# Convex Hulls

Birgit Vogtenhuber

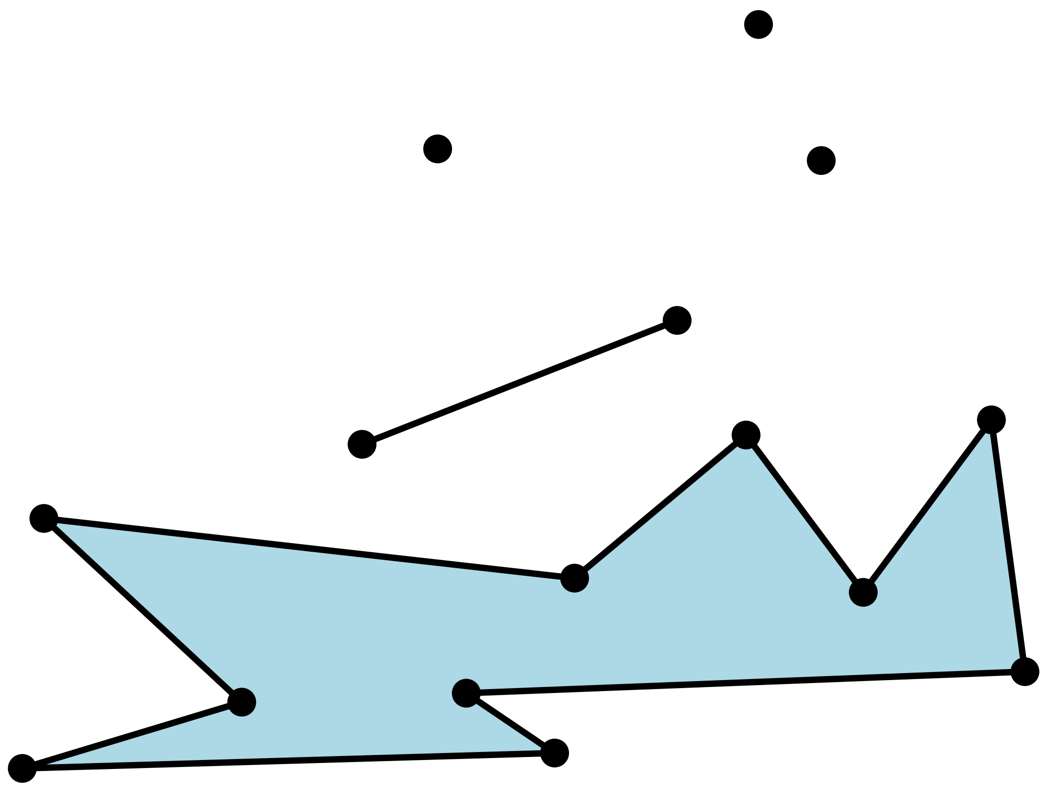
# Overview

- Introduction
- Algorithms for computing convex hulls
- A lower bound for the computational complexity of convex hull computation
- Summary

# Introduction

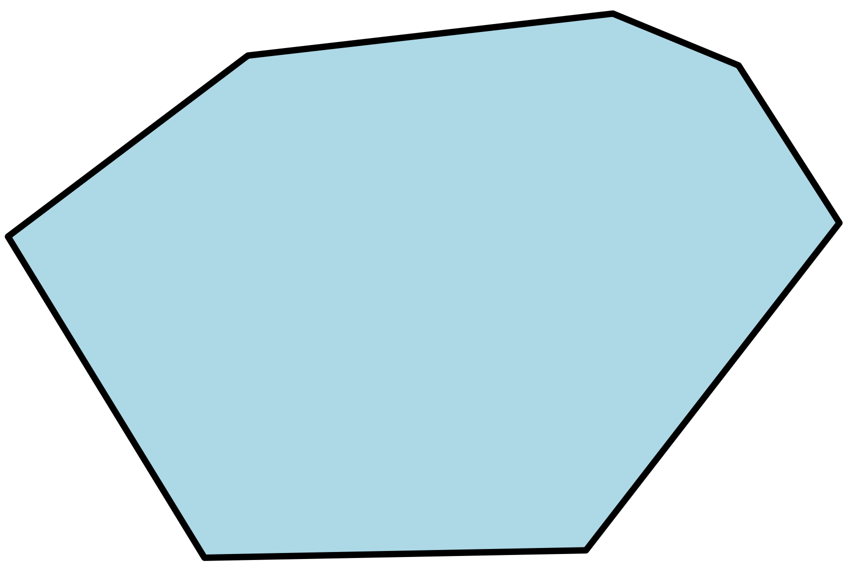
# Representing objects in space

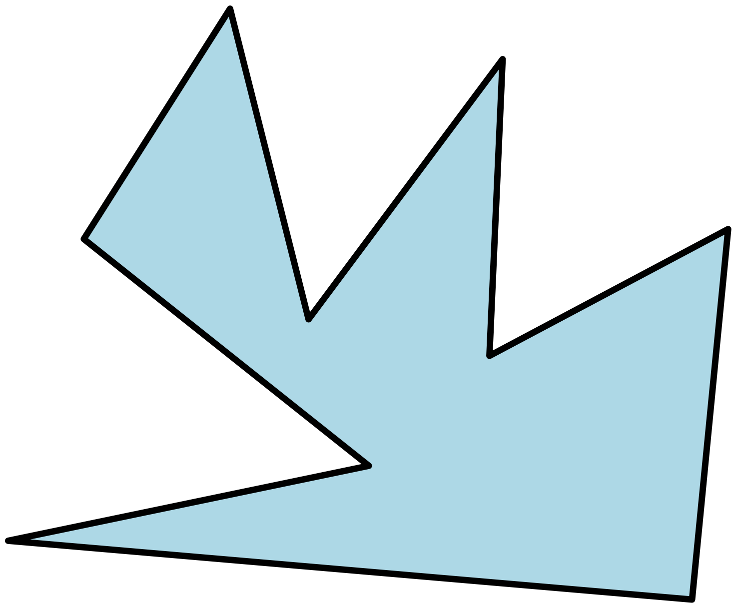
- Algorithmic solution of geometric problems: nice playground for developing algorithms.
- For a start: geometric objects in the plane.
- Constant-size description
  - Points / lines
  - Curves
- Compound objects
  - Line segments
  - Polygons



# Motivation

- Polygons represent 2-dimensional parts of the plane
- Complexity can be rather arbitrary
- Convex sets have many important properties
- Convex hull "simplifies" a set.

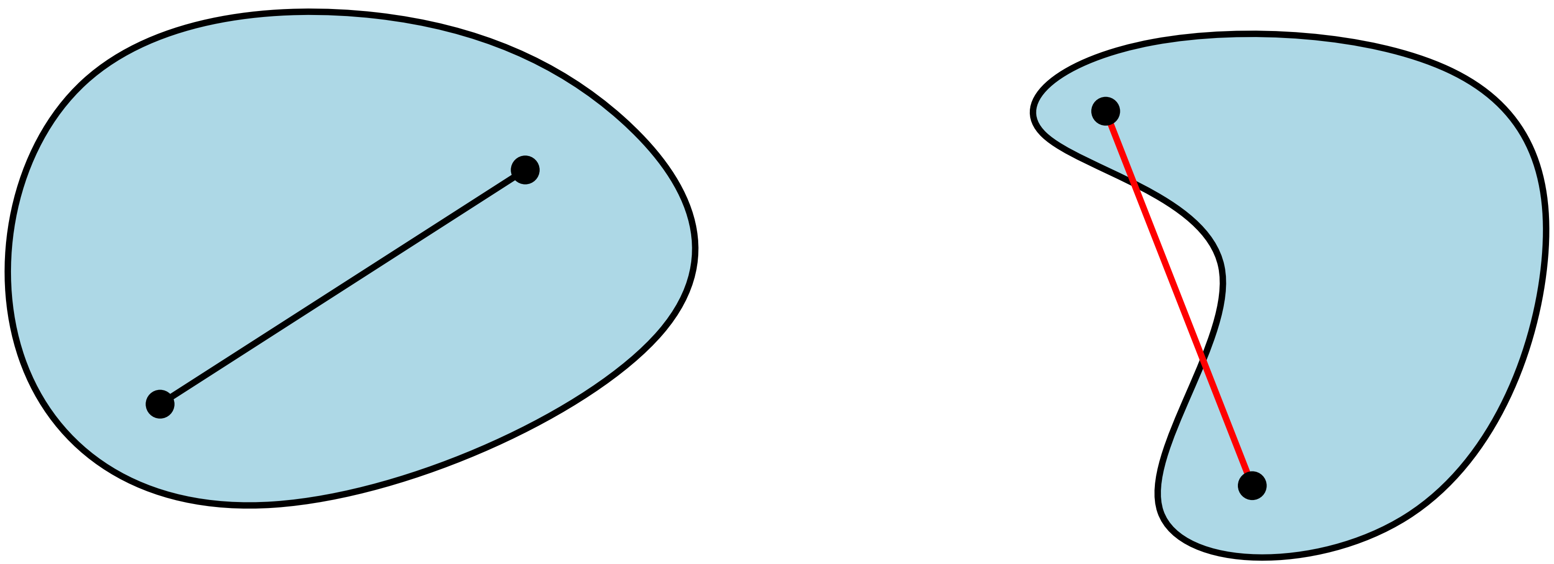




# Convex Sets

**Definition:** A set $P \subseteq \mathbb{R}^d$ is *convex* if the segment between two points of $P$ is also contained in $P$.

- $d$-dimensional real space
- Euclidean metric
- more general / different definitions exist

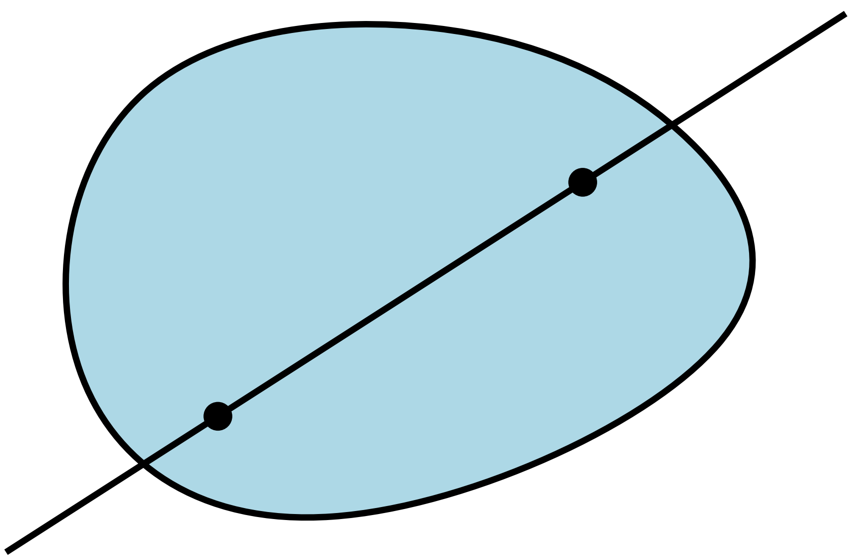


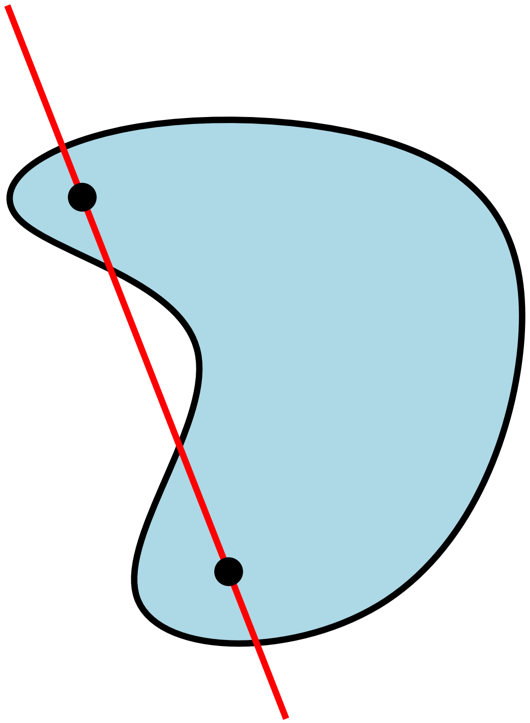
# Convex Sets

**Definition:** A set $P \subseteq \mathbb{R}^d$ is *convex* if the segment between two points of $P$ is also contained in $P$.

Equivalent: intersection of $P$ with a line is connected.

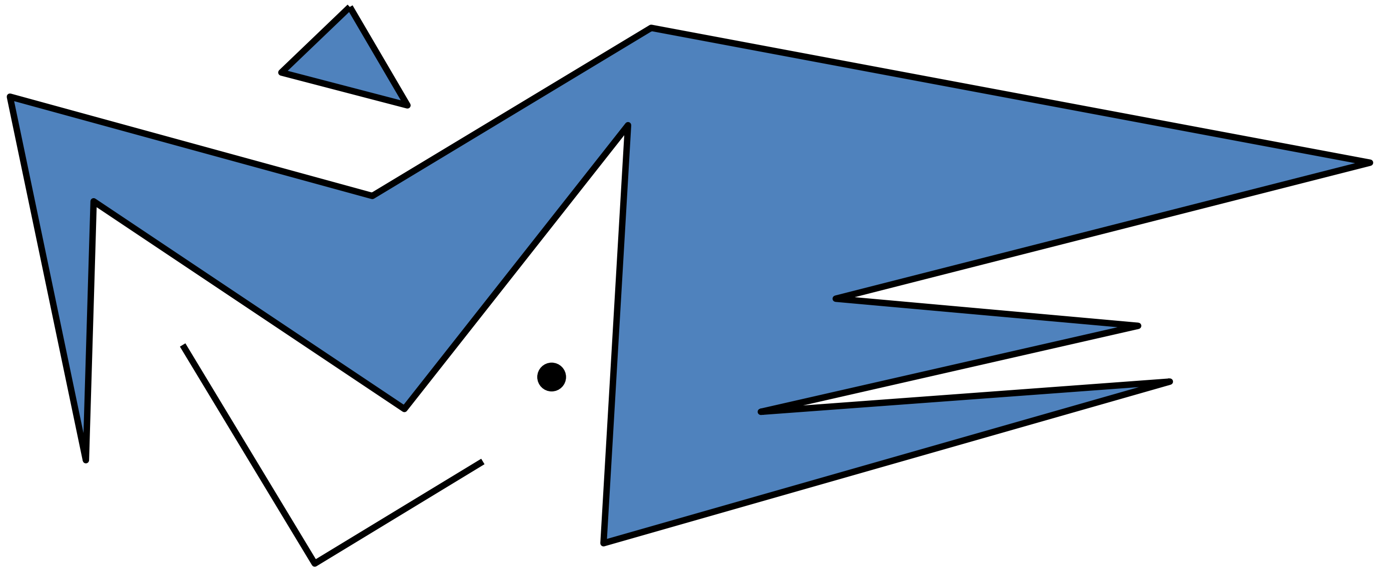
**Observation:** For any family $(P_i)_{i \in I}$ of convex sets, the intersection $\bigcap_{i \in I} P_i$ is convex.





# Convex Sets

- Convex sets are relatively easy to handle.
- We may want to “approximate” a set with a convex set.
- For a given set, we want a small convex set containing it, or actually, a smallest.



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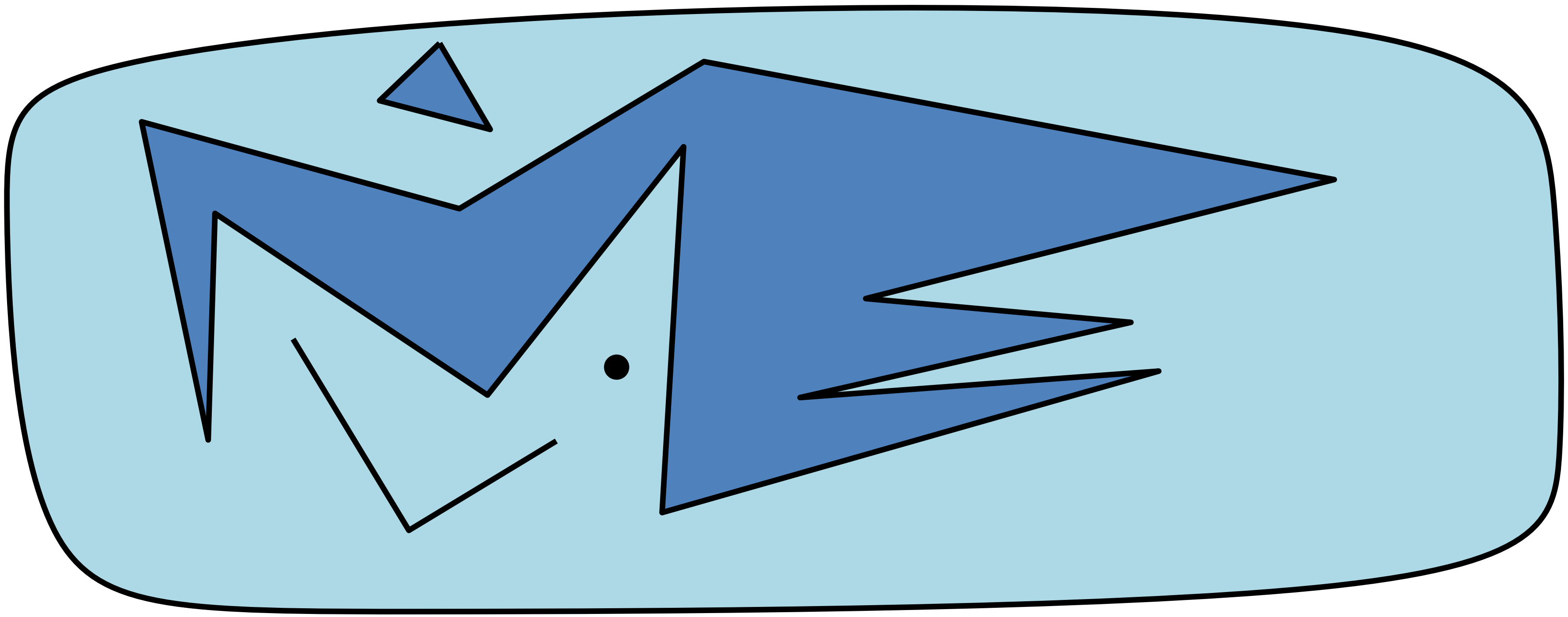

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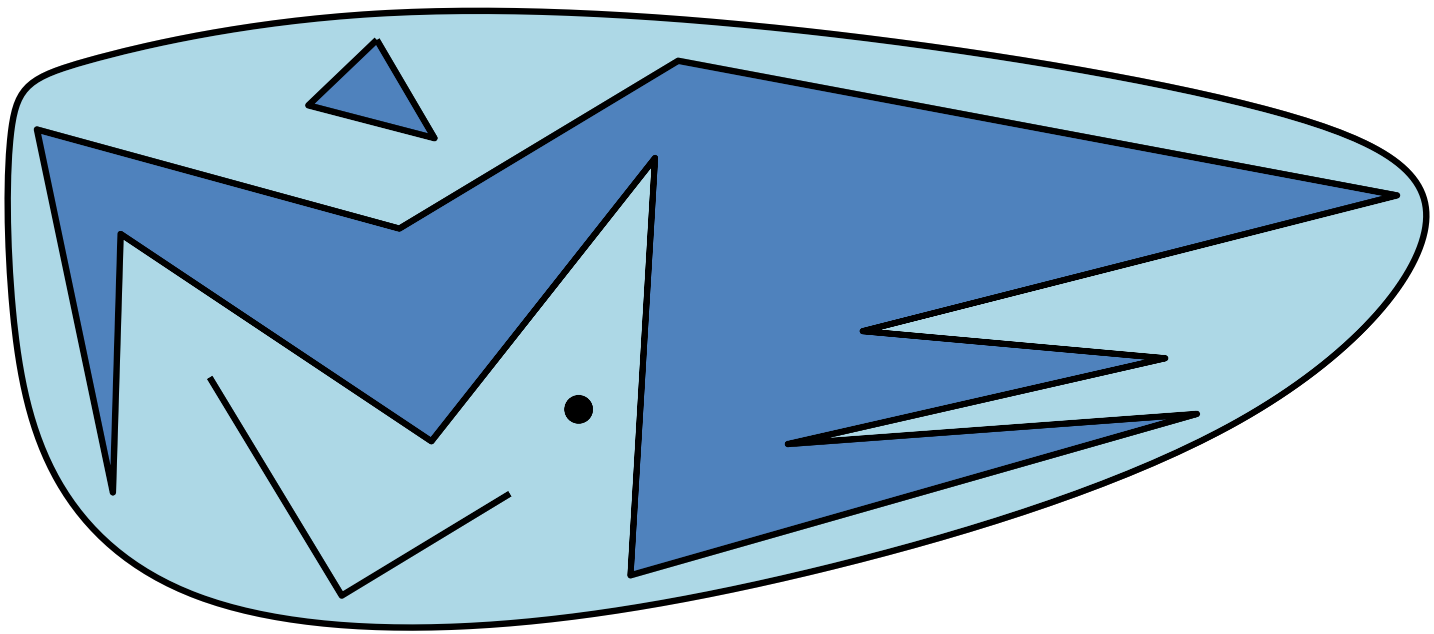

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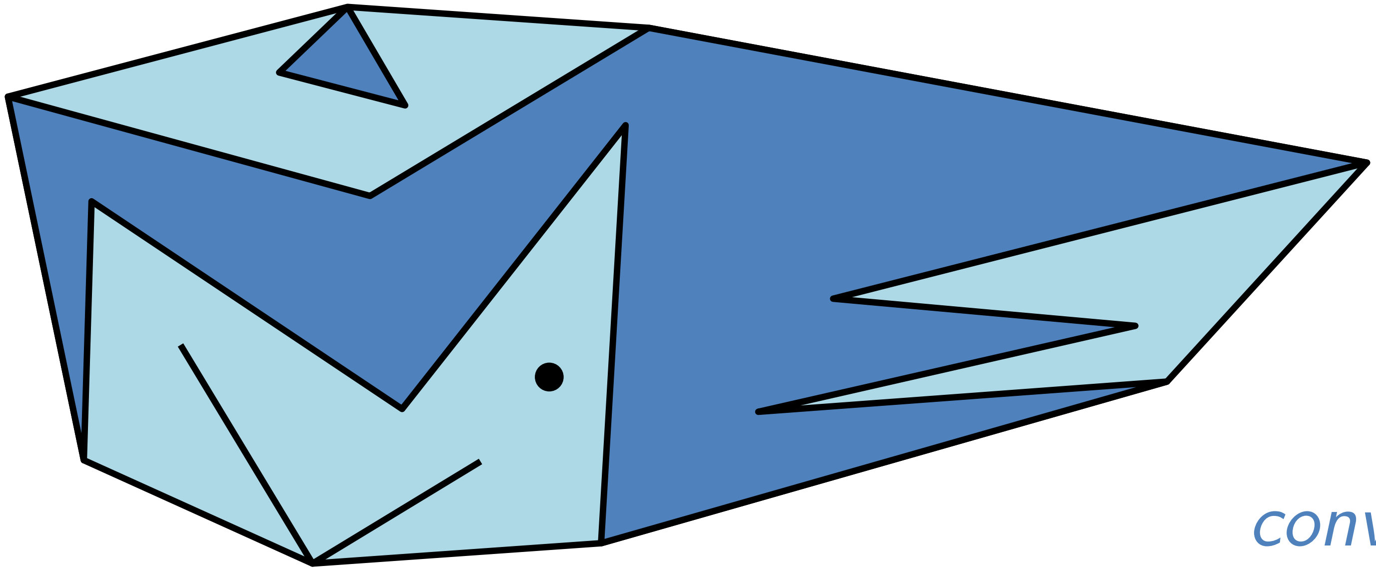
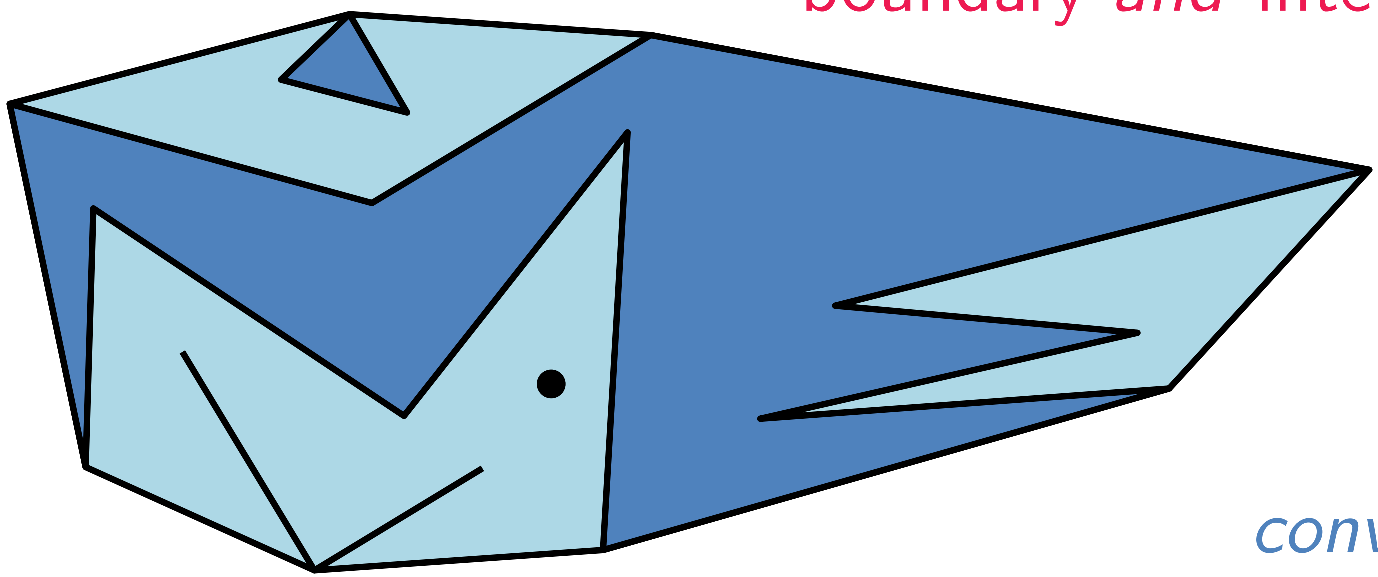


*convex hull*

# Convex Hull

**Definition:** The *convex hull* conv$(P)$ of a set $P \subseteq \mathbb{R}^d$ is the intersection of all convex supersets of $P$.

the "smallest" convex set
boundary *and* interior

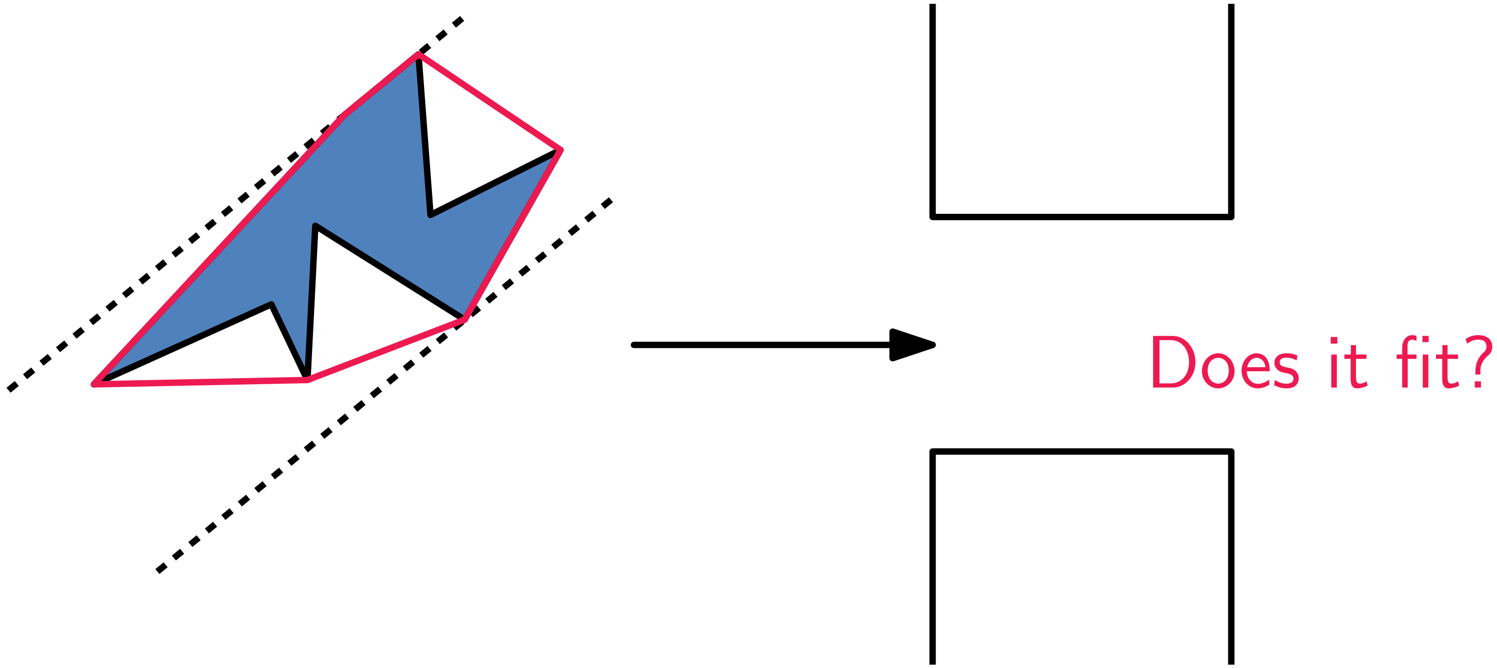


*convex hull*

# Why are we looking at this?

- Paramount in Computational Geomtry
  - geometric data structure
- Applications:

*Width* of a polygon



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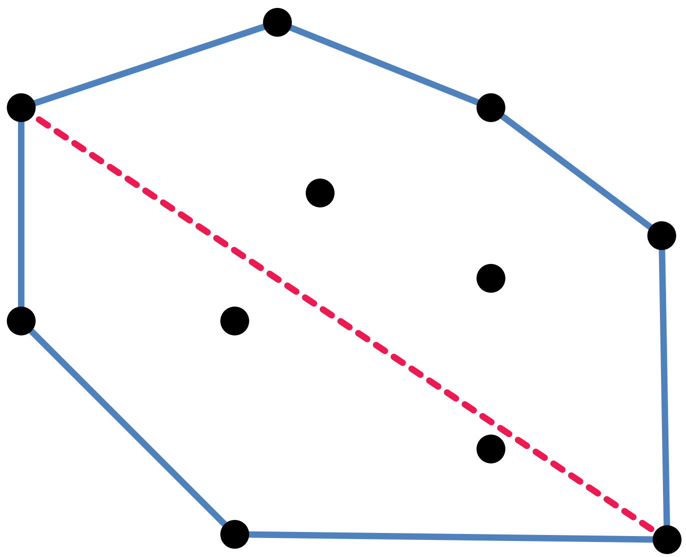
- Applications:

*Diameter* of a point set $S$: maximum distance between two points of $S$.



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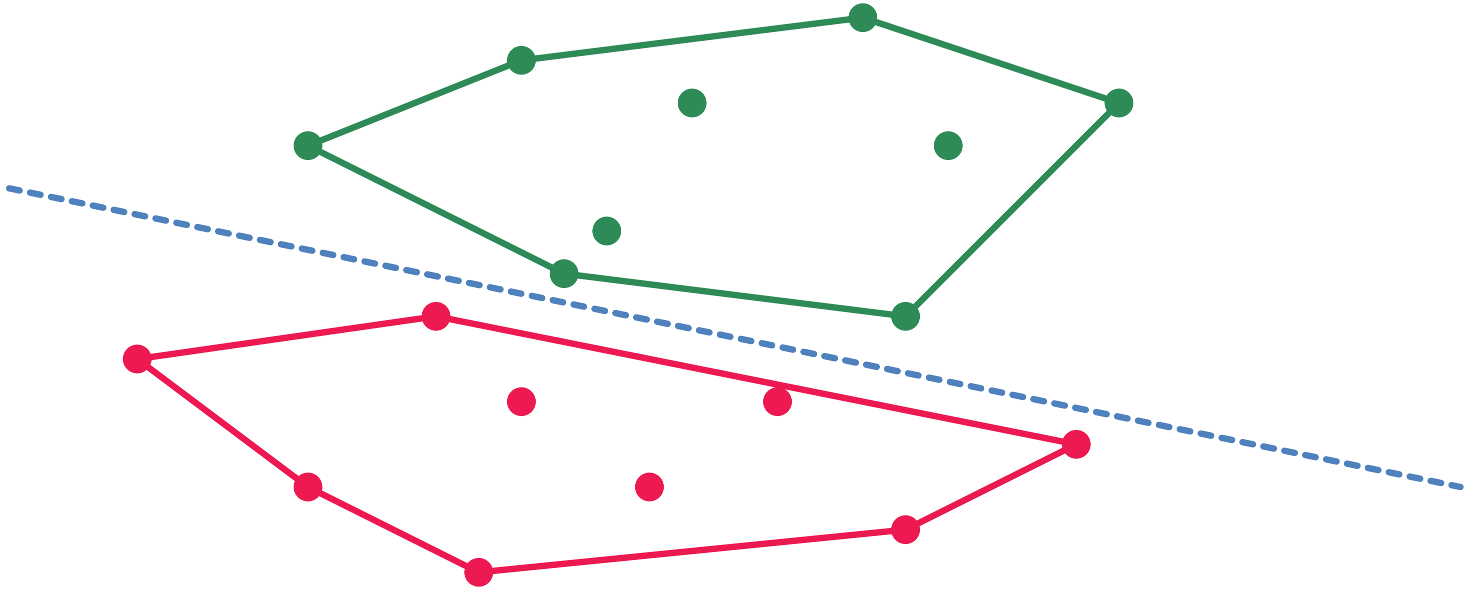
- Applications:

*Linear separation* of data points:



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- Paramount in Computational Geomtry
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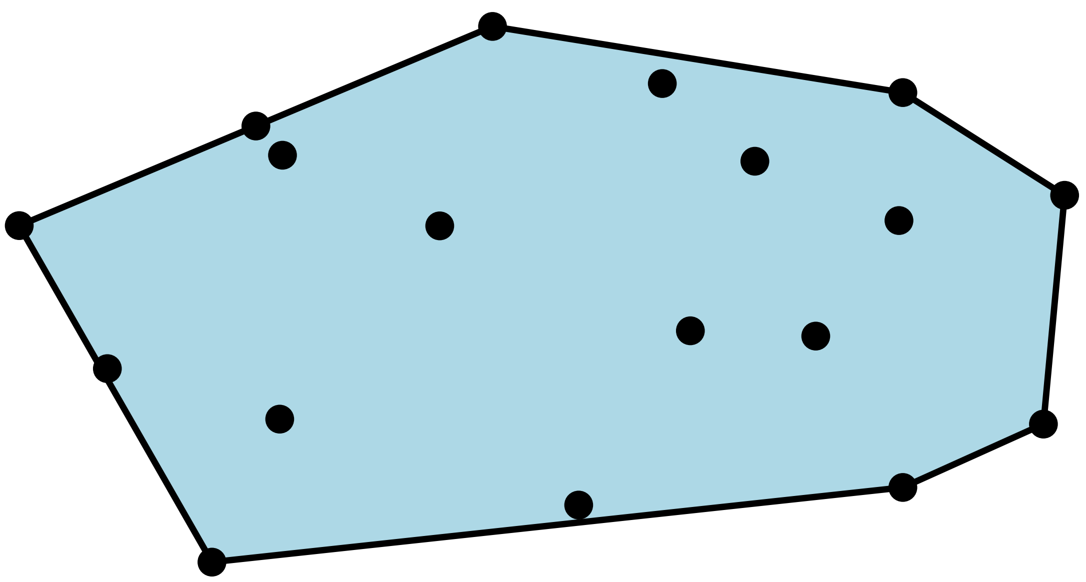
- Applications:

Many algorithms involve computing the convex hull of a point set as a sub-routine.

# Convex Hull Algorithms

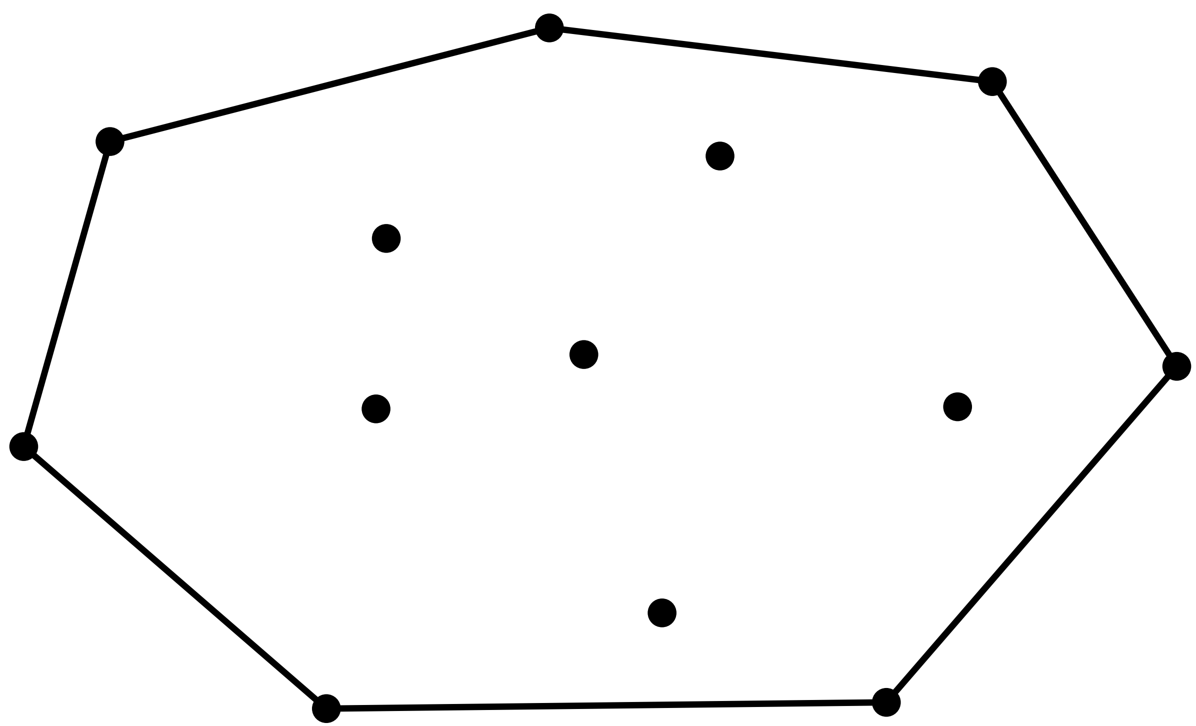
# Planar Convex Hulls

- How should we construct the convex hull of a set?
- Suitable sets needed
- Suitable representation of the hull needed
- Focus: finite point set in $\mathbb{R}^2$



# Planar Convex Hulls

- Convex hull of a point set: *convex polygon*.
- Convex hull of a polygon: convex hull of its vertex set.



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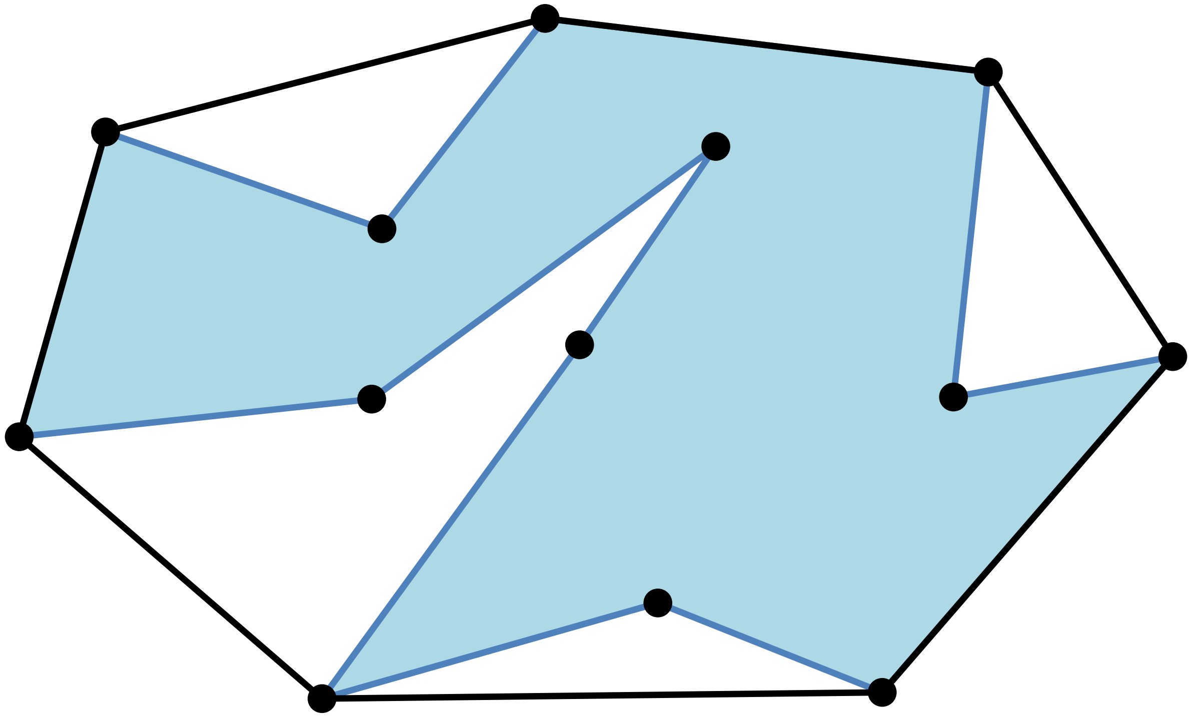
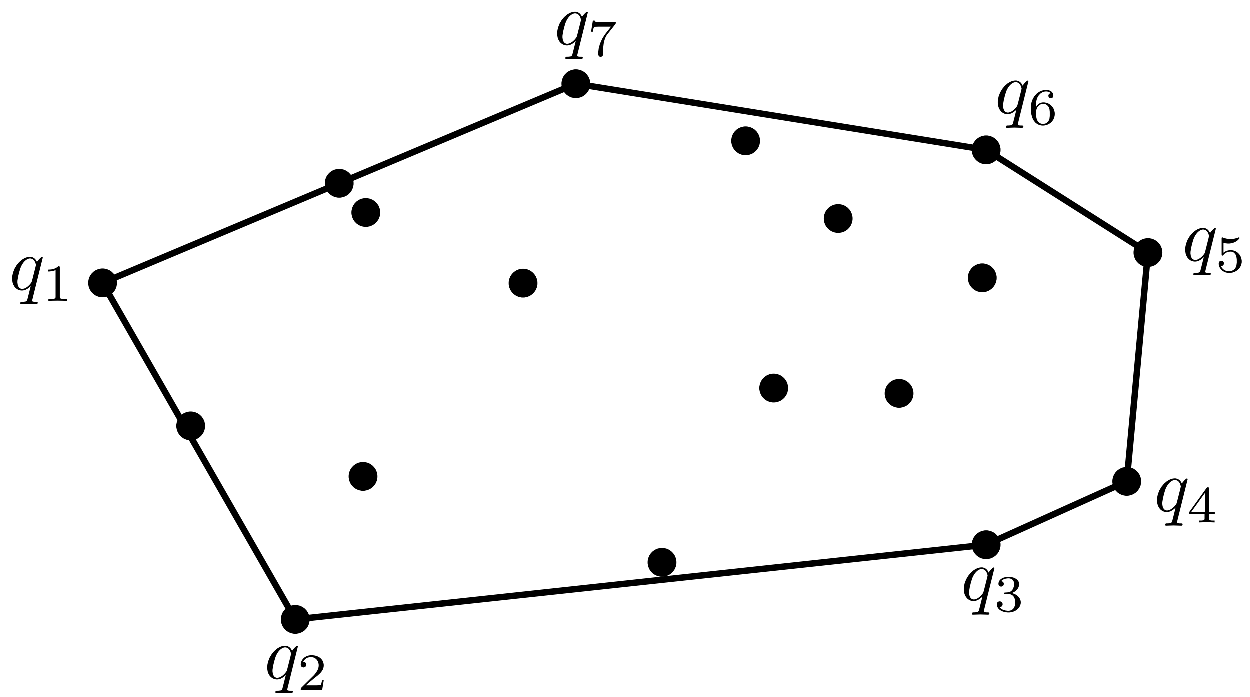


# Planar Convex Hulls

Input:
$P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^2$

Output: Sequence
$(q_1, \ldots, q_h)$ of vertices



# Planar Convex Hulls

maybe only a set?

Input:
$P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^2$

Output: Sequence
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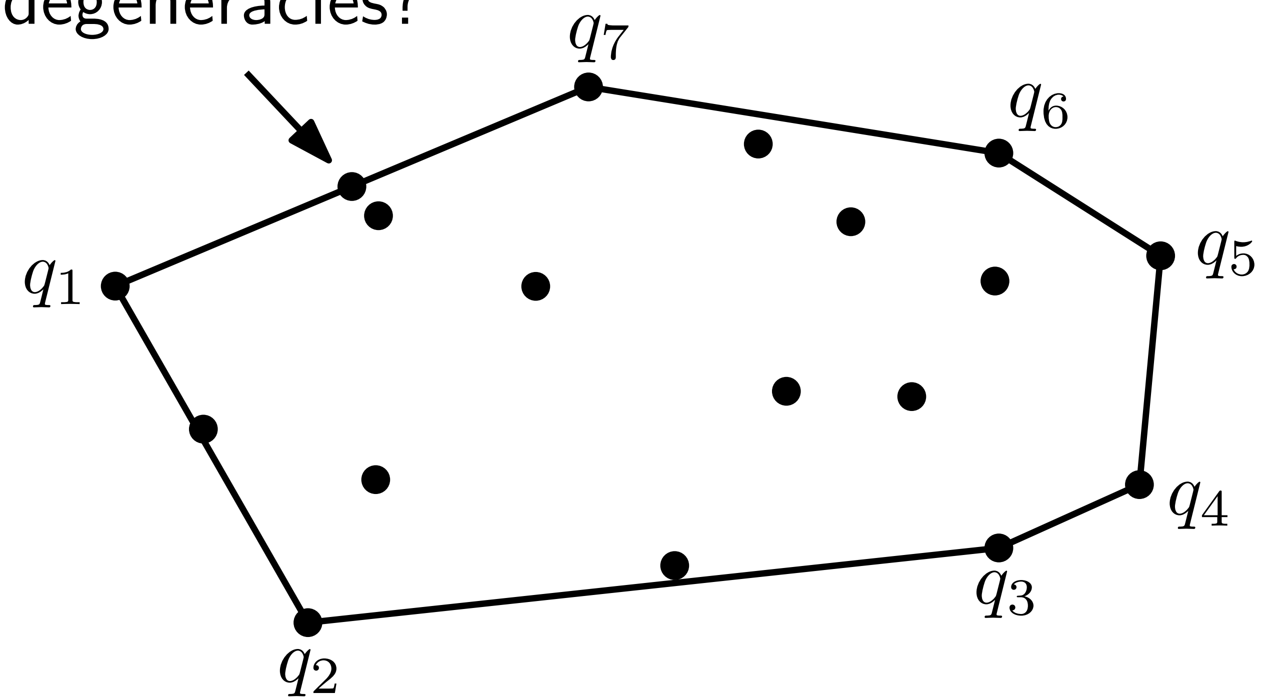
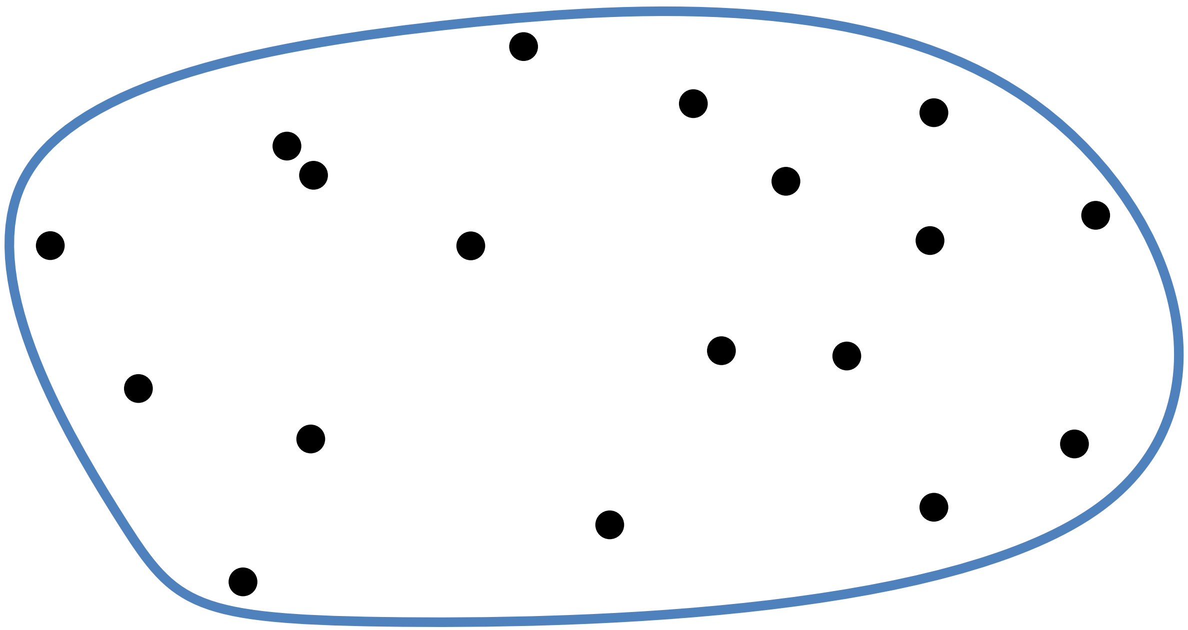
what about degeneracies?



# How does an algorithm “see” this?

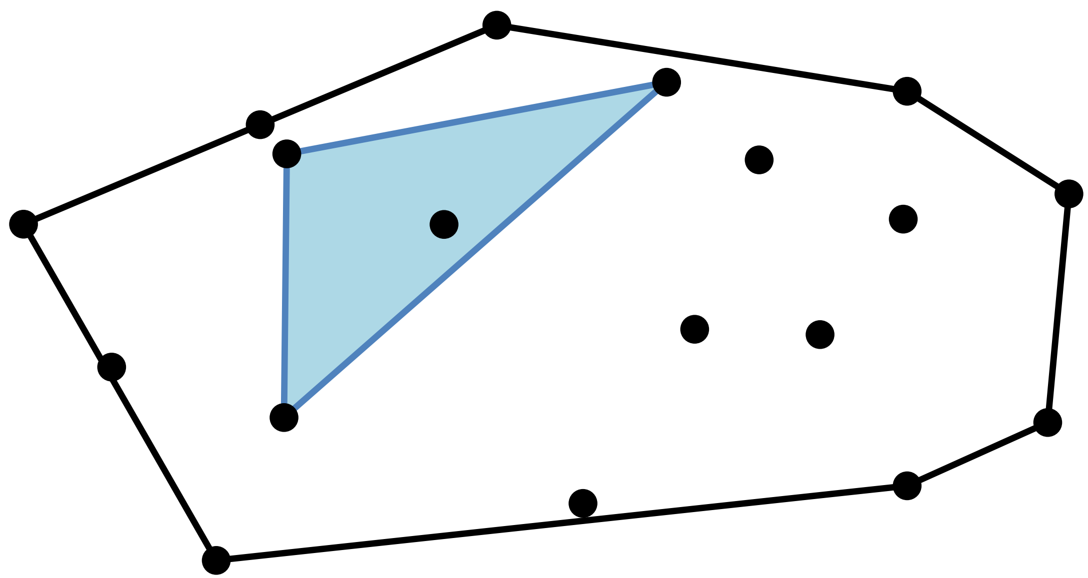


How can you find an extreme point?

| | |
|---|---|
| 1841 | 4167 |
| 5007 | 7880 |
| 8370 | 9968 |
| 9726 | 6983 |
| 1320 | 4617 |
| 860 | 6772 |
| 5694 | 771 |
| 2680 | 1338 |
| 7487 | 4582 |
| 5114 | 7406 |
| 8678 | 5231 |
| 397 | 621 |
| 643 | 5297 |
| 708 | 7428 |
| 2128 | 8275 |
| 1249 | 1906 |
| 4222 | 8732 |
| 9755 | 4487 |

# Properties of the convex hull

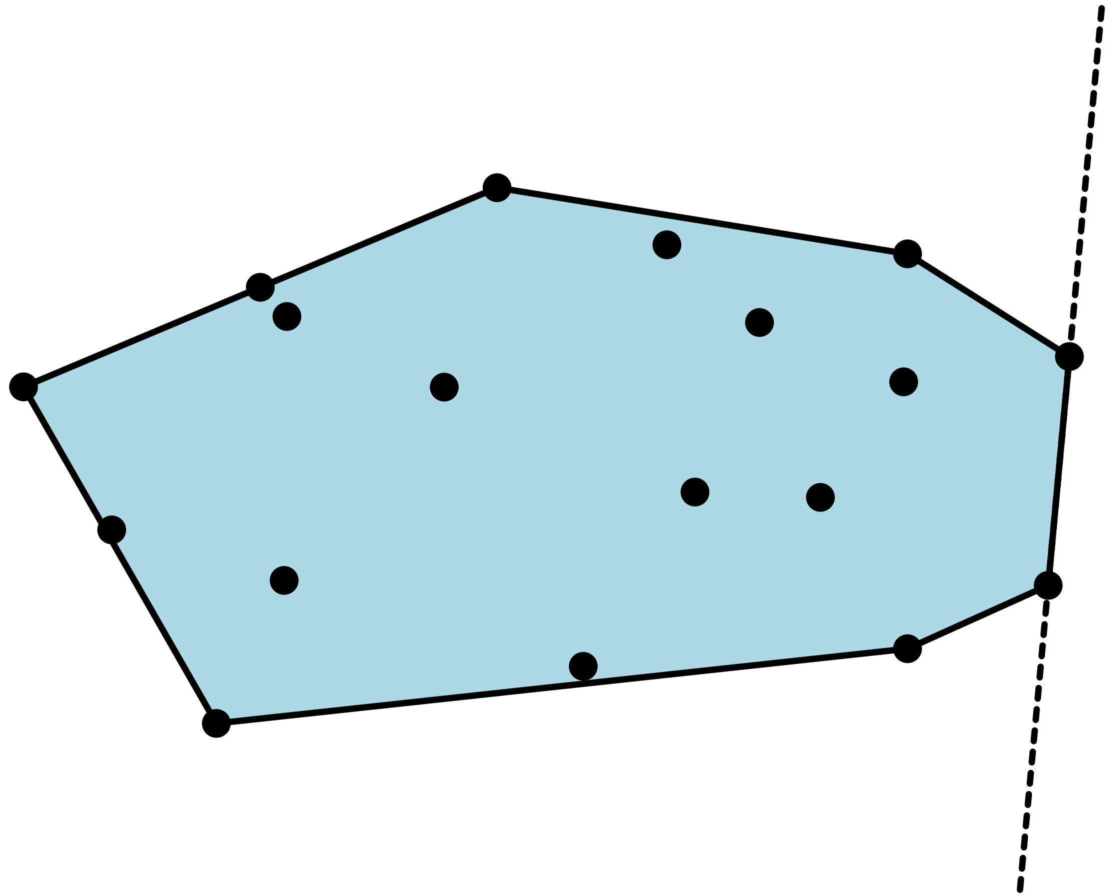
A point is on the convex hull boundary iff it is not contained in a triangle spanned by three points.



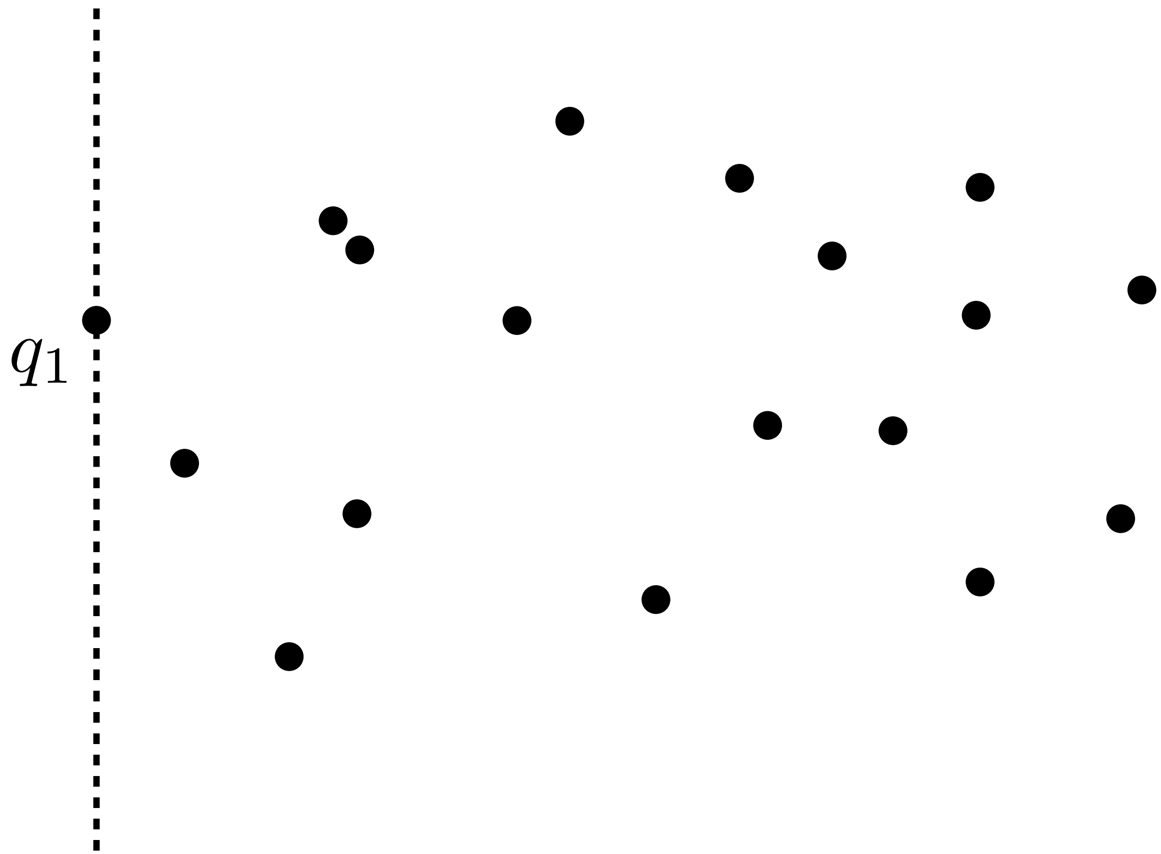
Trivial algorithm:
Check all points with all triangles in $O(n^4)$ time.

# Another Algorithm

Identify edges of the convex hull in $O(n^3)$ time.



# Jarvis' Wrap




- Start with extreme point

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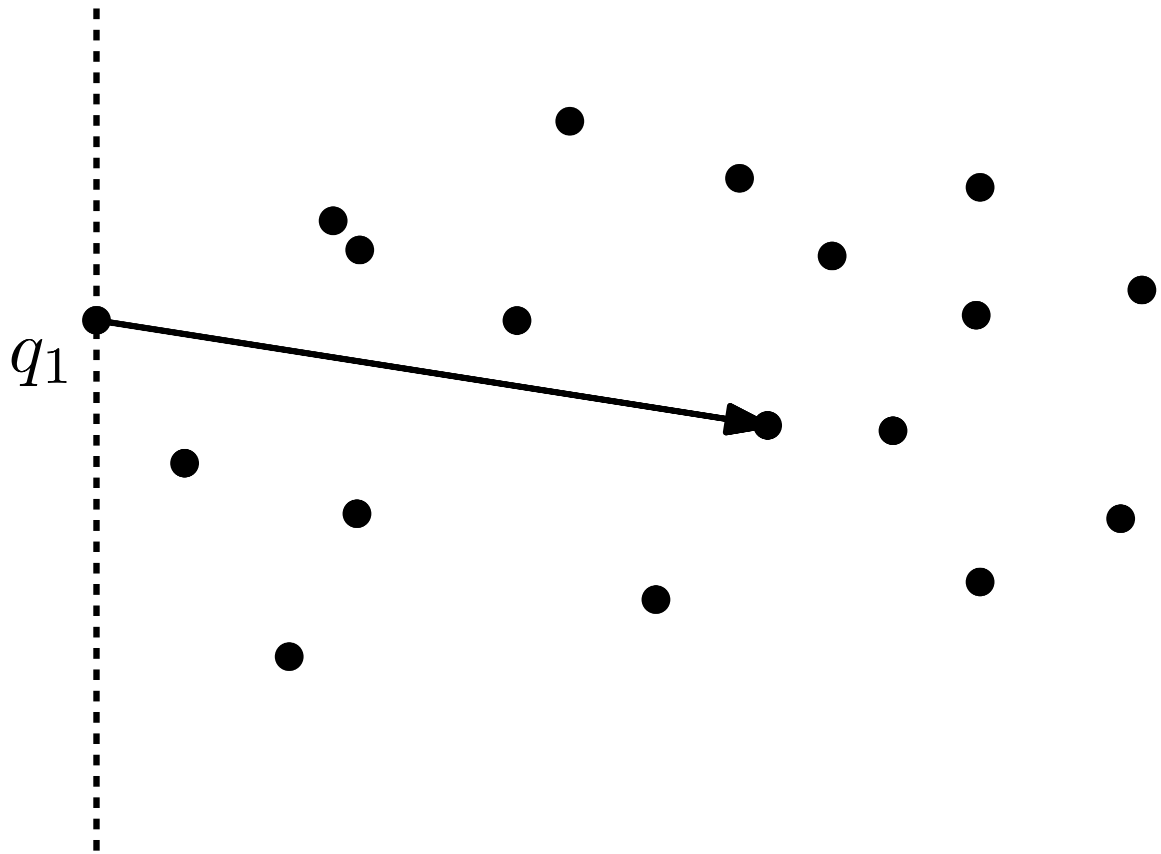


- Start with extreme point
- Find edges that have all other points to the left

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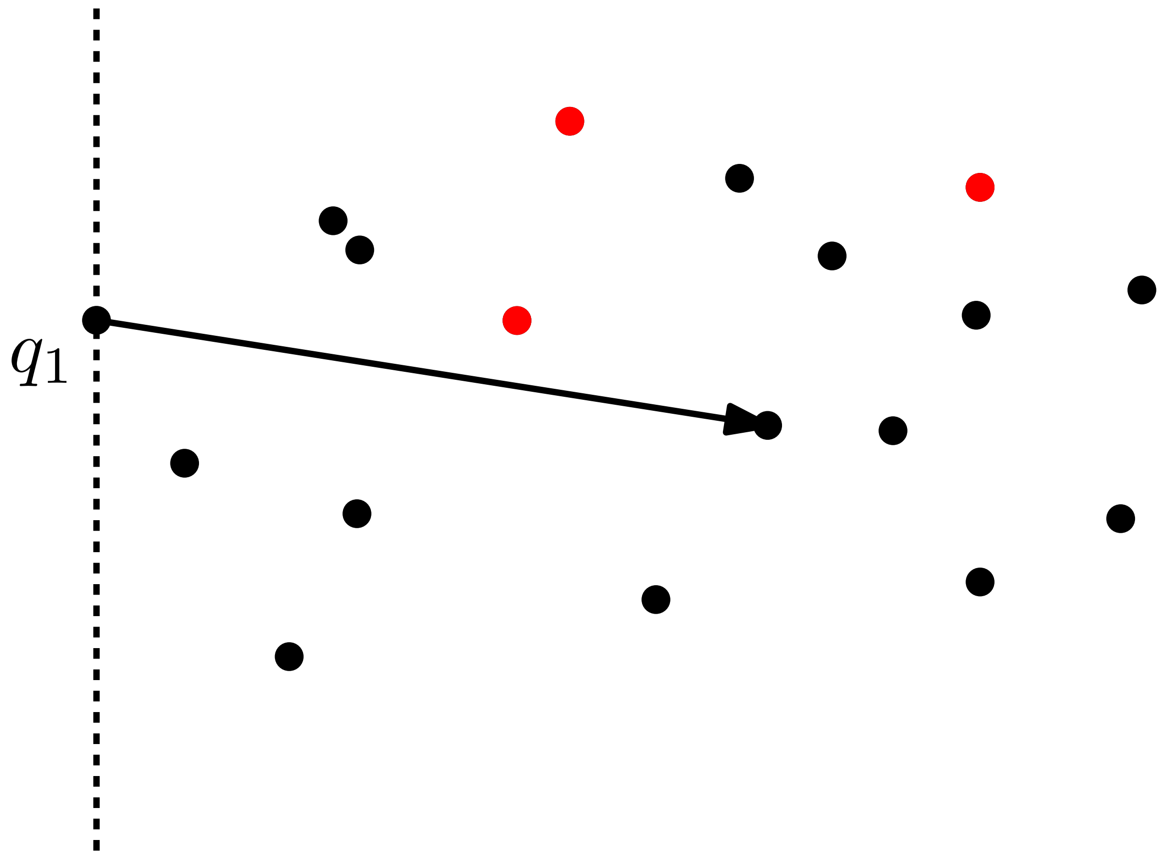


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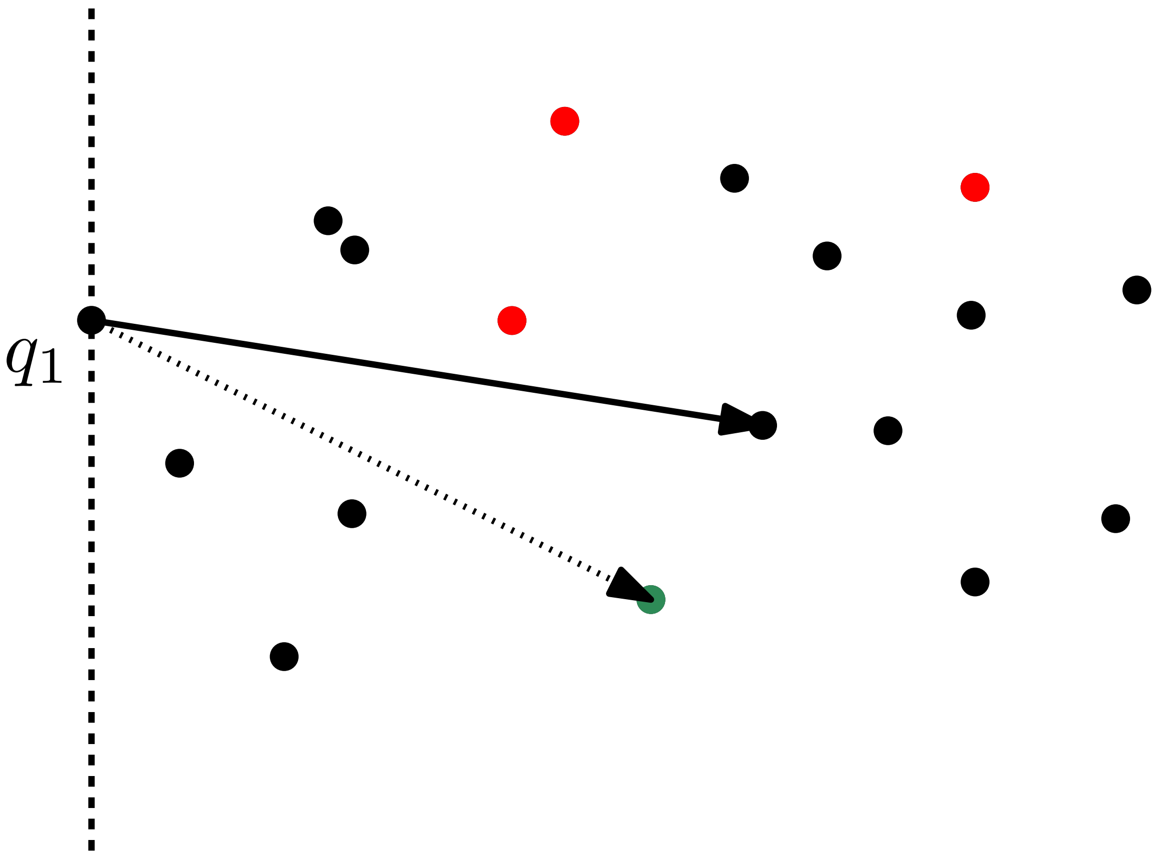


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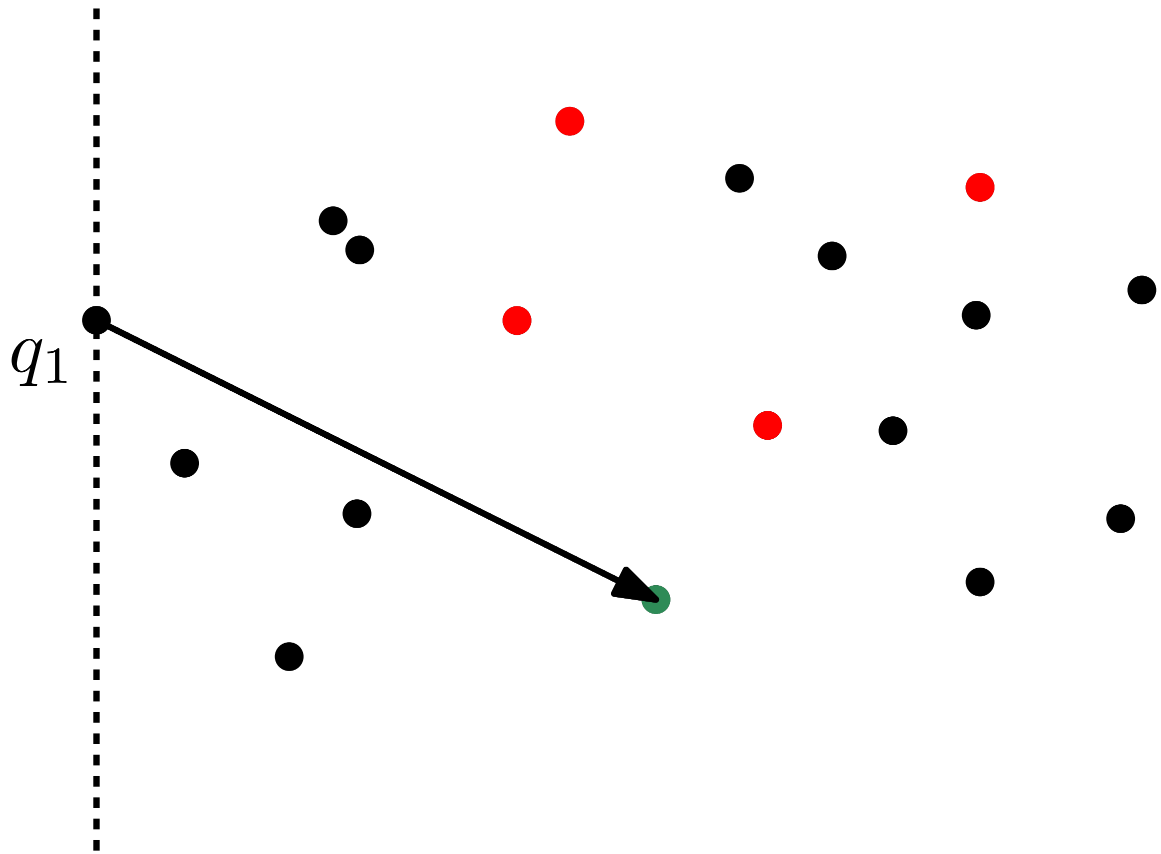


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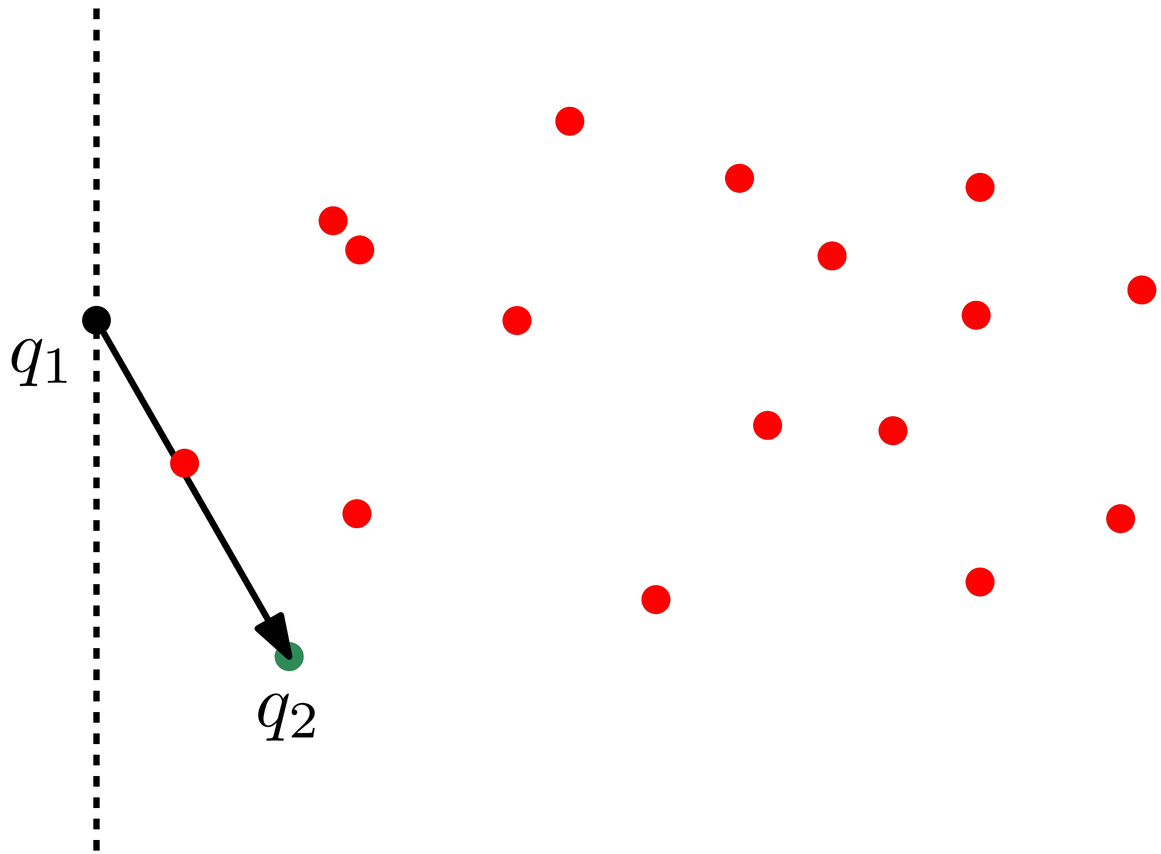


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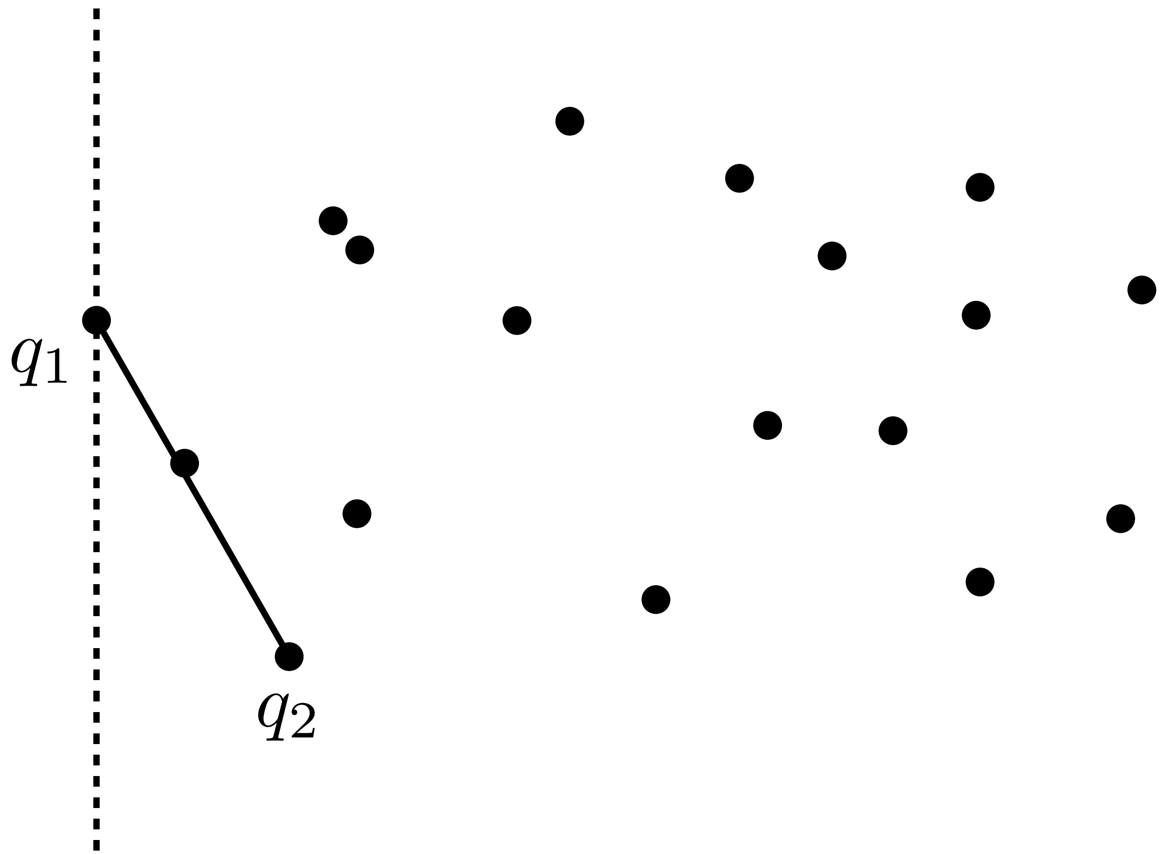


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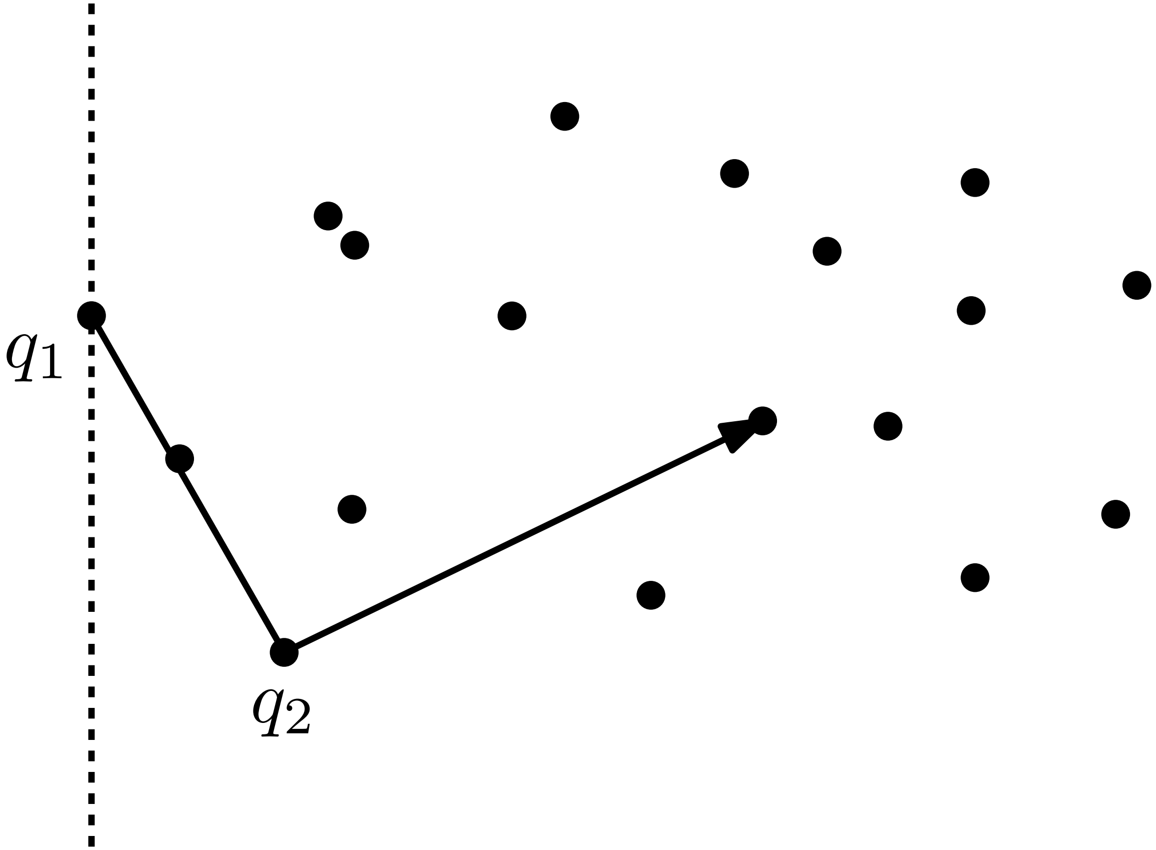


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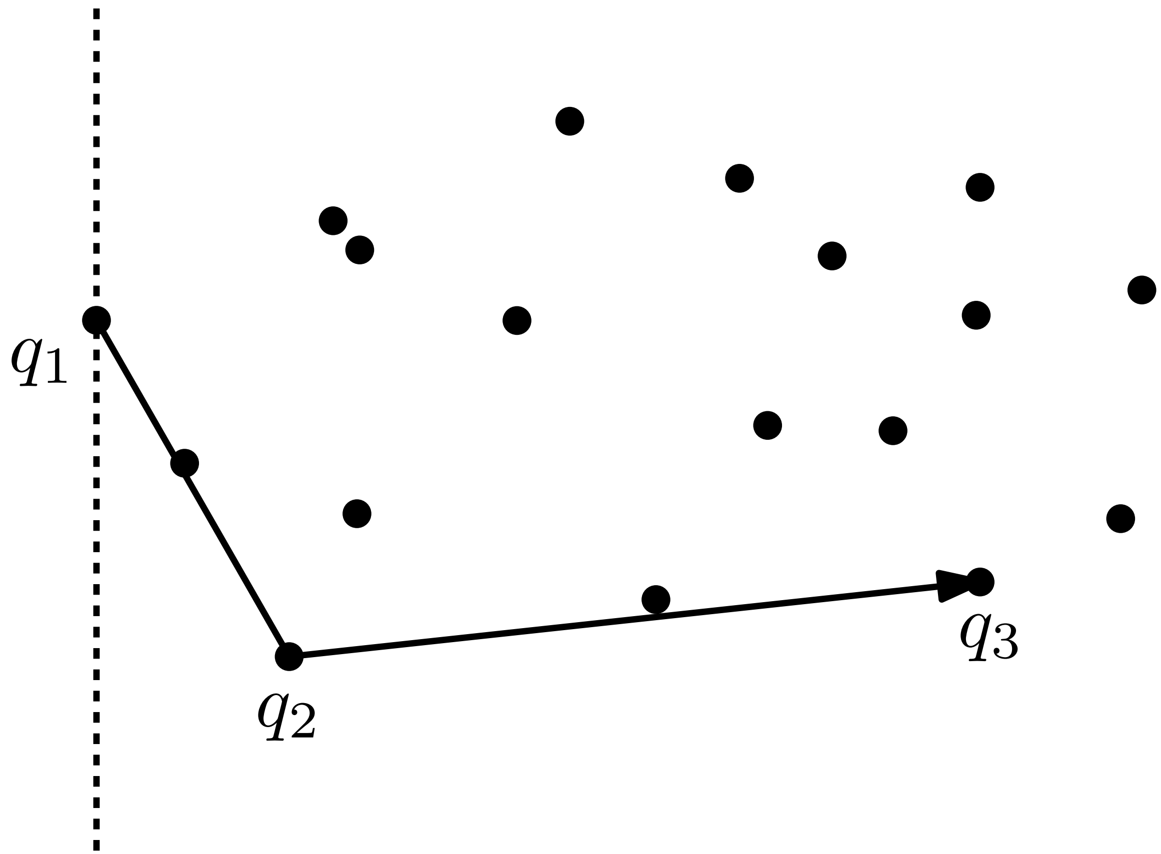


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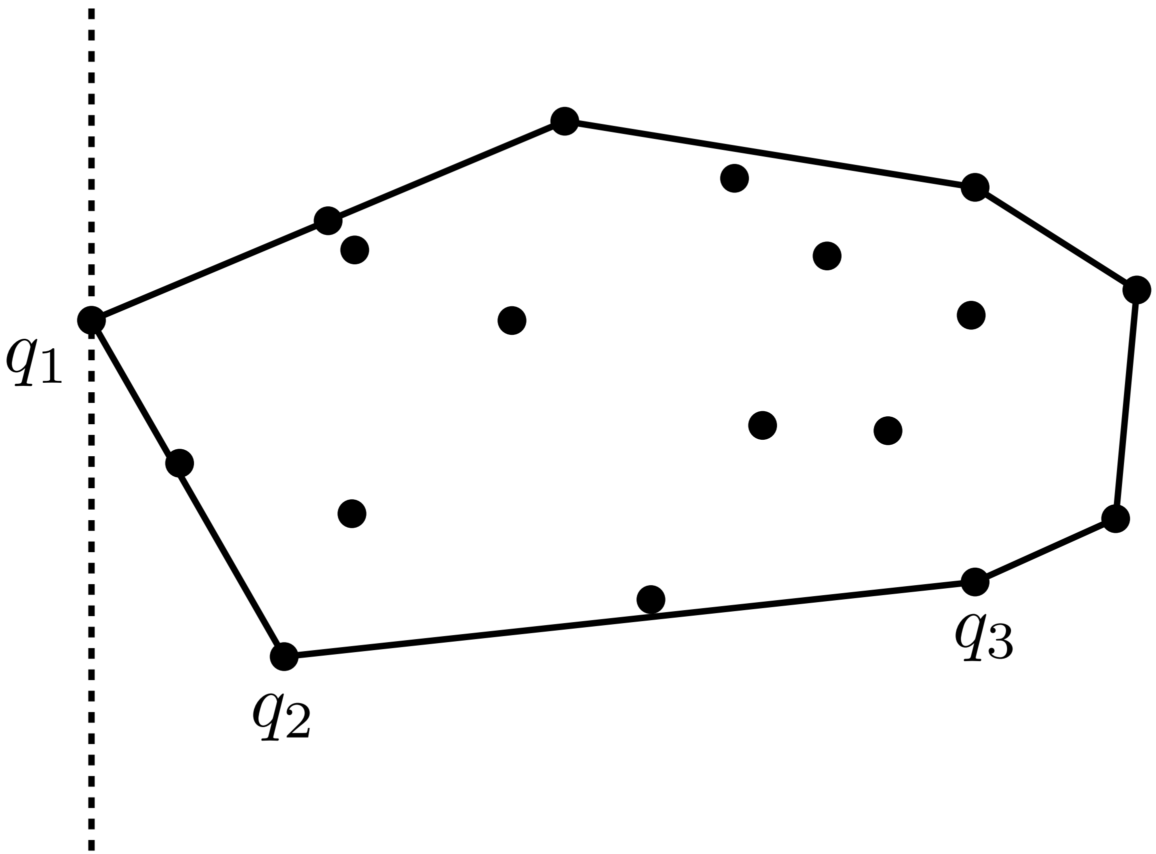


- Start with extreme point
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# Jarvis' Wrap

Input:

- Array `p[1..N]` of points ($\mathtt{N} \geq 3$)

Output:

- Array `q[1..h]` with convex hull vertices in order

Preparation:

- Find the point with smallest $x$-coordinate (`q_now`).
- `q_now` is the first known convex hull point.
- Choose a different point (`q_next`) as the first candidate for the next convex hull point.
- The array `q` is still empty.

# Jarvis' Wrap

```
for (i = 2 to N)
   if (p[i].x < p[1].x)
      swap(p[1], p[i])
q_now = p[1]
q_next = p[2]
h = 0
```

# Jarvis' Wrap

Every round:

- add `q_now` to array q
- find next convex hull point `q_next`:
  point such that no other point is right of the directed line from `q_now` to `q_next`
- replace `q_now` by `q_next`
- replace `q_next` by new candidate different from `q_now`

End:

- When the next found convex hull point is equal to `q[1]` then the convex hull is completed.
- This is the case when `q_now` equals `q[1]` after a round.

# Jarvis' Wrap

```
for (i = 2 to N)
   if (p[i].x < p[1].x)
      swap(p[1], p[i])
q_now = p[1]
q_next = p[2]
h = 0
do
  h = h+1
  q[h] = q_now
  for (i = 2 to N)
    if (rightturn(q_now, q_next, p[i]))
      q_next = p[i]
  q_now = q_next
  q_next = p[1]
while (q_now != q[1])
```

# Jarvis' Wrap Analysis

**Runtime:**

- Preparation: $\Theta(n)$ time
- Outer loop is processed $h \leq n$ times
- Inner loop is processed $\Theta(n)$ times each round.
- $\Theta(nh) = O(n^2)$ rightturn tests
- Worst case: $h = \Theta(n)$
- Output sensitive (good for small hulls)
- Asymptotically not optimal

**Storage:**

- $O(n)$ in addition to input:
  q and constantly many extra variables.

# Jarvis' Wrap Analysis

**Correctness:**

- Before round $k$,
  - `q` contains $k-1$ vertices of the convex hull in order,
  - `q_now` is the next vertex on the convex hull.
- At the end of round $k$, `q_now` is the next convex hull vertex after `q[h]`: edge from `q[h]` to `q_now` has no other point to the right.
- When `q_now` equals `q[1]` the convex hull is complete.

**Degeneracies:**

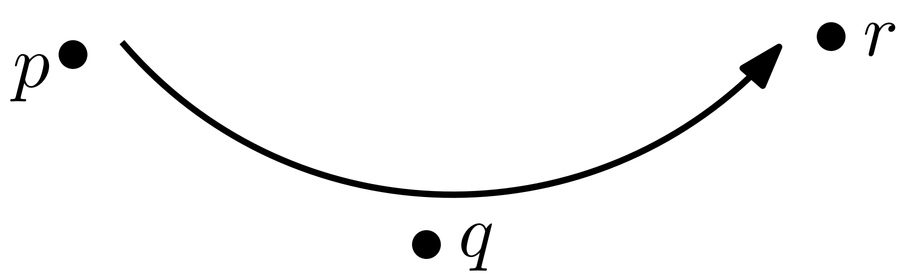
- Several points with smallest $x$-coordinate
- More than two points on a line

# Orientation Test

**Do three points make a left turn?**

- Needed by many algorithms
- Evaluate degree 2 polynomial: *algebraic degree* 2
- Practical relevance

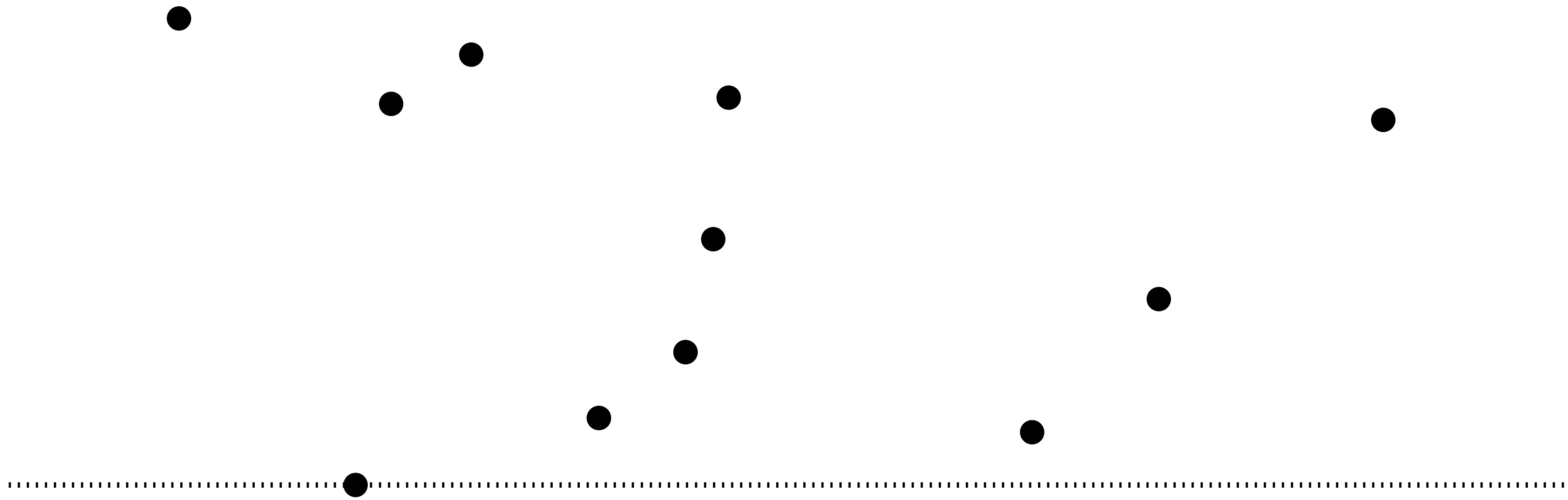
$$\operatorname{sign}\left(\begin{vmatrix} 1 & p_x & p_y \\ 1 & q_x & q_y \\ 1 & r_x & r_y \end{vmatrix}\right)$$




**Do two segments cross?** $\Rightarrow$ four orientation tests

# Graham Scan

- Create hull by “Successive Local Repair”
- sequence sorted around extreme point: remove points not making a left turn



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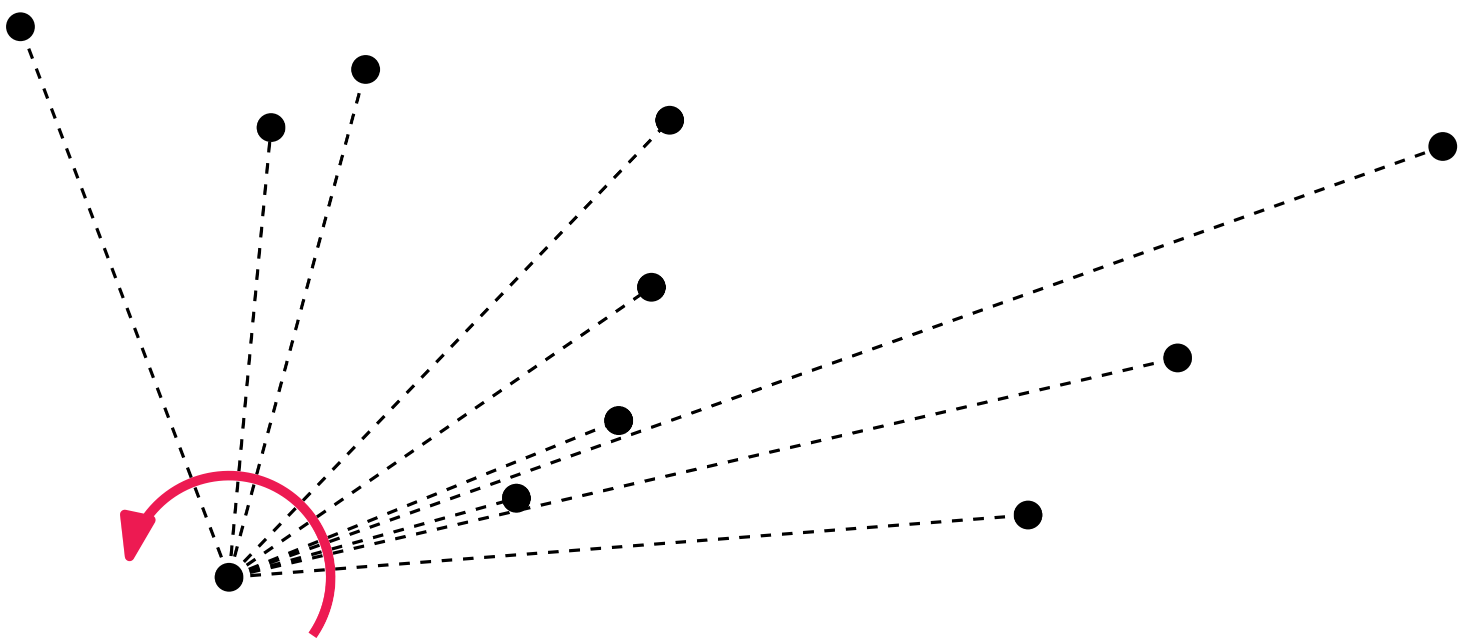

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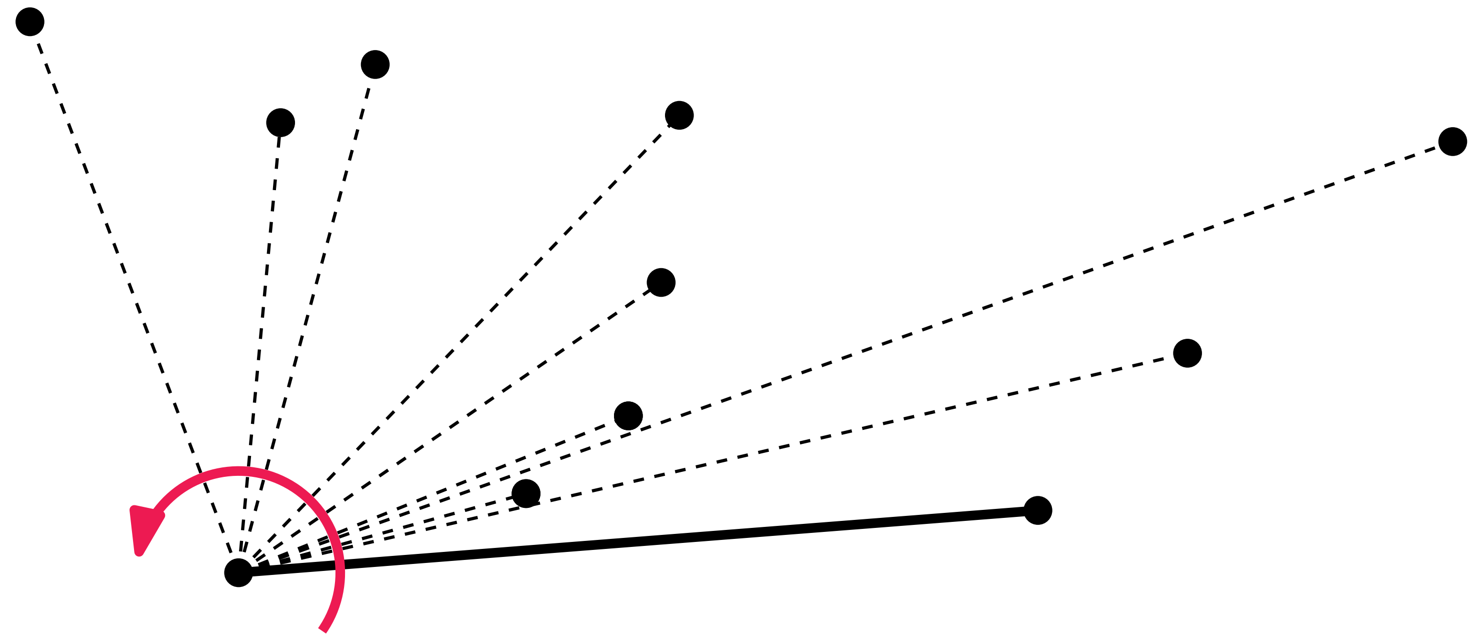

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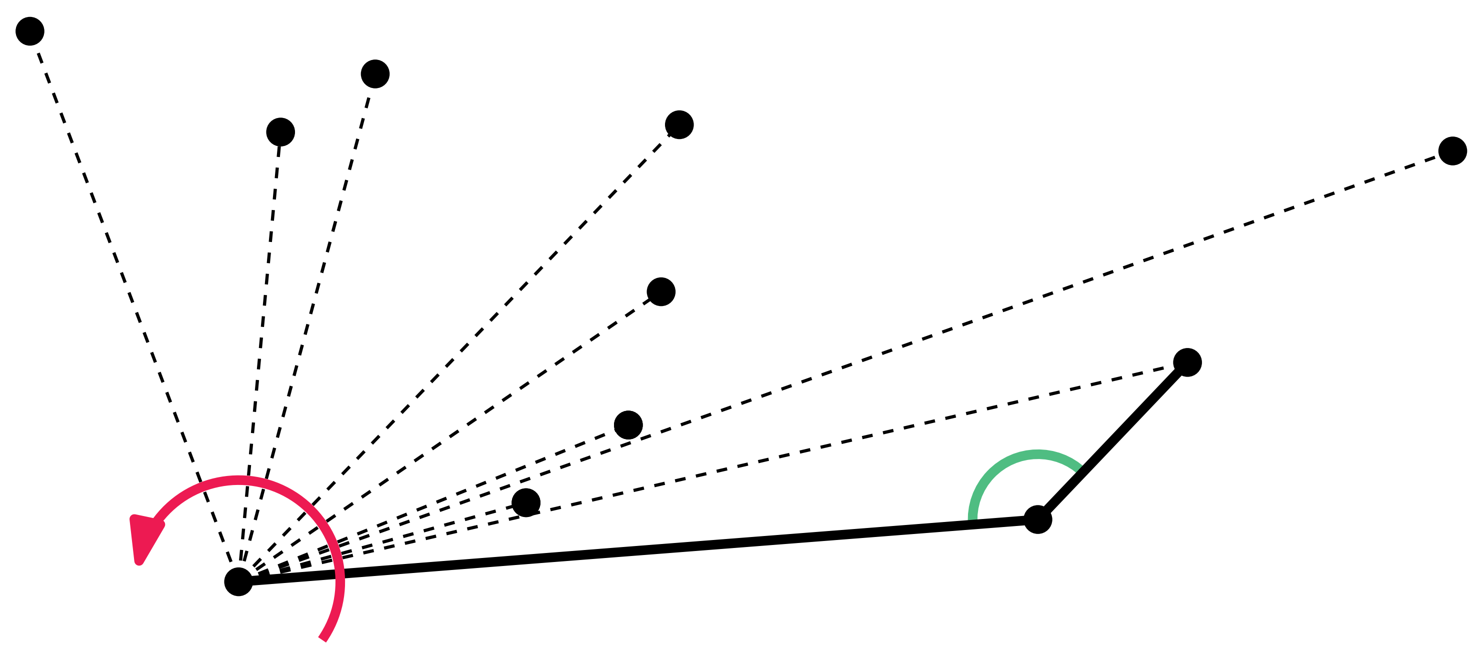

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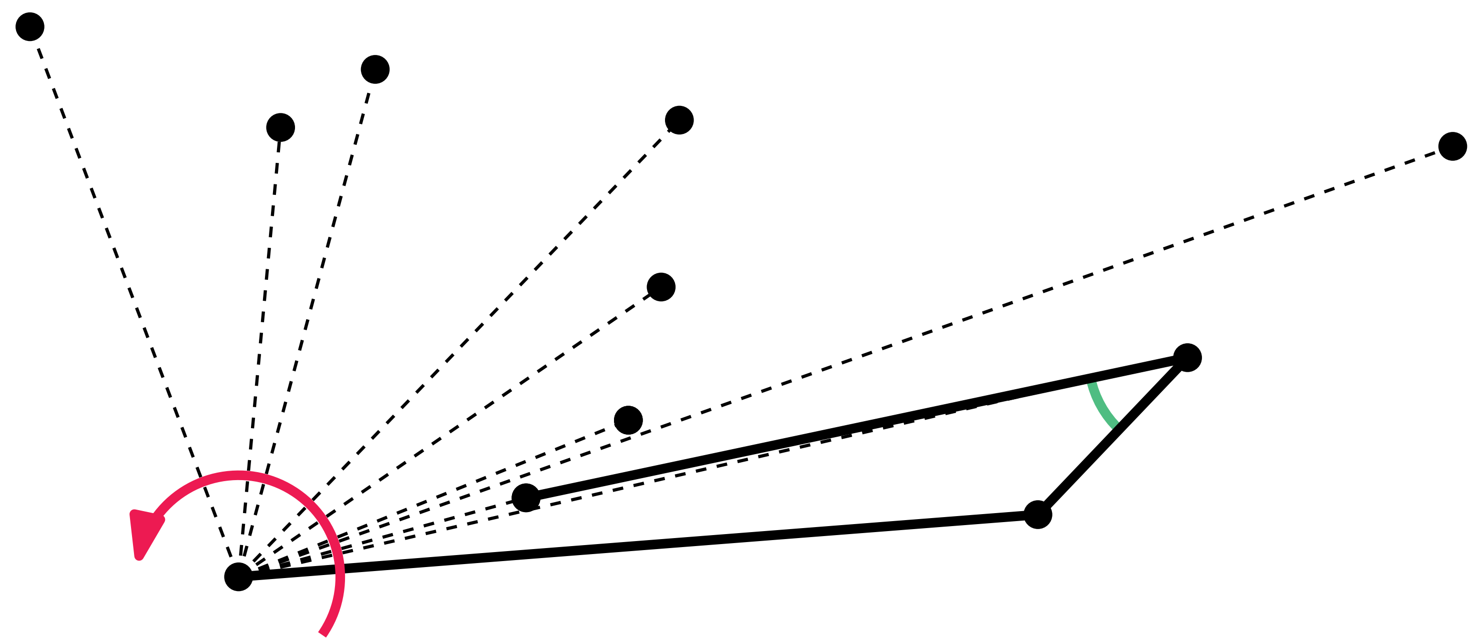

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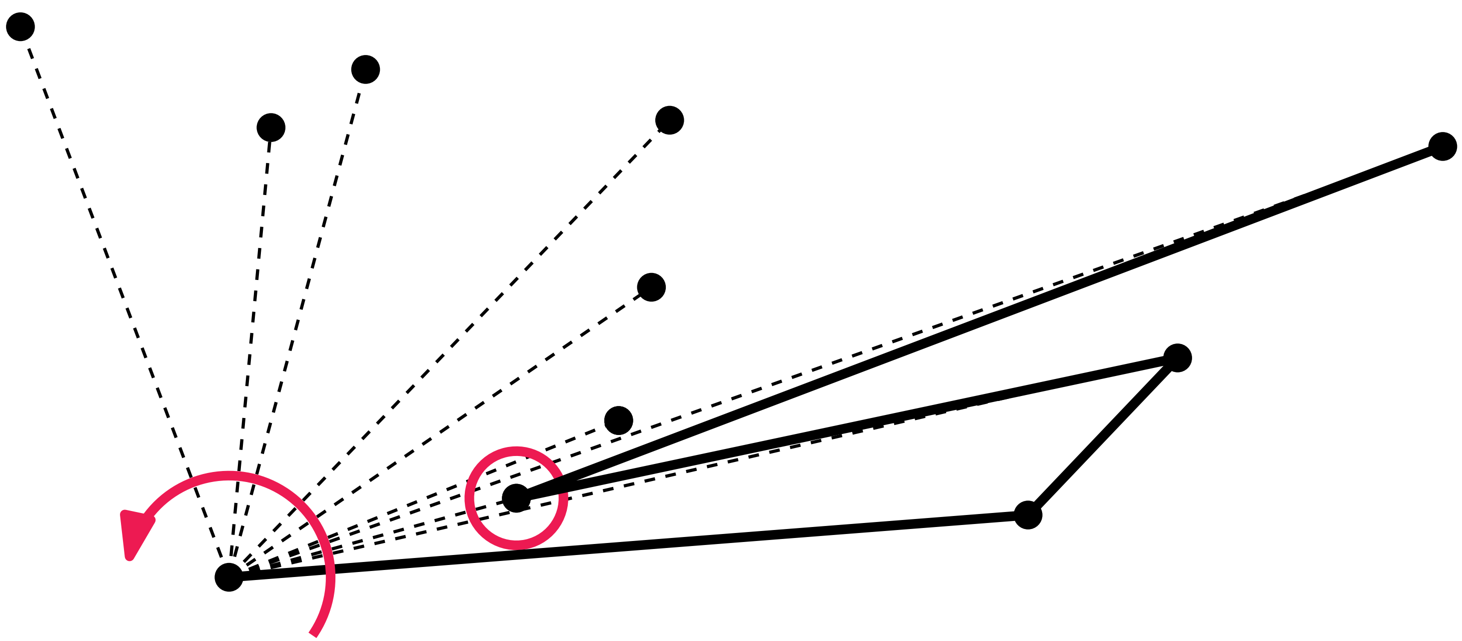

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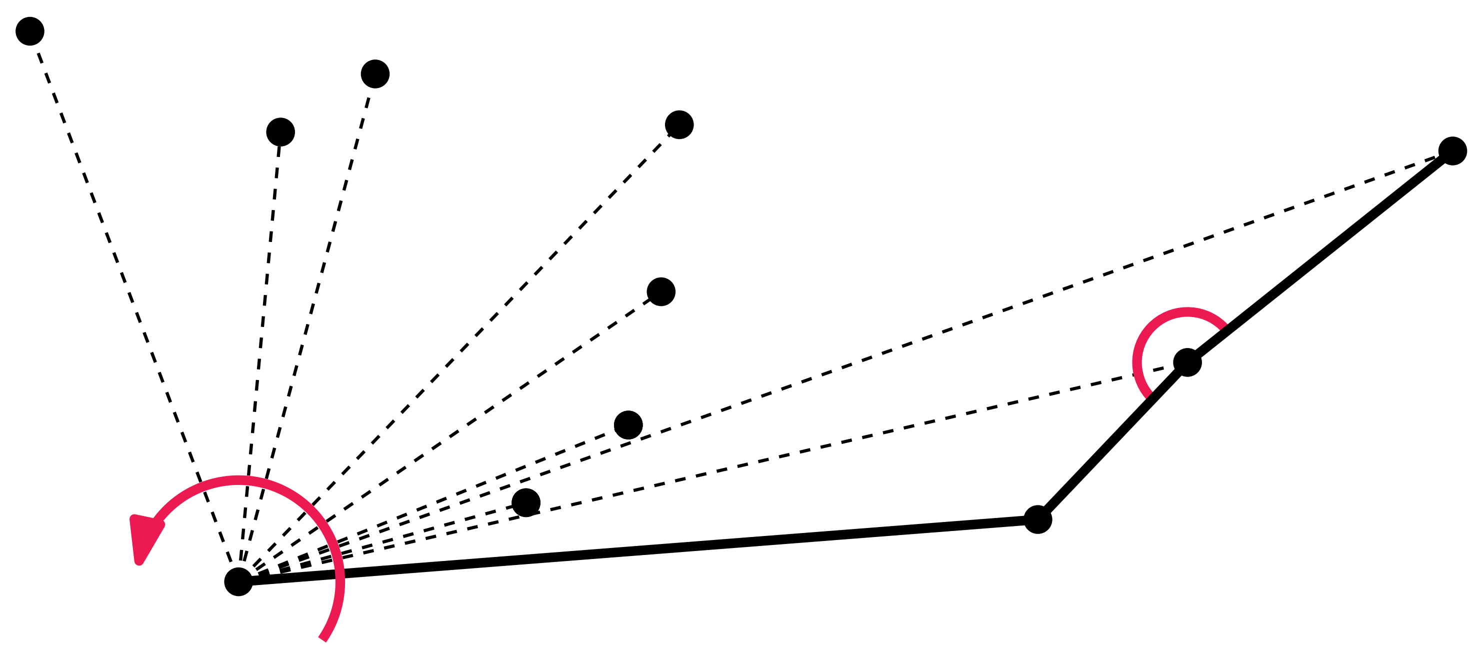

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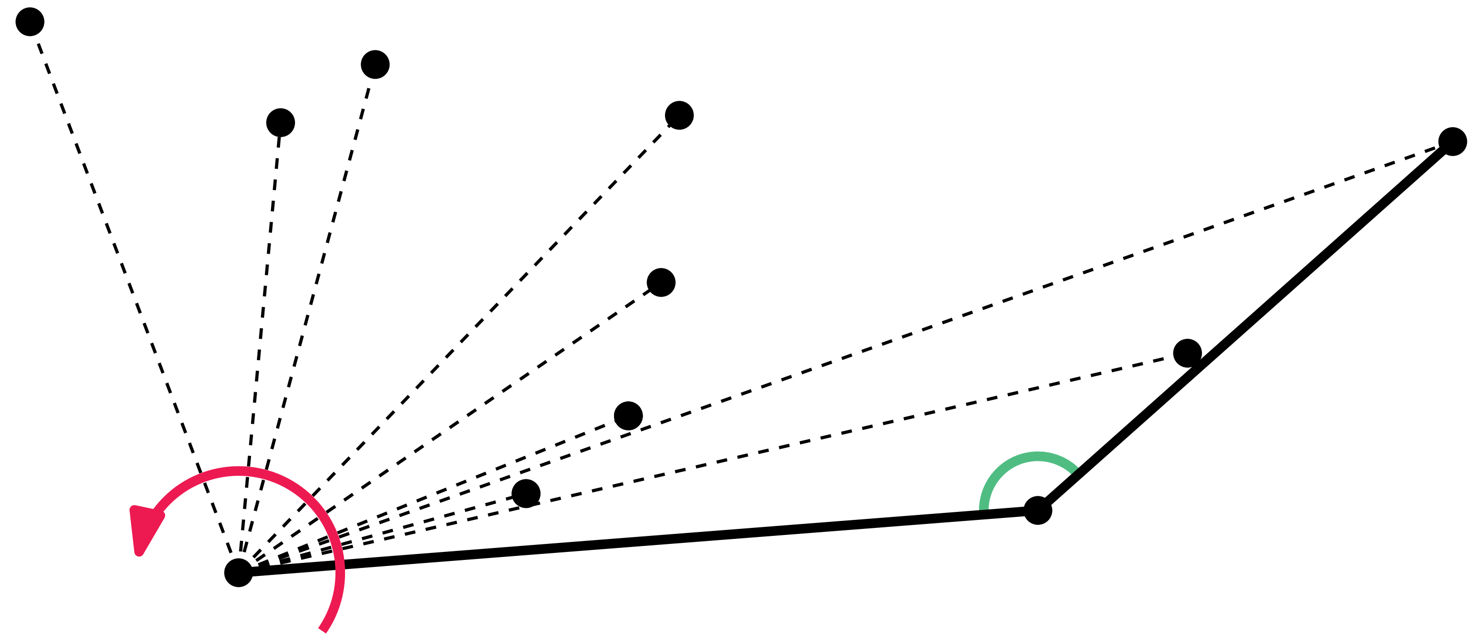

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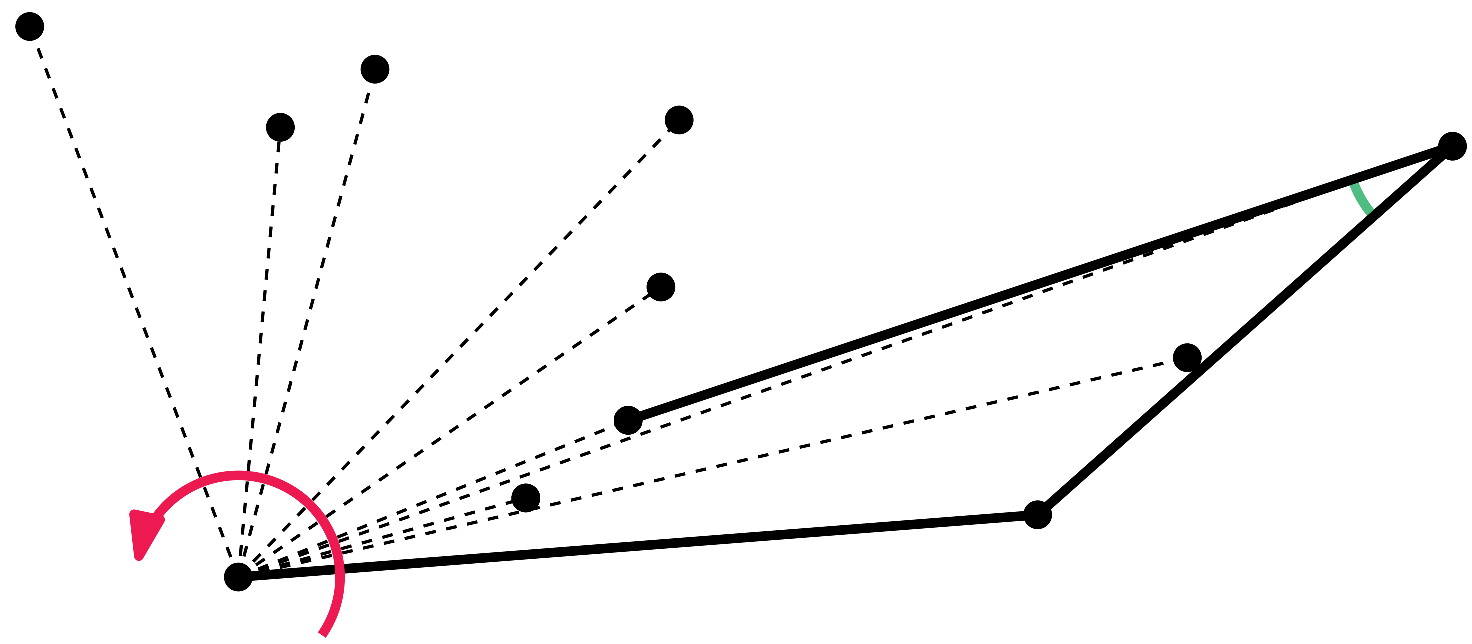

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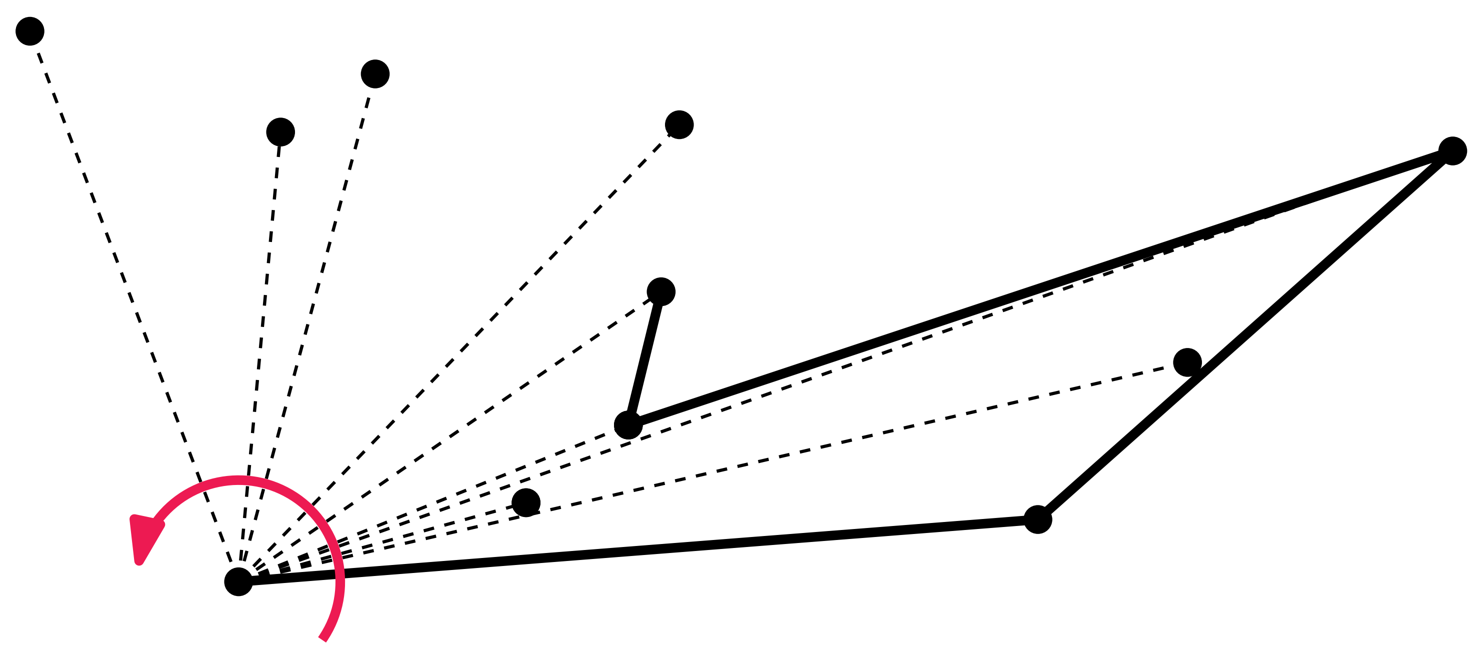

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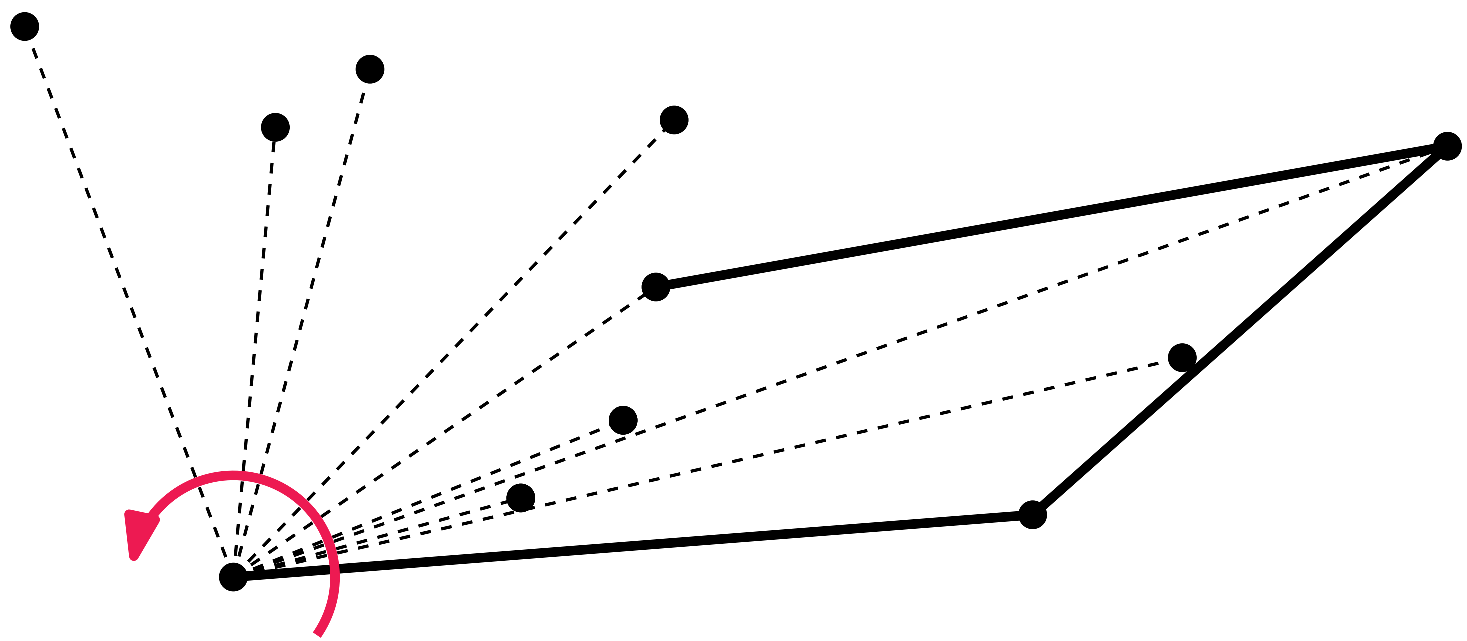

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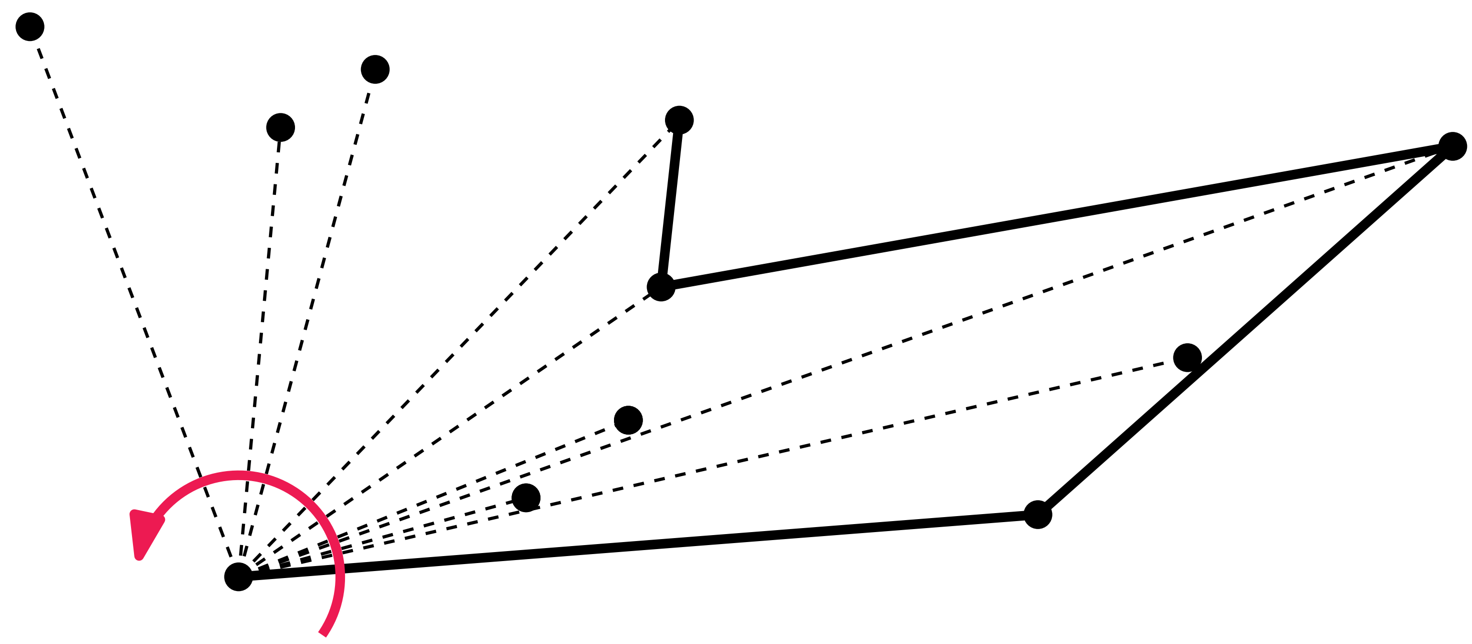

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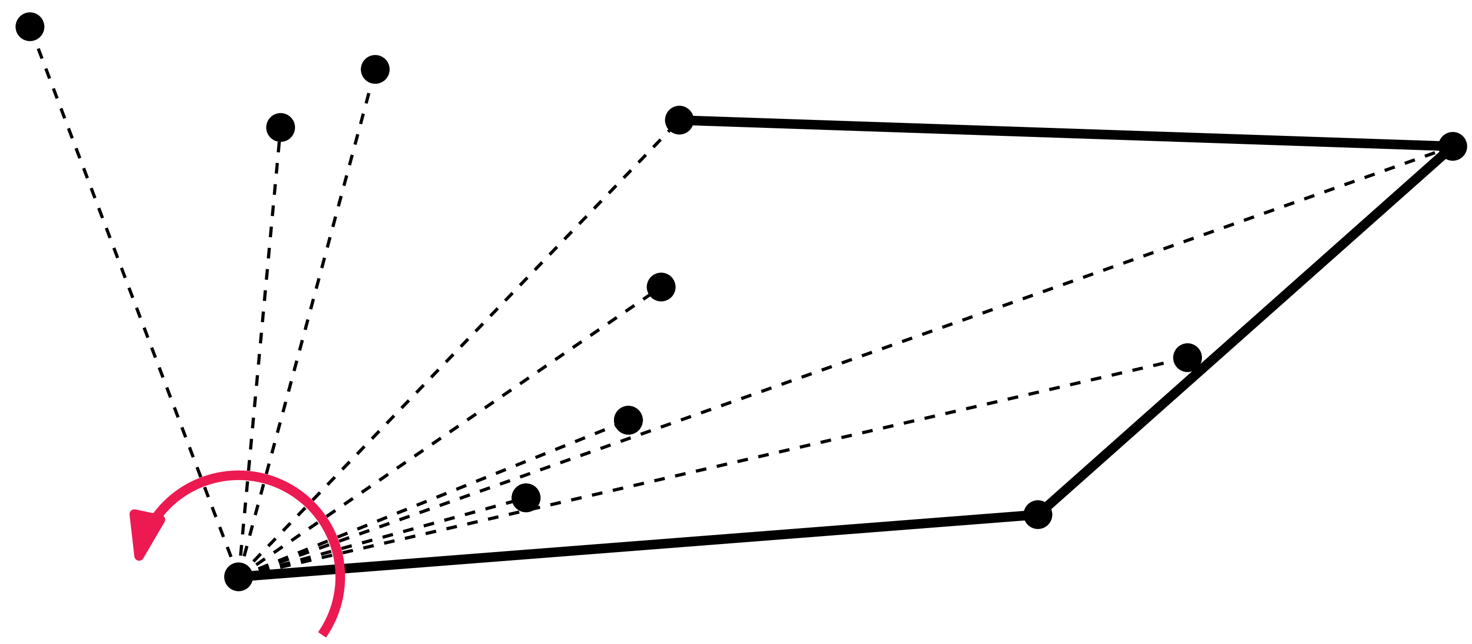

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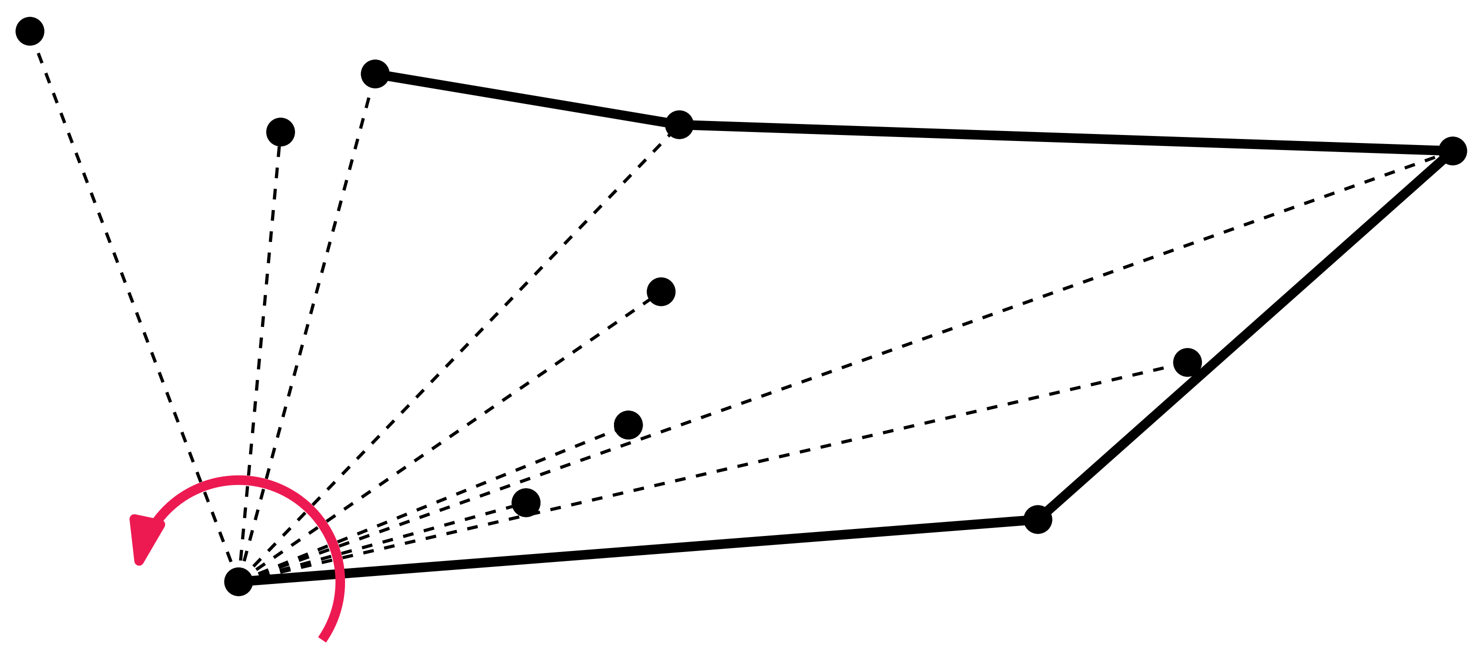

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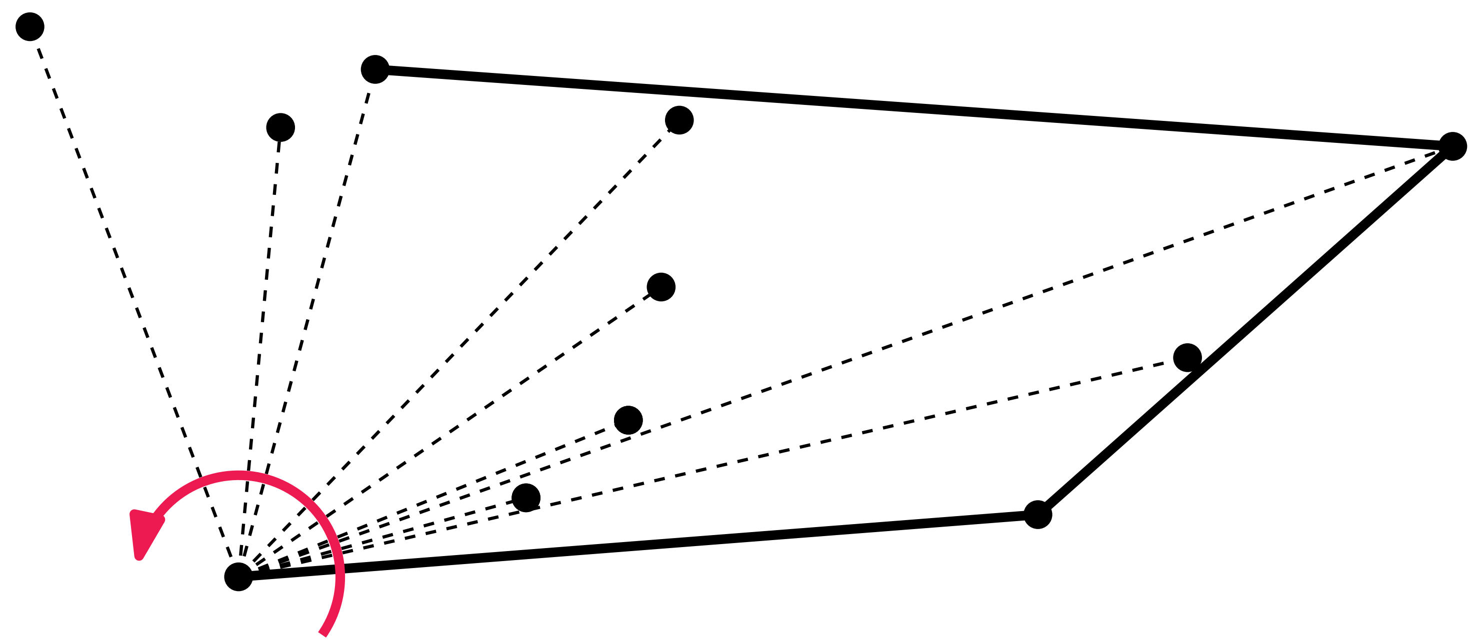
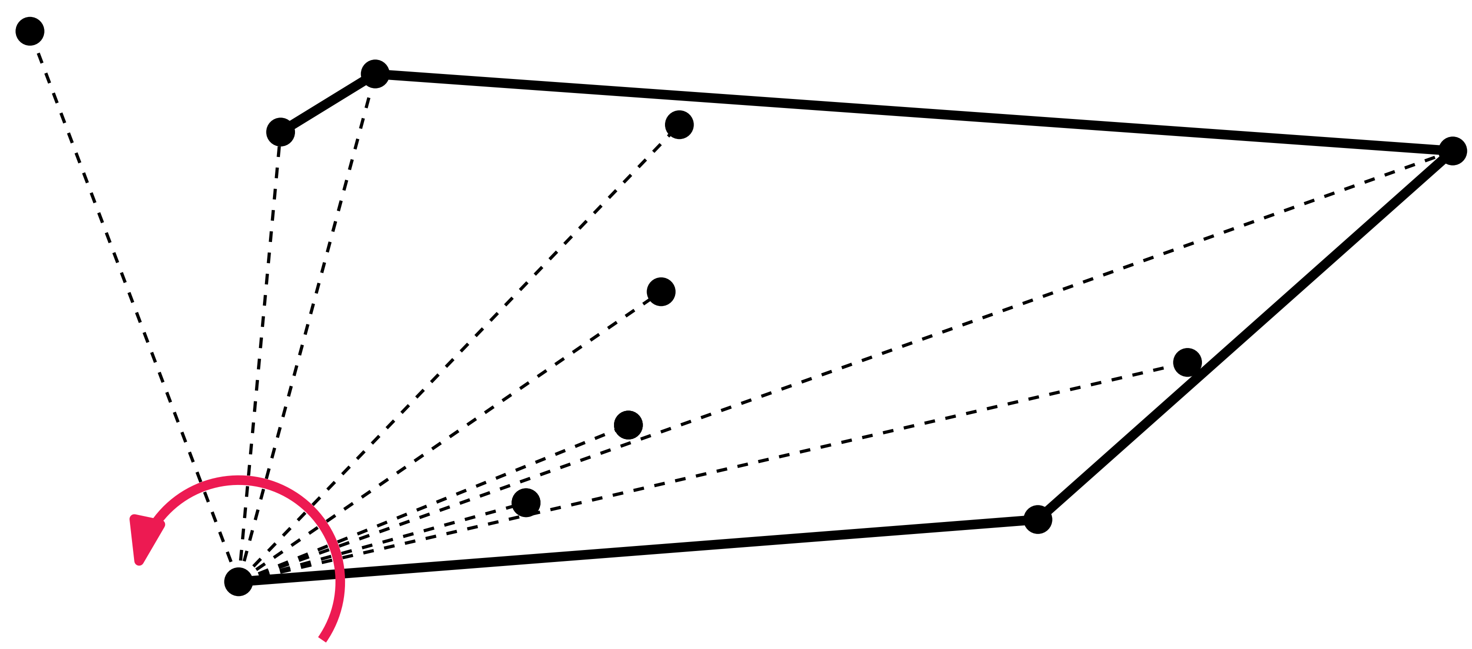


# Graham Scan

- Create hull by “Successive Local Repair”
- sequence sorted around extreme point: remove points not making a left turn



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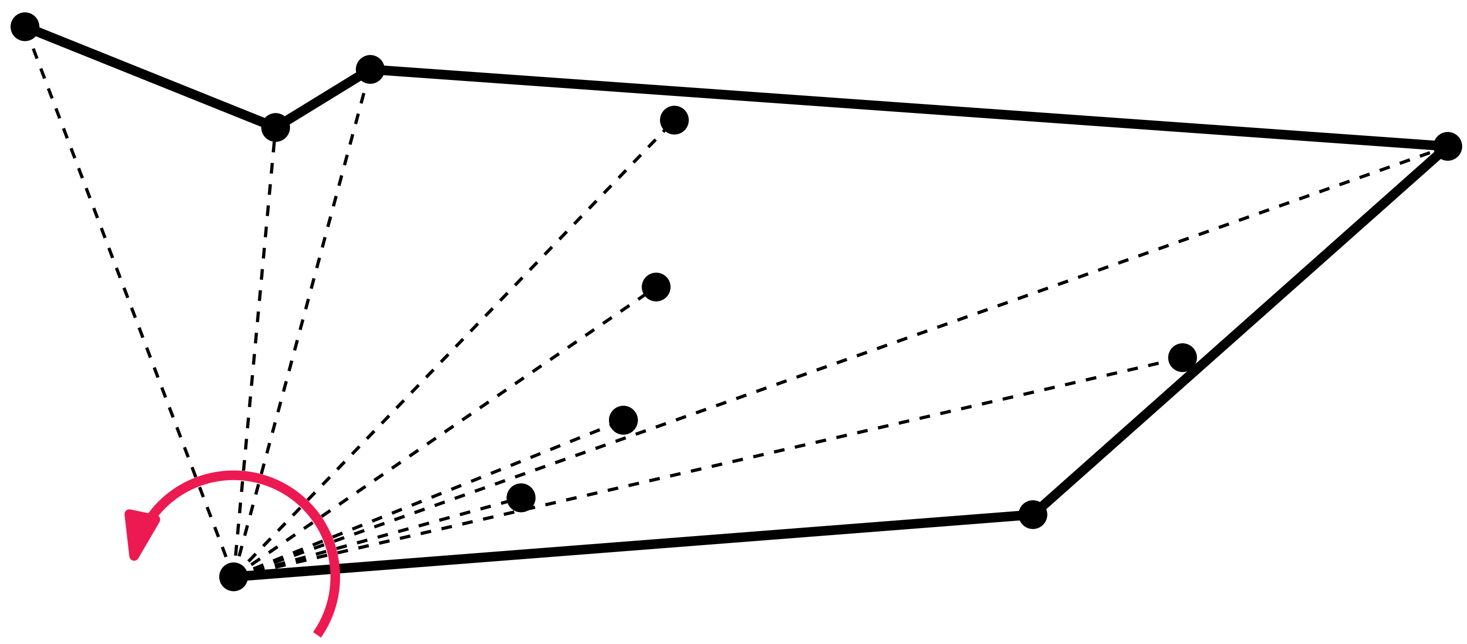

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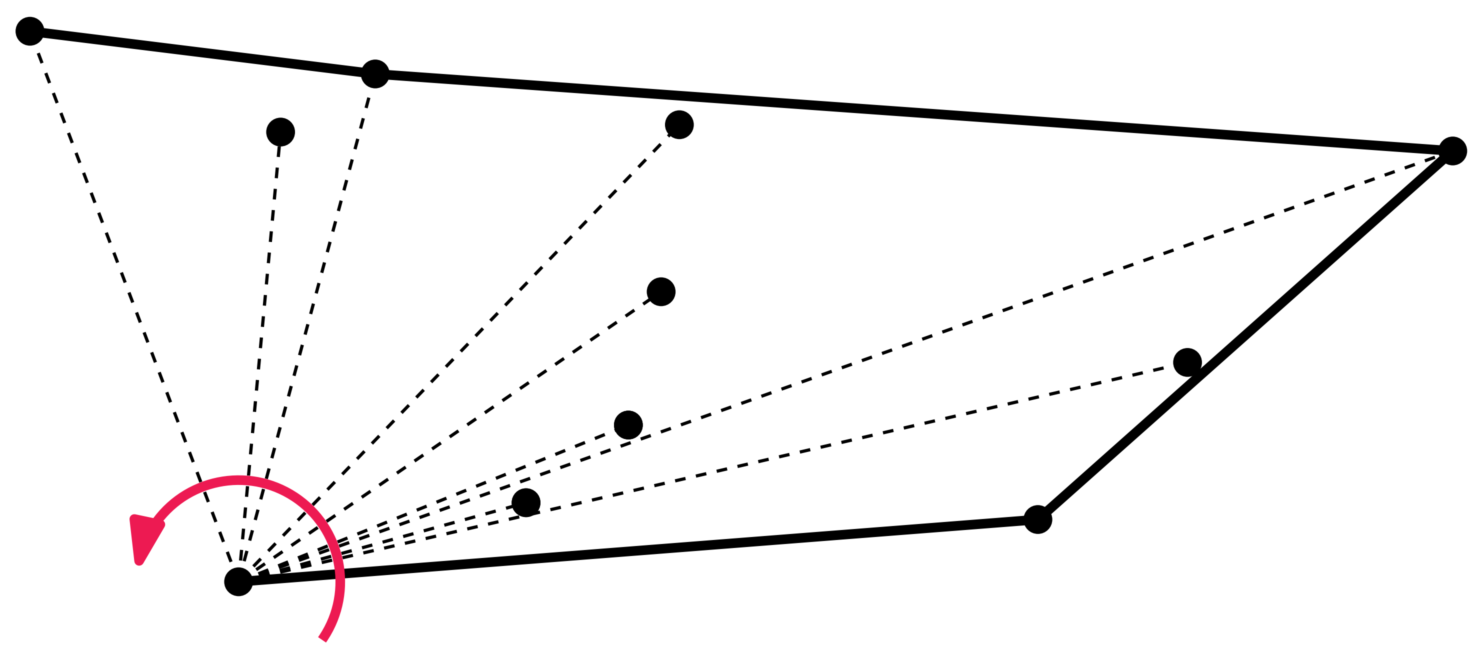

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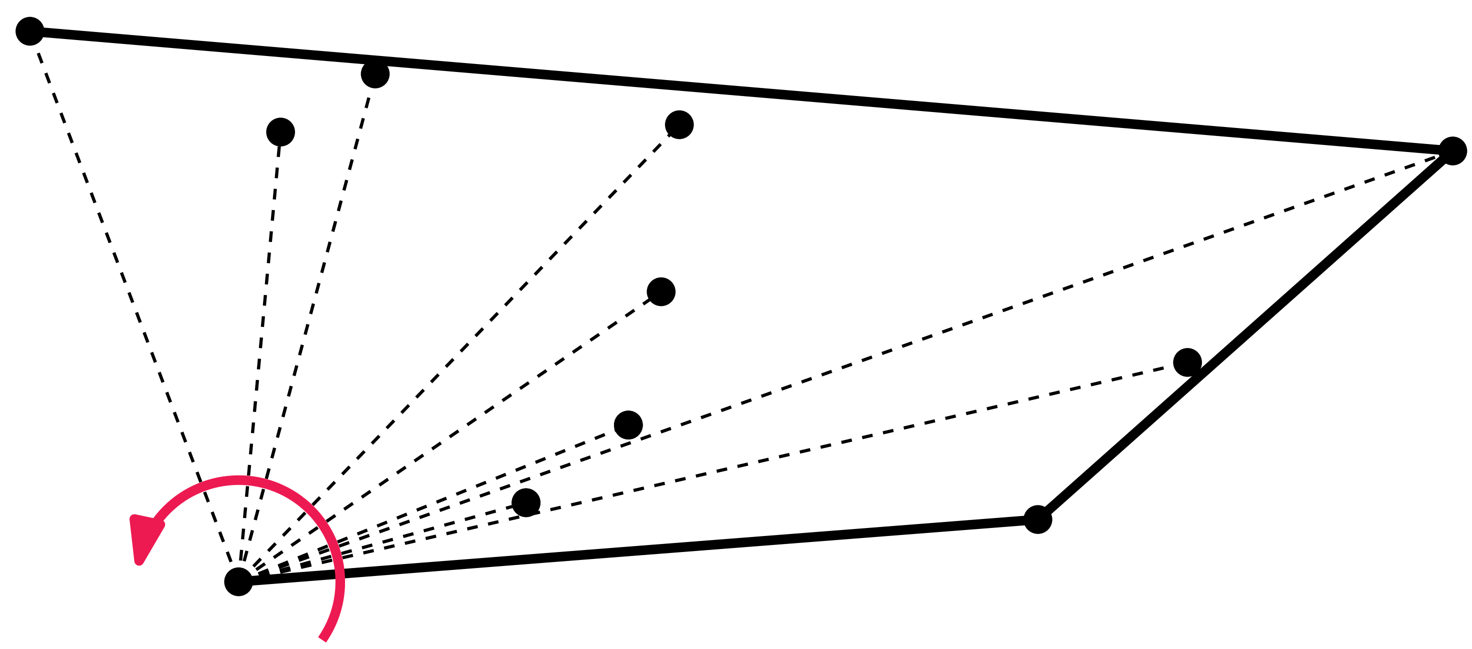

# Graham Scan

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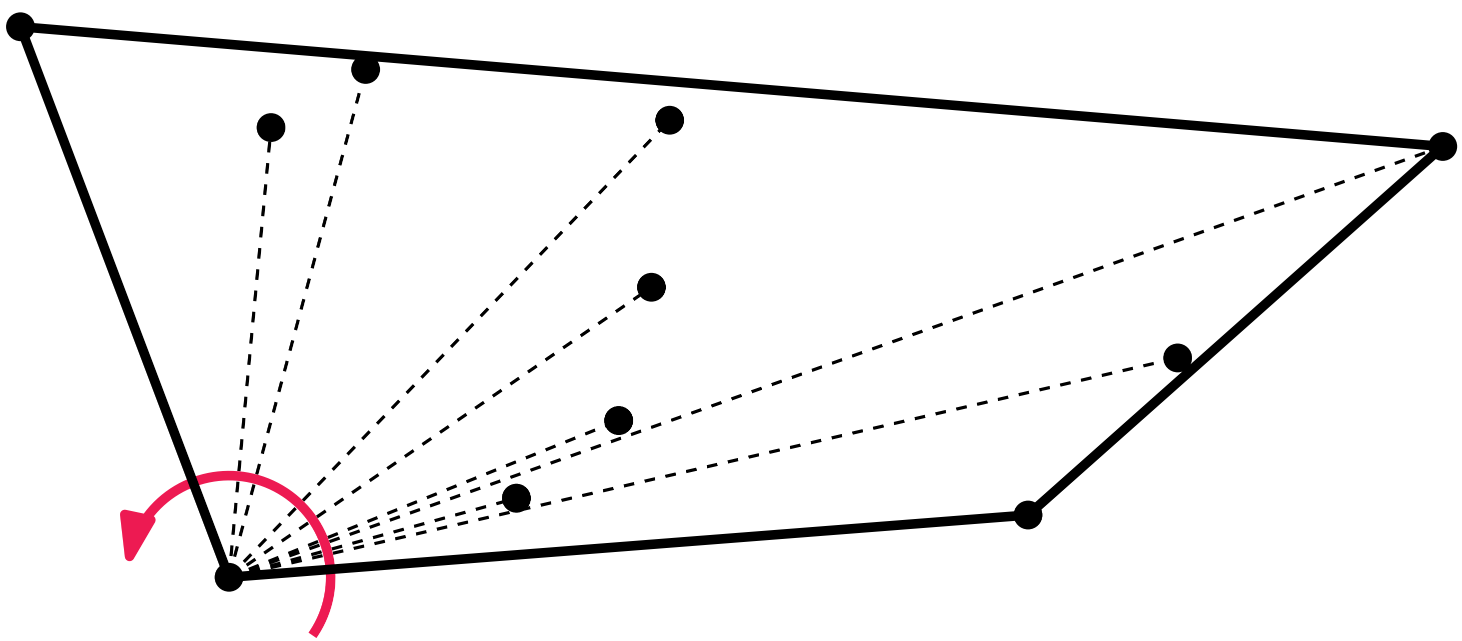


# Graham's Scan

Input:

- Array `p[1..N]` of points ($\texttt{N} \geq 3$)

Output:

- Array `q[1..h]` with convex hull vertices (in order)

Preparation:

- Place the point with smallest $y$-coordinate into `p[1]`
- Sort all other points counterclockwise around `p[1]`: `p[i]` is larger than `p[j]` if `p[i]` is left of the directed line from `p[1]` to `p[j]`
- `p[1]` and `p[2]` are the first two convex hull vertices: Add them in this order to q.

# Graham's Scan

```
for (i = 2 to N)
  if (p[i].y < p[1].y)
    swap(p[1], p[i])
sort p[2..N] counterclockwise around p[1]
q[1] = p[1]
q[2] = p[2]
h = 2
```

# Graham's Scan

Process the remaining points from `p[3]` to `p[n]`.

Processing point `p[i]`:

- While from the last edge of the convex hull there is no left turn to `p[i]`, remove the last point from q.
- Add `p[i]` to q.

End:

- After `p[n]` has been processed, the convex hull vertices are stored in order in q.

# Graham's Scan

```
for (i = 2 to N)
  if (p[i].y < p[1].y)
    swap(p[1], p[i])
sort p[2..N] counterclockwise around p[1]
q[1] = p[1]
q[2] = p[2]
h = 2
for (i = 3 to N)
  while (h>1 and not leftturn(q[h-1],q[h],p[i]))
    h = h - 1
  h = h + 1
  q[h] = p[i]
```

# Graham Scan

**Running time:**

- Preparation in $O(n \log n)$ time due to sorting.
- Buliding the convex hull: $\Theta(n)$ time. Why?

```
for (i = 3 to N)
  while (h>1 and not leftturn(q[h-1],q[h],p[i]))
    h = h - 1
  h = h + 1
  q[h] = p[i]
```

→ in total $O(n \log n)$ time.

**Storage:**

- $O(n)$ in addition to input

# Graham Scan

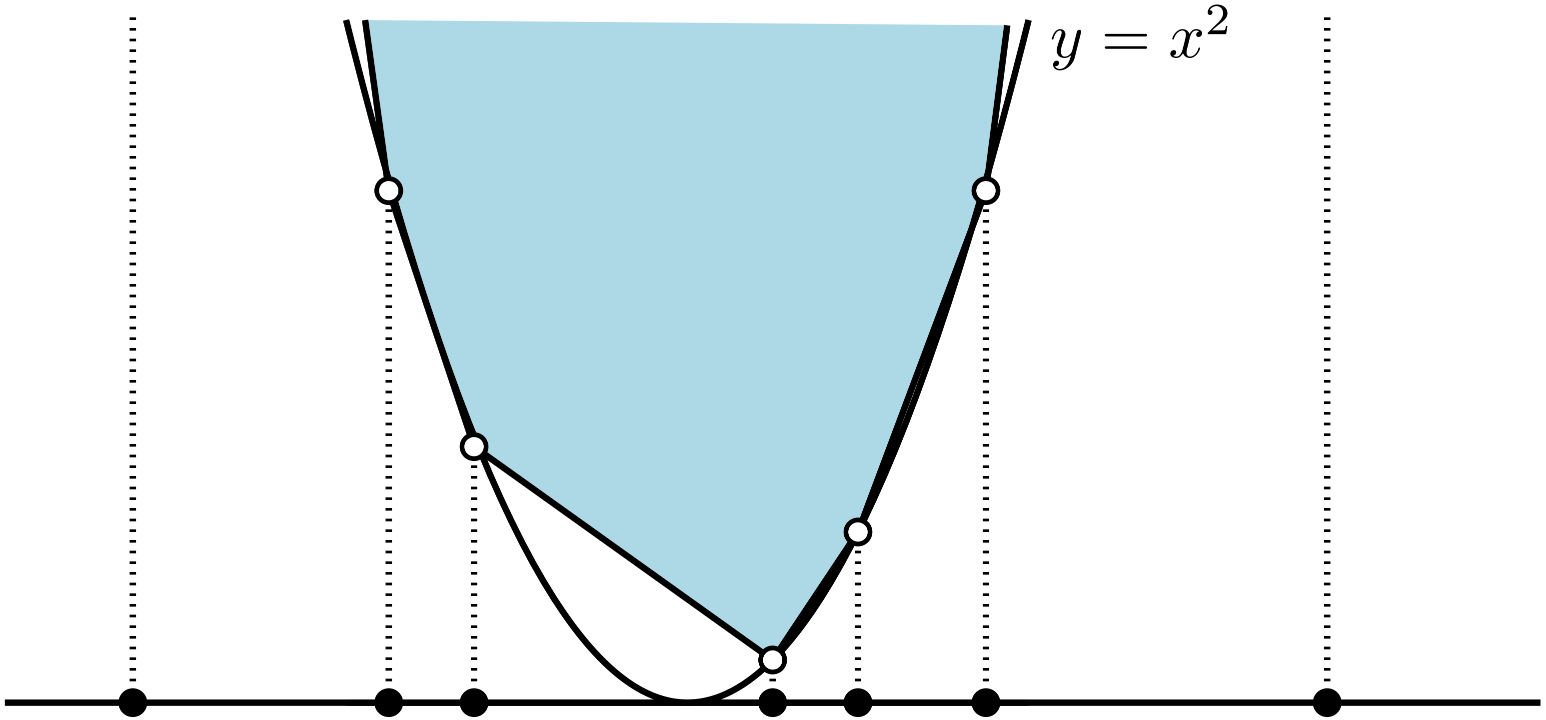
**Correctness:**

- Before the round for `p[i]`, `q` contains the vertices of the convex hull of `p[1..i-1]` in order. $\Leftarrow$ initially true
- After the round for `p[i]`, `q` contains the vertices of the convex hull of `p[1..i]` in order:
  - Point `p[i]` is the "last" vertex of the convex hull.
  - The last point `q[h]` is removed iff it lies in the triangle $\Delta$ `q[1]` `q[h-1]` `p[i]`.
    If it does not lie in this triangle, it is on the convex hull of `p[1..i]`.

# Lower Bound

# Lower Bound

**Theorem:** In the algebraic decision tree model, $\Omega(n \log n)$ operations are needed to construct the convex hull of $n$ points in $\mathbb{R}^d$.



# Conclusion

- The convex hull of $n$ points in $\mathbb{R}^2$ can be computed in
  - $O(nh)$ time by Jarvis' Wrap:
    output sensitive, but quadratic in the worst case.
  - $O(n \log n)$ time by Graham's scan:
    worst-case optimal, but not output-sensitive.
- Lower bound of $\Omega(n \log n)$ time in the worst case follows from $\Omega(n \log n)$ worst case time for sorting.

**Remark:** The convex hull of $n$ points in $\mathbb{R}^2$ can be computed in $O(n \log h)$ time by *Chan's algorithm* (a clever combination of Graham's scan and Jarvis' wrap).

**Question:** Can the convex hull of $n$ points in $\mathbb{R}^2$ be computed faster when the points are sorted in $x$-order?

# Conclusion

- The convex hull of $n$ points in $\mathbb{R}^2$ can be computed in
  - $O(nh)$ time by Jarvis' Wrap: output sensitive, but quadratic in the worst case.
  - $O(n \log n)$ time by Graham's scan: worst-case optimal, but not output-sensitive.
- Lower bound of $\Omega(n \log n)$ time in the worst case follows from $\Omega(n \log n)$ worst case time for sorting.

## Thank you for your attention.

**Question:** Can the convex hull of $n$ points in $\mathbb{R}^2$ be computed faster when the points are sorted in $x$-order?