Cryptography 3: Asymmetric Cryptography

Maria Eichlseder

Information Security – WT 2023/24

You Are Here

Crypto 1	Crypto 2	Crypto 3	Crypto 4
Symmetric Authentication	Symmetric Encryption	Asymmetric Cryptography	Protocols and Applications
Integrity	Confidentiality	Establishing communication	Theory meets Practice
■ Hash functions ■ MACs (Message Authentication)	■ AEAD (Auth. Encryption) ■ Symmetric primitives	■ Key exchange ■ Signatures ■ Asymmetric primitives	■ Protocols ■ Applications

Recap of Last Week (1): Schemes for Encryption

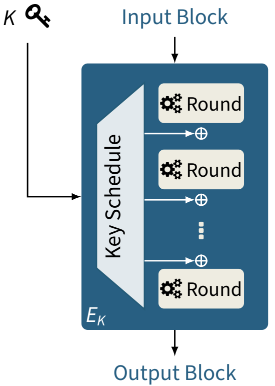
Encryption schemes transform a plaintext Message M of arbitrary length to a Ciphertext C of about the same length based on a Key K of fixed length.

Schemes may accept additional inputs or produce an authentication Tag T.

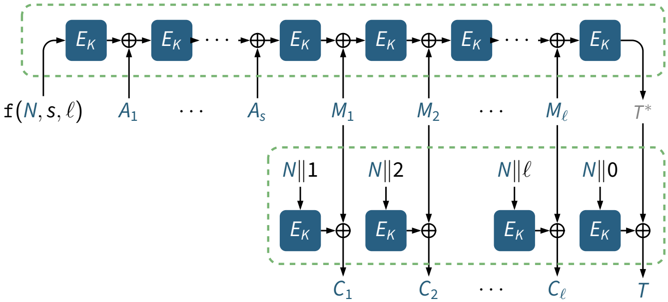
Encryption $\mathcal{E}_{K_{AB}}$	Auth. Enc. $\mathcal{AE}_{K_{AB}}$	Key Encapsulation
Symmetric Key K_{AB}	Symmetric Key K_{AB}	Asymmetric Keys
Confidentiality only	Confid. + Authenticity	Confidentiality
A, B can encrypt	A, B can encrypt + auth	Anyone can encrypt
A, B can decrypt	A, B can decrypt + verify	A can decrypt

Recap of Last Week (2): Layers of the Symmetric Crypto Stack

Primitive (e.g., AES)



Mode of Operation (e.g., AES-CCM)



Outline

Background

- Motivation, Goals, Applications
- Modular Arithmetic and Hard Problems

Key Exchange

- Diffie–Hellman Key Exchange

Asymmetric Encryption

- Trapdoor One-way Functions
- RSA Public-Key Encryption

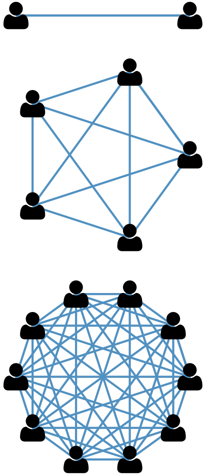
Signatures

- RSA Signatures

Background

Introduction

Limitations of Symmetric Cryptography



Key Distribution

- System with n users needs $\binom{n}{2} = \frac{n \cdot (n-1)}{2}$ key-pairs
- Adding new users is expensive and complicated
- How would this work for securing the internet?!

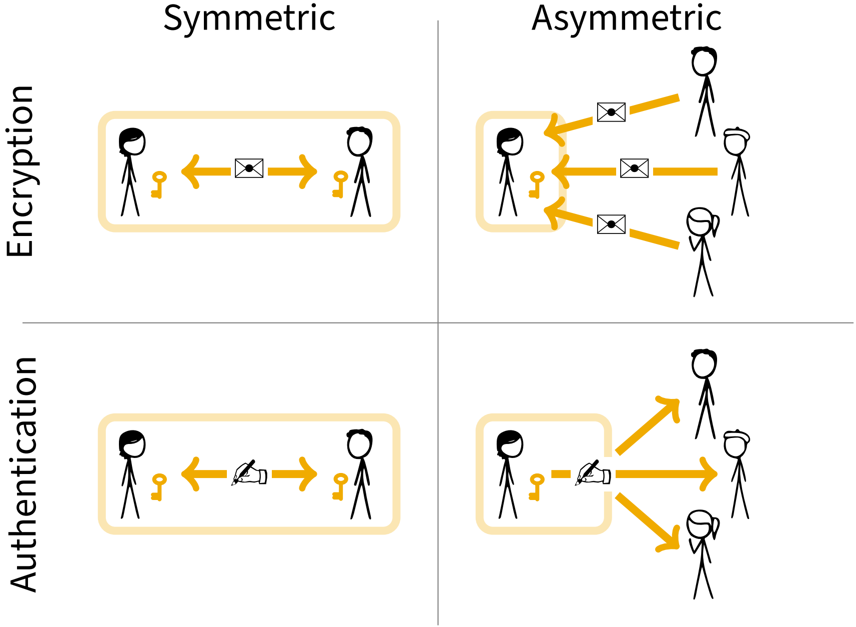
Symmetric Trust Relationships

- Assumes that users trust each other equally
- Does not support establishing new connections
- Does not support properties like non-repudiation

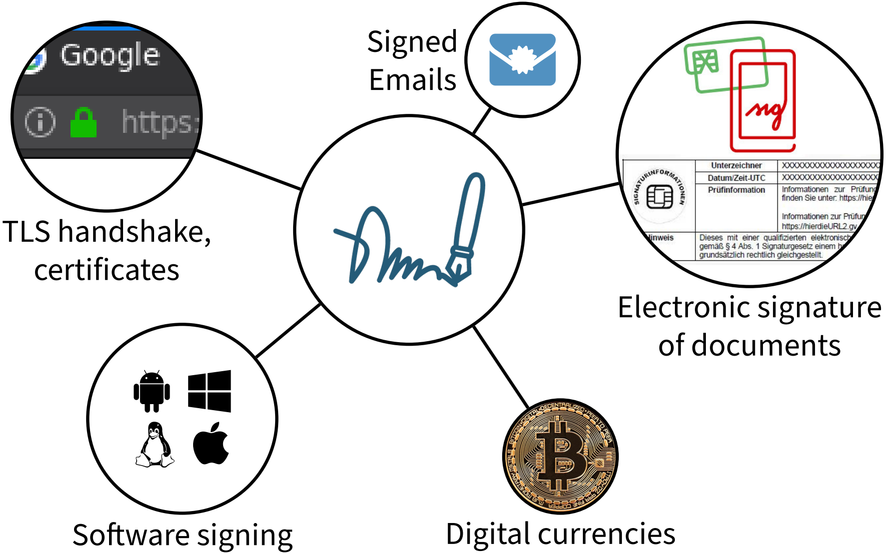
Asymmetric Crypto Schemes

$\overbrace{\text{to establish a new connection}}$ $\overbrace{\text{to authenticate a new connection}}$

Key Exchange	Asym. Encryption	Signature
Two Keypairs K_A, K_B	Asymmetric Keypair K_A	Asymmetric Keypair K_A
A and *B* communicate to agree on a new symmetric key	*A* receives confidential messages (usually an “encapsulated” key)	*A* creates a signature to authenticate messages
A, *B* can influence key	Anyone can encrypt	*A* can authenticate
A, *B* can derive key	*A* can decrypt	Everyone can verify


Symmetric
Asymmetric
Encryption
Authentication

Applications of Digital Signatures



Applications of Key Exchange and Asymmetric Encryption

Key Exchange is used to **agree on a session key** to be used for a symmetrically protected communication channel

- Secure Communication via TLS
- IPsec for protecting VPNs
- SSH Secure Shell
- ...

Asymmetric Encryption is mostly used to **send a session key** for a symmetrically protected message (“key encapsulation”)

- SSH Secure Shell
- Email encryption with PGP or S/MIME
- ...

Recap: Modular Arithmetic and the Set $\mathbb{Z}_n$

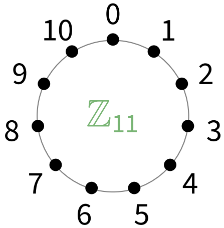
We arrange integers in classes by their remainder after division by the modulus n
(aka "modulo n", "reduce by n")

$\mathbb{Z}_n = \{0, \ldots, n-1\}$ is the set of all classes modulo n.

Integers a, b in the same class are "congruent mod n": "$a \equiv b \pmod{n}$".

Example: mod 11

Class $\in \mathbb{Z}_{11}$	Integers $\subseteq \mathbb{Z}$
0	$\{\ldots, -11, 0, 11, 22, \ldots\}$
1	$\{\ldots, -10, 1, 12, 23, \ldots\}$
2	$\{\ldots, -9, 2, 13, 24, \ldots\}$
$\vdots$	$\vdots$
10	$\{\ldots, -1, 10, 21, 32, \ldots\}$



Computing $\pmod{n}$: The Additive Group $(\mathbb{Z}_n, +)$

The set $\mathbb{Z}_n$ with the operation $+$ (addition modulo n) is a group that satisfies:

1. Associativity: $\forall a, b, c \in \mathbb{Z}_n : \ a + (b + c) = (a + b) + c$
2. Commutativity: $\forall a, b \in \mathbb{Z}_n : \ a + b = b + a$
3. Neutral element 0: $\forall a \in \mathbb{Z}_n : \ a + 0 = a = 0 + a$
4. Inverse element $-a$ for every element $a \in \mathbb{Z}_n$: $\quad a + (-a) = 0$

Example $(\mathbb{Z}_{11}, +)$:

$+$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10	0
2	2	3	4	5	6	7	8	9	10	0	1
3	3	4	5	6	7	8	9	10	0	1	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10	10	0	1	2	3	4	5	6	7	8	9

Computing $\pmod{n}$: The Multiplicative Group $(\mathbb{Z}_n^*, \cdot)$

The set $\mathbb{Z}_n$ with the operation $\cdot$ (multiplication modulo n) is **not** a group:
For example, 0 has no multiplicative inverse b such that $b \cdot 0 \equiv 1$.

But the set $\mathbb{Z}_n^* := \{a \in \mathbb{Z}_n \mid \exists\, b \in \mathbb{Z}_n : \; b \cdot a = 1\}$ of invertible elements is a group.

Example $(\mathbb{Z}_{11}^*, \cdot)$:

$\cdot$	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	1	3	5	7	9
3	3	6	9	1	4	7	10	2	5	8
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10	10	9	8	7	6	5	4	3	2	1

Recap: Invertible Elements modulo n and Euler phi-Function

- **Definition**: Integers a, b are co-prime if they have no common prime factor.

- **Theorem**: Element a has a multiplicative inverse mod n iff a, n are co-prime. This inverse can be found with the Extended Euclidean Algorithm.

- **Definition**: The Euler phi-function $\varphi(n)$ gives the number of integers in the range $1, \ldots, n-1$ which are co-prime to the integer n.
  - p prime: $\varphi(p) = p - 1$
  - $n = p \cdot q$ with p, q prime: $\varphi(n) = \varphi(p \cdot q) = (p-1) \cdot (q-1)$

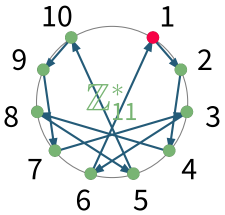
  Example: $\varphi(15) = (3-1) \cdot (5-1) = 8$: numbers $\{1, 2, 4, 7, 8, 11, 13, 14\}$

Recap: Generators and Euler's Theorem

- $\mathbb{Z}_n^*$ contains exactly the $\varphi(n)$ elements in $1, \ldots, n-1$ that are co-prime to n.
- **Euler's Theorem**: For all integers a and n that are co-prime: $a^{\varphi(n)} \equiv 1 \pmod{n}$
- **Definition**: If $\varphi(n)$ is the smallest integer $t > 1$ such that $a^t \equiv 1 \pmod{n}$, then a is called a generator of $\mathbb{Z}_n^*$.

Example: $a = 2$ is a generator of $\mathbb{Z}_{11}^*$, where $\varphi(11) = 10$:

- $2^1 = 2$
- $2^2 = 2 \cdot 2 = 4$
- $2^3 = 2 \cdot 4 = 8$
- $2^4 = 16 \equiv 5$
- $2^5 \equiv 10$
- $2^6 \equiv 20 \equiv 9$
- $2^7 \equiv 18 \equiv 7$
- $2^8 \equiv 14 \equiv 3$
- $2^9 \equiv 6$
- $2^{10} \equiv 12 \equiv 1$



The Discrete Logarithm Problem (DLP)

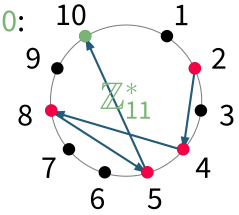
Discrete Logarithm Problem

Given a prime number p, a generator $g \in \mathbb{Z}_p^*$, and an element $y \in \mathbb{Z}_p^*$, find the integer $x \in \{0, \dots, p-2\}$ such that $\underbrace{g \cdot g \cdots g}_{x \text{ times}} = g^x \equiv y \pmod{p}$.

The DLP is believed to be hard in the group $(\mathbb{Z}_p^*, \cdot)$ for large primes p.

Example: Prime modulus $p = 11$, generator $g = 2$, and $y = 10$:

- $2^1 = 2$ ✗
- $2^2 = 4$ ✗
- $2^3 = 8$ ✗
- $2^4 = 16 \equiv 5 \pmod{11}$ ✗
- $2^5 = 32 \equiv 10 \pmod{11}$ ✓



The Integer Factorization Problem (IFP)

Integer Factorization Problem

Given $n \in \mathbb{N}$, find primes p_i and exponents $e_i \in \mathbb{N}$ such that $n = p_1^{e_1} \cdot p_2^{e_2} \cdot \cdots p_k^{e_k}$

The IFP is believed to be hard if n is the product of two large primes: $n = p \cdot q$

Example: $n = 143 \quad \Rightarrow \quad n = p \cdot q = 11 \cdot 13$

Key Exchange

Establishing Secure Communication

Diffie–Hellman (DH) Key Exchange

- In 1976, Diffie and Hellman proposed the first asymmetric cryptosystem.
- DH and its relatives are the most relevant key-exchange algorithms in today's protocols. They allow Alice and Bob to derive a new shared secret key.
- Turing Award 2015
  Sometimes called Diffie–Hellman–Merkle (Merkle invented asymmetric crypto)

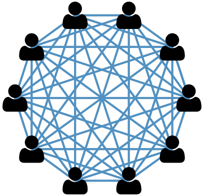
Whitfield Diffie

Martin Hellman

Ralph Merkle

Diffie–Hellman (DH) Key Exchange – Goal

- Solves the key distribution problem



If Alice and Bob want to start communicating, they exchange a few message to generate a shared secret key K to use for AEAD:

- Key agreement → Asymmetric crypto
- Actual communication → Symmetric crypto

Diffie–Hellman (DH) Key Exchange – Definition

Diffie–Hellman Key Exchange

Choose a large prime p and a generator α of $\mathbb{Z}_p^*$ (public system parameters).

$a \in \{2, \ldots, p-2\}$

$\xrightarrow{\alpha^a \pmod p}$

$\xleftarrow{\alpha^b \pmod p}$

$b \in \{2, \ldots, p-2\}$

$K_{AB} \equiv (\alpha^b)^a \pmod p$

$K_{BA} \equiv (\alpha^a)^b \pmod p$

- Correctness: $K_{AB} \equiv (\alpha^b)^a \equiv (\alpha)^{b \cdot a} = (\alpha)^{a \cdot b} \equiv (\alpha^a)^b \equiv K_{BA}$, so both Alice and Bob derive the same key $K \equiv K_{AB} \equiv K_{BA}$
- We call a Alice's private key and α^a her public key (same for Bob's b and α^b)

Diffie–Hellman (DH) Key Exchange – Example

Diffie–Hellman Key Exchange

Choose a large prime $p = 11$ and a generator $\alpha = 2$ of $\mathbb{Z}_p^*$ (public parameters).

$a = 3 \in \{2, \ldots, p-2\}$ $\xrightarrow{\alpha^a \equiv 8}$

$\xleftarrow{\alpha^b \equiv 10}$ $b = 5 \in \{2, \ldots, p-2\}$

$K_{AB} = (\alpha^b)^a \equiv 10^3 = 1000 \equiv 10$

$K_{BA} = (\alpha^a)^b \equiv 8^5 = 32768 \equiv 10$

Diffie–Hellman (DH) Key Exchange – Security

Alice and Bob have no previous shared secrets. Eve knows all exchanged info:

- Parameters p and α
- Alice's public key $\alpha^a \pmod{p}$
- Bob's public key $\alpha^b \pmod{p}$

Eve would like to know the secret $K_{AB} \equiv (\alpha^a)^b \equiv (\alpha^a)^b \equiv \alpha^{a \cdot b}$.
This looks easy, but is generally believed to be a hard problem for large p.

Diffie–Hellman Problem (DHP)

Given generator $\alpha \in \mathbb{Z}_p^*$ and $\alpha^a \pmod{p}$, $\alpha^b \pmod{p}$, find $K_{AB} = \alpha^{a \cdot b}$.

Best known solution to DHP: find a from α^a, or b from α^b (= solve DLP in $\mathbb{Z}_p^*$).

Recommended key size: For 128-bit security, p should be about **3072 bits** long.

Diffie–Hellman (DH) Key Exchange – Remarks

- The prime p and generator $\alpha \in \mathbb{Z}_p^*$ are public system parameters that can be the same for all users.
- Standards (NIST, ISO, ...) define parameters p, α for different security levels, how to encode values, how to use the resulting key K by hashing it to a suitable size, ...
- Modern protocols use ephemeral Diffie–Hellman (DHE) with temporary keypairs for forward secrecy.

Asymmetric Encryption

Confidentiality

Asymmetric Encryption using Trapdoor One-way Functions

Asymmetric cryptography makes extensive use of “**one-way functions**”:

easy to compute, hard to invert.

A “**trapdoor one-way function**” is a one-way function which can be inverted with an additional piece of information, the trapdoor information.

easy to lock

hard to open

unless you have the key

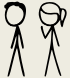
Asymmetric Encryption – Algorithms and Keys

Key Generation

Alice generates a private key and corresponding public key.
She distributes publicly and keeps safe.

Encrypt

With the public key, Bob (or anyone) encrypts a message M to a ciphertext C using $C = \mathcal{E}(M)$ and sends C to Alice.



Decrypt

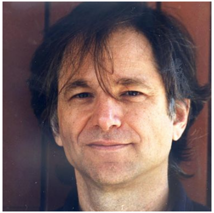
With her private key, Alice decrypts the ciphertext C to recover the message M using $\mathcal{D}(C) = M$

RSA (Rivest–Shamir–Adleman) Public-Key Encryption

- In 1977, Rivest, Shamir, and Adleman proposed one of the first public-key encryption schemes.
- RSA encryption as well as the related signature scheme are widely used.
- Turing Award 2002

Ron Rivest

Adi Shamir



Leonard Adleman

RSA Encryption (Rivest–Shamir–Adleman 1977)

Euler function:
$\varphi(pq) = (p-1)(q-1)$

Euler theorem:
if a, n are coprime, then
$a^{\varphi(n)} \equiv 1 \pmod{n}$

Key Generation

Choose 2 large, random primes p, q
Compute modulus $n = p \cdot q$
Choose public exponent e co-prime to $\varphi(n)$
Compute private exponent $d \equiv e^{-1} \pmod{\varphi(n)}$

public key $= (e, n)$ private key $= (d, n)$

Encrypt $\mathcal{E}(M)$

Encrypt message M:

$$C \equiv M^e \pmod{n}$$

Decrypt $\mathcal{D}(C)$

Decrypt ciphertext C:

$$M \equiv C^d \pmod{n} \equiv M^{e \cdot d} \equiv M^{1+k\varphi(n)} \equiv M$$

RSA Encryption – Example

Key Generation

Choose 2 tiny, random primes $p = 3, q = 11$

Compute modulus $n = p \cdot q = 33$

Choose public exponent $e = 3$ co-prime to
$\varphi(n) = (p-1)(q-1) = 2 \cdot 10 = 20$

Compute private exponent $d \equiv e^{-1} \pmod{\varphi(n)} \equiv 7 \pmod{20}$
since $d \cdot e = 3 \cdot 7 = 21 = 20 + 1 \equiv 1 \pmod{20}$

Euler function:
$\varphi(pq) = (p-1)(q-1)$

Euler theorem:
if a, n are coprime, then
$a^{\varphi(n)} \equiv 1 \pmod{n}$

Encrypt $\mathcal{E}(M = 4)$

$$C \equiv M^e \pmod{n}$$
$$= 4^3 \equiv 31 \pmod{33}$$

Decrypt $\mathcal{D}(C = 31)$

$$M \equiv C^d \pmod{n} \equiv M^{e \cdot d} \equiv M^{1+k\varphi(n)} \equiv M$$
$$= 31^7 \equiv 4 \pmod{33}$$

RSA Encryption – Security

RSA Problem (RSAP)

Given modulus n, exponent e, ciphertext C: find M such that $M^e \equiv C \pmod{n}$.

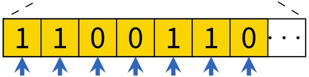
- If we can solve factorization (IFP), we can recover p, q from n and break RSA
- The RSAP is believed to be as hard as the IFP and infeasible for large n.
- The modulus n must be large enough so that the runtime of the best factoring algorithms is not feasible for any attacker.

  Factoring record 2009: 768-bit modulus ($\approx$ 2000 CPU years)
- "Security level of k bits" = we estimate that factoring n takes more than 2^k time

Recommended key size: For 128-bit security, n should be about **3072 bits** long.

RSA Encryption – Implementation: Square-and-Multiply

$$C = M^{\boxed{e}} \bmod n$$



$$\text{Result} = \underbrace{\text{Result} \times \text{Result}}_{\text{square}} \times \underbrace{\text{M}}_{\text{multiply}}$$

- Initialize result to 1
- Scan exponent *e* bit by bit
  - If bit is 0: square result
  - If bit is 1: square result, then multiply by M

RSA Encryption – Semantic Security

- There is a huge problem with this "textbook RSA": It is **deterministic**.
- If the message has low entropy (e.g., $M \in \{\text{yes}, \text{no}, \text{maybe}\}$), the attacker can intercept C, **guess** M **and verify if** $C = \mathbf{RSA}(M)$!
- We need a padding scheme to make RSA "**semantically secure**":

Indistinguishability (under Adaptive Chosen-Ciphertext Attack)

An attacker who knows the public key, chooses 2 messages M_0, M_1, and gets ciphertext C **can not distinguish** whether $C = E(M_0)$ or $C = E(M_1)$, even if they can ask for decryption of any $C^* \neq C$.

RSA Encryption – Padding for Semantic Security

PKCS #1 (**P**ublic-**K**ey **C**ryptography **S**tandard) defines 2 **RSA E**ncryption **S**chemes (**RSAES**):

⚠ **RSAES-PKCS1-v1_5** (⚠ deprecated):

$$C = \textbf{RSA}\left(\boxed{\texttt{00 02}} \boxed{\geq 8 \texttt{ random bytes}} \boxed{\texttt{00}} \boxed{\texttt{message } M} \right)$$

✔ **RSAES-OAEP** (“optimal asymmetric encryption padding”)

Signatures

Authenticity

Signatures – Algorithms in a Signature Scheme

Key Generation

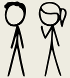
Alice generates a private key and corresponding public key.
She distributes publicly and keeps safe.

Sign

With her private key, Alice computes the signature $\mathcal{S}(M) = S$
of a message M. She transmits , to the recipient(s).

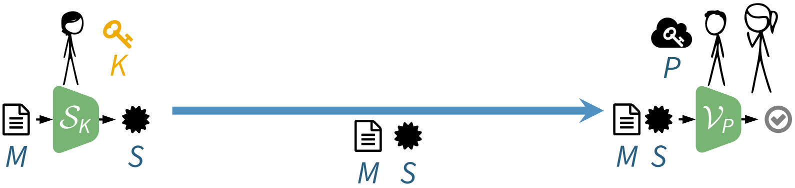
Verify

With the public key, Bob (or anyone) can verify the signature:
$\mathcal{V}(M, S) \in \{\checkmark, \times\}$



Signatures – Definition and Application

Signatures: private key K and public key P



Digital signatures ensure

- Sender authentication
- Message integrity
- Non-repudiation

Signatures – Security

- It must be easy to compute S using the private key
- It must be easy to verify S using the public key
- It must be hard to compute S without the private key (forgery)
  even if the attacker chooses the message and knows previous signatures

This is achieved using complexity-theoretically hard problems such as

- IFP: Integer factorization problem
- DLP: Discrete logarithm problem

RSA Signatures (Rivest–Shamir–Adleman 1977)

Euler function:
$\varphi(pq) = (p-1)(q-1)$

Euler theorem:
if a, n are coprime, then
$a^{\varphi(n)} \equiv 1 \pmod{n}$

Key Generation

- Choose 2 large, random primes p, q
- Compute modulus $n = p \cdot q$
- Choose public exponent e co-prime to $\varphi(n)$
- Compute private exponent $d \equiv e^{-1} \pmod{\varphi(n)}$

public key $= (e, n)$ private key $= (d, n)$

Sign $\mathcal{S}(M)$

Compute signature S:

$$S \equiv M^d \pmod{n}$$

Verify $\mathcal{V}(M, S)$

Verify that

$$M \stackrel{?}{\equiv} S^e \pmod{n} \equiv M^{d \cdot e} \equiv M^{1+k\varphi(n)} \equiv M$$

RSA Signatures – Security

The message M is recoverable from the signature S as $M \equiv S^e$
$\rightarrow$ An attacker could easily generate valid pairs (M, S)!

Solution: Sign the hash of the message (“signature with appendix”)

PKCS #1 defines 2 **RSA S**ignature **S**chemes with **A**ppendix (**RSASSA**):

- **RSASSA-PKCS1-v1_5** (legacy):
  1. Compute **hash**(M)
  2. $S =$ **RSA-Sign**$\left(\boxed{\texttt{00 01}}\boxed{\texttt{FF}\cdots\texttt{F}}\boxed{\texttt{00}}\boxed{\textbf{hash}(M)}\right)$
- **RSASSA-PSS** (provably secure “**p**robabilistic **s**ignature **s**cheme”)

Conclusion

Conclusion

- Establishing a secure communication channel
  - Authentication ➔ Asymmetric crypto
  - Key agreement ➔ Asymmetric crypto
  - Actual communication ➔ Symmetric crypto
- Important asymmetric schemes (key sizes: 3072+ bits)
  - Diffie–Hellman (DH) key exchange
  - RSA encryption
  - RSA signatures
  - DSA signatures