

- $R = [a_1, b_1]x[a_2, b_2]x[a_3, b_3]$
- $\int \int \int_R f(x, y, z) dy dz dx = \int_{x=a_1}^{b_1} \left( \int_{y=a_2}^{b_2} \left( \int_{z=a_3}^{b_3} f(x, y, z) dz \right) dy \right) dx$ 
  - andere Reihenfolgen auch möglich

## Normalbereich im $\mathbb{R}^3$

- $N = (x, y, z) \in \mathbb{R}^3 \mid (x, y) \in M, g(x, y) \leq z \leq h(x, y)$
- $vol(N) = \int_{y=c}^d \left( \int_{x=\psi(y)}^{\varphi(y)} \left( \int_{z=g(x,y)}^{h(x,y)} dz \right) dx \right) dy$
- Beispiel

$$\begin{aligned} \underline{\text{Bsp.}} \quad B &= \{ (x, y, z) \in \mathbb{R}^3 \mid x, y, z \geq 0, x+y+z \leq 1 \} \\ &= \{ (x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1, 0 \leq y \leq 1-x, 0 \leq z \leq 1-x-y \} \\ \underline{\quad} \\ \int \int \int_B x y^2 z^2 dx dy dz &= \int_{x=0}^1 \left( \int_{y=0}^{1-x} \left( \int_{z=0}^{1-x-y} x y^2 z^2 dz \right) dy \right) dx = \int_{x=0}^1 \left( \int_{y=0}^{1-x} \left( x y^2 \frac{z^3}{3} \Big|_{z=0}^{1-x-y} \right) dy \right) dx = \int_{x=0}^1 \left( \int_{y=0}^{1-x} x y^2 (1-x-y)^3 dy \right) dx \end{aligned}$$

## Substitutionsregel (Transformationsformel)

- $T : B \rightarrow \mathbb{R}^3$
- $T$  differenzierbar und injektiv
- $\int \int \int_{T(B)} f(x, y, z) dx dy dz = \int \int \int_B f \circ T(u, v, w) \cdot \left| \det \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw$
- Volumensumrechnungsfaktor - JACOBI-Determinante
  - $x, y, z$  beliebig vertauschbar
- Beispiel: dreidimensionale Polarkoordinaten
  - $x \Rightarrow r \sin(\psi) \cos(\varphi)$
  - $y \Rightarrow r \sin(\psi) \sin(\varphi)$
  - $z \Rightarrow r \cos(\psi)$
  - Bedingungen
    - \*  $r \geq 0$
    - \*  $0 \leq \psi \leq \pi$
    - \*  $0 \leq \varphi \leq 2\pi$

$$\underline{\quad} \quad \frac{\partial(x, y, z)}{\partial(r, \psi, \varphi)} = \begin{pmatrix} \sin(\psi) \cos(\varphi) & r \cos(\psi) \cos(\varphi) & -r \sin(\psi) \sin(\varphi) \\ \sin(\psi) \sin(\varphi) & r \cos(\psi) \sin(\varphi) & r \sin(\psi) \cos(\varphi) \\ \cos(\psi) & -r \sin(\psi) & 0 \end{pmatrix}$$

- Beispiel: Polarkoordinaten bei Kugel

$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \geq 0, x^2 + y^2 + z^2 \leq 9\}$$

$$0 \leq x \leq 3, \quad 0 \leq y \leq \sqrt{9 - x^2}, \quad 0 \leq z \leq \sqrt{9 - x^2 - y^2}$$

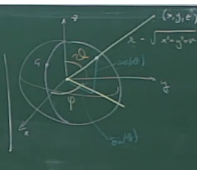
$$\iiint_B xy \, dx \, dy \, dz = \int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} xy \, dz \, dy \, dx$$

$$= \int_0^3 \int_0^{\sqrt{9-x^2}} xy \sqrt{9-x^2-y^2} \, dy \, dx$$

$$= \int_0^3 \left[ -\frac{1}{3} y^3 \sqrt{9-x^2-y^2} + \frac{xy^2}{2} \sqrt{9-x^2-y^2} + \frac{xy^3}{2} \frac{1}{\sqrt{9-x^2-y^2}} \right]_0^{\sqrt{9-x^2}} dx$$

$$= \int_0^3 \left[ -\frac{1}{3} (9-x^2)^{3/2} + \frac{xy^2}{2} \sqrt{9-x^2-y^2} + \frac{xy^3}{2} \frac{1}{\sqrt{9-x^2-y^2}} \right] dx$$

$$= \int_0^3 \left[ -\frac{1}{3} (9-x^2)^{3/2} + \frac{xy^2}{2} \sqrt{9-x^2-y^2} + \frac{xy^3}{2} \frac{1}{\sqrt{9-x^2-y^2}} \right] dx$$

$$= \frac{243}{16}$$


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# • Zylinderkoordinaten

-  $x \Rightarrow r \cos(\varphi)$

-  $y \Rightarrow r \sin(\varphi)$

-  $z \Rightarrow z$

- Volumenelement

\*  $dx dy dz = r dr d\varphi dz$

- Beispiel

$$B = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq 1 - (x^2 + y^2)\}$$

$$\iiint_B z \, dx \, dy \, dz = \int_0^1 \left( \int_0^{2\pi} \left( \int_0^{\sqrt{1-z}} z \, r \, dr \right) d\varphi \right) dz$$

$$= \int_0^1 d\varphi \cdot \int_0^{\sqrt{1-z}} \frac{z}{2} r^2 \bigg|_0^{\sqrt{1-z}} dz = \frac{2\pi}{2} \int_0^1 z (1-z)^2 dz$$

$$= \pi \int_0^1 (1-u)^2 \frac{du}{2} = \frac{\pi}{6}$$

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[[Mehrdimensionale Integralrechnung]]