AALBORG UNIVERSITET

Written examination in the course

Optimisation Theory and Stochastic Processes

Thursday June 7th 2012

kl. 9 - 13 (4 hours)

All usual helping aids are allowed, i.e. books, notes, calculator, computer etc. All communication equipment and computer communication protocols must be turned off.

The questions should be answered in English.

REMEMBER to write your study number and page number on all sheets handed in.

The set consists of seven exercises. The total weighting for each of the exercises is stated in percentage. Sub-questions in each exercise have equal weight. You need 50 % in order to pass the exam.

It should be clear from the solution, which methods are used, and there should be a sufficient number of intermediate calculations, so the line of thought is clear.

Exercise 1: (10 %)

The following optimization problem is considered:

$$\begin{aligned} \text{Minimize} \quad f(\mathbf{x}) &= (x_1 - 1)^2 + (x_2 - 1)^2 \\ \text{Subject to} \quad h(\mathbf{x}) &= x_1 + x_2 - 4 = 0 \end{aligned} \tag{1}$$

a) Set up the Lagrangian function and find point(s) satisfying the KKT necessary conditions.

b) Check if the point(s) is an optimum point using the graphical method (make a simple sketch).

Exercise 2: (15 %)

We will consider gradient-based minimization of the following unconstrained function:

$$f(\mathbf{x}) = (1 - x_1)^2 + (x_2 - 2)^2 + 2 \cdot x_1 \tag{2}$$

The starting point is: $\mathbf{x}^{(0)} = [\ 3 \quad 1\ ]^T$.

a) Complete the first iteration of the steepest descent method for the function. The 1D line search problem should be solved analytically.

b) Determine the search direction for the first iteration of Newton's method for the function.

Exercise 3: (13 %)

An optimisation problem is given as:

$$\text{minimise} \qquad f(\mathbf{x}) = -x_1^2 + 3x_2^2 + x_1 x_2 - 3 \tag{3}$$

Subject to the constraints:

$$\begin{aligned} g_1(\mathbf{x}) &= \tfrac{1}{x_1} - 2x_2 \leq 0 \\ g_2(\mathbf{x}) &= x_1 - 2x_2^3 \leq 0 \\ & x_i \geq 0 \quad \forall \quad x_i = \{1, 2\} \end{aligned} \tag{4}$$

a) Linearise the problem at the point $(x_1, x_2) = (1, 1)$, and write up the linearised subproblem. Note that there is no need to do a normalisation of the problem!

b) Solve the linearised sub-problem using tableaus and the basic steps of the Simplex method

Exercise 4: (12 %)

A multi-objective optimisation problem is formulated as:

$$\text{minimise} \qquad \begin{aligned} f_1(\mathbf{x}) &= (x_1-4)^2+(x_2-2)^2 \\ f_2(\mathbf{x}) &= (x_1-4)^2+(x_2-8)^2 \end{aligned} \tag{5}$$

Subject to the constraints:

$$\begin{aligned} g_1(\mathbf{x}) &= x_2-7 \leq 0 \\ g_2(\mathbf{x}) &= -x_1-x_2+8 \leq 0 \end{aligned} \tag{6}$$

The two contour curves along with the constraints are shown in figure 1.

a) Illustrate the Pareto optimal points in figure 1 (the page should be handed in with the solution).

b) Sketch the Pareto front in the criterion space. The sketch should be based on function values from the contour plot. A coordinate system may be found in figure 2.

c) Indicate where the utopia point is located for the given problem.

*d) A weighting method is used in solving the problem (i.e. minimise $U = w_1 f_1(\mathbf{x}) + w_2 f_2(\mathbf{x})$), with $w_1 = w_2 = 1$. What is the solution to this problem - give both function value U, and design variables x_1 and x_2. **Hint:** Use the results from questions a) and b).*

Page to be handed in with the solution!

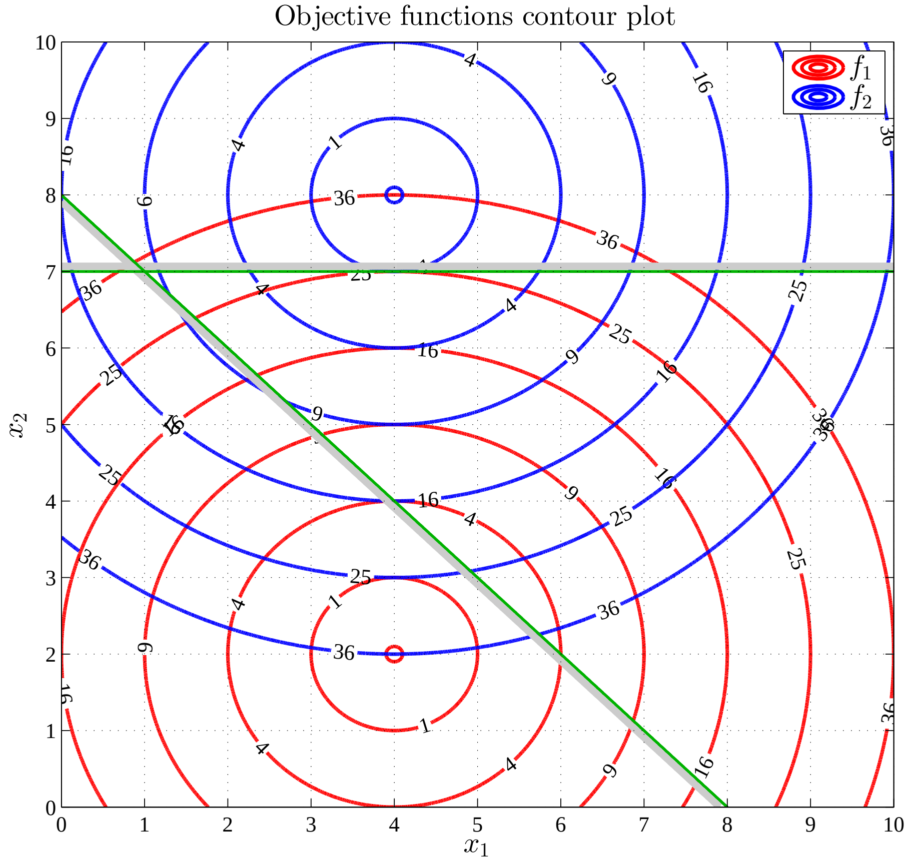

Figure 1: Contour curves for the problem of exercise 4.

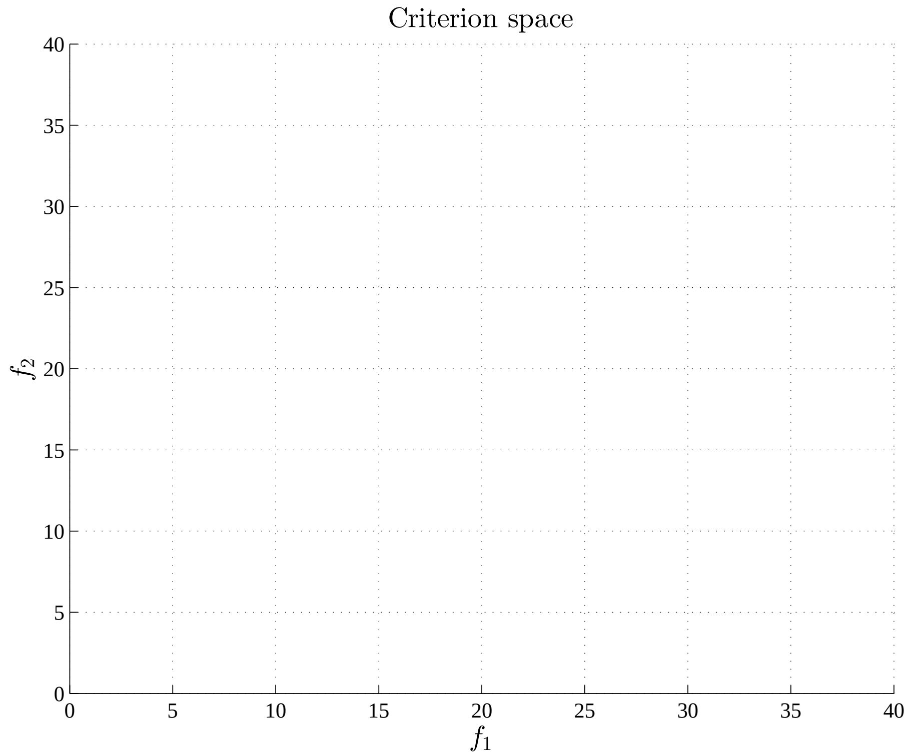

Figure 2: Coordinate system for plotting the criterion space Pareto front in exercise 4.

Exercise 5: Question 1: (points 25%)

A system with an **input** u and an **output** y and zero initial conditions is described by the following differential equation:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y = 5\frac{du}{dt} + u, \tag{7}$$

a) Derive the impulse response of the system.

From now on, we assume: u is a wide-sense stationary, zero-mean Gaussian random process with σ_u^2.

b) Determine the autocorrelation function R_{yy}.

c) Derive the power spectrum S_{yy}.

Assume now that the system is replaced by a square law detector; that is, a nonlinear system without memory. In other words, from now on our system is described by:

$$y = u^2. \tag{8}$$

d) Verify that the output of the system is no longer Gaussian.

e) Determine the autocorrelation function R_{yy} of the output and its variance.

Hint:
For zero-mean Gaussian random variables x_1, x_2, x_3, x_4, the following equality holds: $E\,[x_1x_2x_3x_4] = E\,[x_1x_2]\,E\,[x_3x_4] \;+\; E\,[x_2x_3]\,E\,[x_1x_4] \;+\; E\,[x_1x_3]\,E\,[x_2x_4]$.

Question 6: (points 15%)

In a digital communication system, consider a source whose output under hypothesis H_1 is a constant voltage of value m, while its output under H_0 is zero. The received signal is corrupted by N, an additive white Gaussian noise of zero mean, and variance σ^2.

a) Find the probability density function of the output under both hypotheses.

b) Calculate the log-likelihood function.

c) Find the MAP decision rule for the following a priori probability distributions: $P[H_0] = P[H_1] = 0.5$

Question 7: (points 10%)

We wish to observe a variable Y of a system. But the observation X is actually

$$X = 0.9Y + W$$

where W is a normal random variable which is independent of Y. The variances $\sigma_W = \sigma_Y = 1$ and the mean values $\mu_W = \mu_Y = 0.2$ are given.

Find the best linear estimator of Y

$$\hat{Y} = a + bX \tag{9}$$