

## Written examination in the course

### Optimisation Theory and Stochastic Processes

Thursday June 7th 2012

kl. 9 - 13 (4 hours)

All usual helping aids are allowed, i.e. books, notes, calculator, computer etc. All communication equipment and computer communication protocols must be turned off.

The questions should be answered in English.

**REMEMBER** to write your study number and page number on all sheets handed in.

The set consists of seven exercises. The total weighting for each of the exercises is stated in percentage. Sub-questions in each exercise have equal weight. You need 50 % in order to pass the exam.

*It should be clear from the solution, which methods are used, and there should be a sufficient number of intermediate calculations, so the line of thought is clear.*

### Exercise 4: (12 %)

A multi-objective optimisation problem is formulated as:

$$\text{minimise} \quad \begin{aligned} f_1(\mathbf{x}) &= (x_1 - 4)^2 + (x_2 - 2)^2 \\ f_2(\mathbf{x}) &= (x_1 - 4)^2 + (x_2 - 8)^2 \end{aligned} \quad (5)$$

Subject to the constraints:

$$\begin{aligned} g_1(\mathbf{x}) &= x_2 - 7 \leq 0 \\ g_2(\mathbf{x}) &= -x_1 - x_2 + 8 \leq 0 \end{aligned} \quad (6)$$

The two contour curves along with the constraints are shown in figure 1.

- a) Illustrate the Pareto optimal points in figure 1 (the page should be handed in with the solution).
- b) Sketch the Pareto front in the criterion space. The sketch should be based on function values from the contour plot. A coordinate system may be found in figure 2.
- c) Indicate where the utopia point is located for the given problem.
- d) A weighting method is used in solving the problem (i.e. minimise  $U = w_1 f_1(\mathbf{x}) + w_2 f_2(\mathbf{x})$ ), with  $w_1 = w_2 = 1$ . What is the solution to this problem - give both function value  $U$ , and design variables  $x_1$  and  $x_2$ . Hint: Use the results from questions a) and b).

# Optimisation Theory - Exam 12

## 1 - Optimisation Problem:

$$\text{Minimize } f(x) = (x_1 - 1)^2 + (x_2 - 1)^2$$

$$\text{Subject to } h(x) = x_1 + x_2 - 4 = 0$$

a - Set up the Lagrangian Function

$$L(x, v) = f(x) + \sum_{j=1}^p v_j h_j(x) = f(x) + v^T h(x)$$

$v_j, j = 1 \dots p$  are Lagrangian multipliers

$L$  and  $f$  will have the same optimum,

$$v^T h(x) = 0$$

Thus:

$$L(x) = (x_1 - 1)^2 + (x_2 - 1)^2 + v \cdot (x_1 + x_2 - 4)$$

b - Find the points that satisfy the kKT necessary conditions:

$$\frac{\partial L}{\partial x_i} = \underbrace{\frac{\partial f}{\partial x_i}}_{\nabla f} + \sum_{j=1}^p v_j \underbrace{\frac{\partial h_j}{\partial x_i}}_{\nabla h_j} + \sum_{i=1}^m \underbrace{\frac{\partial g_i}{\partial x_i}}_{\nabla g_i} = 0$$

$$\frac{\partial L}{\partial x_1} = 2x_1 - 2 + v = 0 \quad |$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - 2 + v = 0 \quad |$$

$$\frac{\partial L}{\partial v} = x_1 + x_2 - 4 = h(x) = 0, \text{ since eq. constraint recovered.} \quad |$$

Using 1, 2 & 3 to solve for  $v$ :

$$\frac{(2-v)}{2} + \frac{2-v}{2} - 4 = 0 \\ ||$$

$$v = -2$$

|

# - Optimisation Theory - Exam 12

Inserting  $v = -2$  in 1 & 2 and solves for  $x_1$  and  $x_2$

$$\begin{cases} 2x_1 - 2 + (-2) = 0 \\ 2x_2 - 2 + (-2) = 0 \end{cases}$$

↓

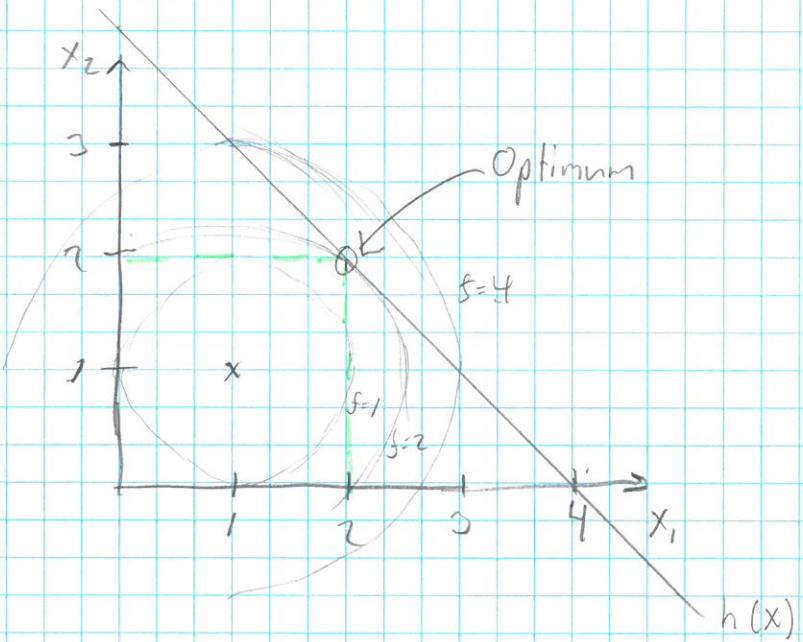
$$x_1 = \frac{4}{2} = 2$$

$$x_2 = \frac{4}{2} = 2$$

Thus the objective function is minimized to:

$$f(x) = (2-1)^2 + (2-1)^2 \Rightarrow f(x^*) = 2$$

- Check by graphical solution:



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## 2 - Gradient based Method

Minimization of:

$$f(x) = (1-x_1)^2 + (x_2 - 2)^2 + 2 \cdot x_1$$

The starting point is:

$$x^{(0)} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}^T$$

- 1. iteration of the steepest descent method:

$$C = \nabla f = \begin{bmatrix} 2x_1 - 2 + 2 \\ 2x_2 - 4 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 - 4 \end{bmatrix}$$

\* The gradient is evaluated at the starting point,  $x^{(0)}$ :

$$C^{(0)} = \begin{bmatrix} 2 \cdot 3 \\ 2 \cdot 1 - 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

\* Check the norm of the gradient:

$$\|C^{(0)}\| = \sqrt{6^2 + (-2)^2} = 6.3246$$

\* Compared with the tolerance,  $\varepsilon$ :

$$6.3246 > \varepsilon, \quad \varepsilon = 0.061$$

\* Continue:

\* The -direction of steepest descent:

$$d^{(0)} = -C^{(0)}$$

$$d^{(0)} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

\* Calculate the step size,  $\alpha_0$ , the minimizes

$$f(\alpha) = f(x^{(0)} + \alpha_0 d^{(0)})$$

$$\begin{aligned} f(\alpha) &= (1 - (3 + \alpha \cdot (-6)))^2 + ((1 + \alpha \cdot 2) - 2)^2 + 2 \cdot (3 + \alpha \cdot (-6)) \\ &= 40\alpha^2 - 40\alpha + 11 \end{aligned}$$

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The 1D line search should be solved analytically.

Utilizing:  $x^{(0)} + \alpha d^{(0)} = [3 + \alpha(-2) \quad 1 + \alpha \cdot 2]^T$

$$\frac{df(\alpha)}{d\alpha} = 2 \cdot 10\alpha - 40 = 80\alpha - 40$$

Setting  $f'(\alpha) = 0$

$$80\alpha - 40 = 0$$

$$\Downarrow \alpha = 1/2$$

\* Update design

$$x^{(0+1)} = x^{(0)} + \alpha_0 d^{(0)}$$

$$x^{(1)} = [3] + 1/2 [-6] = [0]$$

- 1. iteration of Newton's methods:

Minimization of:

$$f(x) = (1 - x_1)^2 + (x_2 - 2)^2 + 2 \cdot x_1$$

The starting point is:

$$x^{(0)} = [3 \quad 1]^T$$

\* The gradient vector:

$$c = \nabla f = \begin{bmatrix} 2x_1 - 2 + 2 \\ 2x_2 - 4 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 - 4 \end{bmatrix}$$

\* The gradient vector evaluated at the starting point:

$$c^{(0)} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

\* Check the norm of  $c^{(0)}$ :

$$\|c^{(0)}\| = \sqrt{6^2 + (-2)^2} = 6,3246$$

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\* Compare w/ tolerance:

$$6,3246 > \epsilon$$

\* Continue:

\* Calculate the Hessian matrix,  $H^{(0)}$

$$H^{(0)} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

$$\frac{\partial^2 f}{\partial x_1^2} = 2$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = 0$$

$$\frac{\partial^2 f}{\partial x_2^2} = 2$$

Thus

$$H^{(0)} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

- constant for all iterations, since no variables.

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\* Calculate the eigenvalues:

$$|A - \lambda I| = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} = 0$$

$$= (2-\lambda)(2-\lambda) = 0$$

$$= 4 - 2\lambda + \lambda^2 - 2\lambda = 0$$

$$= \lambda^2 - 4\lambda + 4 = 0$$

$$= \lambda = \frac{4 \pm \sqrt{16-4 \cdot 4}}{2}$$

$$= \lambda_{1,2} = 2$$

$$= \lambda_1 = 2, \lambda_2 = 2$$

Thus, the eigenvalues are positive, which means that the Hessian is positive definite, i.e. Newtons Methods can be applied.

\* Calculate the new search direction:

$$d^{(0)} = -H^{-1} c^{(0)}$$

$$d^{(0)} = - \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

\* Check the descent condition:

$$c^{(0)} d^{(0)} < 0$$

$$\begin{bmatrix} 6 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -18 \\ -2 \end{bmatrix} < 0$$

The descent condition is satisfied.

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\* Update the design

$$x^{(0+1)} = x^{(0)} + \alpha_0 d^{(0)}$$

-  $\alpha_0 \rightarrow$  Step size

$$\alpha_0 = 1/2$$

$$x^{(1)} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1,5 \\ 1,5 \end{bmatrix}$$

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## 3 - Linearised subproblem

Minimised  $f(x) = -x_1^2 + 3x_2^2 + x_1x_2 - 3$

Subject to  $g_1(x) = x_1 - 2x_2 \leq 0$

$$g_2(x) = x_1 - 2x_2^3 \leq 0$$

$$g_3(x) = x_1 \geq 0$$

$$g_4(x) = x_2 \geq 0$$

Lecture 6,

Slide 11

Book,

P. 502.

- Linearise the problem at the point

$$(x_1, x_2) = (1, 1) = x^{(0)}$$

- Evaluating the cost function at  $x^{(0)}$ :

$$f(1, 1) = -1^2 + 3 \cdot 1^2 + 1 \cdot 1 - 3 = 0$$

- Evaluating the constraints at  $x^{(0)}$ :

$$g_1(1, 1) = 1 - 2 \cdot 1 \leq 0 \Rightarrow -1 \leq 0 \quad (\text{Inactive})$$

$$g_2(1, 1) = 1 - 2 \cdot 1^3 \leq 0 \Rightarrow -1 \leq 0 \quad (\text{Inactive})$$

$$g_3(1, 1) = -1 \leq 0 \quad -\text{Sign change}$$

$$g_4(1, 1) = -1 \leq 0$$

- The gradient of the cost and constraints at  $x^{(0)}$

$$C^{(0)} = \nabla f(1, 1) = \begin{bmatrix} -2 \cdot 1 + 1 \\ 6 \cdot 1 + 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$$

$$\nabla g_1(1, 1) = \begin{bmatrix} -1/2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\nabla g_2(1, 1) = \begin{bmatrix} 1 \\ -6 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

$$\nabla g_3(1, 1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\nabla g_4(1, 1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Scan Values

fn gradients

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- Linearize using 1. order Taylor Approx.

$$\bar{f} = f(x^{(0)}) + \nabla f(x^{(0)}) \cdot d \quad , \quad d = \Delta x^{(0)}$$

$$\bar{g} = g(x^{(0)}) + \nabla g^T(x^{(0)}) \cdot d$$

Linearisation

$$\bar{f} = 0 + \begin{bmatrix} -1 & 7 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = -d_1 - 7d_2$$

$$\bar{g}_1 = -1 + \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = -1 - d_1 + 2d_2 \leq 0 \\ = -d_1 + 2d_2 \leq 1$$

$$g_2 = -1 + \begin{bmatrix} 1 & 6 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = -1 + d_1 + 6d_2 \leq 0 \\ = d_1 + 6d_2 \leq 1$$

$$g_3 = -1 + \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = -1 - d_1 \leq 0 \\ = -d_1 \leq 1$$

$$g_4 = -1 + \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = -1 - d_2 \leq 0 \\ = -d_2 \leq 1$$

Thus:

Minimise  $\bar{f} = c^T d$

$$= \begin{bmatrix} -1 & 7 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

Subject to  $A^T d \leq b$

$$A = \left[ \begin{array}{cc|cc} -1 & 1 & -1 & 0 \\ 2 & 6 & 0 & -1 \end{array} \right]$$

$$b = \left[ \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right]$$

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- Solve the linearised sub-problem:

$$\text{Minimise } f(x) = -x_1 + 7x_2$$

$$\text{Subject to } g_1(x) = -x_1 + 2x_2 \leq 1$$

Book:  
p. 314

$$g_2(x) = x_1 + 6x_2 \leq 1$$

$$x_1, x_2 \geq 0$$

Lecture 5, Slide 4 Since  $g_{1-4} \leq b_{1-4}$ , the slack variables  $x_{3-6}$  are introduced:

$$g_1(x) = -x_1 + 2x_2 + x_3 = 1$$

$$g_2(x) = x_1 + 6x_2 + x_4 = 1$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$-x_1 + x_6 = 1$$

- Simplex Method:

Book,  
p. 314

- Setting up the Tableau:

Basic ↓	$x_1$	$x_2$	$x_3$	$x_4$	b	b/a
1: $x_3$	-1	2	1	0	1	$1/-1 = 1$ - Negative
2: $x_4$	1	6	0	1	1	$1/1 = 1$
3: C	-1	7	0	0	5	

- Basic Variables:  $x_3, x_4$  Due to it only being in the identity matrix

- Non-Basic Variables:  $x_1, x_2$  being in the identity matrix

- Pivot Column:  $x_1$ , since c is lowest

- Pivot Row:  $x_3$ , since it has the lowest ratio

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Gauss-Jordan reduction:

- Pivot under pivot column

Basic ↓	$x_1$	$x_2$	$x_3$	$x_4$	b
$x_1$	0	8	1	1	2
$x_3$	1	6	0	1	1
c	0	13	0	1	5+1

- Add row 2 to 3

- Add row 2 to 1

Basic variable:  $x_1, x_3$

Non-basic variable:  $x_2, x_4$

Since the non-zero entries are non-negative in the cost row, the optimum is reached

Optimum:  $f(x) = -1$ .

# Optimisation Theory - Exam 12

## 4 - Multi-objective optimisation problem

$$\text{Minimise } f_1(x) = (x_1 - 4)^2 + (x_2 - 2)^2$$

$$f_2(x) = (x_1 - 4)^2 + (x_2 - 8)^2$$

Subject to:  $g_1(x) = x_2 - 7 \leq 0$

$$g_2(x) = -x_1 - x_2 + 8 \leq 0$$

### - Pareto point

A point  $x^*$  in the feasible design space  $S$  is Pareto Optimal if there does not exist another point,  $x$ , in the set  $S$ , for which  $f(x) \leq f(x^*)$

(Where the lowest value of each intersects)

### - Criterion Space:

Point	$f_1$	$f_2$
1	25	1
2	16	4
3	9	9
4	4	16
5	2	25

### - Utopia Point

A point where the minimum of each function intersects.

- Utopia point  $\approx (2, 2)$ .

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## - Weighting Method:

Lecture 8,  
Slide 26

minimise  $V = w_1 f_1(x) + w_2 f_2(x)$ , with  
 $w_1 = w_2 = 1$

Point	$f_1$	$f_2$	$V$
1	25	1	26
2	16	4	20
3	9	9	18
4	4	16	20
5	2	25	27

Smallest

Thus  $(9,9)$  is the point closest to  
 the utopian point, hence  
 comprise point  $\Rightarrow$  Pareto Optimal.

or

$$\frac{\partial V}{\partial x_1} = 0 \Rightarrow x_1 =$$

$$\frac{\partial V}{\partial x_2} = 0 \Rightarrow x_2 =$$

Inserting in  $V$  yields:

$$V =$$