Idea

- The permutation test is a resampling method for testing whether two distributions are the same
- This test is exact: it is not based on large sample approximations
- Let $X_1, X_2, \dots, X_n$ be a random sample of size $n$, $X_1 \sim F_X$
- Let $Y_1, Y_2, \dots, Y_m$ be a random sample of size $m$, $Y_1 \sim F_Y$
- With permutation test we are testing:

$$H_0 : F_X = F_Y \text{ against } H_1 : F_X \neq F_Y$$

- Let $T_N = t(X_1, \dots, X_n, Y_1, \dots, Y_m)$ be some test statistic, where $N = n + m$
- E.g. $T_n = |\overline{X}_n - \overline{Y}_m|$
- We consider all $N!$ permutations of the data $X_1, \dots, X_n, Y_1, \dots, Y_m$
- For each permutation we compute the test statistics $T$
- We denote these values with $T_1^*, \dots, T_{N!}^*$

- How likely are each of the $T_1^*, \dots, T_{N!}^*$ under the $H_0$?
- Equally likely!
- The distribution $P_0$ that puts $1/N!$ mass on each $T_j^*$ is called the **permutation distribution** of $T$
- Let $t_n$ be the observed value of the test statistic
- Assuming we reject when $T$ is large, the p-value of the permutation test:

$$p - value = P_0(T* > t_N) = \frac{1}{N!} \sum_{j=1}^{N!} I(T_j^* \geq t_N)$$

- Usually, it is not practical to evaluate all $N!$ permutations
- We can approximate the p-value by simulating random permutations
- The fraction of times $T_j^* > t_N$ among these samples approximate the p-value

Example

- toy
  - Suppose the data are: $(X_1, X_2, Y_1) = (3, 9, 1)$. Let $T(X_1, X_2, Y_1) = |\overline{X}_n - \overline{Y}_m|$, i.e. $t_N = 5$. Compute the p-value of the test statistic $T$.
  - Suppose the data are: $(X_1, X_2, Y_1) = (3, 9, 1)$. Let $T(X_1, X_2, Y_1) = |\overline{X}_n - \overline{Y}_m|$, i.e. $t_N = 5$. Compute the p-value of the test statistic $T$.

| permutation | $T^*$ | $P_0(T^*)$ |
|---|---|---|
| $(3,9,1)$ | 5 | $1/6$ |
| $(1,3,9)$ | 7 | $1/6$ |
| $(1,9,3)$ | 2 | $1/6$ |
| $(3,1,9)$ | 7 | $1/6$ |
| $(9,1,3)$ | 2 | $1/6$ |
| $(9,3,1)$ | 5 | $1/6$ |

p-value: $P(T^* \geq 5) = 4/6$

- driving behavior

  Suppose we have a list of cities with the car driver velocities on weekdays and weekends. Test if the driving behavior in each city is different on weekdays as compared to weekends.

  We take $H_0$: no difference in driving behavior and test it against $H_1$: driving behavior is different at weekdays and weekends for a given city. We use permutation test for the difference in means.