A Simple and General Approach to Determination of Self and Mutual Inductances for AC machines

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Abstract— **Modelling of AC electrical machines plays an important role in electrical engineering education related to electrical machine design and control. One of the fundamental requirements in AC machine modelling is to derive the self and mutual inductances, which could be position dependant. Theories developed so far for inductance determination are often associated with complicated machine magnetic field analysis, which exhibits a difficulty for most students. This paper describes a simple and general approach to the determination of self and mutual inductances of different types of AC machines. A new vector projection method is proposed. In this method, integration of the magnetic field may simply be replaced by vector projection of the field vector. Inductance may then be obtained by observing the magnetic field axes and the phase winding axes. Examples of applying the proposed method for inductance determination are given for a 3-phase, salient-pole synchronous machine, and an induction machine.**

Index Terms — **inductance determination, synchronous machines, induction machines.**

I. INTRODUCTION

Efforts for improving the teaching of electrical machine related courses are often laid on computer-aided simulation and analysis [1]~[5], or purpose-built teaching equipments [6]~[9]. One of the teaching purposes is to enable the students to have a good understanding of the theory, which should enable them to be able to apply properly the theory in different circumstances. The theory-learning stage is unavoidable especially for engineering students, and where, teaching equipments can do little help. To enable the students to have a deep understanding of the theory, it is always preferred that the access to the theory could be made simpler and straightforward to its essential part.

Courses on AC machine modelling are important fundamental courses for students interested in electrical machine design and control. In modelling of electrical machines, the first important task is to determine the self and mutual inductances. Theories developed so far for inductance determination are often associated with integration of magnetic field in the air gap [10], which is not easy to grasp, especially when it comes to position dependant inductances. The dependence of the inductance on the position is not made clear and straightforward in the classical method.

In this paper, a new approach based on vector projection method is introduced for self and mutual inductance determination of different types of AC machines. By using the vector projection method, the influence of the position on the inductance is made clear. Integration of the magnetic field may simply be realized by vector projection of the field vector. Inductance may then be obtained by observing the magnetic field axes and phase winding axes. Determination of the complicated inductance matrices for different types of electrical machines becomes very easy. The proposed method is a general approach, so the same procedure can be used for different types of machines for a quick inductance determination.

In this paper, section II describes the theory behind the vector projection method for inductance determination. A synchronous machine with salient rotor poles is used as an example to demonstrate this method. In section III, application of this method to a 3-phase induction machine is briefly described. Section IV concludes this paper.

II. THEORY FOR INDUCTANCE DETERMINATION USING THE VECTOR PROJECTION METHOD FOR SYNCHRONOUS MACHINES

A. *Basic principle of vector projection method*

Inductance is defined as the ratio of flux linkage vs. current. To calculate the inductance, the flux linkage needs to be determined first. Consider a simple example where the task is to calculate the flux linking the stator phase-a winding of a 3-phase, salient-pole, synchronous machine with full-pitch coils. The flux in the air gap is assumed to be established by the rotor excitation coil. The magnetic field in the air gap linking the stator winding may be assumed to be sinusoidal [10]. A schematic graph for an arbitrary rotor position θ_r, showing the air gap field and the relative rotor position with respect to the winding axis, is shown in Fig. 1.

According to Fig. 1, the flux linking phase-a winding (assuming it has 1-turn) may be calculated by:

$$\lambda_a = L \times \int_{-\pi/2}^{\pi/2} B_m \cos(\gamma - \theta_r) R_g \, d\gamma = \lambda_m \cos(\theta_r) \tag{1}$$

where R_g is the air gap radius and L is the stack length. B_m is the maximum air gap flux density. Angle γ is the space angle measured with respect to phase-a axis. In (1), the maximum flux linkage value λ_m is achieved at position $\theta_r = 0$, and it may be obtained from the integral calculation (1) as:

$$\lambda_m = \frac{2}{\pi} B_m \tau L \tag{2}$$

where τ is the pole pitch, which equals πR_g for a two-pole machine.

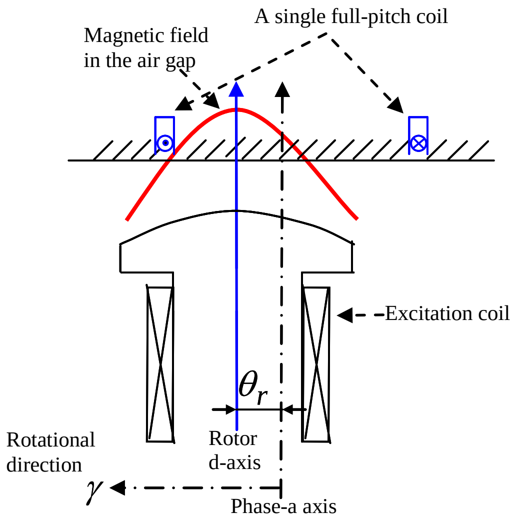

Fig. 1 A schematic graph showing the relative position of the air gap field with respect to the stator phase axis at an arbitrary rotor position.

The flux linkage expression of (1) is obtained from a space integration of the air gap flux density waveform. The same result may be achieved by using the vector projection method. Considering a vector that is aligned with the flux axis (where the peak flux value is), which is $e^{j\theta_r}$ for the position shown in Fig. 1. The axis of phase-a is fixed at zero degrees, which is e^{j0^o} expressed in the vector form. When the flux axis is aligned with the phase-a axis, the flux linking phase-a is at its maximum value (λ_m). For an arbitrary rotor position θ_r, the position dependant term in the phase-a flux linkage expression may be obtained by projecting the flux axis to the phase-a axis, as:

$$\lambda_a = \lambda_m \operatorname{Re}\left(\frac{e^{j\theta_r}}{e^{j0^o}}\right) = \lambda_m \cos\theta_r \tag{3}$$

Equation (3) gives the same flux linkage expression as that obtained from the space integration of the sinusoidal magnetic field (1). The vector projection method gives a much easier way to determine the position dependant term in the flux linkage expressions. For example, being aware of that phase-b axis is located at e^{j120^o}, the flux linkage for phase-b may be easily found to be:

$$\lambda_b = \lambda_m \operatorname{Re}\left(\frac{e^{j\theta_r}}{e^{j120^o}}\right) = \lambda_m \cos\left(\theta_r - 120^o\right) \tag{4}$$

Using the vector projection method, space integration of the magnetic field in the air gap is avoided in the process of obtaining the flux linkage from the air gap magnetic field waveform. Knowing the flux axis and the axis of the phase winding, the position dependant term in the flux linkage expression may be quickly determined.

B. Vector projection method used for inductance determination

The inductance is defined as the ratio of the flux linkage vs. current. The magnetic field in the air gap in the previous discussions is established by the current in the rotor excitation coil. The peak flux linkage value (2) will then be proportional to this rotor current if saturation effect is neglected. Therefore, by dividing the flux linkages (1) and (4) by this rotor current, the mutual inductances between the rotor coil and stator phase windings may be found. These mutual inductances will have the same peak value but differed by 120 electrical degrees, in a manner similar to the flux linkage expressions.

For determining the rotor to stator mutual inductance, it can be always assumed that the field in the air gap is established by the rotor coil. If it is the d-axis rotor coil generates the air gap flux, the flux axis will then always be on the d-axis. But to study the stator self and mutual inductances, the field in the air gap is established by the current in a selected stator phase winding. The axis of the MMF for each phase is always on its phase axis. This phase MMF waveform, which is normally sinusoidal [11], is modulated by the air gap to generate the air gap magnetic field. Therefore, determination of the air gap magnetic field waveform due to a phase winding current requires the knowledge of the air gap reluctance profile. The magnetic field waveform obtained is then used in the integration (as (1)) to find the flux linkage and consequently the inductance in the traditional method [10]. This approach is complicated and may be greatly simplified by adopting the vector projection concept.

The MMF produced by e.g., phase-a current, is aligned with the phase-a axis. This MMF may be decomposed intro two components – the d-component on the rotor d-axis and the q-component on the rotor q-axis, as illustrated in Fig. 2.

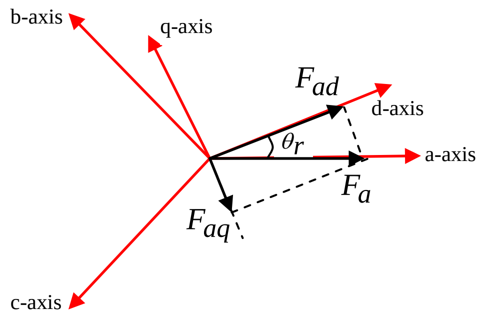

Fig. 2 Decomposition of the phase-a MMF into its d, q-axes components.

If the MMF for phase-a is denoted as F_a, by using the vector projection principle, it is easy to find that its corresponding d, q-components (F_{ad}, F_{aq}) will be:

$$F_{ad} = F_a \operatorname{Re}\left(\frac{e^{j0^o}}{e^{j\theta_r}}\right) = F_a \cos(\theta_r) \tag{5}$$

$$F_{aq} = F_a \operatorname{Re}\left(\frac{e^{j0^o}}{e^{j\left(\theta_r + 90^o\right)}}\right) = -F_a \sin(\theta_r) \tag{6}$$

where $e^{j\theta_r}$, $e^{j(\theta_r+90^o)}$, e^{j0^o} represents the location of the rotor d-axis, the rotor q-axis and the phase-a axis, respectively.

The d-axis MMF component will produce an air gap flux component centred on the d-axis. The corresponding flux linking a selected phase winding may be found by following the vector projection method discussed in subsection A. For example, the flux linkage of phase-a due to the d-axis flux component may be found to be:

$$\lambda_{ad} = \frac{F_{ad}}{\Re_d}\mathrm{Re}\left(\frac{e^{j\theta_r}}{e^{j0^o}}\right) = \frac{F_a}{\Re_d}\mathrm{Re}\left(\frac{e^{j0^o}}{e^{j\theta_r}}\right)\mathrm{Re}\left(\frac{e^{j\theta_r}}{e^{j0^o}}\right) = \frac{F_a}{\Re_d}\cos^2\theta_r \tag{7}$$

where $\Re_d$ is the reluctance experienced by the d-axis flux. Apparently, observed from (7), $F_a/\Re_d$ gives the maximum phase-a flux linkage value produced by the d-axis flux component (assuming the winding has 1-turn). This maximum value is achieved when the d-axis is aligned with the phase-a axis ($\theta_r = 0^o$). Similarly, the flux linkage of phase-a produced by the q-axis flux component may be found as:

$$\lambda_{aq} = \frac{F_{aq}}{\Re_q}\mathrm{Re}\left(\frac{e^{j(\theta_r+90^o)}}{e^{j0^o}}\right) = \frac{F_a}{\Re_q}\sin^2\theta_r \tag{8}$$

The total flux linking phase-a will be the sum of (7) and (8), which gives:

$$\lambda_a = F_a\left(\frac{1}{\Re_q}\cos^2\theta_r + \frac{1}{\Re_q}\sin^2\theta_r\right) \tag{9}$$

Assuming the winding has one-turn, the MMF of phase-a may be related to the phase current I_a as:

$$F_a = k_w I_a \tag{10}$$

where k_w is the winding factor. Dividing (9) by phase-a current I_a and assuming these is no leakage inductance present, the self-inductance of phase-a, denoted as L_{aa}, may be found as:

$$L_{aa} = L_{aad}\cos^2\theta_r + L_{aaq}\sin^2\theta_r \tag{11}$$

where L_{aad} and L_{aaq} are the d-axis inductance and the q-axis inductance respectively. The physical meaning of those two inductance components is clear. Observed from (11), when the rotor d-axis is aligned with phase-a axis ($\theta_r = 0^o$), the self-inductance of phase-a becomes L_{aad} and when the rotor q-axis is aligned with phase-a axis ($\theta_r = 90^o$), the phase-a self-inductance becomes L_{aaq}.

The above analysis gives a simple way for inductance determination. All the stator side inductances will have a similar form, which consists of a d-axis inductance component and a q-axis inductance component. The d (or q)-axis inductance contains a fundamental inductance term, L_{aad} (or L_{aaq}), which is determined by the d (or q)-axis reluctance and the winding factor. They are constants for a given machine. The d (or q)-axis inductance contains also two position dependant terms. One of them is due to the projection of the phase MMF to the d (or q)-axis. The other position dependant term appears in the process of obtaining the phase flux linkage from the d (or q)-axis flux component, as discussed in subsection A. For example, to study the mutual inductance between phase-a and b, it is understood that it contains a d-axis inductance component, $M_{ab,d}$, and a q-axis inductance component, $M_{ab,q}$. The d-axis component contains a constant inductance term, L_{aad} and two position dependant terms. The first position dependant term generated by projecting phase-a MMF to d-axis may be found by the vector projection method, as $\mathrm{Re}\left(e^{j0^o}/e^{j\theta_r}\right) = \cos\theta_r$. The second position dependant term accounting for the d-axis flux linking phase-b, may be found as $\mathrm{Re}\left(e^{j\theta_r}/e^{j120^o}\right) = \cos(\theta_r - 120^o)$. Therefore, the d-axis inductance component in the mutual inductance between phase-a and b becomes:

$$M_{ab,d} = L_{aad}\cos\theta_r\cos(\theta_r - 120^o) \tag{12}$$

The q-axis inductance component involved in the mutual inductance between phase-a and b may be obtained in a similar way. Thus the complete expression of the mutual inductance may be found as:

$$M_{ab} = L_{aad}\cos\theta_r\cos(\theta_r - 120^o) + L_{aaq}\sin\theta_r\sin(\theta_r - 120^o) \tag{13}$$

It is interesting to notice that if the following definitions of new inductances are used:

$$L_1 = \frac{(L_{aad} + L_{aaq})}{2} \tag{14}$$

$$L_2 = \frac{(L_{aad} - L_{aaq})}{2} \tag{15}$$

equation (13) may be transformed to a popular form that is used in many text books (e.g. [10]):

$$M_{ab} = -\frac{1}{2}L_1 - L_2\cos(2\theta_r + 60^o) \tag{16}$$

By using the vector projection method, it has been illustrated that determination of the position dependant inductance becomes very simple. The information needed is the location of the d, q-axes and the phase axes.

III. VECTOR PROJECTION METHOD APPLED TO AN INDUCTION MACHINE

The same principle may be applied to an induction machine for determining different inductances. An induction machine has a uniform air gap. For an arbitrarily defined d, q-axes, the d, q-axes reluctances are equal. It is convenient to define that $L_{aad} = L_{aaq} = L_m$. By making L_{aad} and L_{aaq} equal, the stator self-inductance, as (11), becomes a constant (L_m) and the mutual inductance, as (13) becomes half of the self-inductance ($-L_m/2$). The position dependant inductances for an induction machine will be the mutual inductances between the rotor and the stator windings.

To determine the position dependant terms in the mutual inductance, the information of the location of the phase axes and the d, q-axes should be known, which is illustrated in Fig. 3.

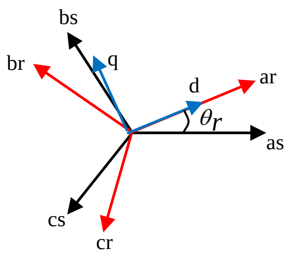

Fig. 3 Illustration of the phase axes for a 3-phase induction machine.

For example, for the mutual inductance between rotor phase-b and stator phase-c, their phase axes may be represented by $e^{j(\theta_r+120^o)}$ (rotor phase-b) and $e^{j(-120^o)}$ (stator phase-c). By following the same method described in the previous section, this mutual inductance contains a d-axis component and a q-axis component. Noticing that by exchanging the positions of the numerator and the denominator, the vector projection calculation will give the same result, as $\mathrm{Re}\left(e^{-j120^o}/e^{j\theta_r}\right) = \mathrm{Re}\left(e^{j\theta_r}/e^{-j120^o}\right)$. Therefore, the d-axis inductance component of this mutual inductance between rotor phase-b and stator phase-c may be found as:

$$M_{csbr,d} = L_{aad}\,\mathrm{Re}\left(\frac{e^{j\theta_r}}{e^{-j120^o}}\right)\mathrm{Re}\left(\frac{e^{j\theta_r}}{e^{j(\theta_r+120^o)}}\right) = L_{aad}\cos\left(\theta_r+120^0\right)\cos\left(120^0\right) \tag{17}$$

The q-axis inductance component may be found in a similar way and the complete mutual inductance expression becomes:

$$M_{csbr} = L_{aad}\cos\left(\theta_r+120^0\right)\cos\left(120^0\right) + L_{aad}\sin\left(\theta_r+120^0\right)\sin\left(120^0\right) \tag{18}$$

Noticing that $L_{aad} = L_{aaq} = L_m$, the mutual inductance (18) may finally be simplified to:

$$M_{csbr} = L_m\cos\left(\theta_r-120^0\right) \tag{19}$$

This is the known mutual inductance expression for rotor phase-b and stator phase-c. The expression of the inductance (19) means that at rotor position $\theta_r = 120^o$, this mutual inductance reaches its maximum value. This is corresponding to the situation that rotor phase-b axis is fully aligned with the stator phase-c axis, as may be observed from Fig. 3.

IV. CONCLUSION

This paper has presented a simple method for determination of the position dependant inductances for different types of electrical machines. By using the vector projection concept, the only information needed for inductance determination is the location of the d, q-axes, and the phase axes. The method proposed in this paper is a simple and general approach for a quick determination of the inductance expressions. Compared to the traditional method, knowledge of the air gap flux waveform is not needed and space integration of the air gap flux is avoided.

REFERENCES

[1] T.-F. Chan, "Analysis of electrical machines using Symphony," IEEE Trans. Educ., vol. 35, no. 1, pp. 76–82, Feb. 1992.

[2] N. A. Demerdash, Z. Luo, M. A. Alhamadi, B. T. Mattingly, "Teaching electric machinery and associated electromagnetic fields-a case for the benefits of academic computing", IEEE Transactions on Education, Vol. 36, Issue 2, May 1993, pp. 240 – 249.

[3] M. H. Nehrir, F. Fatehi, and V. Gerez, "Computer modeling for enhancing instruction of electric machinery," IEEE Trans. Educ., vol. 38, no. 2, pp. 166–170, May 1995.

[4] K. A. Nigim, R. R. DeLyser, "Using MathCad in understanding the induction motor characteristics", IEEE Transactions on Education, Vol. 44, Issue 2, May 2001, pp. 165 – 169.

[5] S. Ayasun, C. O. Nwankpa, "Induction motor tests using MATLAB/Simulink and their integration into undergraduate electric machinery courses", IEEE Transactions on Education, Vol. 48, Issue 1, Feb 2005, pp. 37 – 46.

[6] T. F. Schubert, F. G. Jacobitz, E. M. Kim, "Exploring the Basic Principles of Electric Motors and Generators With a Low-Cost Sophomore-Level Experiment", IEEE Transactions on Education, Vol. 52, Issue 1, Feb. 2009, pp. 57 – 65.

[7] S. A. Shirsavar, Benjamin A. Potter, and Isabel M. L. Ridge, "Three-Phase Machines and Drives—Equipment for a Laboratory-Based Course", IEEE Transactions on Education, Vol. 49, NO. 3, Aug. 2006, pp. 383–388.

[8] B. A. Potter, S. A. Shirsavar, and I. Ridge, "Purpose built teaching equipment for a laboratory based course in three phase machines and drives," presented at the IEEE Int. Conf. Mechatronics, Istanbul, Turkey, 2004.

[9] M. Mazo, J. Urena, F. J. Rodriguez, J. J. Garcia, J. L. Lazaro, E. Santiso, F. Espinosa, R. Garcia, P. Revenga, J. C. Garcia, E. Bueno, and R. Mateos, "Teaching equipment for training in the control of dc, brushless, and stepper servomotors," IEEE Trans. Educ., vol. 41, no. 2, pp. 146–158, May 1998.

[10] P. C. Krause, O. Wasynczuk, S. D. Sudhoff, *Analysis of electric machinery and drive systems*, 2nd edition, Wiley-interscience, ISBN 0-471-14326-X, 2002.

[11] D. Hanselman, *Brushless permanent magnet motor design*, 2nd edition, Writers' collective, ISBN 1-932133-63-1, 2003.