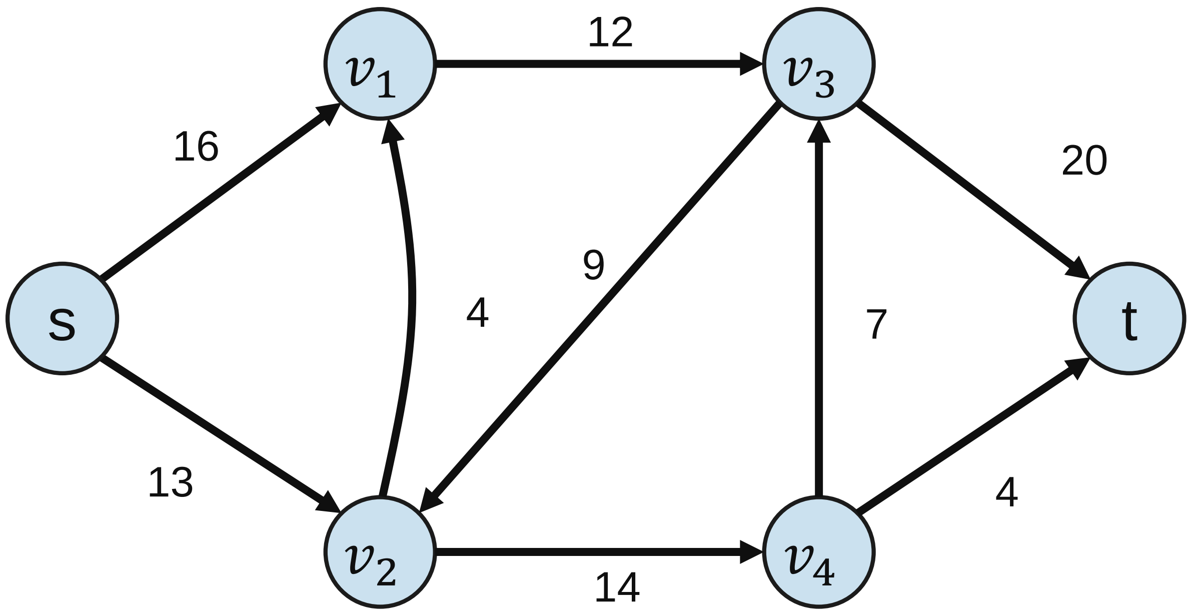


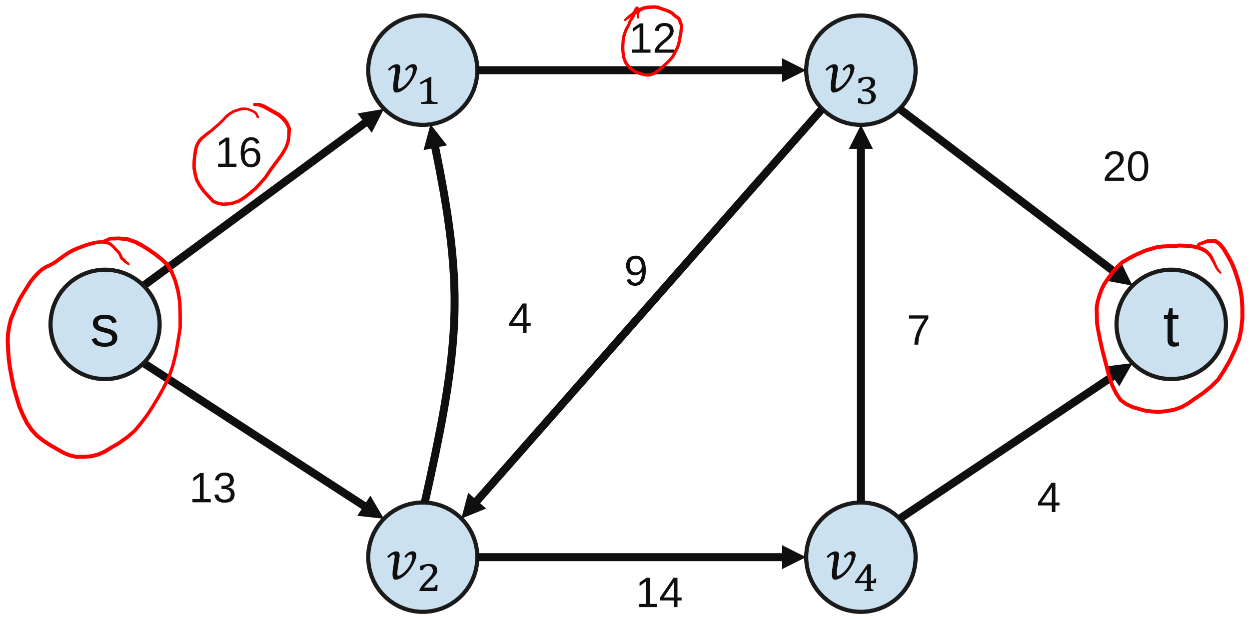
Max Flow

Yannic Maus

Flow network

A **flow network** $(\boldsymbol{G}, \boldsymbol{c})$ consists of a directed graph $G = (V, E)$ together with

- a capacity function $c: E \to \mathbb{N}_{\geq 0}$ **(set to $\boldsymbol{c(u,v) = 0}$ for $\boldsymbol{(u,v) \notin E}$)**
- a source s, a sink t: in-degree$(s) = 0$, out-degree$(t) = 0$



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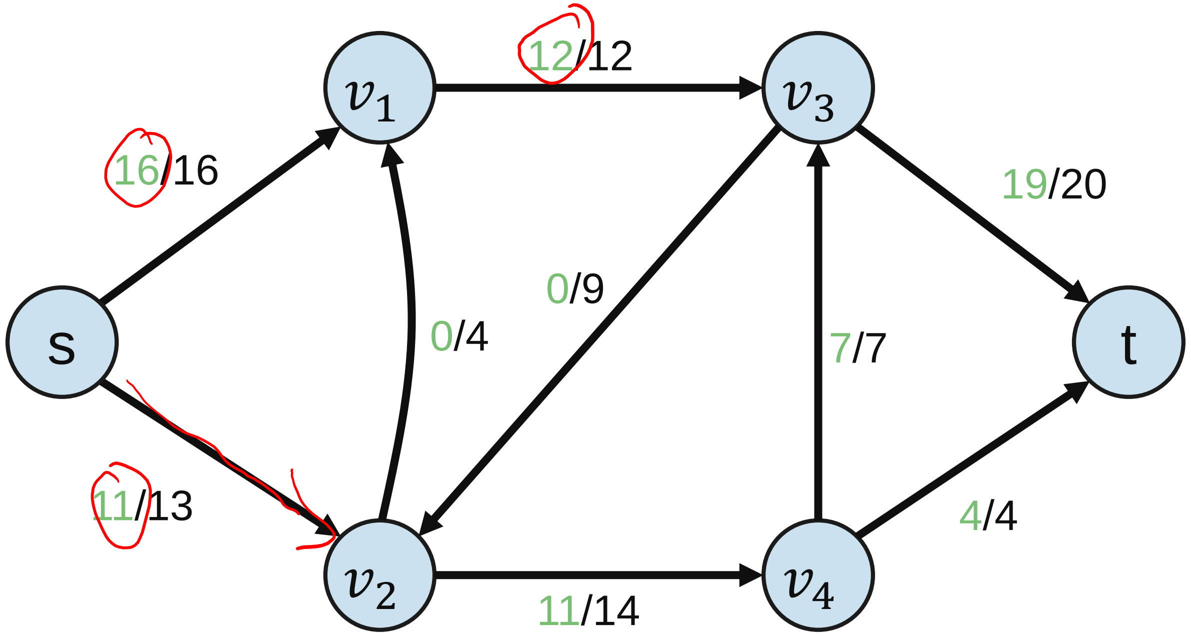
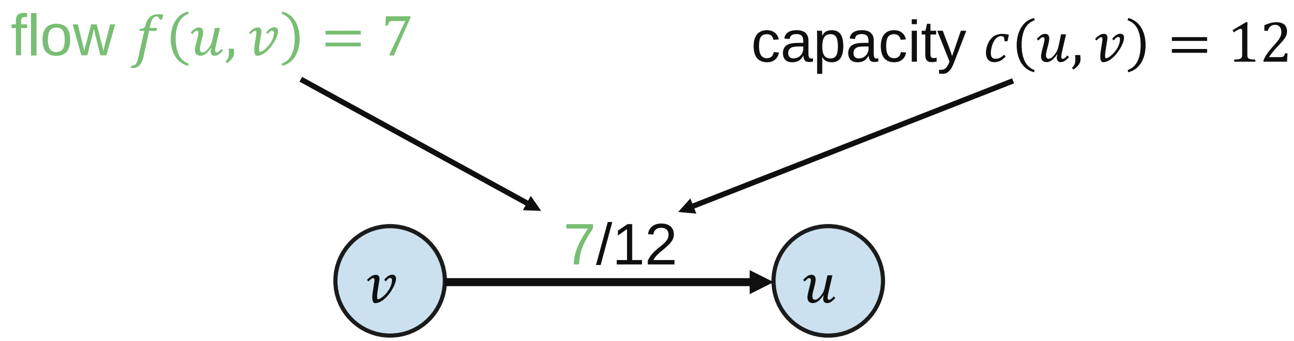


A flow in a flow network (G, c) is a function $f: V \times V \to \mathbb{R}$ that satisfies

- **capacity constraints,**
- **conservation of flow,**
- **skew symmetry.**

Capacity constraint

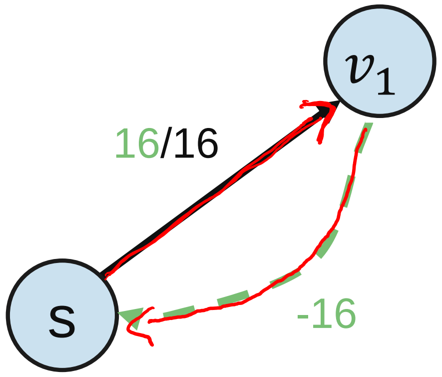
Capacity constraint: $f(u,v) \leq c(u,v)\ \forall (u,v) \in V \times V$



Never exceed the capacity of an edge

(technicality: we might have $c(u,v) = 0$, and $f(u,v) < 0$)

Skew symmetry: $f(u,v) = -f(v,u)$

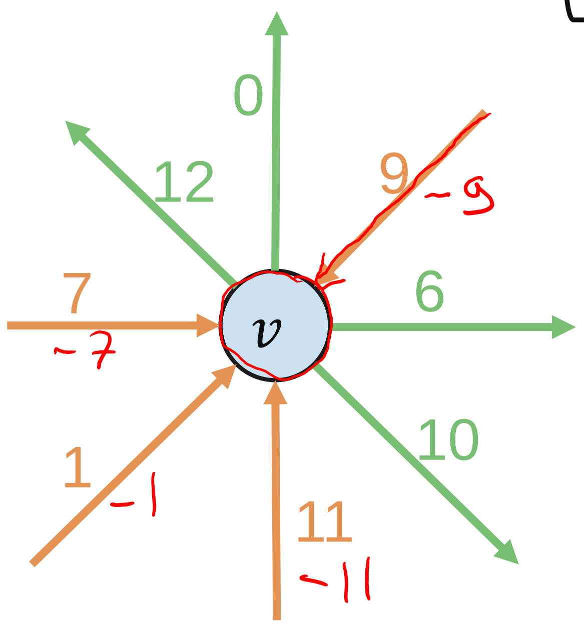


We only note down positive flows.
The other direction of flow is "implicit"

The net flow from u to v must be the opposite of the net flow from v to u.

Conservation of flow

Conservation of flow: $\forall v \in V \setminus \{s,t\}$: $\underbrace{f(v,V) = \sum_{u\in V} f(v,u) = 0}$



$f(v,V) = -f_{in}(v) + f_{out}(v) = 0$

$f_{in}(v)$ =9+7+11+1=28

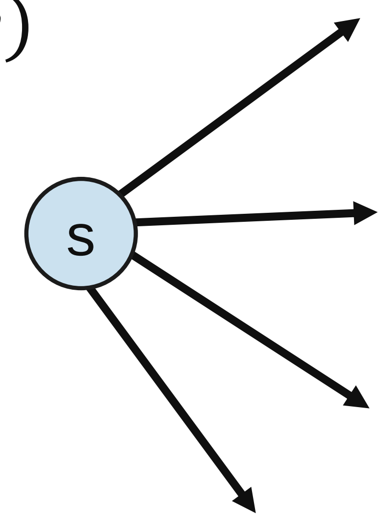
$f_{out}(v)$ =6+10+12=28

The net flow to a node is zero, except for the source, which "produces" flow, and the sink, which "consumes" flow

Value of a flow

The value $|f|$ of a flow f is defined as $|f| = \sum_{v \in V \setminus \{s\}} f(s, v)$

"the flow leaving the source"



Goal: Given a flow network, find a flow with maximum value

Value of a flow

The value $|f|$ of a flow f is defined as $|f| = \sum_{v \in V \setminus \{s\}} f(s, v)$

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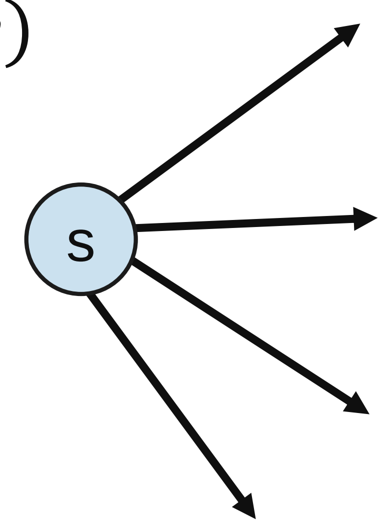
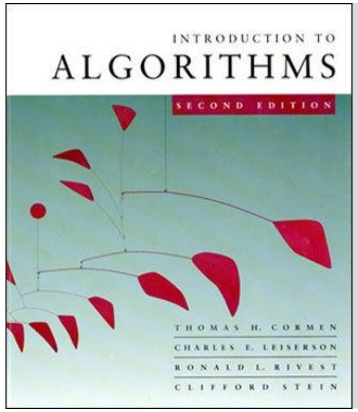


Goal: Given a flow network, find a flow with maximum value



Warning! Not all literature have the exact same definition of a flow, e.g., CLRS requires $f(u, v) \geq 0$ for all $u, v \in V$, while we explicitly allow and make use of $\underline{f(u, v) < 0}$ (recall skew symmetry).

Both definitions lead to a solid theory and arguments are similar, but differ a bit in the details.

Agenda

- Ford-Fulkerson-Method
  - Correctness & Termination
  - Bad Examples
- Edmonds-Karp-Algorithm
  - Runtime analysis
- Outlook: State of the art …
- Application: Maximum Cardinality Bipartite Matchings

Ford-Fulkerson-Method

Ford-Fulkerson Method

Input: Given a flow network (G, c) with source node s, and sink node t
Output: Compute a flow f from s to t of maximum value

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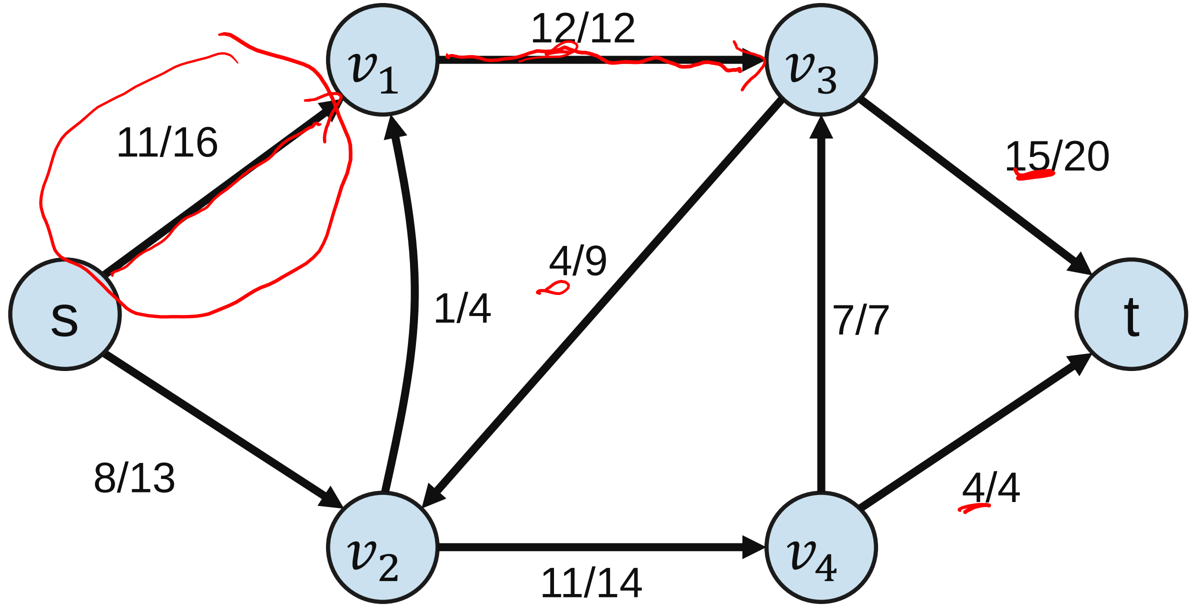
```
Initialize flow f with 0
while there exists an augmenting path p in G_f do
    augment f along p (by f_p)
return f
```

Residual Network

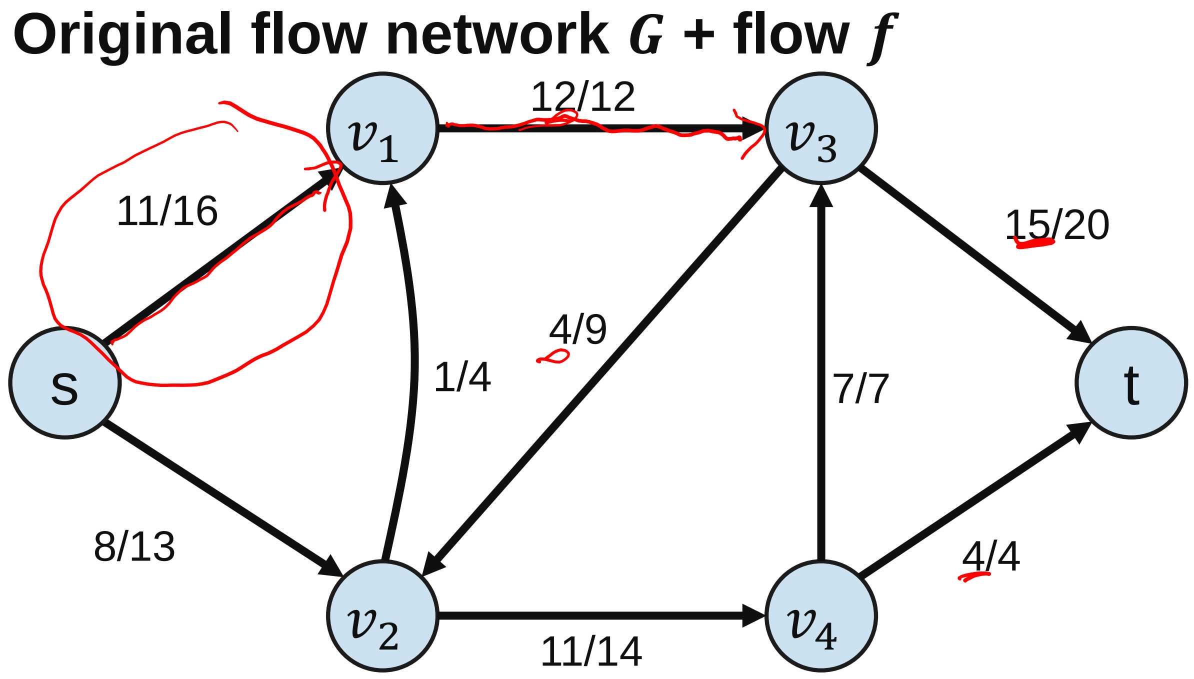
Given a flow network (G, c) and a flow f: for **all pairs** $(u, v) \in V \times V$ define the **residual capacity** of f $\boldsymbol{c_f(u, v) = c(u, v) - f(u, v)}$

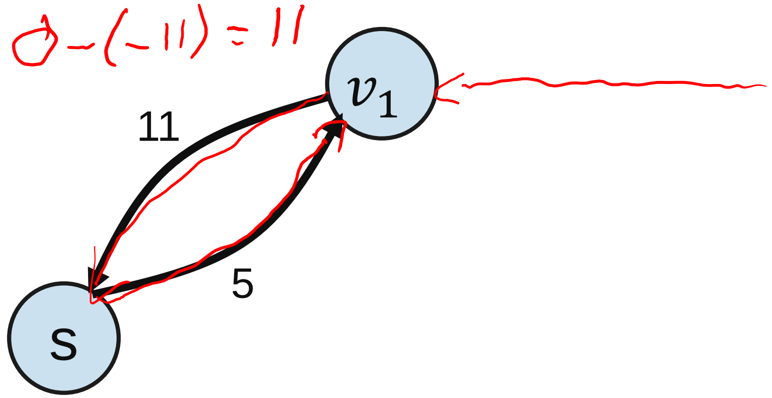
Original flow network G + flow f



$0-(-11)=11$

Given a flow network (G, c) and a flow f: for **all pairs** $(u, v) \in V \times V$ define the **residual capacity** of f $\boldsymbol{c_f(u,v) = c(u,v) - f(u,v)}$





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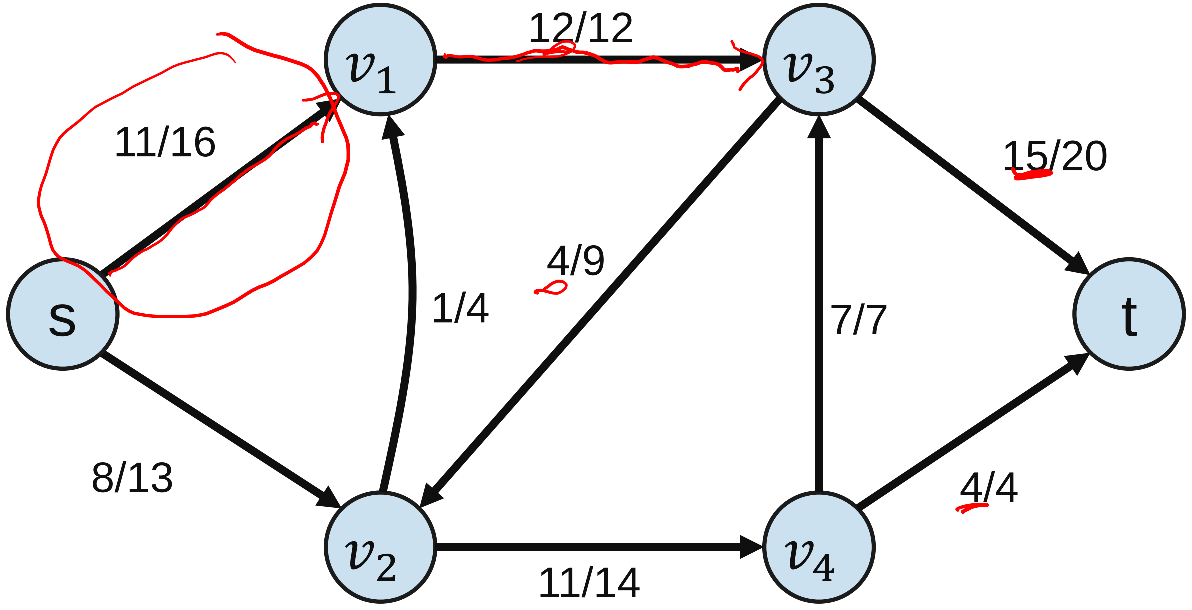
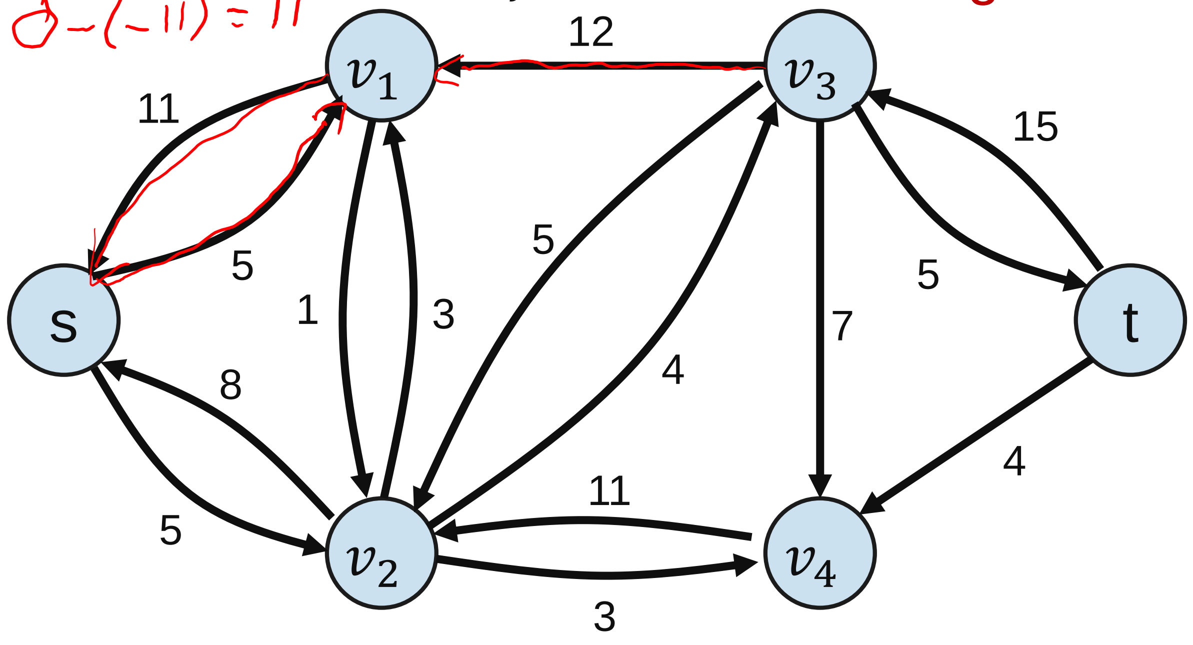


Residual network G_f

backward edges!

$0-(-11)=11$



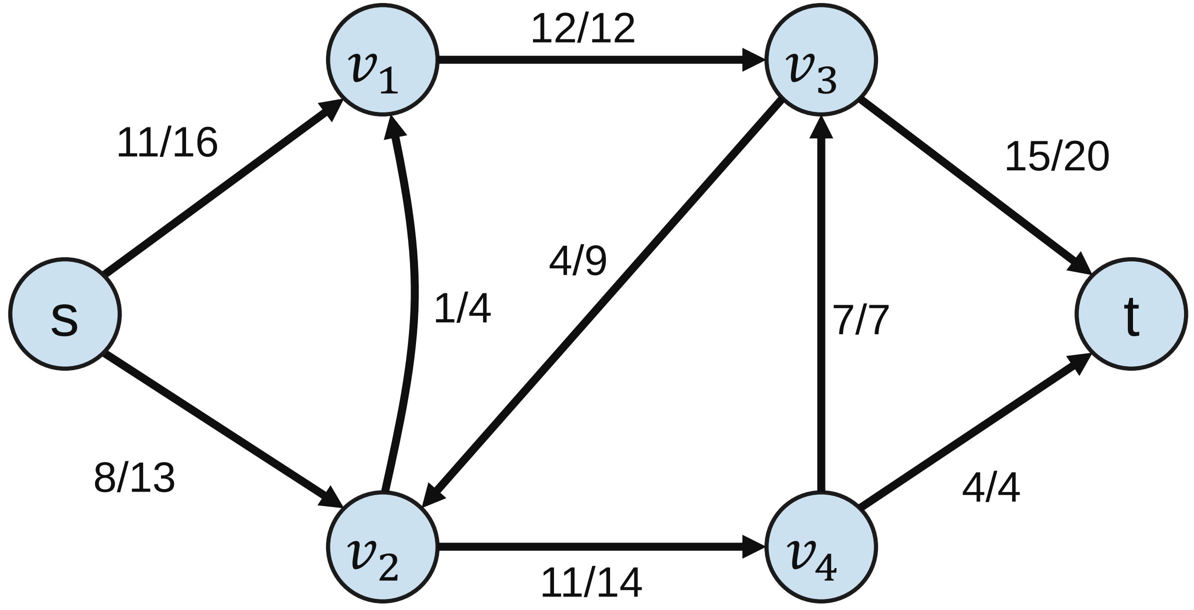
Given a flow network (G, c) and a flow f: for **all pairs** $(u, v) \in V \times V$ define the **residual capacity** of f $\boldsymbol{c_f(u,v) = c(u,v) - f(u,v)}$

The **residual network** $G_f(V, E_f)$ is given via the edge set

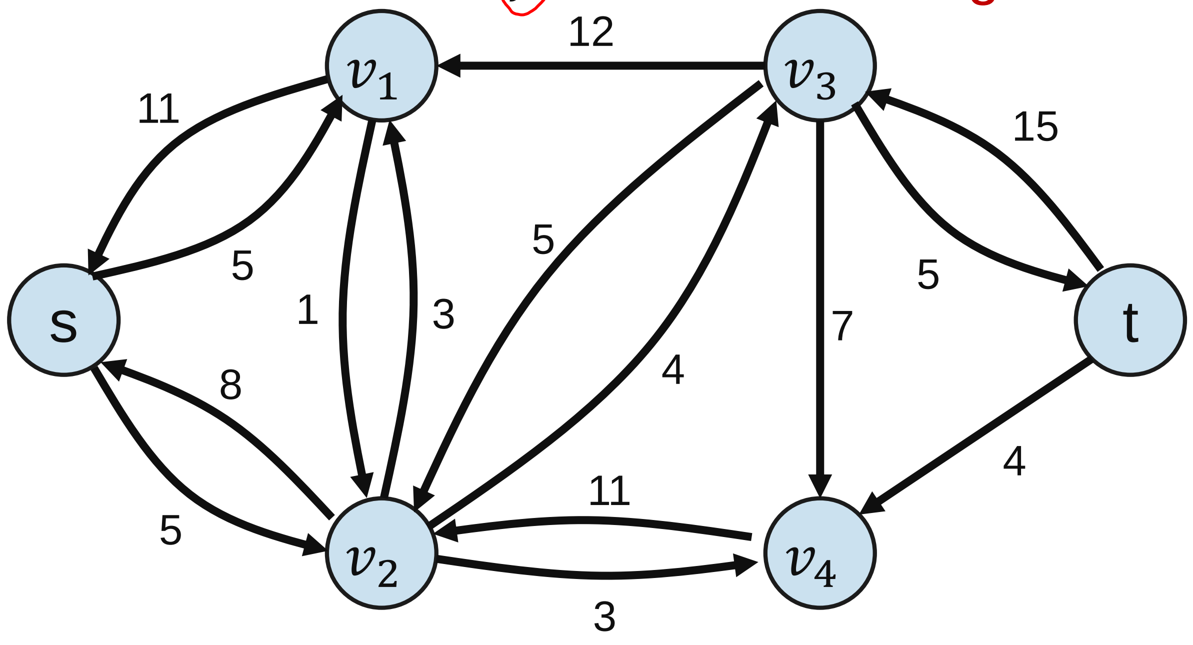
$$E_f = \{(u, v) \in V \times V \mid c_f(u, v) > 0\}$$

G_f is a flow network with capacities c_f.

Original flow network G + flow f



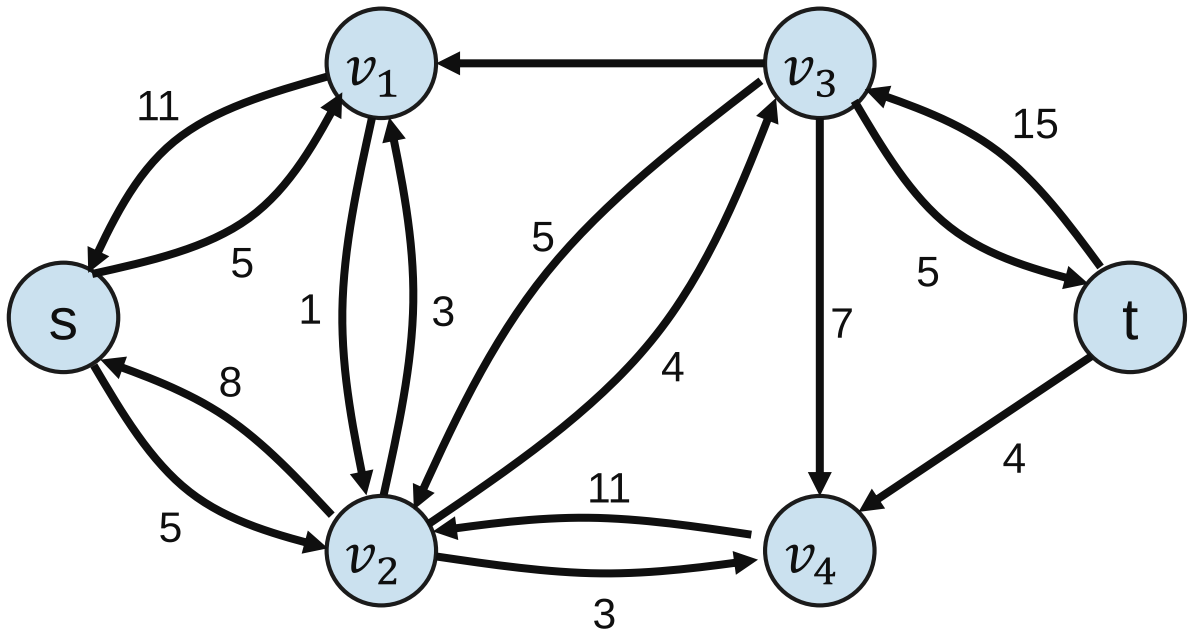
Residual network G_f backward edges!



Augmenting path

An s-t-path p in G_f is an augmenting path.

$$c_f(p) = \min\{\, c_f(u,v) \,|\, (u,v) \in p \,\}$$ is the **residual capacity** of p.



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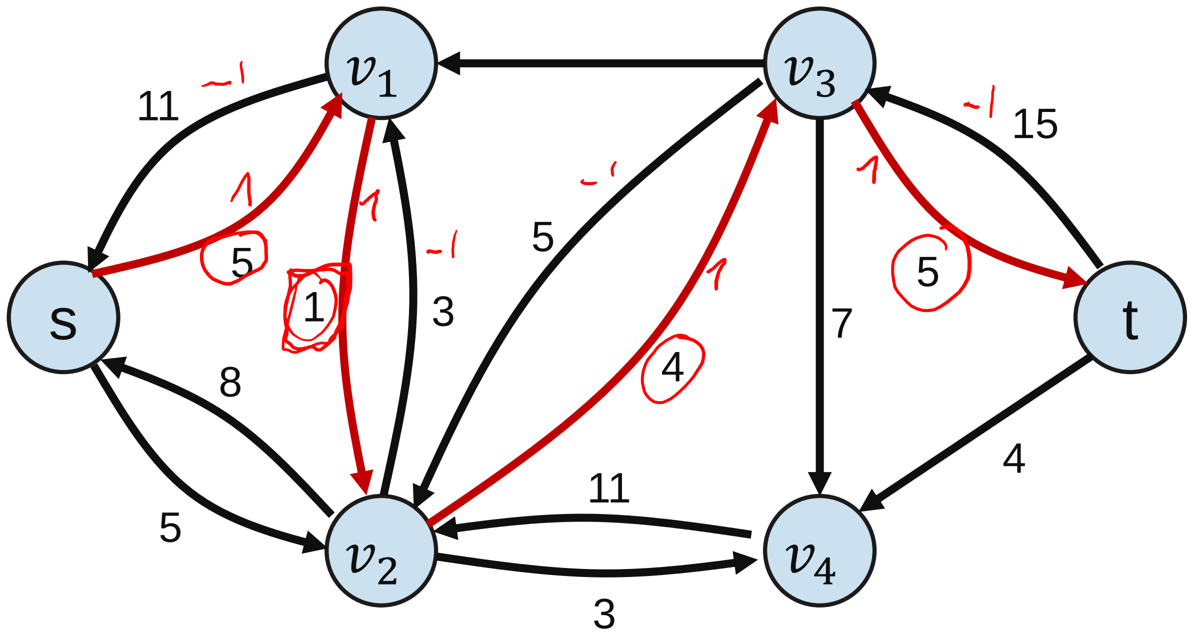


Residual capacity $c_f(p)$ of this path is 1

Ford-Fulkerson Method

Input: Given a flow network (G, c) with source node s, and sink node t
Output: Compute a flow f from s to t of maximum value

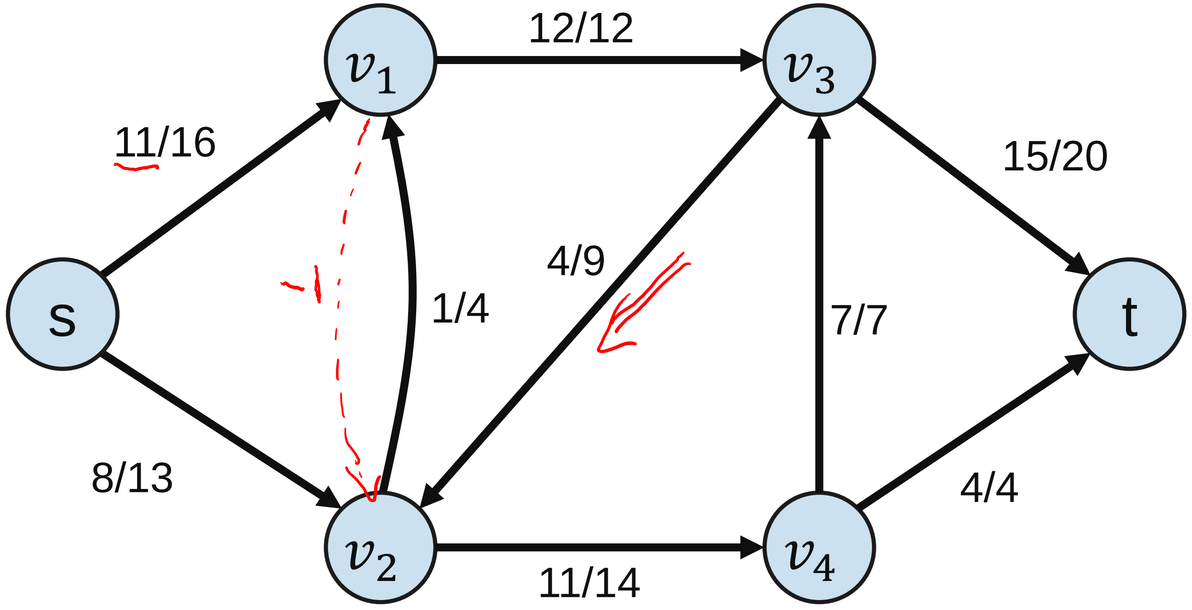
```
Initialize flow f with 0
while there exists an augmenting path p in G_f do
    augment f along p (by f_p)
return f
```

$$f_p(u,v) = \begin{cases} c_p(u,v) & \text{if } (u,v) \in p \\ -c_p(u,v) & \text{if } (v,u) \in p \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ f_p is a flow in G_f

augment f along p: $f + f_p$

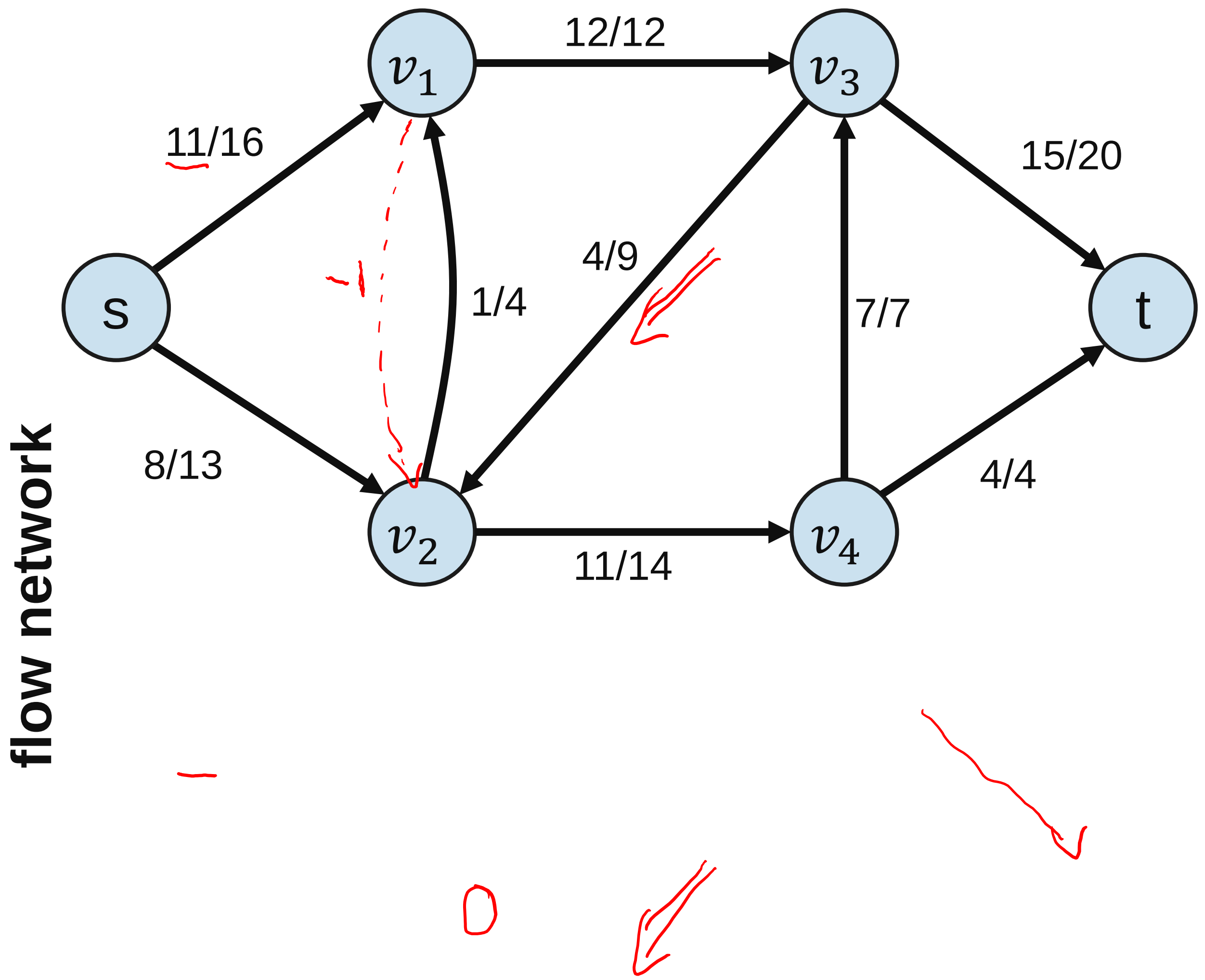
Example: Ford-Fulkerson-Method

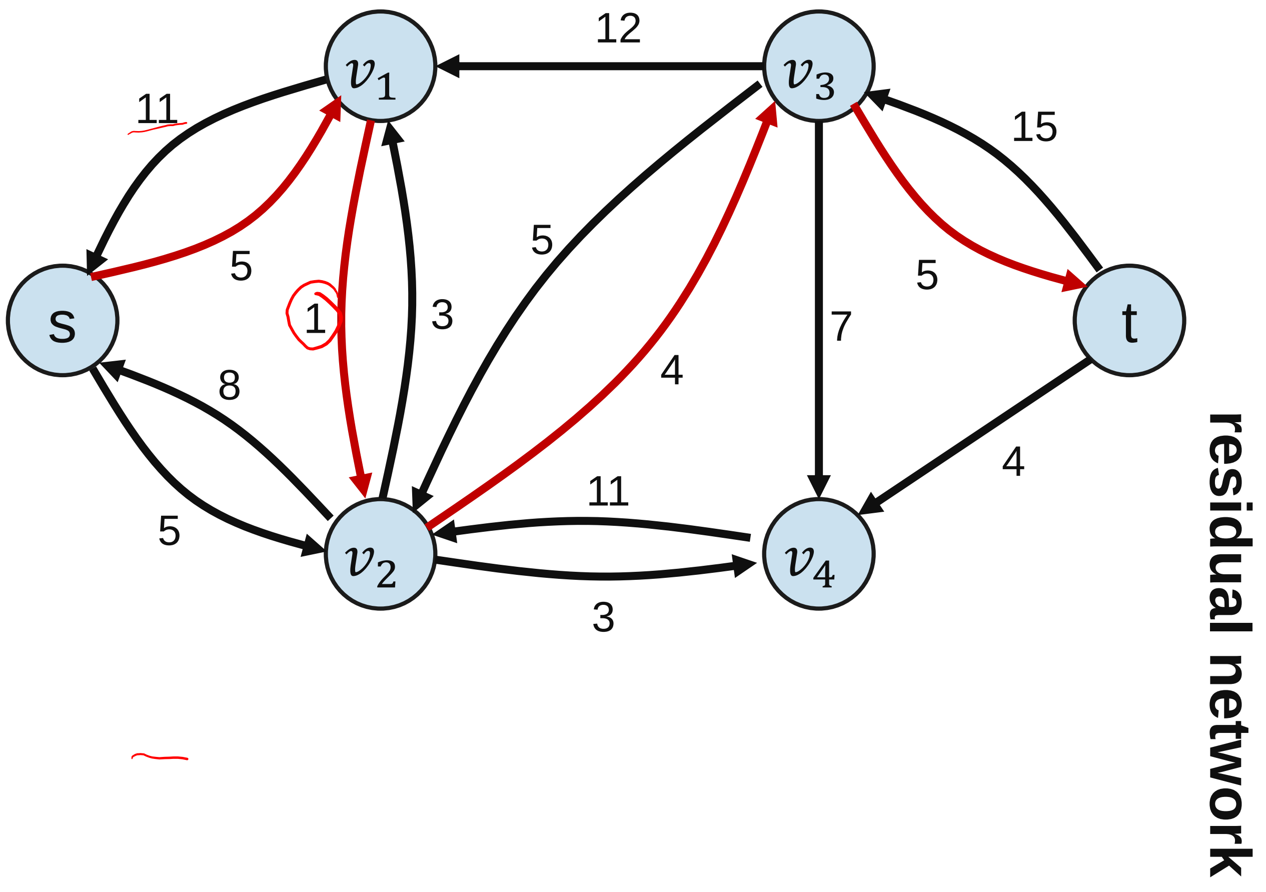


flow network

residual network

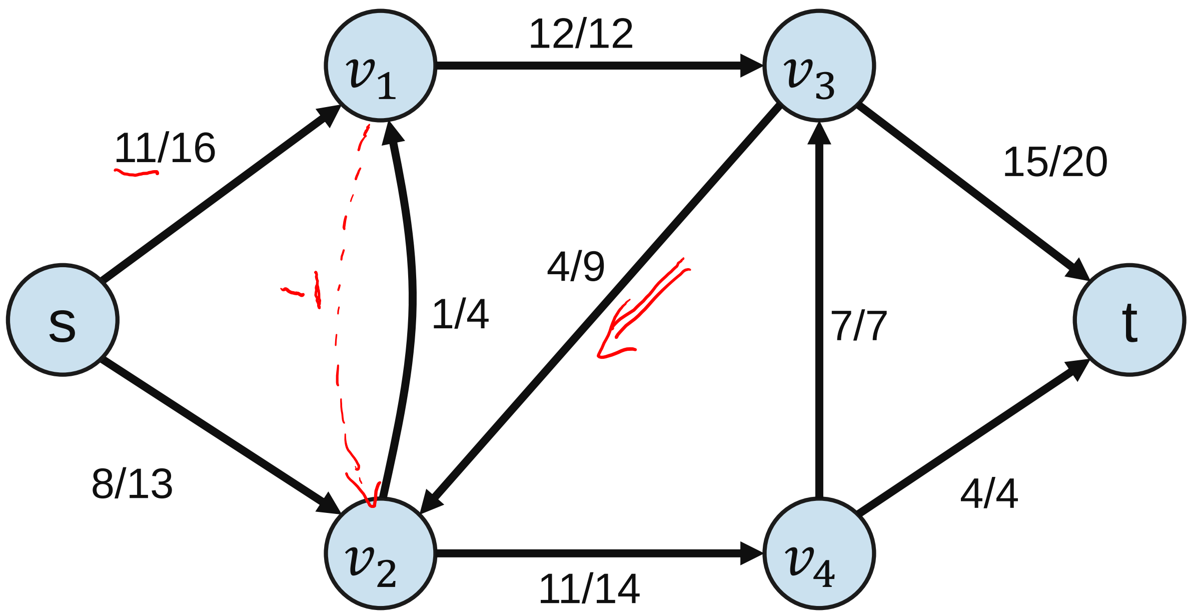
Example: Ford-Fulkerson-Method

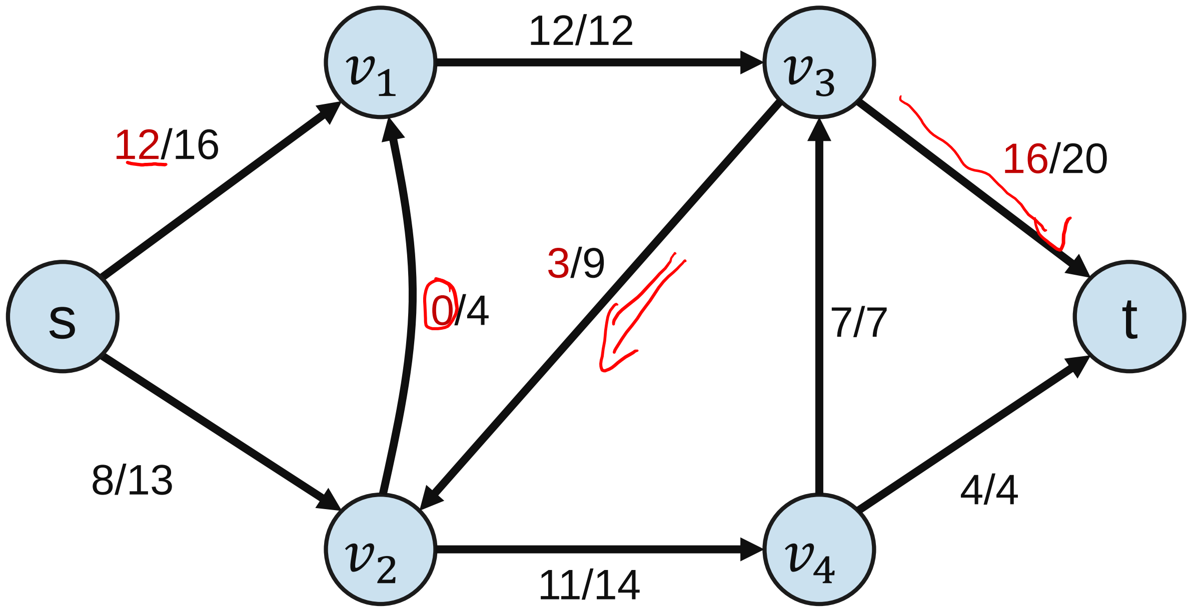




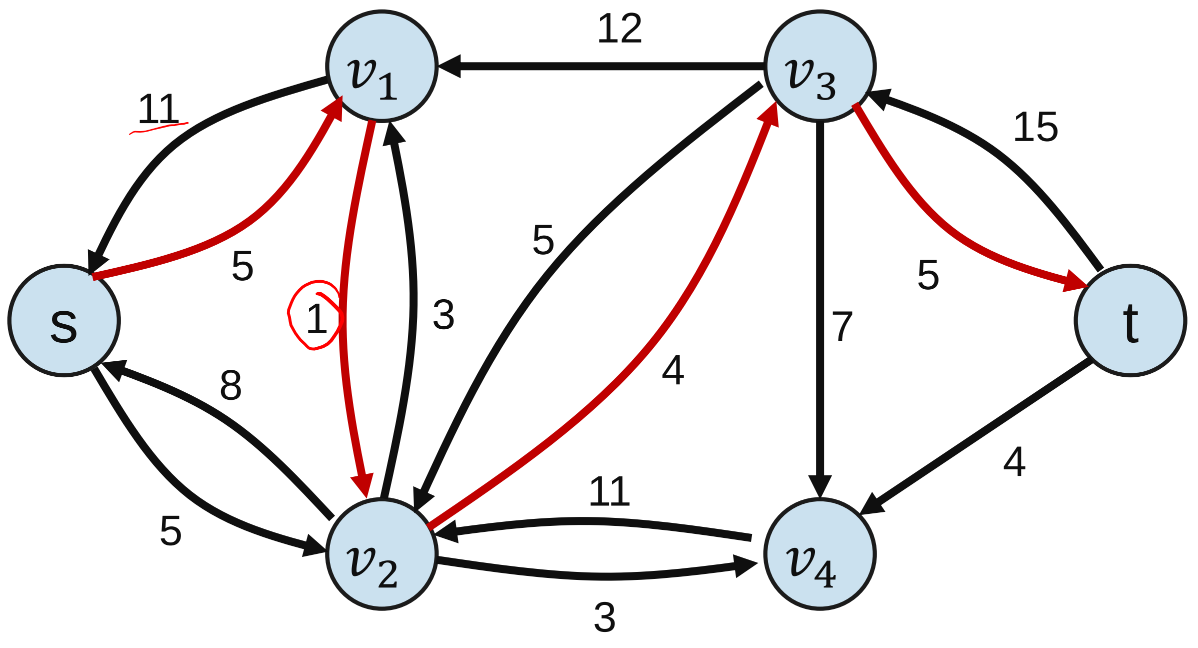
Example: Ford-Fulkerson-Method

flow network

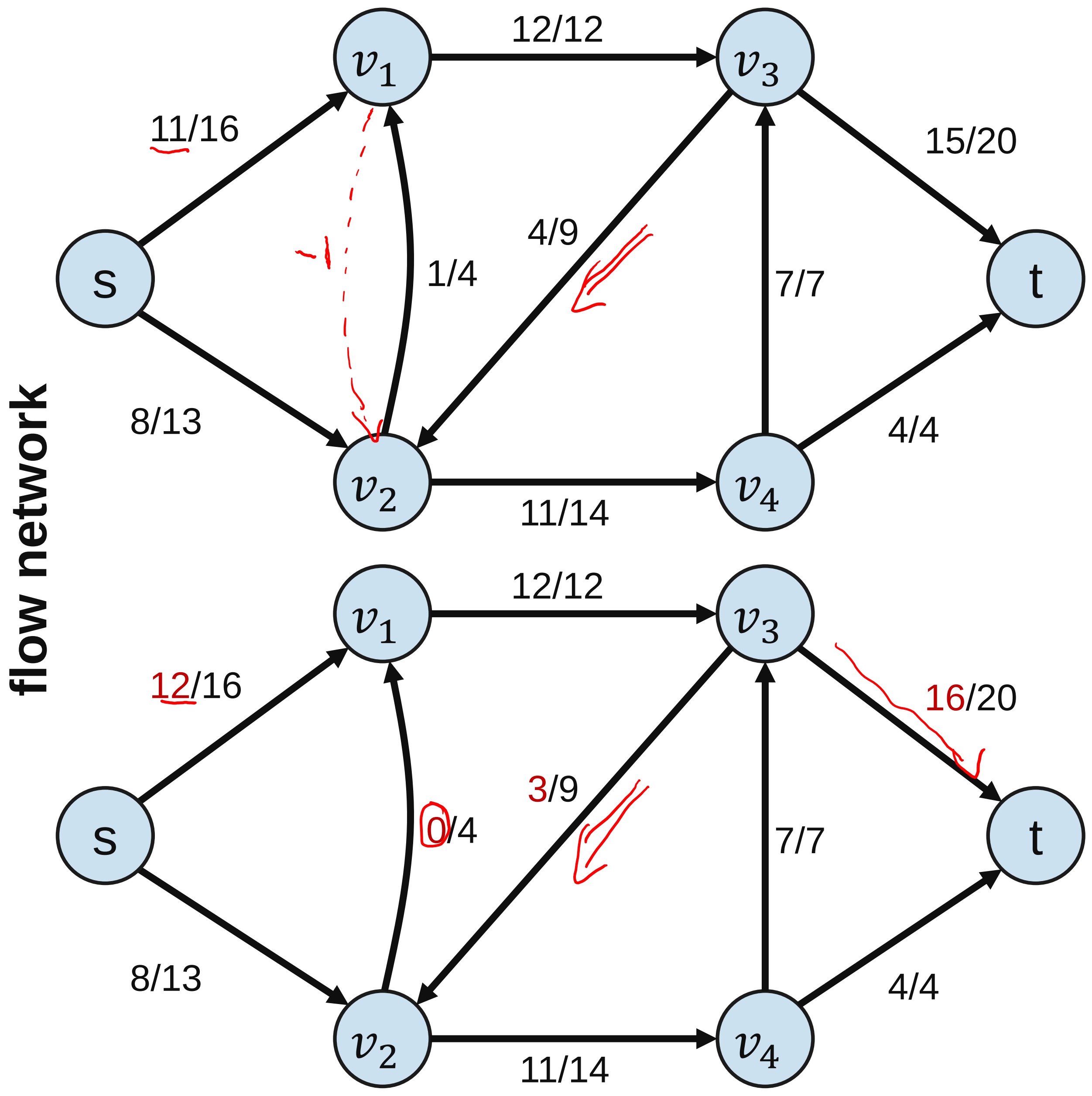


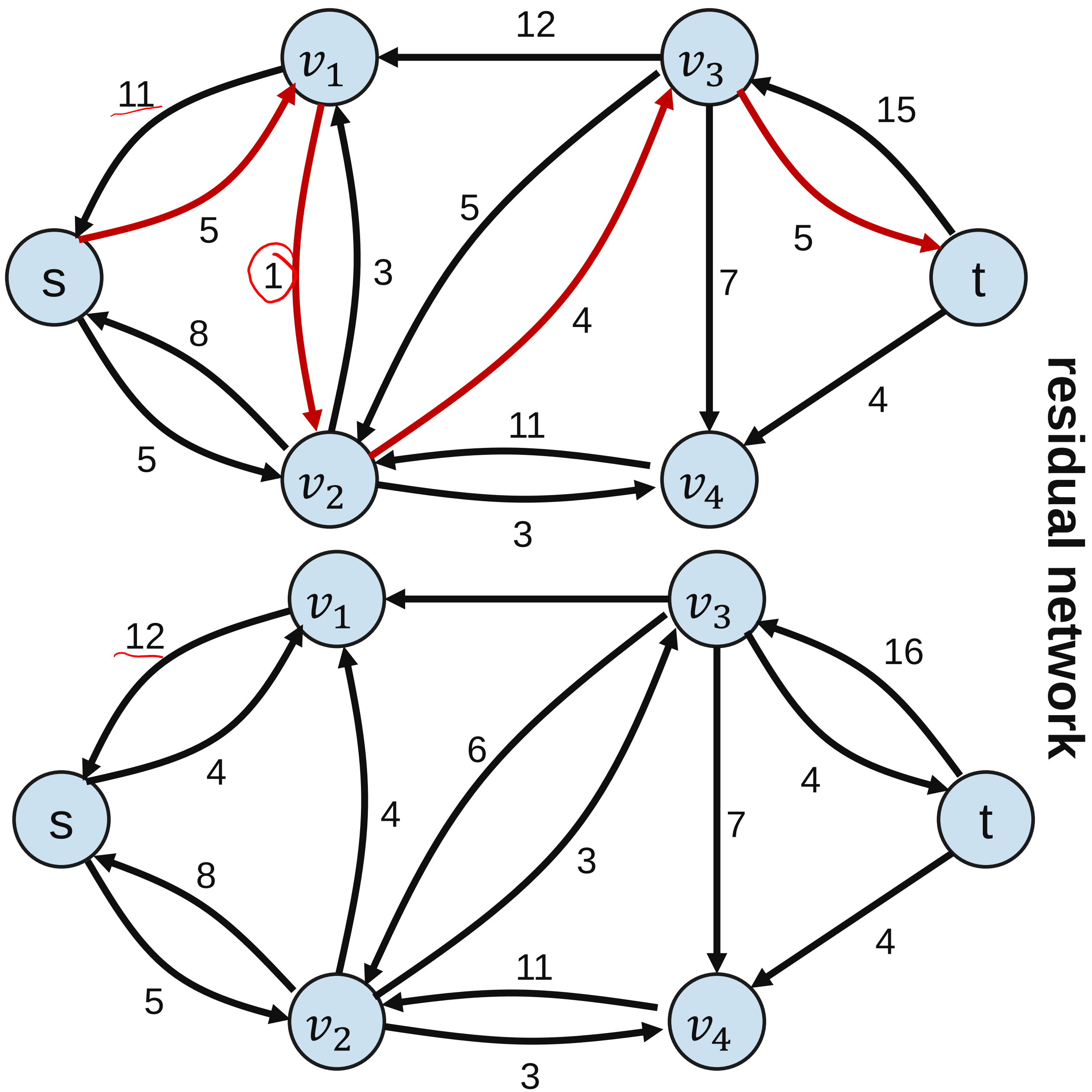


residual network



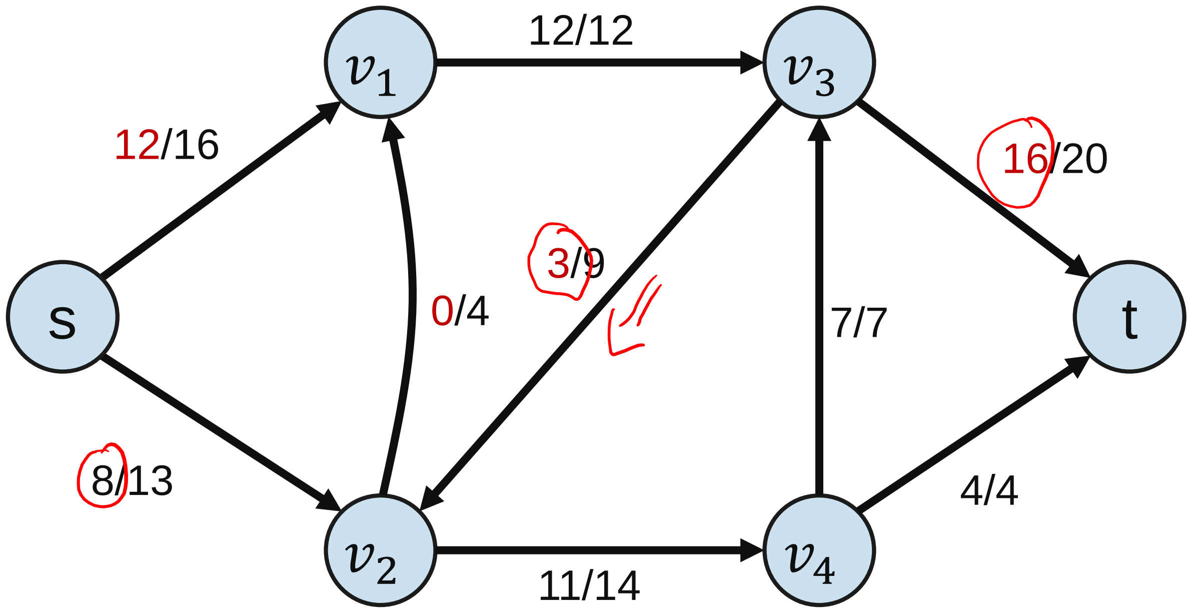
Example: Ford-Fulkerson-Method

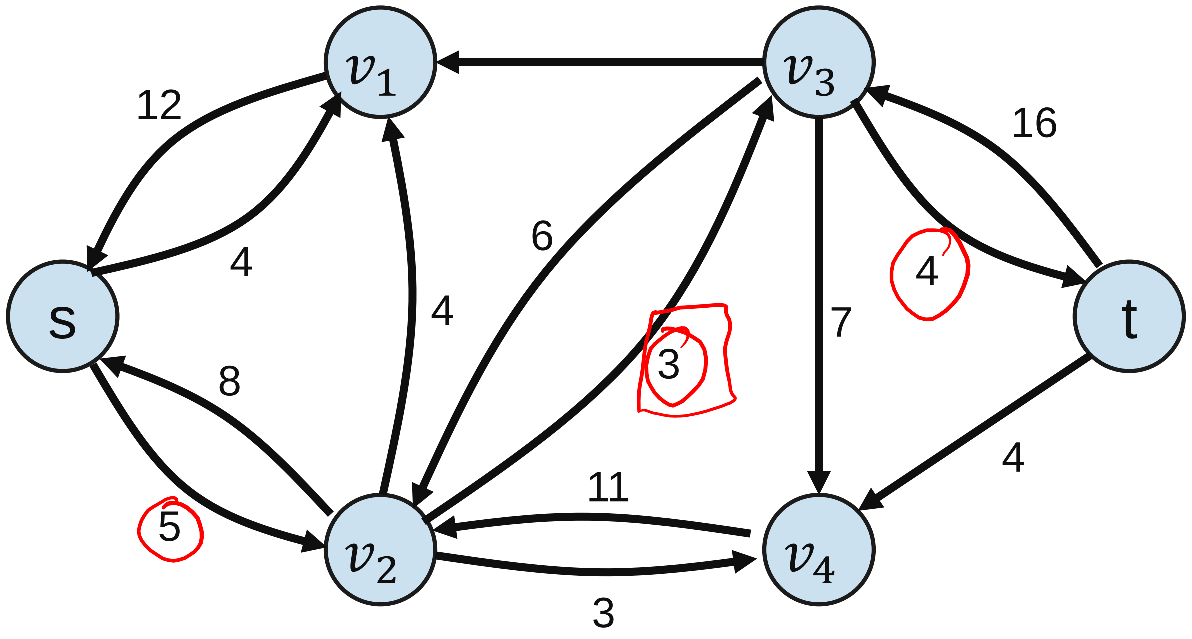




Example: Ford-Fulkerson-Method

flow network

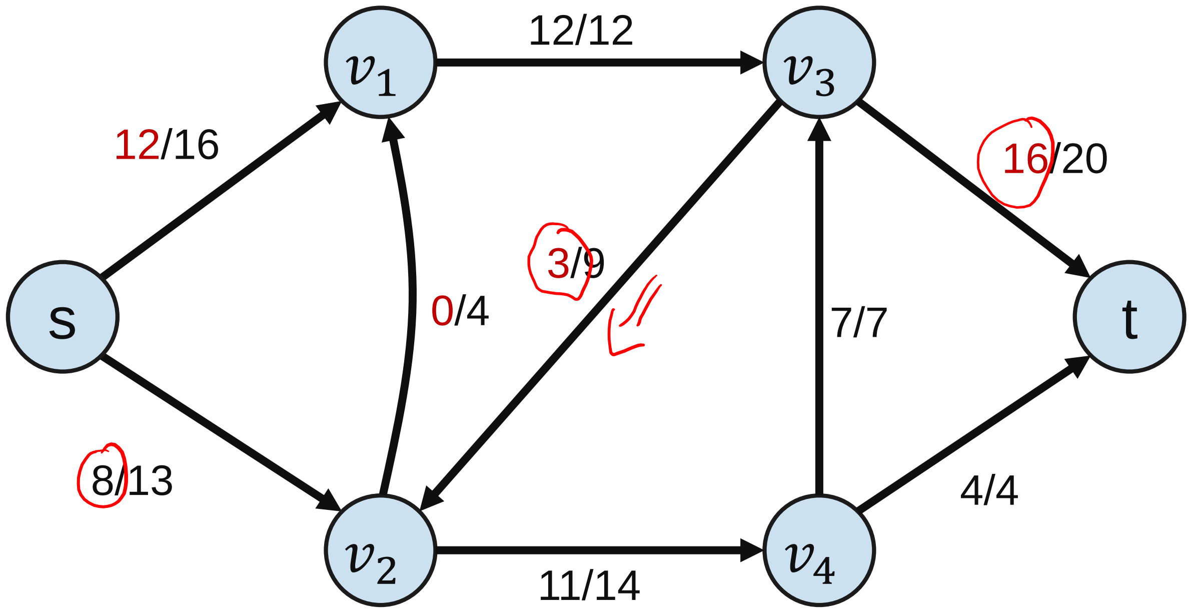


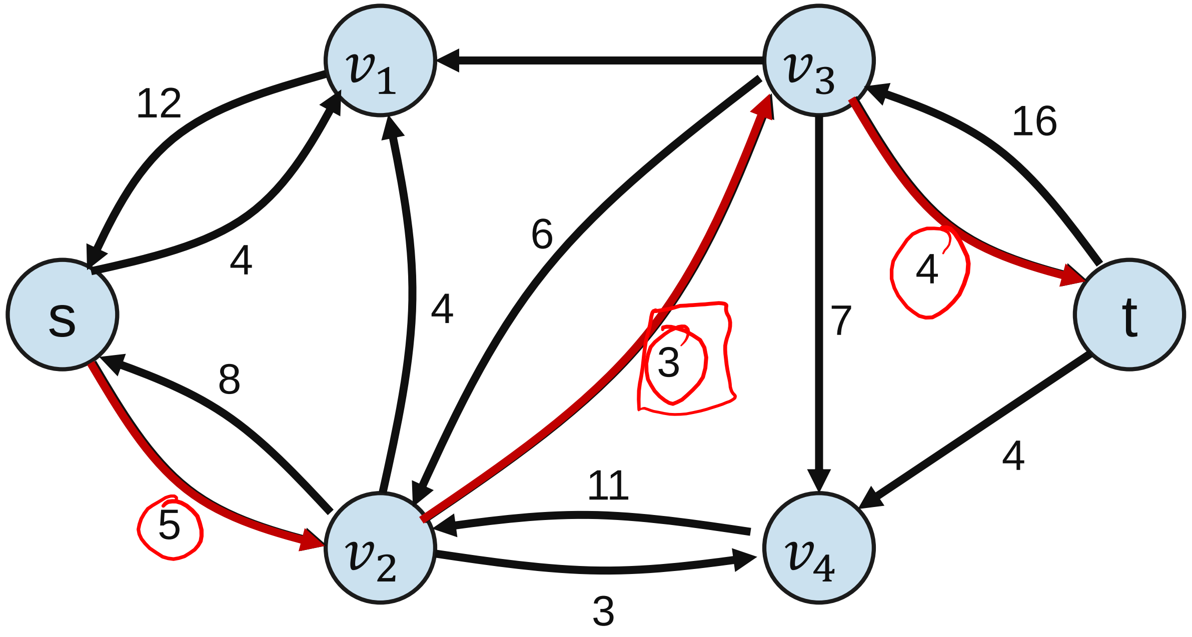


residual network

Example: Ford-Fulkerson-Method

flow network

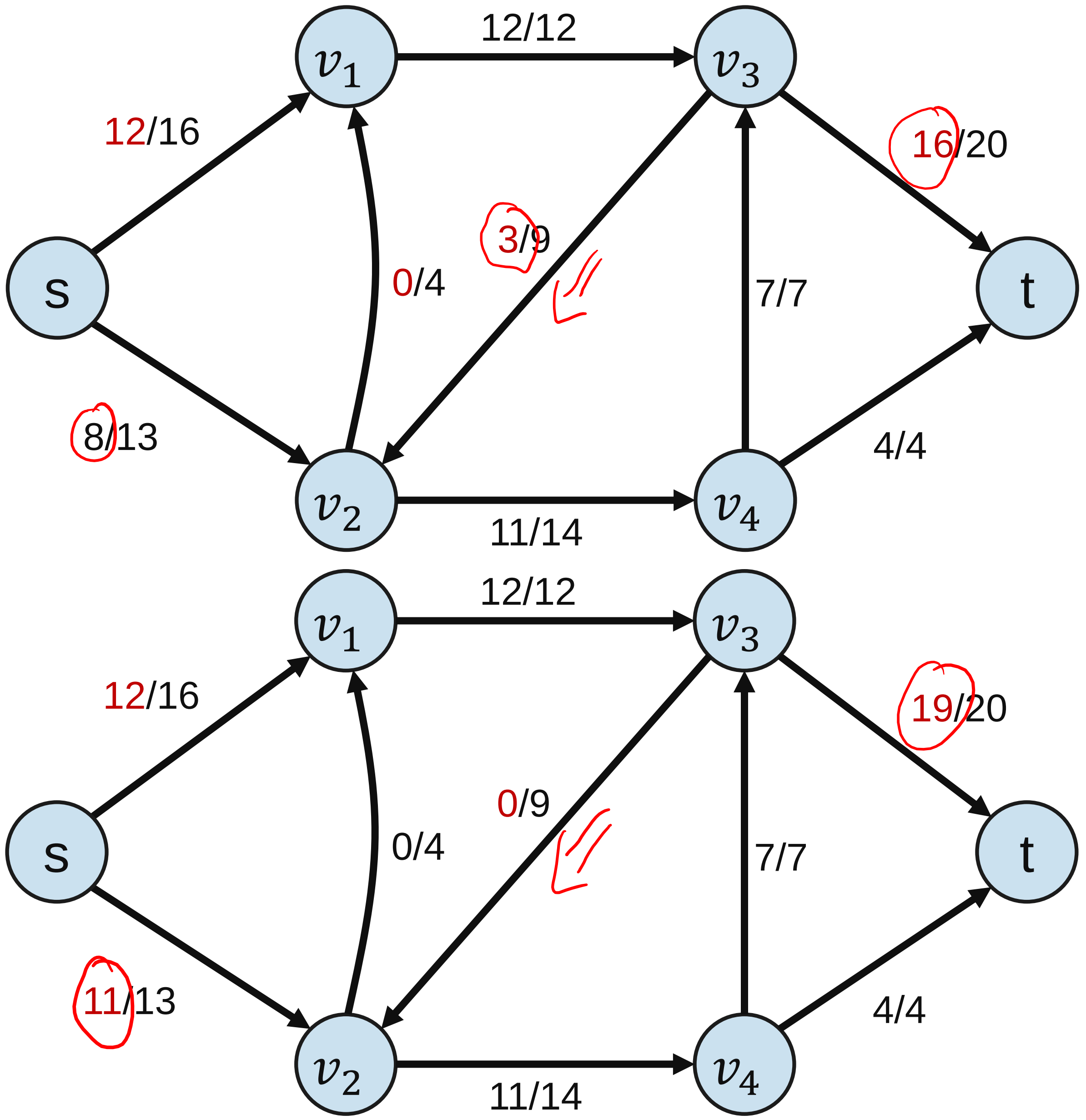




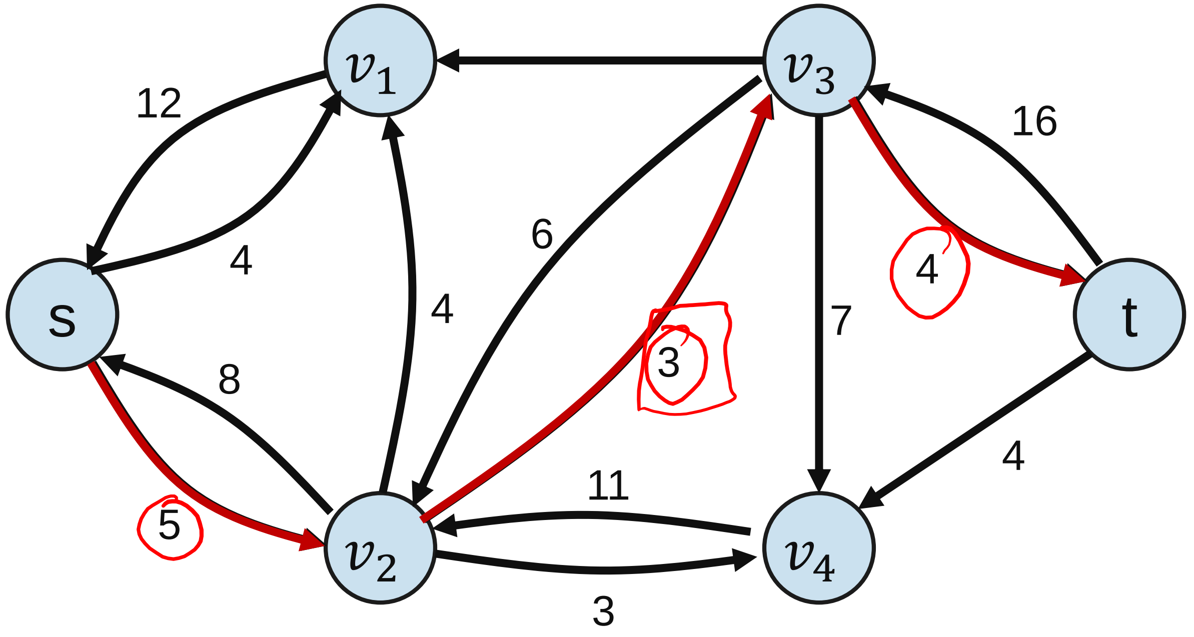
residual network

Example: Ford-Fulkerson-Method

flow network

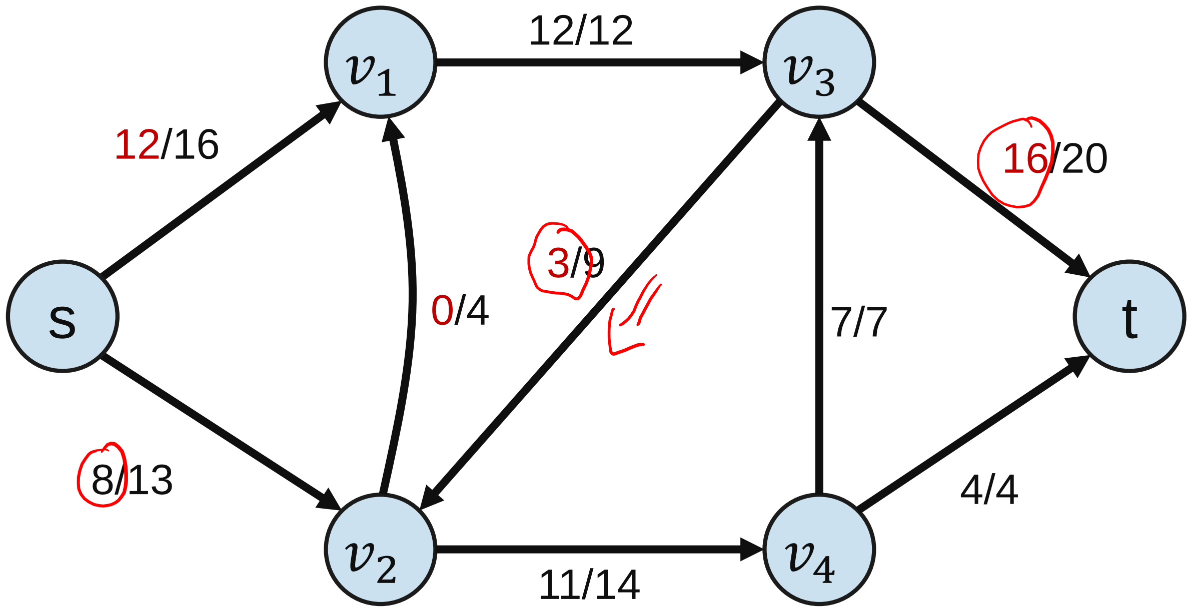


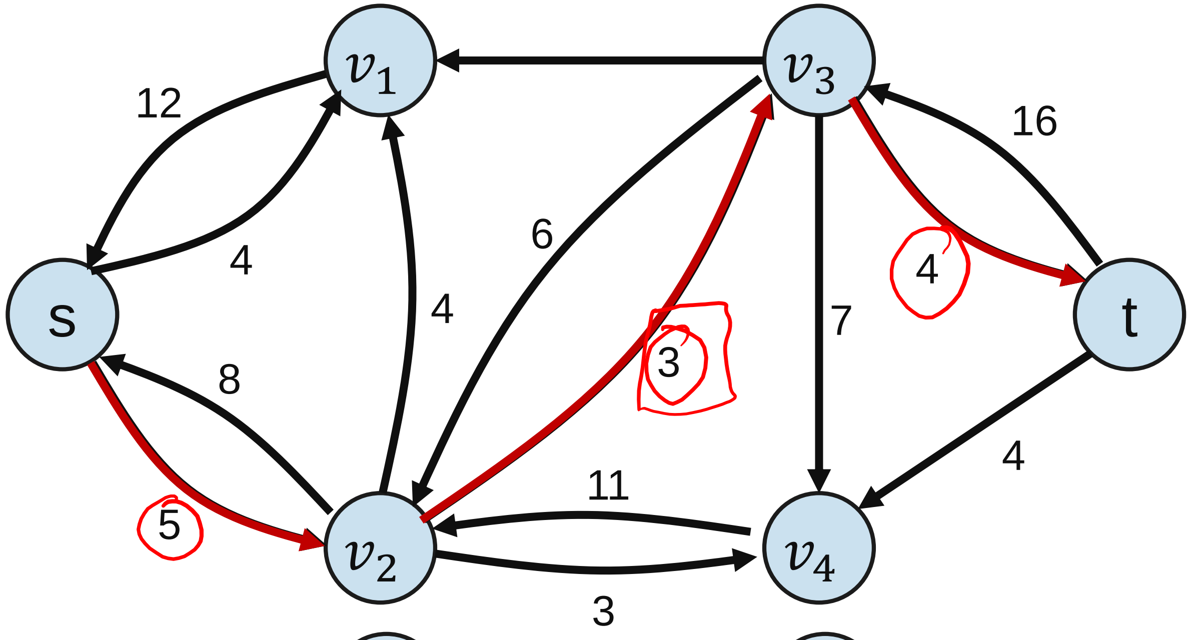
residual network

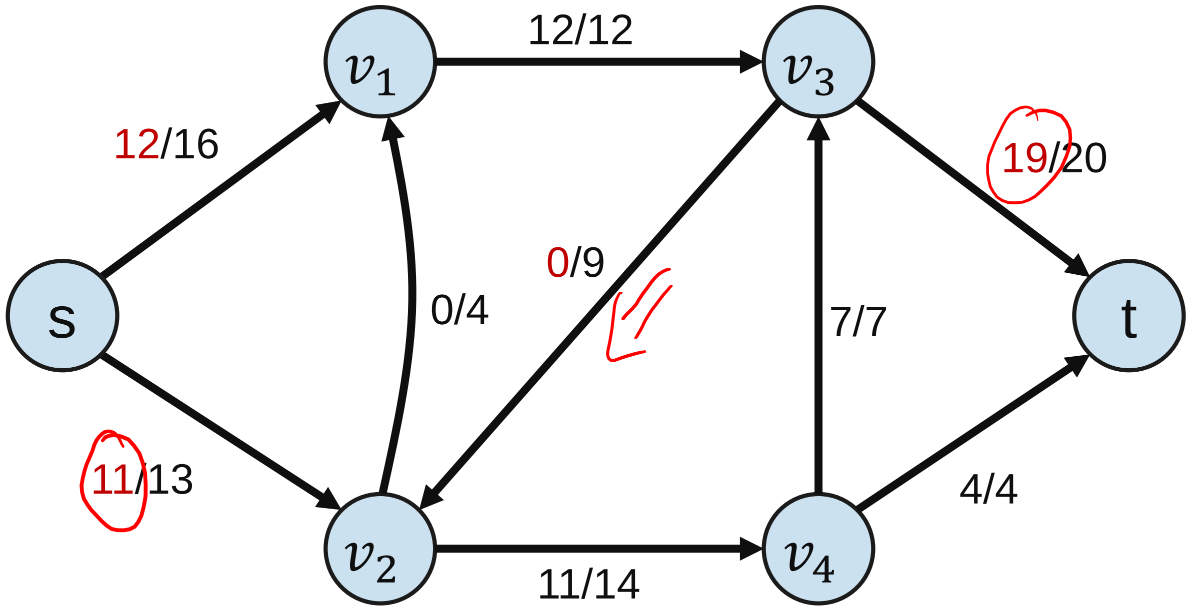


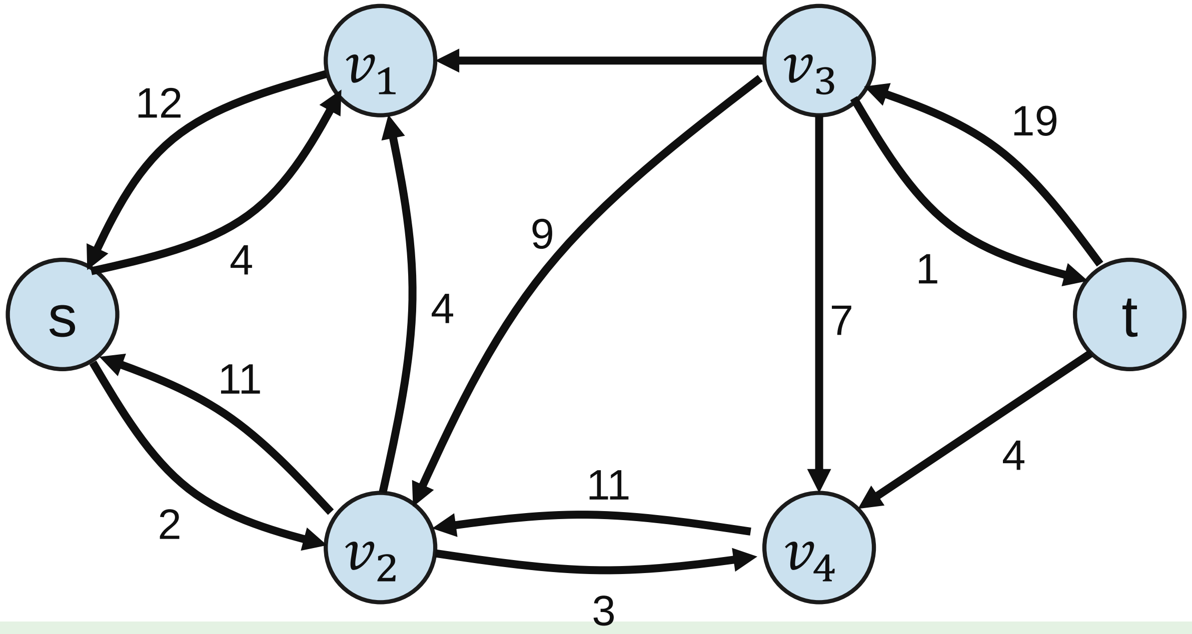
Example: Ford-Fulkerson-Method

flow network



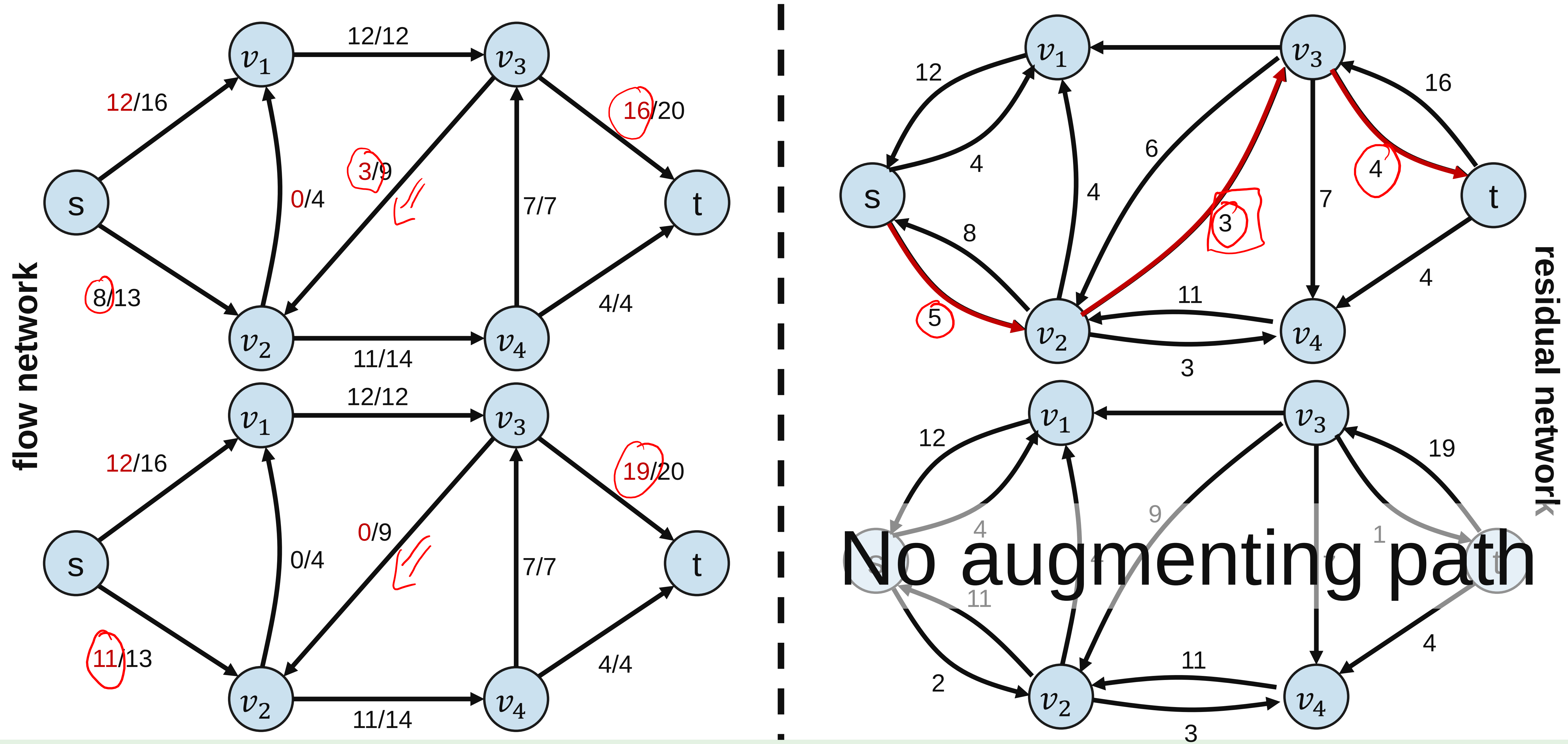






residual network

Example: Ford-Fulkerson-Method



Analysis 1: $f + f_p$ is a flow

Link to animated version: https://algorithms.discrete.ma.tum.de/

Correctness: $f + f_p$ is a flow

The augmented version is a flow:

Capacity constraint:

$$\begin{aligned}(f + f_p)(u,v) &= f(u,v) + f_p(u,v) \leq f(u,v) + c_f(u,v) \\ &\leq f(u,v) + \big(c(u,v) - f(u,v)\big) \qquad = c(u,v)\end{aligned}$$

(arguments use that that f_p is a flow in G_f and that f is a flow in G)

Correctness: $f + f_p$ is a flow

The augmented version is a flow:

Capacity constraint:

$$(f + f_p)(u,v) = f(u,v) + f_p(u,v) \le f(u,v) + c_f(u,v)$$
$$\le f(u,v) + \big(c(u,v) - f(u,v)\big) = c(u,v)$$

Conservation of flow:

Let $u \in V \setminus \{s,t\}$. $(f + f_p)(u,V) = f(u,V) + f_p(u,V) = 0 + 0 = 0$

(arguments use that that f_p is a flow in G_f and that f is a flow in G)

Correctness: $f + f_p$ is a flow

The augmented version is a flow:

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$$(f + f_p)(u,v) = f(u,v) + f_p(u,v) \leq f(u,v) + c_f(u,v)$$
$$\leq f(u,v) + \big(c(u,v) - f(u,v)\big) = c(u,v)$$

Conservation of flow:

Let $u \in V \setminus \{s,t\}$. $(f + f_p)(u,V) = f(u,V) + f_p(u,V) = 0 + 0 = 0$

Skew Symmetry:

$$(f + f_p)(u,v) = f(u,v) + f_p(u,v) = -f(v,u) - f_p(v,u)$$
$$= -(f(v,u) + f_p(v,u)) = -(f + f_p)(v,u)$$

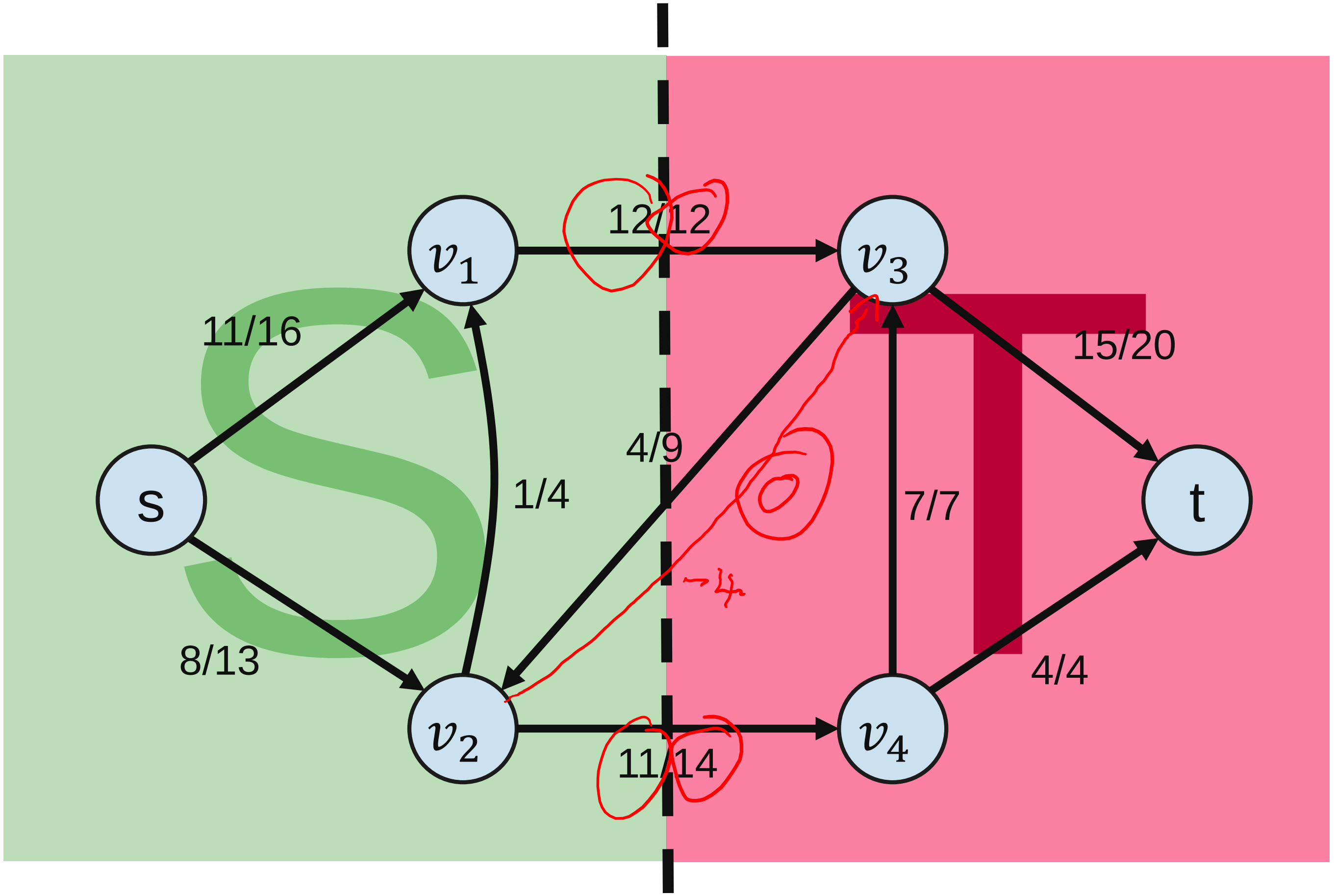
(arguments use that that f_p is a flow in G_f and that f is a flow in G)

Analysis 2: A maximum flow?

Min-cut=Max-flow

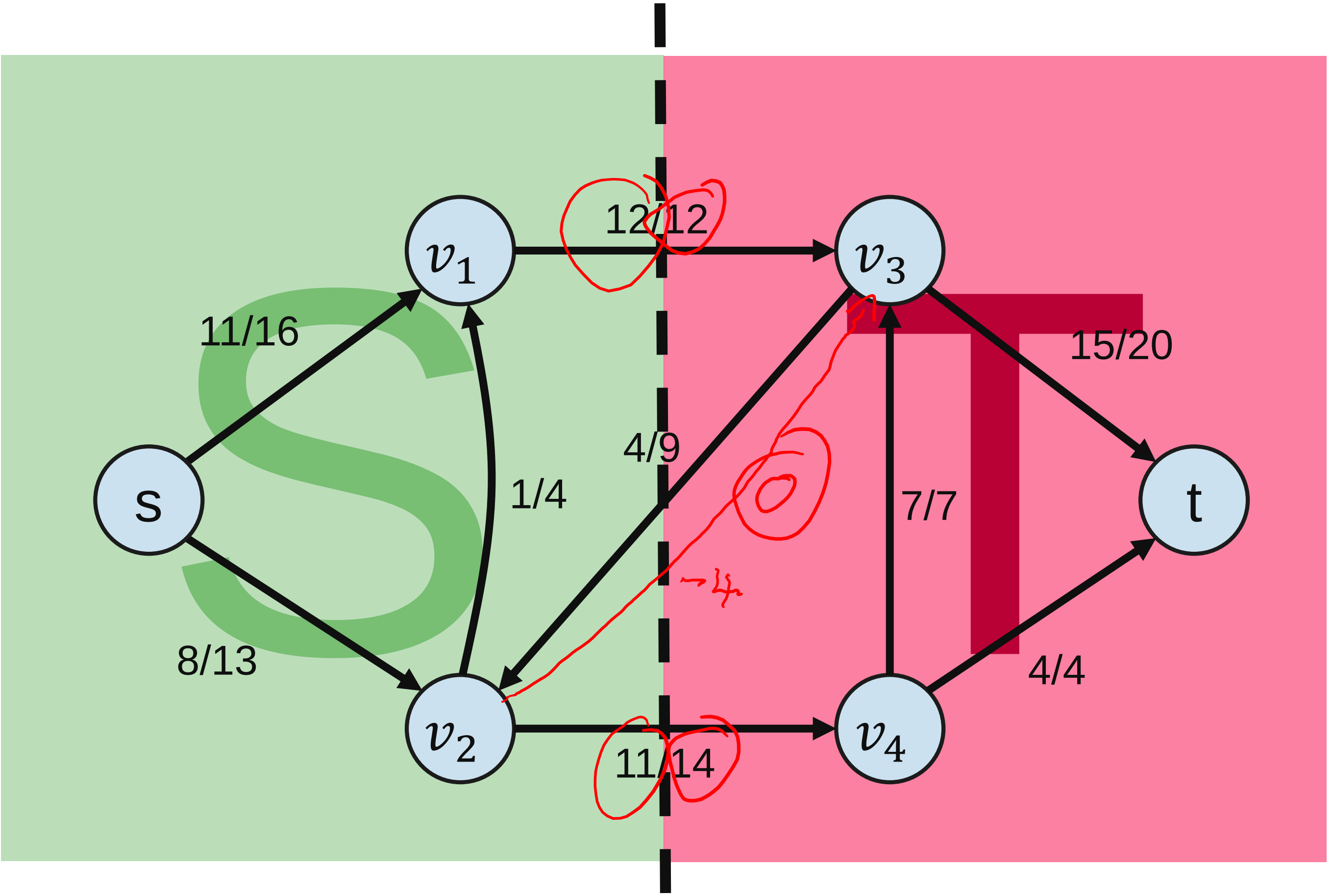
(seemingly)

An s-t-cut (S, T) in G is a partition of V ($V = S \cup T$, $S \cap T = \emptyset$) with $s \in S, t \in T$.



Cut

An s-t-cut (S,T) in G is a partition of V ($V = S \cup T$, $S \cap T = \emptyset$) with $s \in S, t \in T$.



$c(S,T) = 26$

The **capacity** of a cut is

$$c(S,T) = \sum_{u \in S, v \in T} c(u,v)$$

Cut

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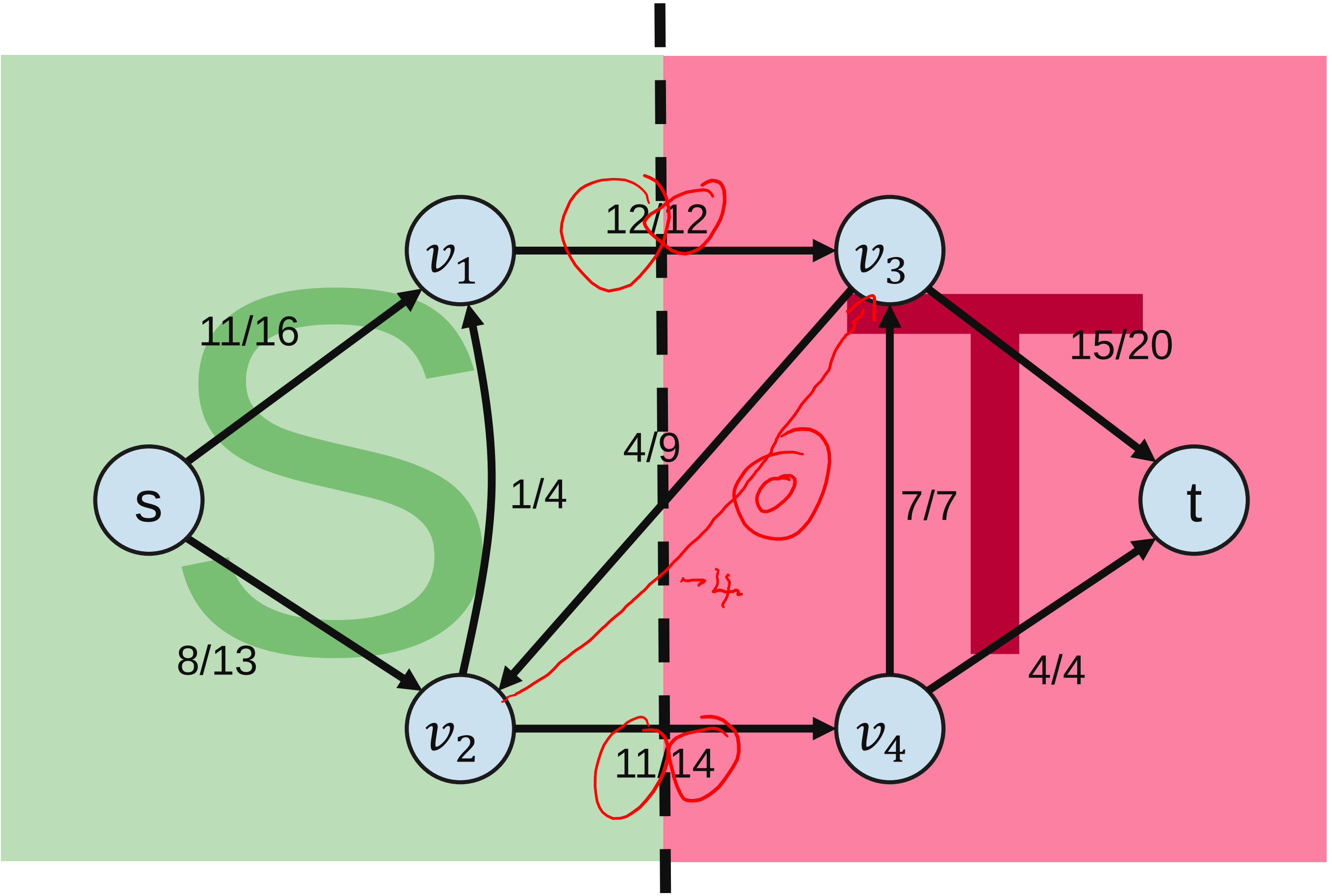


The **capacity** of a cut is

$$c(S, T) = \sum_{u \in S, v \in T} c(u, v)$$

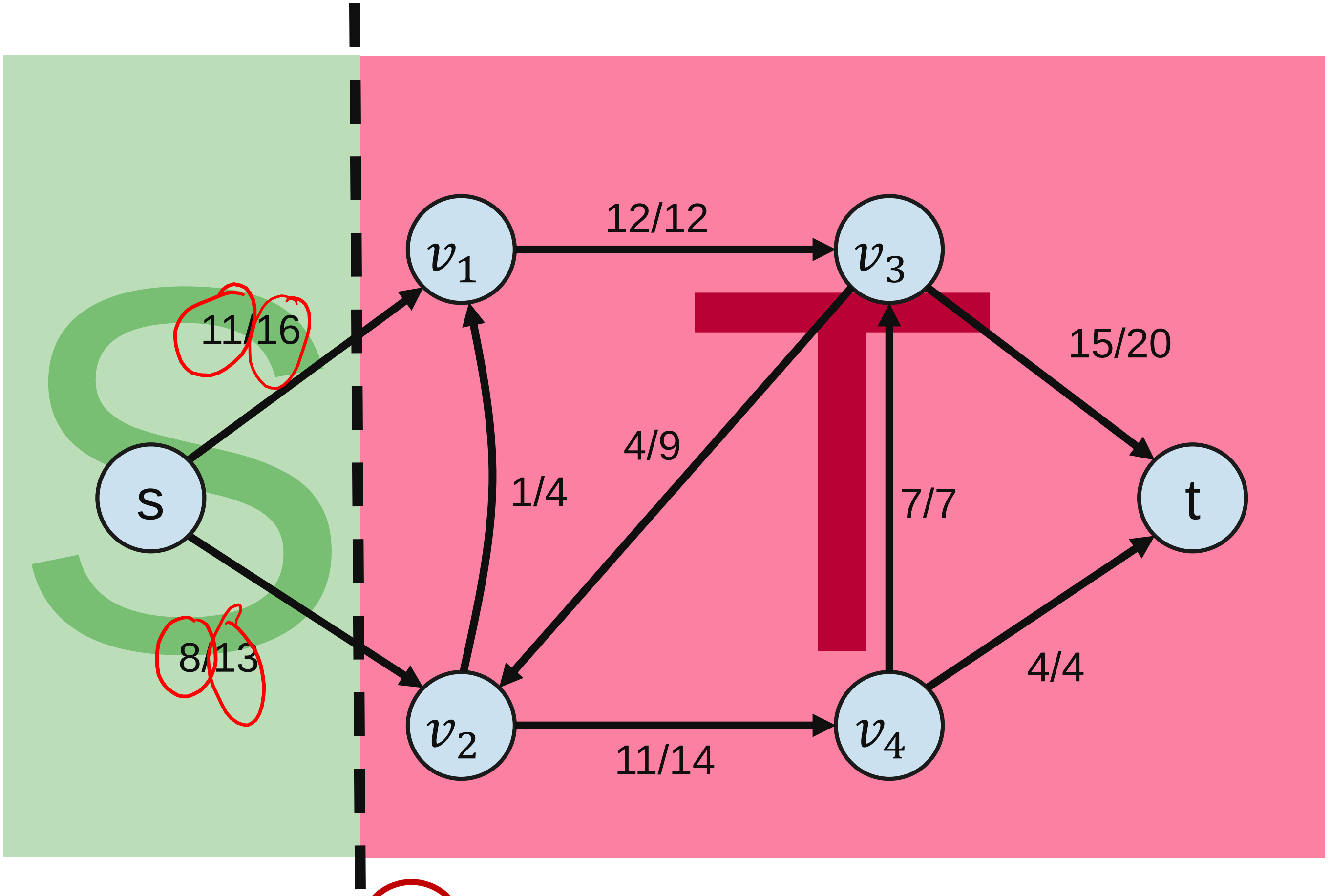
The **flow** over a cut is

$$f(S, T) = \sum_{u \in S, v \in T} f(u, v)$$

$c(S, T) = 26$

$f(S, T) = 12 + 11 + (-4) = 19$

An s-t-cut (S,T) in G is a partition of V ($V = S \cup T$, $S \cap T = \emptyset$) with $s \in S, t \in T$.



The **capacity** of a cut is

$$c(S,T) = \sum_{u \in S, v \in T} c(u,v)$$

The **flow** over a cut is

$$f(S,T) = \sum_{u \in S, v \in T} f(u,v)$$

$c(S,T) = 29$

change!

$f(S,T) = \cancel{12 + 11 + (-4)} = 19$

$11 + 8$

no change!

Lemma A: For each flow f and each cut (S,T) we have $|f| = f(S,T)$.

s

$= f(s,V) + \sum_{u \in S \setminus \{s\}} f(u,V)$

Def

Lemma A: For each flow f and each cut (S,T) we have $|f| = f(S,T)$.

s

$= f(s,V) + \sum_{u \in S \setminus \{s\}} f(u,V)$

Def

Notation:

$$f(x,Y) = \sum_{y \in Y} f(x,y)$$

$$f(X,Y) = \sum_{x \in X} \sum_{y \in Y} f(x,y)$$

Lemma A: For each flow f and each cut (S,T) we have $|f| = f(S,T)$.

Proof:

$$f(S,T) = f(S,V) - f(S,\overbrace{V \setminus T}^{=S})$$

$$= f(S,V) - f(S,S) = f(S,V) = f(s,V) + \sum_{u \in S\setminus\{s\}} f(u,V)$$

$$= f(s,V) + f(S \setminus \{s\}, V) = f(s,V) \overset{\text{Def}}{=} |f|$$

Notation:

$$f(x,Y) = \sum_{y \in Y} f(x,y)$$

$$f(X,Y) = \sum_{x \in X} \sum_{y \in Y} f(x,y)$$

Lemma A: For each flow f and each cut (S,T) we have $|f| = f(S,T)$.

Proof:

insert definition

(omitted simple proof, using symmetry)

$f(S,S) = 0$

$$f(S,T) = f(S,V) - f(S,\overset{S}{V \setminus T})$$

$$= f(S,V) - f(S,S) = f(S,V) = f(s,V) + \sum_{u \in S \setminus \{s\}} f(u,V)$$

$$= f(s,V) + \underline{f(S \setminus \{s\},V)} = f(s,V) \overset{Def}{=} |f|$$

$\underline{V \setminus T = S}$

$f(S \setminus \{s\},V) = 0$ as
$f(u,V) = 0$ for all $u \neq s,t$
due to flow conservation

Notation:

$$f(x,Y) = \sum_{y \in Y} f(x,y)$$

$$f(X,Y) = \sum_{x \in X} \sum_{y \in Y} f(x,y)$$

Max-Flow = Min-Cut Theorem

Theorem: Let f be an s-t-flow in a flow network $(G = (V, E), c)$.

The following are equivalent

1. flow f is a maximum flow
2. The residual network G_f contains **no** augmenting path
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Corollaries:

- If the Ford-Fulkerson-Method terminates, it has computed a maximum flow
- The capacity of the "smallest cut" equals the capacity of the maximum flow

Max-Flow = Min-Cut Theorem: Proof **1→ 2**

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Assume there is an augmenting path p in G_f.

Then $f + f_p$ is a flow with $|f + f_p| > |f|$, a contradiction.

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Adding an augmenting path flow f_p always increases the value of the flow. Why?

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Proof 2→ 3:

Assume that G_f has no s-t path. Define:

$S := \{ v \in V \mid$ there is an s-v path in $G_f\}$, $T := V \setminus S$ (note $T \neq \emptyset$)

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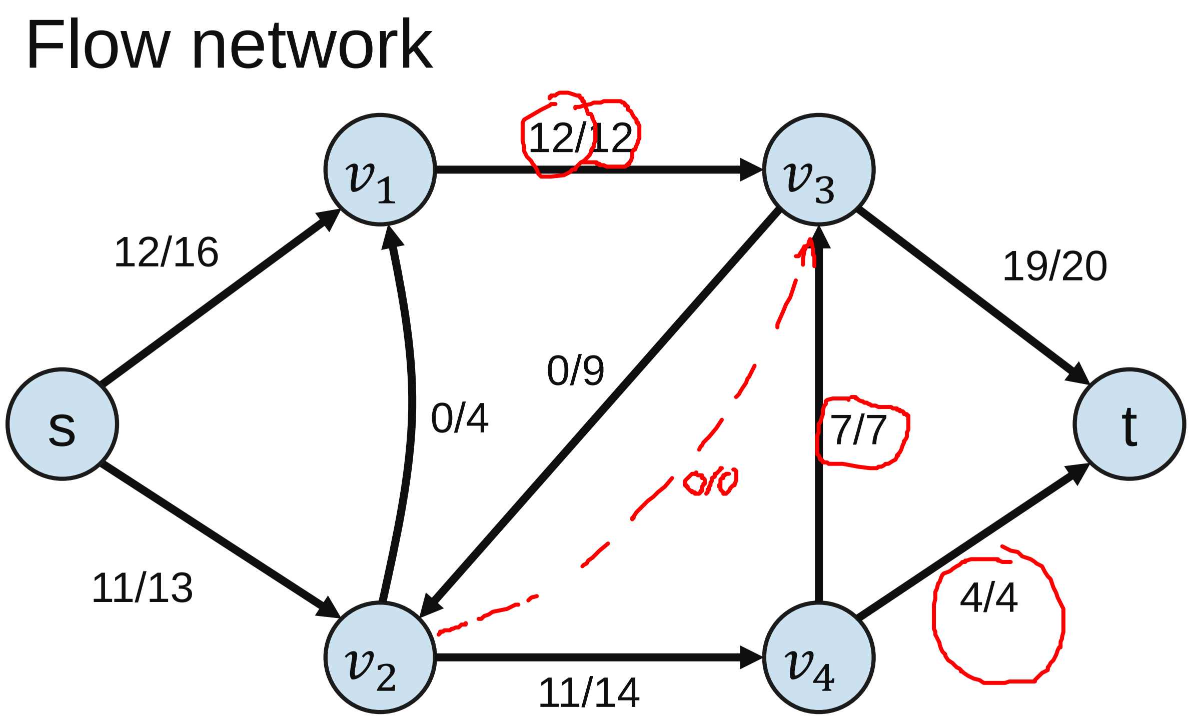
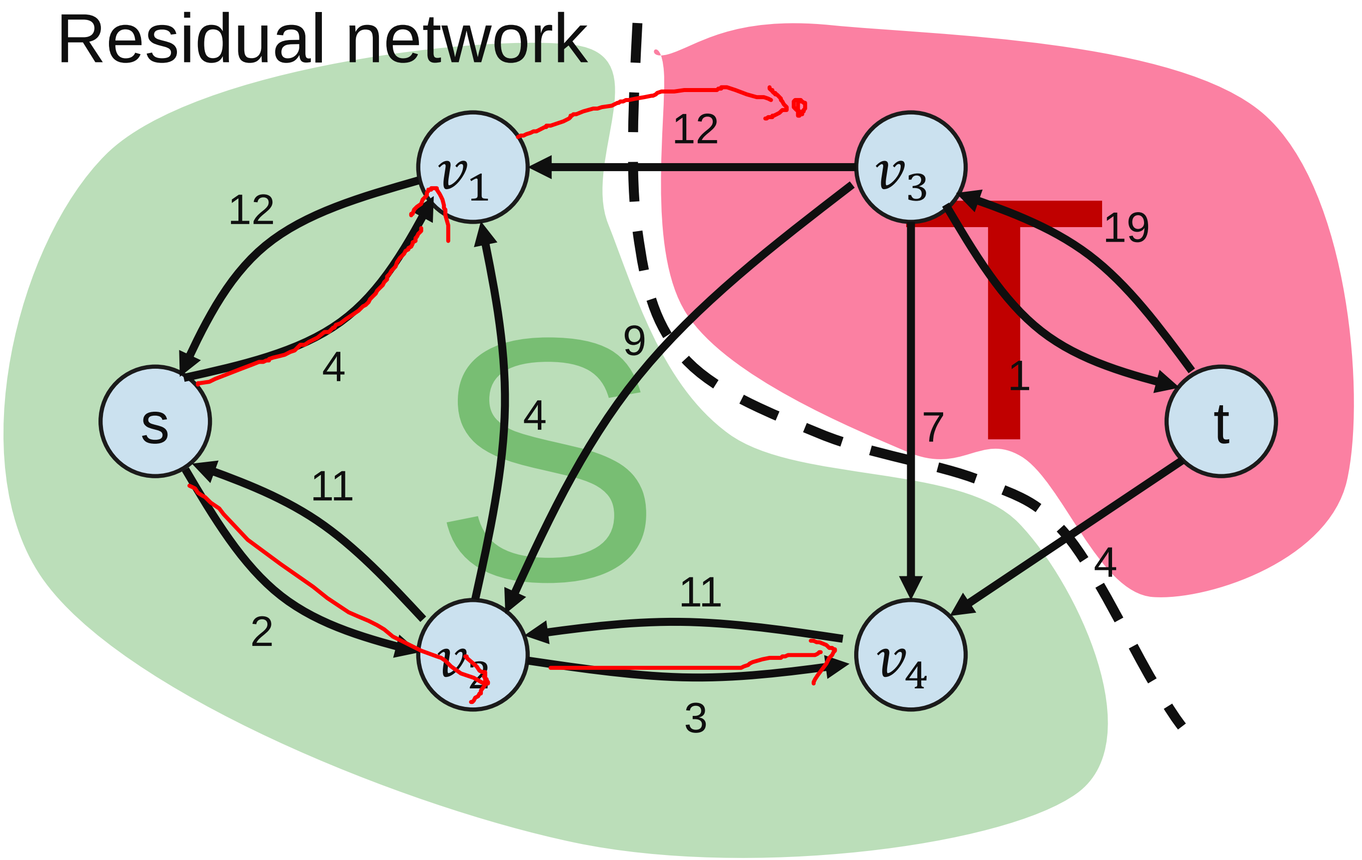
$$|f| \underset{\substack{\uparrow \\ \text{Lemma A}}}{=} f(S, T) = \sum_{u \in S, v \in T} f(u, v) = \sum_{u \in S, v \in T} c(u, v) = c(S, T).$$





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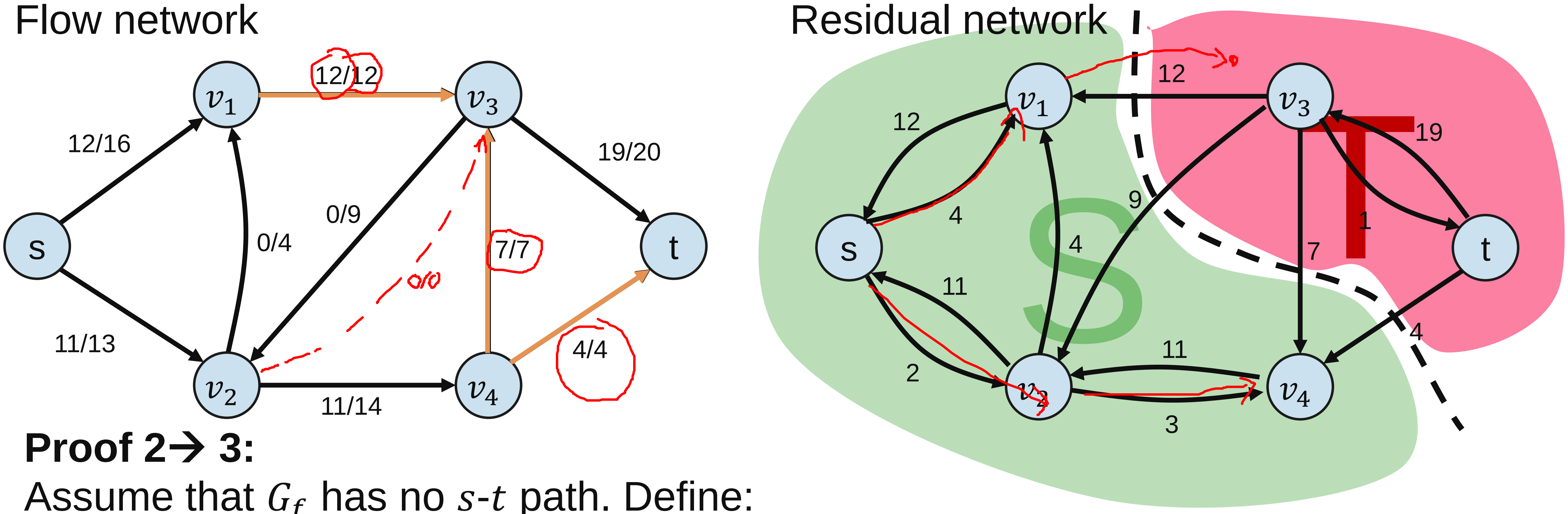
Max-Flow = Min-Cut Theorem: Proof 2→ 3

Flow network   Residual network



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Thus, the value of every flow is at most $c(S, T)$, so, if it equals $c(S, T)$, it has to be maximal!

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Warning! *The theorem statement does not hold for real-valued capacities.*
The ford-Fulkerson-Method might not even terminate!

How to find a min-cut?

1. Compute a max-flow
2. Let S consist of the vertices that one can reach from s via non-critical edges (An edge (u, v) is critical if $c(u, v) = f(u, v)$)
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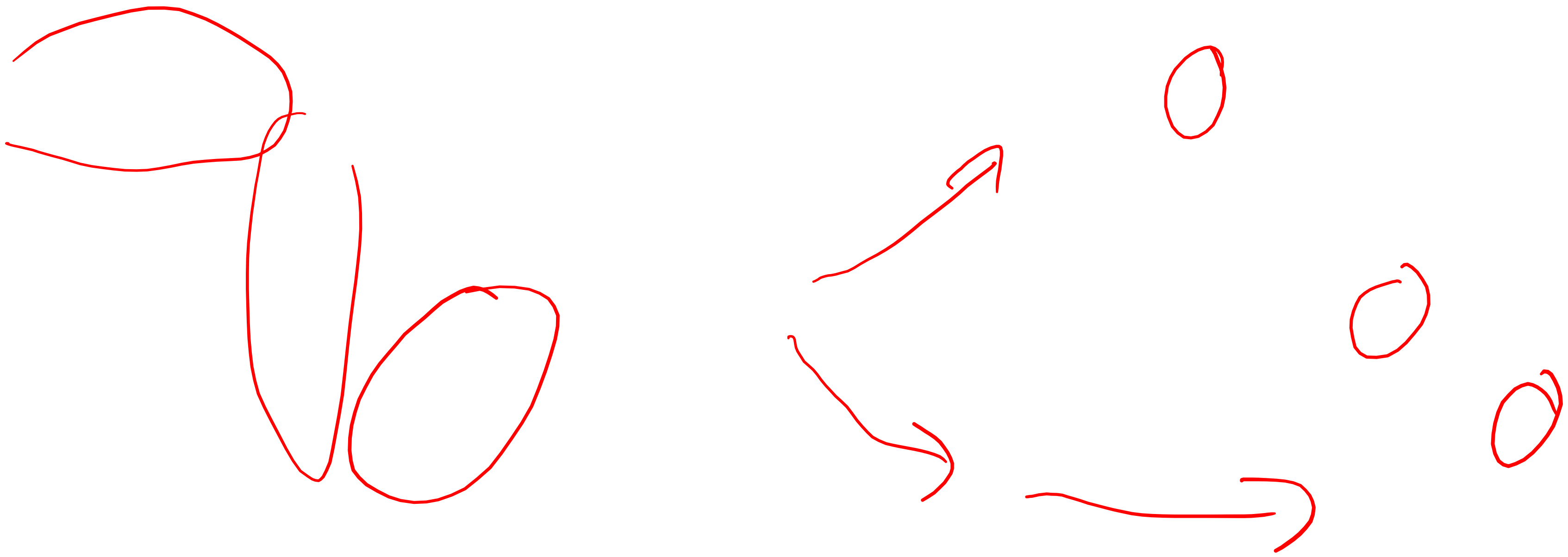

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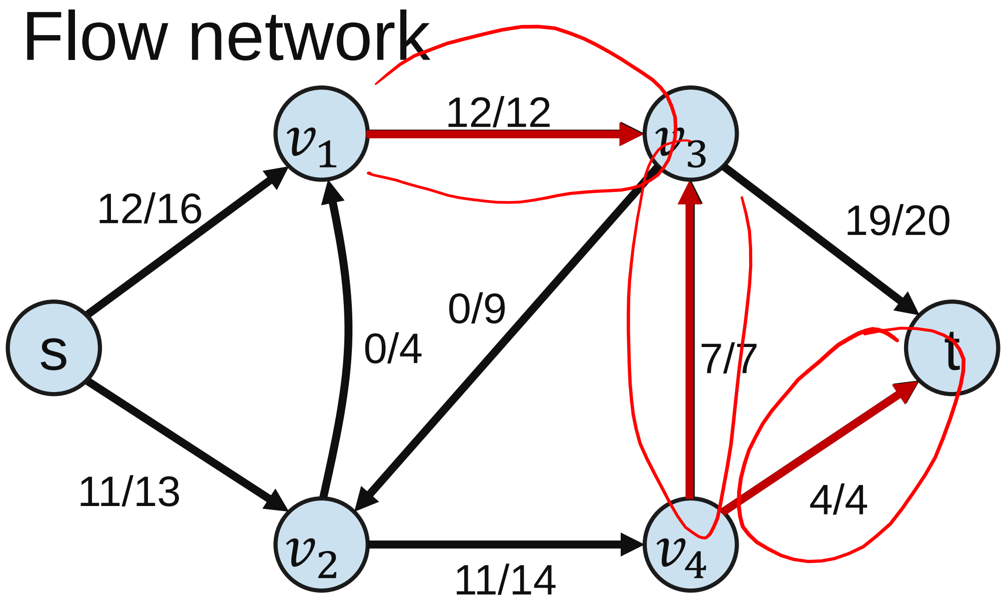

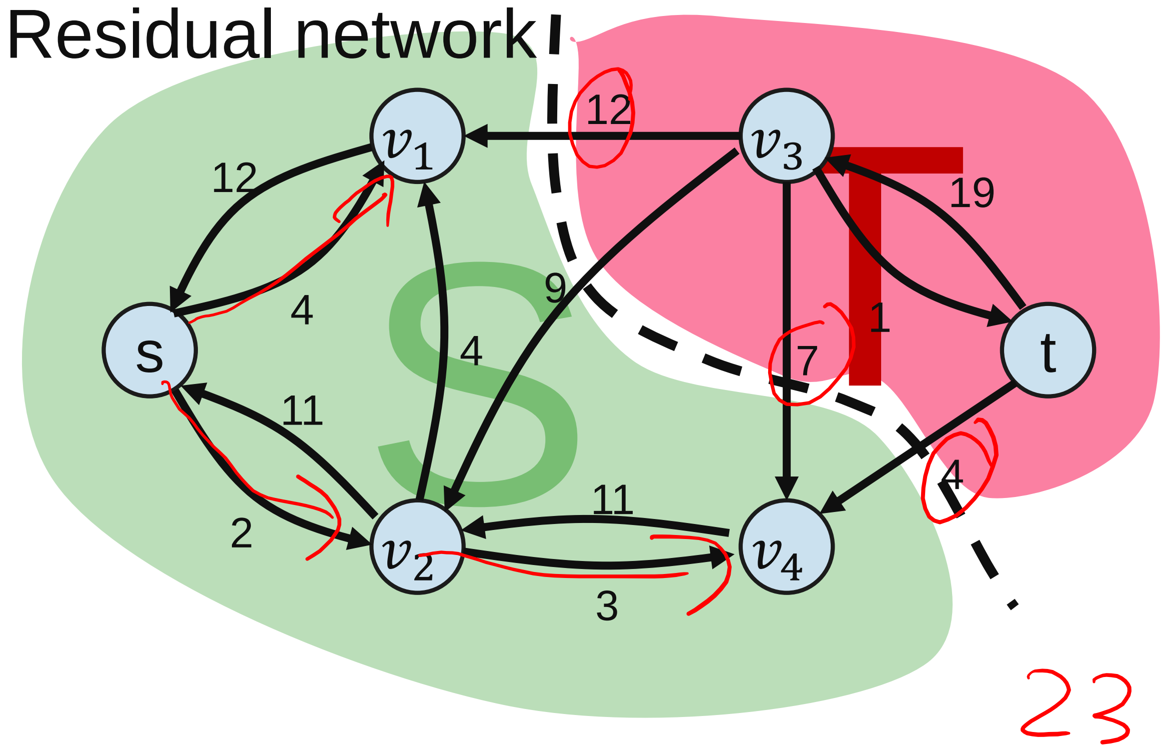

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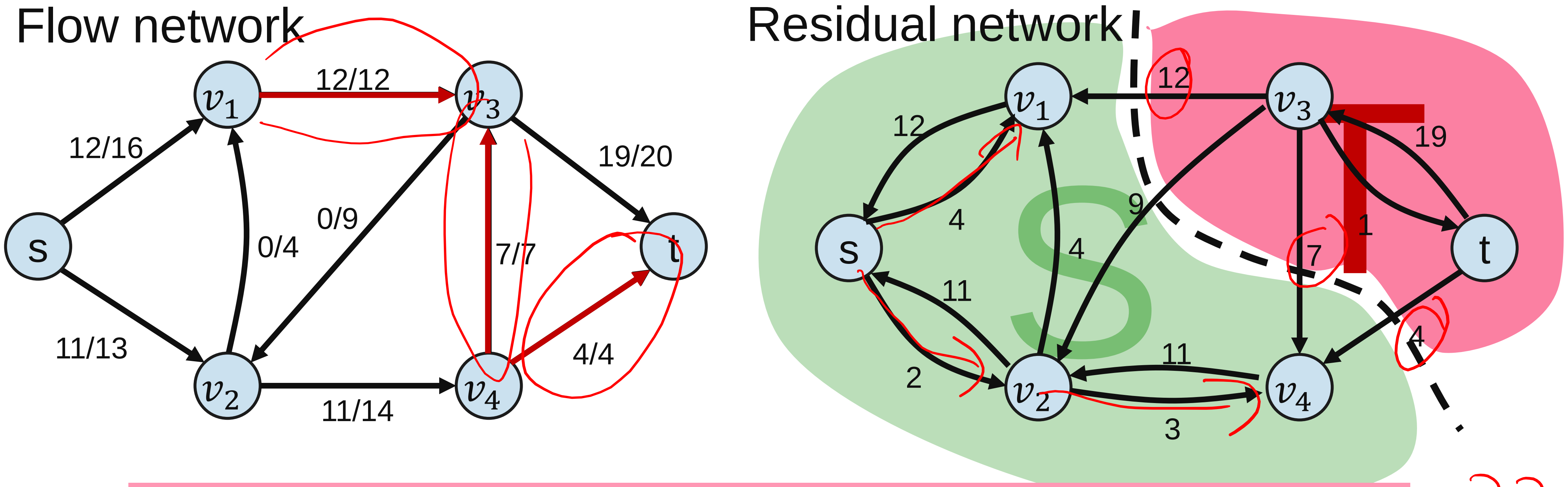

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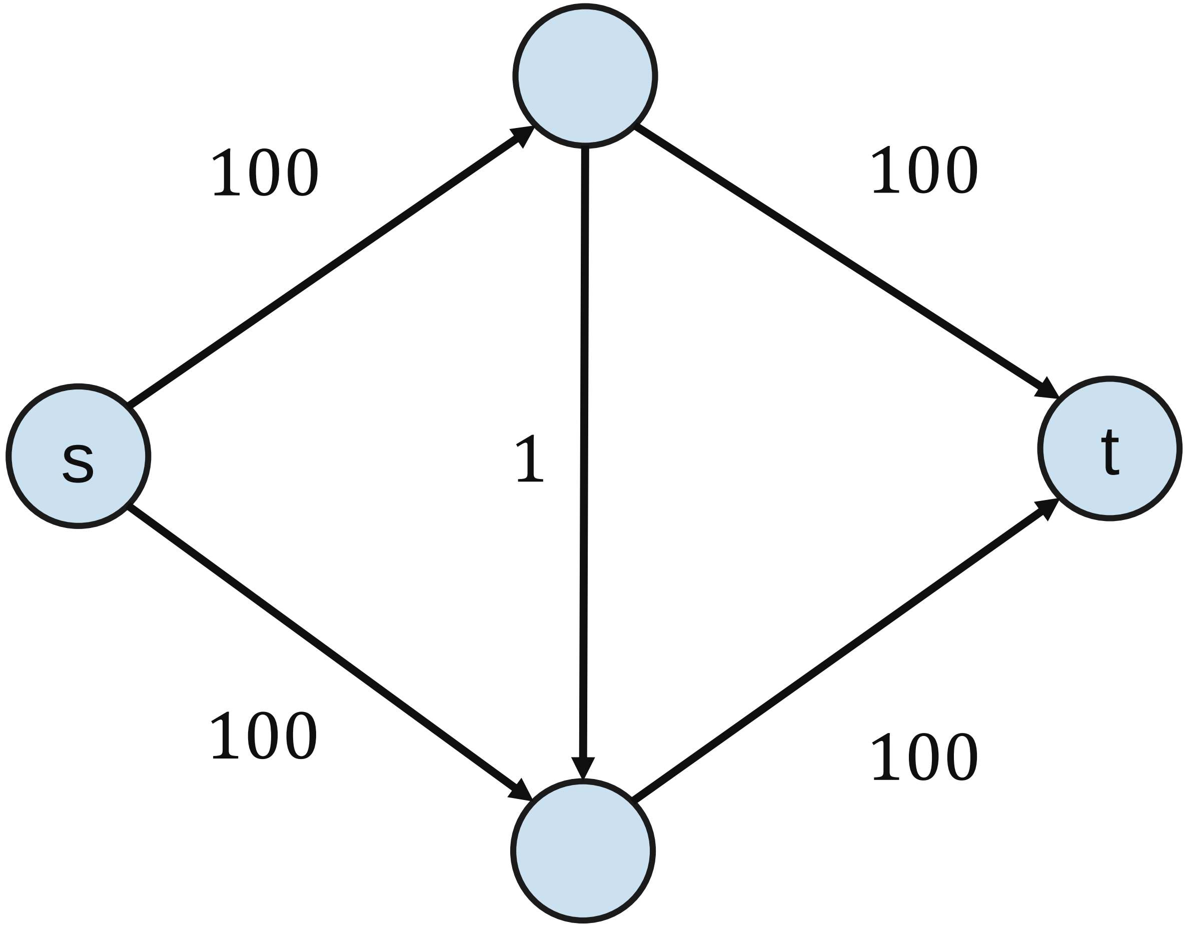
There are better, more direct methods! (not in this lecture)

Analysis 3: Runtime

Example: Ford-Fulkerson can be slow

The Ford-Fulkerson-method uses augmentation steps but f^* can be really large …



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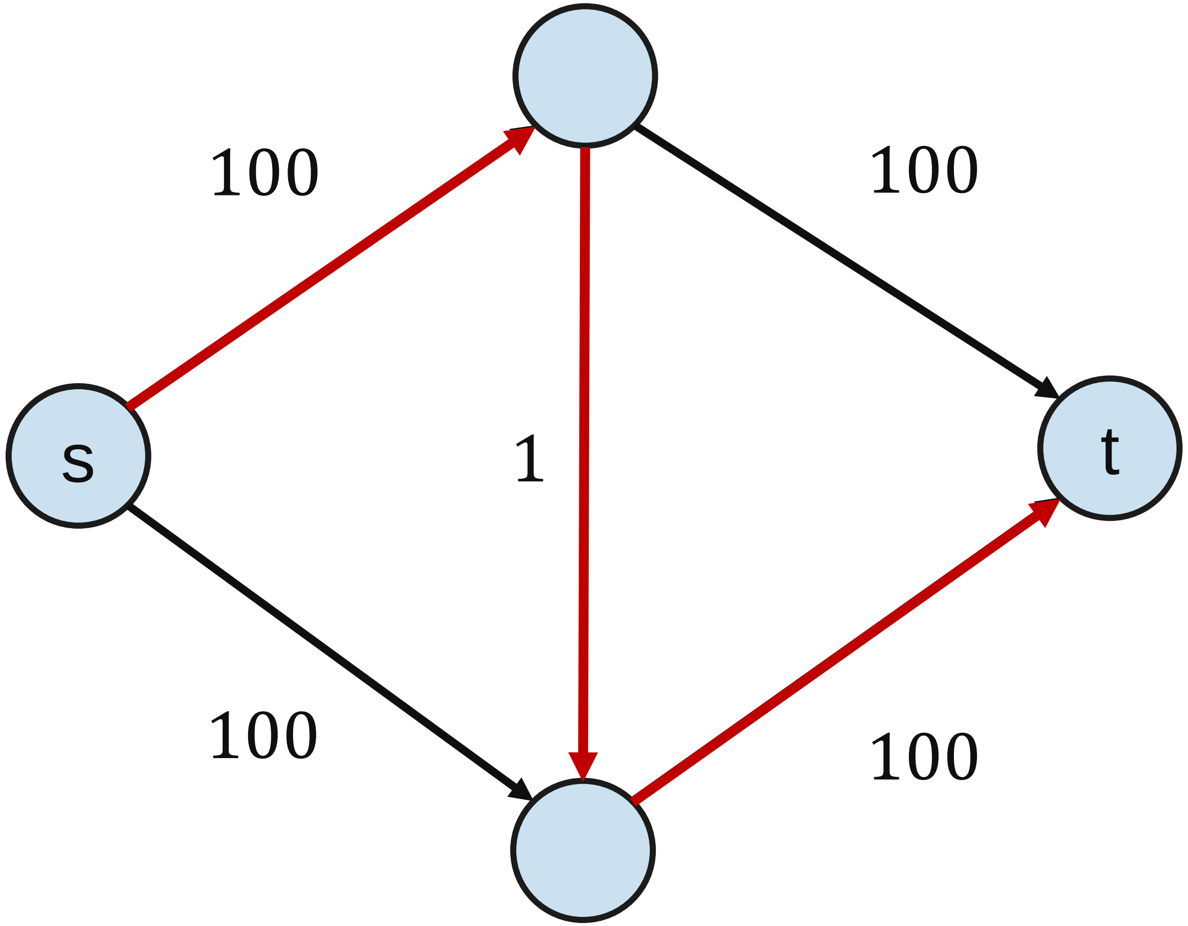

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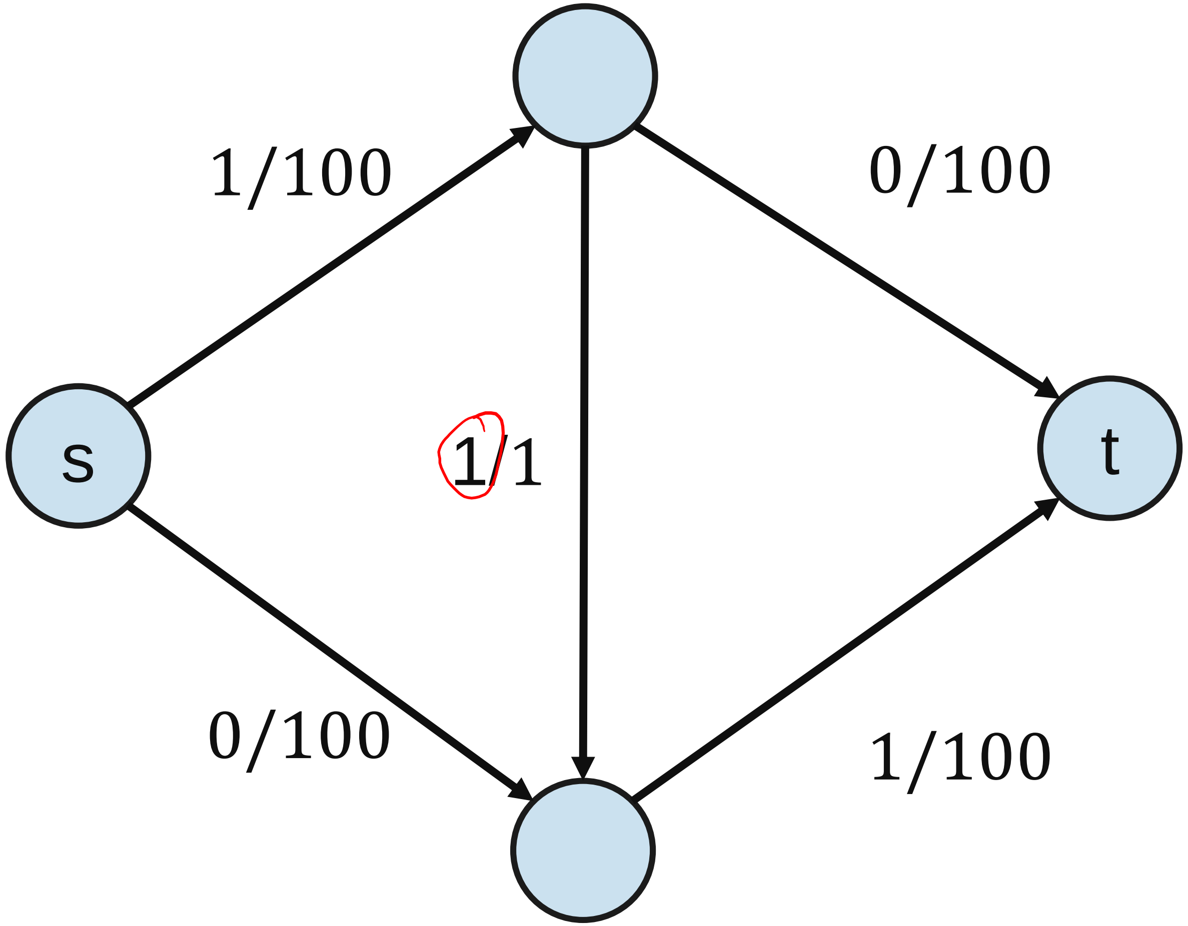

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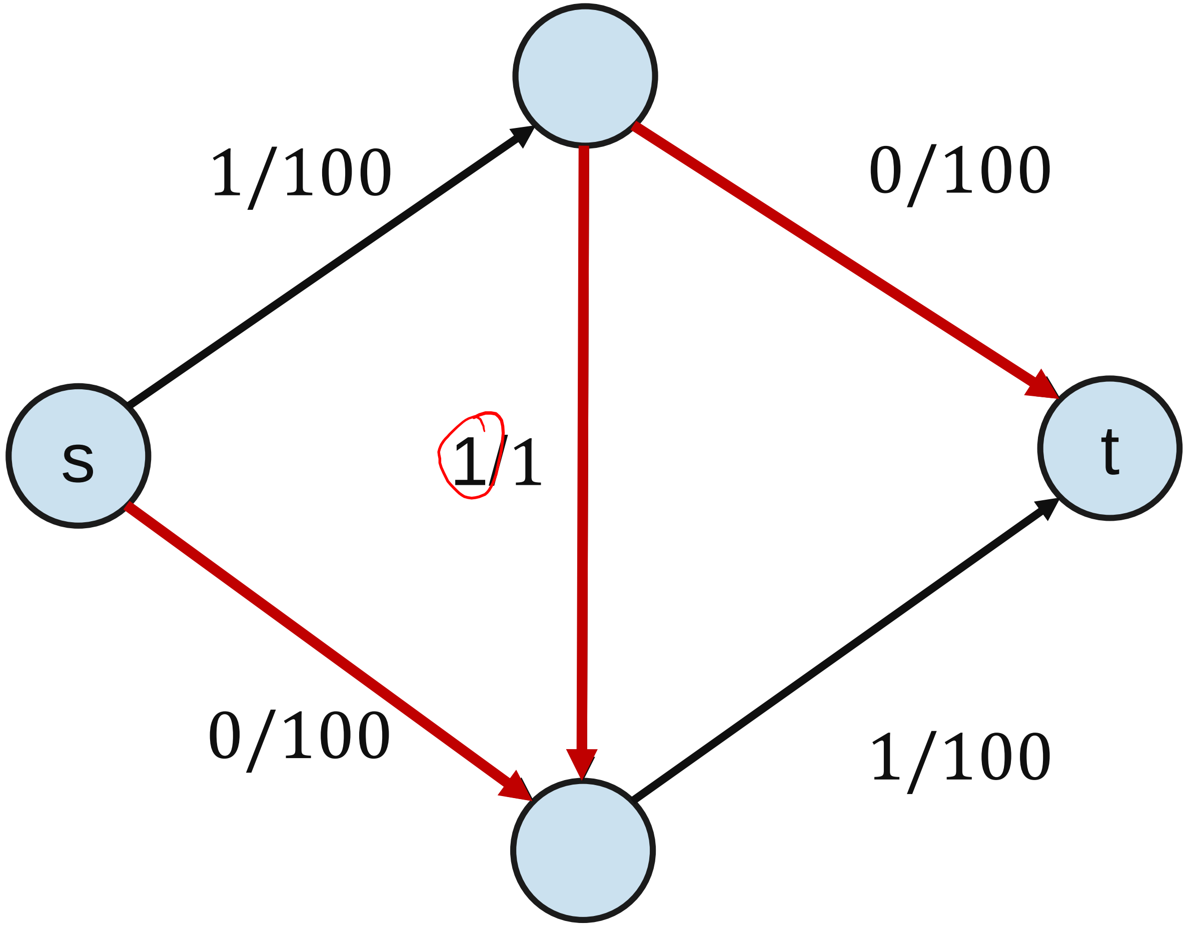

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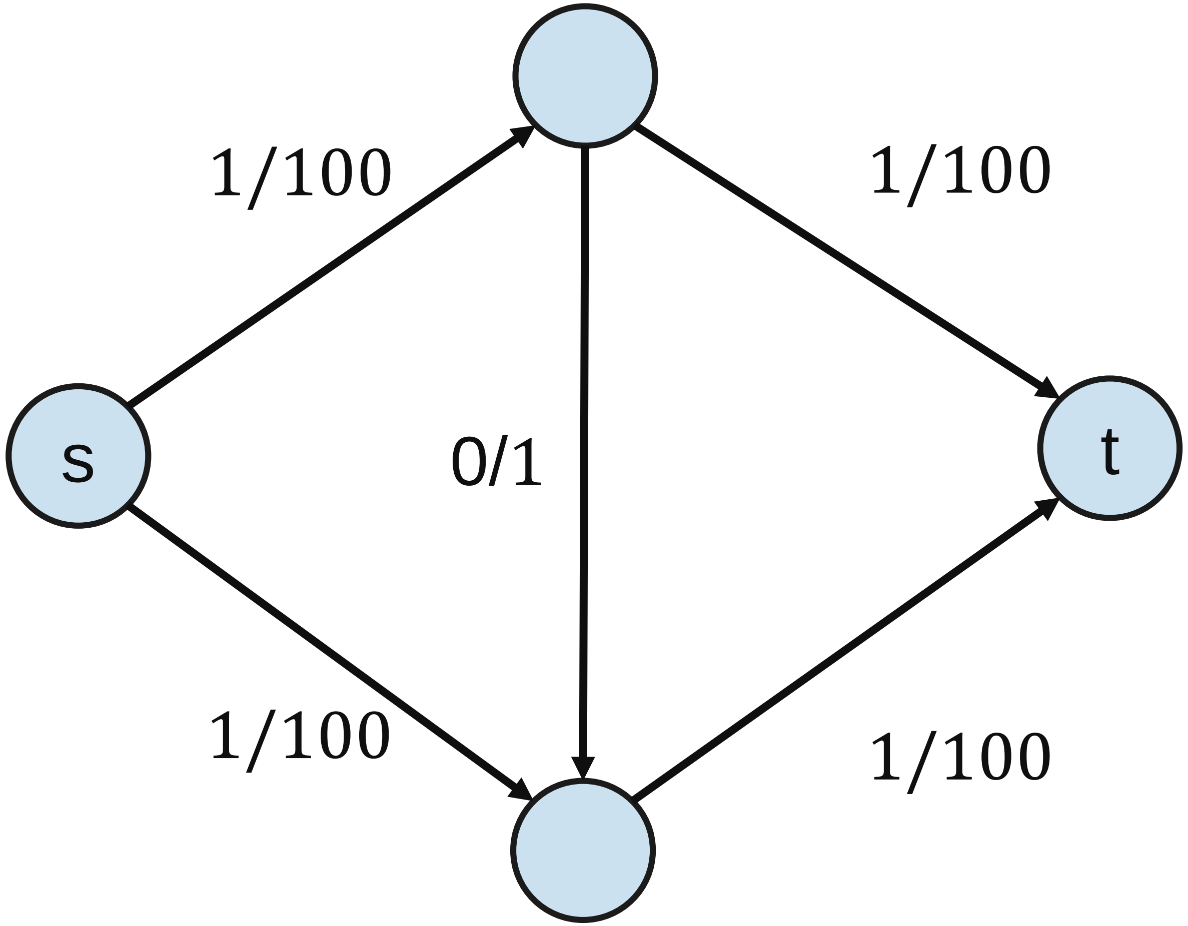

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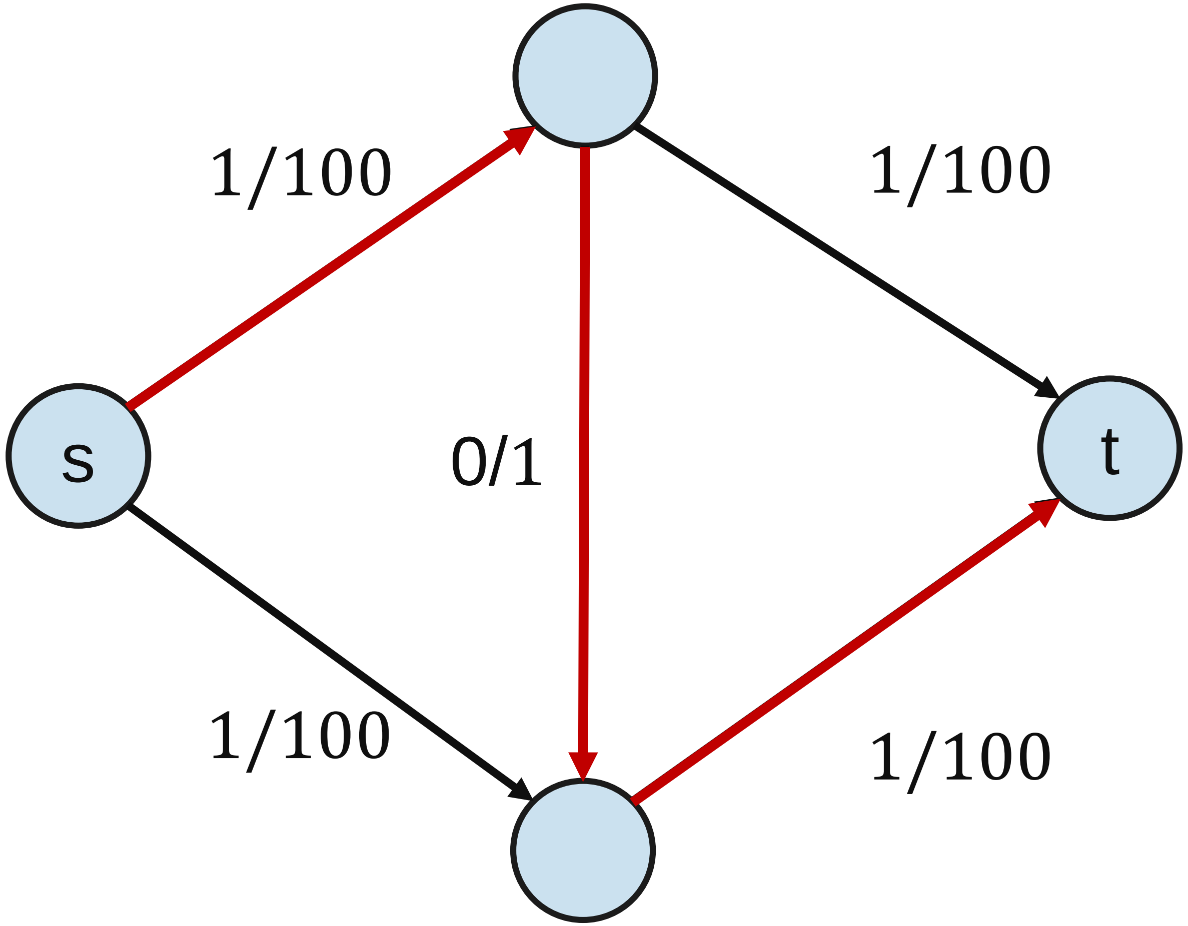

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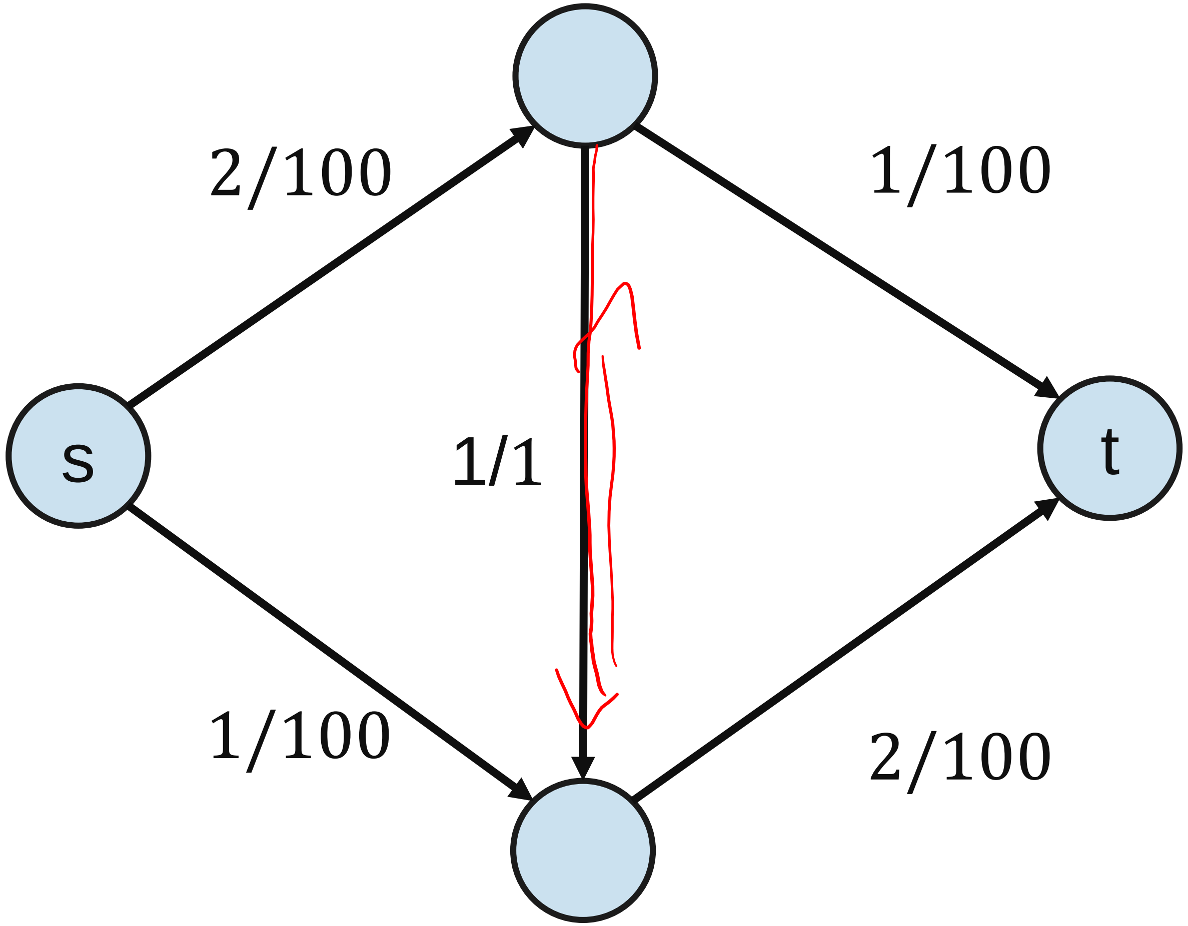

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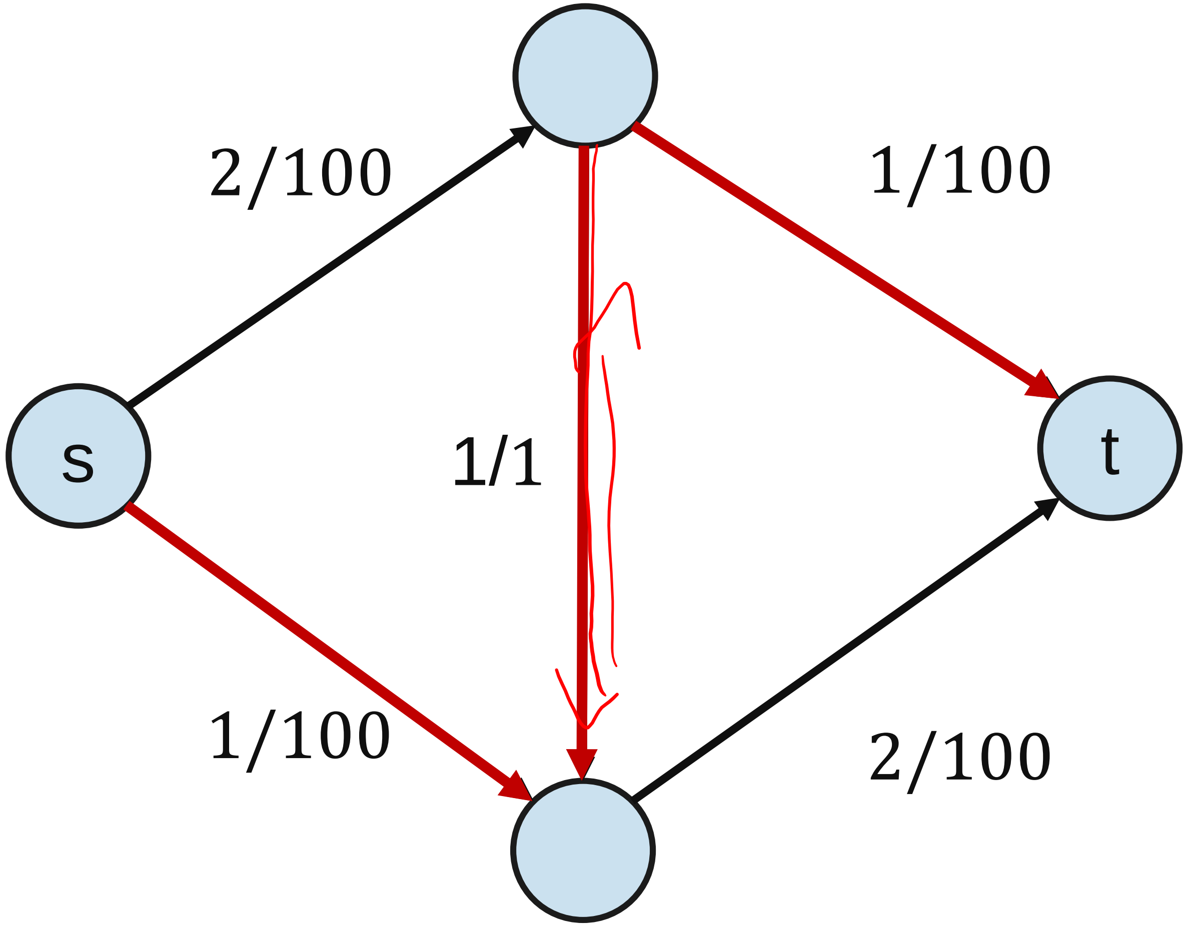

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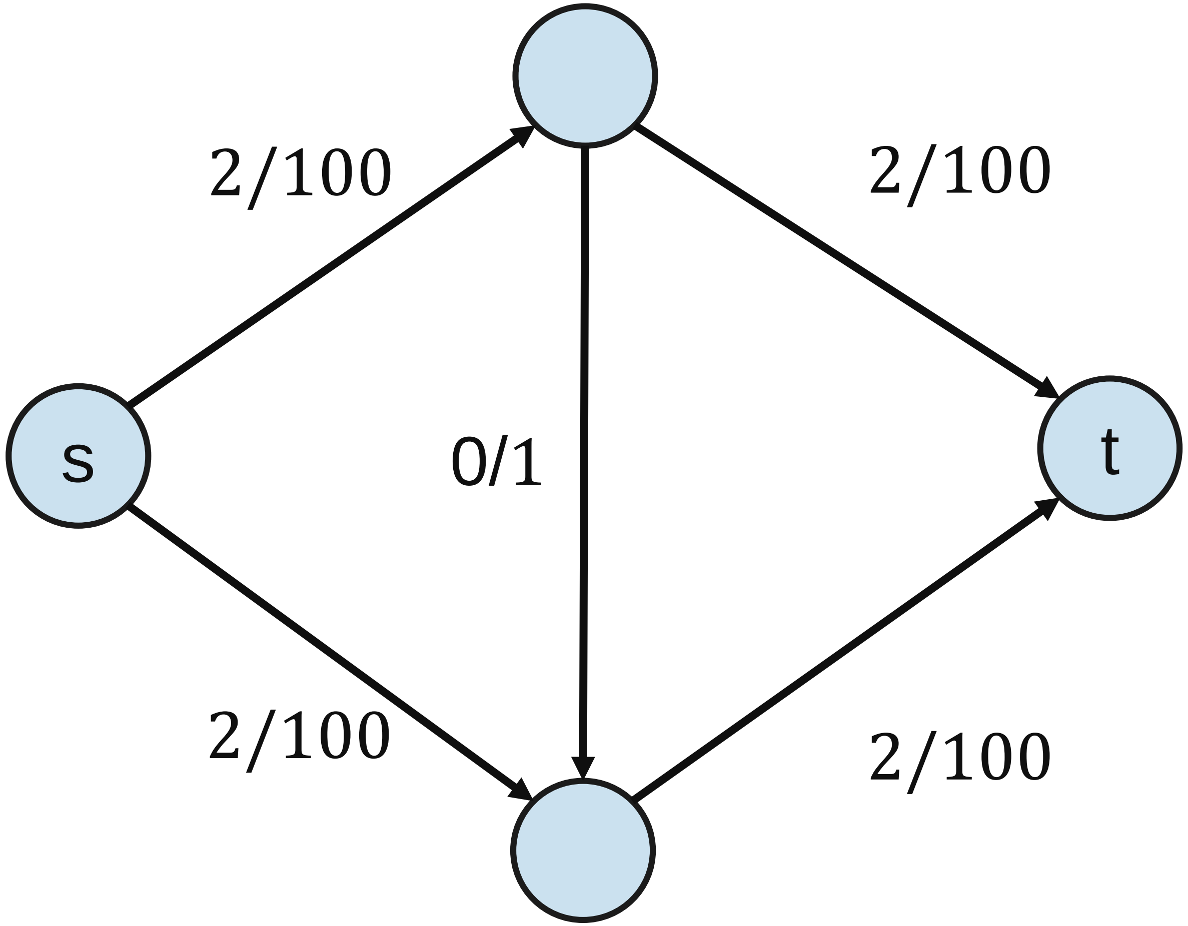


Might be very slow, even if the network is very small.

Edmonds-Karp-Algorithm

credit also goes to Yefim Dinitz (who actually first published the algorithm/analysis 1970)

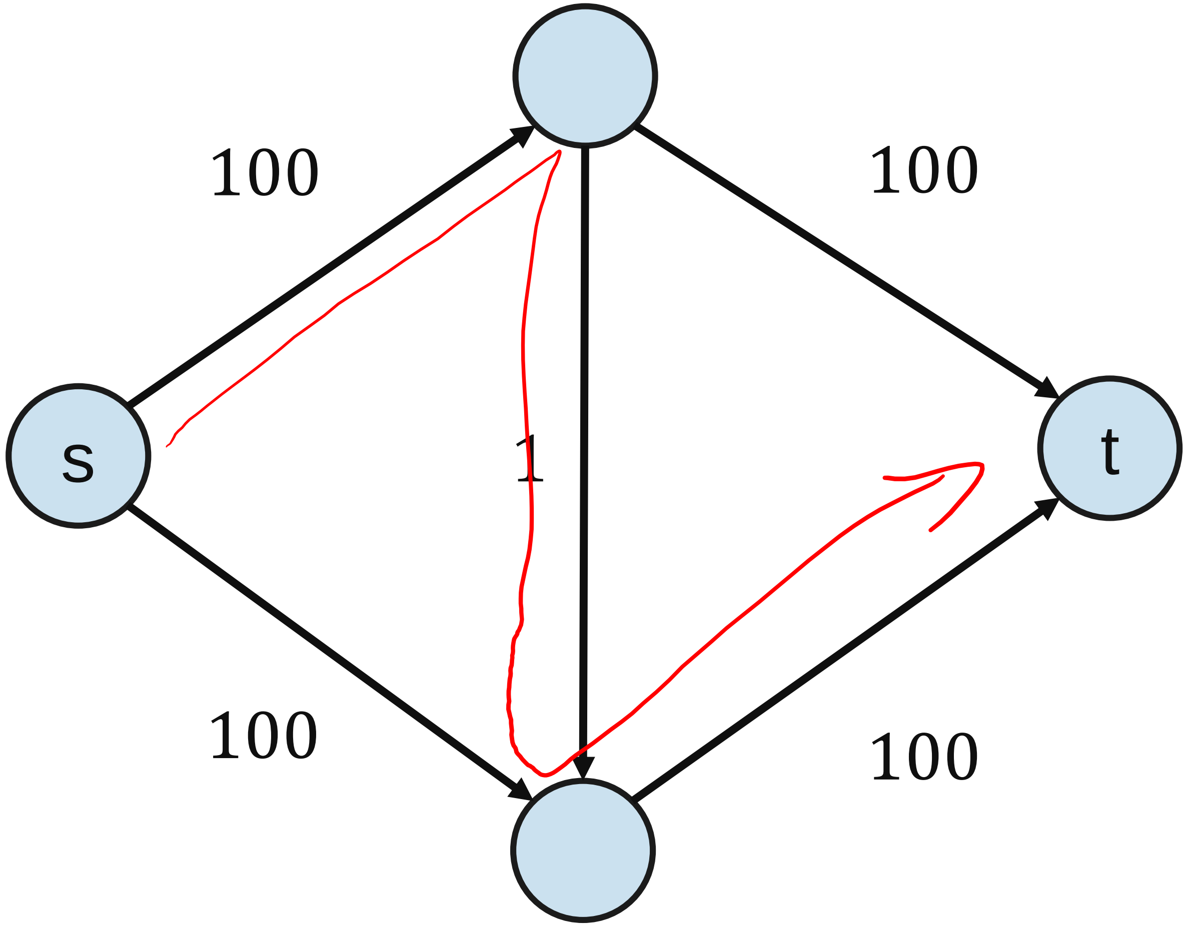
Edmonds-Karp-Algorithm

```
Initialize flow f with 0
while there exists an augmenting path in G_f do
    find a shortest augmenting path p
    augment f along p
return f
```

Difference: We always chose a **shortest** augmenting path.

Theorem: The runtime of the Edmonds–Karp–Algorithm is polynomial in the size of the network, regardless of the value of the capacities.

Recap our Example



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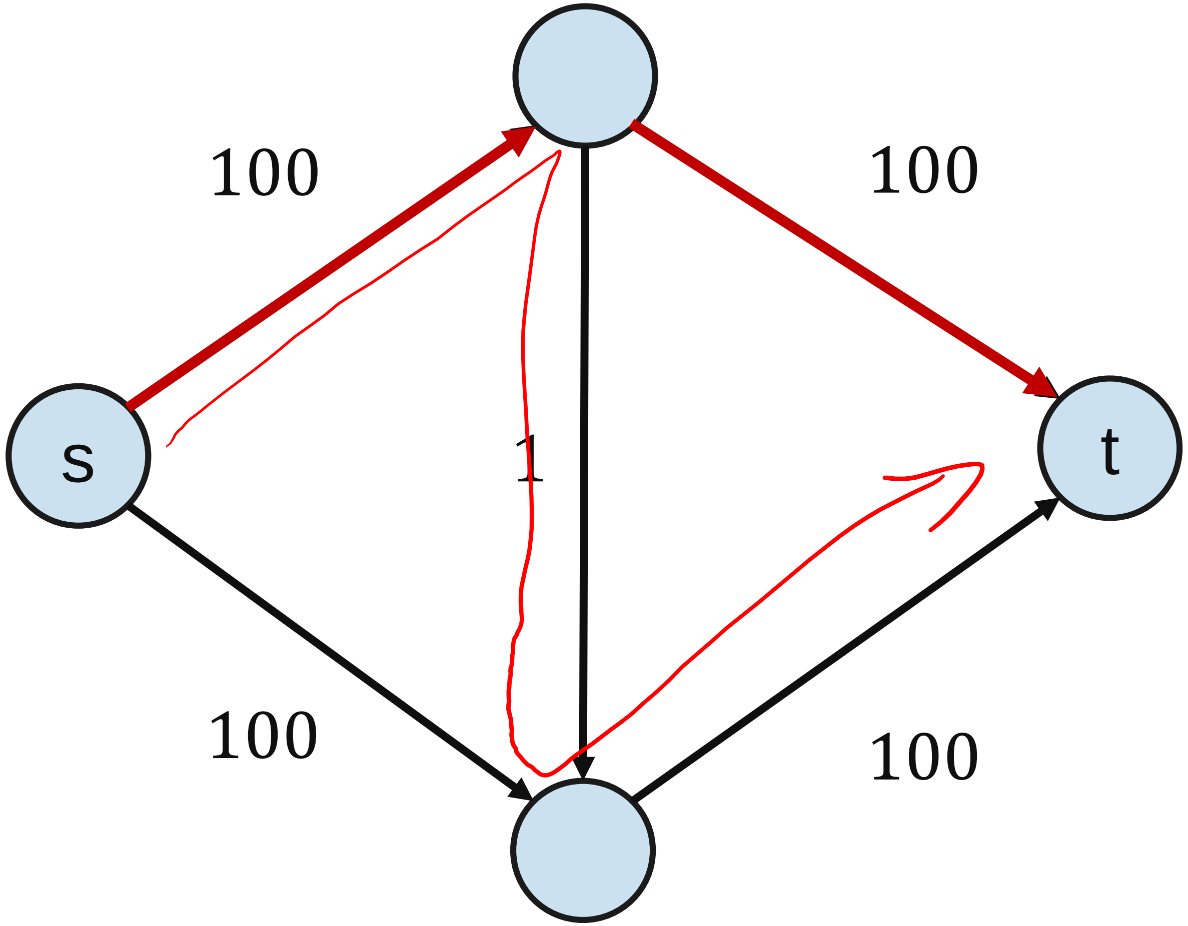

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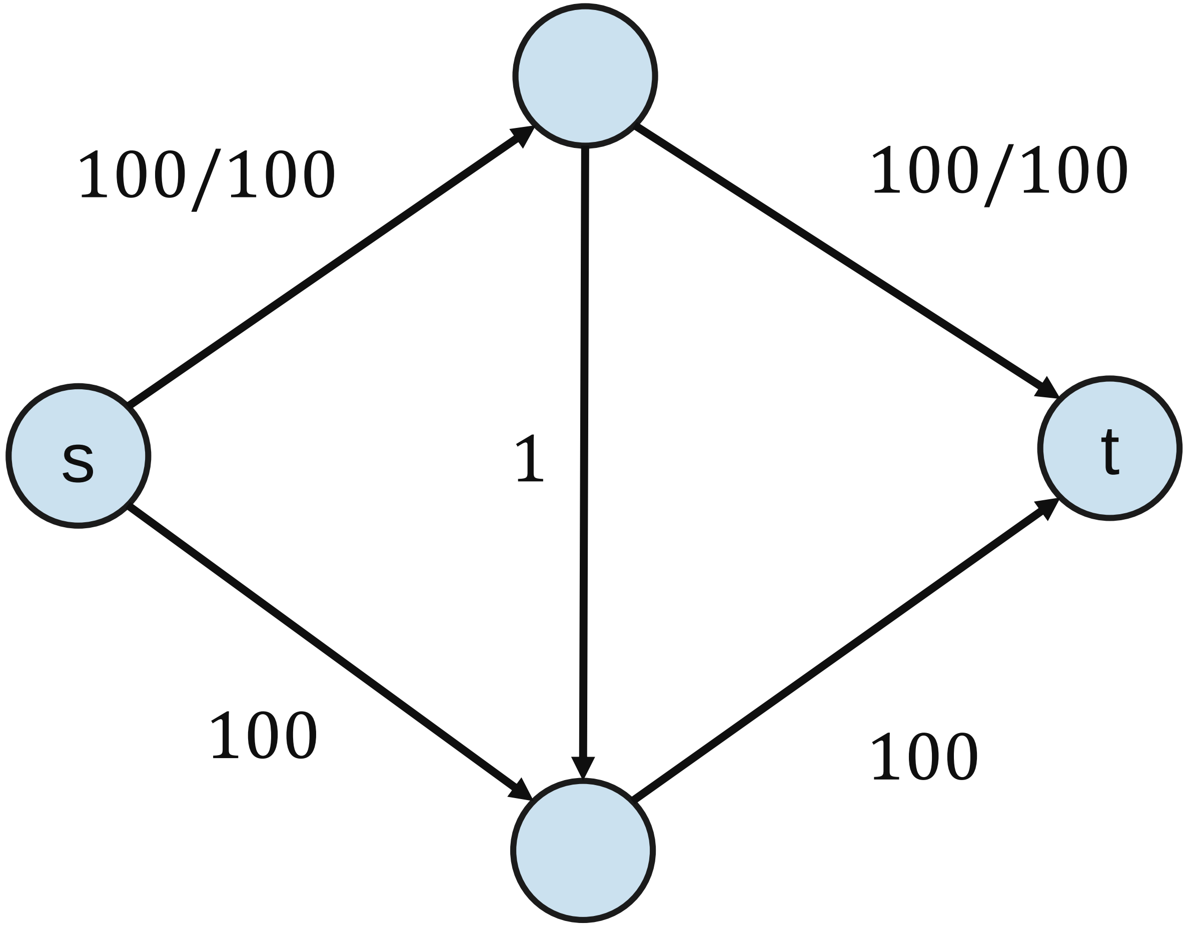

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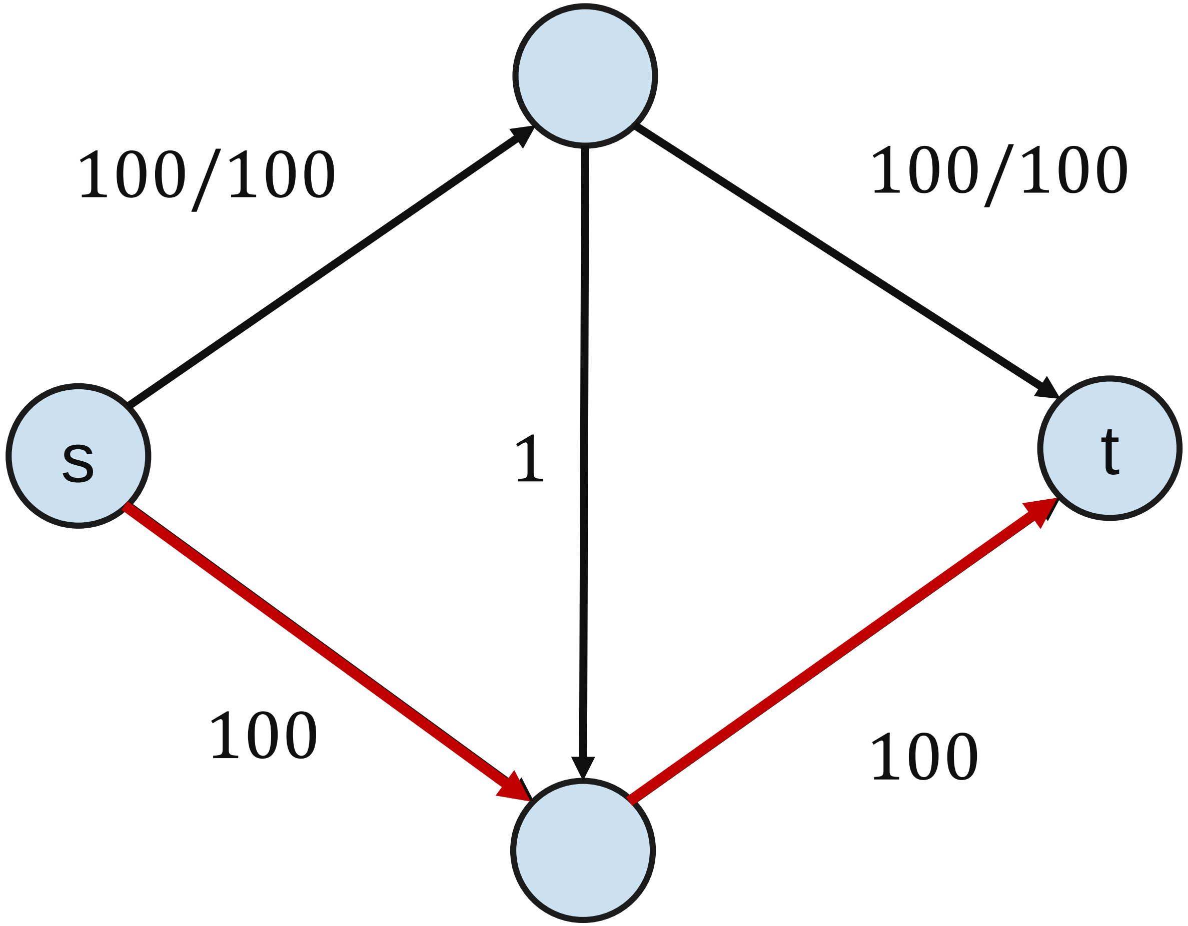

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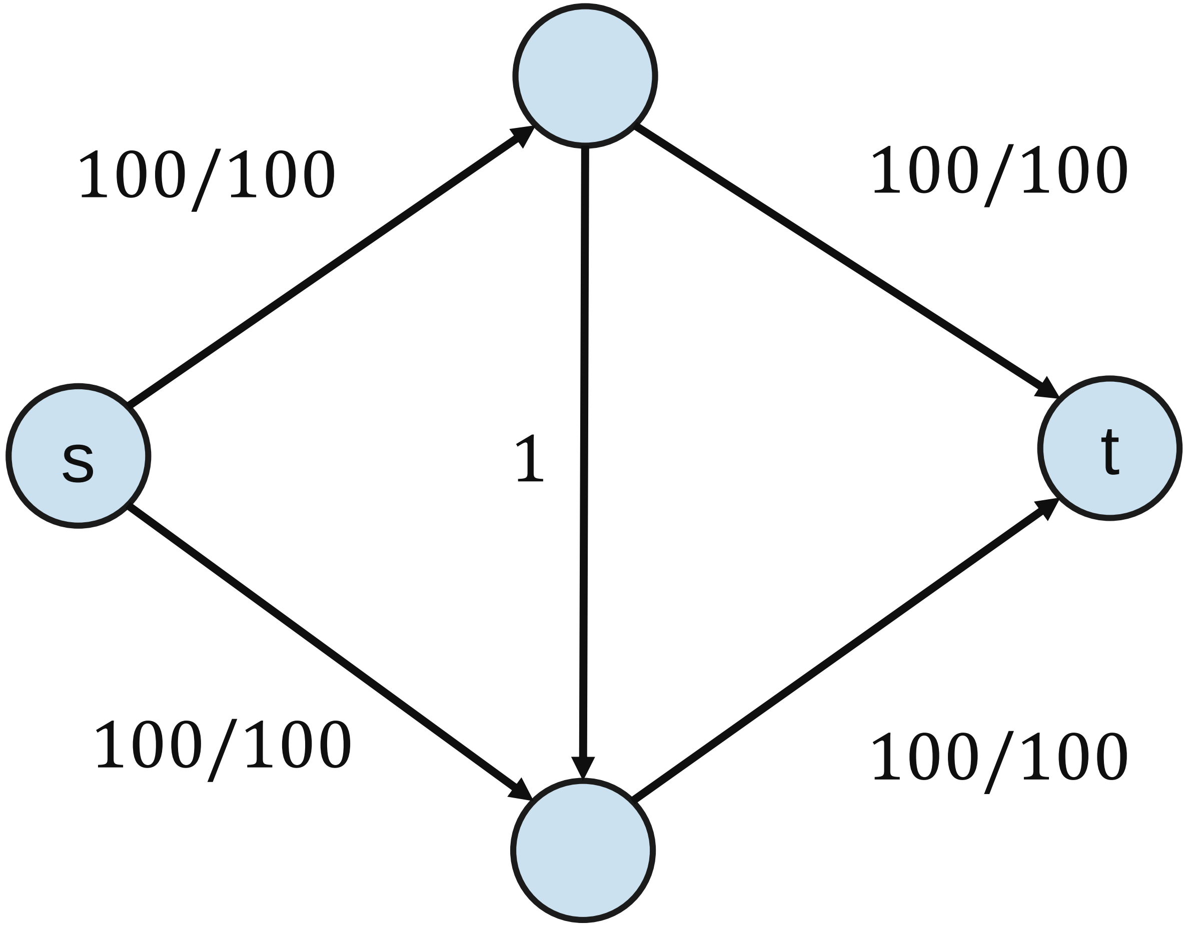


Edmonds-Karp-Algorithm is fast (for this example)!

Analysis of Edmonds-Karp

"We will show that level of a vertex increases if its incident edges appears in several augmentations."

Analysis of Edmonds-Karp

Define:

- f_i: the current flow before the i-th augmentation step.
- $G_i = G_{f_i}$: the residual network of f_i (note that $G_1 = G$)
- $level_i(v)$: the distance of s to v in G_i $\in \{0, 1, \ldots, |V|-1\} \cup \{\infty\}$

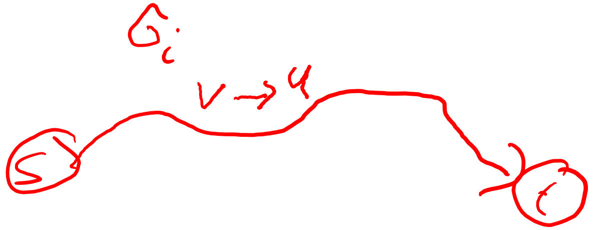
"We will show that level of a vertex increases if its incident edges appears in several augmentations."

Lemma B (levels don't decrease):

For all $v \in V$ and all i we have $level_{i+1}(v) \geq level_i(v)$.

Proof: $level_{i+1}(v) \geq level_i(v)$.

For $v = s$ the claim is trivial as $level_i(s) = 0$.



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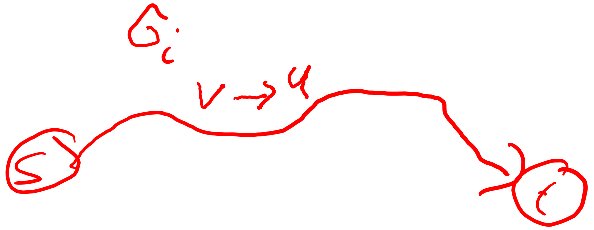
Let $v \neq s$ with minimal $level_{i+1}(v)$, contradicting the claim.



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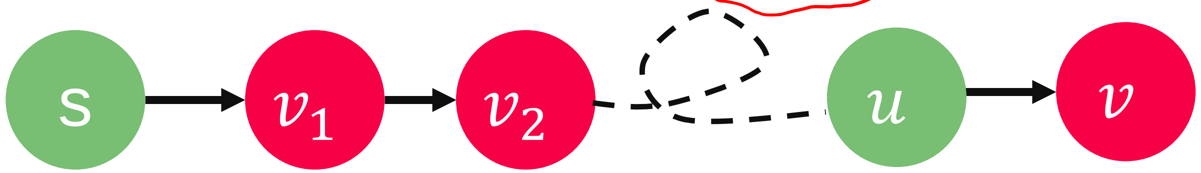
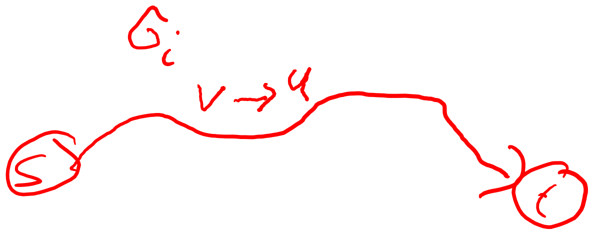


consider a shortest path from s to v in G_{i+1}



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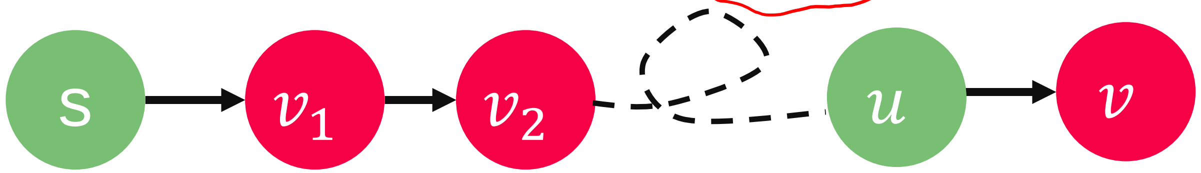

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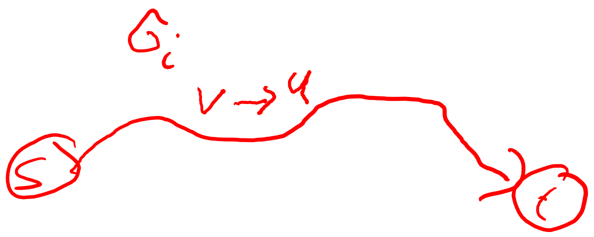
(*) We have $level_{i+1}(v) = level_{i+1}(u) + 1 \geq$ $level_i(u) + 1$

due to choice of v, u doesn't contradict the claim !



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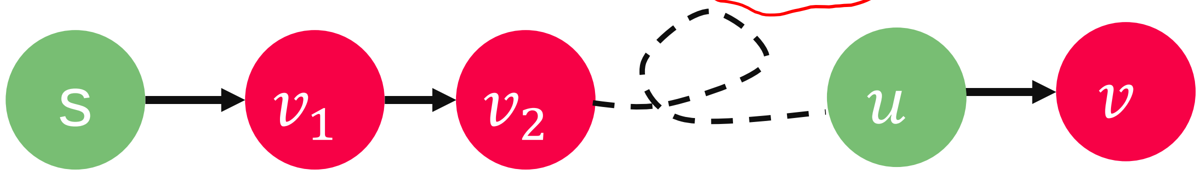

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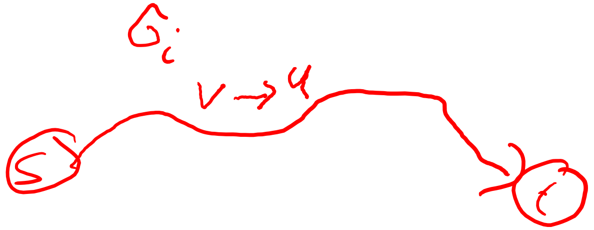
(levels of adjacent vertices can differ by at most one)



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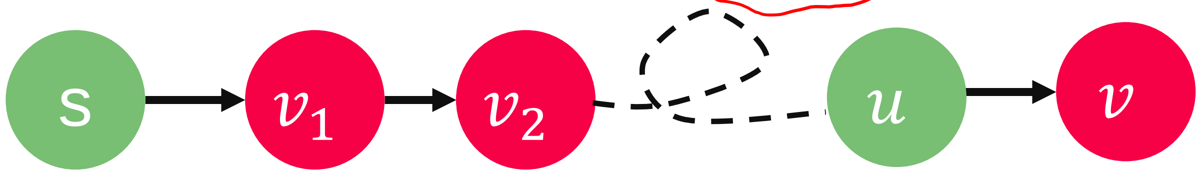

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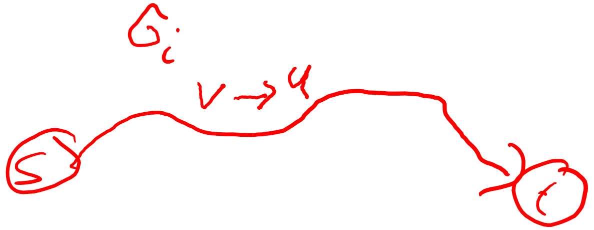

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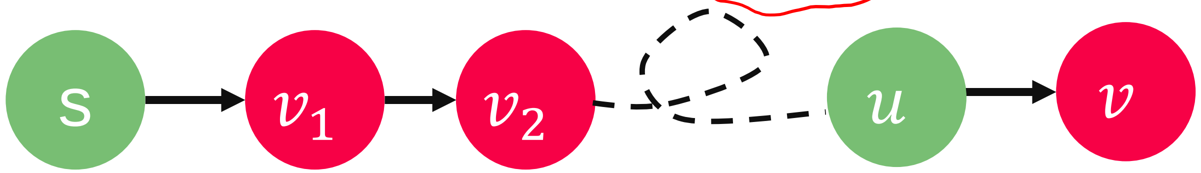

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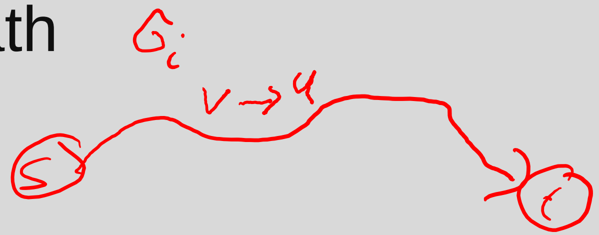
Case $(\boldsymbol{u}, \boldsymbol{v}) \notin \boldsymbol{G_i}$: (v, u) must be an edge in the i-th augmenting path
(v, u) must lie on the shortest s-t-path in G_i as (u, v) in G_{i+1}, hence
$level_i(v) = level_i(u) - 1 \underset{\text{trivial}}{\leq} level_i(u) + 1$



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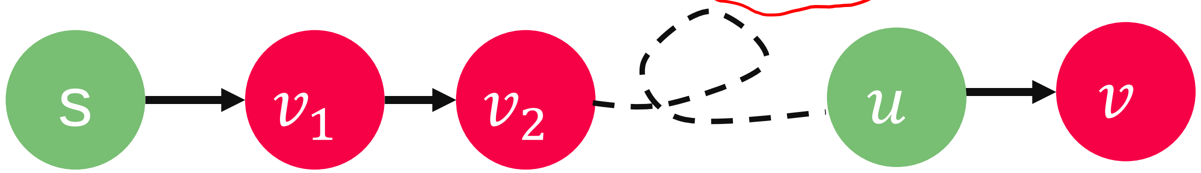


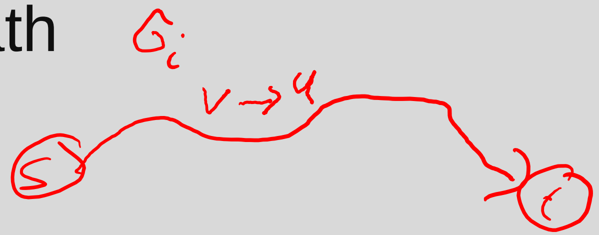
consider a shortest path from s to v in G_{i+1}

(*) We have $level_{i+1}(v) = level_{i+1}(u) + 1 \geq$ $level_i(u) + 1$

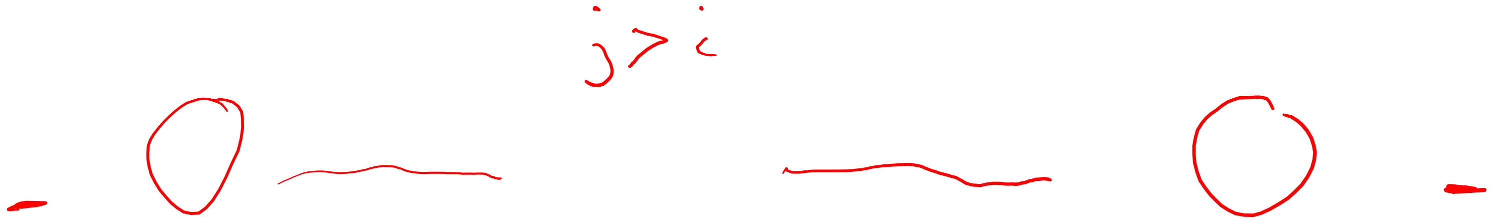
due to choice of v, u doesn't contradict the claim !

Case $(\boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{G_i}$: $level_i(v) \leq level_i(u) + 1 \overset{(*)}{\leq} level_{i+1}(v)$
(levels of adjacent vertices can differ by at most one)

Case $(\boldsymbol{u}, \boldsymbol{v}) \notin \boldsymbol{G_i}$: (v, u) must be an edge in the i-th augmenting path
(v, u) must lie on the shortest s-t-path in G_i as (u, v) in G_{i+1}, hence
$level_i(v) = level_i(u) - 1 \underset{\text{trivial}}{\leq} level_i(u) + 1 \underset{(*)}{\leq} level_{i+1}(v)$



During the execution of the Edmonds-Karp-Algorithm, any edge (u, v) disappears from the residual graph at most $V/2$ times.



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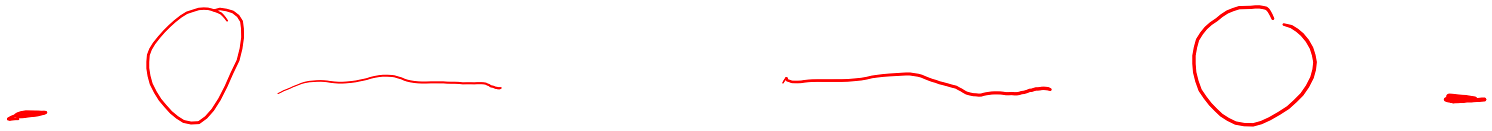
Proof: Suppose $u \rightarrow v$ is in two residual graphs G_i and G_{j+1}, but not in any of the intermediate residual graphs G_{i+1}, G_j for some i < j.

$j > i$



Edmonds-Karp-Algorithm: Runtime Analysis

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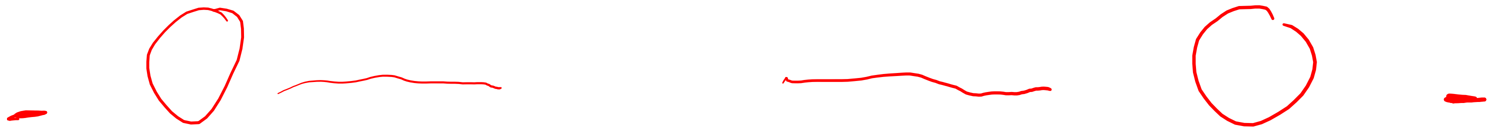

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By Lemma B, we have

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$$level_j(u) = level_j(v) + 1 \geq level_i(v) + 1 = level_i(u) + 2.$$

The distance from s to u increased by at least 2 between the disappearance and reappearance of $u \to v$. Since every level is either less than V or infinite, the number of disappearances is at most $V/2$

Edmonds-Karp-Algorithm

The Edmonds-Karp-Algorithm computes a maximum flow with runtime $O(|E|^2|V|)$.

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- Every augmentation requires $O(|E|)$ steps as a shortest augmenting paths can be found via BFS and also G_f can be built with $O(|E|)$ steps.
- There are at most $O(|E| \cdot |V|)$ augmentations ($\leq |V|/2$ augmentations per edge)

State-of-The-Art

Ford-Fulkerson Naïve, 1956	$O(Ef^*)$
Edmonds-Karp/Dinic's Algorithm, 1970	$O(E^2V)$ $O(V^5)$
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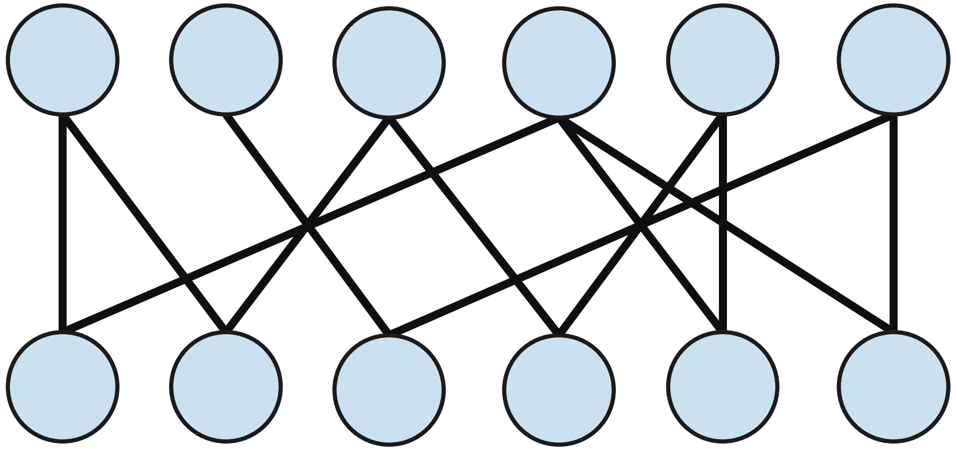
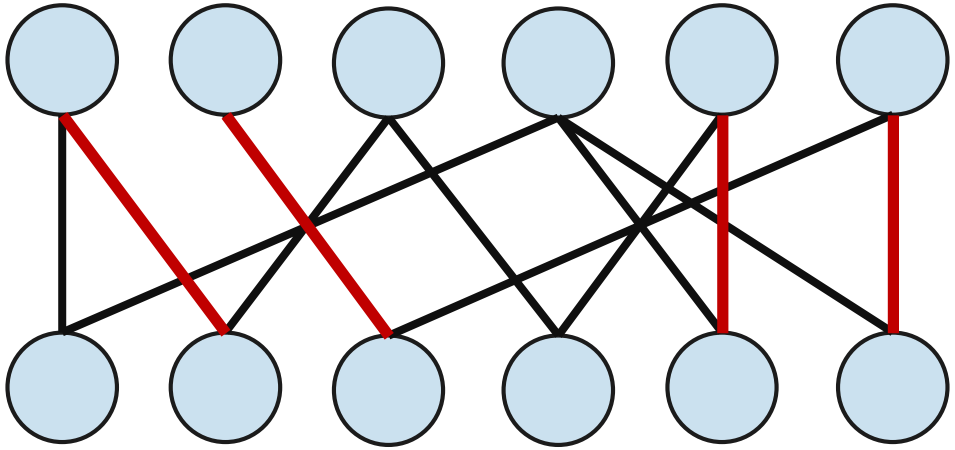
…	
Orlin + KRT, 2013	$O(VE)$
Gao-Liu-Peng, 2021	$\tilde{O}(E^{\frac{3}{2}-\frac{1}{328}} \log maxCapacity)$

Complicated, rely on involved Data structures

Application: Maximum Cardinality Bipartite Matching



A matching of a graph $G = (V, E)$ is a subset $M \subseteq E$ of the edges such any vertex $v \in V$ has at most one adjacent edge in M.



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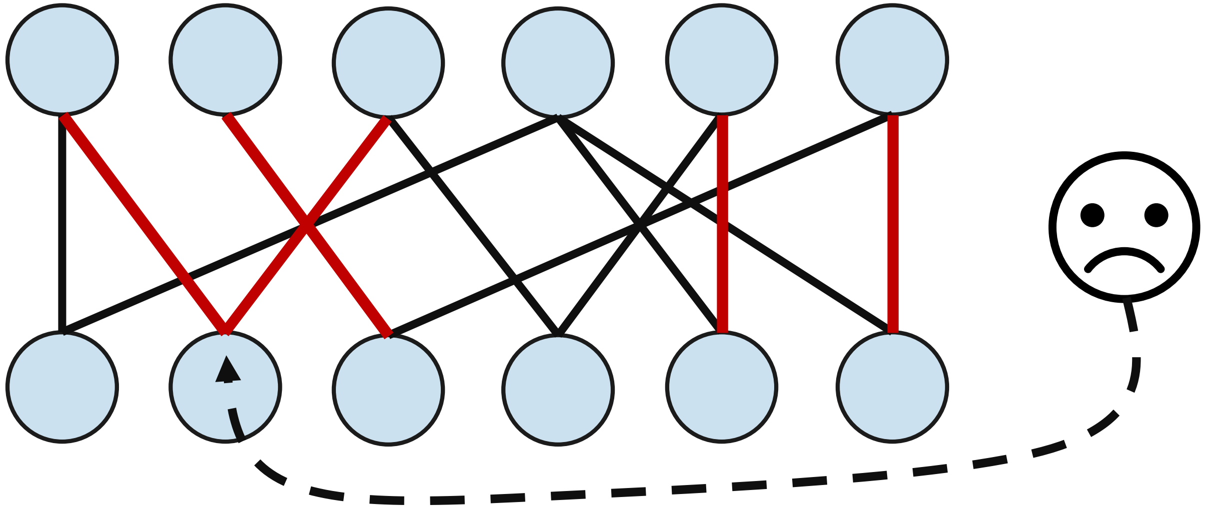

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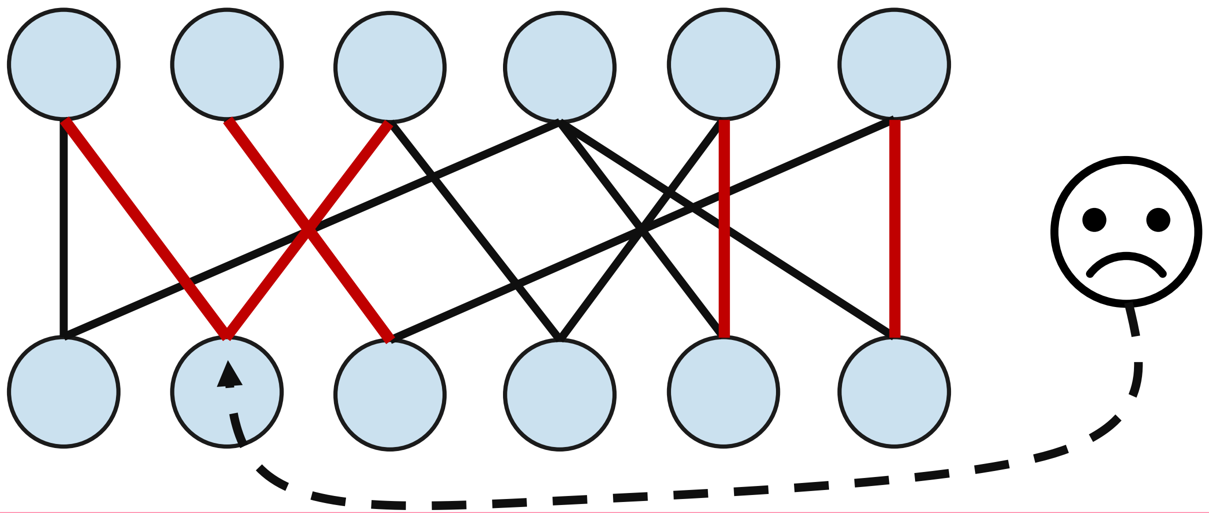

Maximum Bipartite Cardinality Matching: Given a bipartite graph, compute a matching that contains as many edges as possible.

Here, important special case: The graph is bipartite!

(match customers to products, ads to customers, …)

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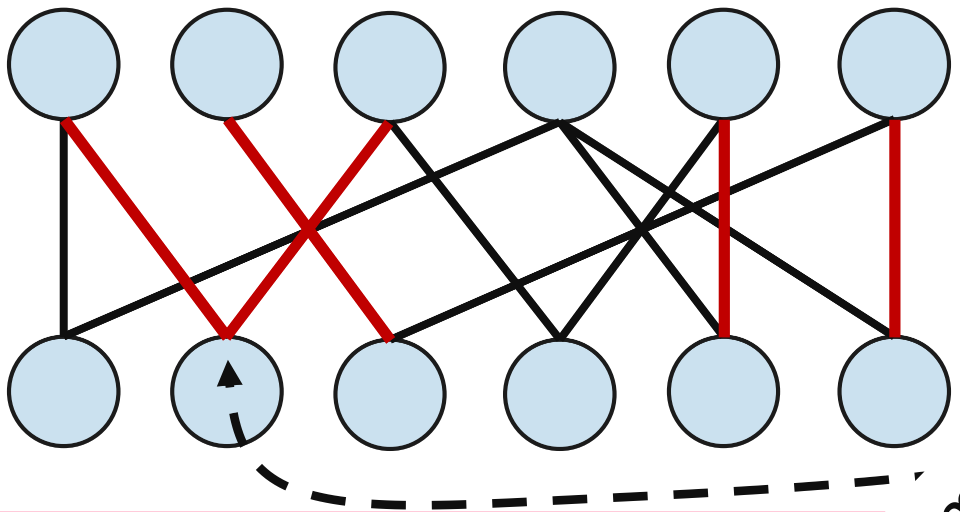

Many extensions exist: minimum cost bipartite matchings, …

Today: The simplest version

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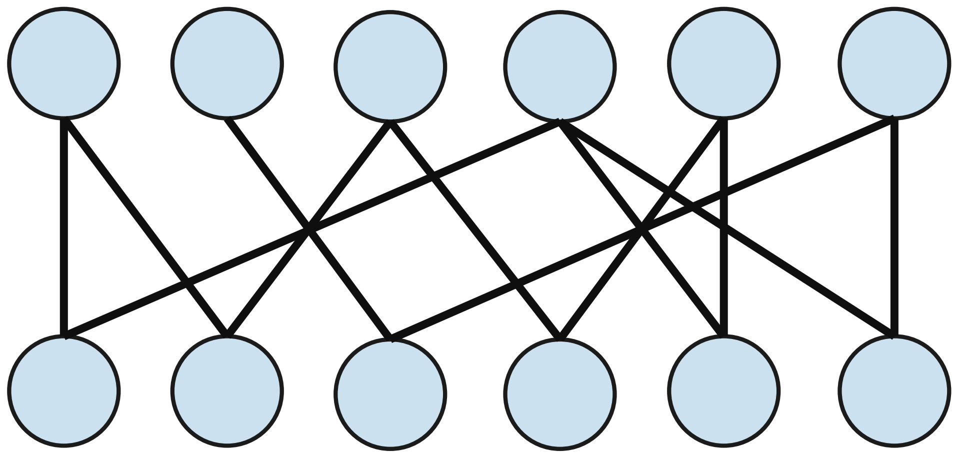

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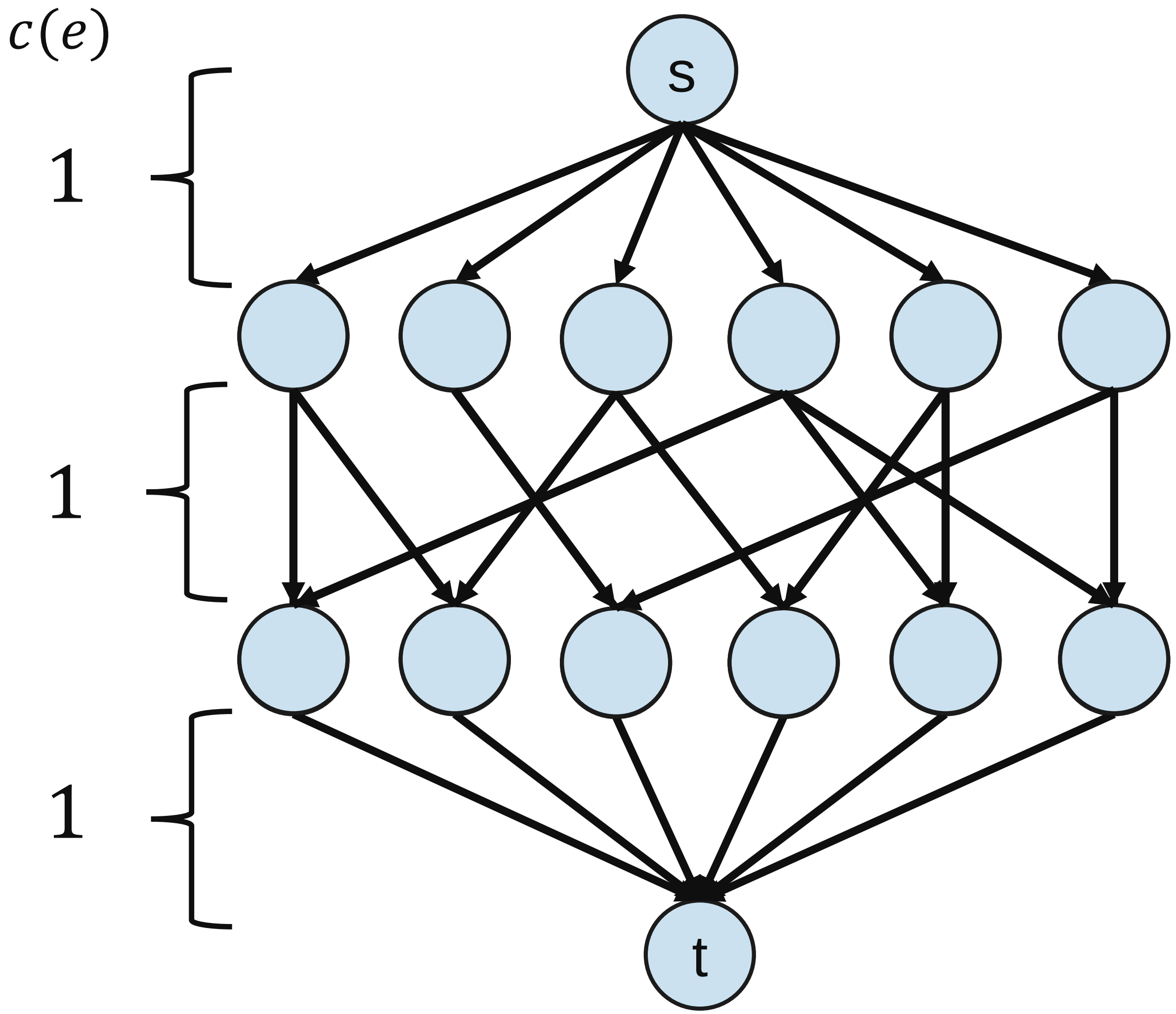

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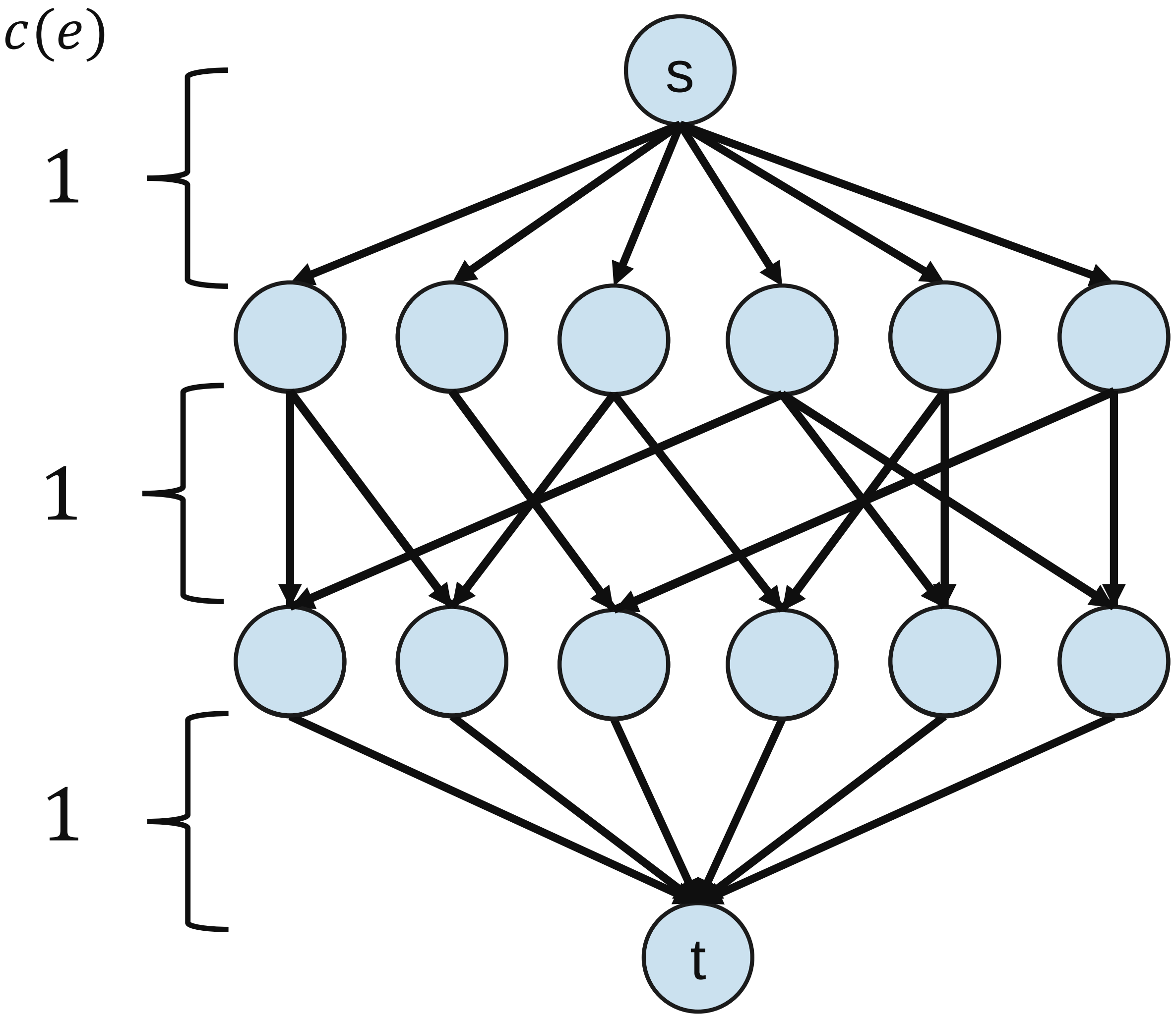

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Given: Bipartite graph $B = (X \cup Y, E)$.

1. Build flow network

$$\tilde{B} = (X \cup Y \cup \{s,t\}, E', c) \text{ with}$$

$$E' = \{(s,x) | x \in X\} \cup \{(y,t) | y \in Y\} \cup \{(x,y) | \{x,y\} \in E\}$$

and $c(e') = 1$ for all $e' \in E'$.

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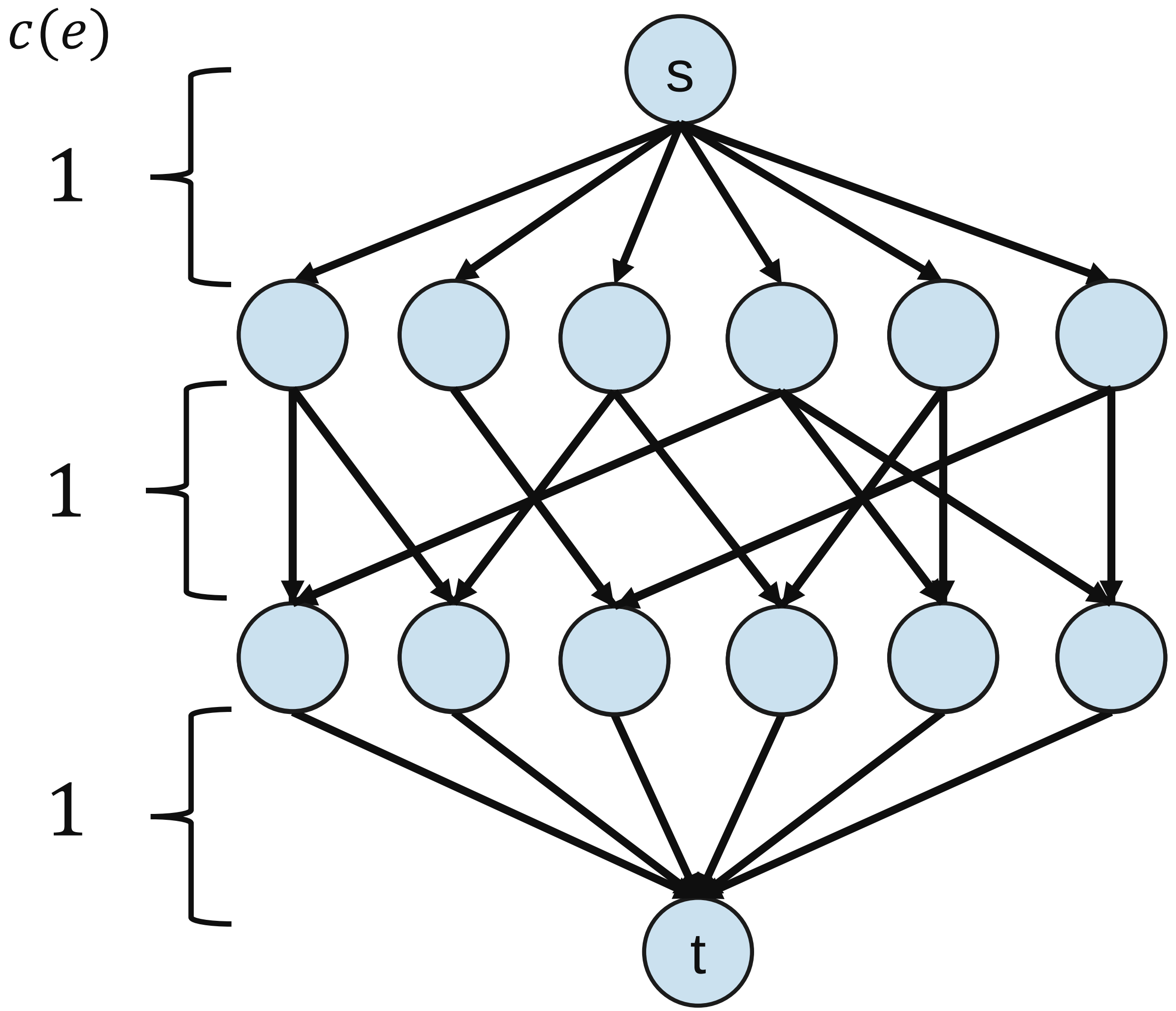

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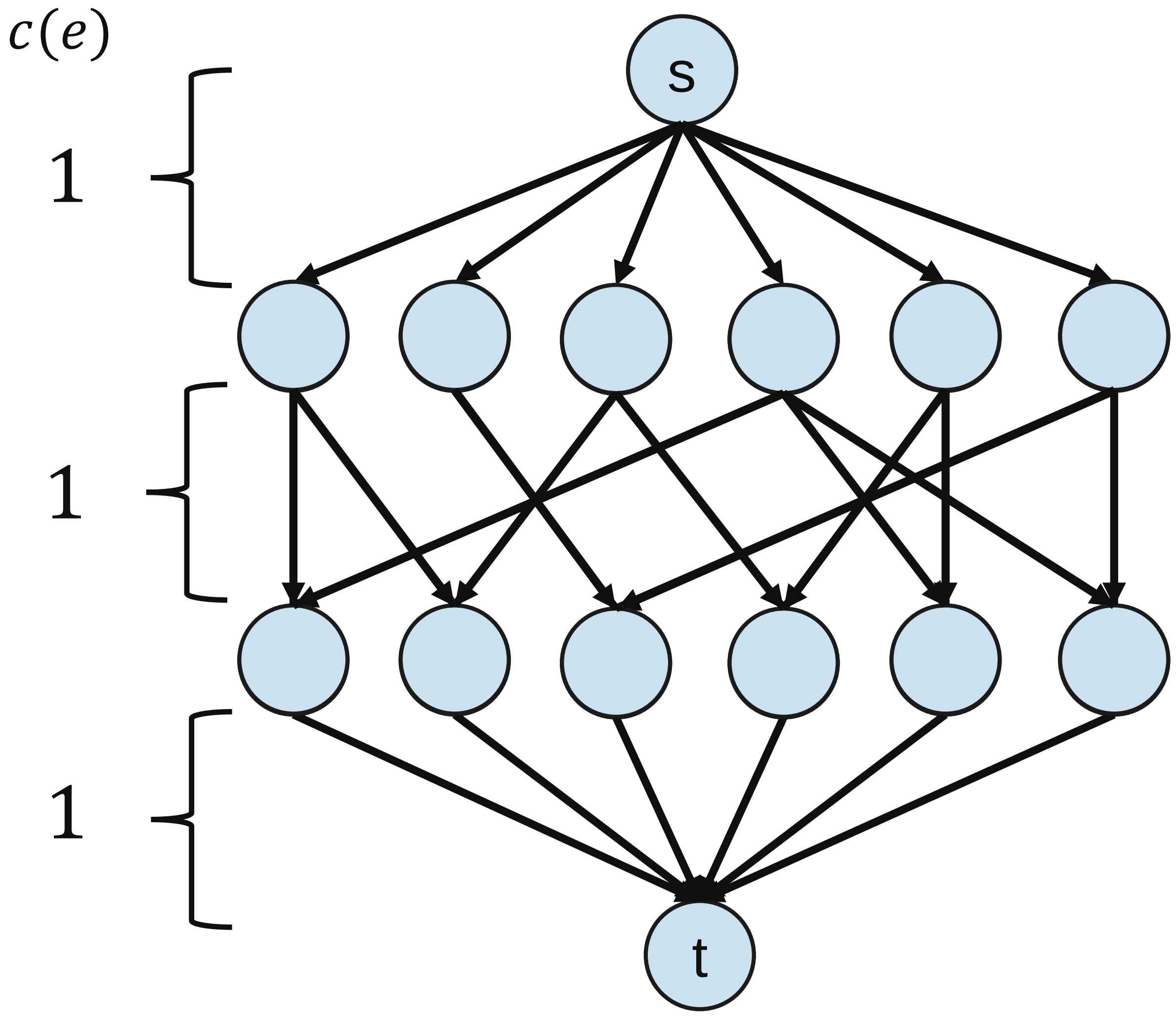

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Proof: Omitted, see, e.g., CLRS.

What else?

Link to animated version: https://algorithms.discrete.ma.tum.de/

Thank you