

Written examination in the course
Optimisation Theory and
Modern Reliability from a Practical Approach
Wednesday June 11th 2014
kl. 9 - 13 (4 hours)

The examination set consist of two parts, one for »Optimisation Theory« and one for »Modern Reliability from a Practical Approach«.

For the part in Modern Reliability from a Practical Approach: No computers or other helping aids as slides, textbooks etc. may be used. This means that you should hand in your solution for this part, before starting up your computer or bringing out your notes if you wish to use these for the Optimisation Theory part. The pages in the reliability part are meant to be handed in.

For the Optimisation Theory part: All usual helping aids are allowed, i.e. books, notes, calculator, computer etc. All communication equipment and computer communication protocols must be turned off.

The questions should be answered in English.

REMEMBER to write your study number and page number on all sheets handed in.

The set consists of four exercises in the optimisation part (totalling 50% of the examination set) and 25 questions in the reliability part. The weighting for each exercise is stated in percentage. You need in total 50 % of the entire set (optimisation and reliability part) correct in order to pass the exam.

It should be clear from the solution, which methods are used, and there should be a sufficient number of intermediate calculations, so the line of thought is clear.

Exercise 1: (25 %)

A company produces two products: 1) a MP3 player and 2) a Watch-TV (a wristwatch TV).

The production process for each product is similar in that both require a certain number of hours of electronic work and a certain number of labor-hours in the assembly department. Each MP3 player takes 4 hours of electronic work and 2 hours in the assembly shop. Each Watch-TV requires 3 hours in electronics and 1 hour in assembly.

During the current production period, 240 hours of electronic time are available, and 100 hours of assembly department time are available.

Each MP3 player sold yields a profit of 7 Euro; each Watch-TV produced may be sold for a 5 Euro profit.

a) Formulate the problem of determining the best possible combination of MP3 players and Watch-TVs to manufacture to reach the maximum profit.

b) Solve the problem formulated in a) using graphical methods.

c) Set up the Lagrangian function for the problem formulated in a) and find point(s) satisfying the KKT necessary conditions. You may make use of results from b) when checking/solving the KKT necessary conditions.

Exercise 2: (7 %)

The following multi-objective optimisation problem is considered:

$$\begin{aligned} \text{minimise} \quad & f_1(\mathbf{x}) = (x_1 - 8)^2 + (x_2 - 2)^2 + 3 \\ & f_2(\mathbf{x}) = (x_1 - 1)^2 + (x_2 - 7)^2 + 10 \end{aligned}$$

Figure 1 shows the Pareto optimal points in the design space and figure 2 shows the Pareto optimal set in the criterion space.

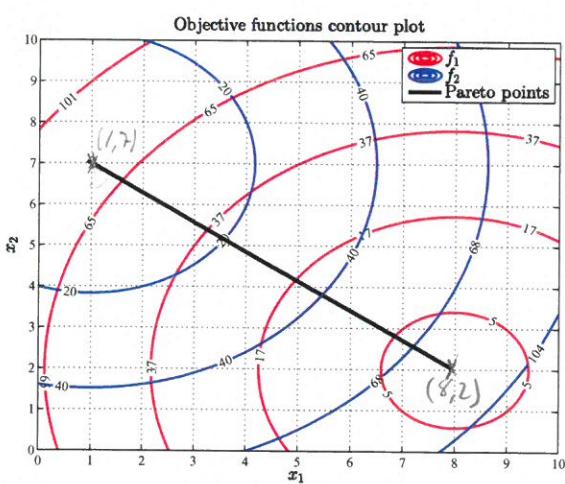


Figure 1: Contour curves and Pareto optimal points for the problem of exercise 3.

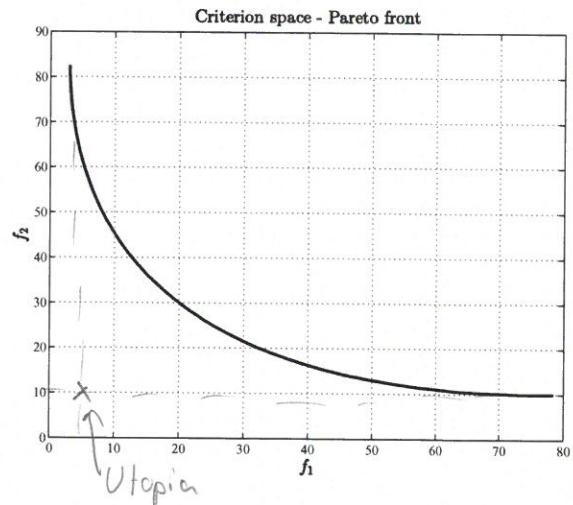


Figure 2: Pareto set in criterion space for the problem of exercise 3.

a) Determine the objective function values of the utopia point.

b) Assume that the multi-objective problem is solved as single objective problem using the weighting method (i.e. minimise $U(\mathbf{x}) = w_1 f_1(\mathbf{x}) + w_2 f_2(\mathbf{x})$), with $w_1 = w_2 = 1$. Determine the minimum objective function value $U(\mathbf{x}^*)$, and the optimum set of design variables \mathbf{x}^* .

Exercise 3: (10 %)

An optimisation problem is formulated as:

$$\text{minimise} \quad f(\mathbf{x}) = 4x_1^2 + 2x_2^2 - 2x_1x_2 + 2x_1 - x_2$$

Subject to:

$$\begin{aligned} g_1(\mathbf{x}) &= x_1 - 12 \leq 0 \\ g_2(\mathbf{x}) &= x_1^2 - x_1 - 2x_2 - 8 \leq 0 \\ g_3(\mathbf{x}) &= x_1 + x_2 - 20 \leq 0 \end{aligned}$$

The initial starting point for an optimisation is $\mathbf{x}^{(0)} = (-1, -2)$, for which the search direction is determined to $\mathbf{d}^{(0)} = [2 \ 7]^T$ and the vector of Lagrange multipliers for the constraints is $\mathbf{u} = [0 \ 0 \ 0]^T$. Let $R_0 = 1$ and $\gamma = 0.5$. Choose the trial step according to the sequence $t_0 = 1, t_1 = \frac{1}{2}, t_2 = \frac{1}{4}, t_3 = \frac{1}{8} \dots$

Calculate the step size using the inexact line search procedure (approximate step size procedure) and determine the new design variables, $\mathbf{x}^{(1)}$, for the next iteration.

Exercise 4: (8 %)

Solve the following problem by setting up the solution tree and using the Branch & Bound Method with Local Minimization:

$$\text{minimise} \quad f(\mathbf{x}) = -3x_1 - 8x_2$$

Subject to:

$$\begin{aligned} g_1(\mathbf{x}) &= -0.5x_1 + x_2 - 20 \leq 0 \\ g_2(\mathbf{x}) &= 2x_1 + x_2 - 20 \leq 0 \\ g_3(\mathbf{x}) &= 4x_1 - x_2 - 20 \leq 0 \end{aligned}$$

Both x_1 and x_2 should be integer values

As a help for solving the problem the objective function contours and constraints are plotted in figure 3.

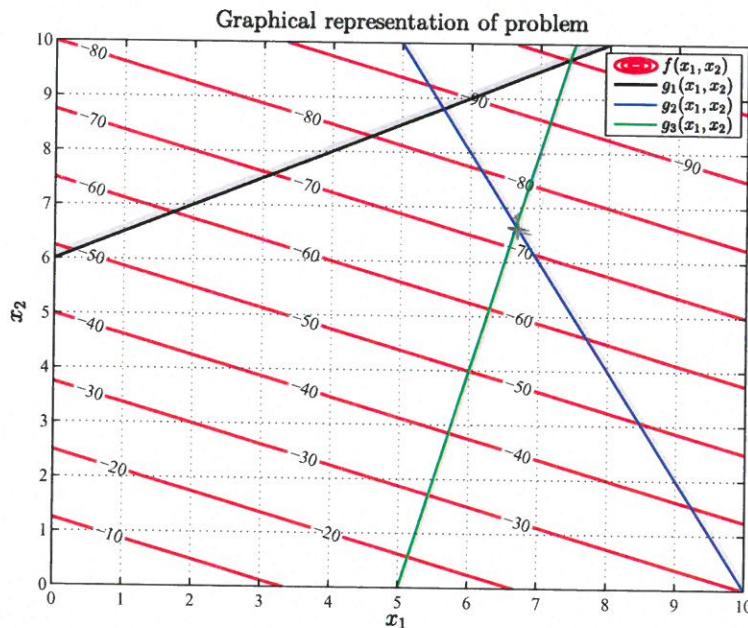


Figure 3: Graphical representation of problem in exercise 4.

Optimisation Theory - Exam 14

2 product.

	Work	Assembly	Profit
Product 1 :	4	1	7
Product 2 :	3	1	5

- Determine the best combination of 1 & 2 to reach max. profit:

Design Variables:

x_1 : Product 1

x_2 : Product 2

Profit Sch:

$$f(x_1, x_2) = 7x_1 + 5x_2$$

Constraints:

240 h of work and 100 h of assembly are available

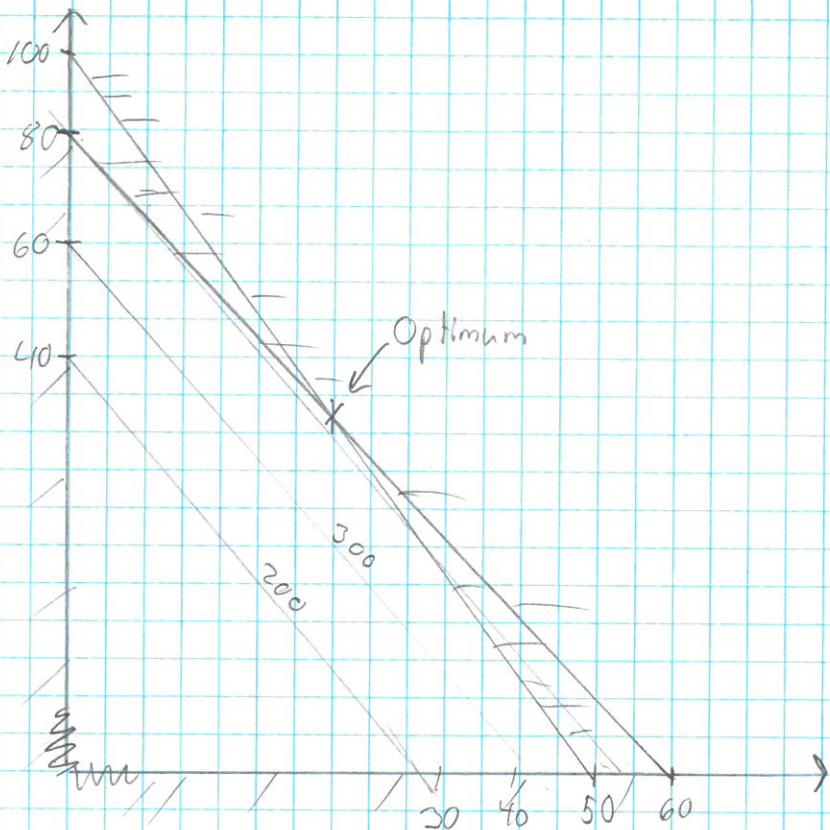
$$4x_1 + 3x_2 \leq 240$$

$$2x_1 + x_2 \leq 100$$

$$x_1, x_2 \geq 0$$

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- Graphical Solution



Optimum at intersection of the two constraints

$$\begin{cases} 4x_1 + 3x_2 = 240 \\ 2x_1 + x_2 = 100 \end{cases}$$

$$\begin{array}{l} \text{---} \\ \text{---} \end{array} \quad \begin{array}{l} 7x_1 = 140 \\ x_1 = 20 \end{array}$$

$$x_1 = 20$$

$$x_2 = 60$$

$$f = 5 \cdot 20 + 7 \cdot 60 = 430$$

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- Set up the Lagrangian function:

Lecture 6,
Slide 3
Book,
p. 190

$$\text{minimise } f(x_1, x_2) = -7x_1 - 5x_2$$

$$\text{Subject to } g_1(x) = 4x_1 + 3x_2 \leq 240$$

$$g_2(x) = 2x_1 + x_2 \leq 100$$

$$x_1, x_2 \geq 0$$

$$L(x, v) = f(x) + v^T g(x)$$

$$\Rightarrow f(x) + u_1 g_1(x) + u_2 g_2(x)$$

Thus:

$$L(x, v) = -7x_1 - 5x_2 + u_1(4x_1 + 3x_2 - 240) + u_2(2x_1 + x_2 - 100)$$

- Gradient:

$$\frac{\partial L}{\partial x_1} = -7 + 4u_1^* + 2u_2^* \quad 1$$

$$\frac{\partial L}{\partial x_2} = -5 + 3u_1^* + u_2^* \quad 2$$

- Feasibility Check:

$$g_1(x^*) \leq 0 \Rightarrow (4)$$

$$g_2(x^*) \leq 0$$

- Slack Condition

$$u_1 g_1 = 0 \Rightarrow u_1(4x_1^* + 3x_2^* - 240) = 0 \quad 3$$

$$u_2 g_2 = 0 \Rightarrow u_2(2x_1^* + x_2^* - 100) = 0 \quad 4$$

- Non-negativity check:

$$u_1^* \geq 0$$

$$u_2^* \geq 0$$

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- Guess from Graphical Solution:

$$\{ g_1(x^*) = 0 \rightarrow \text{Active}$$

$$\{ g_2(x^*) = 0 \rightarrow \text{Active}$$



$$\{ 4x_1^* + 3x_2^* - 240 = 0$$

$$\{ 2x_1^* + x_2^* - 100 = 0$$



$$x_1^* = 30$$

$$x_2^* = 40$$

- Non-negative

$$\{ -7 + 4u_1^* + 2u_2^* = 0$$

$$\{ -5 + 3u_1^* + u_2^* = 0$$



$$u_1^* = 3/2$$

$$u_2^* = 1/2$$

Since $u_{1,2} > 0$, the necessary condition
of KKT is satisfied.

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2 - Multi - Objective Optimisation Problem:

$$\text{minimise } f_1(x) = (x_1 - 8)^2 + (x_2 - 2)^2 + 3$$

$$f_2(x) = (x_1 - 1)^2 + (x_2 - 7)^2 + 10$$

- The objective function values of the utopian point:

$$f_1(8, 2) = 3$$

$$f_2(1, 7) = 10$$

- The minimum objective function value:

$$\text{Minimise } U(x) = w_1 f_1(x) + w_2 f_2(x), \quad w_1 = w_2 = 0$$

f_1	f_2	U
65	5	70
35	20	65
15	40	65
3	65	78

Smallest.

$$\begin{cases} 15 = (x_1 - 8)^2 + (x_2 - 2)^2 + 3 \\ 40 = (x_1 - 1)^2 + (x_2 - 7)^2 + 10 \end{cases}$$

$$\Downarrow$$

$$x_1 = x_2 = 4,5$$

$$U = 55$$

Or

$$\frac{\partial U}{\partial x_1} = 0 \Rightarrow x_1 =$$

$$\frac{\partial U}{\partial x_2} = 0 \Rightarrow x_2 =$$

Inserting in U yields:

$U =$

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3 - Optimisation Problem

$$\text{minimise } f(x) = 4x_1^2 + 2x_2^2 - 2x_1x_2 + 2x_1 - x_2$$

$$\text{Subject to } g_1(x) = x_1 - 12 \leq 0$$

$$g_2(x) = x_1^2 - x_1 - 2x_2 - 8 \leq 0$$

$$g_3(x) = x_1 + x_2 - 20 \leq 0$$

- Initial Starting point: $x^{(0)} = [-1, -2]^T$

- Search direction: $d^{(0)} = [2 \ 7]^T$

- Vector of Lagrange multipliers $\lambda = [0 \ 0 \ 0]^T$

- Let $R_0 = 1$ and $\gamma = 0,5$,

- Choose a trial step according to the Lecture 7 Sequence, $t_0 = 1, t_1 = 1/2, t_2 = 1/4, t_3 = 1/8$

Book

p.542 - Calculate the step size using the inexact line search procedure (Approximate Step Size procedure):

$$r_0 = \sum_{i=0}^m u_i^{(0)} = 0$$

$$R = \max \{ R_0, r_0 \} \Rightarrow \max \{ 1, 0 \} = 1$$

$$V_0 = \max \{ g_1^{(0)}, g_2^{(0)}, g_3^{(0)} \}$$

$$\max \{ -13, -4, -25 \} \Rightarrow \text{Set to } 0$$

- Value of the descent function:

$$\bar{f}_0 = f_0 + R V_0 \Rightarrow$$

$$\Rightarrow 4 \cdot (-1)^2 + 2 \cdot (-2)^2 - 2 \cdot (-1)(-2) + 2 \cdot (-1) - (-2) + 1 \cdot 0$$

$$\Rightarrow 8 + 1 + 0 = 8$$

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- The constant:

$$\beta_0 = \gamma \|d^{(0)}\|^2 \Rightarrow 0,5 \cdot (\sqrt{2^2+7^2})^2 = 26,5$$

- Evaluate the new value of the descent function to check the descent conditions:

$$x^{(1,0)} = x^{(0)} + t_{0,3} d^{(0)}, \quad t_{0,3} - \text{Trial Steps}$$

$$x^{(1,0)} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} + [1 \ 1/2 \ 1/4 \ 1/8] \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

$$x^{(1,0)} = \begin{bmatrix} 1 & 0 & -0,5 & -0,75 \\ 5 & 1,5 & -0,25 & -1,125 \\ t_0 & t_1 & t_2 & t_3 \end{bmatrix}$$

- Evaluate the cost and constraint function:

$$f^{(1,0)} = [41 \quad 5 \quad 0,125 \quad 2,71988]$$

$$V_{1,0} = V(t_0) = \max \{-11, -18, -14\} \Rightarrow 0$$

$$V_{1,0} = V(t_1) = \max \{-12, -11, -18,5\} \Rightarrow 0$$

$$V_{1,0} = V(t_2) = \max \{-11,5, -7,75, -19,25\} \Rightarrow 0$$

$$V_{1,0} = V(t_3) = \max \{-12,75, -5,93, -13\} \Rightarrow 0$$

- Evaluate the maximum constraint violation to calculate the descent function:

$$\phi_{1,0} = f_{1,0} + \gamma V_{1,0} \Rightarrow [41 \quad 5 \quad 0,125 \quad 2,71988]$$

$$\phi_0 - t_{0,3} \beta_0 = 9 - [1 \ 1/2 \ 1/4 \ 1/8] \cdot 26,5 \\ = [-17,5 \quad -4,25 \quad -2,375 \quad -5,6875]$$

The descent condition, $\phi_{1,0} \leq \phi_0 - t_0 \beta_0$, is at $t_0 = 1/4$, since $0,125 \in [-2,375]$

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The design is updated:

$$x^{(1,1)} = x^{(0)} + t_2 d^{(0)}$$

$$= \begin{bmatrix} -1 \\ -2 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} -0,5 \\ -0,25 \end{bmatrix}$$

4 - The solution tree / Branch & Bound Method

Book, with Local Minimisation

P. 622 Minimise $f(x) = -3x_1 - 8x_2$

Subject to: $g_1(x) = -0,5x_1 + x_2 - 20 \leq 0$

$$g_2(x) = 2,5x_1 + x_2 - 20 \leq 0$$

$$g_3(x) = 4x_1 - x_2 - 20 \leq 0$$

x_1 & x_2 are integers

- Start at the optimal continuous solution:

- From graphical solution:

$$x_1 = 6,7, \quad x_2 = 6,7, \quad f(6,7,6,7) = -73,7$$

- Solution Tree: - Local Minimisation - Ignore integer constraints for one variable during.

Since all constraints are "less than or equal"

↳ Round down

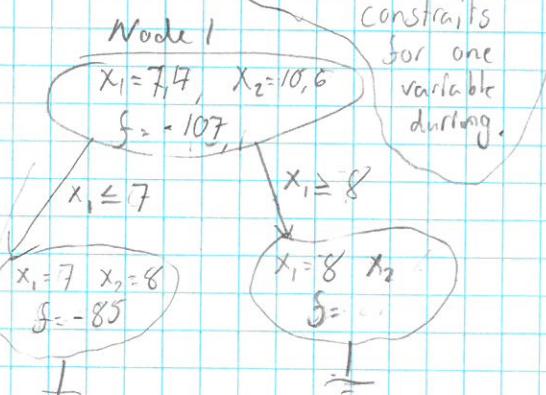
At Node 1, the best known solution

$$\text{is: } x_1 = 7, \quad x_2 = 10 \\ f = -101.$$

Restriction on x_1
↳ If is the largest value below 7 that is an integer
↳ the smallest above.

Since no integer solution between 6 & 7.
So here the algorithm eliminate the space in between 7 and 8.

Thus, the optimum integer solution is from the feasible region.



$x_1 = 7, \quad x_2 = 8, \quad \text{for which } f = -85$