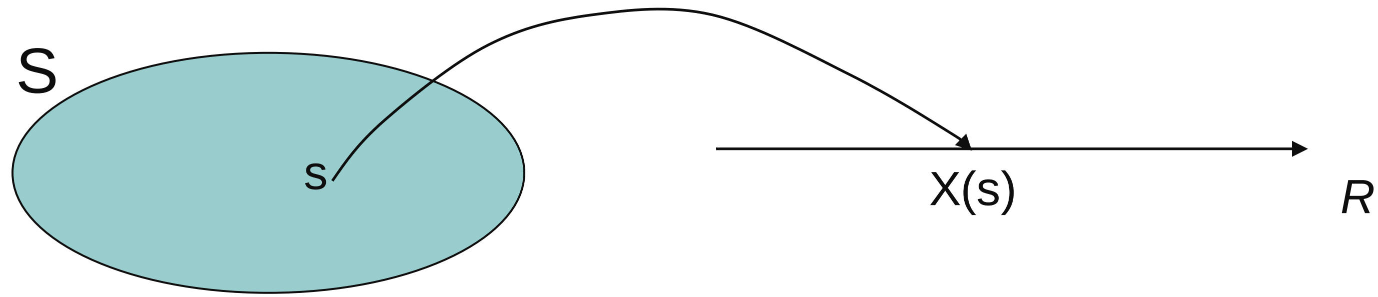
Probability theory
Random variables

In an experiment a number is often attached to each outcome.



Definition:
A **random variable** X is a function defined on S, which takes values on the real axis

$$X: \; S \rightarrow R$$

Sample space   Real numbers

Probability theory
Random variables

Example:

Random variable	Type	
Number of eyes when rolling a die	discrete	counting
The sum of eyes when rolling two dice	discrete	counting
Number of children in a family	discrete	counting
Age of first-time mother	discrete	counting
Time of running 5 km	continuous	measure
Amount of sugar in a coke	continuous	measure
Height of males	continuous	measure

Discrete: can take a finite number of values or an infinite but countable number of values.

Continuous: takes values from the set of real numbers.

Discrete random variable
Probability function

Definition:

Let $X : S \to R$ be a discrete random variable.

The function f(x) is a **probability function** for X, if

1. $f(x) \geq 0$ for all x
2. $\sum_{x} f(x) = 1$
3. $P(X = x) = f(x)$,

   where $P(X=x)$ is the probability for the outcomes $s \in S : X(s) = x$.

Discrete random variable
Probability function

Example: Flip three coins   X : # heads $X: S \to \{0,1,2,3\}$

Outcome	**Value of X**	**Probability function**
TTT	X=0	f(0) = P(X=0) = 1/8
HTT, TTH, THT	X=1	f(1) = P(X=1) = 3/8
HHT, HTH, THH	X=2	f(2) = P(X=2) = 3/8
HHH	X=3	f(3) = P(X=3) = 1/8

Notice! The definition of a probability function is fulfilled:

1. $f(x) \geq 0$
2. $\sum f(x) = 1$
3. $P(X=x) = f(x)$

Discrete random variable
Cumulative distribution function

Definition:
Let $X : S \to R$ be a discrete random variable with probability function $f(x)$.

The cumulative distribution function for X, $F(x)$, is defined by

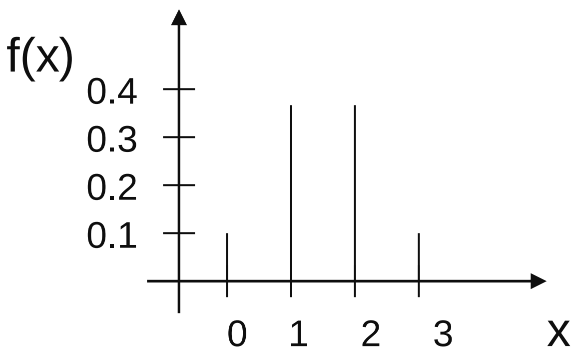
$$F(x) = P(X \le x) = \sum_{t \le x} f(t) \qquad \text{for } -\infty < x < \infty$$

Discrete random variable
Cumulative distribution function

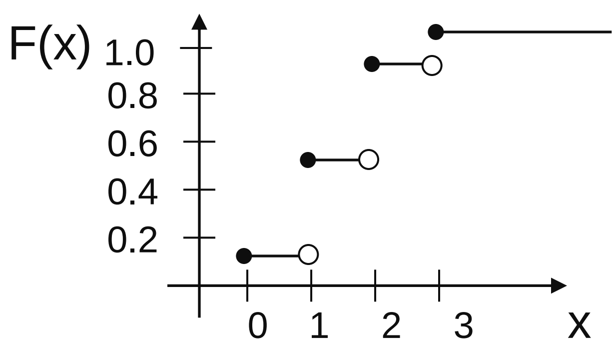
Example: Flip three coins X : # heads $X : S \rightarrow \{0,1,2,3\}$

Outcome	Value of X	Probability function	Cumulative dist. Func.
TTT	X=0	$f(0) = P(X=0) = 1/8$	$F(0) = P(X \leq 0) = 1/8$
HTT, TTH, THT	X=1	$f(1) = P(X=1) = 3/8$	$F(1) = P(X \leq 1) = 4/8$
HHT, HTH, THH	X=2	$f(2) = P(X=2) = 3/8$	$F(2) = P(X \leq 2) = 7/8$
HHH	X=3	$f(3) = P(X=3) = 1/8$	$F(3) = P(X \leq 3) = 1$

Probability function:



Cumulative distribution function:



Continuous random variable

A continuous random variable X has probability 0 for all outcomes!!

Mathematically: $P(X = x) = f(x) = 0$ for all x

Hence, we cannot represent the probability function $f(x)$ by a table or bar chart as in the case of discrete random variables.

Instead we use a continuous function – a density function.

Continuous random variable
Density function

Definition:

Let X: S $\to$ R be a continuous random variables.

A **probability density function** f(x) for X is defined by:

1. $f(x) \geq 0$ for all x
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$
3. $P(a < X < b) = \int_a^b f(x)\,dx$

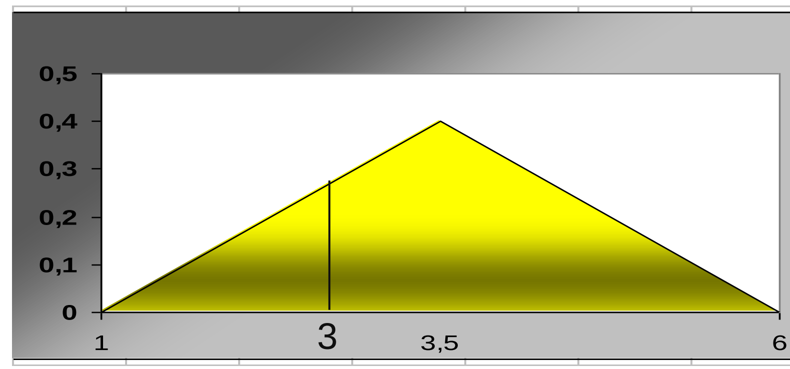
Note!! Continuity: $P(a < X < b) = P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b)$

Continuous random variable
Density function

Example: X: service life of car battery in years (continuous)

Density function:

$$f(x) = \begin{cases} -0.16 + 0.16x & \text{for} \quad 1 \le x \le 3.5 \\ 0.96 - 0.16x & \text{for} \quad 3.5 < x \le 6 \\ 0 & \text{otherwise} \end{cases}$$



Probability of a service life longer than 3 years:

$$P(X > 3) = \int_3^\infty f(x)\,dx$$

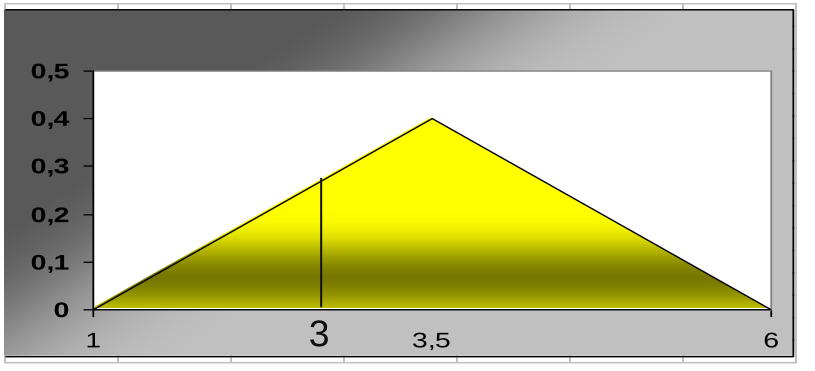
$$= \int_3^{3.5} (-0.16 + 0.16x)\,dx + \int_{3.5}^{6} (0.96 - 0.16x)\,dx$$

$$= \ldots = 0.68$$

Continuous random variable
Density function

Alternatively:

$$f(x)=\begin{cases} -0.16+0.16x & \text{for} \quad 1\le x\le 3.5 \\ 0.96-0.16x & \text{for} \quad 3.5< x\le 6 \\ 0 & \text{otherwise} \end{cases}$$



Probability of a service life longer than 3 years:

$$P(X>3)=1-P(X\le 3)$$

$$=1-\int_{-\infty}^{3} f(x)\,dx$$

$$=1-\int_{1}^{3}(-0.16+0.16x)\,dx$$

$$=\ldots=1-0.32=0.68$$

Continuous random variable

Density function

Definition:

Let $X : S \to R$ be a continuous random variable with density function f(x).

The cumulative distribution function for X, F(x), is defined by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt \quad \text{for } -\infty < x < \infty$$

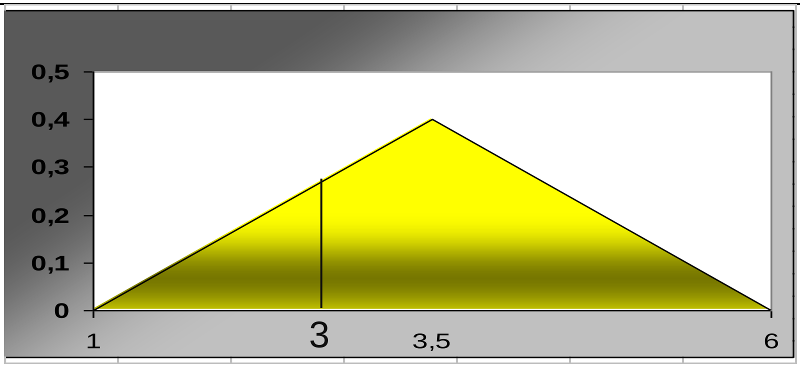
Note: $F'(x) = f(x)$

$$F(3) = P(X \leq 3)$$

$$= \int_{-\infty}^{3} f(x)\,dx$$

$$= \int_{1}^{3} -0.16 + 0.16x\,dx$$

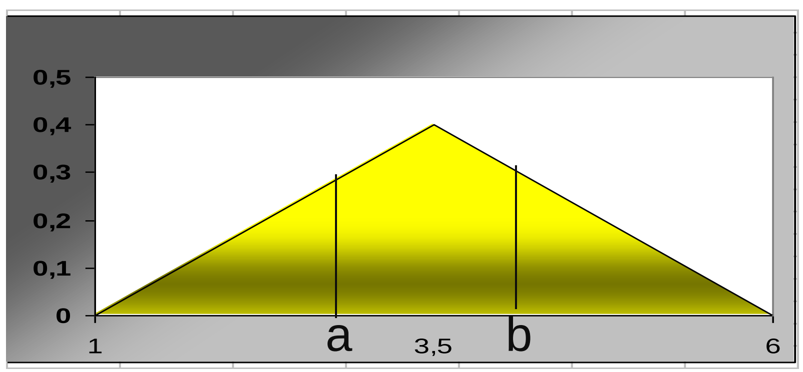
$$= \ldots = 0.32$$



Continuous random variable
Density function

From the definition of the cumulative distribution funct. we get:

$$P(\, a < X < b\,) = P(\, a \leq X \leq b\,) = P(\, a \leq X < b\,) = P(\, a < X \leq b\,)$$
$$= P(X \leq b\,) - P(X \leq a\,)$$
$$= F(b) - F(a)$$



Continuous random variable
Density function

Discrete random variable

- Sample space is finite or has countable many outcomes
- Probability function f(x) is often given by table
- Calculation of probabilities

$$P(a < X < b) = \sum_{a<t<b} f(t)$$

Continuous random variable

- The sample space contains infinitely many outcomes
- Density function f(x) is a continuous function
- Calculation of probabilities

$$P(a < X < b) = \int_a^b f(t)\, dt$$

Joint distribution
Joint probability function

Definition:
Let X and Y be two **discrete** random variables.
The **joint probability function** $f(x,y)$ for X and Y
is defined by

1. $f(x,y) \geq 0$ for all x and y
2. $\sum_x \sum_y f(x,y) = 1$
3. $P(X = x, Y = y) = f(x,y)$ (the probability that both X = x and Y=y)

For a set A in the xy-plane: $P((X,Y) \in A) = \sum \sum_{(x,y) \in A} f(x,y)$

Joint distribution
Marginal probability function

Definition:
Let X and Y be two **discrete** random variables with joint probability function f(x,y).
The **marginal probability function** for X is given by

$$g(x) = \sum_{y} f(x,y) \quad \text{for all } x$$

The **marginal probability function** for Y is given by

$$h(y) = \sum_{x} f(x,y) \quad \text{for all } y$$

Joint distribution
Marginal probability function

Example 3.16 (modified):
The joint probability function f(x,y) for X and Y is given by

y \ x	0	1	2
0	3/28	9/28	3/28
1	3/14	3/14	0
2	1/28	0	0

- $h(1) = P(Y = 1) = 3/14+3/14+0 = 3/7$
- $g(2) = P(X = 2) = 3/28+0+0 = 3/28$
- $P(X+Y < 2) = 3/28+9/28+3/14 = 18/28 = 9/14$

Joint distribution
Joint density function

Definition:
Let X and Y be two **continuous** random variables.
The **joint density function** f(x,y) for X and Y is defined by

1. $f(x,y) \geq 0$ for all x and y
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$
3. $P(a < X < b,\ c < Y < d) = \int_{c}^{d}\int_{a}^{b} f(x,y)\,dx\,dy$

For a region A in the xy-plane: $P[(X,Y)\in A] = \iint_A f(x,y)\,dxdy$

Joint distribution
Marginal density function

Definition:
Let X and Y be two **continuous** random variables with joint density function f(x,y).
The **marginal density function** for X is given by

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy \quad for\ all\ x$$

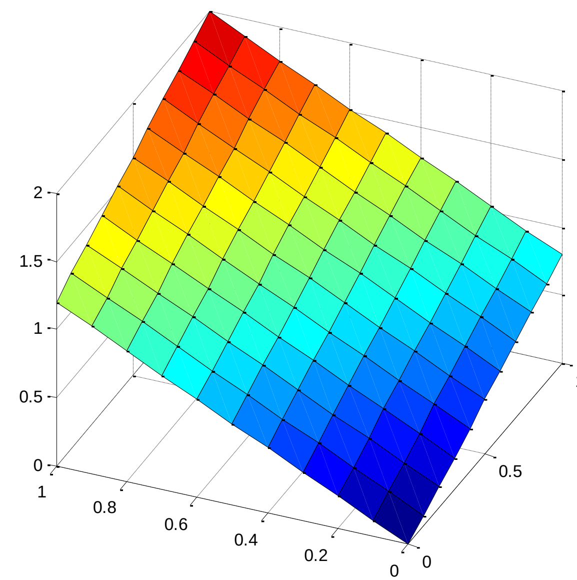
The **marginal density function** for Y is given by

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx \quad for\ all\ y$$

Joint distribution
Marginal density function

Example 3.15 + 3.17 (modified):
Joint density f(x,y) for X and Y:

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$



Marginal density function for X:

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_0^1 \frac{2}{5}(2x+3y)\, dy$$

$$= \left[\frac{2}{5} 2xy + \frac{1}{5} 3y^2 \right]_0^1 = \frac{4}{5} x + \frac{3}{5}$$

Joint distribution
Conditional density and probability functions

Definition:
Let X and Y be random variables (continuous or discrete) with joint density/probability function $f(x,y)$. Then the Conditional density/probability function for Y given X=x is

$$f(y|x) = f(x,y) / g(x) \quad g(x) \neq 0$$

where $g(x)$ is the marginal density/probability function for X, and the conditional density/probability function for X given Y=y is

$$f(x|y) = f(x,y) / h(y) \quad h(y) \neq 0$$

where $h(y)$ is the marginal density/probability function for Y.

Joint distribution
Conditional probability function

Examples 3.16 + 3.18 (modified):
Joint probability function
f(x,y) for X and Y is given by:

y \ x	0	1	2
0	3/28	9/28	3/28
1	3/14	3/14	0
2	1/28	0	0

- marginal pf. $g(x) = \begin{cases} \frac{10}{28} & \text{for} \quad x = 0 \\ \frac{15}{28} & \text{for} \quad x = 1 \\ \frac{3}{28} & \text{for} \quad x = 2 \end{cases}$

- P(Y=1 | X=1) = f(1|1)
  = f(1,1) / g(1)
  = (3/14) / (15/28)
  = 6/15

Joint distribution
Independence

Definition:
Two random variables X and Y (continuous or discrete) with joint density/probability functions $f(x,y)$ and marginal density/probability functions $g(x)$ and $h(y)$, respectively, are said to be independent if and only if

$$f(x,y) = g(x)\,h(y) \qquad \text{for all } x,y$$

or if $f(x|y) = g(x)$ (x indep. of y) or $f(y|x)=h(y)$ (y indep. of x)