Università degli studi di Padova

Physical Models of Living Systems

Master Degree in Physics of Data, 2022-2023

Zomer Nicola

October 18, 2022

Contents

1 Homeworks Week 01

Tasks:

1. Solve the Quasi Stationary Approximation of the Consumer Resource Model with 1 species and 1 abiotic resource and compare it numerically with the full solution. Optional: find a regime of parameters where the QSA is good. Remember to check that parameters you choose and initial condition for R and N should be so that R* in the QSA is not negative.

2. Write the Fokker Plank Equation associated to the stochastic logistic equation with environmental noise and solve for the stationary solution P*. Optional: compare analytical and numerical simulation of the SDE.

1.1 Exercise 1

Let $N(t)$ be the size of the population and $R(t)$ the amount of resources available. The Consumer Resource Model with 1 species and 1 abiotic resource is described by the following system of differential equations:

$$\frac{dR}{dt} = \mu(R) - cRN \tag{1}$$
$$\frac{dN}{dt} = (\gamma cR - d)N \tag{2}$$

For abiotic resources, $\mu(R)$ is given by the Monod function:

$$\mu(R) = \frac{R}{k_s + R} \tag{3}$$

In order to solve the system using the QSA, first we need to find the stationary solution of equation (1), which corresponds to assuming R is fixed at equilibrium.

$$0 \stackrel{!}{=} \frac{dR}{dt} = \frac{R}{k_s + R} - cRN\bigg|_{R=R^*} \tag{4}$$

Using the fact that $R^* = \text{const} \neq 0$ we can solve it for R^*:

$$(1 - cNk_s)R - cNR^2|_{R=R^*} = 0 \Longrightarrow R^* = \frac{1 - cNk_s}{cN} \tag{5}$$

This solution can be substituted in equation (2), leading to a linear ordinary differential equation for the population $N(t)$:

$$\frac{dN}{dt} = \left[\frac{\gamma(1 - cNk_s)}{N} - d\right] N \tag{6}$$

If we define $\tilde{a} = \gamma$ and $\tilde{b} = \gamma ck_s + d$ we can rewrite the equation as:

$$\frac{dN}{dt} = \tilde{a} - \tilde{b}N \tag{7}$$

The general solution can be found solving the corresponding homogeneous equation and adding a particular solution, which can be found for example by looking at the stationary state.

$$\text{Homogeneous: } \dot{N} = -\tilde{b}N \Rightarrow N(t) = Ce^{-\tilde{b}t}$$
$$\text{Stationary state: } 0 \stackrel{!}{=} \dot{N} = \tilde{a} - \tilde{b}N \Rightarrow N(t) = \frac{\tilde{a}}{\tilde{b}}$$

So the full solution of the QSA of the Consumer Resource Model with 1 species and 1 abiotic resource is given by:

$$N(t) = Ce^{-\tilde{b}t} + \frac{\tilde{a}}{\tilde{b}} \tag{8}$$

and the multiplicative constant C can be determined imposing the initial condition $N(t = 0) = N_0$, which leads to:

$$C = N_0 - \frac{\tilde{a}}{\tilde{b}} \tag{9}$$

1.1.1 Numerical comparison

Let's consider the following set of parameters:

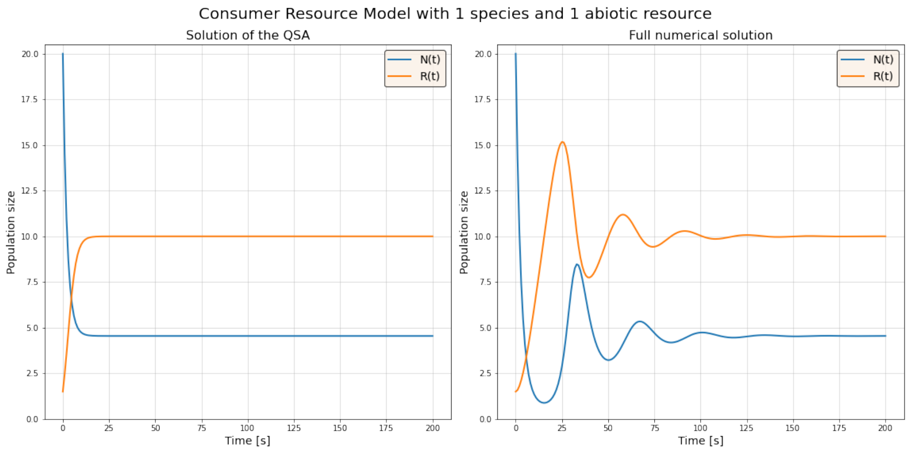
$$k_s = 1, \quad c = 0.02, \quad d = 0.4, \quad \gamma = 2 \tag{10}$$

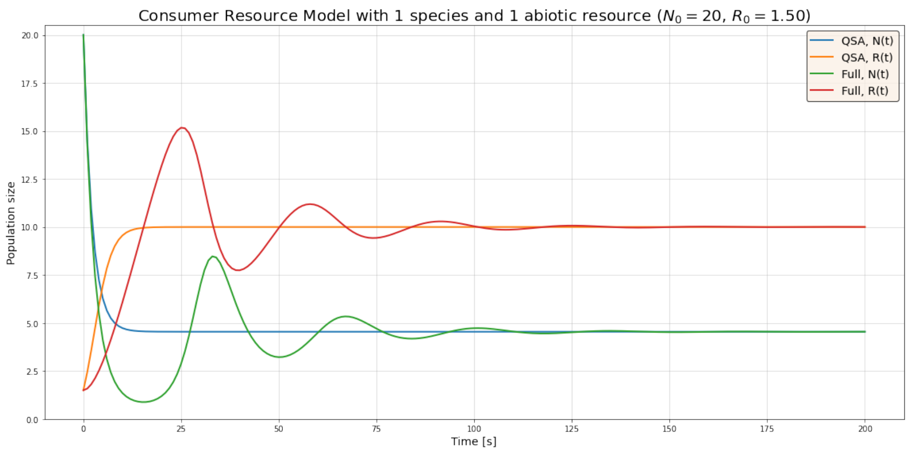
and the initial condition:

$$N(t = 0) = N_0 = 20 \tag{11}$$

$$R(t = 0) = \frac{1 - cN_0 k_s}{cN_0} = 1.5 \tag{12}$$

The following plots are obtained by simulating the evolution of $N(t)$ and $R(t)$ for $t \in [0, 200]$ with the above set of parameters and initial condition. Through them it is possible to compare numerically the QSA solution with the full one.





1.2 Exercise 2

Given the stochastic logistic equation with environmental noise:

$$\frac{dx}{dt} = \frac{x}{\tau}\left(1 - \frac{x}{K}\right) + \sqrt{\frac{\sigma}{\tau}}x\xi(t) \tag{13}$$

where $\xi(t)$ is Gaussian white noise, it is possible to identify $A(x)$ and $B(x)$ such that the previous equation can be reformulated as:

$$\frac{dx}{dt} = A(x) + \sqrt{B(x)}\xi(t) \tag{14}$$

In this case, we have that:

$$A(x) = \frac{x}{\tau}\left(1 - \frac{x}{K}\right) \tag{15}$$
$$B(x) = \frac{\sigma}{\tau}x^2 \tag{16}$$

At this point it is possible to write the associated Fokker Plank equation:

$$\begin{aligned} \frac{\partial P}{\partial t} &= -\frac{\partial}{\partial x}[A(x)P(x)] + \frac{1}{2}\frac{\partial^2}{\partial^2 x}[B(x)P(x)] = \\ &= -\frac{\partial}{\partial x}\left[\frac{x}{\tau}\left(1 - \frac{x}{K}\right)P(x)\right] + \frac{1}{2}\frac{\partial^2}{\partial^2 x}\left[\frac{\sigma}{\tau}x^2 P(x)\right] \end{aligned} \tag{17}$$

To find the stationary solution P^* we have to impose that $\frac{\partial P}{\partial t} \overset{!}{=} 0$. Defining the flux:

$$J(x) = -\frac{x}{\tau}\left(1 - \frac{x}{K}\right)P(x) + \frac{1}{2}\frac{\partial}{\partial x}\left[\frac{\sigma}{\tau}x^2 P(x)\right] \tag{18}$$

we have that $\frac{\partial P}{\partial t} = 0 \iff \frac{\partial J(x)}{\partial x} = 0 \iff J(x) = \text{const.} = 0$, where the last equal is achieved by imposing the boundary condition $J(0) = 0$. We are left with a first order differential equation, $J(x)|_{P(x)=P^*} = 0$, which can be solved by separation of variables.

$$\begin{aligned} &J(x) = 0 \\ &\Rightarrow -\frac{x}{\tau}\left(1 - \frac{x}{K}\right)P(x) + \frac{\sigma}{\tau}xP(x) + \frac{1}{2}\frac{\sigma}{\tau}x^2\frac{dP(x)}{dx} = 0 \\ &\Rightarrow \frac{1}{P(x)}\frac{dP(x)}{dx} = \frac{2}{x}\left(\frac{1}{\sigma} - 1\right) - \frac{2}{\sigma K} \end{aligned}$$

Integrating on both sides we obtain:

$$\ln P(x) = 2\left(\frac{1}{\sigma} - 1\right)\ln x - \frac{2}{\sigma K}x + C \tag{19}$$

and, finally, taking the exponential and redefining the constant C we have the stationary solution:

$$P^*(x) = C\exp\left\{2\left(\frac{1}{\sigma} - 1\right)\ln x - \frac{2}{\sigma K}x\right\} \tag{20}$$

To find C we have to impose the normalization condition on $P^*(x)$ and solve the corresponding equation:

$$\int P^*(x)dx = 1 \tag{21}$$

The integral can be solved using the Gamma function and the final expression of the stationary distribution is:

$$P^*(x) = \frac{1}{\Gamma\left(\frac{2}{\sigma}\right)}\exp\left\{2\left(\frac{1}{\sigma} - 1\right)\ln x - \frac{2}{\sigma K}x\right\} \tag{22}$$