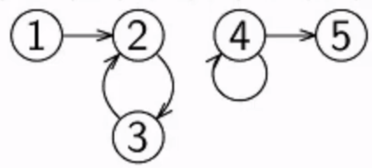
Terminology

- **Directed graph** $G = (V, E)$:
  V is the set of vertices and
  E is the set of edges: $E \subseteq V \times V$
  edge (u, v): ordered pair of vertices,
  u = startpoint, v = endpoint

- Example: $V = \{1, 2, 3, 4, 5\}$,
  $E = \{(1,2), (2,3), (3,2), (4,5), (4,4)\}$



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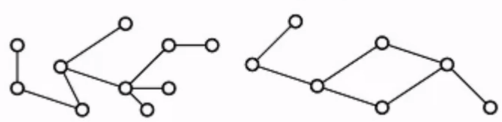
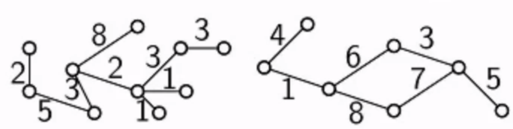
- A **loop** at a vertex $v \in V$ is an edge (v, v)
- A directed Graph is called **simple** if it has no loops

- **Undirected graph**: like a simple directed graph where E is symmetric, that is, $(v, w) \in E \Leftrightarrow (w, v) \in E$

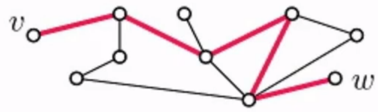


$\Leftrightarrow$ E is a set of unordered pairs (v, w) with $v \neq w \in V$

- A (directed or undirected) **weighted graph** is a triple $G = (V, E, g)$ where g assigns a weight to each edge



- A **path** in $G = (V, E)$ from v to w is a sequence of vertices $v = v_0, v_1, \ldots, v_k = w \in V$ with edges $(v_i, v_{i+1}) \in E$ for all $0 \leq i < k$



- **Simple** path: contains every vertex in V at most once
- **Length** of a path: number of its edges (v_i, v_{i+1})
  length in *weighted graphs*: sum of the edge weights
- **Distance** $d_G(v, w)$ from v to w: length of the *shortest path* from v to w in G (or ∞ if no such path exists)

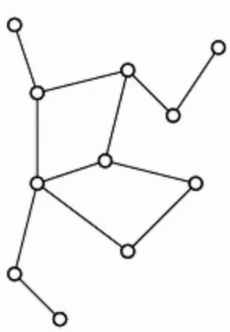
- A **cycle** in $G = (V, E)$ is a path $v_0, v_1, \ldots, v_k$ with $v_0 = v_k$

- **Simple** cycle: contains every vertex in V at most once, except for $v_0 = v_k$
- **Length** of a cycle: number of its edges (v_i, v_{i+1})
  length in *weighted graphs*: sum of the edge weights
- **Trivial** cycle: contains only one vertex: 
  or two for undirected graphs: 

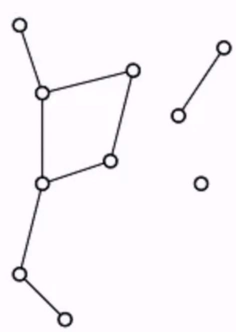
- A **subgraph** $G' = (E', V')$ of $G = (E, V)$ is a graph with $V' \subseteq V$ and $E' \subseteq E$. Note that for every edge $(u', v') \in E'$ we must have $u', v' \in V'$.

$Example:$

Graph G



Subgraph G'



- A graph is **acyclic** if does not contain any subgraph that is a non-trivial simple cycle

acyclic :

not acyclic :

- A graph is **connected** if it contains a path from v to w for every pair $v, w \in V$ (not necessarily an edge (v, w))

connected :

disconnected :

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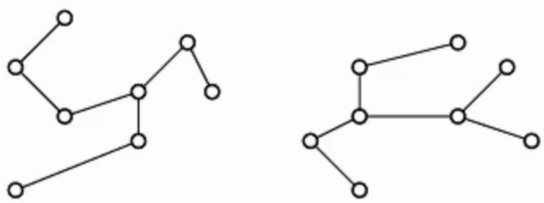
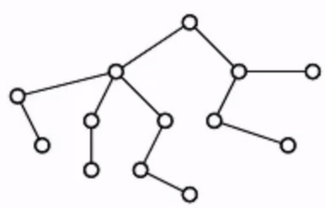
- An undirected graph is a **forest** if it is acyclic



- An undirected graph is a **tree** if it is acyclic and connected



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- The **in-degree (out-degree)** of a vertex $v \in V$ is the number of edges ending (starting) in v:

  in-degree $(v) := |\{w \mid (w, v) \in E\}|$

  out-degree $(v) := |\{w \mid (v, w) \in E\}|$

- For undirected graphs we have

  in-degree(v) = out-degree(v) $\quad \forall\, v \in V$.

  In this case it is called the **degree** of the vertex v.

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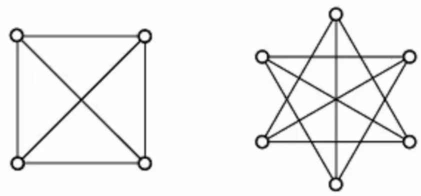
Degrees of vertices in an undirected graph $G = (V, E)$:

- degree $(v) = 0$: v is called an **isolated vertex** of G
- degree $(v) = 1$: v is called a **leaf** (end vertex) of G
- degree $(v) = r \quad \forall\, v \in V$: The graph G is called a **regular graph** of degree r

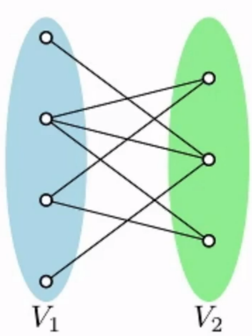
  Example: regular graph(s) of degree 3



- An undirected **complete graph** on n vertices contains all $\binom{n}{2}$ possible edges (regular of degree $n-1$)

- An **empty graph** has no edges (regular of degree 0)

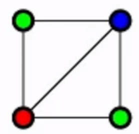
A graph $G(V, E)$ is called **bipartite**, if there exists a partition of V into V_1, V_2 such that all $(u, v) \in E$ have one endpoint in V_1 and the other one in V_2.



In other words:
G contains no edges within V_1 or V_2

k-Colorability of a Graph

A graph $G(V, E)$ is called **k-colorable** if its nodes can be colored with $\leq k$ colors, so that no nodes of the same color share an edge.

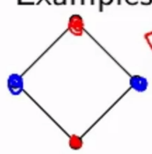


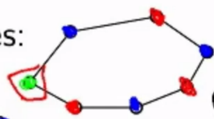
Example: How many colors are needed?

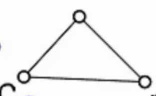
$k = 3$ colors are enough

The **chromatic number** $\chi(G)$ (spoken 'Chi of G') of a graph G is the minimum k such that G is k-colorable.

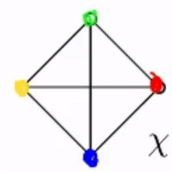
Examples:







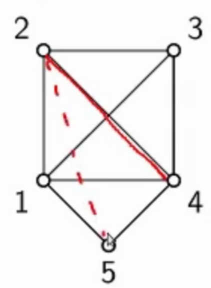
$\chi(\text{cycle}) = \begin{cases} 2 & \text{even \# vtcs} \\ 3 & \text{odd \# vtcs} \end{cases}$



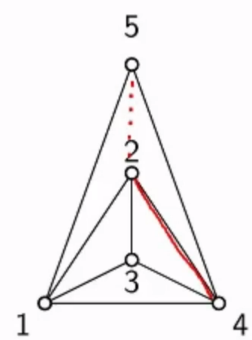
$\chi(K_4) = 4$

- Two graphs $G = (V, E)$ and $G'(V', E')$ are called **isomorphic** if there exists a bijection $\phi : V \rightarrow V'$, so that $(v, w) \in E \Leftrightarrow (\phi(v), \phi(w)) \in E'$

Example:



$\Leftrightarrow$



- **Definition 1:**
A graph $G = (V, E)$ is called **planar** if there exists an injective mapping of V to $n = |V|$ distinct points in the Euclidian plane and a mapping of E to pairwise disjoint open curves in the Euclidian plane, so that:
  1. $e = (x_i, x_j) \Rightarrow$ curve (e) connects point (x_i) with point (x_j)
  2. $e = (x_i, x_j) \Rightarrow$ curve (e) does not contain any other point $(x_k), k \neq i, j$

Definition 2:
A graph is called **planar** if it can be drawn in the plane without intersections between its edges (except in endpoints).

A drawing of a graph is called **plane** if it is free of intersections.

A non-plane drawing of a graph is called **embeddable** if it is isomorphic to a planar graph.

A **plane graph** is a plane drawing of a planar graph.

Example of an embeddable / plane drawing:

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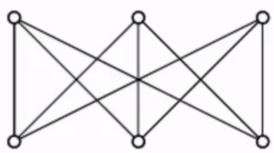
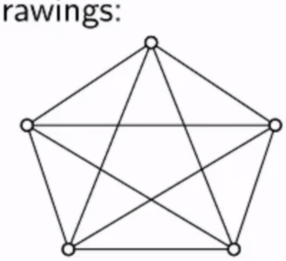
Examples of non-embeddable drawings:

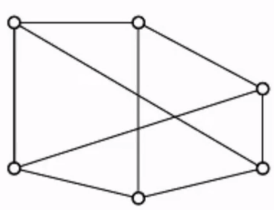


$K_{3,3}$

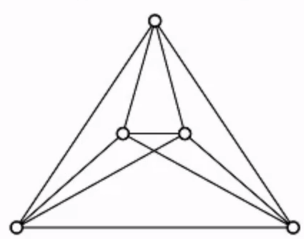


K_5

Non-embeddable drawing ⇔ drawing of a non-planar graph



$K_{3,3}$



K_5

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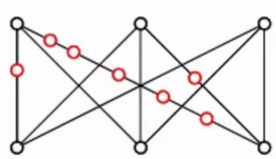
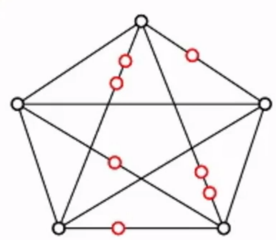
embeddability is invariant under subdivision of edges



$K_{3,3}$

additional vertices: degree 2



K_5

Kuratowski's Theorem: A graph G is planar if and only if it does not contain a subgraph G' that is isomorphic to a *subdivision* of $K_{3,3}$ or K_5, respectively. [without proof]

Subdivision of a graph G: some vertices are added "on edges" of G (some edges of G replaced by paths)

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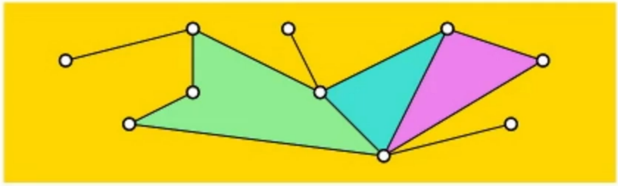
A plane graph (= plane drawing of a planar graph) divides the plane into regions called the *faces* of the drawing

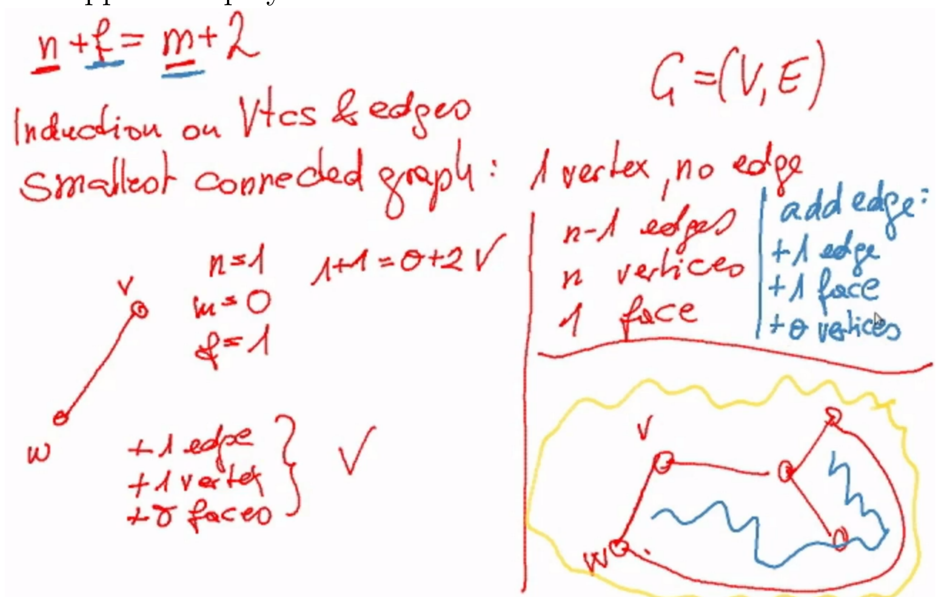


4 faces
12 edges
10 vertices

Theorem: Euler's Formula (on plane graphs)
Given a connected, plane graph with exactly f faces, $n = |V|, m = |E|$. Then it holds that:

$$n + f = m + 2$$

- In other words: vertices - edges + faces = 2
  - also applies for polyhedrons



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  – Planarity

Theorem: For a planar connected graph $G = (V, E)$ with $|V| = n \geq 3$ and $|E| = m$ it holds that $m \leq 3 \cdot n - 6$

Proof: Consider a plane drawing of G with f faces. Let $e_i,\ i = 1, \ldots, f$ be the number of edges which bound face i.

- $\sum_{i=1}^{f} e_i \geq \sum_{i=1}^{f} 3 = 3 \cdot f$ (since $e_i \geq 3$)
- $\sum_{i=1}^{f} e_i = 2 \cdot m$
  (since each edge is counted twice – once from each side)

$\Rightarrow$ Combine the two results: $3 \cdot f \leq 2 \cdot m$

$\Rightarrow$ Combine with Euler's Formula (multiplied by 3):

$$3 \cdot m + 6 = 3 \cdot n + 3 \cdot f \leq 3 \cdot n + 2 \cdot m$$
$$\Leftrightarrow \qquad m \leq 3 \cdot n - 6 \qquad ✓$$

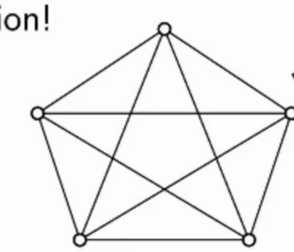
*

Claim: K_5 is not planar.

Proof:
- K_5 has $n = 5$ vertices and $m = \binom{5}{2} = 10$ edges
- Assume K_5 is planar

  $\Rightarrow$ We can apply the previous theorem
  $m \leq 3 \cdot n - 6$, with $n = 5, m = 10$
  $10 \leq 3 \cdot 5 - 6 = 9$ contradiction!

  $\Rightarrow$ K_5 cannot be planar. ✓



◆

Theorem: For a bipartite planar connected graph $G = (V, E)$ with $|V| = n \geq 3$ and $|E| = m$ it holds that $m \leq 2 \cdot n - 4$

Proof: Consider a plane drawing of G with f faces. Let e_i, $i = 1, \ldots, f$ be the number of edges which bound face i.

- G bipartite $\Rightarrow$ every face is 2-colorable $\Rightarrow e_i \geq 4$
- $\sum_{i=1}^{f} e_i \geq \sum_{i=1}^{f} 4 = 4 \cdot f$, $\qquad \sum_{i=1}^{f} e_i = 2 \cdot m$

$\Rightarrow$ Combine the two results: $4 \cdot f \leq 2 \cdot m \Leftrightarrow 2 \cdot f \leq m$

$\Rightarrow$ Combine with Euler's Formula (multiplied by 2):

$$2 \cdot m + 4 = 2 \cdot n + 2 \cdot f \leq 2 \cdot n + m$$
$$\Leftrightarrow \qquad m \leq 2 \cdot n - 4$$
✓

*

Claim: K_5 is not planar.

Proof:
- K_5 has $n = 5$ vertices and $m = \binom{5}{2} = 10$ edges
- Assume K_5 is planar

  $\Rightarrow$ We can apply the previous theorem
  $m \leq 3 \cdot n - 6$, with $n = 5, m = 10$
  $10 \leq 3 \cdot 5 - 6 = 9$ contradiction!

  $\Rightarrow$ K_5 cannot be planar. ✓

by this, $K_{3,3}$ could be planar!

Question:
Can we show in the same way that $K_{3,3}$ is not planar?

$n=6$ vtcs, $m = 3 \cdot 3 = 9$ edges
$9 \leq 3 \cdot 6 - 6 = 12$ ✓

◆

Claim: $K_{3,3}$ is not planar.

Proof:

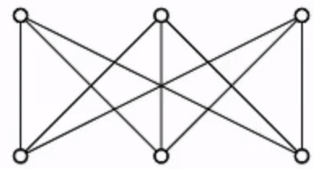
- $K_{3,3}$ has $n = 6$ vertices and $m = 3 \cdot 3 = 9$ edges
- Assume $K_{3,3}$ is planar

$\Rightarrow$ We can apply the previous theorem

$m \leq 2 \cdot n - 4$, with $n = 6, m = 9$

$9 \leq 2 \cdot 6 - 4 = 8$ contradiction!

$\Rightarrow$ $K_{3,3}$ cannot be planar.



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