Università degli studi di Padova

Physical Models of Living Systems

Master Degree in Physics of Data, 2022-2023

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Contents

1 Homework Week 01

Tasks:

1. Solve the Quasi Stationary Approximation of the Consumer Resource Model with 1 species and 1 abiotic resource and compare it numerically with the full solution. Optional: find a regime of parameters where the QSA is good. Remember to check that parameters you choose and initial condition for R and N should be so that R* in the QSA is not negative.

2. Write the Fokker Plank Equation associated to the stochastic logistic equation with environmental noise and solve for the stationary solution P*. Optional: compare analytical and numerical simulation of the SDE.

1.1 Exercise 1

Let $N(t)$ be the size of the population and $R(t)$ the amount of resources available. The Consumer Resource Model with 1 species and 1 abiotic resource is described by the following system of differential equations:

$$\frac{dR}{dt} = \mu(R) - cRN \tag{1}$$
$$\frac{dN}{dt} = (\gamma cR - d)N \tag{2}$$

For abiotic resources, $\mu(R)$ is given by the Monod function:

$$\mu(R) = \frac{R}{k_s + R} \tag{3}$$

In order to solve the system using the QSA, first we need to find the stationary solution of equation (1), which corresponds to assuming R is fixed at equilibrium.

$$0 \overset{!}{=} \frac{dR}{dt} = \frac{R}{k_s + R} - cRN\bigg|_{R=R^*} \tag{4}$$

Using the fact that $R^* = \text{const} \neq 0$ we can solve it for R^*:

$$(1 - cNk_s)R - cNR^2|_{R=R^*} = 0 \Longrightarrow R^* = \frac{1 - cNk_s}{cN} \tag{5}$$

This solution can be substituted in equation (2), leading to a linear ordinary differential equation for the population $N(t)$:

$$\frac{dN}{dt} = \left[\frac{\gamma(1 - cNk_s)}{N} - d\right] N \tag{6}$$

If we define $\tilde{a} = \gamma$ and $\tilde{b} = \gamma ck_s + d$ we can rewrite the equation as:

$$\frac{dN}{dt} = \tilde{a} - \tilde{b}N \tag{7}$$

The general solution can be found solving the corresponding homogeneous equation and adding a particular solution, which can be found for example by looking at the stationary state.

$$\text{Homogeneous: } \dot{N} = -\tilde{b}N \Rightarrow N(t) = Ce^{-\tilde{b}t}$$
$$\text{Stationary state: } 0 \overset{!}{=} \dot{N} = \tilde{a} - \tilde{b}N \Rightarrow N(t) = \frac{\tilde{a}}{\tilde{b}}$$

So the full solution of the QSA of the Consumer Resource Model with 1 species and 1 abiotic resource is given by:

$$N(t) = Ce^{-\tilde{b}t} + \frac{\tilde{a}}{\tilde{b}} \tag{8}$$

and the multiplicative constant C can be determined imposing the initial condition $N(t = 0) = N_0$, which leads to:

$$C = N_0 - \frac{\tilde{a}}{\tilde{b}} \tag{9}$$

1.1.1 Numerical comparison

Let's consider the following set of parameters:

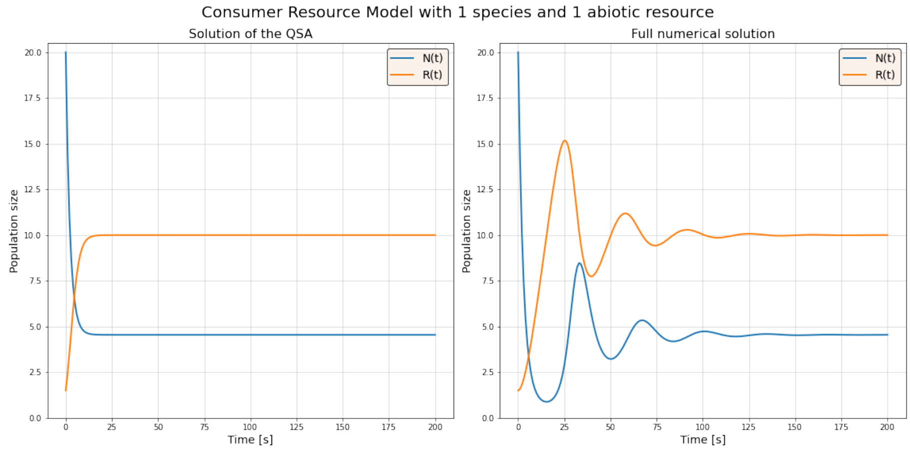
$$k_s = 1, \quad c = 0.02, \quad d = 0.4, \quad \gamma = 2 \tag{10}$$

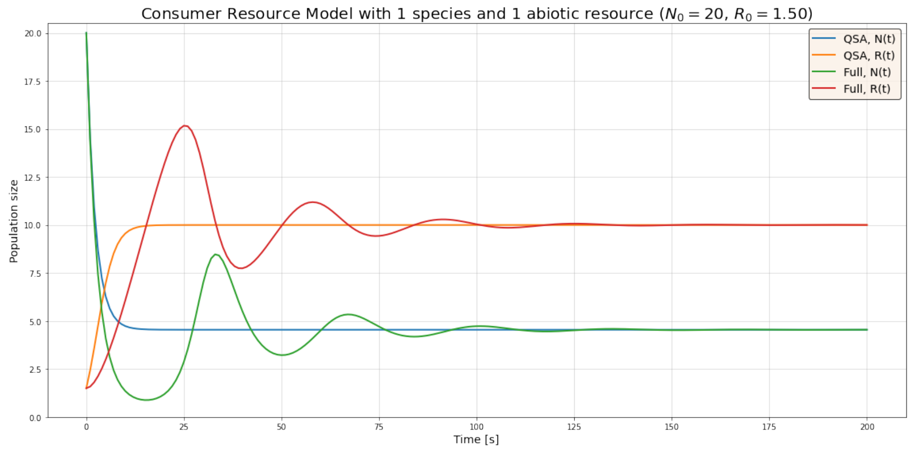
and the initial condition:

$$N(t=0) = N_0 = 20 \tag{11}$$

$$R(t=0) = \frac{1 - cN_0 k_s}{cN_0} = 1.5 \tag{12}$$

The following plots are obtained by simulating the evolution of $N(t)$ and $R(t)$ for $t \in [0, 200]$ with the above set of parameters and initial condition. Through them it is possible to compare numerically the QSA solution with the full one.





For more details, see the corresponding notebook: hw01_Dynamics_of_Single_Species.ipynb.

1.2 Exercise 2

Given the stochastic logistic equation with environmental noise:

$$\frac{dx}{dt} = \frac{x}{\tau}\left(1 - \frac{x}{K}\right) + \sqrt{\frac{\sigma}{\tau}} x \xi(t) \tag{13}$$

where $\xi(t)$ is Gaussian white noise, it is possible to identify $A(x)$ and $B(x)$ such that the previous equation can be reformulated as:

$$\frac{dx}{dt} = A(x) + \sqrt{B(x)}\xi(t) \tag{14}$$

In this case, we have that:

$$A(x) = \frac{x}{\tau}\left(1 - \frac{x}{K}\right) \tag{15}$$

$$B(x) = \frac{\sigma}{\tau}x^2 \tag{16}$$

At this point it is possible to write the associated Fokker Plank equation:

$$\begin{aligned}\frac{\partial P}{\partial t} &= -\frac{\partial}{\partial x}[A(x)P(x)] + \frac{1}{2}\frac{\partial^2}{\partial^2 x}[B(x)P(x)] = \\ &= -\frac{\partial}{\partial x}\left[\frac{x}{\tau}\left(1 - \frac{x}{K}\right)P(x)\right] + \frac{1}{2}\frac{\partial^2}{\partial^2 x}\left[\frac{\sigma}{\tau}x^2 P(x)\right]\end{aligned} \tag{17}$$

To find the stationary solution P^* we have to impose that $\frac{\partial P}{\partial t} \stackrel{!}{=} 0$. Defining the flux:

$$J(x) = -\frac{x}{\tau}\left(1 - \frac{x}{K}\right)P(x) + \frac{1}{2}\frac{\partial}{\partial x}\left[\frac{\sigma}{\tau}x^2 P(x)\right] \tag{18}$$

we have that $\frac{\partial P}{\partial t} = 0 \iff \frac{\partial J(x)}{\partial x} = 0 \iff J(x) = \text{const.} = 0$, where the last equal is achieved by imposing the boundary condition $J(0) = 0$. We are left with a first order differential equation, $J(x)|_{P(x)=P^*} = 0$, which can be solved by separation of variables.

$$\begin{aligned}&J(x) = 0 \\ &\Rightarrow -\frac{x}{\tau}\left(1 - \frac{x}{K}\right)P(x) + \frac{\sigma}{\tau}xP(x) + \frac{1}{2}\frac{\sigma}{\tau}x^2\frac{dP(x)}{dx} = 0 \\ &\Rightarrow \frac{1}{P(x)}\frac{dP(x)}{dx} = \frac{2}{x}\left(\frac{1}{\sigma} - 1\right) - \frac{2}{\sigma K}\end{aligned}$$

Integrating on both sides we obtain:

$$\ln P(x) = 2\left(\frac{1}{\sigma} - 1\right)\ln x - \frac{2}{\sigma K}x + C \tag{19}$$

and, finally, taking the exponential and redefining the constant C we have the stationary solution:

$$P^*(x) = C\exp\left\{2\left(\frac{1}{\sigma} - 1\right)\ln x - \frac{2}{\sigma K}x\right\} \tag{20}$$

To find C we have to impose the normalization condition on $P^*(x)$ and solve the corresponding equation:

$$\int P^*(x)dx = 1 \tag{21}$$

The integral can be solved using the Gamma function and the final expression of the stationary distribution is:

$$P^*(x) = \frac{1}{\Gamma\left(\frac{2}{\sigma}\right)}\exp\left\{2\left(\frac{1}{\sigma} - 1\right)\ln x - \frac{2}{\sigma K}x\right\} \tag{22}$$

2 Homework Week 02

Infer the number of species from data of a forest sampled in 1% of the total area. Each row of the file represent a different species, and the number indicates the species abundance (i.e., the number of individuals). Perform the analysis to infer the number of species at the whole scale (p=1). You can work in small group of 2/3 people.

2.1 Solution

The solution can be found in the repository of the course ([1]). The notebook is developed together with Tommaso Amico and Andrea Lazzari, and it can be accessed via the following link: hw02_Spatial_Scaling_RSA.ipynb.

3 Homework Week 03

Consider the Lotka-Volterra equations:

$$\frac{dx}{dt} = ax - pxy \tag{23}$$

$$\frac{dy}{dt} = -cy + pxy \tag{24}$$

Tasks:

1. Find the stationary solutions.
2. Do the stability analysis of the stationary solutions. Is there any stable solution?
3. (optional) Simulate Eqs.(23)-(24) with different parameters. Is there a range of parameters where do you observe sustained oscillations?

3.1 Stationary solutions

Stationary solutions are characterized by:

$$\begin{cases} \frac{dx}{dt} \stackrel{!}{=} 0 \\ \frac{dy}{dt} \stackrel{!}{=} 0 \end{cases} \implies \begin{cases} 0 = ax - pxy \\ 0 = -cy + pxy \end{cases} \tag{25}$$

The solutions of this system are:

- $x_0^* = y_0^* = 0$
- $x_1^* = \frac{c}{p}$, $y_1^* = \frac{a}{p}$

so we found 2 stationary points, $(x_0^*, y_0^*) = (0, 0)$ and $(x_1^*, y_1^*) = (\frac{c}{p}, \frac{a}{p})$.

3.2 Stability analysis

For each stationary point, we have to compute the Jacobian of the system in it:

$$J_i = \begin{pmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{pmatrix} \Bigg|_{x_i^*} = \begin{pmatrix} a - py & -px \\ py & px - c \end{pmatrix} \Bigg|_{x_i^*} \tag{26}$$

We obtain:

$$J_0 = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix} \qquad J_1 = \begin{pmatrix} 0 & -c \\ a & 0 \end{pmatrix} \tag{27}$$

Writing the linearized equations around the stationary point, we know that stability is determined by the largest real part of the Jacobian's eigenvalues. The eigenvalues corresponding to J_0 and J_1 are, respectively:

$$\lambda_1^0 = a \qquad \lambda_2^0 = -c \tag{28}$$

$$\lambda_1^1 = -i\sqrt{ac} \qquad \lambda_2^1 = i\sqrt{ac} \tag{29}$$

where we used the fact that $a, c > 0$. As $\lambda_1^0 = a > 0$, the first stationary point x_0^* is unstable. On the other side, in the second stationary point we have that $Re(\lambda_1^1) = Re(\lambda_2^1) = 0$, which corresponds to a neutral or critical situation. Since the imaginary part of x_1^* is nonzero, or in other words x_1^* is complex, we expect an oscillatory behaviour around the stationary point. Recall indeed that the general case $Re(\lambda) = 0$ and $Im(\lambda) = \pm k$ corresponds to an Harmonic oscillator.

3.3 Simulation results

To do...

References

[1] Nicola Zomer. *Github Repository of Physical Models of Living Systems*. Aug. 31, 2022. URL: https://github.com/NicolaZomer/Physical_Models_of_Living_Systems.