Probability Theory and Statistics

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Literature:

Walpole, Myers, Myers & Ye:
Probability and Statistics for Engineers and Scientists,
Prentice Hall, 9th ed.

Slides and lecture overview:

http://ses.moodle.aau.dk/course/view.php?id=627

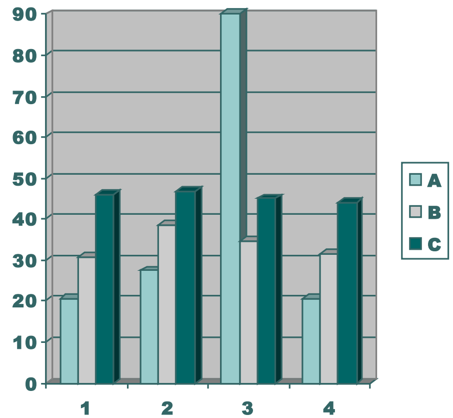
Lecture format:

2x45 min lecturing followed by exercises in group rooms

STATISTICS
What is it good for?

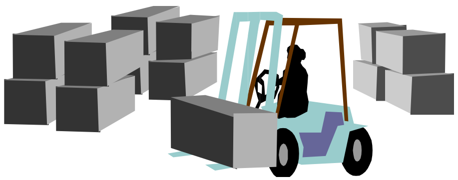
Forecasting:

- Expectations for the future?
- How will the stock markets behave??



Analysis of sales:

- How much do we sell, and when?
- Should we change or sales strategy?



Quality control:

- What is my rate of defective products?
- How can I best manage my production?
- What is the best way to sample?

STATISTICS
Why is it important?

Statistics is used in order to make inference based on data

Why do we need statistics/statistical models?

• The major reason for using statistics is variation(!) in the observed data

• Statistics is used to model the patterns of variation in order to learn from them.

Reasons for variation in data:

• The real-world problems is not completely predictable at levels of precision that are important for our goal (intrinsic variability of the real-world)

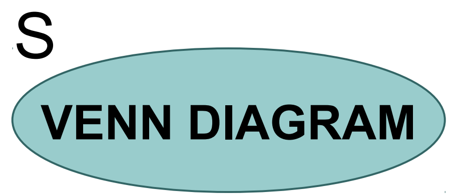
• Variability induced in the data collection process (measurement, “sampling”, errors)

Probability theory

Sample space and events

Consider an experiment

Sample space S:

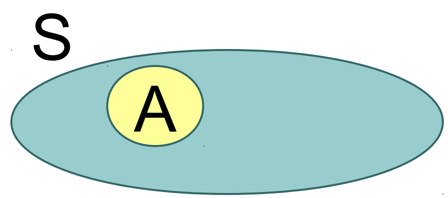


Example:

S={1,2,…,6} rolling a die

S={tail,head} flipping a coin

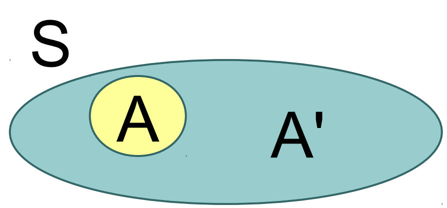
Event A:



Example:

A={1,6} when rolling a die

Complementary event A':



Example:

A'={2,3,4,5} rolling a die

Probability theory

Events

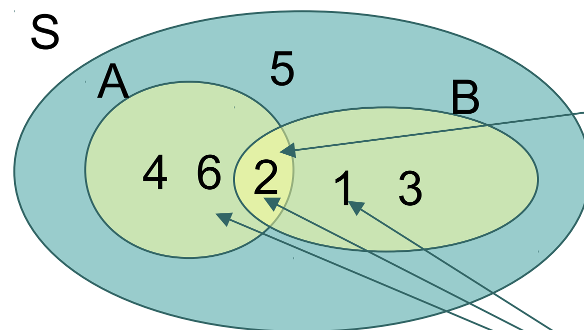
Example:

Rolling a die

$S=\{1,2,3,4,5,6\}$

$A=\{2,4,6\}$

$B=\{1,2,3\}$



Intersection:

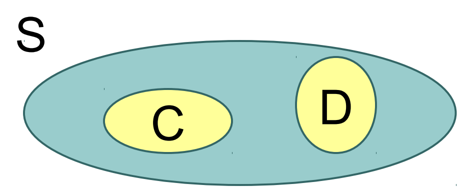
$A \cap B=\{2\}$

Union:

$A \cup B=\{1,2,3,4,6\}$

Disjoint events: $C \cap D = \emptyset$

$C=\{1,3,5\}$ and $D=\{2,4,6\}$ are disjoint



Probability theory

Counting sample points

Ways of placing your bets: Guess the results of 13 matches

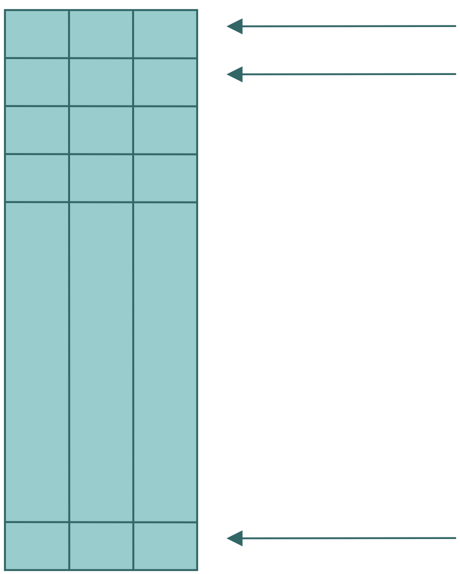
Possible outcomes:

Home win

Draw

Away win

1 X 2



3 possibilities

3 possibilities

.
.
.

3 possibilities

Answer: $3 \cdot 3 \cdot 3 \cdot \cdots \cdot 3 = 3^{13}$

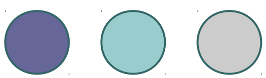
The multiplication rule

Probability theory

Counting sample points

Ordering *n* different objects

Number of permutations ???



There are

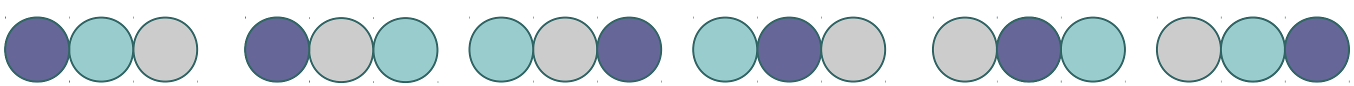
- n ways of selecting the first object
- n -1 ways of selecting second object

  ⋮
- 1 way of selecting the last object

"n factorial"

$n \cdot (n-1) \cdot \cdots \cdot 1 = n!$ ways

The multiplication rule



$3! = 6$

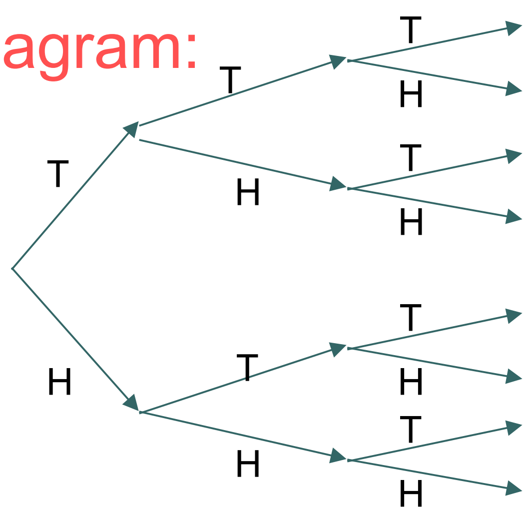
Probability theory
Counting sample points

Multiplication rule:

If k independent operations can be performed in $n_1, n_2, \ldots, n_k$ ways, respectively, then the k operations can be performed in

$$n_1 \cdot n_2 \cdot \cdots \cdot n_k \text{ ways}$$

Tree diagram:



Flipping a coin three times (Head/Tail)

$2^3 = 8$ possible outcomes

Probability theory
Counting sample points

Number of possible ways of selecting *r* objects from a set of *n* distinct elements:

	Without replacement	With replacement
Ordered	${}_{n}P_{r} = \frac{n!}{(n-r)!}$	n^{r}
Unordered	$\binom{n}{r} = \frac{n!}{r!(n-r)!}$	-

Probability theory
Counting sample points

Example:
Ann, Barry, Chris, and Dan should form a committee consisting of two persons, i.e. unordered without replacement.

Number of possible combinations:

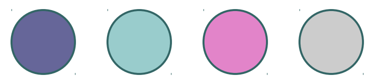
$$\binom{4}{2} = \frac{4!}{2!\,2!} = 6$$

Writing it out : AB AC AD BC BD CD

Probability theory

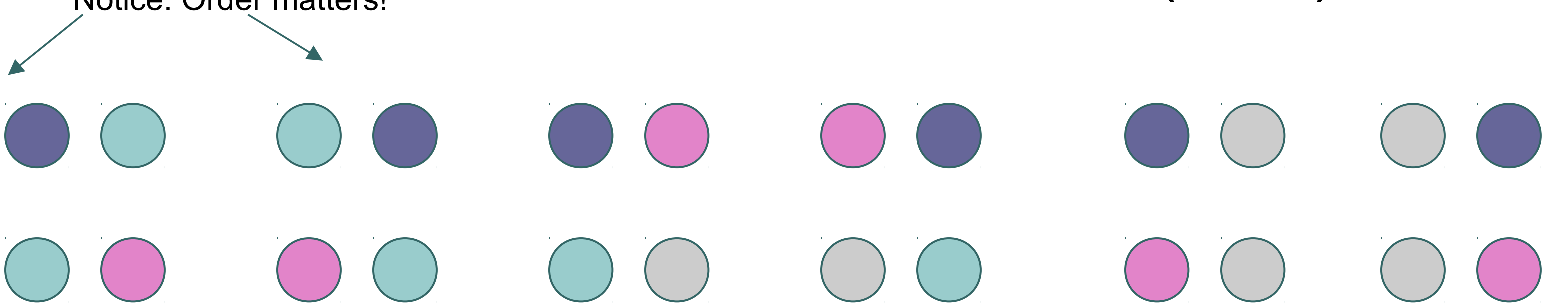
Counting sample points

Example:
Select 2 out of 4 different balls ordered and without replacement



Number of possible combinations: $_4P_2 = \dfrac{4!}{(4-2)!} = 12$

Notice: Order matters!

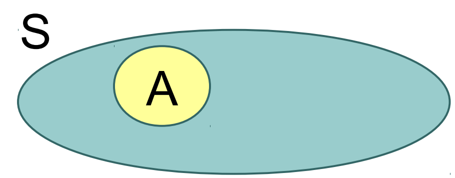


Probability theory

Probability

Let A be an event, then we denote

$P(A)$ the probability for A



It always hold that $0 \leq P(A) \leq 1$ $P(\emptyset) = 0$ $P(S) = 1$

Consider an experiment which has N equally likely outcomes, and let exactly n of these events correspond to the event A. Then

$$P(A) = \frac{n}{N} = \frac{\text{\# successful outcomes}}{\text{\# possible outcomes}}$$

Example:
Rolling a die

$$P(\text{even number}) = \frac{3}{6} = \frac{1}{2}$$

Probability theory
Probability

Example: Quality control
A batch of 20 units contains 8 defective units.

Select 6 units (unordered and without replacement).

Event A: no defective units in our random sample.

Number of possible samples: $N=\binom{20}{6}$ (# possible)

Number of samples without defective units: $n=\binom{12}{6}$ (# successful)

$$P(A)=\frac{\binom{12}{6}}{\binom{20}{6}}=\frac{12!\,6!\,14!}{6!\,6!\,20!}=\frac{77}{3230}=0.024$$

Probability theory
Probability

Example: continued

Event B: exactly 2 defective units in our sample

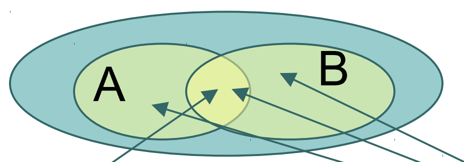
Number of samples with exactly 2 defective units: $n = \binom{12}{4}\binom{8}{2}$

(# successful)

$$P(B) = \frac{\binom{12}{4}\binom{8}{2}}{\binom{20}{6}} = \frac{\frac{12!}{4!8!}\,\frac{8!}{2!6!}}{\frac{20!}{6!14!}} = \frac{12!8!6!14!}{4!8!2!6!20} = 0.3576$$

Probability theory

Rules for probabilities



Intersection: $A \cap B$

Union: $A \cup B$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(B) = P(B \cap A) + P(B \cap A')$$

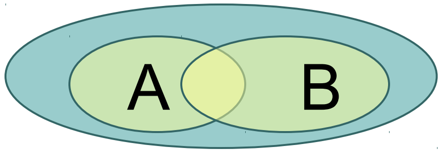
If A and B are disjoint: $P(A \cup B) = P(A) + P(B)$

In particular: $P(A) + P(A') = 1$

Probability theory
Conditional probability

Conditional probability for A given B:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$ where P (B) > 0



Bayes' Rule: $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$

Rewriting Bayes' rule:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A')P(A')}$$

Probability theory
Conditional probability

Example page 63:
The distribution of employed/unemployed amongst men and women in a small town.

	Employed	Unemployed	Total
Man	460	40	500
Woman	140	260	400
Total	600	300	900

$$P(\text{man}|\text{employed}) = \frac{P(\text{man \& employed})}{P(\text{employed})} = \frac{460/900}{600/900} = \frac{460}{600} = \frac{23}{30} = 0.767$$

$$P(\text{man}|\text{unemployed}) = \frac{P(\text{man \& unemployed})}{P(\text{unemployed})} = \frac{40/900}{300/900} = \frac{40}{300} = \frac{2}{15} = 0.133$$

Probability theory

Bayes' rule

Example: Lung disease & Smoking

According to ”The American Lung Association” 7% of the population suffers from a lung disease, and 90% of these are smokers. Amongst people without any lung disease 25.3% are smokers.

Events:

A: person has lung disease

B: person is a smoker

Probabilities:

P(A) = 0.07

P(B|A) = 0.90

P(B|A´) = 0.253

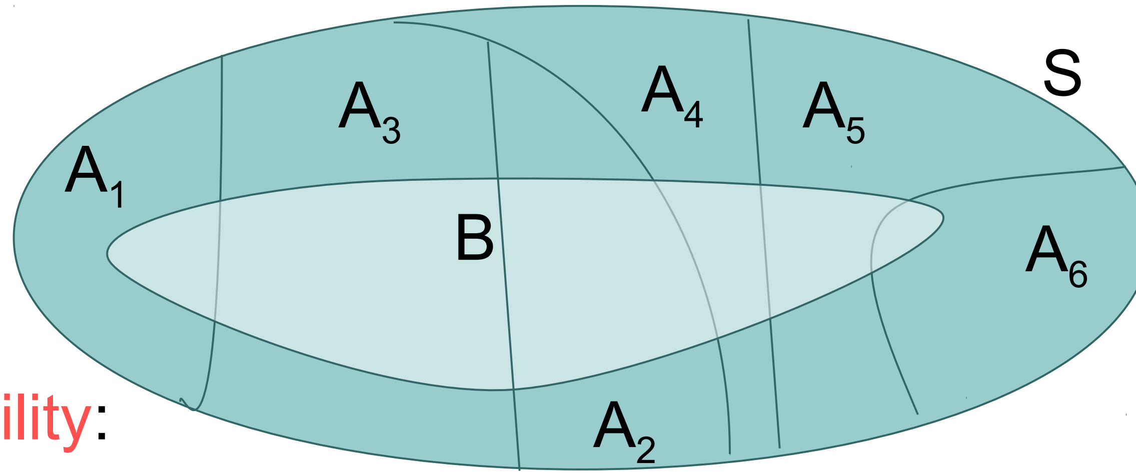
What is the probability that at smoker suffers from a lung disease?

$$P(A|B)=\frac{P(B|A)P(A)}{P(B|A)P(A)+P(B|A')P(A')}=\frac{0.9\cdot 0.07}{0.9\cdot 0.07+0.253\cdot 0.93}=0.211$$

Probability theory

Bayes' rule – extended version

$A_1, \ldots, A_k$ are a partitioning of S



Law of total probability:

$$P(B) = \sum_{i=1}^{k} P(B|A_i)\,P(A_i)$$

Bayes' extended rule:

$$P(A_r|B) = \frac{P(B|A_r)\,P(A_r)}{\sum_{i=1}^{k} P(B|A_i)\,P(A_i)}$$

Probability theory
Independence

Definition:
Two events A and B are said to be independent if and only if

$$P(B|A) = P(B) \quad \text{or} \quad P(A|B) = P(A)$$

Alternative Definition:
Two events A and B are said to be independent if and only if

$$P(A \cap B) = P(A)P(B)$$

Notice: Disjoint event (mutually exclusive event) are dependent!

Probability theory
Conditional probability

Example:

	Employed	Unemployed	Total
Man	460	40	500
Woman	140	260	400
Total	600	300	900

$$P(\text{man} \mid \text{employed}) = \frac{460/900}{600/900} = 0.767$$

$$P(\text{man}) = 500/900 = 0.556$$

Conclusion: the two events “man” and “employed” are dependent.

Probability theory
Rules for conditional probabilities

Probability of events A and B happening simultaneously

$$P(A \cap B) = P(A|B)\,P(B)$$

Probability of events A, B and C happening simultaneously

$$P(A \cap B \cap C) = P(A|B \cap C)\,P(B|C)\,P(C)$$

Proof:

$$P(A \cap B \cap C) = P(A|B \cap C)\,P(B \cap C) = P(A|B \cap C)\,P(B|C)\,P(C)$$

General rule:

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) =$$
$$P(A_1|A_2 \cap \cdots \cap A_k) \cdot P(A_2|A_3 \cap \cdots \cap A_k) \cdots P(A_{k-1}|A_k)\,P(A_k)$$