

- $R = [a_1, b_1]x[a_2, b_2]x[a_3, b_3]$
- $\int \int \int_R f(x, y, z) dy dz dx = \int_{x=a_1}^{b_1} \left(\int_{y=a_2}^{b_2} \left(\int_{z=a_3}^{b_3} f(x, y, z) dz \right) dy \right) dx$
– andere Reihenfolgen auch möglich

Normalbereich im \mathbb{R}^3

- $N = (x, y, z) \in \mathbb{R}^3 | (x, y) \in M, g(x, y) \leq z \leq h(x, y)$
- $vol(N) = \int_{y=c}^d \left(\int_{x=\psi(y)}^{\varphi(y)} \left(\int_{z=g(x,y)}^{h(x,y)} dz \right) dx \right) dy$
- Beispiel

$$\begin{aligned} \underline{\text{Bsp:}} \quad B &= \{ (x, y, z) \in \mathbb{R}^3 \mid x, y, z \geq 0, \quad x+y+z \leq 1 \} \\ &= \{ (x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1, \quad 0 \leq y \leq 1-x, \quad 0 \leq z \leq 1-x-y \} \\ \int \int \int_B x y z^2 dx dy dz &= \int_{x=0}^1 \left(\int_{y=0}^{1-x} \left(\int_{z=0}^{1-x-y} x y z^2 dz \right) dy \right) dx = \int_{x=0}^1 \left(\int_{y=0}^{1-x} \left(x y \frac{z^3}{3} \Big|_{z=0}^{1-x-y} \right) dy \right) dx = \int_{x=0}^1 \left(\int_{y=0}^{1-x} x y \frac{(1-x-y)^3}{3} dy \right) dx \end{aligned}$$

Substitutionsregel (Transformationsformel)

- $T : B \rightarrow \mathbb{R}^3$
- T differenzierbar und injektiv
- $\int \int \int_{T(B)} f(x, y, z) dx dy dz = \int \int \int_B f \circ T(u, v, w) * \left| \det \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw$
- Volumensumrechnungsfaktor - JACOBI-Determinante
– x,y,z beliebig vertauschbar
- Beispiel: dreidimensionale Polarkoordinaten
– $x \Rightarrow r \sin(\psi) \cos(\varphi)$
– $y \Rightarrow r \sin(\psi) \sin(\varphi)$
– $z \Rightarrow r \cos(\psi)$
– Bedingungen
* $r \geq 0$
* $0 \leq \psi \leq \pi$
* $0 \leq \varphi \leq 2\pi$

$$\frac{\partial(x, y, z)}{\partial(r, \psi, \varphi)} = \begin{pmatrix} \frac{\partial}{\partial r}(r \sin(\psi) \cos(\varphi)) & \frac{\partial}{\partial \psi}(r \sin(\psi) \cos(\varphi)) & \frac{\partial}{\partial \varphi}(r \sin(\psi) \cos(\varphi)) \\ \frac{\partial}{\partial r}(r \sin(\psi) \sin(\varphi)) & \frac{\partial}{\partial \psi}(r \sin(\psi) \sin(\varphi)) & \frac{\partial}{\partial \varphi}(r \sin(\psi) \sin(\varphi)) \\ \frac{\partial}{\partial r}(r \cos(\psi)) & \frac{\partial}{\partial \psi}(r \cos(\psi)) & \frac{\partial}{\partial \varphi}(r \cos(\psi)) \end{pmatrix}$$

- Beispiel: Polarkoordinaten bei Kugel

$B = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \geq 0, x^2 + y^2 + z^2 \leq 9\}$
 $0 \leq r \leq 3, 0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq \theta \leq \frac{\pi}{2}$

$$\iiint_B xy^2 \, dx \, dy \, dz = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 r^3 \sin^2(\theta) \cos(\varphi) \sin^2(\varphi) \sin(\theta) \cos(\theta) r^2 \, dr \, d\varphi \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4} \sin^2(\theta) \cos(\varphi) \sin^2(\varphi) \sin(\theta) \cos(\theta) \right]_0^3 d\varphi \, d\theta$$

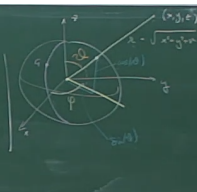
$$= \frac{81}{4} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin^2(\theta) \cos(\varphi) \sin^2(\varphi) \sin(\theta) \cos(\theta) \, d\varphi \, d\theta$$

$$= \frac{81}{4} \int_0^{\frac{\pi}{2}} \left[\frac{\sin^2(\varphi)}{2} \cos(\varphi) \sin^2(\theta) \sin(\theta) \cos(\theta) \right]_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{81}{8} \int_0^{\frac{\pi}{2}} \sin^2(\theta) \sin^3(\theta) \cos(\theta) \, d\theta$$

$$= \frac{81}{8} \int_0^{\frac{\pi}{2}} \sin^5(\theta) \cos(\theta) \, d\theta$$

$$= \frac{81}{8} \left[\frac{\sin^6(\theta)}{6} \right]_0^{\frac{\pi}{2}} = \frac{81}{8} \cdot \frac{1}{6} = \frac{27}{8}$$



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• Zylinderkoordinaten

– $x \implies r \cos(\varphi)$

– $y \implies r \sin(\varphi)$

– $z \implies z$

– Volumenelement

* $dx \, dy \, dz = r \, dr \, d\varphi \, dz$

– Beispiel

$B = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq 1 - (x^2 + y^2)\}$

$$\iiint_B z \, dx \, dy \, dz = \int_0^1 \left(\int_0^{2\pi} \left(\int_0^{\sqrt{1-z}} z \, dr \right) r \, d\varphi \right) dz$$

$$= \int_0^1 d\varphi \cdot \int_0^{\sqrt{1-z}} \left[\frac{z}{2} \right]_0^{\sqrt{1-z}} r \, dr = \frac{2\pi}{2} \int_0^1 z (1-z)^2 \, dz$$

$$= \pi \int_0^1 z (1-z)^2 \, dz$$

$$= \pi \int_0^1 (1-z)^2 \frac{du}{2} = \frac{\pi}{6}$$

$$= \pi \cdot \left[-\frac{1}{3} (1-z)^3 \right]_0^1 = \frac{\pi}{6}$$

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[[Mehrdimensionale Integralrechnung]]