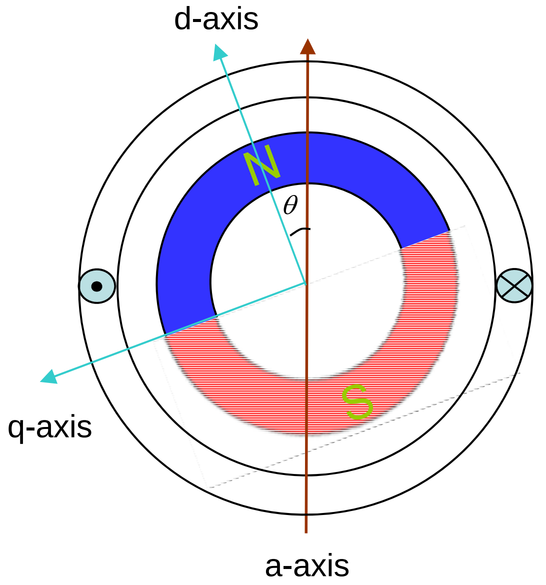
Lecture 6 - contents

Torque analysis of electrical machines

- A simple example – basic analysis method for torque analysis
- Torque equation for synchronous machines
- Torque equation for induction machines

Basic torque analysis method

A simple example - A single phase PM machine



Voltage equation $$u = Ri + \frac{d\lambda}{d\theta}\omega$$

Total flux linkage $$\lambda = \lambda_{pm} + \lambda_a$$

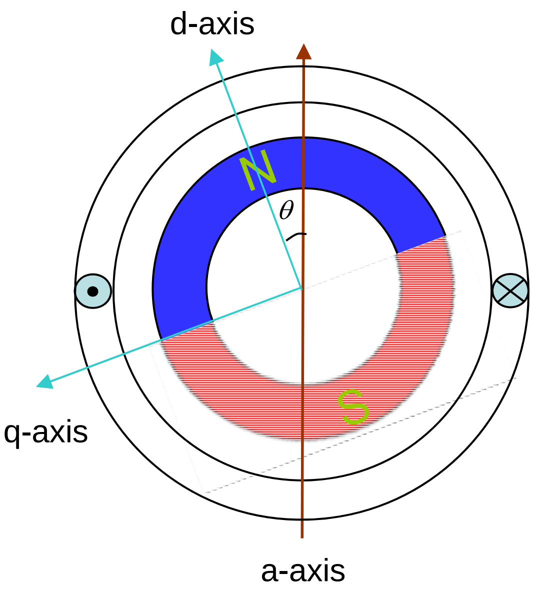
$$\lambda_{pm} = \lambda_{mpm}\cos\theta$$

$$\lambda_a = L_a i$$

Using the power balance equation to derive the torque equation

Basic torque analysis method

Current waveform? Assuming an arbitrary sinusoidal armature current



$$i = I_m \cos(\theta + \varphi)$$

For motoring operation, air gap power ≥ 0

$$P_g = \frac{1}{2\pi}\int_0^{2\pi}(-e \cdot i)\,d\theta \geq 0,$$

$$-e = \omega \frac{d\lambda_{pm}}{d\theta} = -\omega\lambda_{mpm}\sin\theta$$

$$i = -I_m \sin(\theta + \boxed{\theta_t})$$

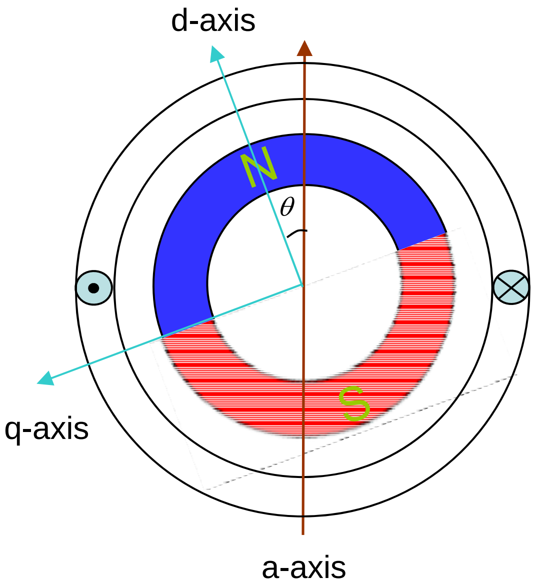
A constant

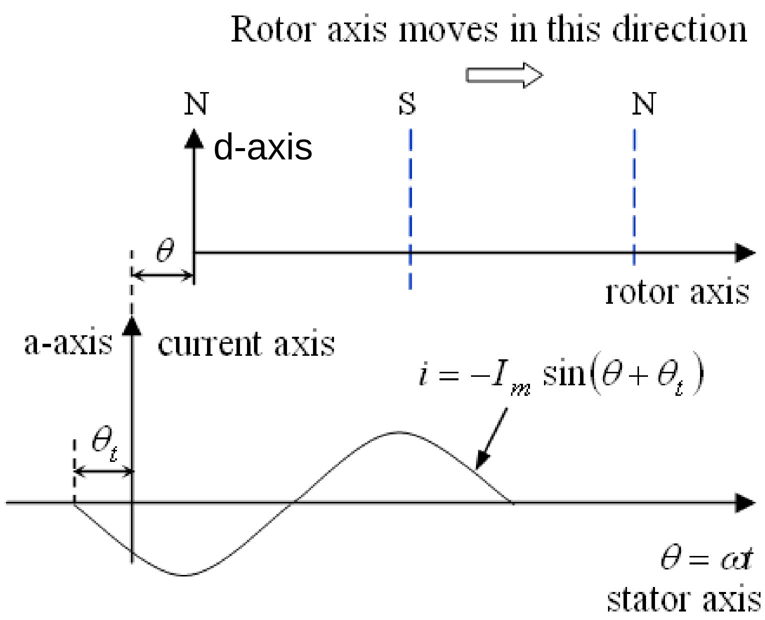
$$-\frac{\pi}{2} < \theta_t < \frac{\pi}{2}$$

Basic torque analysis method

What does the current waveform mean?

$$i = -I_m \sin(\theta + \theta_t) \qquad -\frac{\pi}{2} < \theta_t < \frac{\pi}{2}$$





Basic torque analysis method

Basics for torque calculation – power balance equation

Instantaneous input power

$$P_{in} = ui = Ri^2 + L_a \frac{di}{dt} i + i \frac{d\lambda_{pm}}{dt}$$

Copper loss

Output mechanical power!!!

Change of the energy stored in the magnetic field

Energy stored in the magnetic field $W_e = \frac{1}{2} L_a i^2 \Rightarrow \boxed{\frac{dW_e}{dt} = L_a \frac{di}{dt} i}$

Basic torque analysis method

Instantaneous torque

Number of pole pairs

$$\tau = \frac{1}{\Omega} P_{mec} = \frac{1}{\Omega} i \frac{d\lambda_{pm}}{dt} = \boxed{p} i \lambda_{mpm} (-\sin\theta)$$

electrical rotor angle!

$$i = -I_m \sin(\theta + \theta_t)$$

$$\tau = p I_m \lambda_{mpm} \sin\theta \sin(\theta + \theta_t) = \frac{1}{2} p I_m \lambda_{mpm} [\cos\theta_t - \cos(2\theta + \theta_t)]$$

What is this torque waveform?

Basic torque analysis method

Average torque

- Average torque

$$T_{ave} == \frac{1}{2} p I_m \lambda_{mpm} \cos\theta_t$$

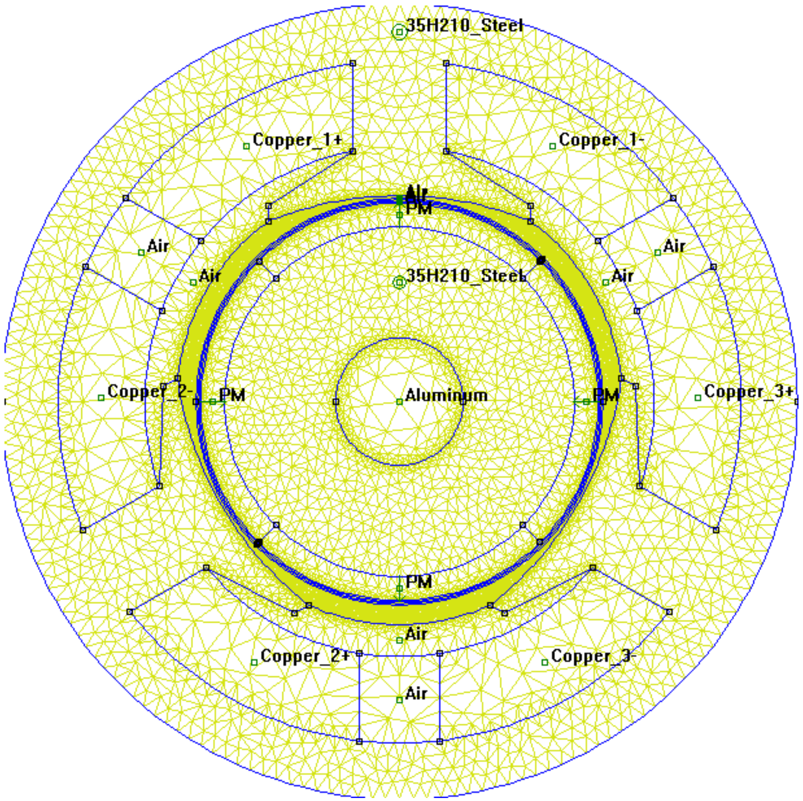
- The maximum average torque occurs when $\theta_t = 0$

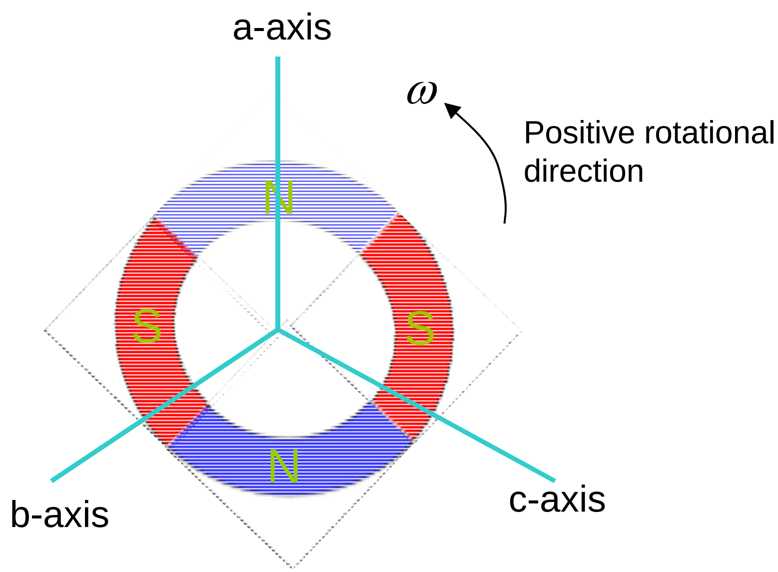
$$T_{ave} = \frac{1}{2} p I_m \lambda_{mpm}$$

- What is the instantaneous / average torque of a 3/2 surface mounted PM motor?

Basic torque analysis method

Torque for a 3/4 surface mounted PM motor?





Torque for synchronous machines

$$u_{qs} = R_s i_{qs} + \frac{d}{dt}\lambda_{qs} + \omega_\theta \lambda_{ds}$$

$$u_{ds} = R_s i_{ds} + \frac{d}{dt}\lambda_{ds} - \omega_\theta \lambda_{qs}$$

E.g. q-axis, stator side winding analysis:

$$\boxed{P_{inq}} = i_{qs} u_{qs} = \boxed{R_s i^2{}_{qs}} + \boxed{i_{qs}\frac{d}{dt}\lambda_{qs}} + \boxed{\omega_\theta \lambda_{ds} i_{qs}}$$

Output mechanical power!!!

Rate change of the stored field energy

Copper loss

Input power of the q-axis winding

Torque for synchronous machines

The stator q-axis output mechanical power

$$P_{inq} = \omega_\theta \lambda_{ds} i_{qs}$$

Similar for the d-axis

Lecture 2

$$\boxed{\frac{2}{3}} P_{mec} = \omega_\theta \left(\lambda_{ds} i_{qs} - \lambda_{qs} i_{ds} \right)$$

$$P_{mec} = \frac{3}{2} \omega_r \left(\lambda_{ds} i_{qs} - \lambda_{qs} i_{ds} \right) \Rightarrow \boxed{\tau = \frac{3}{2} p \cdot \left(\lambda_{ds} i_{qs} - \lambda_{qs} i_{ds} \right)}$$

Number of pole pairs

Torque for synchronous machines

Why the power associated with the 0 sequence circuit is not involved in the mechanical power?

$$u_0 = Ri_0 + \frac{d\lambda_0}{dt}$$

$$i_0 = \frac{1}{3}\left(i_a + i_b + i_c\right)$$

$$\lambda_0 = L_0 i_0$$

$$L_0 = L_{aal} = L_{bbl} = L_{ccl}$$

Torque for induction machines

Instantaneous torque equation for the IM

- General rule from the energy point of view

$$P_{in} = P_{rloss} + P_{fchg} + P_{mec}$$

- We may let the resistance to be zero when deriving the torque equations

$$P_{in} = \begin{bmatrix} u_{as} \\ u_{bs} \\ u_{cs} \end{bmatrix}^T \cdot \begin{bmatrix} i_{as} \\ i_{bs} \\ i_{cs} \end{bmatrix} + \begin{bmatrix} u_{ar} \\ u_{br} \\ u_{cr} \end{bmatrix}^T \cdot \begin{bmatrix} i_{ar} \\ i_{br} \\ i_{cr} \end{bmatrix}$$

- The only thing that we need to do is to find the power related to the change of the energy stored in the magnetic field

- The shaft torque could be calculated by

$$\tau = \frac{P_{mec}}{\Omega_r} = p\frac{P_{mec}}{\omega_r}$$

The rate change of the power stored in the magnetic field

- For a single inductance system, the power stored in the field is

$$P_f = \frac{1}{2} L i^2 = \frac{1}{2} L \lambda \quad \Rightarrow \quad \frac{dP_f}{dt} = L i \frac{di}{dt} = i \frac{d\lambda}{dt}$$

If the inductance is position and current independent

- For a IM, the stored magnetic field power may be expressed as

$$P_f = \frac{1}{2}\left(\underline{i}_{abcs}\right)^T \underline{L}_s \underline{i}_{abcs} + \boxed{\frac{1}{2}\left(\underline{i}_{abcs}\right)^T \underline{L}_{sr} \underline{i}_{abcr}} + \boxed{\frac{1}{2}\left(\underline{i}_{abcr}\right)^T \left(\underline{L}_{sr}\right)^T \underline{i}_{abcs}} + \frac{1}{2}\left(\underline{i}_{abcr}\right)^T \underline{L}_r \underline{i}_{abcr}$$

They are equal

$$P_f = \frac{1}{2}\left(\underline{i}_{abcs}\right)^T \underline{L}_s \underline{i}_{abcs} + \left(\underline{i}_{abcs}\right)^T \underline{L}_{sr} \underline{i}_{abcr} + \frac{1}{2}\left(\underline{i}_{abcr}\right)^T \underline{L}_r \underline{i}_{abcr}$$

The leakage inductance should be included

Take account of all the inductances!

$$\frac{dP_f}{dt} = \left(\underline{i}_{abcs}\right)^T \boxed{\underline{L}_s \frac{d\left(\underline{i}_{abcs}\right)}{dt}} + \left(\underline{i}_{abcr}\right)^T \boxed{\underline{L}_r \frac{d\left(\underline{i}_{abcr}\right)}{dt}}$$

$$+ \boxed{\frac{d\left(\underline{i}_{abcs}\right)^T}{dt} \underline{L}_{sr}} \underline{i}_{abcr} + \left(\underline{i}_{abcs}\right)^T \boxed{\frac{d\,\underline{L}_{sr}}{dt} \underline{i}_{abcr}} + \left(\underline{i}_{abcs}\right)^T \boxed{\underline{L}_{sr} \frac{d\left(\underline{i}_{abcr}\right)}{dt}}$$

$$P_{in} = \left(\underline{i}_{abcs}\right)^T \underline{u}_{abcs} + \left(\underline{i}_{abcr}\right)^T \underline{u}_{abcr}$$

$$\underline{u}_{abcs} = p\left(\underline{L}_s \underline{i}_{abcs} + \underline{L}_{sr} \underline{i}_{abcr}\right) = \boxed{\underline{L}_s \frac{d\left(\underline{i}_{abcs}\right)}{dt}} + \boxed{\frac{d\,\underline{L}_{sr}}{dt} \underline{i}_{abcr}} + \boxed{\underline{L}_{sr} \frac{d\left(\underline{i}_{abcr}\right)}{dt}}$$

$$\underline{u}_{abcr} = p\left(\underline{L}_r \underline{i}_{abcr} + \left(\underline{L}_{sr}\right)^T \underline{i}_{abcs}\right) = \boxed{\underline{L}_r \frac{d\left(\underline{i}_{abcr}\right)}{dt}} + \frac{d\left(\underline{L}_{sr}\right)^T}{dt} \underline{i}_{abcs} + \boxed{\left(\underline{L}_{sr}\right)^T \frac{d\left(\underline{i}_{abcs}\right)}{dt}}$$

$$P_{mec} = P_{in} - \frac{dP_f}{dt} = \left(\underline{i}_{abcr}\right)^T \frac{d\left(\underline{L}_{sr}\right)^T}{dt} \underline{i}_{abcs} = \left(\underline{i}_{abcs}\right)^T \frac{d\left(\underline{L}_{sr}\right)}{dt} \underline{i}_{abcr} = \left(\underline{i}_{abcs}\right)^T \frac{d\left(\underline{L}'_{sr}\right)}{dt} \underline{i}'_{abcr}$$

Instantaneous torque equation for the IM becomes:

$$P_{mec} = P_{in} - \frac{dP_f}{dt} = \left(\underline{i}_{abcs}\right)^T \frac{d\left(\underline{L}'_{sr}\right)}{dt} \underline{i}'_{abcr} = \left(\underline{i}_{qd0s}\right)^T \boxed{\left(\left(K_s\right)^{-1}\right)^T \frac{d\left(\underline{L}'_{sr}\right)}{dt} \left(K_r\right)^{-1} \underline{i}'_{qd0r}}$$

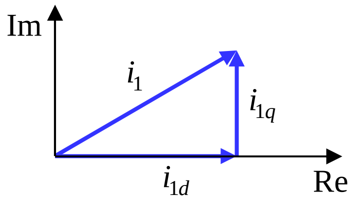
$$\boxed{\omega_r \frac{3}{2} \begin{bmatrix} 0 & L_m & 0 \\ -L_m & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

$$\boxed{\tau = p \frac{1}{\omega_r} P_{mec} = \frac{3}{2} p L_m \left(i_{qs} i'_{dr} - i_{ds} i'_{qr}\right)}$$

We may find more torque expressions based on the followings

$$\bar{i}_1 = i_{1d} + ji_{1q}$$

$$\bar{i}_2 = i_{2d} + ji_{2q}$$



$$\mathrm{Im}\left(\bar{i}_1 \cdot \bar{i}^*_2\right) = i_{1q}i_{2d} - i_{1d}i_{2q}$$

$$\left|\bar{i}_1 \times \bar{i}_2\right| = \left|\bar{i}_1\right| \cdot \left|\bar{i}_2\right| \sin\left[\sin^{-1}\left(\frac{i_{1q}}{\left|\bar{i}_1\right|}\right) - \sin^{-1}\left(\frac{i_{2q}}{\left|\bar{i}_2\right|}\right)\right] = i_{1q}i_{2d} - i_{1d}i_{2q}$$

$$\boxed{\mathrm{Im}\left(\bar{i}_1 \cdot \bar{i}^*_2\right) = \left|\bar{i}_1 \times \bar{i}_2\right|}$$

Using stator flux linkages instead of stator currents

$$\bar{\lambda}_{qds} = (L_{ls} + L_m)\bar{i}_{qds} + L_m \bar{i}'_{qdr}$$

$$\bar{\lambda}'_{qdr} = (L'_{lr} + L_m)\bar{i}'_{qdr} + L_m \bar{i}_{qds}$$

$$\bar{f}_{qd} = f_q - jf_d$$

$$\boxed{\operatorname{Im}\left(\bar{i}_1 \cdot \bar{i}^{*}_{2}\right) = \left|\bar{i}_1 \times \bar{i}_2\right|}$$

$$\tau = \frac{3}{2} pL_m \left(i_{qs} i'_{dr} - i_{ds} i'_{qr}\right) = \frac{3}{2} pL_m \left|\bar{i}_{qds} \times \bar{i}_{qdr}\right| = \frac{3}{2} pL_m \operatorname{Im}\left(\bar{i}_{qds} \cdot \bar{i}^{*}_{qdr}\right) \quad \text{......(1)}$$

$$\tau = \frac{3}{2} pL_m \left|\bar{i}_{qds} \times \bar{i}_{qdr}\right| = \frac{3}{2} p \frac{L_m}{L_s} \left|\bar{\lambda}_{qds} \times \bar{i}_{qdr}\right| = \frac{3}{2} p \frac{L_m}{L_s} \operatorname{Im}\left(\bar{\lambda}_{qds} \cdot \bar{i}^{*}_{qdr}\right) \quad \text{......(2)}$$

$$\boxed{\begin{aligned} L_s &= L_{ls} + L_m \\ L'_r &= L'_{lr} + L_m \end{aligned}}$$

$$\boxed{\begin{aligned} \bar{i}_{qds} \times \bar{i}_{qds} &= 0 \\ \bar{i}_{qdr} \times \bar{i}_{qdr} &= 0 \end{aligned}}$$

More other expressions:

$$\overline{\lambda}_{qds} = (L_{ls} + L_m)\overline{i}_{qds} + L_m \overline{i}'_{qdr}$$
$$\overline{\lambda}'_{qdr} = (L'_{lr} + L_m)\overline{i}'_{qdr} + L_m \overline{i}_{qds}$$

$$\Downarrow$$

$$\left|\overline{\lambda}_{qds} \times \overline{\lambda}'_{qdr}\right| = \left|L_s L'_r \overline{i}_{qds} \times \overline{i}'_{qdr} + L^2_m \overline{i}_{qdr} \times \overline{i}'_{qds}\right| = \left(L_s L'_r - L^2_m\right)\left|\overline{i}_{qds} \times \overline{i}'_{qdr}\right| = \left(L_s L'_r - L^2_m\right)\frac{\tau}{\frac{3}{2} p L_m}$$

$$\Downarrow$$

$$\tau = \frac{3}{2} p L_m \frac{1}{\left(L_s L'_r - L^2_m\right)}\left|\overline{\lambda}_{qds} \times \overline{\lambda}'_{qdr}\right| = \frac{3}{2} p L_m \frac{1}{\left(L_s L'_r - L^2_m\right)} \mathrm{Im}\left(\overline{\lambda}_{qds} \cdot \overline{\lambda}'^{*}_{qdr}\right) \qquad \text{......(3)}$$

$$\boxed{\overline{\lambda}_{qdm} = L_m\left(\overline{i}_{qds} + \overline{i}_{qdr}\right)}$$

$$\tau = \frac{3}{2} p L_m \left|\overline{i}_{qds} \times \overline{i}_{qdr}\right| = \frac{3}{2} p \left|\overline{\lambda}_{qdm} \times \overline{i}_{qdr}\right| = \frac{3}{2} p\, \mathrm{Im}\left(\overline{\lambda}_{qdm} \cdot \overline{i}^{*}_{qdr}\right) \qquad \text{......(4)}$$

and more ……

$$\tau = \frac{3}{2} p \,\mathrm{Im}\left(\bar{i}_{qds} \cdot \overline{\lambda}^{*}_{qdm} \right) \qquad \tau = \frac{3}{2} p \frac{L_m}{L_r^{'}} \mathrm{Im}\left(\bar{i}_{qds} \cdot \overline{\lambda}^{'*}_{qdr} \right)$$

$$\tau = \frac{3}{2} p \,\mathrm{Im}\left(\bar{i}_{qds} \cdot \overline{\lambda}^{*}_{qds} \right) \qquad \tau = \frac{3}{2} p \,\mathrm{Im}\left(\overline{\lambda}^{'}_{qdr} \cdot \bar{i}^{'*}_{qdr} \right)$$

Instantaneous torque for the IM – another view

$$u_{qs} = R_s i_{qs} + \frac{d}{dt}\lambda_{qs} + \omega_\theta \lambda_{ds} \qquad u'_{qr} = R'_r i'_{qr} + \frac{d}{dt}\lambda'_{qr} + (\omega_\theta - \omega_r)\lambda'_{dr}$$

$$u_{ds} = R_s i_{ds} + \frac{d}{dt}\lambda_{ds} - \omega_\theta \lambda_{qs} \qquad u'_{dr} = R'_r i'_{dr} + \frac{d}{dt}\lambda'_{dr} - (\omega_\theta - \omega_r)\lambda'_{qr}$$

$$\boxed{P_{inq}} = i_{qs} u_{qs} = \boxed{R_s i^2_{qs}} + \boxed{i_{qs}\frac{d}{dt}\lambda_{qs}} + \boxed{\omega_\theta \lambda_{ds} i_{qs}}$$

Output mechanical power

Rate change of the stored field energy, P5, P13

Copper loss

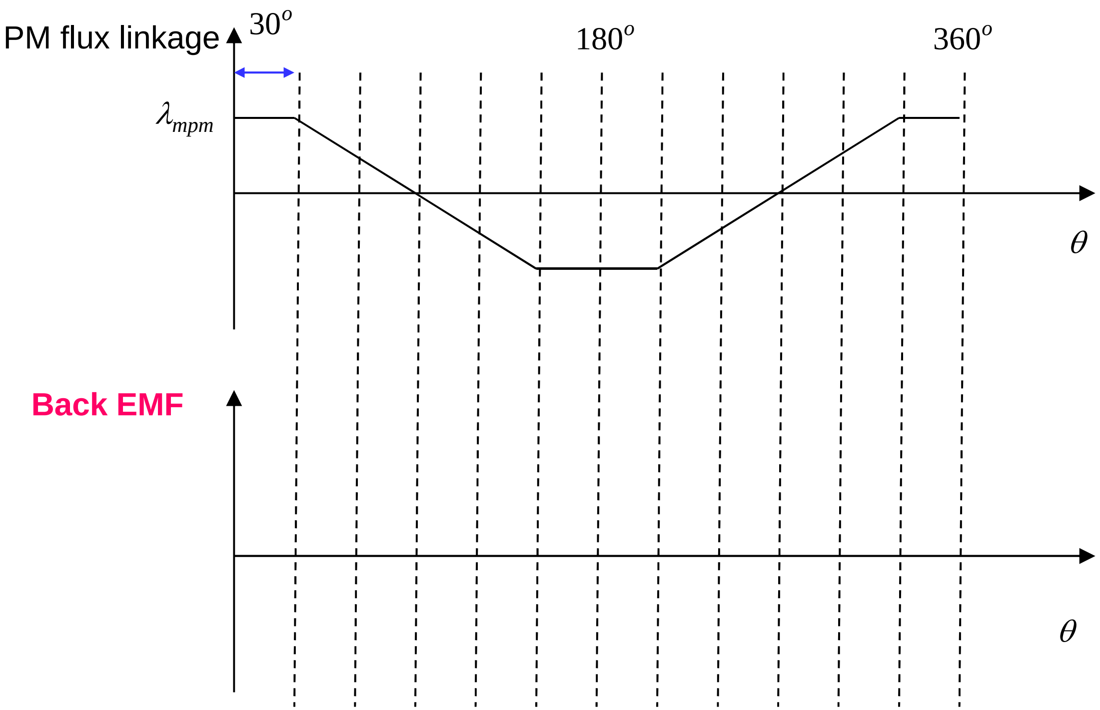
Input power of the q-axis winding

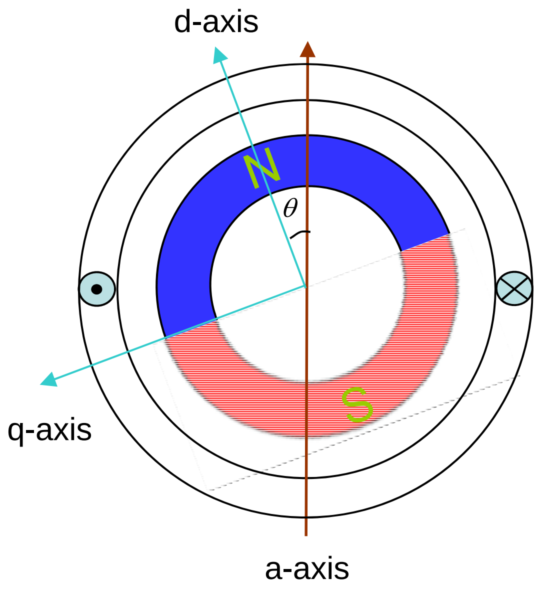
$$\frac{2}{3}P_{mec} = \omega_\theta(\lambda_{ds} i_{qs} - \lambda_{qs} i_{ds}) + (\omega_\theta - \omega_r)(\lambda'_{dr} i'_{qr} - \lambda'_{qr} i'_{dr}) = \cancel{\omega_\theta} L_m (i_{qs} i'_{dr} - i_{ds} i'_{qr}) + (\omega_r - \cancel{\omega_\theta}) L_m (i_{qs} i'_{dr} - i_{ds} i'_{qr})$$

$$\boxed{P_{mec} = \frac{3}{2}\omega_r L_m (i_{qs} i'_{dr} - i_{ds} i'_{qr}) \Longrightarrow \tau = \frac{3}{2} p L_m (i_{qs} i'_{dr} - i_{ds} i'_{qr})}$$

Exercises

1. A single-phase permanent magnet (PM) motor, illustrated as below.
   Suppose the PM flux linkage waveform as a function of the position looks like:





- <u>Sketch</u> the back EMF (differentiation of the PM flux linkage) waveform
- Find a proper current waveform (ideal waveform will be enough), for torque production and having a highest torque/Ampere ratio.
- Please give a analytical torque equation for this motor using armature current and PM flux linkage variables