

Sprague-Grundy-Theory

- First/Second-Player Win depends on number of coins
 - for a single pile

Take away 1≤i≤k coins 1,2,3 coins	1,2 coins: O 2nd O 1st O 1st O 2nd O 1st O 1st O 1st O 1st
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- Nimbers

Nimbers $*i, i \geq 0$, are a 'code' used for game-positions:

 - $*i, i \neq 0 \Rightarrow$ 1st player win (the player to move)
 - $*0 \Rightarrow$ 2nd player win (the one just moved)

A nimber code implies:

- From a $*0$ situation no legal move leads to another $*0$ situation
 - Interpretation: If I made a winning move, my opponent can not
- From any $*i, i \neq 0$, situation there is a legal move to a $*0$ situation
 - Interpretation: If my opponent gives me a (for them) non-optimal situation, I can make a winning move

- nimber depends on height

MOVE	MEX	RULE
$\text{High} \leftarrow i$	\star_2	
	\star_4	
$\rightarrow \star_2$	\star_3	
$\rightarrow \star_1$	\star_1	
$\rightarrow \star_0$	\star_2	
$\rightarrow \star_1$	\star_2	
$\rightarrow \star_0$	\star_1	

NIM Rules

MEX-rule (Minimal Excluded):

- The nimber of a position P is the smallest value which is NOT a nimber of any position which is reachable by a valid move from P .

- The MEX-rule guarantees a good code!
- From a \star_0 situation no legal move leads to another \star_0 position
- From any $\star_i, i \neq 0$, situation there is a legal move to a \star_0 situation.

- XOR-rule:

- The nimber of a set of positions is the XOR-sum of the nimbers of the positions.

- Simplifies computation of nimbers for several piles.
- That is, using the MEX-rule AND the XOR-rules makes computations much more efficient!
- For correctness the MEX-rule is sufficient.

*3	O	O	O
1st player	O	O	O
2nd	O	O O	O
*4 *3 *1 *5			
4 ≈ 100		XOR:	
3 ≈ 011		011 ≈ *3	
1 ≈ 001			
5 ≈ 101			

Optimal Strategy

- compute each pile's nimber
 - XOR to find out if a winning move exists
 - zero \Rightarrow no winning move
 - compute for each pile if there is a winning move for this pile
 - execute a optimal move
 - repeat after opponent's move

NIM:		High Number	A: 7 ≈ 111	Optimal moves:
0	7		→ 100 ≈ *4 ✓	
0			B: 2 ≈ 010	→ 001 ≈ *1 ✓
0	5	9	C: 5 ≈ 101	→ 110 ≈ *6 ↴
0	0	8	D: 3 ≈ 011	→ 000 ≈ *∅ ✓
0	0	7		
0	0	6	011 ≈ 3 *3	
0	2 0 0	5	Goul: 000 ≈ *∅	
0	0 0 0	4		⇒ winning moves:
0	0 0 0	3	A: remove 3 coins from A	
		2	B: -II- 1 coin -II- B	
		1	C: -II- 3 coins -II- D	
		0		
*3 1st player win				

Variations of [[NIM-type Games]]

- take a limited number of coins each turn

$\begin{matrix} 5 & 0 \\ 0 & 0 \end{matrix}$ $\begin{matrix} 0 & 3 \\ 0 & 0 \end{matrix}$ $\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$ $\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$	$\begin{matrix} 6 \\ : \\ 6 \\ 5 \\ 4 \end{matrix}$ <small>HIGH Number</small>	<i>Take 1, 2, or 3 coins from one pile</i> $A: *1 \triangleq 001$ $B: *3 \triangleq 011$ $C: *2 \triangleq 010$ <hr/> $A \ B \ C$ $*1 \ *3 \ *2$ $\underbrace{\quad}_{\Rightarrow *0}$ $3 \ *3$ $2 \ *2$ $1 \ *1$ $0 \ ; *0$	\oplus $xor: 000 \triangleq *0$ <i>2nd player win</i> $\Rightarrow NO$ winning moves
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- take 1 coin or split a pile into two (non-empty) piles

Take 1 coin OR split a pile into two piles

$\begin{matrix} 0 & *0 \\ 0 & *2 \\ 0 & *4 \\ 0 & *2 \\ 0 & *0 \\ 0 & *2 \\ 0 & *0 \\ 0 & *2 \\ 0 & *1 \\ 0 & *0 \end{matrix}$	$H=2: \rightarrow H=1 \rightarrow *1$ $\rightarrow H=1, H=1: *1, *1 \triangleq *0$ $H=3: \rightarrow H=2 \rightarrow *2$ $\rightarrow H=2, H=1: *2, *1 \triangleq *3$ $H=4: \rightarrow H=3 \rightarrow *0$ $\rightarrow H=3, H=1: *0, *1 \triangleq *1$ $H=2, H=1: *2, *2 \triangleq *0$	$H=5:$ $\rightarrow H=4: *2$ $\rightarrow H=4, H=1: *2, *1 \triangleq *3$ $\rightarrow H=3, H=2: *2, *2 \triangleq *0$ $\rightarrow H=3, H=1: *1 \triangleq *0$ $\rightarrow H=2, H=3: *0, *2 \triangleq *2$ \dots
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- Laskers NIM

- NIM or split a pile into two (non-empty) piles

LASKERS NIM $\begin{matrix} 0 & 10 & *10 \\ 0 & 9 & *9 \\ 0 & 8 & *7 \\ 0 & 7 & *8 \\ 0 & 6 & *6 \\ 0 & 5 & *5 \\ 0 & 4 & *3 \\ 0 & 3 & *4 \\ 0 & 2 & *2 \\ 0 & 1 & *1 \\ 0 & 0 & *0 \end{matrix}$	$H=5: \rightarrow H=4: *3 \quad H=4, 1: *3, *1$ $\rightarrow H=3: *4 \quad *2$ $H=2: *2 \quad H=3, 2: *4, *2$ $H=1: *1 \quad =*6$ $H=0: *0 \quad$ $H=6: *5, *3, *4, *2, *1, *0$ $H=5, 1: *5, *1 \triangleq *4$ $H=4, 2: *3, *2 \triangleq *1$ $H=3, 3: *4, *4 \triangleq *0$ $H=7: *0, \dots, *6$ $H=6, 1: *6, *1 \triangleq *7$ $H=5, 2: *5, *2 \triangleq *7$ $H=4, 3: *3, *4 \triangleq *7$
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LASKERS NIM		position/number to reach:	H : *
A	0	$0100 \triangleq *4$	12 : *11
B	0	$1100 \triangleq *12$	11 : *12
C	9	$1000 \triangleq *11$	10 : *10
D	0	$1100 \triangleq *12$	9 : *9
E	0	$1110 \triangleq *14$	8 : *7
F	6	$1001 \triangleq *1$	7 : *8
G	0	$0001 = *1$	6 : *6
H	0	$\overline{XOR: 0111 \triangleq *7}$	5 : *5
I	0	1st player wins	4 : *3
J	0	$\textcircled{1} \text{ Take 5 coins from pile D}$	3 : *4
K	0	$\textcircled{2} \text{ Take 1 coin from pile A}$	2 : *2
L	0	$\textcircled{3} \text{ Take 1 coin from pile B}$	1 : *1
M	7	$\textcircled{4} \text{ Split pile D into 4 and 2}$	0 : *0

- Kayles and Dawson's Kayles
 - no repeating nimber patterns

[[Algorithms and Games]]