

## Potentialgleichung

- $\Delta u = 0$ 
  - Lösung dieser Gleichung kommt mit zweiter Funktion  $v$ 
    - \*  $v$  erfüllt ebenso Gleichung
    - \*  $v$  heißt konjugiert harmonische Funktion
    - \*  $v$  ist mit  $u$  über CR-Gleichungen verbunden
      - ♦ bzw.  $f(x + iy) = u(x, y) + iv(x, y)$
  - jede Lösung ergibt quellenfreies Gradientenfeld  $\text{grad}(u)$ 
    - \*  $\text{grad}(u)$  senkrecht auf  $\text{grad}(v) \iff \langle \text{grad}(u), \text{grad}(v) \rangle = 0$
    - \* Äquipotentiallinien (Niveaulinien) von  $u$  und  $v$  senkrecht aufeinander
      - ♦ außer  $\text{grad}(v) = \text{grad}(u) = 0$

♦  $\{(x, y) \in U \mid u(x, y) = \cos(x)\} \quad \{(x, y) \in U \mid v(x, y) = \sin(x)\}$

## Bestimmen von $v(x, y)$

- Gradientenfeld gegeben  $\implies v(x, y)$  als Integral

$$v(x, y) = \int_{(x_0, y_0)}^{(x, y)} \underbrace{\frac{\partial v}{\partial x}(\xi, \eta)}_{\frac{\partial u}{\partial y}(\xi, \eta)} d\xi + \underbrace{\frac{\partial v}{\partial y}(\xi, \eta)}_{-\frac{\partial u}{\partial x}(\xi, \eta)} d\eta = \int_{(x_0, y_0)}^{(x, y)} -\frac{\partial u}{\partial y}(\xi, \eta) d\xi + \frac{\partial u}{\partial x}(\xi, \eta) d\eta$$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} \iff \Delta u = 0$

- Integrabilitätsbedingung prüfen  $\implies$  wegunabhängig

- \*  $u_{xx} = u_{yy}$
- $u_x = v_y$
- $u_y = -v_x$
- Beispiel WTF?

Bsp:  $u(x, y) = e^{-y}(x \cos(x) - (y+1) \sin(x))$

\*  $\frac{\partial u}{\partial x} = e^{-y}(\cos(x) - x \sin(x) - (y+1) \sin(x)) = e^{-y}(-y \cos(x) - x \sin(x))$   
 $\frac{\partial^2 u}{\partial x^2} = e^{-y}(-y \sin(x) - \sin(x) - x \cos(x)) = e^{-y}(-(y+1) \sin(x) - x \cos(x))$   
 $\frac{\partial^2 u}{\partial y^2} = -e^{-y}(x \cos(x) - (y+1) \sin(x)) = e^{-y}(-(y+1) \sin(x) - x \cos(x))$   
 $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} \implies \Delta u = 0$

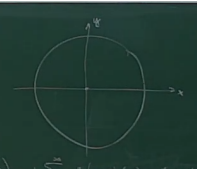
\*  $\frac{\partial u}{\partial x} = e^{-y}(-y \cos(x) - x \sin(x)) \stackrel{!}{=} \frac{\partial v}{\partial y} = e^{-y}(-y \sin(x) - x \cos(x))$   
 $\frac{\partial u}{\partial y} = -e^{-y}(x \cos(x) - (y+1) \sin(x)) \stackrel{!}{=} -\frac{\partial v}{\partial x} = -e^{-y}(x \sin(x) - (y+1) \cos(x))$   
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## Randwertaufgabe

$\Delta u = 0$   $U = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \}$  Einheitskreisscheibe  
 Randwerte/gegeben:  $u(\cos(t), \sin(t)) = g(t)$  gegeben  $0 \leq t < 2\pi$   
 Platz C/F  
 $u(x,y) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n} = \sum_{n=0}^{\infty} r^n \left( \alpha_n \cos(n\varphi) + \beta_n \sin(n\varphi) \right)$   
 Randbedingung:  $x = r \cos(\varphi)$   $y = r \sin(\varphi)$   $z = r e^{i\varphi}$   $\left| \begin{aligned} &= \sum_{n=0}^{\infty} r^n \left( \alpha_n \cos(n\varphi) + \beta_n \sin(n\varphi) \right) + i \sum_{n=1}^{\infty} r^n \left( \alpha_n \sin(n\varphi) - \beta_n \cos(n\varphi) \right) \\ &= g(\varphi) \end{aligned} \right.$



## Poissonsche Integralformel

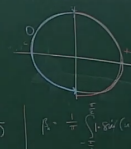
- sei  $g: [0, 2\pi] \rightarrow \mathbb{R}$  eine Funktion
  - $u(r \cos(\varphi), r \sin(\varphi)) = \frac{1}{2\pi} \int_0^{2\pi} g(t) \frac{1-r^2}{1-2r \cos(t-\varphi) + r^2} dt$
  - Lösung der Potentialgleichung  $\Delta u = 0$  mit

$"u(\cos(\varphi), \sin(\varphi)) = g(\varphi)" \text{ für } \varphi \in [0, 2\pi]$   
 $\lim_{r \rightarrow 1-} u(r \cos(\varphi), r \sin(\varphi)) = g(\varphi) \text{ f. u. u.}$   
 in allen Stetigkeitspunkten von  $g$ .

\*

- Beispiel

Bsp:  $g(t) = \begin{cases} 1 & \text{für } -\frac{\pi}{2} < t < \frac{\pi}{2} \\ 0 & \text{sonst} \end{cases} \quad t \in [-\pi, \pi]$   
 $\alpha_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 dt = \frac{1}{2}$   
 $\alpha_n = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 \cdot \cos(nt) dt = \frac{2 \sin(n \cdot \frac{\pi}{2})}{\pi n}$   
 $\alpha_{2k+1} = \frac{2 \sin((2k+1) \cdot \frac{\pi}{2})}{\pi (2k+1)} = \frac{2(-1)^k}{\pi (2k+1)}$   
 $\beta_n = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 \cdot \sin(nt) dt = 0$   
 $\beta_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(t) dt = 0$   
 $u(r \cos(\varphi), r \sin(\varphi)) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{(-1)^k}{\pi (2k+1)} r^{2k+1} \cos((2k+1)\varphi)$   
 $g(\varphi) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{(-1)^k}{\pi (2k+1)} \cos((2k+1)\varphi)$   
 $\text{für } \varphi \in (-\frac{\pi}{2}, \frac{\pi}{2}) \quad \varphi = \frac{\pi}{2}$   
 $g(\varphi) = 1 = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{(-1)^k}{\pi (2k+1)}$



## Fourier-Reihe

- sei  $g: I \rightarrow \mathbb{R}$ 
  - $I \dots$  Intervall der Länge  $2\pi$
  - Koeffizienten
    - \*  $\alpha_0 = \frac{1}{2\pi} \int_I g(t) dt$
    - \*  $\alpha_n = \frac{1}{\pi} \int_I g(t) \cos(nt) dt$
    - \*  $\beta_n = \frac{1}{\pi} \int_I g(t) \sin(nt) dt$
- wenn Reihe konvergiert, dann gilt in allen Stetigkeitspunkten von  $g$ 
  - $g(\varphi) = \alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \cos(n\varphi) + \beta_n \sin(n\varphi))$
- Lösung der Potentialgleichung
  - $u(r \cos(\varphi), r \sin(\varphi)) = \alpha_0 + \sum_{n=1}^{\infty} r^n (\alpha_n \cos(n\varphi) + \beta_n \sin(n\varphi))$

$u(r \cos(\varphi), r \sin(\varphi)) = \alpha_0 + \sum_{n=1}^{\infty} r^n (\alpha_n \cos(n\varphi) + \beta_n \sin(n\varphi))$

- Beispiele:

$$\begin{aligned}
 \gamma &= \gamma_1 \cup \gamma_2 \quad \gamma(t) = |t| \quad \text{für } |t| \leq \tau \\
 \gamma_1 &= \frac{1}{\tau} \int_{-\tau}^{\tau} |t| dt = \frac{2}{\tau} \int_0^{\tau} t dt = \frac{1}{\tau} \cdot \frac{t^2}{2} \Big|_0^{\tau} = \frac{\tau^2}{2\tau} = \frac{\tau}{2} \\
 \gamma_2 &= \frac{1}{\tau} \int_{-\tau}^{\tau} |t| \cos(ut) dt = \frac{2}{\tau} \int_0^{\tau} t \cos(ut) dt = \frac{2}{\tau} \left[ t \cdot \frac{\sin(ut)}{u} - \int_0^{\tau} \frac{\sin(ut)}{u} dt \right] \\
 &= \frac{2}{\tau} \left[ \frac{\cos(ut)}{u^2} \right]_0^{\tau} = \frac{2}{\tau} \left[ \frac{\cos(u\tau) - 1}{u^2} \right] = \frac{2}{\tau} \cdot \frac{\cos(u\tau) - 1}{u^2} \\
 \gamma_2 &= 0 \quad \gamma_{2,1} = \frac{2}{\tau} \cdot \frac{1}{(2u+1)^2} \\
 |t| &= \frac{\tau}{2} - \frac{4}{\tau} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos((2n+1)t) \quad \text{für } |t| \leq \tau \\
 u(x \cos(\varphi), y \sin(\varphi)) &= \frac{\tau}{2} - \frac{4}{\tau} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)^2} \cos((2n+1)\varphi) \\
 &= \frac{\tau}{2} - \frac{4}{\tau} \left( \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \dots \right)
 \end{aligned}$$

[[Komplexe Kurvenintegrale]]