Application

- Idea: "guess" an asymptotic bound ($\mathcal{O}$, Ω) and prove it by mathematical induction

  Example: $T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + n$

  Guess: $T(n) = \mathcal{O}(n \log n)$

  We have to show: $\exists c > 0, n_0 \in \mathbb{N}$ such that

  $T(n) \leq c \cdot n \log n$ for all $n \geq n_0$

  Induction hypothesis: $T(n) \leq c \cdot n \log_2 n$ for some const. $c > 0$

  Induction base: $T(k) = \Theta(1) \leq d$ for small constant k and some constant $d > 0$ (by def.)

  Induction step: $T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + n$

  $$\leq 2 \cdot (c \cdot \lfloor \tfrac{n}{2} \rfloor \log_2 \lfloor \tfrac{n}{2} \rfloor) + n \overset{?}{\leq} c \cdot n \log_2 n$$

- Example: $T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + n$, $\quad T(n) \overset{?}{=} \mathcal{O}(n \log n)$

  $$\begin{aligned}
  T(n) &= 2T(\lfloor \tfrac{n}{2} \rfloor) + n \\
  &\leq 2 \cdot \left(c \lfloor \tfrac{n}{2} \rfloor \log(\lfloor \tfrac{n}{2} \rfloor)\right) + n \\
  &\leq c \cdot n \log_2(\tfrac{n}{2}) + n \\
  &= c \cdot n \log_2 n - c \cdot n \log_2 2 + n \\
  &= c \cdot n \log_2 n - c \cdot n + n \\
  &= c \cdot n \log_2 n + (1 - c) \cdot n \\
  &\leq c \cdot n \log_2 n \qquad\qquad \text{for } c \geq 1
  \end{aligned}$$

  Choose $k \geq 3$ for induction base, $c \geq \max\{1, d\}$, $n_0 \geq 2$

  from induction base

Pitfalls

Don't use asymptotic notation in the induction step!

Example again: $T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + n$
Guess: $T(n) = \mathcal{O}(n)$

$\Rightarrow$ Show: $\exists c > 0, n_0 \in \mathbb{N} : T(n) \leq c \cdot n$ for $n \geq n_0$

Induction hypothesis: $T(n) \leq c \cdot n$ for some const. $c > 0$

Induction step:

$$
\begin{aligned}
T(n) &= 2T(\lfloor \tfrac{n}{2} \rfloor) + n \\
&\leq 2(c \cdot \lfloor \tfrac{n}{2} \rfloor) + n \\
&\leq c \cdot n + n = (c+1) \cdot n \cancel{= \mathcal{O}(n)} \quad \textbf{WRONG !!}
\end{aligned}
$$

- Sometimes the "obvious" induction hypothesis doesn't work:

Example: $T(n) = T(\lfloor \frac{n}{2} \rfloor) + T(\lceil \frac{n}{2} \rceil) + 1$
Guess: $T(n) = \mathcal{O}(n)$

$\Rightarrow$ Show: $\exists c > 0, n_0 \in \mathbb{N} : T(n) \leq c \cdot n$ for $n \geq n_0$

Induction hypothesis: $T(n) \leq c \cdot n$ for some const. $c > 0$

Induction step:

$$
\begin{aligned}
T(n) &= T(\lfloor \tfrac{n}{2} \rfloor) + T(\lceil \tfrac{n}{2} \rceil) + 1 \\
&\leq c \cdot \lfloor \tfrac{n}{2} \rfloor + c \cdot \lceil \tfrac{n}{2} \rceil + 1 \\
&= c \cdot n + 1 > c \cdot n \qquad \Rightarrow \text{Induction fails}
\end{aligned}
$$

But: weaker hypothesis $T(n) \leq c \cdot n - d$ with $d \in \mathbb{R}$ works

- Induction hypothesis: $\cancel{T(n) \leq c \cdot n}$ for some const. $c > 0, d \in \mathbb{R}$
  $T(n) \leq c \cdot n - d$

Induction step:

$$
\begin{aligned}
T(n) &= T(\lfloor \tfrac{n}{2} \rfloor) + T(\lceil \tfrac{n}{2} \rceil) + 1 \\
&\leq c \cdot \lfloor \tfrac{n}{2} \rfloor + c \cdot \lceil \tfrac{n}{2} \rceil + 1 \qquad \boxed{-d} \\
&= c \cdot n + 1 \cancel{> c \cdot n \qquad \Rightarrow \text{Induction fails}} \qquad \boxed{-2 \cdot d} \\
&= c \cdot n - d + (1 - d) \leq c \cdot n - d \quad \text{for } d \geq 1
\end{aligned}
$$

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Properties

- advantage: more powerful than the other two methods

  disadvantage: two proofs needed for Θ ($\mathcal{O}$ and Ω)

  How to make the right guess?
  - similarity to known recurrence relations
  - recursion tree

  How to get the right approach?
  - look at additional function
  - in case of doubt, try a second time