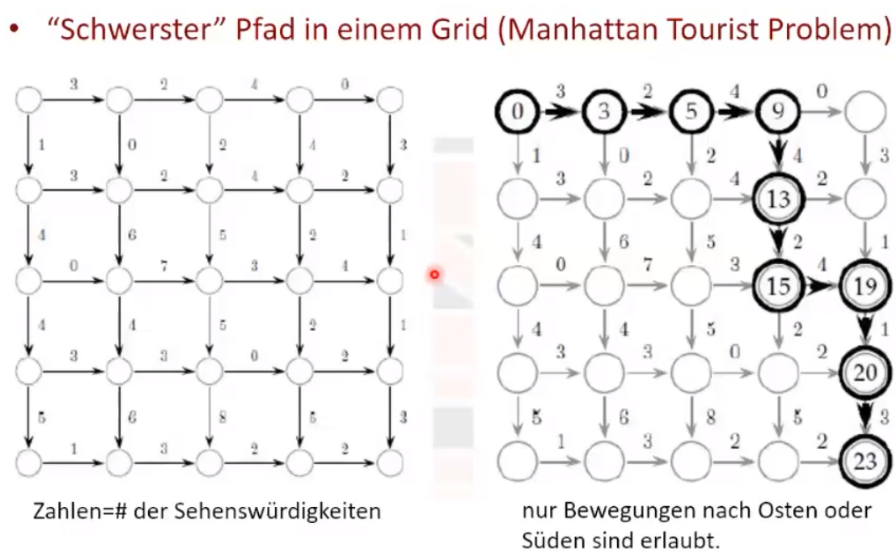
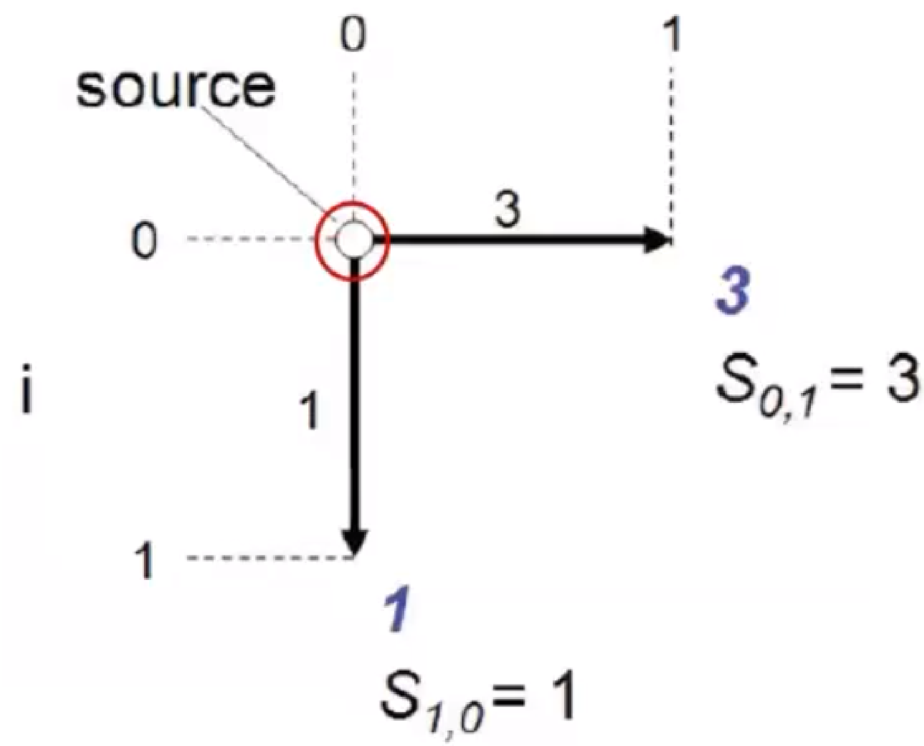
Eigenschaften

- Gesamtlösung baut auf Teillösungen eines Problems auf
- base cases
  - non-overlapping subproblems
- Teilprobleme müssen nicht voneinander unabhängig sein
  - unllike [[Divide & Conquer]] (only base cases)
- Memorization
  - Teilergebnisse werden wiederverwendet
  - use subproblem's solution if available otherwise compute recursively

Manhattan Tourist Problem

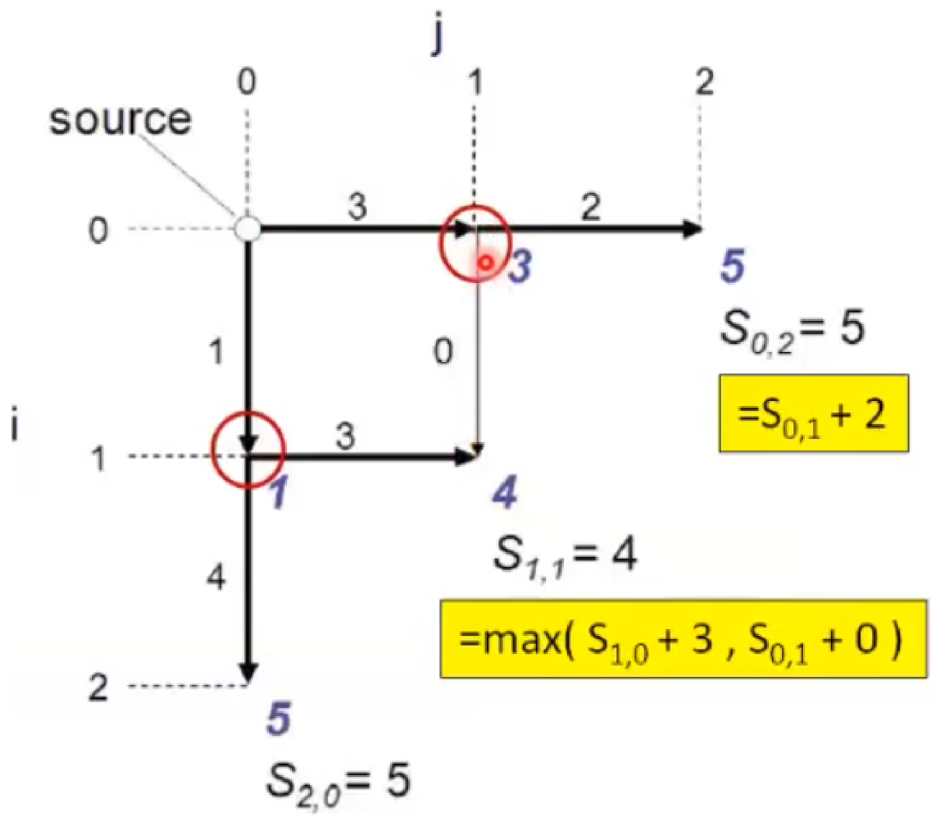


-



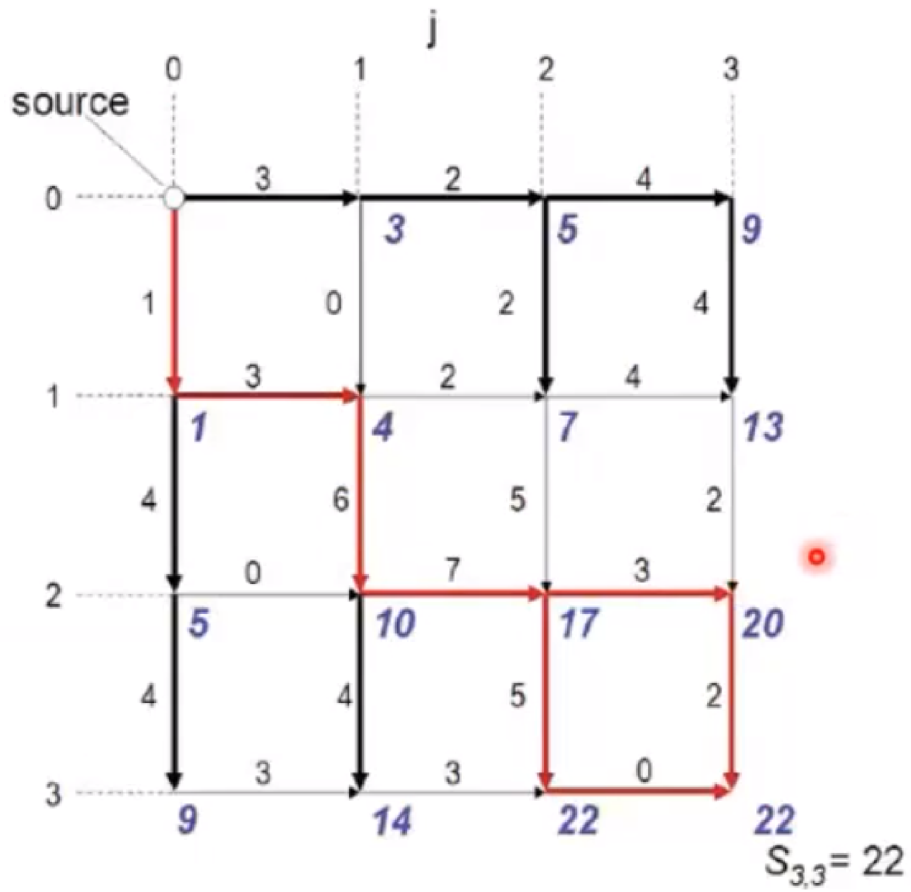
0
1
source
3
0
3
i
1
$S_{0,1} = 3$
1
1
$S_{1,0} = 1$

-



j
0
1
2
source
3
2
0
3
5
1
0
$S_{0,2} = 5$
i
$=S_{0,1} + 2$
1
3
1
4
4
$S_{1,1} = 4$
$=\max(S_{1,0} + 3 , S_{0,1} + 0)$
2
5
$S_{2,0} = 5$

-

- 

Relate Computation

- time complexity to solve SP
- if solutions to other SPs already known
- overall runtime = sum of relate computation of all SPs

Fibonacci subproblems: F(0),F(1), … ,F(n)

Relate computation = adding two numbers

`F(i)=F(i-1) + F(i-2)`

Fibonacci numbers are large, use up to O(n) bits:

$$O(n/w)$$

Runtime:
- #subproblems = n

$\Rightarrow$ runtime $O\left(\#subproblems \cdot \frac{n}{w}\right) = O\left(\frac{n^2}{w}\right)$

- assumption: wordsize w addition in $O(1)$ time

Bowling

- **Input:**
  sequence of $\boldsymbol{n}$ **bowling pins**, numbered with integers $\in [-K, K]$

  You can either hit **single pins**, or **two adjacent pins**.
  Throw as many balls as you want (you don't have to hit all pins)

  Goal: Maximize your points
  - hit a single pin with number $x_i \rightarrow x_i$ points
  - hit two adjacent pins with numbers x and $y \rightarrow x \cdot y$ points

  - **Subproblems:** $S[i]$: Optimal number of points that we can get with the first i pins (prefix problem)
  - **Original problem:** $S[n]$
  - **Base case:** $S[0] = 0$ points (empty prefix)

  Relation: if $i \geq 2$: $S[i] = \max\{S[i-1], S[i-1] + x_i, S[i-2] + x_{i-1} \cdot x_i\}$
  if $i = 1$: $S[i] = \max\{S[i-1],\ S[i-1] + x_i\}$

- - **Topological order:** S[0], S[1], S[2], …, S[n]

  ```
  S[0]=0, S[1]=max{0,x_1}    //base case
  FOR i=2 to n
        S[i]= max{S[i-1],S[i-1]+x_i,S[i-2]+x_{i-1}·x_i}
  Return S[n]
  ```

  "for

- **Runtime:** $\sum_{S}$ relate computation of $S = O(n) \cdot\ O(1) = O(n)$
  subproblem

Rucksack problem

- **Knapsack**
  Input:
  n items: i-th item has weight $\boldsymbol{w_i}$ and value $\boldsymbol{v_i}$
  W: the weight capacity of the knapsack.

  Question: What's the maximum value of items you can pack such that their weights' sum does not exceed W?

  Subproblems: $S[i, x]$, $0 \leq i \leq n$, $0 \leq x \leq W$:
  best value we can get with first i items with "using" capacity x
  Original problem: $S[n, W]$ (all items, full capacity)
  Base case: $S[0, x] = 0$ for $0 \leq x \leq W$; $S[i, 0] = 0$ for $0 \leq i \leq n$
  Relate: $S[i, x] = \textbf{if } w_i \leq x\text{: } \max\{S[i-1, x], S[i-1, x - w_i] + v_i\}$ **else**: $S[i-1, x]$

- **Topological order:** `For i=0 to n: For x=0 to W:` $S[i, x]$

  ```
  For i from 0 to n:
  For x from 0 to W:
        If i == 0 or x == 0: S(i,x) = 0
        else if: w_{i-1} <= x: S(i,x)= max(v_{i-1} + S(i-1,x - w_{i-1}), S(i-1,x))
        else: S(i,x) = S(i-1,x)
  ```

- `Return S(n,W)`

		0	1	2	3	4	5	6	7	8	9	10
	0	0	0	0	0	0	0	0	0	0	0	0
$w_1 = 2, v_1 = 3$	**1**	0	0	3	3	3	3	3	3	3	3	3
$w_2 = 3, v_2 = 4$	**2**	0	0	3	4	4	7	7	7	7	7	7
$w_3 = 4 , v_3 = 5$	**3**	0	0	3	4	5	7	8	9	9	12	12
$w_4 = 5 , v_4 = 6$	**4**	0	0	3	4	5	7	8	9	10	11	13
$w_5 = 6 , v_4 = 7$	**5**	0	0	3	4	5	7	8	9	10	11	13

- **Runtime:**
  Relate computation: $O(1)$
  #subproblems: $O(n \cdot W)$

- Is this polynomial time?

  No. W can be exponential in the input.

  Knapsack is **NP-complete.**