Application

Idea: "guess" an asymptotic bound ($\mathcal{O}$, Ω) and prove it by mathematical induction

Example: $T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + n$

Guess: $T(n) = \mathcal{O}(n \log n)$

We have to show: $\exists c > 0, n_0 \in \mathbb{N}$ such that
$\quad\quad T(n) \leq c \cdot n \log n$ for all $n \geq n_0$

Induction hypothesis: $T(n) \leq c \cdot n \log_2 n$ for some const. $c > 0$

Induction base: $T(k) = \Theta(1) \leq d$ for small constant k
$\quad\quad$ and some constant $d > 0$ (by def.)

Induction step: $T(n) = 2\underline{T(\lfloor \frac{n}{2} \rfloor)} + n$
$$\leq 2 \cdot (c \cdot \underline{\lfloor \tfrac{n}{2} \rfloor \log_2 \lfloor \tfrac{n}{2} \rfloor}) + n \overset{?}{\leq} c \cdot n \log_2 n$$

Example: $T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + n, \quad T(n) \overset{?}{=} \mathcal{O}(n \log n)$

$$
\begin{aligned}
T(n) &= 2T(\lfloor \tfrac{n}{2} \rfloor) + n \\
&\leq 2 \cdot \left(c \lfloor \tfrac{n}{2} \rfloor \log(\lfloor \tfrac{n}{2} \rfloor) \right) + n \\
&\leq c \cdot n \log_2 (\tfrac{n}{2}) + n \\
&= c \cdot n \log_2 n - c \cdot n \log_2 2 + n \\
&= c \cdot n \log_2 n - c \cdot n + n \\
&= c \cdot n \log_2 n + (1 - c) \cdot n \\
&\leq c \cdot n \log_2 n \quad\quad\quad\quad \text{for } c \geq 1
\end{aligned}
$$

Choose $k \geq 3$ for induction base, $c \geq \max\{1, d\}$, $n_0 \geq 2$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ from induction base

Pitfalls

Don't use asymptotic notation in the induction step!

Example again: $T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + n$
Guess: $T(n) = \mathcal{O}(n)$

$\Rightarrow$ Show: $\exists c > 0, n_0 \in \mathbb{N} : T(n) \le c \cdot n$ for $n \ge n_0$

Induction hypothesis: $T(n) \le c \cdot n$ for some const. $c > 0$

Induction step:
$$T(n) = 2T(\lfloor \tfrac{n}{2} \rfloor) + n$$
$$\le 2(c \cdot \lfloor \tfrac{n}{2} \rfloor) + n$$
$$\le c \cdot n + n = (c+1) \cdot n = \cancel{\mathcal{O}(n)} \quad \textbf{WRONG !!}$$

Sometimes the "obvious" induction hypothesis doesn't work:

Example: $T(n) = T(\lfloor \frac{n}{2} \rfloor) + T(\lceil \frac{n}{2} \rceil) + 1$
Guess: $T(n) = \mathcal{O}(n)$

$\Rightarrow$ Show: $\exists c > 0, n_0 \in \mathbb{N} : T(n) \le c \cdot n$ for $n \ge n_0$

Induction hypothesis: $T(n) \le c \cdot n$ for some const. $c > 0$

Induction step:
$$T(n) = T(\lfloor \tfrac{n}{2} \rfloor) \; + \; T(\lceil \tfrac{n}{2} \rceil) \; + 1$$
$$\le c \cdot \lfloor \tfrac{n}{2} \rfloor \; + \; c \cdot \lceil \tfrac{n}{2} \rceil \; + 1$$
$$= c \cdot n \; + 1 \; > \; c \cdot n \qquad \Rightarrow \text{Induction fails}$$

But: weaker hypothesis $T(n) \le c \cdot n - d$ with $d \in \mathbb{R}$ works

Induction hypothesis: $\cancel{T(n) \le c \cdot n}$ for some const. $c > 0, d \in \mathbb{R}$
$$T(n) \le c \cdot n - d$$
Induction step:
$$T(n) = T(\lfloor \tfrac{n}{2} \rfloor) \; + \; T(\lceil \tfrac{n}{2} \rceil) \; + 1$$
$$\le c \cdot \lfloor \tfrac{n}{2} \rfloor \; + \; c \cdot \lceil \tfrac{n}{2} \rceil \; + 1 \quad \boxed{-d}$$
$$\boxed{-2 \cdot d}$$
$$= c \cdot n + 1 \cancel{> c \cdot n} \qquad \cancel{\Rightarrow \text{Induction fails}}$$
$$= c \cdot n - d + (1 - d) \; \le \; c \cdot n - d \quad \text{for } d \ge 1$$

Properties

advantage: more powerful than the other two methods

disadvantage: two proofs needed for Θ ($\mathcal{O}$ and Ω)

How to make the right guess?
○ similarity to known recurrence relations
○ recursion tree

How to get the right approach?
○ look at additional function
○ in case of doubt, try a second time
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