

Solution of the exercises

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1 Lecture 1

1.1 Proof that $A(u, v)$ and $F(v)$ in the Poisson problem satisfy the hypotheses of the Lax-Milgram lemma

We need to show that the bilinear form

$$A(u, v) = \int_0^1 u'(x)v'(x) dx.$$

is continuous:

$$\exists \gamma > 0 : |A(u, v)| \leq \gamma \|u\|_V \|v\|_V \quad \forall u, v \in V.$$

and coercive:

$$\exists \alpha_0 > 0 : A(u, u) \geq \alpha_0 \|u\|_V^2 \quad \forall u \in V.$$

We also need to show that the linear functional

$$F(v) = \int_0^1 f(x)v(x) dx.$$

is continuous:

$$\exists \beta > 0 : |F(v)| \leq \beta \|v\|_V \quad \forall v \in V.$$

1.1.1 Continuity of $A(u, v)$

Recall that the norm on $V = H_0^1(0, 1)$ is given by

$$\|u\|_V = \left(\int_0^1 |u'(x)|^2 dx \right)^{1/2}.$$

Now, consider the absolute value of $A(u, v)$:

$$|A(u, v)| = \left| \int_0^1 u'(x)v'(x) dx \right|.$$

Using the Cauchy-Schwarz inequality:

$$|A(u, v)| \leq \left(\int_0^1 |u'(x)|^2 dx \right)^{1/2} \left(\int_0^1 |v'(x)|^2 dx \right)^{1/2}.$$

But this is exactly $\|u\|_V \|v\|_V$. Therefore, the bilinear form $A(u, v)$ is continuous with $\gamma = 1$:

$$|A(u, v)| \leq \|u\|_V \|v\|_V.$$

1.1.2 Coercivity of $A(u, v)$

$$A(u, u) = \int_0^1 |u'(x)|^2 dx = \|u\|_V^2.$$

If we take $\alpha = 1$ we get:

$$A(u, u) = \|u\|_V^2 \geq \alpha \|u\|_V^2.$$

Thus $A(u, v)$ is coercive with $\alpha = 1$.

1.1.3 Continuity of $F(v)$

Using the Cauchy-Schwarz inequality:

$$|F(v)| = \left| \int_0^1 f(x)v(x) dx \right| \leq \left(\int_0^1 |f(x)|^2 dx \right)^{1/2} \left(\int_0^1 |v(x)|^2 dx \right)^{1/2}$$

and recognizing the L^2 -norm:

$$\|f\|_{L^2(0,1)} = \left(\int_0^1 |f(x)|^2 dx \right)^{1/2}$$

we obtain:

$$|F(v)| \leq \|f\|_{L^2(0,1)} \|v\|_{L^2(0,1)}.$$

Now from the Poincaré inequality we know that for $v \in H_0^1(0, 1)$:

$$\|v\|_{L^2(0,1)} \leq C_p \|v'\|_{L^2(0,1)} = C_p \|v\|_V$$

Thus, $F(v)$ is continuous with $\beta = \|f\|_{L^2(0,1)} C_p$:

$$|F(v)| \leq \|f\|_{L^2(0,1)} C_p \|v\|_V.$$

1.2 Proof that if u minimizes the energy functional then is solution of the weak problem

Given the potential energy functional

$$J(v) = \frac{1}{2}A(v, v) - F(v)$$

we want to prove that if a function u minimizes such functional, is then solution of the weak problem $A(u, v) = F(v)$. Let's introduce the auxiliary $u_\epsilon = u + \epsilon v$ and compute

$$\frac{d}{d\epsilon} [J(u + \epsilon v)]_{\epsilon=0} = 0$$

- substitute the expression for J :

$$\frac{d}{d\epsilon} \left[\frac{1}{2}A(u + \epsilon v, u + \epsilon v) - F(u + \epsilon v) \right]_{\epsilon=0} = 0$$

- expand both A and F :

$$\frac{d}{d\epsilon} \left[\frac{1}{2}A(u, u) + \epsilon A(u, v) + \frac{1}{2}\epsilon^2 A(v, v) - F(u) - \epsilon F(v) \right]_{\epsilon=0} = 0$$

- compute the derivative:

$$[A(u, v) + \epsilon A(v, v) - F(v)]_{\epsilon=0} = 0$$

- evaluating at $\epsilon = 0$ and rearranging we get

$$A(u, v) = F(v)$$

which is the weak formulation we wanted.

2 Lecture 2

2.1 Proof of the Strong maximum principle for harmonic functions

Let Ω be a connected open subset of \mathbb{R}^n and u be a harmonic function, i.e., $\Delta u = 0$, such that $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Then:

1. $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$.
2. If $\exists x_0 \in \Omega$ such that $u(x_0) = \max_{\overline{\Omega}} u = M$, then u is constant in Ω .

By contradiction suppose that u attains its maximum at some point $x_0 \in \Omega$, i.e. $u(x_0) = \max_{\overline{\Omega}} u$; now letting $B(x_0, r) \subset \Omega$, it follows that since u is harmonic, by the mean value property its value at a point is equal to the average integral over a sphere of any radius centred at that point:

$$M = u(x_0) = \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} u(x) dx.$$

For the average to be equal to the maximum value it must be that $u(x) = M \forall x \in B(x_0, r)$.

Now consider the set

$$\{x \in \Omega : u(x) = M\}$$

this set is open because inside the open ball $B(x_0, r)$ where $u(x_0) = M$, all nearby points satisfy also $u(y) = M$. Since u is continuous, if a sequence of points in $B(x_0, r)$ converges to a point in Ω , that point must also be in $B(x_0, r)$; this implies that the set is also relatively closed in Ω . If a non-empty set S is both open and closed within a connected space U , then it must be the entire space:

1. suppose S is not the entire space U ; then $U \setminus S$ is non-empty
2. since S is open and relatively closed, $U \setminus S$ is both relatively open and relatively closed
3. this would imply that U can be divided into two disjoint, non-empty, open subsets, which contradicts the assumption that it is connected

We conclude that the set

$$\{x \in \Omega : u(x) = M\}$$

is the entire space Ω and thus u is constant in Ω .

2.2 Proof of the corollary (uniqueness for Poisson)

Suppose there exist two solutions u and v to the Poisson problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

and

$$\begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = g & \text{on } \partial\Omega. \end{cases}$$

defining $w = u - v$ we get:

$$\begin{cases} -\Delta w = -\Delta u + \Delta v = f - f = 0 & \text{in } \Omega, \\ w = u - v = g - g = 0 & \text{on } \partial\Omega. \end{cases}$$

so w is a harmonic function that is null on $\partial\Omega$ and we can apply the strong maximum principle: since w achieves both its maximum and minimum on the boundary, it must be constant on Ω , thus proving $u = v$.

3 Lecture 3

3.1 Convergence of the truncation error

Given the finite difference discretization

$$-\frac{a_i}{h^2}(u_{i+1} - 2u_i + u_{i-1}) + \frac{b_i}{2h}(u_{i+1} - u_{i-1}) = f_i$$

we want to show that the truncation error $|T| = |Lu(x_i) - L_h(u(x_i))|$ is bounded by

$$|T| \leq \frac{h^2}{12} \|a\| \|u^{(4)}\| + \frac{h^2}{6} \|b\| \|u^{(3)}\|$$

3.1.1 Second derivative term

Using Taylor expansion around x_i :

$$u_{i+1} = u(x_i + h) = u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i) + \frac{h^3}{6}u^{(3)}(x_i) + \frac{h^4}{24}u^{(4)}(x_i) + O(h^5)$$

$$u_{i-1} = u(x_i - h) = u(x_i) - hu'(x_i) + \frac{h^2}{2}u''(x_i) - \frac{h^3}{6}u^{(3)}(x_i) + \frac{h^4}{24}u^{(4)}(x_i) + O(h^5)$$

$$\text{Thus } u_{i+1} - 2u_i + u_{i-1} = h^2u''(x_i) + \frac{h^4}{12}u^{(4)}(x_i) + O(h^6)$$

3.1.2 First derivative term

Using the same Taylor expansion around x_i we get:

$$u_{i+1} - u_{i-1} = 2hu'(x_i) + \frac{2h^3}{6}u^{(3)}(x_i) + O(h^5)$$

3.1.3 Combine the errors

Combining the truncation errors for both terms, we have:

$$T_i = -a_i \left(\frac{h^2}{12} u^{(4)}(x_i) \right) + b_i \left(\frac{h^2}{6} u^{(3)}(x_i) \right)$$

Taking the absolute value and using the norms $\|a\|$ and $\|b\|$, we get:

$$|T_i| \leq \frac{h^2}{12} \|a\| \|u^{(4)}\| + \frac{h^2}{6} \|b\| \|u^{(3)}\|.$$

3.2 Construction of a 3-point scheme: method of undetermined coefficients

We want to find the best combination of values for the coefficients $\alpha, \beta, \gamma \in \mathbb{R}$ that appear in the finite difference operator

$$L = \alpha u_{i+1} + \beta u_i + \gamma u_{i-1}$$

so that the approximation $u''(x_i) = u_i''$ is the best possible. We start by expanding using Taylor series:

$$L = \alpha \left(u_i + h_{i+1}u_i' + \frac{h_{i+1}^2}{2}u_i'' + \frac{h_{i+1}^3}{6}u_i'''(\xi_{i+1}) \right) + \beta u_i + \gamma \left(u_i - h_i u_i' + \frac{h_i^2}{2}u_i'' + \frac{h_i^3}{6}u_i'''(\xi_i) \right)$$

The terms of order 0 and 1 must cancel out, while the second order term must have coefficient 1, so we get the following conditions on the coefficients:

$$\begin{cases} \alpha + \beta + \gamma = 0 \\ \alpha h_{i+1} - \gamma h_i = 0 \\ \alpha \frac{h_{i+1}^2}{2} + \gamma \frac{h_i^2}{2} = 1 \end{cases}$$

from the second equation we get $\alpha = \frac{h_i}{h_{i+1}}\gamma$; by substituting in the third find $\gamma = \frac{2}{h_i(h_{i+1}+h_i)}$ which gives $\alpha = \frac{2}{h_{i+1}(h_{i+1}+h_i)}$ and $\beta = -\frac{2}{h_{i+1}+h_i}$. The resulting FD scheme is therefore:

$$\begin{cases} u_0 = 0 \\ \frac{2}{h_{i+1}(h_{i+1}+h_i)}u_{i+1} - \frac{2}{h_{i+1}+h_i}u_i + \frac{2}{h_i(h_{i+1}+h_i)}u_{i-1} = f_i \quad \text{for } i = 1, \dots, N-1 \\ u_N = 0 \end{cases}$$

4 Lecture 4

4.1 Truncation error bound for 2D Poisson b.v.p.

If $u \in C^4(\Omega) \cap C^0(\Omega)$ then the truncation error of the 5-point scheme is bounded by:

$$|T(x)| \leq \frac{h^2}{12} (\|u_{xxxx}\|_{C(\bar{\Omega})} + \|u_{yyyy}\|_{C(\bar{\Omega})})$$

To prove this we remember that the truncation error in this case is defined as:

$$T_{ij} = \frac{1}{h^2} (u(x_{i+1}, y_j) + u(x_{i-1}, y_j) + u(x_i, y_{j+1}) + u(x_i, y_{j-1}) - 4u(x_i, y_j)) - f(x_i, y_j)$$

which we rewrite as:

$$T_{ij} = \frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j))}{h^2} + \frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1}))}{h^2} - f(x_i, y_j)$$

using Taylor expansion we can prove that the first two terms are equivalent to:

$$u_{xx}(x_i, y_j) + \frac{h^2}{24} (u_{xxxx}(\xi_1, y_j) + u_{xxxx}(\zeta_1, y_j))$$

and

$$u_{yy}(x_i, y_j) + \frac{h^2}{24} (u_{yyyy}(x_i, \xi_2) + u_{yyyy}(x_i, \zeta_2))$$

for some $\xi_1, \zeta_1 \in [x_{i-1}, x_{i+1}]$ and $\xi_2, \zeta_2 \in [y_{i-1}, y_{i+1}]$ respectively. Thus, combining the result and taking the maximum of the absolute value for each derivative we get the following bound:

$$|T_i| \leq \frac{h^2}{12} \left(\max_{(x,y) \in \Omega} |u_{xxxx}(x,y)| + \max_{(x,y) \in \Omega} |u_{yyyy}(x,y)| \right)$$

which is equivalent to the result we want to prove if we recall the definition of L^∞ norm.

4.2 Derivation of FD scheme for the mixed operator $\delta_{2h}^x \delta_{2h}^y u_{ij}$

$$(\delta_h^x u)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h}, \quad (\delta_h^y u)_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2h}.$$

Applying δ_h^x first and then δ_h^y :

$$(\delta_h^y \delta_h^x u)_{i,j} = \frac{(\delta_h^x u)_{i,j+1} - (\delta_h^x u)_{i,j-1}}{2h}.$$

Substitute $(\delta_h^x u)_{i,j \pm 1}$ into the above and simplify:

$$(\delta_h^y \delta_h^x u)_{i,j} = \frac{u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}}{4h^2}.$$

5 Lecture 5

5.1 Proof that $\|v\|_1$ is a full norm for the space H_0^1

1. Definiteness: show that

$$\|v\|_1 = 0 \iff v \equiv 0 \text{ (in } \Omega \text{)}.$$

(\implies) If $\|\nabla v\|_{L^2(\Omega)} = 0$, then $\nabla v = 0$ almost everywhere. Thus v is (almost everywhere) a constant C . Since $v \in H_0^1(\Omega)$, it satisfies $v = 0$ on $\partial\Omega$. Hence that constant must be $C = 0$. Therefore $v \equiv 0$ in Ω .

(\impliedby) Clearly if $v \equiv 0$, then $\nabla v \equiv 0$ and thus $\|\nabla v\|_{L^2(\Omega)} = 0$.

Hence

$$\|v\|_1 = \|\nabla v\|_{L^2(\Omega)} = 0 \iff v \equiv 0.$$

2. Homogeneity: $\forall \alpha \in \mathbb{R}$ and $\forall v \in H_0^1(\Omega)$ show that

$$\|\alpha v\|_1 = |\alpha| \|v\|_1.$$

By definition,

$$\|\alpha v\|_1 = \|\nabla(\alpha v)\|_{L^2(\Omega)} = \|\alpha \nabla v\|_{L^2(\Omega)} = |\alpha| \|\nabla v\|_{L^2(\Omega)} = |\alpha| \|v\|_1.$$

3. Triangle inequality: $\forall u, v \in H_0^1(\Omega)$ show that

$$\|u + v\|_1 \leq \|u\|_1 + \|v\|_1.$$

Indeed,

$$\|u+v\|_1 = \|\nabla(u+v)\|_{L^2(\Omega)} = \|\nabla u + \nabla v\|_{L^2(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)} = \|u\|_1 + \|v\|_1.$$

6 Lecture 6

6.1 Proof of the continuity of the bilinear form in Lax-Milgram

The bilinear form A is continuous if:

$$|A(u, v)| \leq \|u\| \|v\| \forall u, v \in V.$$

We study the three terms separately.

6.1.1 First term

Assume A is a bounded, symmetric and positive definite matrix with components $A_{ij}(x)$, such that:

$$\lambda_{min} |\xi|^2 \leq A(x) \xi \cdot \xi \leq \lambda_{max} |\xi|^2$$

$\forall \xi \in \mathbb{R}^n$ and $\forall x \in \Omega$, where λ_{min} and λ_{max} are the minimum and maximum eigenvalues of A . Then taking the absolute value the following inequality holds:

$$\left| \int_{\Omega} A \nabla u \nabla v dx \right| \leq \lambda_{max} \int_{\Omega} |\nabla u| |\nabla v| dx$$

using the Cauchy-Schwarz inequality:

$$\int_{\Omega} |\nabla u| |\nabla v| dx \leq \left(\int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla v|^2 dx \right)^{\frac{1}{2}} = \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}$$

so

$$\left| \int_{\Omega} A \nabla u \nabla v dx \right| \leq \lambda_{max} \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}$$

6.1.2 Second term

Assume $b(x)$ is a bounded vector field. To obtain boundedness it is sufficient to assume that $b(x) \in L^\infty(\Omega)$. Using then also Cauchy-Schwarz we get:

$$\left| \int_{\Omega} (b \cdot \nabla u) v dx \right| \leq \int_{\Omega} |b \cdot \nabla u| |v| \leq \|b\|_{L^\infty(\Omega)} \int_{\Omega} |\nabla u| |v| \leq \|b\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

6.1.3 Third term

Assume $c(x)$ is a bounded scalar field and use Cauchy-Schwarz as before:

$$\left| \int_{\Omega} cuv dx \right| \leq \|c\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

Now combining the results:

$$|A(u, v)| \leq \lambda_{max} \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|b\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|c\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

Now recalling the definition of the H^1 norm $\|u\|_{H^1(\Omega)}$, which is defined as

$$\|u\|_{H^1(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}$$

we can bound each term with the H^1 norm of u or v , for example:

$$\lambda_{max} \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \leq \lambda_{max} \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$

so we can factor out $\|u\|_{H^1(\Omega)}$ and $\|v\|_{H^1(\Omega)}$ from each term and obtain:

$$|A(u, v)| \leq C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$

where the constant C is given by:

$$C = \max \left(\lambda_{max}, \|b\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)} \right)$$

which proves the continuity of $A(u, v)$.

6.2 Proof of special case of Ceà's lemma

Let V be a Hilbert space and V_h a finite-dimensional subspace of V . Consider a bilinear form $a : V \times V \rightarrow \mathbb{R}$ and a linear functional $F : V \rightarrow \mathbb{R}$. Assume a is coercive, continuous and symmetric. Let $u \in V$ be the solution of the variational problem: $a(u, v) = F(v) \forall v \in V$, and let $u_h \in V_h$ be the solution of the corresponding finite-dimensional problem $a(u_h, v_h) = F(v_h) \forall v_h \in V_h$. Then, we have:

$$\|u - u_h\|_V \leq \sqrt{\frac{\gamma}{\alpha_0}} \min_{v_h \in V_h} \|u - v_h\|$$

Proof

adding symmetry to the properties of the form a makes it an inner product on the space V , which allows us to define the norm $\|u\|_a = \sqrt{a(u, u)}$. Thus,

$$\|u - u_h\|_a^2 = a(u - u_h, u - u_h) = a(u - u_h, u - v_h) \leq \|u - u_h\|_a \|u - v_h\|_a \quad \forall v_h \in V_h$$

which leads to

$$\|u - u_h\|_a \leq \|u - v\|_a \quad \forall v \in V_h$$

Using this result together with continuity and coercivity we can derive:

$$\alpha \|u - u_h\|^2 \leq a(u - u_h, u - u_h) = \|u - u_h\|_a^2 \leq \|u - v_h\|_a^2 \leq \gamma \|u - v_h\|^2 \quad \forall v_h \in V_h$$

from which follows:

$$\|u - u_h\| \leq \sqrt{\frac{\gamma}{\alpha}} \|u - v_h\| \quad \forall v_h \in V_h$$

7 Lecture 11

7.1 Local truncation error (consistency) of the θ -method

Prove that the local truncation error of the θ -method for the heat equation $u_t = a u_{xx}$ is given by:

$$T_i^{n+\theta} = \delta_k^t u_i^{n+\theta} - \theta a \delta_h^2 u_i^{n+1} - (1 - \theta) a \delta_h^2 u_i^n$$

and that:

- For $\theta \neq \frac{1}{2}$, the scheme is **first order** in time:

$$T_i^{n+\theta} = O(k) + O(h^2)$$

- For $\theta = \frac{1}{2}$ (the Crank–Nicolson case), it is **second order** in time:

$$T_i^{n+1/2} = O(k^2) + O(h^2)$$

Proof

Let $k = t_{n+1} - t_n$, $h = x_i - x_{i-1}$, $u_i^n = u(x_i, t_n)$. Our scheme is:

$$\frac{U_i^{n+1} - U_i^n}{k} = \theta a \delta_h^2 U_i^{n+1} + (1 - \theta) a \delta_h^2 U_i^n,$$

where δ_h^2 is the centered-difference operator in space:

$$\delta_h^2 v_i = \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2}.$$

The local truncation error is obtained by substituting the true PDE solution u into the discrete scheme and measuring the residual:

1. Replace U_i^n by $u_i^n = u(x_i, t_n)$.
2. Define

$$T_i^{n+\theta} := \underbrace{\frac{u_i^{n+1} - u_i^n}{k}}_{\delta_k^t u_i^{n+\theta}} - \left[\theta a \delta_h^2 u_i^{n+1} + (1 - \theta) a \delta_h^2 u_i^n \right].$$

Because u satisfies $u_t - a u_{xx} = 0$, we can write:

$$\delta_k^t u_i^{n+\theta} - a u_{xx}(t_{n+\theta}, x_i) = 0.$$

Now using Taylor expansion we will show that:

- If $\theta \neq \frac{1}{2}$, then $T_i^{n+\theta} = \mathcal{O}(k) + \mathcal{O}(h^2)$.
- If $\theta = \frac{1}{2}$ (the Crank–Nicolson/trapezoid rule in time), then $T_i^{n+1/2} = \mathcal{O}(k^2) + \mathcal{O}(h^2)$.

To see the orders cleanly, we rewrite the truncation error as follows:

$$T_i^{n+\theta} = [\delta_k^t u_i^{n+\theta} - u_t(t_{n+\theta}, x_i)] - a [\theta \delta_h^2 u_i^{n+1} + (1-\theta) \delta_h^2 u_i^n - u_{xx}(t_{n+\theta}, x_i)].$$

and name the two main brackets as:

$$T_i^{n+\theta} = R_{\text{time}} - a R_{\text{space}}.$$

7.1.1 The time part R_{time}

$$R_{\text{time}} = \frac{u(x_i, t_{n+1}) - u(x_i, t_n)}{k} - u_t(t_{n+\theta}, x_i)$$

Case (a): $\theta \neq \frac{1}{2}$

We expand $u(t_{n+1}, x_i)$ and $u(t_n, x_i)$ in Taylor series around $t_{n+\theta}$. Let $t_{n+\theta} = t_n + \theta k$. Then

$$t_{n+1} = t_{n+\theta} + (1-\theta)k, \quad t_n = t_{n+\theta} - \theta k.$$

which gives

$$u(t_{n+1}, x_i) = u(t_{n+\theta}, x_i) + (1-\theta)k u_t(t_{n+\theta}, x_i) + \frac{(1-\theta)^2 k^2}{2} u_{tt}(\xi_1, x_i),$$

for some $\xi_1 \in (t_{n+\theta}, t_{n+1})$, and similarly

$$u(t_n, x_i) = u(t_{n+\theta}, x_i) - \theta k u_t(t_{n+\theta}, x_i) + \frac{\theta^2 k^2}{2} u_{tt}(\xi_2, x_i).$$

Subtracting and dividing by k :

$$\frac{u_i^{n+1} - u_i^n}{k} = u_t(t_{n+\theta}, x_i) + \frac{[(1-\theta)^2 + \theta^2] k}{2} u_{tt}(\xi_3, x_i).$$

Therefore

$$R_{\text{time}} = \frac{u_i^{n+1} - u_i^n}{k} - u_t(t_{n+\theta}, x_i) = \frac{[(1-\theta)^2 + \theta^2] k}{2} u_{tt}(\xi_3, x_i) = \mathcal{O}(k).$$

Hence for $\theta \neq \frac{1}{2}$, R_{time} is first-order in k .

Case (b): $\theta = \frac{1}{2}$

In this special case, the method is exactly the trapezoid rule in time (Crank–Nicolson):

$$\frac{u(t_{n+1}, x_i) - u(t_n, x_i)}{k} - u_t\left(\frac{t_n + t_{n+1}}{2}, x_i\right) = \mathcal{O}(k^2).$$

The leading error term (proportional to k) cancels, leaving a remainder of order k^2 . Thus

$$R_{\text{time}} = \mathcal{O}(k^2) \quad \text{when } \theta = \frac{1}{2}.$$

7.1.2 The space part R_{space}

$$R_{\text{space}} = \theta \delta_h^2 u_i^{n+1} + (1 - \theta) \delta_h^2 u_i^n - u_{xx}(t_{n+\theta}, x_i).$$

In space, recalling the standard second-order finite-difference analysis, we know that:

$$\delta_h^2 u(t, x_i) - u_{xx}(t, x_i) = \mathcal{O}(h^2)$$

Now we must also do a Taylor expansion in time so that $\delta_h^2 u(t_n, x_i)$ can be compared to $\delta_h^2 u(t_{n+\theta}, x_i)$:

$$\delta_h^2 u(t_n, x_i) = \delta_h^2 u(t_{n+\theta}, x_i) - \theta k \frac{\partial}{\partial t} [\delta_h^2 u(t_{n+\theta}, x_i)] + \mathcal{O}(k^2)$$

$$\delta_h^2 u(t_{n+1}, x_i) = \delta_h^2 u(t_{n+\theta}, x_i) + (1 - \theta)k \frac{\partial}{\partial t} [\delta_h^2 u(t_{n+\theta}, x_i)] + \mathcal{O}(k^2),$$

which when substituting back cancel out and give:

$$\theta \delta_h^2 u(t_{n+1}, x_i) + (1 - \theta) \delta_h^2 u(t_n, x_i) = \delta_h^2 u(t_{n+\theta}, x_i) + \mathcal{O}(k^2).$$

Now putting everything together:

$$\theta \delta_h^2 u_i^{n+1} + (1 - \theta) \delta_h^2 u_i^n = \delta_h^2 u_i^{n+\theta} + \mathcal{O}(k^2),$$

and

$$\delta_h^2 u_i^{n+\theta} - u_{xx}(t_{n+\theta}, x_i) = \mathcal{O}(h^2).$$

so

$$R_{\text{space}} = \delta_h^2 u_i^{n+\theta} - u_{xx}(t_{n+\theta}, x_i) + \mathcal{O}(k^2) = \mathcal{O}(h^2) + \mathcal{O}(k^2)$$

which holds for any value of θ , thus we arrive at the desired result:

$$\text{Local truncation error} = \begin{cases} \mathcal{O}(k + h^2), & \theta \neq \frac{1}{2}, \\ \mathcal{O}(k^2 + h^2), & \theta = \frac{1}{2}. \end{cases}$$

7.2 Sufficient condition for stability of the θ -method

We want to prove that the condition:

$$\mu(1 - \theta) \leq 1/2$$

is sufficient for the stability of the following θ -method scheme:

$$-\mu\theta U_{i+1}^{n+1} + (1 + 2\mu\theta)U_i^{n+1} - \mu\theta U_{i-1}^{n+1} = -\mu\theta U_{i+1}^n + (1 - 2\mu(1 - \theta))U_i^n - \mu\theta U_{i-1}^n$$

Let's start by bounding the right hand side. By taking the absolute value we get:

$$RHS \leq |\mu(1 - \theta)||U_{i+1}^n| + |1 - 2\mu(1 - \theta)||U_i^n| + |\mu(1 - \theta)||U_{i-1}^n|$$

Now remember that by definition $\mu = \frac{h}{k^2} > 0$ and $0 \leq \theta \leq 1$, and let $\mu(1 - \theta) \leq 1/2$ (our hypothetical sufficient condition). This means that the coefficients $\mu(1 - \theta)$ and $1 - 2\mu(1 - \theta)$ are both non-negative, so we can remove the absolute value. Let's also define $U_{max}^n = \max_i(U_i^n)$ so we can write:

$$RHS \leq |U_{max}^n|[\mu(1 - \theta) + 1 - 2\mu(1 - \theta) + \mu(1 - \theta)] = |U_{max}^n| = \|U^n\|_\infty$$

Now let's bound the left hand side as well:

$$LHS \leq -|\mu\theta||U_{i+1}^{n+1}| + |1 + 2\mu\theta||U_i^{n+1}| - |\mu\theta||U_{i-1}^{n+1}|$$

by similar consideration we obtain

$$LHS \leq |U_{max}^{n+1}|(-\mu\theta + 1 + 2\mu\theta - \mu\theta) = |U_{max}^{n+1}| = \|U^{n+1}\|_\infty$$

hence $LHS \leq RHS$ translates to $\|U^{n+1}\|_\infty \leq \|U^n\|_\infty \forall n$, which by extension implies $\|U^n\|_\infty \leq \|U^0\|_\infty$.

7.3 Fourier (von Neumann) Analysis for the θ -method in 1D

We consider the 1D heat equation

$$u_t = a u_{xx}, \quad x \in (0, 1), \quad t > 0,$$

with homogeneous Dirichlet boundary conditions. We discretize spatially with a uniform grid

$$x_j = j h, \quad j = 0, \dots, M, \quad h = \frac{1}{M},$$

and discretize time with

$$t_n = n k.$$

Let U_j^n approximate $u(x_j, t_n)$. The θ -method for the heat equation in 1D is

$$\frac{U_j^{n+1} - U_j^n}{k} = \theta a \delta_h^2 U_j^{n+1} + (1 - \theta) a \delta_h^2 U_j^n,$$

where

$$\delta_h^2 U_j^n = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2},$$

for $j = 1, \dots, M-1$, with boundary conditions $U_0^n = U_M^n = 0$. Define

$$\mu = \frac{a k}{h^2}.$$

We look for solutions of the form

$$U_j^n = \lambda^n \exp(i \alpha x_j) = \lambda^n \exp(i \alpha j h),$$

where

- j is the spatial index (integer),
- i is the imaginary unit,
- α is a wavenumber

Our goal is to find λ (the amplification factor) in terms of α , μ , and θ .

Now we compute the second difference $\delta_h^2 U_j^n$

$$\delta_h^2 U_j^n = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}$$

substitute $U_j^n = \lambda^n \exp(i \alpha j h)$:

$$U_{j+1}^n = \lambda^n \exp(i \alpha (j+1) h) = \lambda^n \exp(i \alpha j h) \exp(i \alpha h)$$

$$U_j^n = \lambda^n \exp(i \alpha j h)$$

$$U_{j-1}^n = \lambda^n \exp(i \alpha (j-1) h) = \lambda^n \exp(i \alpha j h) \exp(-i \alpha h)$$

hence

$$\delta_h^2 U_j^n = \frac{\lambda^n \exp(i \alpha j h)}{h^2} [\exp(i \alpha h) - 2 + \exp(-i \alpha h)]$$

but

$$\exp(i \alpha h) + \exp(-i \alpha h) = 2 \cos(\alpha h)$$

so

$$\exp(i \alpha h) - 2 + \exp(-i \alpha h) = -4 \sin^2\left(\frac{\alpha h}{2}\right)$$

therefore,

$$\delta_h^2 U_j^n = -\frac{4 \sin^2\left(\frac{\alpha h}{2}\right)}{h^2} \lambda^n \exp(i \alpha j h)$$

which when substituted into the θ -method gives:

$$\frac{U_j^{n+1} - U_j^n}{k} = \theta a \delta_h^2 U_j^{n+1} + (1 - \theta) a \delta_h^2 U_j^n.$$

Now we look at the time difference (LHS):

$$U_j^{n+1} = \lambda^{n+1} \exp(i \alpha j h), \quad U_j^n = \lambda^n \exp(i \alpha j h).$$

$$\frac{U_j^{n+1} - U_j^n}{k} = \frac{\lambda^{n+1} - \lambda^n}{k} \exp(i \alpha j h) = \exp(i \alpha j h) \frac{\lambda^n}{k} (\lambda - 1)$$

and at the space difference (RHS):

$$\begin{aligned} \theta a \delta_h^2 U_j^{n+1} + (1-\theta) a \delta_h^2 U_j^n &= \theta a \left[-\frac{4 \sin^2(\frac{\alpha h}{2})}{h^2} \right] \lambda^{n+1} \exp(i \alpha j h) + (1-\theta) a \left[-\frac{4 \sin^2(\frac{\alpha h}{2})}{h^2} \right] \lambda^n \exp(i \alpha j h) \\ &= -\frac{4 a \sin^2(\frac{\alpha h}{2})}{h^2} \exp(i \alpha j h) [\theta \lambda^{n+1} + (1-\theta) \lambda^n] \\ &= -\frac{4 a \sin^2(\frac{\alpha h}{2})}{h^2} \exp(i \alpha j h) \lambda^n [\theta \lambda + (1-\theta)] \end{aligned}$$

and finally equating them we find that the original:

$$\frac{U_j^{n+1} - U_j^n}{k} = \theta a \delta_h^2 U_j^{n+1} + (1-\theta) a \delta_h^2 U_j^n$$

becomes

$$\exp(i \alpha j h) \frac{\lambda^n}{k} (\lambda - 1) = -\frac{4 a \sin^2(\frac{\alpha h}{2})}{h^2} \exp(i \alpha j h) \lambda^n [\theta \lambda + (1-\theta)]$$

and after cancelling $\exp(i \alpha j h) \lambda^n$ we get:

$$\frac{\lambda - 1}{k} = -\frac{4 a \sin^2(\frac{\alpha h}{2})}{h^2} [\theta \lambda + (1-\theta)].$$

Now by multiplying both sides by k and remembering the substitution $\mu = \frac{a k}{h^2}$ we get:

$$\lambda + 4 \mu \theta \sin^2(\frac{\alpha h}{2}) \lambda = 1 - 4 \mu (1-\theta) \sin^2(\frac{\alpha h}{2}).$$

from which we obtain the amplification factor λ :

$$\lambda(\alpha) = \frac{1 - 4 \mu (1-\theta) \sin^2(\frac{\alpha h}{2})}{1 + 4 \mu \theta \sin^2(\frac{\alpha h}{2})}.$$

The stability condition is then $|\lambda(\alpha)| \leq 1$ for all relevant α .

- If $\theta \geq 1/2$: the denominator exceeds or equals the numerator in absolute value for any $\mu > 0$, so $|\lambda(\alpha)| \leq 1$ unconditionally.

- If $0 \leq \theta < 1/2$: the worst case is $\sin^2(\frac{\alpha h}{2}) = 1$, then

$$\lambda(\alpha) = \frac{1 - 4\mu(1 - \theta)}{1 + 4\mu\theta}$$

and requiring $|\lambda(\alpha)| \leq 1$ forces

$$\mu(1 - 2\theta) \leq \frac{1}{2}.$$

8 Lecture 14

8.1 Wave equation equivalent system

Consider the classical wave equation:

$$u_{tt} - u_{xx} = 0.$$

Introduce an auxiliary variable v and consider the system:

$$u_t + v_x = 0, \tag{1}$$

$$u_x + v_t = 0. \tag{2}$$

This system is equivalent to the wave equation.

Proof that u satisfies the wave equation:

1. Differentiate equation (1) with respect to t :

$$u_{tt} + v_{xt} = 0.$$

2. Differentiate equation (2) with respect to x :

$$u_{xx} + v_{tx} = 0.$$

3. Since partial derivatives commute (i.e. $v_{xt} = v_{tx}$), subtract the second equation from the first:

$$u_{tt} - u_{xx} = 0.$$

Thus, u satisfies the wave equation.

Proof that v satisfies the wave equation:

1. Differentiate equation (2) with respect to t :

$$u_{xt} + v_{tt} = 0.$$

2. Differentiate equation (1) with respect to x :

$$u_{tx} + v_{xx} = 0.$$

3. Notice that $u_{xt} = u_{tx}$ (same as before), subtract the second equation from the first:

$$v_{tt} - v_{xx} = 0.$$

Thus, v also satisfies the wave equation.

8.2 Conservation of energy for the wave equation

Consider the classical wave equation:

$$u_{tt} - u_{xx} = 0, \quad (t, x) \in (0, T] \times \mathbb{R}.$$

Show that the energy

$$E(t) = \frac{1}{2} \int_{\mathbb{R}} \left[(u_t(x, t))^2 + (u_x(x, t))^2 \right] dx$$

is conserved, i.e., $E(t)$ is constant in time.

Solution:

1. Multiply the wave equation by $u_t(x, t)$ to obtain:

$$u_t u_{tt} - u_t u_{xx} = 0.$$

2. Integrate the above expression over \mathbb{R} with respect to x :

$$\int_{\mathbb{R}} (u_t u_{tt} - u_t u_{xx}) dx = 0.$$

3. Process the time derivative term:

recognize that

$$u_t u_{tt} = \frac{1}{2} \frac{\partial}{\partial t} (u_t^2)$$

thus,

$$\int_{\mathbb{R}} u_t u_{tt} dx = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} u_t^2 dx.$$

4. Process the spatial derivative term:

integrate by parts in x by setting

$$v = u_t \quad \text{and} \quad dw = u_{xx} dx$$

so that $dv = u_{tx} dx$ and $w = u_x$. Then,

$$-\int_{\mathbb{R}} u_t u_{xx} dx = -[u_t u_x]_{-\infty}^{+\infty} + \int_{\mathbb{R}} u_{tx} u_x dx.$$

Assuming that u_t and u_x vanish at infinity, the boundary term is zero.

Notice that

$$u_{tx} u_x = \frac{1}{2} \frac{\partial}{\partial t} (u_x^2)$$

hence

$$\int_{\mathbb{R}} u_{tx} u_x dx = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} u_x^2 dx.$$

5. Combine the results:

putting the two parts together, we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} u_t^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} u_x^2 dx = 0$$

this can be rewritten as:

$$\frac{d}{dt} \left[\frac{1}{2} \int_{\mathbb{R}} (u_t^2 + u_x^2) dx \right] = 0$$

therefore, the energy

$$E(t) = \frac{1}{2} \int_{\mathbb{R}} (u_t^2 + u_x^2) dx$$

is conserved in time.

9 Lecture 15

9.1 Consistency proof for the Lax-Wendroff scheme

We consider the linear advection PDE:

$$\begin{cases} u_t + a u_x = 0, \\ u(0, x) = u_0(x), \\ u(b, x) = u_b(x), \end{cases}$$

where a is taken as a constant for simplicity. The **Lax–Wendroff scheme** for $u_i^n \approx u(x_i, t_n)$ is written as:

$$\frac{u_i^{n+1} - u_i^n}{k} + a \frac{u_{i+1}^n - u_{i-1}^n}{2h} - \frac{k(a)^2}{2h^2} (u_{i+1}^n - 2u_i^n + u_{i-1}^n) = 0.$$

We set $\nu := \frac{k}{h}$ (the Courant number). Our goal is to show that the *local truncation error* T_i^n is

$$|T_i^n| \leq \frac{k^2}{6} \|u_{ttt}\|_{\infty} + |a| \frac{h^2}{6} \|u_{xxx}\|_{\infty},$$

i.e. it is $\mathcal{O}(k^2 + h^2)$.

Local truncation error

1. Denoting the exact PDE solution at grid points as $U_i^n = u(x_i, t_n)$, substituting into the Lax-Wendroff scheme we get the residual:

$$T_i^n := \frac{U_i^{n+1} - U_i^n}{k} + a \frac{U_{i+1}^n - U_{i-1}^n}{2h} - \frac{k a^2}{2h^2} (U_{i+1}^n - 2U_i^n + U_{i-1}^n).$$

2. We expand the time shift:

$$U_i^{n+1} = U_i^n + k U_t^n + \frac{k^2}{2} U_{tt}^n + \frac{k^3}{6} U_{ttt}(x_i, t_n) + \mathcal{O}(k^4).$$

3. and the space shifts:

$$\begin{aligned} U_{i+1}^n &= U_i^n + h U_x^n + \frac{h^2}{2} U_{xx}^n + \frac{h^3}{6} U_{xxx}(x_i, t_n) + \mathcal{O}(h^4), \\ U_{i-1}^n &= U_i^n - h U_x^n + \frac{h^2}{2} U_{xx}^n - \frac{h^3}{6} U_{xxx}(x_i, t_n) + \mathcal{O}(h^4), \end{aligned}$$

Substitute into the scheme

1.

$$\frac{U_i^{n+1} - U_i^n}{k} = U_t^n + \frac{k}{2} U_{tt}^n + \frac{k^2}{6} U_{ttt}(x_i, t_n) + \mathcal{O}(k^3)$$

2.

$$a \frac{U_{i+1}^n - U_{i-1}^n}{2h} = a \left(U_x^n + \frac{h^2}{6} U_{xxx}^n \right) + \mathcal{O}(h^3)$$

3.

$$\frac{k a^2}{2} \frac{U_{i+1}^n - U_i^n + U_{i-1}^n}{h^2} = \frac{k a^2}{2} U_{xx}^n + \mathcal{O}(h^2)$$

Notice that from the PDE we have:

$$U_t = -a U_x, U_{tt} = a^2 U_{xx}, U_{ttt} = -a^3 U_{xxx}$$

so the expression for the truncation error is dominated by $k^2 U_{ttt}$ and $a h^2 U_{xxx}$ terms, hence $|T_i^n|$ is bounded by a combination of $k^2 |U_{ttt}|$ and $|a| h^2 |U_{xxx}|$, plus higher order terms. Consequently,

$$|T_i^n| \leq \frac{k^2}{6} M_{ttt} + |a| \frac{h^2}{6} M_{xxx} + \dots,$$

which shows $T_i^n = \mathcal{O}(k^2 + h^2)$.

9.2 Leap frog scheme for wave equation

We consider the model problem

$$u_{tt} - u_{xx} = 0,$$

and recall its standard leap-frog discretization:

$$\frac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{k^2} = \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2},$$

where k and h denote the time and space step sizes, respectively.

9.2.1 First-order system formulation

To rewrite the second-order wave equation as a first-order system we define the auxiliary variable v such that:

$$u_t + v_x = 0, \quad (1)$$

$$v_t + u_x = 0. \quad (2)$$

Now we show how u solves $u_{tt} - u_{xx} = 0$:

- differentiating (1) w.r.t. t we get:

$$u_{tt} = -\frac{\partial}{\partial t}(v_x) = -v_{xt}$$

- differentiating (2) w.r.t. x we get:

$$v_{xt} = -u_{xx}$$

- combining, $u_{tt} = -v_{xt} = -[-u_{xx}] = u_{xx}$, hence $u_{tt} - u_{xx} = 0$.

9.2.2 Discretize system by leap-frog method

We now discretize the system (1) and (2) using a leap-frog method with staggered grids. In particular, we choose:

- the solution u is approximated at integer grid points in space and time:

$$U_i^n \approx u(x_i, t^n)$$

with $x_i = i h$ and $t^n = n k$;

- the auxiliary variable v is approximated at half-grid points in both space and time:

$$V_{i+\frac{1}{2}}^{n+\frac{1}{2}} \approx v\left(x_{i+\frac{1}{2}}, t^{n+\frac{1}{2}}\right),$$

where $x_{i+\frac{1}{2}} = (i + \frac{1}{2}) h$ and $t^{n+\frac{1}{2}} = (n + \frac{1}{2}) k$.

We then have:

$$\frac{U_i^{n+1} - U_i^n}{k} + \frac{V_{i+\frac{1}{2}}^{n+\frac{1}{2}} - V_{i-\frac{1}{2}}^{n+\frac{1}{2}}}{h} = 0, \quad (3)$$

$$\frac{V_{i+\frac{1}{2}}^{n+\frac{1}{2}} - V_{i+\frac{1}{2}}^{n-\frac{1}{2}}}{k} + \frac{U_{i+1}^n - U_i^n}{h} = 0. \quad (4)$$

9.2.3 Derivation of the leap frog scheme for U

The goal now is to eliminate the variable V from (3) and (4) to derive a single scheme for U :

- from (3) we have:

$$\frac{U_i^{n+1} - U_i^n}{k} = -\frac{V_{i+\frac{1}{2}}^{n+\frac{1}{2}} - V_{i-\frac{1}{2}}^{n+\frac{1}{2}}}{h},$$

so that

$$U_i^{n+1} = U_i^n - \frac{k}{h} \left(V_{i+\frac{1}{2}}^{n+\frac{1}{2}} - V_{i-\frac{1}{2}}^{n+\frac{1}{2}} \right) \quad (5)$$

- from (4) we have:

$$\frac{V_{i+\frac{1}{2}}^{n+\frac{1}{2}} - V_{i+\frac{1}{2}}^{n-\frac{1}{2}}}{k} = -\frac{U_{i+1}^n - U_i^n}{h},$$

so that

$$V_{i+\frac{1}{2}}^{n+\frac{1}{2}} = V_{i+\frac{1}{2}}^{n-\frac{1}{2}} - \frac{k}{h} \left(U_{i+1}^n - U_i^n \right) \quad (6)$$

- similarly, for the point $i - \frac{1}{2}$ we have:

$$V_{i-\frac{1}{2}}^{n+\frac{1}{2}} = V_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \frac{k}{h} \left(U_i^n - U_{i-1}^n \right) \quad (7)$$

- now subtract (7) from (6):

$$\begin{aligned} V_{i+\frac{1}{2}}^{n+\frac{1}{2}} - V_{i-\frac{1}{2}}^{n+\frac{1}{2}} &= \left(V_{i+\frac{1}{2}}^{n-\frac{1}{2}} - V_{i-\frac{1}{2}}^{n-\frac{1}{2}} \right) - \frac{k}{h} \left[(U_{i+1}^n - U_i^n) - (U_i^n - U_{i-1}^n) \right] \\ &= \left(V_{i+\frac{1}{2}}^{n-\frac{1}{2}} - V_{i-\frac{1}{2}}^{n-\frac{1}{2}} \right) - \frac{k}{h} \left(U_{i+1}^n - 2U_i^n + U_{i-1}^n \right) \end{aligned} \quad (8)$$

- Now we want to express $V_{i+\frac{1}{2}}^{n-\frac{1}{2}} - V_{i-\frac{1}{2}}^{n-\frac{1}{2}}$ in terms of U . Notice that the same discretization (3) applied at the previous time level gives:

$$\frac{U_i^n - U_i^{n-1}}{k} + \frac{V_{i+\frac{1}{2}}^{n-\frac{1}{2}} - V_{i-\frac{1}{2}}^{n-\frac{1}{2}}}{h} = 0$$

so that

$$V_{i+\frac{1}{2}}^{n-\frac{1}{2}} - V_{i-\frac{1}{2}}^{n-\frac{1}{2}} = -\frac{h}{k} \left(U_i^n - U_i^{n-1} \right) \quad (9)$$

- using (9) in (8) yields

$$V_{i+\frac{1}{2}}^{n+\frac{1}{2}} - V_{i-\frac{1}{2}}^{n+\frac{1}{2}} = -\frac{h}{k} \left(U_i^n - U_i^{n-1} \right) - \frac{k}{h} \left(U_{i+1}^n - 2U_i^n + U_{i-1}^n \right) \quad (10)$$

- recall from (5):

$$U_i^{n+1} = U_i^n - \frac{k}{h} \left(V_{i+\frac{1}{2}}^{n+\frac{1}{2}} - V_{i-\frac{1}{2}}^{n+\frac{1}{2}} \right).$$

- substitute (10) into this expression and rearrange:

$$\begin{aligned} U_i^{n+1} &= U_i^n - \frac{k}{h} \left[-\frac{h}{k} \left(U_i^n - U_i^{n-1} \right) - \frac{k}{h} \left(U_{i+1}^n - 2U_i^n + U_{i-1}^n \right) \right] \\ &= U_i^n + \left(U_i^n - U_i^{n-1} \right) + \frac{k^2}{h^2} \left(U_{i+1}^n - 2U_i^n + U_{i-1}^n \right) \\ &= 2U_i^n - U_i^{n-1} + \frac{k^2}{h^2} \left(U_{i+1}^n - 2U_i^n + U_{i-1}^n \right). \end{aligned}$$

- Finally, rewriting this in the standard leap-frog form yields:

$$\frac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{k^2} = \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2}.$$