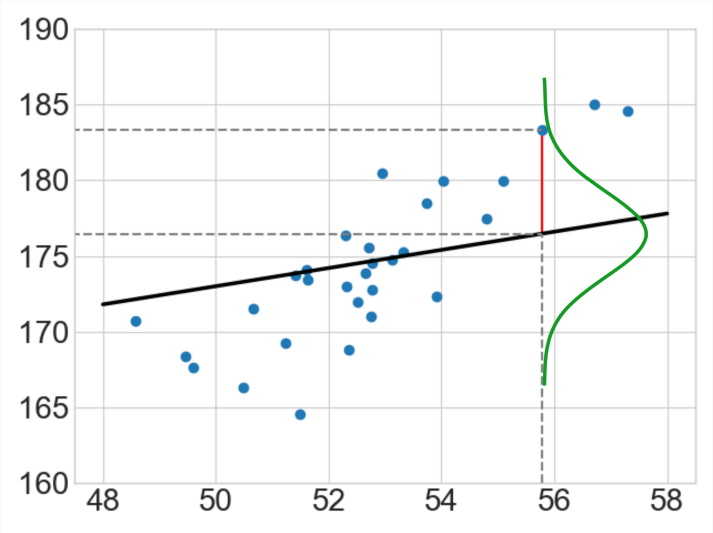
Equivalently, maximize the number of correctly classified examples, or the **classification accuracy**:

$$\max_{\theta} \; ACC = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1} \left[\hat{y}^{(i)} = y^{(i)} \right]$$

Good idea, but non-convex and hard to optimize. . .

Recall that least-squares regression is equivalent to **maximum likelihood** under a **conditional Gaussian model**:

$$\theta^* = \arg\max_{\theta} \sum_{i=1}^{N} \log p(y^{(i)} \mid \phi^{(i)}, \theta)$$



For classification, we could simply take a different distribution, reflecting that $y^{(i)}$ are binary. 🤔

Bernoulli Distribution

Y is a RV with state space $\mathcal{Y} = \{-1, 1\}$ and PMF

$$p_Y(y; \pi) = \begin{cases} 1 - \pi & \text{if } y = -1 \\ \pi & \text{if } y = 1 \end{cases}$$

Bernoulli distribution with **success probability** π.

Jakob Bernoulli, 1654–1705

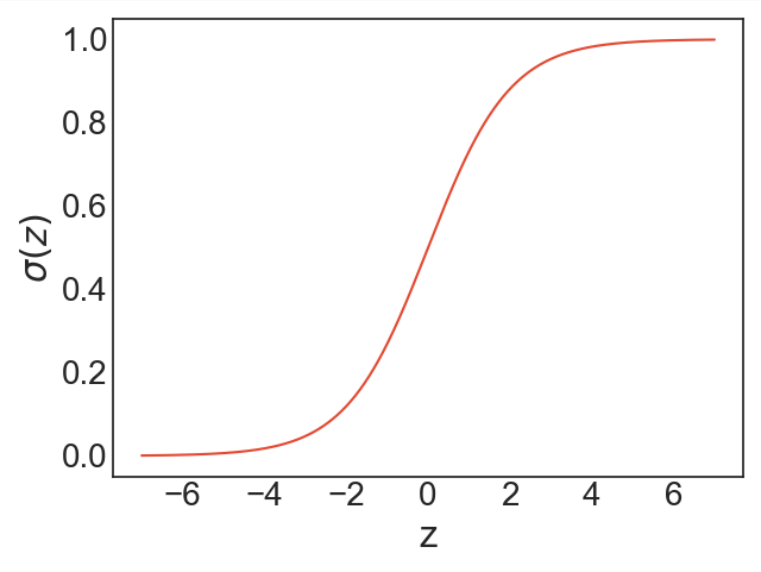
- We want to make the Bernoulli distribution conditional on input ϕ
- That is, the success probability π should be a function of $\theta^T \phi$
- Note that we require $0 \leq \pi \leq 1$

Logistic Function (Sigmoid Function) Definition

The **logistic function** (**sigmoid function**) is defined as

$$\sigma(z) = \frac{1}{1 + \exp(-z)}$$

It maps from the real numbers to the interval $[0, 1]$, thus it is also sometimes called **squashing function**:



The name **logistic regression** stems from this function.

- The output for the i^{th} sample is $\boldsymbol{\theta}^T \boldsymbol{\phi}^{(i)}$
- The outputs are often called **logits**
- We use the logistic function to convert logits into Bernoulli parameters (success probabilities):

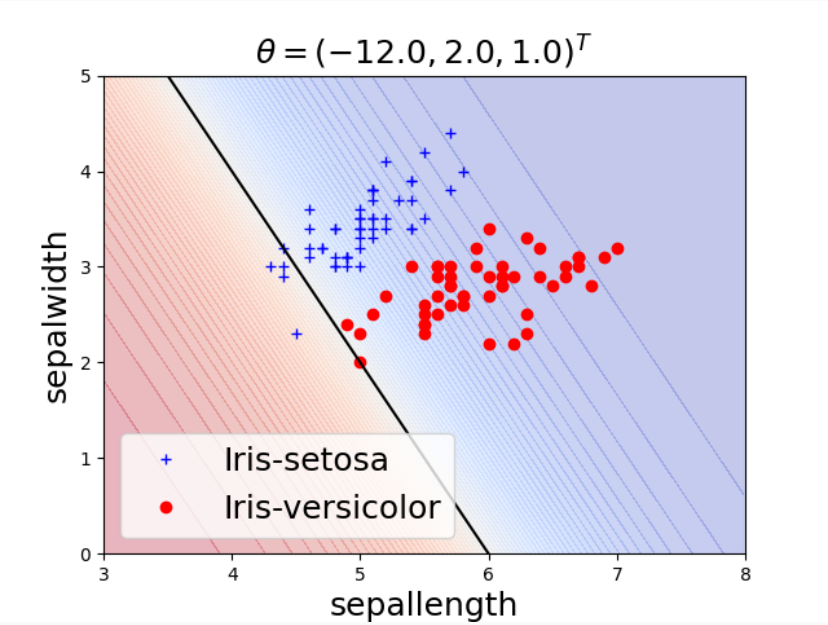
$$\pi^{(i)} := p_{\boldsymbol{\theta}}(y^{(i)} = 1) = \frac{1}{1 + \exp(-\boldsymbol{\theta}^T \boldsymbol{\phi}^{(i)})}$$

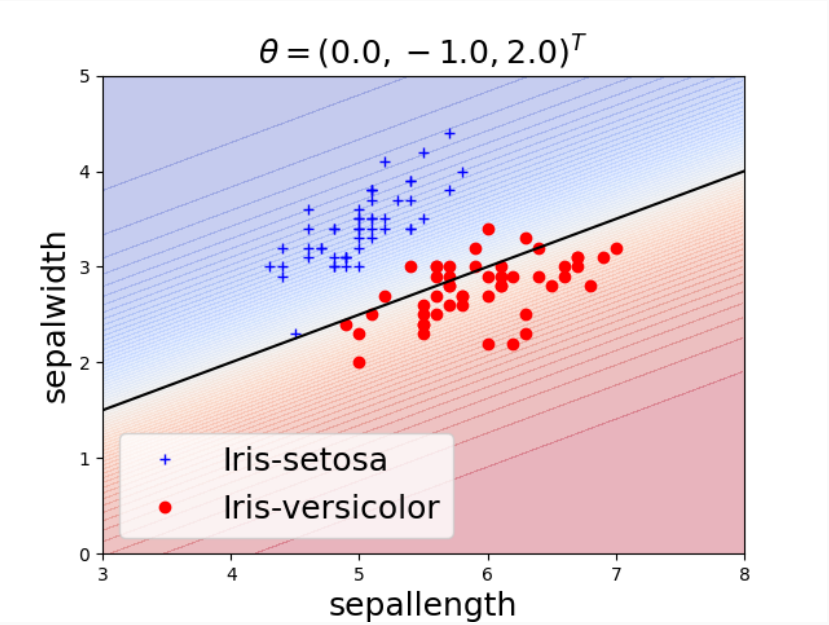
- $\pi^{(i)}$ the **model's probability** (**belief**) that the input sample belongs to the **positive class**
- For **negative class:** $p_{\boldsymbol{\theta}}(y^{(i)} = -1) = 1 - \pi^{(i)}$

i	$\boldsymbol{\theta}^T \boldsymbol{\phi}^{(i)}$	$\pi^{(i)}$	$1 - \pi^{(i)}$
1	3	0.953	0.047
2	1.5	0.818	0.182
3	-2	0.12	0.88
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Model Probabilities

Example





blue shade where $\pi = p(y = 1) = \sigma(\boldsymbol{\theta}^T \boldsymbol{\phi})$ is high

red shade where $\pi = p(y = 1) = \sigma(\boldsymbol{\theta}^T \boldsymbol{\phi})$ is low (or $1 - \pi$ is high)

Maximum Likelihood Objective

The log-likelihood (under i.i.d. assumption) is

$$\log p(\mathcal{D}; \boldsymbol{\theta}) = \sum_{i=1}^{N} \log p(y^{(i)} \mid \boldsymbol{\phi}^{(i)}, \boldsymbol{\theta})$$

$$= \sum_{i=1}^{N} \mathbb{1}[y^{(i)} = 1] \log \pi^{(i)} + \mathbb{1}[y^{(i)} = -1] \log(1 - \pi^{(i)})$$

$$= \sum_{i=1}^{N} \mathbb{1}[y^{(i)} = 1] \log \sigma(\boldsymbol{\theta}^T \boldsymbol{\phi}^{(i)}) + \mathbb{1}[y^{(i)} = -1] \log(1 - \sigma(\boldsymbol{\theta}^T \boldsymbol{\phi}^{(i)}))$$

We might also write this as

$$\log p(\mathcal{D}; \boldsymbol{\theta}) = \underbrace{\sum_{i:\, y^{(i)}=1} \log \sigma(\boldsymbol{\theta}^T \boldsymbol{\phi}^{(i)})}_{\text{sum over positive samples}} + \underbrace{\sum_{i:\, y^{(i)}=-1} \log(1 - \sigma(\boldsymbol{\theta}^T \boldsymbol{\phi}^{(i)}))}_{\text{sum over negative samples}}$$

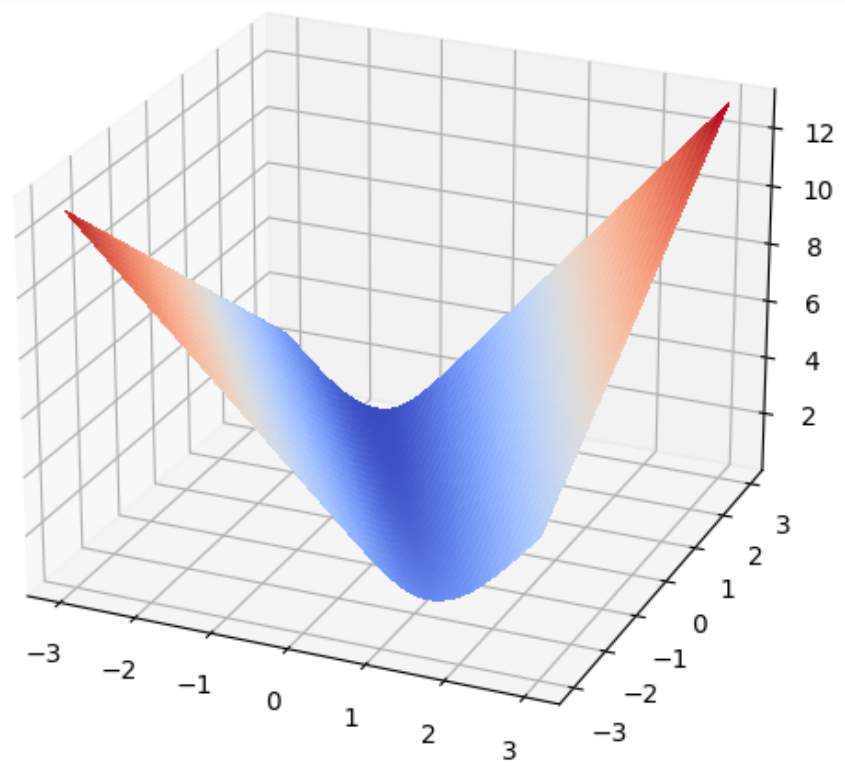
Cross-Entropy

- Usually, one performs two cosmetic modifications of the log-likelihood objective:
  - multiply times -1 and replace maximization by minimization
  - divide by N
- The resulting **equivalent** objective is called **cross-entropy (CE)** loss:

$$\mathcal{L}_{CE}(\boldsymbol{\theta}) = -\frac{1}{N}\sum_{i=1}^{N} \mathbb{1}[y^{(i)}=1]\log\sigma(\boldsymbol{\theta}^T\boldsymbol{\phi}^{(i)}) + \mathbb{1}[y^{(i)}=-1]\log(1-\sigma(\boldsymbol{\theta}^T\boldsymbol{\phi}^{(i)}))$$

- $\mathcal{L}_{CE}$ is the most widely used objective for binary classification
- In general, cross-entropy compares two probability distributions—here the label is interpreted as an "extreme" distribution, assigning all probability to a single class

- No closed form solution for $\mathcal{L}_{CE}$
- But, $\mathcal{L}_{CE}$ is differentiable and convex, thus we can use gradient descent to find a **global minimum**
- For example: cross-entropy loss for two input features and two classes of the Iris dataset:



Gradient Descent for Logistic Regression

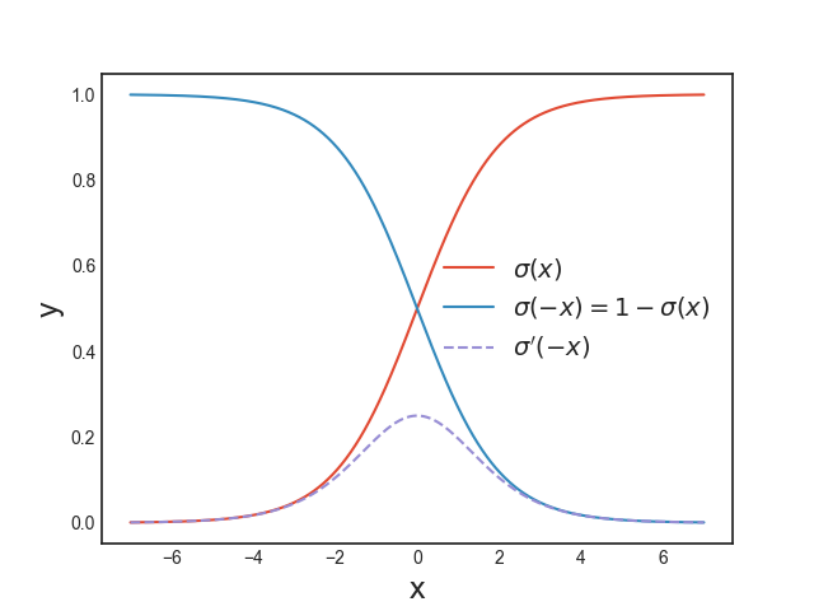
- We are going to derive the gradient of the cross-entropy loss for the logistic regression model, i.e. the vector of all partial derivatives $\frac{\partial \mathcal{L}_{CE}}{\partial \theta_i}$
- We will use the **chain rule of calculus**, since the loss is of the form

$$\text{“}\mathcal{L}_{CE} = \textit{crossentropy}(\textit{logistic function}(\boldsymbol{\theta}^T \boldsymbol{x}))\text{”}$$

- To this end, we first need the derivative of the logistic function $\sigma(\cdot)$

Derivative of Sigmoid Function

$$\sigma'(z) = \left(\frac{1}{1+e^{-z}}\right)' = -\left(1+e^{-z}\right)^{-2}\left(-e^{-z}\right)$$



$$= \frac{e^{-z}}{\left(1+e^{-z}\right)^2}$$

$$= \frac{1}{1+e^{-z}}\,\frac{e^{-z}}{1+e^{-z}}$$

$$= \sigma(z)\,\frac{1}{e^{z}+1}$$

$$= \underline{\sigma(z)\,\sigma(-z)}$$

$$= \underline{\sigma(z)\,(1-\sigma(z))}$$

Gradient of Cross Entropy Loss

$$\mathcal{L}_{CE} = -\frac{1}{N}\left[\sum_{i\,:\,y^{(i)}=1} \log\sigma\left(\boldsymbol{\theta}^T\boldsymbol{\phi}^{(i)}\right) + \sum_{i\,:\,y^{(i)}=-1} \log\left(1-\sigma\left(\boldsymbol{\theta}^T\boldsymbol{\phi}^{(i)}\right)\right)\right]$$

$$\frac{\partial\mathcal{L}_{CE}}{\partial\theta_k} = -\frac{1}{N}\left[\sum_{i\,:\,y^{(i)}=1} \frac{\partial}{\partial\theta_k}\log\sigma\left(\boldsymbol{\theta}^T\boldsymbol{\phi}^{(i)}\right) + \sum_{i\,:\,y^{(i)}=-1} \frac{\partial}{\partial\theta_k}\log\left(1-\sigma\left(\boldsymbol{\theta}^T\boldsymbol{\phi}^{(i)}\right)\right)\right]$$

$$\frac{\partial}{\partial\theta_k}\log\sigma\left(\boldsymbol{\theta}^T\boldsymbol{\phi}^{(i)}\right) = \overbrace{\frac{1}{\sigma\left(\boldsymbol{\theta}^T\boldsymbol{\phi}^{(i)}\right)}}^{\log'}\overbrace{\sigma\left(\boldsymbol{\theta}^T\boldsymbol{\phi}^{(i)}\right)\left(1-\sigma\left(\boldsymbol{\theta}^T\boldsymbol{\phi}^{(i)}\right)\right)}^{\sigma'}\phi_k^{(i)}$$

$$= \overbrace{\left(1-\sigma\left(\boldsymbol{\theta}^T\boldsymbol{\phi}^{(i)}\right)\right)}^{\text{prediction error}}\phi_k^{(i)}$$

Gradient of Cross Entropy Loss cont'd

$$\frac{\partial}{\partial \theta_k} \log\left(1 - \sigma\left(\boldsymbol{\theta}^T \boldsymbol{\phi}^{(i)}\right)\right) = \overbrace{\frac{1}{1 - \sigma\left(\boldsymbol{\theta}^T \boldsymbol{\phi}^{(i)}\right)}}^{\log'} \overbrace{-\sigma(\boldsymbol{\theta}^T \boldsymbol{\phi}^{(i)}) \cancel{\left(1 - \sigma(\boldsymbol{\theta}^T \boldsymbol{\phi}^{(i)})\right)}}^{-\sigma'} \phi_k^{(i)}$$

$$= \overbrace{\left(0 - \sigma(\boldsymbol{\theta}^T \boldsymbol{\phi}^{(i)})\right)}^{\text{prediction error}} \phi_k^{(i)}$$

Collecting all partial derivatives gives the sample-wise the gradient:

- positive: $\nabla_{\boldsymbol{\theta}} \log \sigma\left(\boldsymbol{\theta}^T \boldsymbol{\phi}^{(i)}\right) = \left(1 - \sigma(\boldsymbol{\theta}^T \boldsymbol{\phi}^{(i)})\right) \boldsymbol{\phi}^{(i)}$
- negative: $\nabla_{\boldsymbol{\theta}} \log\left(1 - \sigma\left(\boldsymbol{\theta}^T \boldsymbol{\phi}^{(i)}\right)\right) = \left(0 - \sigma(\boldsymbol{\theta}^T \boldsymbol{\phi}^{(i)})\right) \boldsymbol{\phi}^{(i)}$

Gradient Descent for Cross Entropy

The gradient of the overall loss is given as:

$$\nabla_{\boldsymbol{\theta}} \mathcal{L}_{CE} = -\frac{1}{N} \left[\sum_{i\,:\,y^{(i)}=1} \left(1 - \sigma\left(\boldsymbol{\theta}^T \boldsymbol{\phi}^{(i)}\right)\right) \boldsymbol{\phi}^{(i)} + \sum_{i\,:\,y^{(i)}=-1} -\sigma\left(\boldsymbol{\theta}^T \boldsymbol{\phi}^{(i)}\right) \boldsymbol{\phi}^{(i)} \right]$$

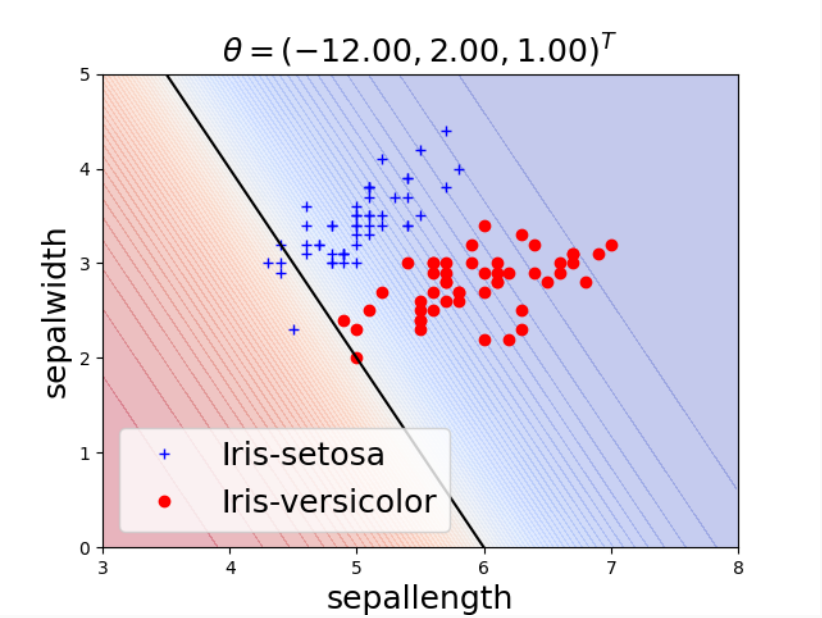
- Initialize $\boldsymbol{\theta}$ arbitrarily
- Repeat until convergence

$$\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} - \eta \nabla_{\boldsymbol{\theta}} \mathcal{L}_{CE}$$

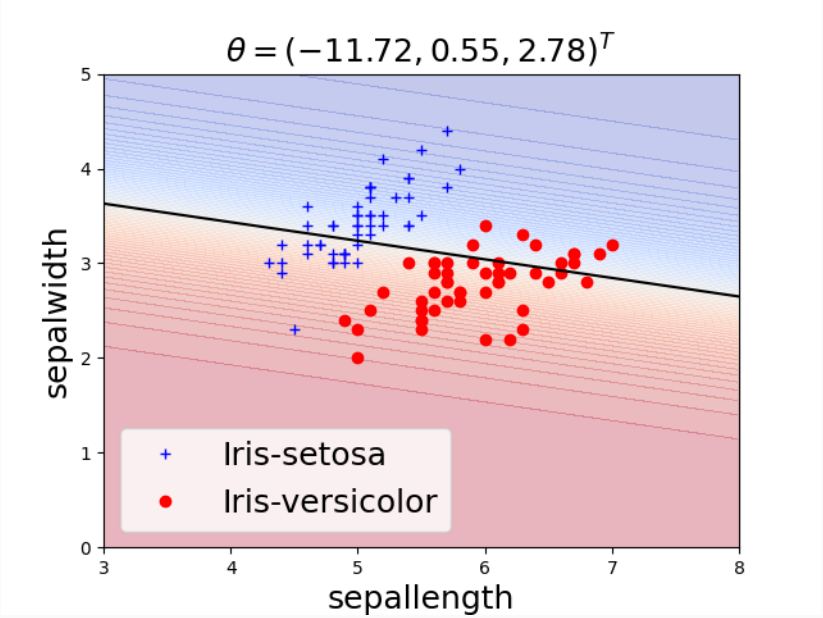
Minimizing Cross-Entropy: Iris Dataset

Example

- using 2 features and 2 classes
- gradient descent with step-size $\eta = 0.1$



Iteration 0

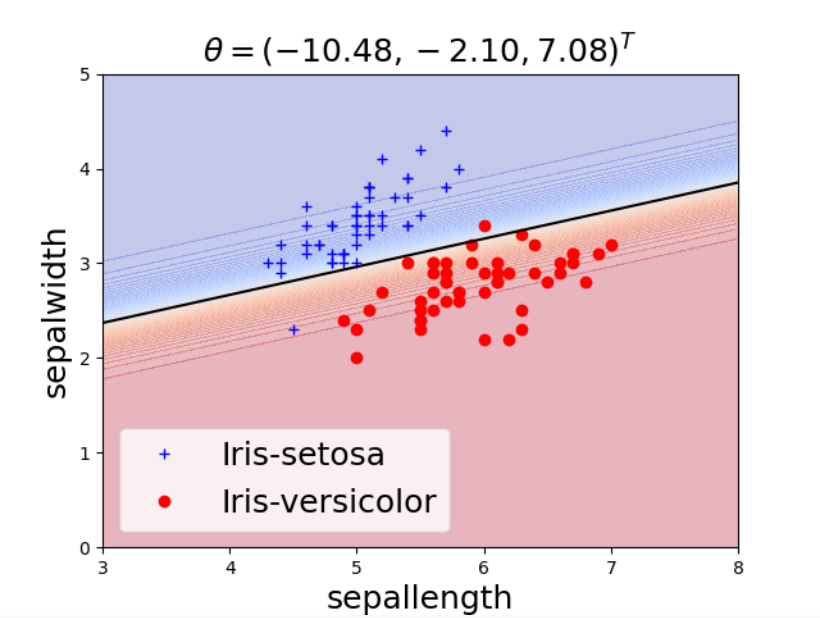


Iteration 100

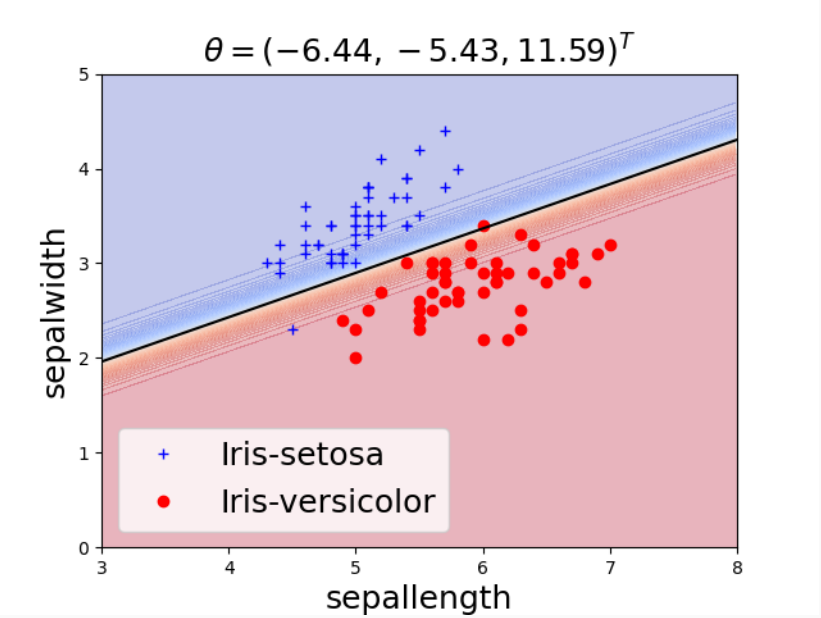
Minimizing Cross-Entropy: Iris Dataset

Example

- using 2 features and 2 classes
- gradient descent with step-size $\eta = 0.1$



Iteration 1000

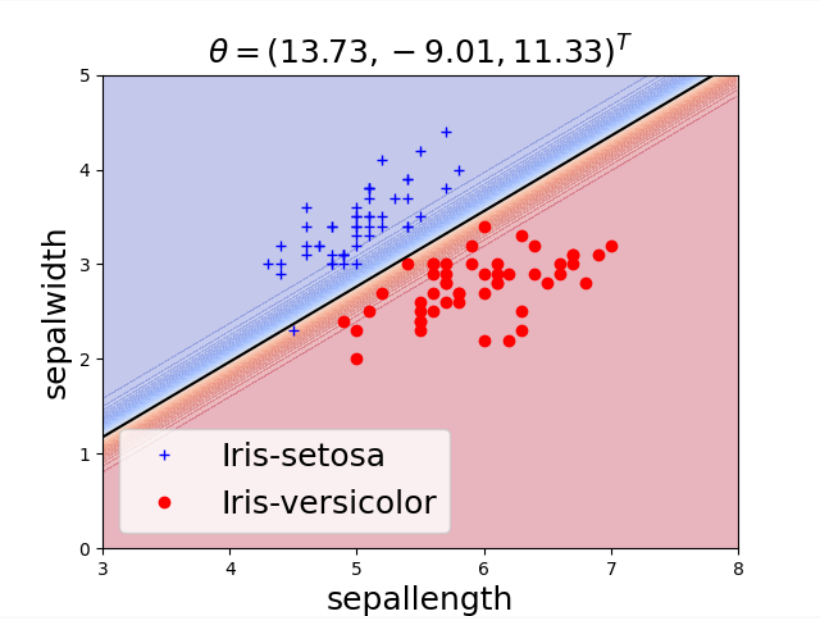


Iteration 10000

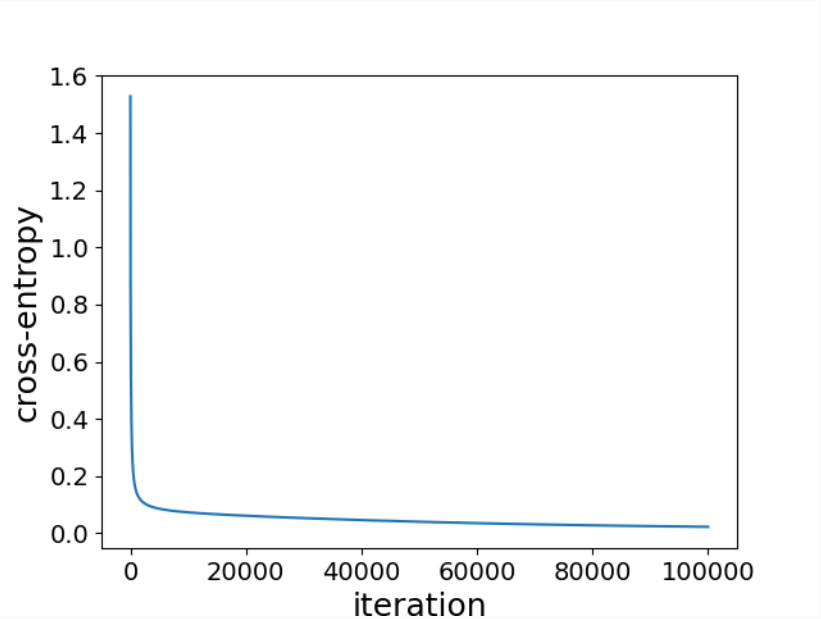
Minimizing Cross-Entropy: Iris Dataset

Example

- using 2 features and 2 classes
- gradient descent with step-size $\eta = 0.1$



Iteration 100000



cross-entropy over iteration

Summary

- Linear models with non-linear features give non-linear models
- For linear regression, the closed form solution is the same – simply replace design matrix $\boldsymbol{X}$ with a design matrix $\boldsymbol{\Phi}$ including non-linear features
- **Logistic regression**: linear model for classification
- Linear **decision boundary** in feature space $\boldsymbol{\phi}$ (non-linear in $\boldsymbol{x}$ if $\boldsymbol{\phi}(\boldsymbol{x})$ is non-linear)
- **Cross-entropy loss**: equivalent to maximum likelihood for conditional Bernoulli, using the **logistic function** to convert **logits** into Bernoulli parameters
- No closed-form solution for logistic regression, so we used gradient descent