Overview

- written as A* Algorithm
- [[Shortest Path Algorithms]] to find a single destination
- based on [[Breadth-First Search]] and [[Dijkstra's Algorithm]]
- informed
  - does not search uniformly
  - uses heuristics
  - prioritizes towards the direction of the goal

Heuristics

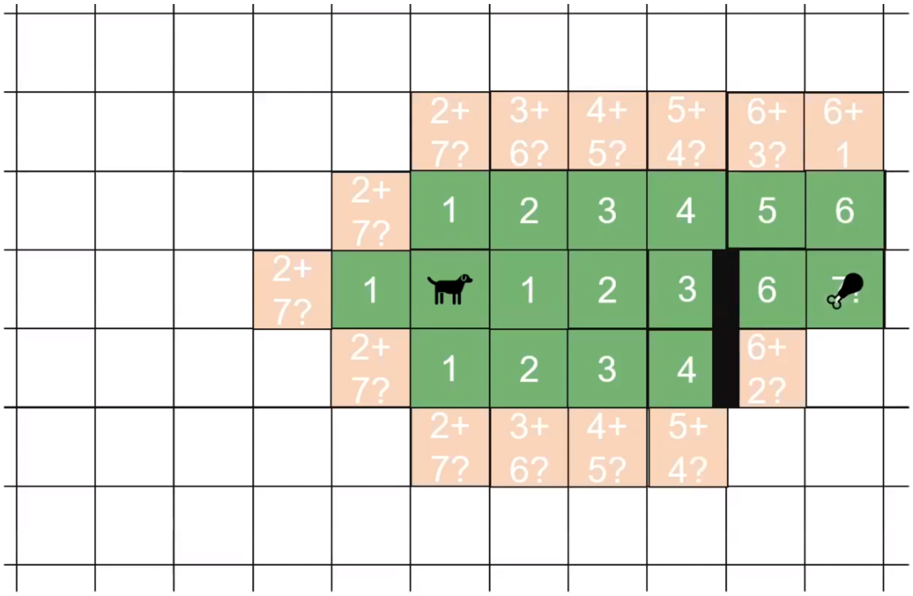
- **Def:** A heuristic is **consistent** if for every edge $\{u, v\} \in E$ we have
  $$h(u) \leq w(u, v) + h(v)$$
- perfect heuristic
  - border line impossible
  - requires perfect knowledge lol
- overestimate
  - fast
  - not admissible
  - might not find path even if one exists

A* Heuristics

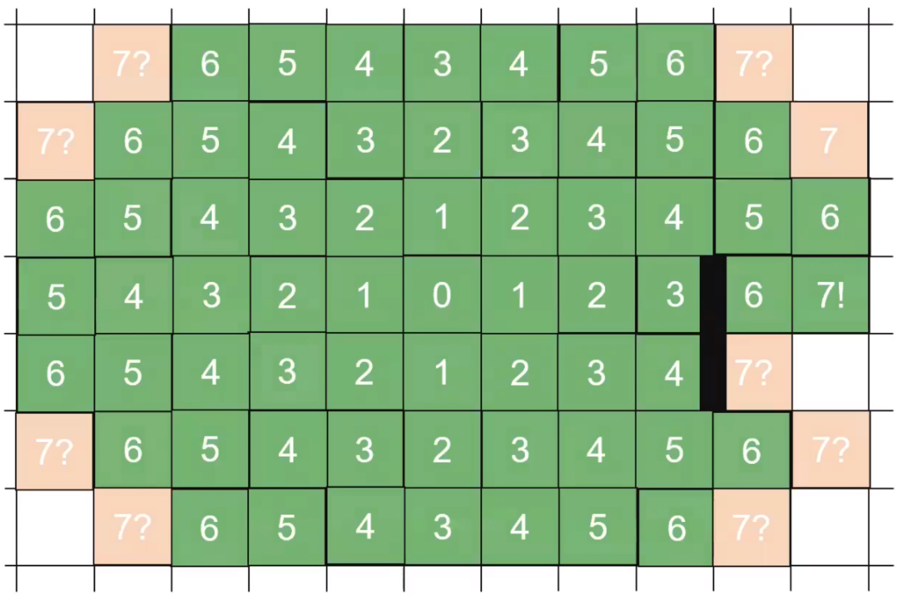
- $g(v)$
  - distance from start to current vertex
- $h(u)$
  - distance from current vertex to end
    - "Luftlinie" - as the crow flies
  - ignores obstacles which may block the path
  - underestimates the future cost
    - good characteristic
- therefore pretty efficient

Comparison between A* and Dijkstra's

- A*



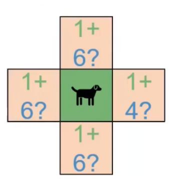
- Dijkstra's

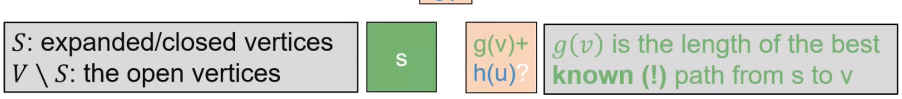


Algorithm

`Input: G(V,E,W) start point: s, end point: t`

`Initialize S={s},` $g(s)=0$, $g(V\backslash N(s))=\infty$, $g(v)=w(s,v)$, $v\in N(s)$





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While t ∉ S do
    u = argmin_{u∈V\S}{g(u) + h(u,t)}
    For v s.t. {u,v}∈ E do
        temp=min{g(v),g(u)+w(u,v)}
        If temp<g(v) then:
            g(v)=temp
            S=S\{v}        //does nothing if v ∉ S
    S= S ∪{u}
```

g(v)+h(u)?

$g(v)$ is the length of the best **known (!)** path from s to v

$h(u,t)$ is a heuristic guess for the path from u to t.

S might decrease!

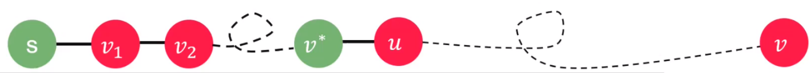
- – red parts differ from [[Dijkstra's Algorithm]]

Properties

- nodes may expand more than once
  - – $g(v)$ heuristic value can change
  - – always terminates if a path exists
- optimal if a path exists

Lemma 1: Always, for every *open* node v and every optimal path P from s to v, there exists an ***open node*** u on P with $\boxed{g(u)}$ **=** $\boxed{\text{distance}(s, u)}$

Proof: Let $P = (s = v_0, v_1, v_2, \ldots, v = v_k)$.



$C = \{v_i \in P \mid v_i$ **closed**$, g(v_i) = d(s,u)\} \neq \emptyset$,

- Let v^*, the vertex in C with highest index. $v^* \neq v$.
- Let u be the successor of v^* in P (possibly $u = v$).

P is optimal path

$$g(\boldsymbol{u}) \leq \boldsymbol{g}(\boldsymbol{v}^*) + \boldsymbol{w}(\boldsymbol{v}^*, \boldsymbol{u}) = \boldsymbol{d}(\boldsymbol{s}, \boldsymbol{v}^*) + \boldsymbol{w}(\boldsymbol{v}^*, \boldsymbol{u}) = \boldsymbol{d}(\boldsymbol{s}, \boldsymbol{u}) \leq \boldsymbol{g}(\boldsymbol{u})$$

v^* expanded   definition of v^* always

u has to be open by definition of C

–

Corollary: Suppose h is admissible and A* has not terminated. Then, for any optimal path P from s to t, there is an open node u with

$\boxed{g(u)} + \boxed{\text{heuristic}(u,t)} \leq \boxed{\text{distance}(s,t)}$

Proof: By Lemma 1, we have open node $u \in P$ *with* $\boxed{g(u)}$ **=** $\boxed{\text{distance}(s, u)}$

$\boxed{g(u)} + \boxed{\text{heuristic}(u,t)}$ **=** $\boxed{\text{distance}(s, u)} + \boxed{\text{heuristic}(u,t)}$

h admissible $\leq \boxed{\text{distance}(s, u)} + \boxed{\text{futureCost}(u,t)}$

–

Proof (Optimality):

Suppose A* terminates at t with a suboptimal path, i.e., in the last step we expanded t with

$\boxed{g(t)} + \overbrace{\boxed{\text{heuristic}(t,t)}}^{0} > \boxed{\text{distance}(s, t)}$

But, by the corollary, there existed just before "expanding t", there is an open node u on an optimal path with

–

$\boxed{g(u)} + \boxed{\text{heuristic}(u,t)} \leq \boxed{\text{distance}(s, t)}$ Contradiction!

- optimally efficient
  - with regards to the number of vertices expanded
- space as bottle neck
  - $g(v) + h(v)$ is stored for each visited v
  - **15-Puzzle:** Search space has a node for each configuration: 16!=20.922.789.888.000 vertices!
  - motivation for memory bounded heuristic search
    - Iterative deepening A*