AALBORG UNIVERSITET

Written examination in the course

Optimisation Theory and Stochastic Processes

Wednesday June 1th 2011

kl. 8.30 - 11.30 (3 hours)

All usual helping aids are allowed, i.e. books, notes, calculator, computer etc. All communication equipment and computer communication protocols must be turned off.

The questions should be answered in English.

REMEMBER to write your study number and page number on all sheets handed in.

The set consists of 6 exercises. The total weighting for each of the exercises is stated in percentage. Sub-questions in each exercise have equal weight.

It should be clear from the solution, which methods are used, and there should be a sufficient number of intermediate calculations, so the line of thought is clear.

Exercise 1: (10 %)

a) Find stationary points for the following function:

$$f(\mathbf{x}) = -2 \cdot x_1^2 + 3 \cdot x_1 \cdot x_2 - 2 \cdot x_2^2 + 2 \tag{1}$$

b) Determine the local minimum, local maximum, or inflection (saddle) points for the function.

Exercise 2: (20 %)

We will consider gradient based minimization of the following unconstrained function:

$$f(\mathbf{x}) = 2 \cdot x_1^2 + x_2^2 + 2 \cdot x_1 \cdot x_2 - 4 \cdot x_2 \tag{2}$$

The starting point is: $\mathbf{x}^{(0)} = [\ 1 \quad 1\ ]^T$.

a) Complete the first iteration of the steepest descent method for the function. The 1D line search problem should be solved analytically.

b) Determine the search direction for the first iteration of the modified Newton's method for the function.

Exercise 3: (10 %)

Solve the following linear optimisation problem using the basic steps of the Simplex method and tableau's:

$$\text{minimise} \qquad f(\mathbf{x}) = -3x_1 + 2x_2 \tag{3}$$

Subject to the constraints:

$$\begin{array}{rcccc} g_1(\mathbf{x}) & = & -\frac{x_1}{2} + x_2 & \leq & 2 \\ g_2(\mathbf{x}) & = & x_1 + x_2 & \leq & 3 \\ & & x_i \geq 0 & \forall & x_i = \{1, 2\} \end{array} \tag{4}$$

Exercise 4: (10 %)

A multi-objective optimisation problem is formulated as:

$$\text{minimise} \qquad \begin{array}{l} f_1(\mathbf{x}) = (x_1 - 3)^2 + (x_2 - 2)^2 \\ f_2(\mathbf{x}) = (x_1 - 6)^2 + (x_2 - 5)^2 \end{array} \tag{5}$$

The two contour curves are shown in figure 1.

a) Illustrate the Pareto optimal points in figure 1 (the page should be handed in with the solution).

b) Sketch the Pareto front in the criterion space. The sketch may be based on function values from the contour plot. A coordinate system may be found in figure 2.

Page to be handed in with the solution!

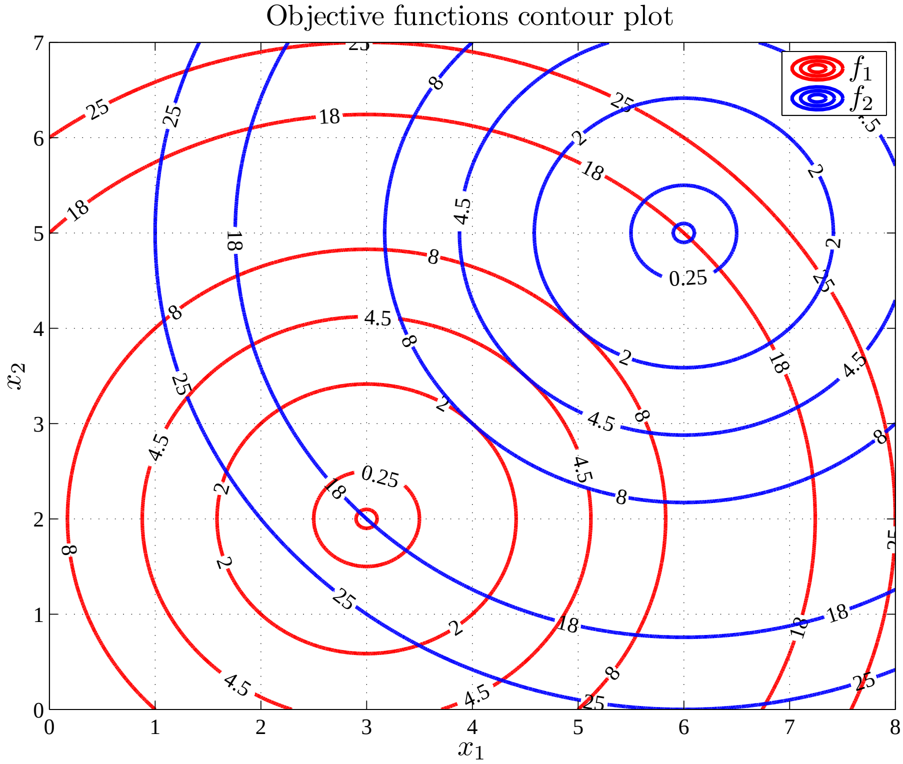

Figure 1: Contour curves for the problem of exercise 4.

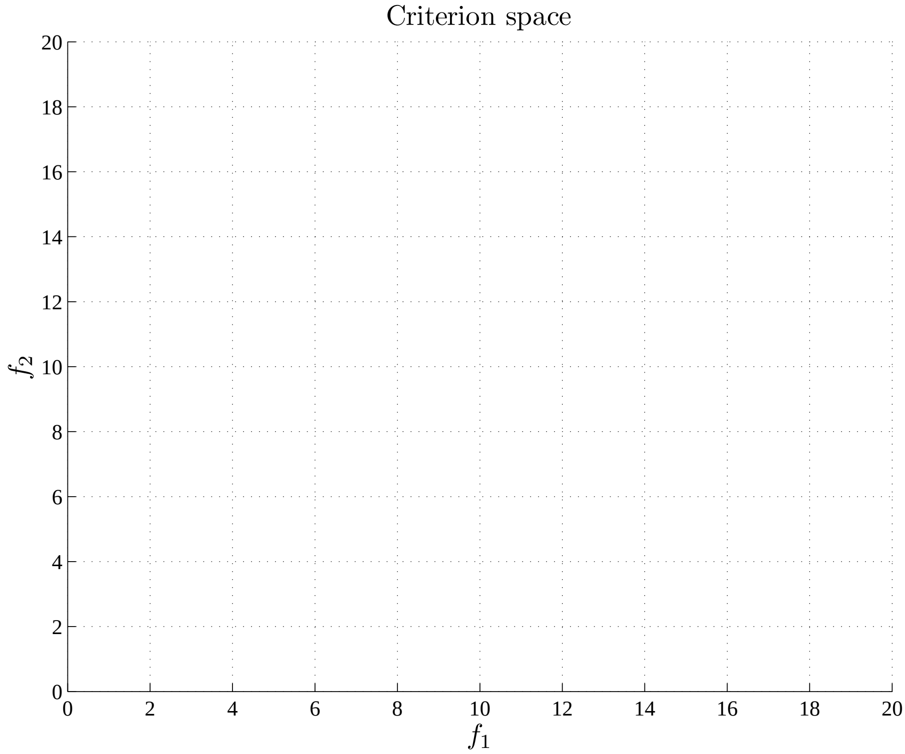

Figure 2: Coordinate system for plotting the criterion space Pareto front in exercise 4.

Exercise 5: Linear systems and ARMA(1,1) process (30 %)

Let us consider the following linear system:

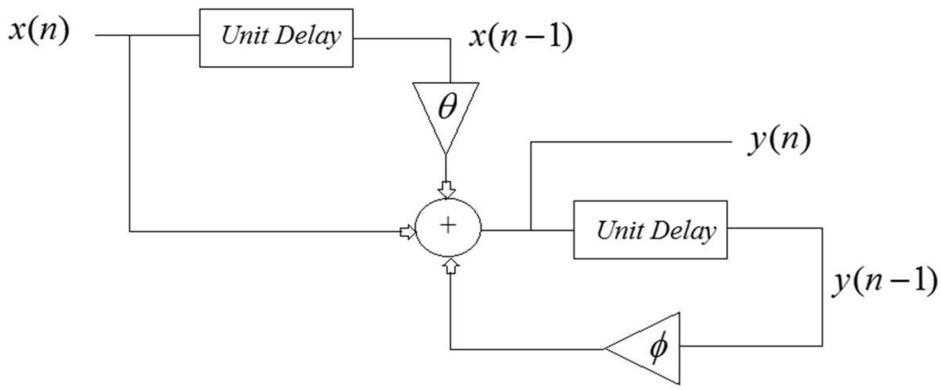

Figure 3: System for exercise 5.

the input-output relationship of which is given by:

$$y(n) = \phi y(n-1) + x(n) + \theta x(n-1) \tag{6}$$

a) Derive the impulse response $h(n)$ of the linear system for initialization $x(n) = y(n) = 0,\ n < 0$.

b) Determine the range of values of ϕ and θ for which the linear system is stable.

From now on, we assume that $\phi = \theta = 0.5$.

c) Derive the transfer function $H(f)$ of the linear system.

Let us now assume that the input sequence is a white noise sequence $\{X(n)\}$ with unit variance, i.e. $E[X(n)] = 0, R_{XX}(k) = [X(n)X(n+k)] = \sigma_X^2\delta(k)$ with $\sigma_X^2 = 1$.

d) Show that $\{Y(n)\}$ is wide–sense stationary only if $E[Y(n)] = 0$ for any n.

e) Derive the power spectrum $S_{YY}(f)$ of $\{Y(n)\}$.

f) What is the interpretation of surface under graph of $S_{YY}(f)$, i. e. the quantity $\int_{-0.5}^{+0.5} S_{YY}(f)df$.

Notice: you do not have to calculate this integral, but only specify the manner it can be interpreted.

Exercise 6: Detection of a Gaussian random signal in background noise (20 %)

Detection of a Gaussian random signal in background noise:

H_0: Only Gaussian noise W is present

H_1: A random Gaussian signal X plus Gaussian noise W is present.

More specifically the received signal under both hypotheses reads:

$$H_0 \,:\, Y = W \tag{7}$$

$$H_1 \,:\, Y = X + W \tag{8}$$

Where

A. W is a zero-mean Gaussian random variable with variance $\sigma_W^2 = 1$, i.e. $W \sim N(0, 1)$.

B. X is a zero-mean Gaussian random variable with σ_X^2, i.e. $X \sim N(0, \sigma_X^2)$.

We further assume that X and W are independent. The signal to noise ratio is:

$$\eta = \left(\frac{\sigma_X}{\sigma_W}\right)^2 = 3 \tag{9}$$

a) Find the probability density function of Y *under both hypotheses, i. e.* $f(y|H_0)$ *and* $f(y|H_1)$.

Hint: Notice that the variance of the sum of two independent random variables equals to the sum of their individual variances.

b) Show that $L(y) = \frac{f(y|H_1)}{f(y|H_0)}$ *is given by:*

$$L(y) = \frac{1}{\sqrt{1+\eta}} \exp\left(\frac{1}{2}\frac{\eta}{(1+\eta)}\left|\frac{y}{\sigma_W}\right|^2\right) = \frac{1}{2}\exp\left(\frac{3}{8}|y|^2\right) \tag{10}$$

c) Calculate the log-likelihood function: $l(y) = \ln\left(\frac{f(Y|H_1)}{f(Y|H_0)}\right)$

d) Find the MAP decision rule for the following a priori probability distributions: $P[H_0] = P[H_1] = \frac{1}{2}$.