

**Supplementary Material for: N-representability of the target density in
Frozen-Density Embedding Theory based methods: numerical significance and
relation to electronic polarisation**

Niccolò Ricardi,^{1, a)} Cristina E. González-Espinoza,^{1, b)} and Tomasz Adam Wesołowski^{1, c)}

*Department of Physical Chemistry, University of Geneva,
Geneva (Switzerland)*

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^{a)}Electronic mail: Niccolo.Ricardi@unige.ch

^{b)}Electronic mail: Cristina.GonzalezEspinoza@unige.ch

^{c)}Electronic mail: Tomasz.Wesolowski@unige.ch

S1. CONSTRAINED SEARCH DEFINITIONS OF DENSITY FUNCTIONALS USED IN FDET

A. Functionals in Kohn-Sham formulation of DFT

$$T_s[\rho] = \min_{\Phi \rightarrow \rho} \langle \Phi | \hat{T} | \Phi \rangle = \langle \Phi^o[\rho] | \hat{T} | \Phi^o[\rho] \rangle = T_s[\rho] \quad (\text{S1})$$

$$\begin{aligned} E_{xc}[\rho] &= \min_{\Psi \rightarrow \rho} \langle \Psi | \hat{T} + \hat{V}^{ee} | \Psi \rangle - T_s[\rho] - J[\rho] \\ &= \langle \Psi^o[\rho] | \hat{T} + \hat{V}^{ee} | \Psi^o[\rho] \rangle - T_s[\rho] - J[\rho], \end{aligned} \quad (\text{S2})$$

where $\hat{V}^{ee} = \sum_{i=1}^N \sum_{j>i}^N \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}$ is the electron-electron repulsion operator, $J[\rho] = \frac{1}{2} \int \int \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' d\mathbf{r}$, Φ and Ψ are N -electron wave-functions of the single-determinant or Full Configuration Interaction form, respectively.

B. Functionals used in FDET

For a) embedded wavefunctions of the Full Configuration Interaction form, b) embedded non-interacting reference (Kohn-Sham-like) system, and c) embedded one-particle spinless reduced density matrix, FDET uses the non-additive kinetic-exchange-correlation bi-functional defined as:

$$\begin{aligned} E_{xct}^{nad}[\rho_A, \rho_B] &\equiv E_{xc}[\rho_A + \rho_B] - E_{xc}[\rho_A] - E_{xc}[\rho_B] \\ &\quad + T_s[\rho_A + \rho_B] - T_s[\rho_A] - T_s[\rho_B]. \end{aligned} \quad (\text{S3})$$

For embedded wavefunctions of truncated Configuration Interaction form, FDET uses another functional:

$$E_{xct}^{nad(truncCI)}[\rho_A, \rho_B] \equiv E_{xct}^{nad}[\rho_A, \rho_B] + E_c^{truncCI}[\rho] \leq E_{xct}^{nad}[\rho_A, \rho_B], \quad (\text{S4})$$

where

$$\begin{aligned} E_c^{truncCI}[\rho] &= \min_{\Psi \rightarrow \rho} \langle \Psi | \hat{T} + \hat{V}^{ee} | \Psi \rangle - \min_{\Psi^{truncCI} \rightarrow \rho} \langle \Psi^{truncCI} | \hat{T} + \hat{V}^{ee} | \Psi^{truncCI} \rangle \\ &= \langle \Psi^o[\rho] | \hat{T} + \hat{V}^{ee} | \Psi^o[\rho] \rangle - \langle \Psi^{o(truncCI)}[\rho] | \hat{T} + \hat{V}^{ee} | \Psi^{o(truncCI)}[\rho] \rangle, \end{aligned} \quad (\text{S5})$$

If the embedded wavefunction has a single-determinant form:

$$E_{xct}^{nad(SD)}[\rho_A, \rho_B] \equiv E_{xct}^{nad}[\rho_A, \rho_B] + E_c[\rho] \leq E_{xct}^{nad}[\rho_A, \rho_B], \quad (S6)$$

where

$$\begin{aligned} E_c[\rho] &= \min_{\Psi \rightarrow \rho} \langle \Psi | \hat{T} + \hat{V}^{ee} | \Psi \rangle - \min_{\Phi \rightarrow \rho} \langle \Phi | \hat{T} + \hat{V}^{ee} | \Phi \rangle \\ &= \langle \Psi^o[\rho] | \hat{T} + \hat{V}^{ee} | \Psi^o[\rho] \rangle - \langle \Phi^o[\rho] | \hat{T} + \hat{V}^{ee} | \Phi^o[\rho] \rangle, \end{aligned} \quad (S7)$$

is the correlation functional defined also in the Kohn-Sham formulation of DFT.

Using variationally obtained embedded single determinant AND neglecting $E_c[\rho]$ leads to the violation of the basic FDET equality relating FDET energy functional to the Hohenberg-Kohn functional (Eq. 2) leading to the increase of the energy. The energy consistent with the Hohenberg-Kohn functional correcting this increase is given in Eq. 6.

S2. PROOF THAT $M[\rho_B - \rho_v^o] \leq P[\rho_A + \rho_B - \rho_v^o]$

We start with the obvious equality,

$$\int \rho_A + \rho_B - \rho_v^o = 0, \quad (S8)$$

from which it follows that:

$$\int_{\rho_v^o < \rho_A + \rho_B} \rho_A + \rho_B - \rho_v^o = \int_{\rho_v^o > \rho_A + \rho_B} \rho_v^o - \rho_A - \rho_B. \quad (S9)$$

The above relation used in the definition of P leads to:

$$\begin{aligned} P[\rho_A + \rho_B - \rho_v^o] &= \frac{1}{2} \int |\rho_A + \rho_B - \rho_v^o| \\ &= \frac{1}{2} \int_{\rho_v^o < \rho_A + \rho_B} \rho_A + \rho_B - \rho_v^o + \\ &\quad \frac{1}{2} \int_{\rho_v^o > \rho_A + \rho_B} \rho_v^o - \rho_A - \rho_B \\ &= \int_{\rho_v^o < \rho_A + \rho_B} \rho_A + \rho_B - \rho_v^o. \end{aligned} \quad (S10)$$

Splitting the domain of integration of the final integral above leads to:

$$P[\rho_A + \rho_B - \rho_v^o] = \int_{\rho_v^o < \rho_B} \rho_A + \rho_B - \rho_v^o + \quad (S11)$$

$$\begin{aligned} & \int_{\rho_B \leq \rho_v^o < \rho_B + \rho_A} \rho_A + \rho_B - \rho_v^o \\ &= \int_{\rho_v^o < \rho_B} \rho_B - \rho_v^o + \quad (S12) \\ &+ \int_{\rho_v^o < \rho_B} \rho_A + \int_{\rho_B \leq \rho_v^o < \rho_B + \rho_A} \rho_A + \rho_B - \rho_v^o \end{aligned}$$

The first integral in the right-hand-side of the equation above is equal to $M[\rho_v^o - \rho_B]$ whereas the second and third are non-negative. As a result:

$$P[\rho_A + \rho_B - \rho_v^o] \geq M[\rho_v^o - \rho_B], \quad (S13)$$

which ends the proof.

S3. DATA

A. cc-pVDZ

complex	ρ_B	$E_k[\Delta\rho_{v'_A}^c, \rho'_A, \rho_B]$ [a]	$E_k[\Delta\rho_{v'_B}^c, \rho'_B, \rho_A]$ [a]	$E_{int}^{FDET(\rho_B)}$ [b]	E_{int}^{ref} [c]	Δ_{CP} [d]	M [e]	P [f]	P_{cmpl} [g]
7HQ-2MeOH	ρ_B^{isol}	0.063		-11.200	-14.482	11.159	0.108	0.216	0.350
7HQ-2MeOH	ρ_B^{FAT}	0.053	0.096	-14.216	-14.482	11.159	0.007	0.066	0.350
7HQ-formate	ρ_B^{isol}	-0.009		-25.717	-38.148	11.902	0.176	0.285	0.648
7HQ-formate	ρ_B^{FAT}	-0.036	0.033	-36.933	-38.148	11.902	0.007	0.070	0.648
uracil-5H ₂ O	ρ_B^{isol}	0.166		-32.320	-34.396	23.111	0.203	0.394	0.600
uracil-5H ₂ O	ρ_B^{FAT}	0.147	0.162	-39.794	-34.396	23.111	0.014	0.126	0.600
PyrBnz-2HCOOH	ρ_B^{isol}	0.130		-26.794	-32.300	13.907	0.175	0.406	0.583
PyrBnz-2HCOOH	ρ_B^{FAT}	0.103	0.108	-33.852	-32.300	13.907	0.012	0.110	0.583

[a] defined in Eq. 7 in the manuscript

[b] defined in Eq. 12, and Eq. 17 in the manuscript for ρ_B^{isol} and ρ_B^{FAT} respectively.

[c] Counterpoise corrected, i.e. $E_{int}^{ref} = E_v^{(AB)} - E_{v_A}^{(AB)} - E_{v_B}^{(AB)}$, where all values are obtained with MP2, the subscript denotes the subsystem, and the superscript denotes the centres involved in the basis set expansion.

[d] Counterpoise correction: $\Delta_{CP} = E_{v_A}^{(A)} + E_{v_B}^{(B)} - E_{v_A}^{(AB)} - E_{v_B}^{(AB)}$, where all values are obtained with MP2, the subscript denotes the subsystem, and the superscript denotes the centres involved in the basis set expansion.

[e] $M = M[\rho_v^{o(ref)} - \rho_B^{FDET(FAT)}]$ with $M[\rho]$ defined in Eq. 18 in the manuscript

[f] $P_{cmpl} = P[\rho_A^{isol} + \rho_B^{isol} - \rho_v^{o(ref)}]$ (cf. Eq. 21), with $P[\rho]$ defined in Eq. 19 in the manuscript

[g] $P = P[\rho_v^{o(ref)} - \rho_{tot}^{FDET(FAT)}]$ with $P[\rho]$ defined in Eq. 19 in the manuscript

[h] $E_{int}^{FDET(FAT)}$ is given in Eq. 17 in the manuscript

TABLE S1. Supplementary data for results obtained with the *supermolecular expansion* and the cc-pVDZ basis set. Energies are given in kcal/mol and quantities related to electron densities are given in atomic units.

complex	ρ_B	$E_k[\Delta\rho_{v'_A}^c, \rho'_A, \rho_B]$ ^[a]	$E_k[\Delta\rho_{v'_B}^c, \rho'_B, \rho_A]$ ^[a]	$E_{int}^{FDET(\rho_B)}$ ^[b]	E_{int}^{ref} ^[c]	Δ_{CP} ^[d]	M ^[e]	P ^[f]	P_{cmtl} ^[g]
7HQ-2MeOH	ρ_B^{isol}	0.023		-10.724	-14.482	11.159	0.131	0.228	0.354
7HQ-2MeOH	$\rho_B^{pp(Mulliken)}$	0.021		-12.063	-14.482	11.159	0.091	0.171	0.354
7HQ-2MeOH	$\rho_B^{pp(ChelPG)}$	0.021		-12.343	-14.482	11.159	0.063	0.137	0.354
7HQ-2MeOH	ρ_B^{FAT}	0.020	0.036	-12.062	-14.482	11.159	0.045	0.103	0.354
7HQ-formate	ρ_B^{isol}	-0.021		-27.189	-38.148	11.902	0.226	0.306	0.645
7HQ-formate	$\rho_B^{pp(Mulliken)}$	-0.021		-29.673	-38.148	11.902	0.181	0.253	0.645
7HQ-formate	$\rho_B^{pp(ChelPG)}$	-0.021		-30.350	-38.148	11.902	0.154	0.222	0.645
7HQ-formate	ρ_B^{FAT}	-0.021	0.040	-29.917	-38.148	11.902	0.116	0.177	0.645
uracil-5H ₂ O	ρ_B^{isol}	0.068		-30.102	-34.396	23.111	0.264	0.413	0.605
uracil-5H ₂ O	$\rho_B^{pp(Mulliken)}$	0.063		-33.053	-34.396	23.111	0.183	0.303	0.605
uracil-5H ₂ O	$\rho_B^{pp(ChelPG)}$	0.061		-33.678	-34.396	23.111	0.144	0.254	0.605
uracil-5H ₂ O	ρ_B^{FAT}	0.057	0.051	-32.892	-34.396	23.111	0.120	0.206	0.605
PyrBnz-2HCOOH	ρ_B^{isol}	0.088		-24.896	-32.300	13.907	0.204	0.422	0.582
PyrBnz-2HCOOH	$\rho_B^{pp(Mulliken)}$	0.079		-28.615	-32.300	13.907	0.106	0.265	0.582
PyrBnz-2HCOOH	$\rho_B^{pp(ChelPG)}$	0.077		-29.135	-32.300	13.907	0.090	0.233	0.582
PyrBnz-2HCOOH	ρ_B^{FAT}	0.071	0.025	-28.491	-32.300	13.907	0.060	0.162	0.582

^[a] defined in Eq. 7 in the manuscript

^[b] defined in Eq. 12, and Eq. 17 in the manuscript for ρ_B^{isol} and ρ_B^{FAT} respectively, and in Eq. 14 for $\rho_B^{pp(Mulliken)}$ and $\rho_B^{pp(ChelPG)}$

^[c] Counterpoise corrected, i.e. $E_{int}^{ref} = E_v^{(AB)} - E_{v_A}^{(AB)} - E_{v_B}^{(AB)}$, where all values are obtained with MP2, the subscript denotes the subsystem, and the superscript denotes the centres involved in the basis set expansion.

^[d] Counterpoise correction: $\Delta_{CP} = E_{v_A}^{(A)} + E_{v_B}^{(B)} - E_{v_A}^{(AB)} - E_{v_B}^{(AB)}$, where all values are obtained with MP2, the subscript denotes the subsystem, and the superscript denotes the centres involved in the basis set expansion.

^[e] $M = M[\rho_v^{o(ref)} - \rho_B^{FDET(FAT)}]$ with $M[\rho]$ defined in Eq. 18 in the manuscript

^[f] $P_{cmtl} = P[\rho_A^{isol} + \rho_B^{isol} - \rho_v^{o(ref)}]$ (cf. Eq. 21), with $P[\rho]$ defined in Eq. 19 in the manuscript

^[g] $P = P[\rho_v^{o(ref)} - \rho_{tot}^{FDET(FAT)}]$ with $P[\rho]$ defined in Eq. 19 in the manuscript

^[h] $E_{int}^{FDET(FAT)}$ is given in Eq. 17 in the manuscript

TABLE S2. Supplementary data for results obtained with the *monomer expansion* and the cc-pVDZ basis set. Energies are given in kcal/mol and quantities related to electron densities are given in atomic units.

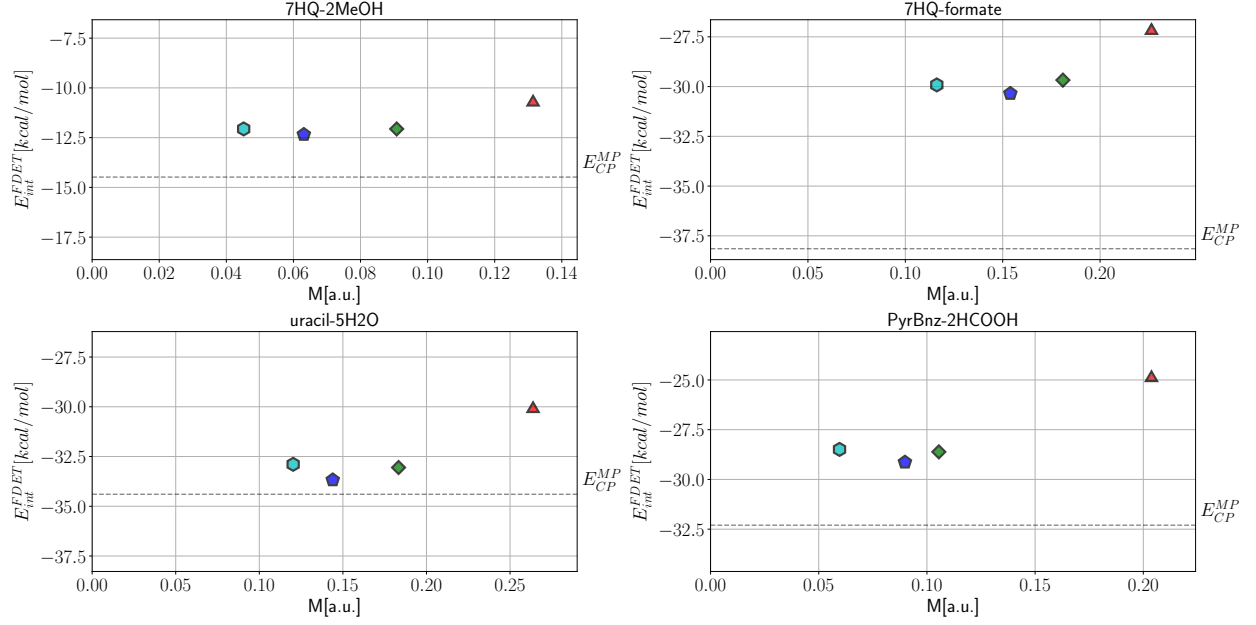


FIG. S1. Integrated negative density M and the FDET-MP2 interaction energy for various choices of ρ_B : a) ρ_B^{isol} (orange triangles), b) ρ_B^{FAT} (light blue hexagons), c) $\rho_B^{pp(Mulliken)}$ (green diamonds), and d) $\rho_B^{pp(ChelPG)}$ (dark blue pentagons). Data obtained using the *monomer expansion* and the cc-pVDZ basis set. Horizontal lines indicate the reference interaction energy.

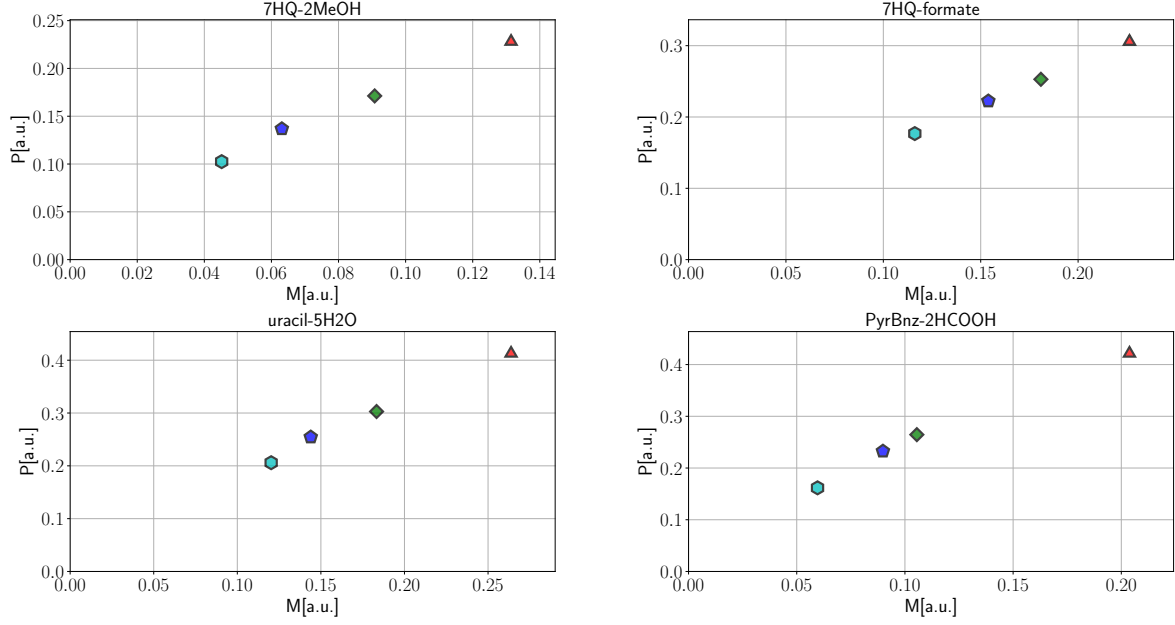


FIG. S2. Integrated negative density M and the total density error P for various choices of ρ_B : a) ρ_B^{isol} (orange triangles), b) ρ_B^{FAT} (light blue hexagons), c) $\rho_B^{pp(Mulliken)}$ (green diamonds), and d) $\rho_B^{pp(ChelPG)}$ (dark blue pentagons). Data obtained using the *monomer expansion* and the cc-pVDZ basis set.

B. aug-cc-pVDZ

complex	ρ_B	$E_k[\Delta\rho_{v'_A}^c, \rho'_A, \rho_B]$ [a]	$E_k[\Delta\rho_{v'_B}^c, \rho'_B, \rho_A]$ [a]	$E_{int}^{FDET(\rho_B)}$ [b]	E_{int}^{ref} [c]	Δ_{CP} [d]	M [e]	P [f]	P_{cmpl} [g]
7HQ-2MeOH	ρ_B^{isol}	0.092		-10.983	-17.473	5.632	0.121	0.228	0.362
7HQ-2MeOH	ρ_B^{FAT}	0.081	0.127	-14.269	-17.473	5.632	0.007	0.059	0.362
7HQ-formate	ρ_B^{isol}	0.020		-23.037	-36.485	3.522	0.206	0.316	0.670
7HQ-formate	ρ_B^{FAT}	-0.008	0.068	-33.325	-36.485	3.522	0.007	0.066	0.670
uracil-5H ₂ O	ρ_B^{isol}	0.224		-32.065	-38.620	7.839	0.234	0.426	0.633
uracil-5H ₂ O	ρ_B^{FAT}	0.204	0.236	-39.695	-38.620	7.839	0.014	0.114	0.633
PyrBnz-2HCOOH	ρ_B^{isol}	0.160		-26.948	-36.532	7.318	0.184	0.416	0.600
PyrBnz-2HCOOH	ρ_B^{FAT}	0.131	0.164	-34.894	-36.532	7.318	0.013	0.104	0.600

[a] defined in Eq. 7 in the manuscript

[b] defined in Eq. 12, and Eq. 17 in the manuscript for ρ_B^{isol} and ρ_B^{FAT} respectively.

[c] Counterpoise corrected, i.e. $E_{int}^{ref} = E_v^{(AB)} - E_{v_A}^{(AB)} - E_{v_B}^{(AB)}$, where all values are obtained with MP2, the subscript denotes the subsystem, and the superscript denotes the centres involved in the basis set expansion.

[d] Counterpoise correction: $\Delta_{CP} = E_{v_A}^{(A)} + E_{v_B}^{(B)} - E_{v_A}^{(AB)} - E_{v_B}^{(AB)}$, where all values are obtained with MP2, the subscript denotes the subsystem, and the superscript denotes the centres involved in the basis set expansion.

[e] $M = M[\rho_v^{o(ref)} - \rho_B^{FDET(FAT)}]$ with $M[\rho]$ defined in Eq. 18 in the manuscript

[f] $P_{cmpl} = P[\rho_A^{isol} + \rho_B^{isol} - \rho_v^{o(ref)}]$ (cf. Eq. 21), with $P[\rho]$ defined in Eq. 19 in the manuscript

[g] $P = P[\rho_v^{o(ref)} - \rho_{tot}^{FDET(FAT)}]$ with $P[\rho]$ defined in Eq. 19 in the manuscript

[h] $E_{int}^{FDET(FAT)}$ is given in Eq. 17 in the manuscript

TABLE S3. Supplementary data for results obtained with the *supermolecular expansion* and the aug-cc-pVDZ basis set. Energies are given in kcal/mol and quantities related to electron densities are given in atomic units.

complex	ρ_B	$E_k[\Delta\rho_{v'_A}^c, \rho'_A, \rho_B]$ ^[a]	$E_k[\Delta\rho_{v'_B}^c, \rho'_B, \rho_A]$ ^[a]	$E_{int}^{FDET(\rho_B)}$ ^[b]	E_{int}^{ref} ^[c]	Δ_{CP} ^[d]	M ^[e]	P ^[f]	P_{cmtl} ^[g]
7HQ-2MeOH	ρ_B^{isol}	0.087		-10.702	-17.473	5.632	0.123	0.231	0.368
7HQ-2MeOH	$\rho_B^{pp(Mulliken)}$	0.084		-10.688	-17.473	5.632	0.121	0.221	0.368
7HQ-2MeOH	$\rho_B^{pp(ChelPG)}$	0.084		-13.492	-17.473	5.632	0.037	0.118	0.368
7HQ-2MeOH	ρ_B^{FAT}	0.081	0.128	-13.181	-17.473	5.632	0.013	0.073	0.368
7HQ-formate	ρ_B^{isol}	0.021		-22.927	-36.485	3.522	0.205	0.316	0.673
7HQ-formate	$\rho_B^{pp(Mulliken)}$	0.016		-27.202	-36.485	3.522	0.148	0.242	0.673
7HQ-formate	$\rho_B^{pp(ChelPG)}$	0.015		-28.678	-36.485	3.522	0.096	0.186	0.673
7HQ-formate	ρ_B^{FAT}	0.011	0.117	-28.028	-36.485	3.522	0.036	0.114	0.673
uracil-5H ₂ O	ρ_B^{isol}	0.212		-31.629	-38.620	7.839	0.237	0.427	0.643
uracil-5H ₂ O	$\rho_B^{pp(Mulliken)}$	0.201		-35.221	-38.620	7.839	0.173	0.305	0.643
uracil-5H ₂ O	$\rho_B^{pp(ChelPG)}$	0.205		-38.010	-38.620	7.839	0.058	0.206	0.643
uracil-5H ₂ O	ρ_B^{FAT}	0.196	0.234	-37.182	-38.620	7.839	0.024	0.129	0.643
PyrBnz-2HCOOH	ρ_B^{isol}	0.161		-25.795	-36.532	7.318	0.185	0.419	0.606
PyrBnz-2HCOOH	$\rho_B^{pp(Mulliken)}$	0.141		-31.448	-36.532	7.318	0.152	0.281	0.606
PyrBnz-2HCOOH	$\rho_B^{pp(ChelPG)}$	0.146		-32.291	-36.532	7.318	0.057	0.215	0.606
PyrBnz-2HCOOH	ρ_B^{FAT}	0.139	0.164	-32.418	-36.532	7.318	0.016	0.127	0.606

^[a] defined in Eq. 7 in the manuscript

^[b] defined in Eq. 12, and Eq. 17 in the manuscript for ρ_B^{isol} and ρ_B^{FAT} respectively, and in Eq. 14 for $\rho_B^{pp(Mulliken)}$ and $\rho_B^{pp(ChelPG)}$

^[c] Counterpoise corrected, i.e. $E_{int}^{ref} = E_v^{(AB)} - E_{v_A}^{(AB)} - E_{v_B}^{(AB)}$, where all values are obtained with MP2, the subscript denotes the subsystem, and the superscript denotes the centres involved in the basis set expansion.

^[d] Counterpoise correction: $\Delta_{CP} = E_{v_A}^{(A)} + E_{v_B}^{(B)} - E_{v_A}^{(AB)} - E_{v_B}^{(AB)}$, where all values are obtained with MP2, the subscript denotes the subsystem, and the superscript denotes the centres involved in the basis set expansion.

^[e] $M = M[\rho_v^{o(ref)} - \rho_B^{FDET(FAT)}]$ with $M[\rho]$ defined in Eq. 18 in the manuscript

^[f] $P_{cmtl} = P[\rho_A^{isol} + \rho_B^{isol} - \rho_v^{o(ref)}]$ (cf. Eq. 21), with $P[\rho]$ defined in Eq. 19 in the manuscript

^[g] $P = P[\rho_v^{o(ref)} - \rho_{tot}^{FDET(FAT)}]$ with $P[\rho]$ defined in Eq. 19 in the manuscript

^[h] $E_{int}^{FDET(FAT)}$ is given in Eq. 17 in the manuscript

TABLE S4. Supplementary data for results obtained with the *monomer expansion* and the aug-cc-pVDZ basis set. Energies are given in kcal/mol and quantities related to electron densities are given in atomic units.

C. cc-pVTZ

complex	ρ_B	$E_k[\Delta\rho_{v'_A}^c, \rho'_A, \rho_B]$ ^[a]	$E_k[\Delta\rho_{v'_B}^c, \rho'_B, \rho_A]$ ^[a]	$E_{int}^{FDET(\rho_B)}$ ^[b]	E_{int}^{ref} ^[c]	Δ_{CP} ^[d]	M ^[e]	P ^[f]	P_{cmpl} ^[g]
7HQ-2MeOH	ρ_B^{isol}	0.064		-11.127	-17.858	5.681	0.117	0.225	0.356
7HQ-2MeOH	ρ_B^{FAT}	0.054	0.101	-14.717	-17.858	5.681	0.007	0.062	0.356
7HQ-formate	ρ_B^{isol}	0.014		-24.519	-39.521	6.011	0.196	0.307	0.663
7HQ-formate	ρ_B^{FAT}	-0.002	0.062	-35.289	-39.521	6.011	0.007	0.069	0.663
uracil-5H ₂ O	ρ_B^{isol}	0.174		-32.777	-40.078	11.802	0.222	0.415	0.619
uracil-5H ₂ O	ρ_B^{FAT}	0.156	0.179	-41.188	-40.078	11.802	0.014	0.118	0.619

^[a] defined in Eq. 7 in the manuscript

^[b] defined in Eq. 12, and Eq. 17 in the manuscript for ρ_B^{isol} and ρ_B^{FAT} respectively.

^[c] Counterpoise corrected, i.e. $E_{int}^{ref} = E_{AB}^{(AB)} - E_{v_A}^{(AB)} - E_{v_B}^{(AB)}$, where all values are obtained with MP2, the subscript denotes the subsystem, and the superscript denotes the centres involved in the basis set expansion.

^[d] Counterpoise correction: $\Delta_{CP} = E_{v_A}^{(A)} + E_{v_B}^{(B)} - E_{v_A}^{(AB)} - E_{v_B}^{(AB)}$, where all values are obtained with MP2, the subscript denotes the subsystem, and the superscript denotes the centres involved in the basis set expansion.

^[e] $M = M[\rho_v^{o(ref)} - \rho_B^{FDET(FAT)}]$ with $M[\rho]$ defined in Eq. 18 in the manuscript

^[f] $P_{cmpl} = P[\rho_A^{isol} + \rho_B^{isol} - \rho_v^{o(ref)}]$ (cf. Eq. 21), with $P[\rho]$ defined in Eq. 19 in the manuscript

^[g] $P = P[\rho_v^{o(ref)} - \rho_{tot}^{FDET(FAT)}]$ with $P[\rho]$ defined in Eq. 19 in the manuscript

^[h] $E_{int}^{FDET(FAT)}$ is given in Eq. 17 in the manuscript

TABLE S5. Supplementary data for results obtained with the *supermolecular expansion* and the cc-pVTZ basis set. Energies are given in kcal/mol and quantities related to electron densities are given in atomic units.

complex	ρ_B	$E_k[\Delta\rho_{v'_A}^c, \rho'_A, \rho_B]$ ^[a]	$E_k[\Delta\rho_{v'_B}^c, \rho'_B, \rho_A]$ ^[a]	$E_{int}^{FDET(\rho_B)}$ ^[b]	E_{int}^{ref} ^[c]	Δ_{CP} ^[d]	M ^[e]	P ^[f]	P_{cmtl} ^[g]
7HQ-2MeOH	ρ_B^{isol}	0.040		-10.716	-17.858	5.681	0.125	0.226	0.358
7HQ-2MeOH	$\rho_B^{pp(Mulliken)}$	0.038		-12.684	-17.858	5.681	0.071	0.153	0.358
7HQ-2MeOH	$\rho_B^{pp(ChelPG)}$	0.037		-13.254	-17.858	5.681	0.045	0.119	0.358
7HQ-2MeOH	ρ_B^{FAT}	0.035	0.068	-13.132	-17.858	5.681	0.021	0.074	0.358
7HQ-formate	ρ_B^{isol}	0.002		-25.786	-39.521	6.011	0.214	0.310	0.661
7HQ-formate	$\rho_B^{pp(Mulliken)}$	0.003		-29.829	-39.521	6.011	0.145	0.230	0.661
7HQ-formate	$\rho_B^{pp(ChelPG)}$	0.003		-30.768	-39.521	6.011	0.116	0.196	0.661
7HQ-formate	ρ_B^{FAT}	0.003	0.064	-30.578	-39.521	6.011	0.057	0.127	0.661
uracil-5H ₂ O	ρ_B^{isol}	0.119		-32.283	-40.078	11.802	0.247	0.413	0.617
uracil-5H ₂ O	$\rho_B^{pp(Mulliken)}$	0.112		-37.341	-40.078	11.802	0.123	0.249	0.617
uracil-5H ₂ O	$\rho_B^{pp(ChelPG)}$	0.111		-37.944	-40.078	11.802	0.091	0.212	0.617
uracil-5H ₂ O	ρ_B^{FAT}	0.104	0.099	-37.489	-40.078	11.802	0.054	0.139	0.617
PyrBnz-2HCOOH	ρ_B^{isol}	0.117		-25.481	-37.236	7.135	0.192	0.416	0.594
PyrBnz-2HCOOH	$\rho_B^{pp(Mulliken)}$	0.106		-30.000	-37.236	7.135	0.083	0.249	0.594
PyrBnz-2HCOOH	$\rho_B^{pp(ChelPG)}$	0.101		-31.343	-37.236	7.135	0.067	0.209	0.594
PyrBnz-2HCOOH	ρ_B^{FAT}	0.091	0.078	-31.420	-37.236	7.135	0.025	0.122	0.594

^[a] defined in Eq. 7 in the manuscript

^[b] defined in Eq. 12, and Eq. 17 in the manuscript for ρ_B^{isol} and ρ_B^{FAT} respectively, and in Eq. 14 for $\rho_B^{pp(Mulliken)}$ and $\rho_B^{pp(ChelPG)}$

^[c] Counterpoise corrected, i.e. $E_{int}^{ref} = E_v^{(AB)} - E_{v_A}^{(AB)} - E_{v_B}^{(AB)}$, where all values are obtained with MP2, the subscript denotes the subsystem, and the superscript denotes the centres involved in the basis set expansion.

^[d] Counterpoise correction: $\Delta_{CP} = E_{v_A}^{(A)} + E_{v_B}^{(B)} - E_{v_A}^{(AB)} - E_{v_B}^{(AB)}$, where all values are obtained with MP2, the subscript denotes the subsystem, and the superscript denotes the centres involved in the basis set expansion.

^[e] $M = M[\rho_v^{o(ref)} - \rho_B^{FDET(FAT)}]$ with $M[\rho]$ defined in Eq. 18 in the manuscript

^[f] $P_{cmtl} = P[\rho_A^{isol} + \rho_B^{isol} - \rho_v^{o(ref)}]$ (cf. Eq. 21), with $P[\rho]$ defined in Eq. 19 in the manuscript

^[g] $P = P[\rho_v^{o(ref)} - \rho_{tot}^{FDET(FAT)}]$ with $P[\rho]$ defined in Eq. 19 in the manuscript

^[h] $E_{int}^{FDET(FAT)}$ is given in Eq. 17 in the manuscript

TABLE S6. Supplementary data for results obtained with the *monomer expansion* and the cc-pVTZ basis set. Energies are given in kcal/mol and quantities related to electron densities are given in atomic units.

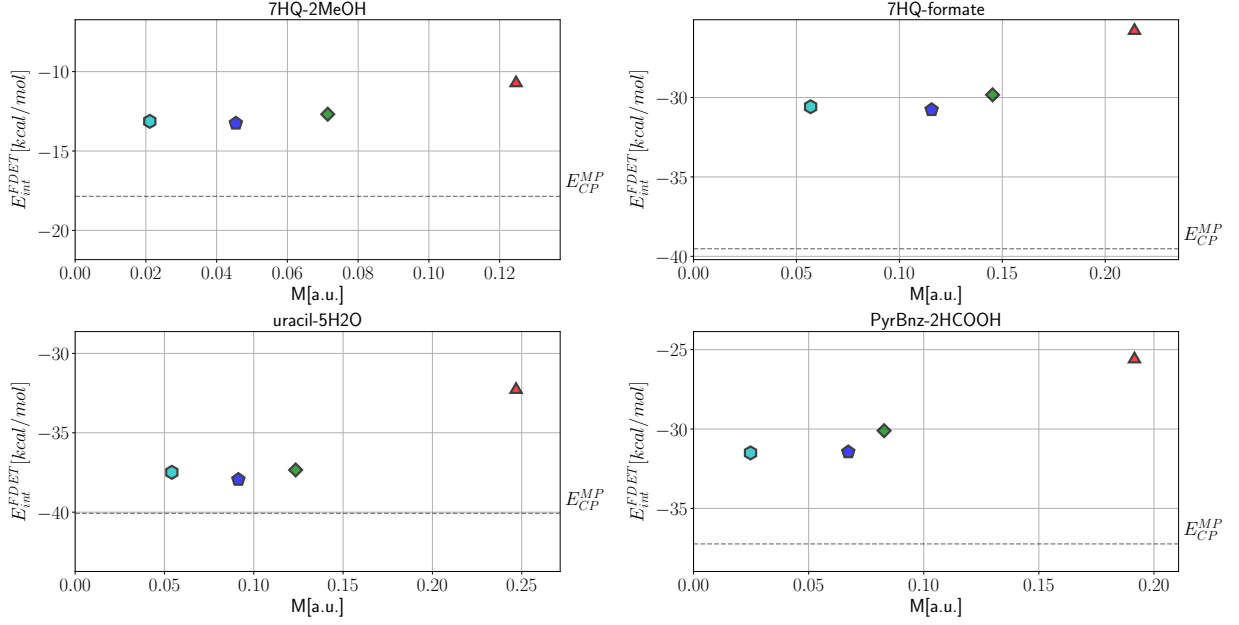


FIG. S3. Integrated negative density M and the FDET-MP2 interaction energy for various choices of ρ_B : a) ρ_B^{isol} (orange triangles), b) ρ_B^{FAT} (light blue hexagons), c) $\rho_B^{pp(Mulliken)}$ (green diamonds), and d) $\rho_B^{pp(ChelPG)}$ (dark blue pentagons). Data obtained using the *monomer expansion* and the cc-pVTZ basis set. Horizontal lines indicate the reference interaction energy.

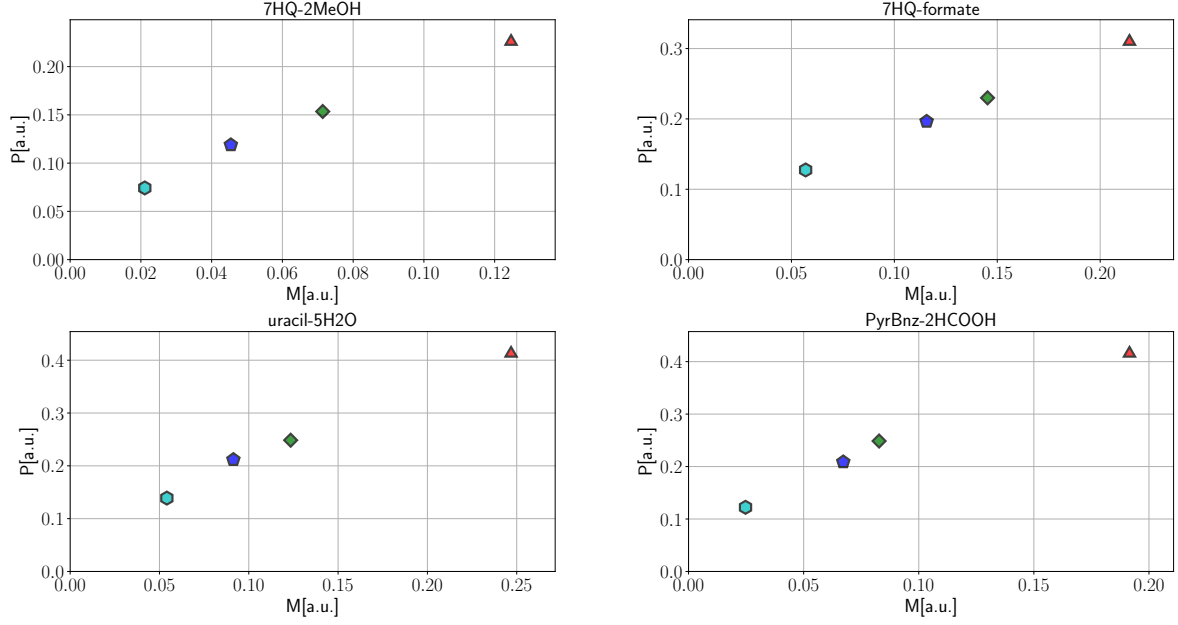


FIG. S4. Integrated negative density M and the total density error P for various choices of ρ_B : a) ρ_B^{isol} (orange triangles), b) ρ_B^{FAT} (light blue hexagons), c) $\rho_B^{pp(Mulliken)}$ (green diamonds), and d) $\rho_B^{pp(ChelPG)}$ (dark blue pentagons). Data obtained using the *monomer expansion* and the cc-pVTZ basis set.

D. aug-cc-pVTZ

complex	ρ_B	$E_k[\Delta\rho_{v'_A}^c, \rho'_A, \rho_B]$ ^[a]	$E_k[\Delta\rho_{v'_B}^c, \rho'_B, \rho_A]$ ^[a]	$E_{int}^{FDET(\rho_B)}$ ^[b]	E_{int}^{ref} ^[c]	Δ_{CP} ^[d]	M ^[e]	P ^[f]	P_{cmpt} ^[g]
uracil-5H ₂ O	ρ_B^{isol}	0.000303		-31.994	-41.753	7.450	0.230	0.426	0.630
uracil-5H ₂ O	$\rho_B^{pp(Mulliken)}$	0.000286		-39.046	-41.753	7.450	0.087	0.218	0.630
uracil-5H ₂ O	$\rho_B^{pp(ChelPG)}$	0.000293		-39.444	-41.753	7.450	0.053	0.209	0.630
uracil-5H ₂ O	ρ_B^{FAT}	0.000276	0.000324	-39.061	-41.753	7.450	0.016	0.128	0.630

^[a] defined in Eq. 7 in the manuscript

^[b] defined in Eq. 12, and Eq. 17 in the manuscript for ρ_B^{isol} and ρ_B^{FAT} respectively, and in Eq. 14 for $\rho_B^{pp(Mulliken)}$ and $\rho_B^{pp(ChelPG)}$

^[c] Counterpoise corrected, i.e. $E_{int}^{ref} = E_v^{(AB)} - E_{v_A}^{(AB)} - E_{v_B}^{(AB)}$, where all values are obtained with MP2, the subscript denotes the subsystem, and the superscript denotes the centres involved in the basis set expansion.

^[d] Counterpoise correction: $\Delta_{CP} = E_{v_A}^{(A)} + E_{v_B}^{(B)} - E_{v_A}^{(AB)} - E_{v_B}^{(AB)}$, where all values are obtained with MP2, the subscript denotes the subsystem, and the superscript denotes the centres involved in the basis set expansion.

^[e] $M = M[\rho_v^{o(ref)} - \rho_B^{FDET(FAT)}]$ with $M[\rho]$ defined in Eq. 18 in the manuscript

^[f] $P_{cmpt} = P[\rho_A^{isol} + \rho_B^{isol} - \rho_v^{o(ref)}]$ (cf. Eq. 21), with $P[\rho]$ defined in Eq. 19 in the manuscript

^[g] $P = P[\rho_v^{o(ref)} - \rho_{tot}^{FDET(FAT)}]$ with $P[\rho]$ defined in Eq. 19 in the manuscript

^[h] $E_{int}^{FDET(FAT)}$ is given in Eq. 17 in the manuscript

TABLE S7. Supplementary data for results obtained with the *monomer expansion* and the aug-cc-pVTZ basis set. Energies are given in kcal/mol and quantities related to electron densities are given in atomic units.

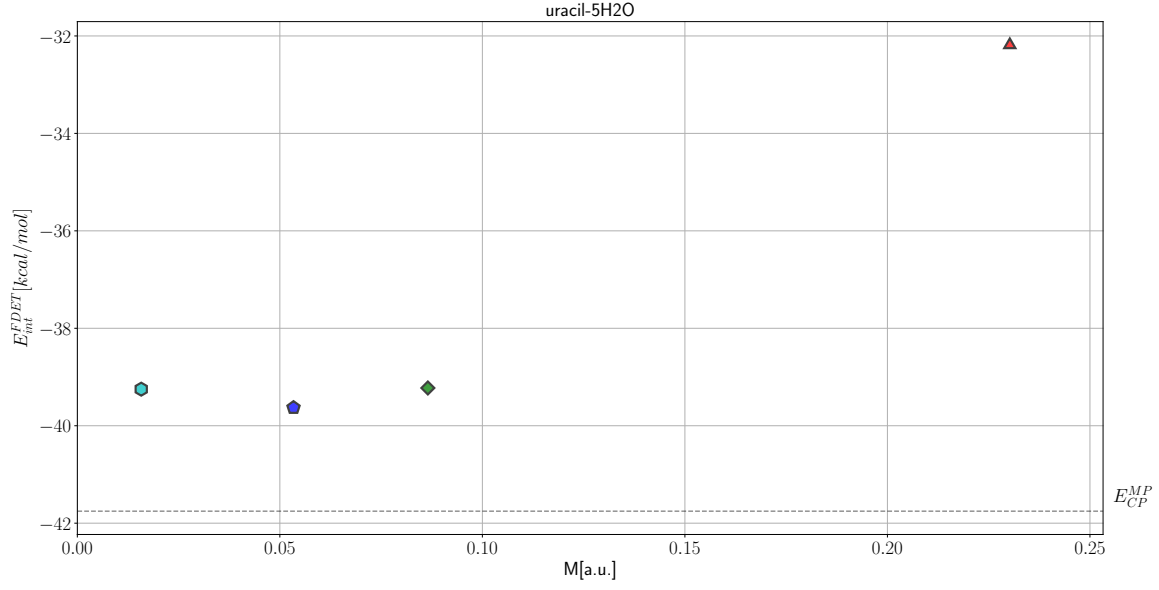


FIG. S5. Integrated negative density M and the FDET-MP2 interaction energy for various choices of ρ_B : a) ρ_B^{isol} (orange triangles), b) ρ_B^{FAT} (light blue hexagons), c) $\rho_B^{pp(Mulliken)}$ (green diamonds), and d) $\rho_B^{pp(ChelPG)}$ (dark blue pentagons). Data obtained using the *monomer expansion* and the aug-cc-pVTZ basis set. Horizontal lines indicate the reference interaction energy.

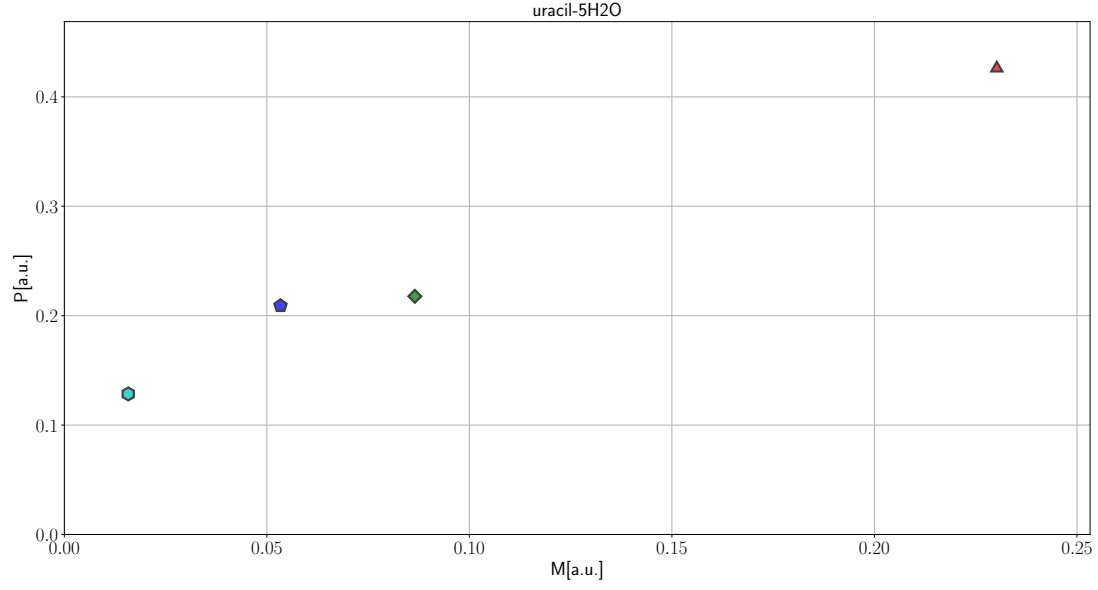


FIG. S6. Integrated negative density M and the total density error P for various choices of ρ_B : a) ρ_B^{isol} (orange triangles), b) ρ_B^{FAT} (light blue hexagons), c) $\rho_B^{pp(Mulliken)}$ (green diamonds), and d) $\rho_B^{pp(ChelPG)}$ (dark blue pentagons). Data obtained using the *monomer expansion* and the aug-cc-pVTZ basis set.