



Continuous distributions

Five important **continuous** distributions:

1. **uniform** distribution (continuous)
2. **Normal** distribution
3. χ^2 —distribution [“ki-square”]
4. **t**-distribution
5. **F**-distribution

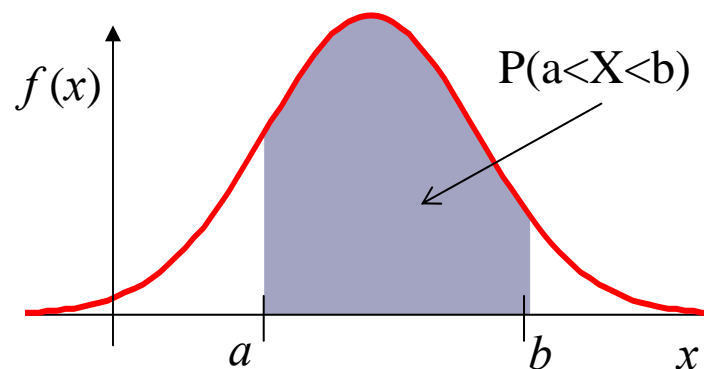
A reminder

Definition:

Let $X: S \rightarrow \mathbb{R}$ be a continuous random variable.

A density function for X , $f(x)$, is defined by:

1. $f(x) \geq 0$ for all x
2. $\int_{-\infty}^{\infty} f(x) dx = 1$
3. $P(a < X < b) = \int_a^b f(x) dx$



Uniform distribution

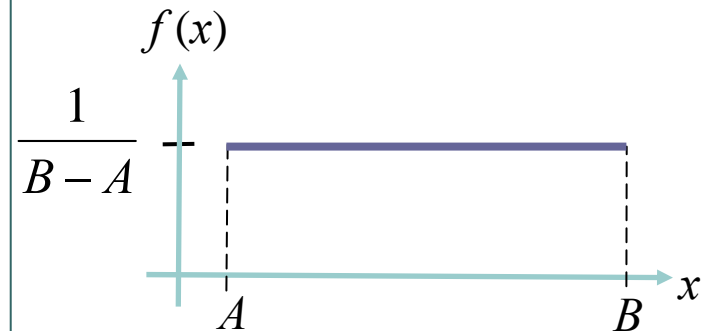
Definition

Definition:

Let X be a random variable. If the **density function** is given by

$$f(x) = \frac{1}{B - A} \quad A \leq x \leq B$$

then the distribution of X is the (continuous) **uniform distribution** on the interval $[A, B]$.





Uniform distribution

Mean & variance

Theorem:

Let X be **uniformly distributed** on the interval $[A, B]$.
Then we have:

- **mean** of X :
$$E(X) = \frac{A + B}{2}$$

- **variance** of X :
$$\text{Var}(X) = \frac{(B - A)^2}{12}$$



Normal distribution

Definition

Definition:

Let X be a continuous random variable.

If the **density function** is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \quad -\infty < x < \infty$$

then the distribution of X is called the **normal distribution** with parameters μ and σ^2 (known).

My notation: $X \sim N(\mu, \sigma^2)$

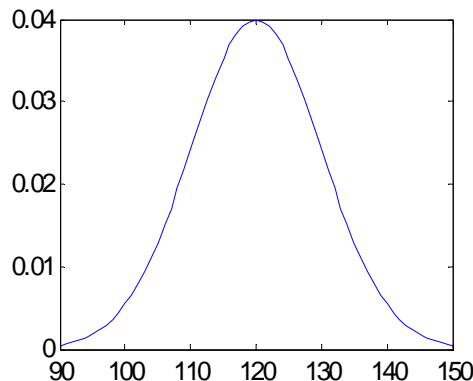
The book's: $n(x; \mu, \sigma)$ for density function

Normal distribution

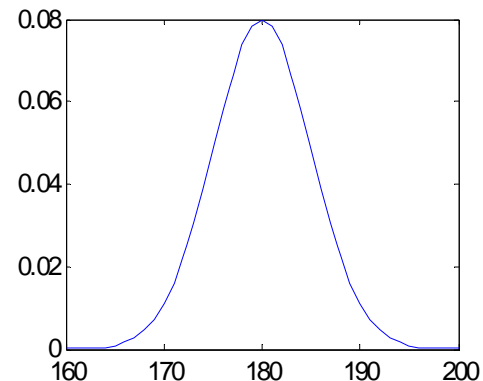
Examples

The normal distribution is without doubt the most important **continuous** distribution, since many phenomena are well described by it.

IQ among AAU students



Height among AAU students



Plotting in Matlab:

```
>> x=90:1:150; y=normpdf(x,120,10); plot(x,y)
```

μ σ

Normal distribution

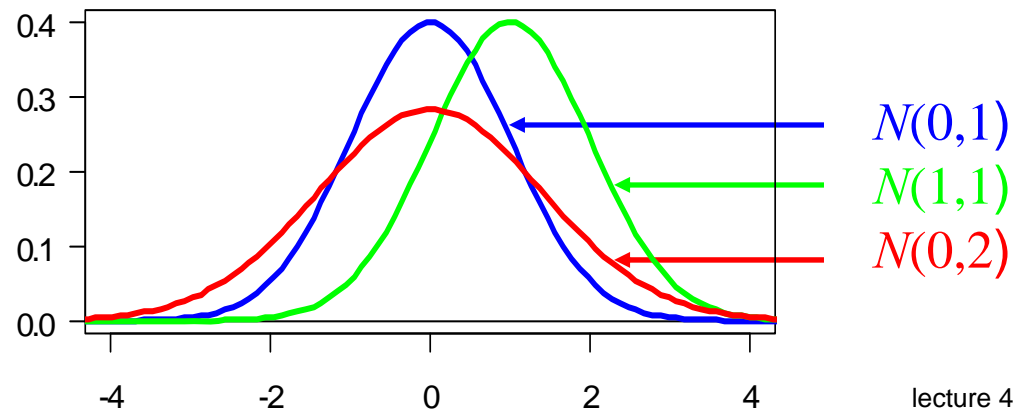
Mean & variance

Theorem:

If $X \sim N(\mu, \sigma^2)$ then

- **mean** of X : $E(X) = \mu$
- **variance** of X : $Var(X) = \sigma^2$

Density function:



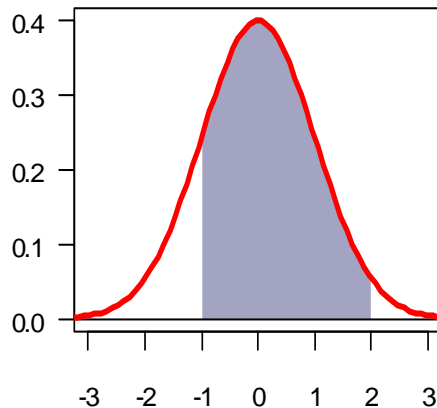
Normal distribution

Standard normal distribution

Standard normal distribution: $Z \sim N(0,1)$

Density function:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$



$$P(-1 \leq Z \leq 2) = \int_{-1}^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

Distribution function:

$$F(z) = P(Z \leq z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

(see Table A.3)

Notice!! Due to symmetry

$$P(Z \leq -z) = 1 - P(Z \leq z)$$

Normal distribution

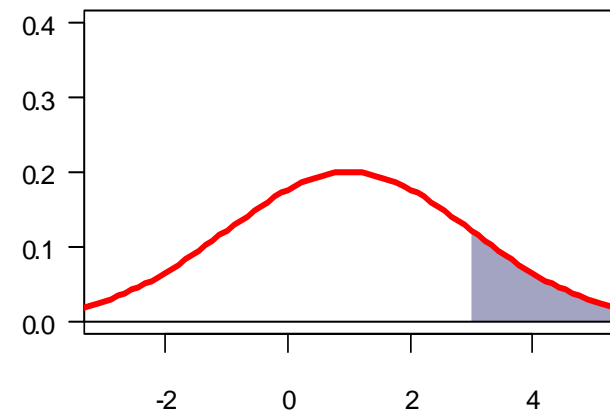
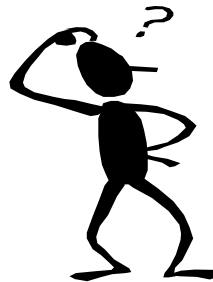
Standard normal distribution

Standard normal distribution, $N(0,1)$, is the only normal distribution for which the distribution function is tabulated.

We typically have $X \sim N(\mu, \sigma^2)$ where $\mu \neq 0$ and $\sigma^2 \neq 1$.

Example: $X \sim N(1,4)$

What is $P(X > 3)$?



Normal distribution

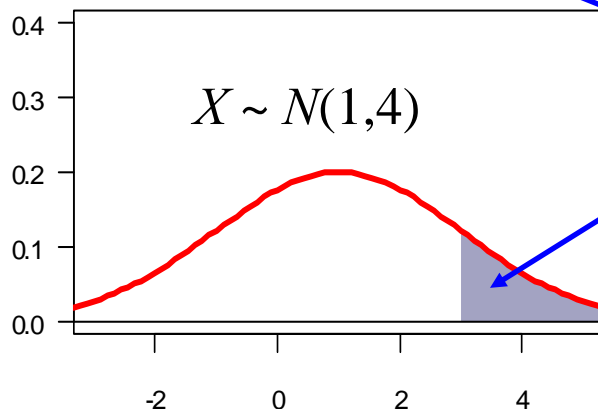
Standard normal distribution

Theorem: Standardise

If $X \sim N(\mu, \sigma^2)$ then $\frac{X - \mu}{\sigma} \sim N(0, 1)$

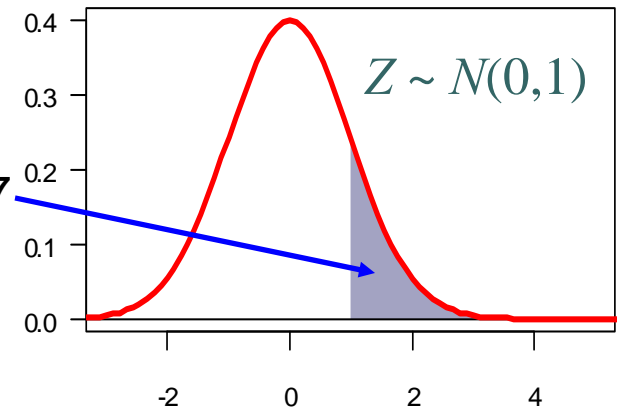
Example cont.: $X \sim N(1, 4)$. What is $P(X > 3)$?

$$P(X > 3) = P\left(\frac{X - 1}{2} > \frac{3 - 1}{2}\right) = P(Z > 1) = 1 - P(Z \leq 1) = 0.1587$$



$Z \sim N(0, 1)$

Equal areas = 0.1587

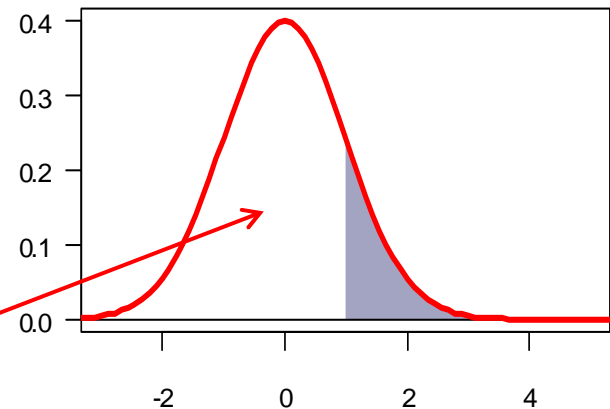


Normal distribution

Standard normal distribution

Table A.3 (continued) Areas under the Normal Curve

<i>z</i>	.00	.01	.02	.03	.04	.05	.06			
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.		
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.		
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.		
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.		
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.		
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319



$$P(Z \leq 1) = 0.8413 \quad \Rightarrow \quad P(X > 3) = 1 - P(Z \leq 1) = 1 - 0.8413 = 0.1587$$



Normal distribution

Standard normal distribution

Example cont.: $X \sim N(1,4)$. What is $P(X > 3)$?

$$P(X > 3) = 1 - P(X \leq 3) = 0.1587$$

Solution in Matlab:

```
>> 1 - normcdf(3,1,2)
ans =
    0.1587
```

Cumulative distribution function
`normcdf(x,μ,σ)`

Solution in R:

```
> 1 - pnorm(3,1,2)
[1] 0.1586553
```

Cumulative distribution function
`pnorm(x,μ,σ)`

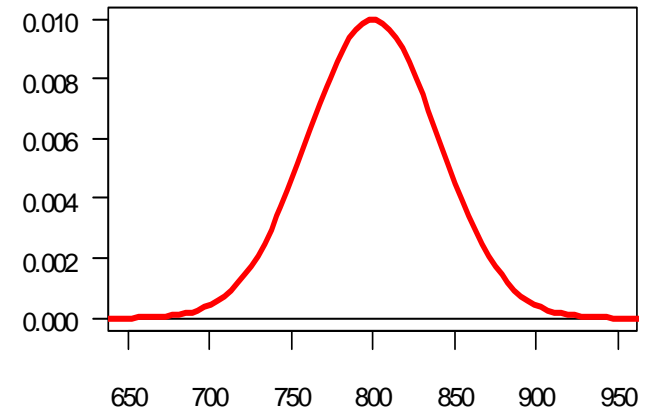
Normal distribution

Example



Problem:

The lifetime of a light bulb is normal distributed with mean 800 hours and standard deviation 40 hours:



1. Find the probability that the lifetime of a bulb is between 750-850 hours.
2. Find the number of hours b , such that the probability of a bulb having a lifetime longer than b is 90%.
3. Find a time period *symmetric* around the mean so that the probability of a lifetime in this interval has probability 95%



Normal distribution

Example

Solution in Matlab:

1. $P(750 < X < 850) = ?$

```
>> normcdf(850,800,40)-normcdf(750,800,40)
```

2. ?

3. ...

Solution in R:

1. $P(750 < X < 850) = ?$

```
> pnorm(850,800,40)-pnorm(750,800,40)
```

2. $P(X > b) = 0.90 \Leftrightarrow P(X \leq b) = 0.10$

```
> qnorm(0.1,800,40)
```

3. ...

Normal distribution

Relation to the binomial distribution

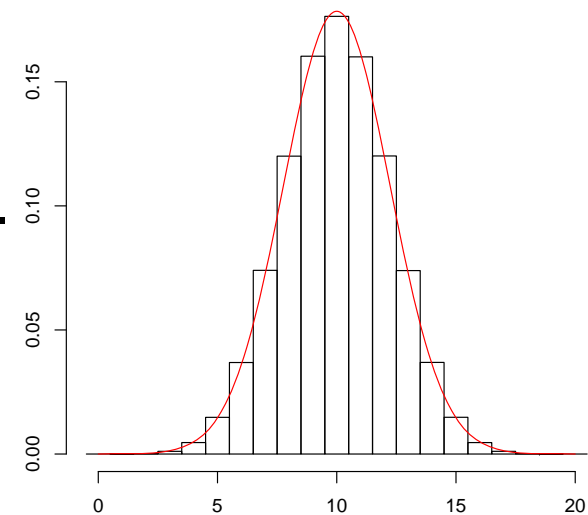
If X is binomial distributed with parameters n and p , then

$$Z = \frac{X - np}{\sqrt{np(1-p)}}$$

is approximately normal distributed.

Rule of thumb:

If $np > 5$ and $n(1-p) > 5$, then the approximation is good





Normal distribution

Linear combinations

Theorem: linear combinations

If X_1, X_2, \dots, X_n are independent random variables, where

$$X_i \sim N(\mu_i, \sigma_i^2), \text{ for } i = 1, 2, \dots, n,$$

and a_1, a_2, \dots, a_n are constant, then the linear combination

$$Y = a_1 X_1 + a_2 X_2 + \dots + a_n X_n \sim N(\mu_Y, \sigma_Y^2),$$

where

$$\mu_Y = a_1 \mu_1 + a_2 \mu_2 + \dots + a_n \mu_n$$

$$\sigma_Y^2 = a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + \dots + a_n^2 \sigma_n^2$$



The χ^2 distribution

Definition

Definition: (alternative to Walpole, Myers, Myers & Ye)

If Z_1, Z_2, \dots, Z_n are **independent** random variables, where

$$Z_i \sim N(0,1), \quad \text{for } i = 1, 2, \dots, n,$$

then the distribution of

$$Y = Z_1^2 + Z_2^2 + \dots + Z_n^2 = \sum_{i=1}^n Z_i^2$$

is the **χ^2 -distribution** with n degrees of freedom.

Notation: $Y \sim \chi^2(n)$

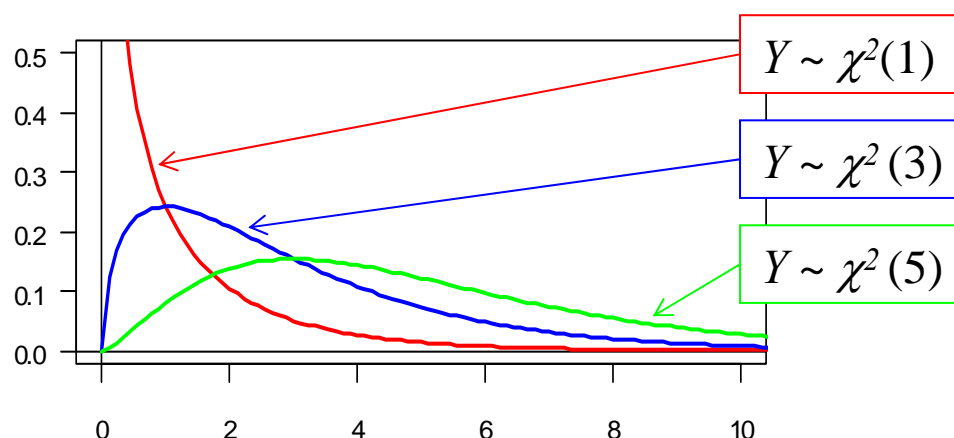
Critical values: Table A.5

The χ^2 distribution

Definition

Definition: A continuous random variable X follows a χ^2 -distribution with n degrees of freedom if it has **density function**

$$f(x) = \frac{1}{2^{n/2} \Gamma(n/2)} x^{n/2-1} e^{-x/2} \quad \text{for } x > 0$$



Assume $Y \sim \chi^2(n)$

$$E(Y) = n$$

$$\text{Var}(Y) = 2n$$

$$E(Y/n) = 1$$

$$\text{Var}(Y/n) = 2/n$$



t-distribution

Definition

Definition:

Let $Z \sim N(0,1)$ and $V \sim \chi^2(n)$ be two **independent** random variables. Then the distribution of

$$T = \frac{Z}{\sqrt{V/n}}$$

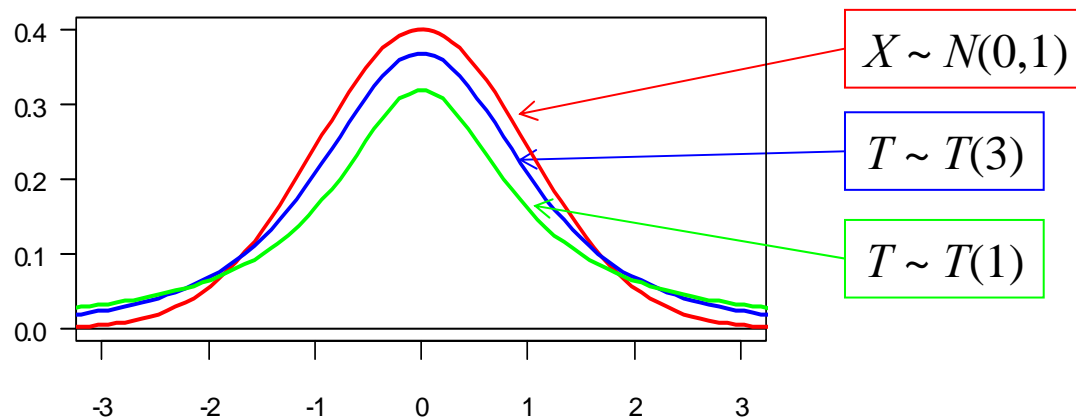
is called the ***t*-distribution** with n degrees of freedom.

Notation: $T \sim t(n)$

Critical values: Table A.4

t -distribution

Compared to standard normal



- The t -distribution is **symmetric** around 0
- The t -distribution is **more flat** than the standard normal
- The more **degrees of freedom** the more the t -distribution looks like a standard normal



F -distribution

Definition

Definition:

Let $U \sim \chi^2(n_1)$ and $V \sim \chi^2(n_2)$ be two **independent** random variables. Then the distribution of

$$F = \frac{U/n_1}{V/n_2}$$

is called the **F -distribution** with n_1 and n_2 degrees of freedom.

Notation: $F \sim F(n_1, n_2)$

Critical values: Table A.6

F-distribution Example

