Re-eksamination 22. Februar 2013 TOA.

Dynamic Models of Electrical Machines and Control Systems.

Problem 4:

Given (*) $\frac{d^2x}{dt^2} + \left(f(x) + x\frac{dx}{dt}\right)\frac{dx}{dt} + g(x) = 0$

$f: R \to R$, $g: R \to R$ C^1-functions

$f(x) > 0 \quad \forall x \in R$; $xg(x) > 0 \quad \forall x \in R \setminus \{0\}$

1: state space formulation (**)

$x_1 = x$, $x_2 = dx/dt \Rightarrow \dot{x}_2 = \frac{d^2x}{dt^2} = -\left(f(x) + x\frac{dx}{dt}\right)\frac{dx}{dt} - g(x)$

i.e. $\underline{\dot{x}} = \underline{f}(\underline{x})$; $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}x_2\\ -(f(x_1) + x_1x_2)x_2 - g(x_1)\end{bmatrix}$ (**)

given $V(x_1, x_2) = \frac{1}{2}x_2^2\varphi(x_1) + \int_0^{x_1}\varphi(u)g(u)\,du$;

show that $V(\underline{x})$ is positive definite (PD) for any C^1-function $\varphi: R \to R_+$

- first term is positive and only zero for x_2 zero.
- for the second term $\int_0^{x_1}\varphi(u)g(u)\,du = x_1 \cdot \varphi(\eta)\cdot g(\eta)$ where η is between 0 and x_1 for $x_1 \neq 0$ it follows that the integral is positive for $0 < |x_1|$ so $V(\underline{x})$ is positive definite.

(the second term is only zero for $x_1 = 0$).

- The derivative of $V(\underline{x})$ is

$$\dot V(\underline{x}) = x_2 \dot x_2 \varphi(x_1) + \tfrac{1}{2} x_2^2 \dot\varphi(x_1)\dot x_1 + \varphi(x_1) g(x_1)\cdot \dot x_1$$

$$= x_2\left(-g(x_1) - (f(x_1) + x_1 x_2) x_2\right)\varphi(x_1) + \tfrac{1}{2} x_2^2 \dot\varphi(x_1) x_2$$

$$+ x_2 \varphi(x_1) g(x_1)$$

$$= x_2^3\left(\tfrac{1}{2}\dot\varphi(x_1) - x_1\varphi(x_1)\right) - x_2^2 f(x_1)\varphi(x_1)$$

The last term is negative semi-definite. (NSD) so if $\varphi(x_1)$ satisfy

$\frac{1}{2}\dot\varphi(x_1) - x_1\varphi(x_1) = 0$ $V(\underline{x})$ is a Lyapunov-function.

The equation $\frac{1}{2}\dot\varphi(x_1) - x_1\varphi(x_1) = 0$ is a 1. order ordinary differential equation that can be solved by separation of variables.

$$\frac{1}{2}\frac{d\varphi}{dx_1} = x_1\varphi \quad \Rightarrow \quad \frac{1}{\varphi}d\varphi = 2x_1 dx_1 \quad \Rightarrow$$

$$\ln|\varphi| = x_1^2 + c \quad \Rightarrow \quad \underline{\varphi = e^{(x_1^2 + c)}}, \ c \text{ arbitrary constant.}$$

2: Show that (0,0) is an asymptotically stable singular point for (**)

- need only to show that (0,0) is a singular point while $V(\underline{x})$ is PD and $\dot V(\underline{x})$ is NSD. This shows stability.

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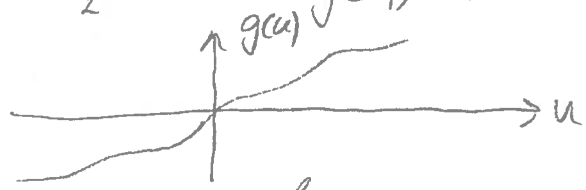
singular points are found by

$\underline{0} = \underline{f}(\underline{x}^0)$ i.e. $x_2 = 0 \wedge g(x_1) = 0$

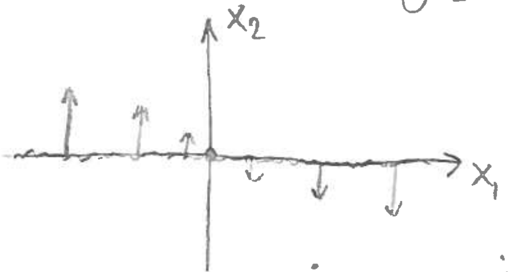


while $ug(u) > 0$
it follows from
continuity that $g(u) = 0$ for $u = 0$ so
$\underline{0}$ is a stable singular point.

To show the asymptotic behaviour we use
invariant set theory.

The set $E = \{(x_1, x_2) \in \mathbb{R} \mid \dot{V}(x_1, x_2) = 0\}$ is
given by.

$0 = -x_2^2 f(x_1) \cdot \varphi(x_1) \Rightarrow E = \{\underline{x} \in \mathbb{R} \mid x_2 = 0\}$.



on $x_2 = 0$ we have
$\dot{x}_1 = 0 \wedge \dot{x}_2 = -g(x_1)$

$\dot{x}_1 = 0 \wedge \dot{x}_2 = 0$ only at $\underline{x}^0$ being the
largest invariant set so all solutions will
converge asymptotically to $\underline{x} = \underline{0}$.

3. for (0,0) being a globally asymptotically
stable singular point we need

$V(\underline{x}) \to \infty$ for $\|\underline{x}\| \to \infty$.

i.e. $\int_0^{x_1} \varphi(u) g(u) du$ being unbounded as $|x_1| \to \infty$

follows directly from question 1

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