AALBORG UNIVERSITET

Written examination in the course

Optimisation Theory and Stochastic Processes

Thursday June 7th 2012

kl. 9 - 13 (4 hours)

All usual helping aids are allowed, i.e. books, notes, calculator, computer etc. All communication equipment and computer communication protocols must be turned off.

The questions should be answered in English.

REMEMBER to write your study number and page number on all sheets handed in.

The set consists of seven exercises. The total weighting for each of the exercises is stated in percentage. Sub-questions in each exercise have equal weight. You need 50 % in order to pass the exam.

It should be clear from the solution, which methods are used, and there should be a sufficient number of intermediate calculations, so the line of thought is clear.

Exercise 1: (10 %)

The following optimization problem is considered:

$$\begin{aligned} \text{Minimize} \quad f(\mathbf{x}) &= (x_1-1)^2+(x_2-1)^2 \\ \text{Subject to} \quad h(\mathbf{x}) &= x_1+x_2-4 \;=\; 0 \end{aligned} \tag{1}$$

a) Set up the Lagrangian function and find point(s) satisfying the KKT necessary conditions.

b) Check if the point(s) is an optimum point using the graphical method (make a simple sketch).

Exercise 2: (15 %)

We will consider gradient-based minimization of the following unconstrained function:

$$f(\mathbf{x}) \;=\; (1-x_1)^2 \;+\; (x_2-2)^2 \;+\; 2\cdot x_1 \tag{2}$$

The starting point is: $\mathbf{x}^{(0)} = [\;3 \quad 1\;]^T$.

a) Complete the first iteration of the steepest descent method for the function. The 1D line search problem should be solved analytically.

b) Determine the search direction for the first iteration of Newton's method for the function.

Exercise 3: (13 %)

An optimisation problem is given as:

$$\text{minimise} \qquad f(\mathbf{x}) = -x_1^2+3x_2^2+x_1x_2-3 \tag{3}$$

Subject to the constraints:

$$\begin{aligned} g_1(\mathbf{x}) &= \tfrac{1}{x_1}-2x_2 \;\leq\; 0 \\ g_2(\mathbf{x}) &= x_1-2x_2^3 \;\leq\; 0 \\ & x_i \geq 0 \quad \forall \quad x_i = \{1,2\} \end{aligned} \tag{4}$$

a) Linearise the problem at the point $(x_1,x_2)=(1,1)$, and write up the linearised subproblem. Note that there is no need to do a normalisation of the problem!

b) Solve the linearised sub-problem using tableaus and the basic steps of the Simplex method

Exercise 4: (12 %)

A multi-objective optimisation problem is formulated as:

$$\text{minimise} \qquad \begin{aligned} f_1(\mathbf{x}) &= (x_1-4)^2+(x_2-2)^2 \\ f_2(\mathbf{x}) &= (x_1-4)^2+(x_2-8)^2 \end{aligned} \tag{5}$$

Subject to the constraints:

$$\begin{aligned} g_1(\mathbf{x}) &= x_2 - 7 \leq 0 \\ g_2(\mathbf{x}) &= -x_1 - x_2 + 8 \leq 0 \end{aligned} \tag{6}$$

The two contour curves along with the constraints are shown in figure 1.

a) Illustrate the Pareto optimal points in figure 1 (the page should be handed in with the solution).

b) Sketch the Pareto front in the criterion space. The sketch should be based on function values from the contour plot. A coordinate system may be found in figure 2.

c) Indicate where the utopia point is located for the given problem.

d) A weighting method is used in solving the problem (i.e. minimise $U = w_1 f_1(\mathbf{x}) + w_2 f_2(\mathbf{x})$), with $w_1 = w_2 = 1$. What is the solution to this problem - give both function value U, and design variables x_1 and x_2. ***Hint:*** *Use the results from questions a) and b).*

Page to be handed in with the solution!

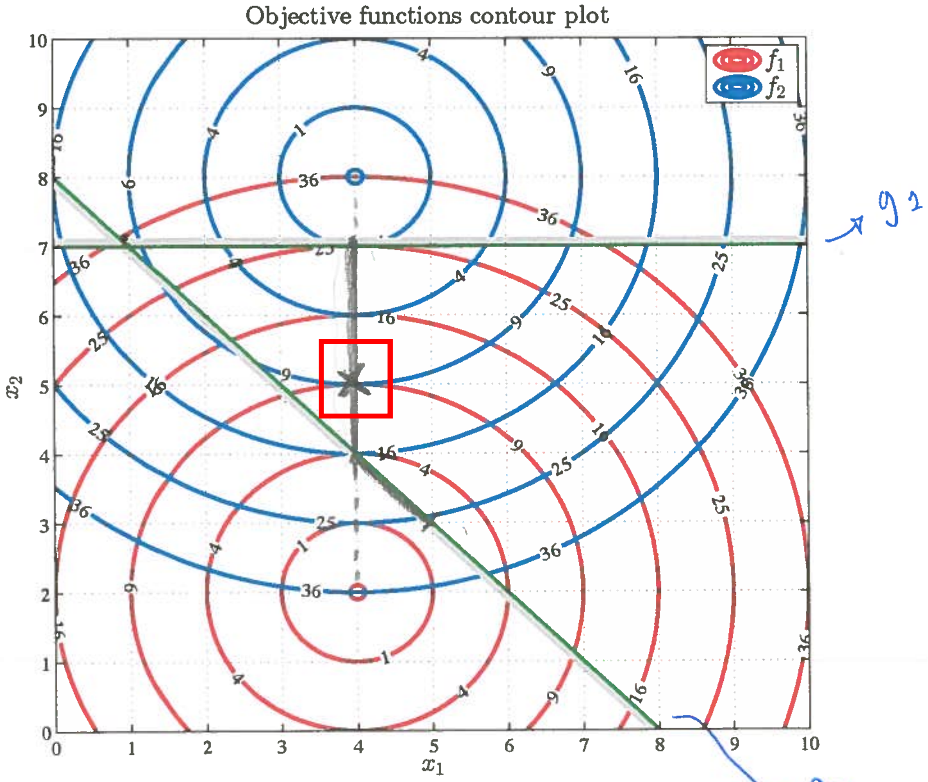


Figure 1: Contour curves for the problem of exercise 4.

Utopia point (4,1) from Victor.

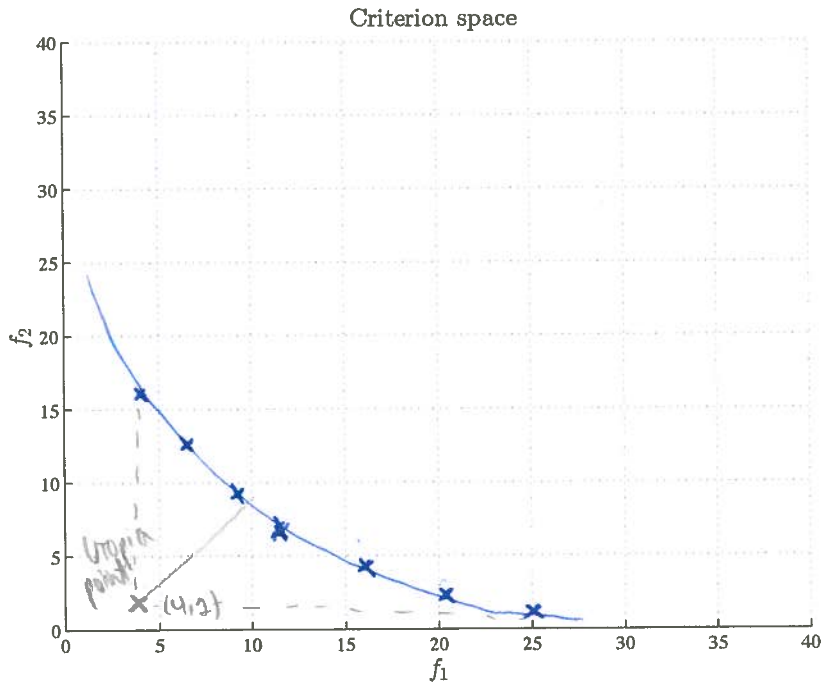


Figure 2: Coordinate system for plotting the criterion space Pareto front in exercise 4.

Exercise 5: Question 1: (points 25%)

A system with an **input** u and an **output** y and zero initial conditions is described by the following differential equation:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y = 5\frac{du}{dt} + u, \tag{7}$$

a) Derive the impulse response of the system.

From now on, we assume: u is a wide-sense stationary, zero-mean Gaussian random process with σ_u^2.

b) Determine the autocorrelation function R_{yy}.

c) Derive the power spectrum S_{yy}.

Assume now that the system is replaced by a square law detector; that is, a nonlinear system without memory. In other words, from now on our system is described by:

$$y = u^2. \tag{8}$$

d) Verify that the output of the system is no longer Gaussian.

e) Determine the autocorrelation function R_{yy} of the output and its variance.

Hint:
For zero-mean Gaussian random variables x_1, x_2, x_3, x_4, the following equality holds: $E\,[x_1x_2x_3x_4] = E\,[x_1x_2]\,E\,[x_3x_4] + E\,[x_2x_3]\,E\,[x_1x_4] + E\,[x_1x_3]\,E\,[x_2x_4]$.

Question 6: (points 15%)

In a digital communication system, consider a source whose output under hypothesis H_1 is a constant voltage of value m, while its output under H_0 is zero. The received signal is corrupted by N, an additive white Gaussian noise of zero mean, and variance σ^2.

a) Find the probability density function of the output under both hypotheses.

b) Calculate the log-likelihood function.

c) Find the MAP decision rule for the following a priori probability distributions: $P[H_0] = P[H_1] = 0.5$

Question 7: (points 10%)

We wish to observe a variable Y of a system. But the observation X is actually

$$X = 0.9Y + W$$

where W is a normal random variable which is independent of Y. The variances $\sigma_W = \sigma_Y = 1$ and the mean values $\mu_W = \mu_Y = 0.2$ are given.

Find the best linear estimator of Y

$$\hat{Y} = a + bX \tag{9}$$

system with an input u and an output y and zero initial onditions is described by the following differential equation:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y = 5\frac{du}{dt} + u$$

a) Derive the impulse of the system:

Laplace:

$$y(s^2+2s+3) = u(5s+1)$$

$$\frac{y}{u} = \frac{5s+1}{s^2+2s+3} = H(s)$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = 5e^{-t}\cdot\left(\cos(\sqrt{2}t) - \frac{2\sqrt{2}\cdot\sin(\sqrt{2}\cdot t)}{5}\right)$$

b) Determine the autocorrelation R_{yy}.

$$R_{yy} = \mathcal{L}^{-1}\{S_{yy}\}$$

$$S_{yy} = S_{uu}\,|H(s)|^2$$

$S_{uu} = \sigma_u^2$ since u is a zero-mean white noise random sequence. slide 30 lect. 4.

$$= \sigma_u^2\cdot\frac{(5s+1)^2}{(s^2+2s+3)^2}$$

$$R_{yy} = \mathcal{L}^{-1}\left\{\sigma_u^2\,\frac{(5s+1)^2}{(s^2+2s+3)^2}\right\}$$

$$= \frac{1}{4}\sigma_u^2\, e^{-t}\left(34\cdot t\,\cos(\sqrt{2}t) - \sin(\sqrt{2}t)\sqrt{2}(-33+40t)\right)$$

c) Determine the power spectrum S_{yy}

$$S_{yy} = \sigma_u^2\,\frac{(5s+1)^2}{(s^2+2s+3)^2}$$

Question 6 ... (cut off)

$H_1: \; Y = m + N$

$H_0: \; Y = N$

N is an additive white Gaussian noise of zero mean and variance σ^2

a) Find the probability density function of Y ~~of~~ under bothhypothes

$$f(N) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{1}{2\sigma^2} N^2\right\}$$

$$f(y|H_1) = f(N)\Big|_{N=Y-m} = \frac{1}{\sqrt{2\pi}\,\sigma_y} \exp\left\{-\frac{1}{2\sigma_y^2}(Y-m)^2\right\}$$

$$f(y|H_0) = f(N)\Big|_{N=Y} = \frac{1}{\sqrt{2\pi}\,\sigma_y} \exp\left\{-\frac{1}{2\sigma_y^2}(Y)^2\right\}$$

b) Calculate the log-likelyhood function:

$$L(y) = \frac{f(y|H_1)}{f(y|H_0)} = \frac{\frac{1}{\sqrt{2\pi}\,\sigma_y}\exp\left\{-\frac{1}{2\sigma_y^2}(y-m)^2\right\}}{\frac{1}{\sqrt{2\pi}\,\sigma_y}\exp\left\{-\frac{1}{2\sigma_y^2}y^2\right\}}$$

$$= \exp\left\{-\frac{1}{2\sigma_y^2}\left[y^2-(y-m)^2\right]\right\}$$

$$\ell(y) = \ln\{L(y)\} = -\frac{1}{2\sigma_y^2}\left[y^2-(y-m)^2\right]$$

c) Find the MAP decision rule for the following a priori probability distributions: $P[H_0] = P[H_1] = 0,5$

$$\ell(y) \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{P[H_0]}{P[H_1]}\right) = 0$$

$$-\frac{1}{2\sigma_y^2}\left[y^2 - \underbrace{(y-m)^2}_{=y^2+m^2-2my}\right] \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$m^2 - 2my \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$\boxed{y \underset{H_0}{\overset{H_1}{\gtrless}} \frac{m^2}{2m}}$$

We wish to observe a variable Y of a system. But the observation is actually:

$X = 0{,}9\,Y + W$

where W is a normal random variable which is independent of Y. $\sigma_W = \sigma_Y = 1$, $\mu_W = \mu_Y = 0{,}2$ ~~$\mu_X = 0{,}4$~~

Find the best linear estimator of Y

$$\hat{Y} = a + bX$$

$$\hat{Y} = h_0 + \sum_{m=1}^{M} h_m X(m)$$

$$h_0 = a = \mu_Y - (\bar{h})^T \mu_X$$

$$\bar{h} = b = (\Sigma_{XX})^{-1} \Sigma_{XY}$$

slide 16, lect. 7

$$\Sigma_{XX} = E\left[(X-\mu_X)(X-\mu_X)^T\right] = \sigma_X^2 = 0{,}9^2\sigma_Y^2 + \sigma_W^2$$

$$\Sigma_{XY} = E\left[(X-\mu_X)(Y-\mu_Y)\right]$$ where $X = 0{,}9Y + W$ and $\mu_X = 0{,}9\mu_Y + \mu_W$

$$= E\left[\left((0{,}9Y+W)-(0{,}9\mu_Y+\mu_W)\right)(Y-\mu_Y)\right]$$

$$= E\left[\left(0{,}9(Y-\mu_Y)+W-\mu_W\right)(Y-\mu_Y)\right]$$

$$= 0{,}9\,E\left[(Y-\mu_Y)^2\right] + \underbrace{E\left[(W-\mu_W)(Y-\mu_Y)\right]}_{=0\text{ ? why?}}$$

from solution, lect. 7.

$$= 0{,}9\,\sigma_Y^2$$

$$\bar{h} = \Sigma_{XX}^{-1} \cdot \Sigma_{XY} = \frac{0{,}9\sigma_Y^2}{\sigma_X^2} = \frac{0{,}9\,\sigma_Y^2}{0{,}9^2\sigma_Y^2 + \sigma_W^2} = \frac{0{,}9}{0{,}9^2+1} = 0{,}4972$$

$$h^0 = \mu_Y - (\bar{h})^T\mu_X = \mu_Y - (\bar{h})^T(0{,}9\mu_Y + \mu_W)$$

$$= 0{,}2 - 0{,}4972\,(0{,}9 \cdot 0{,}2 + 0{,}2) = 0{,}011$$

EXAM 2012

Exercise 1:

LECTURE 2/4.45

The following optimization problem is considered:

$$\text{Minimize} \quad f(x) = (x_1 - 1)^2 + (x_2 - 1)^2$$

$$\text{Subject to} \quad h(x) = x_1 + x_2 - 4 = 0.$$

a) Set up the Lagrangian function and find point(s) satisfying the KKT necessary conditions.

b) Check if the point(s) is an optimum using the graphical method (simple sketch)

The Lagrangian is formulated as,

$$L(x) = f(x) + v \cdot h(x) = (x_1 - 1)^2 + (x_2 - 1)^2 + v \cdot (x_1 + x_2 - 4)$$

Next step is to check the KKT necessary conditions:

$$\frac{\partial L}{\partial x_1} = 2 \cdot (x_1 - 1) + v = 0 \qquad ① \; 2x_1 + v - 2 = 0 \rightarrow v = 2 - 2x_1 \; ④$$

$$\frac{\partial L}{\partial x_2} = 2 \cdot (x_2 - 1) + v \qquad ② \; 2x_2 + v - 2 = 0 \qquad v = 2 - 4 = -2 \quad \boxed{v = -2}$$

$$\frac{\partial L}{\partial v} = h(x) = x_1 + x_2 - 4 = 0 \qquad ③ \; x_1 + x_2 - 4 = 0$$

$$② + ④ \rightarrow 2x_2 + \cancel{2} - 2x_1 - \cancel{2} = 0 \; ; \; 2x_2 = 2x_1 \qquad x_2 = x_1$$

$$③ \; x_1 + x_1 - 4 = 0 \; ; \; 2x_1 = 4 \rightarrow \boxed{x_1 = 2} \qquad \boxed{x_2 = 2}$$

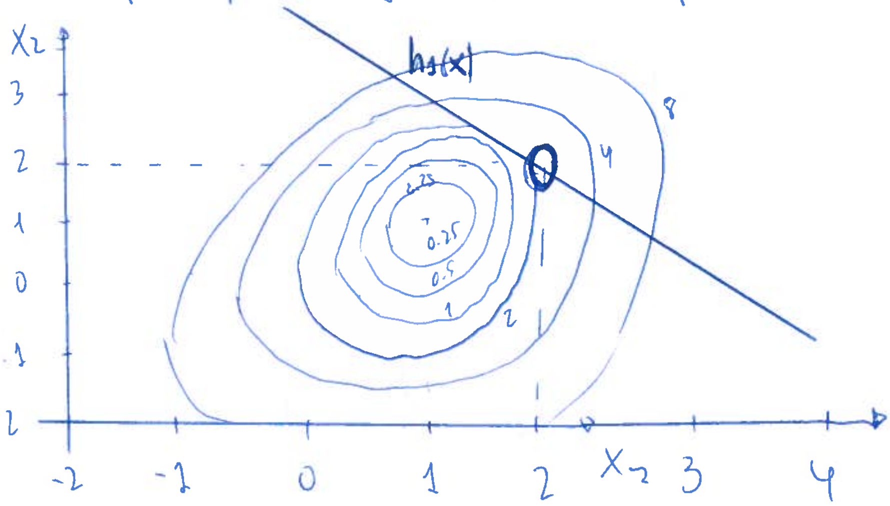
After solving (by hand or using the MATLAB script), the following solution is obtained

$$x_1 = 2 \qquad x_2 = 2 \qquad v = -2$$

Therefore, the objective is minimised to:

$$f(x^*) = (2-1)^2 + (2-1)^2 = 1^2 + 1^2 = 2 \rightarrow f(x^*) = 2.$$

To check if the solution is indeed an optimum point, we need to solve graphically. Using MATLAB script:



We can observe that the optimum point is also a GLOBAL MINIMUM! ✓ ☺

LECTURE 3 - Ex 10.52

Exercise 2:

We will consider a gradient-based minimization of the following unconstrained function:

$$f(x_1, x_2) = (1 - x_1)^2 + (x_2 - 2)^2 + 2 \cdot x_1 = x_1^2 + x_2^2 - 4x_2 + 5.$$

The starting point is: $x^{(0)} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \rightarrow x^{(0)} = (3, 1)$

a) Complete the 1st iteration of the steepest descent method for the function. The 1D line search problem should be solved analytically.

b) Determine the search direction for the 1st iteration of Newton's method.

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a) Function value at $x^{(0)} \rightarrow f(x^{(0)}) = 11$.

The procedure to apply the steepest descent method is shown in Arora p432

1) Calculate the gradient:

$$\nabla f(x) = c(x) = \begin{bmatrix} 2 \cdot x_1 \\ 2 \cdot x_2 - 4 \end{bmatrix}$$

Evaluated at the starting point $x^{(0)}$

$$c(x^0) = \begin{bmatrix} 2 \cdot 3 \\ 2 \cdot 1 - 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \rightarrow c^0 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

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Exercise 2 (continuation):

a) The direction of the steepest descent can be computed from the gradient:

$$d^{(0)} = -c(x^{(0)}) = -c^{(0)} = \begin{bmatrix} -6 \\ 2 \end{bmatrix} \rightarrow d^{(0)} = (-6, 2)$$

Next step is to compute the step size (analytically): 1st, we transformed the problem into a 1-D line search problem:

$$f(\alpha) = f(x^{(0)} + \alpha . d^{(0)}) \quad \text{where} \quad x^{(0)} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad d^{(0)} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

$$f(\alpha) = (3 - 6\alpha)^2 + (1 + 2\alpha)^2 - 4 \cdot (1 + 2\alpha) + 5.$$

$$f(\alpha) = 40\alpha^2 - 40\alpha + 11$$

The optimal size is determined by differentiating:

$$f'(\alpha) = \frac{\partial f(\alpha)}{\partial \alpha} = 80\alpha - 40 \qquad f'(\alpha) = 0 \rightarrow 80\alpha = 40 \rightarrow \alpha = \frac{4}{8} = \frac{1}{2} = 0{,}5$$

$$\boxed{\alpha = 0.5}$$

Therefore, the new design can be computed as,

$$x^{(1)} = x^{(0)} + \alpha \cdot d^{(0)} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 0.5 \begin{bmatrix} -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Being,
$$x^{(1)} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \rightarrow x^{(1)} = (0, 2)$$

Which gives the cost function: $f(x^{(1)}) = 1$.

The process may be repeated until the optimum is found.

b) 1st iteration of Newton's method → Determine search direction:

To determine the search direction for Newton's method

$$d^{(k)} = -\left[H(x^{(k)})\right]^{-1} \cdot \nabla f(x^{(k)}) \to d^{(k)} = -H^{k^{-1}} \cdot c(x^k)$$

Using the previous gradient and function evaluations for the starting point,

$$c^{(0)} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \qquad f(x^0) = 11 \qquad x^{(0)} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

We need to calculate the Hessian:

$$H(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$ which means that is constant!

and we should evaluate the eigen values.

$$|A - \lambda \cdot I| = \begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = \left[(2-\lambda)\cdot(2-\lambda) - 0\right] = 4 - 2\lambda - 2\lambda + \lambda^2$$

$$\lambda^2 - 4\lambda + 4 = 0 \to \lambda = \frac{4 \pm \sqrt{16 - 4 \cdot 4}^{\,0}}{2}$$

$\lambda_1 = 2$, $\lambda_2 = 2$

$\lambda \geq 0$ P.S.D

$\lambda \leq 0$ N.S.D

$\boxed{\lambda_{1,2} = 2}$

λ_1 & $\lambda_2 > 0$ P.D.

Both eigen values are positive

Calculating the search direction, we need the inverse of the Hessian:

$$H^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

Newton's → $x^{(1)} = x^{(0)} + \Delta x$

$x^{(0)} + \alpha \cdot d^{(0)}$

The search direction is then,

$$d^{(0)} = -H^{-1} \cdot c^{(0)} = -\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ +1 \end{bmatrix}$$

$$\boxed{d^{(0)} = \begin{bmatrix} -3 \\ +1 \end{bmatrix}}$$

The descent condition is checked,

$$c^{(0)} \cdot d^{(0)} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -18 \\ -2 \end{bmatrix} \to -20 < 0 \ \checkmark$$

EXAM 2012

LECTURE 6 - 12.15

Exercise 3: An optimisation problem is given as:

$$\text{Minimise} \qquad f(x) = -x_1^2 + 3x_2^2 + x_1 \cdot x_2 - 3$$

Subject to:

$$g_1(x) = \frac{1}{x_1} - 2x_2 \leq 0$$

$$g_2(x) = x_1 - 2x_2^3 \leq 0$$

$$x_i \geq 0 \quad \forall \; x_i = \{1, 2\}$$

a) Linearise the problem at the point $x^* = (x_1, x_2) = (1, 1)$ and write up the linearised subproblem. (Note that there's no need to normalise)

b) Solve the linearised sub-problem using Tableaus and the basic steps of the Simplex method.

---

a) Evaluate cost function at start point $f(x^0) = 0$.

Compute the gradient:

$$\nabla f(x) = c(x) = \begin{bmatrix} -2x_1 + x_2 \\ 6x_2 + x_1 \end{bmatrix}$$

Evaluated at $x^{(0)}$

$$c^{(0)} = \begin{bmatrix} -2 \cdot 1 + 1 \\ 6 \cdot 1 + 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix} \qquad c^{(0)} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$$

The gradients of the constraints are: $\qquad d = x - x^0$

$$\nabla g_1(x_1, x_2) = \begin{bmatrix} -1/x_1^2 \\ -2 \end{bmatrix}_{x^0} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}_{x^0} \rightarrow \bar{g}_1 = -1 \cdot (x_1 - 1) - 2 \cdot (x_2 - 1) = -x_1 - 2x_2 + 3$$

$$\nabla g_2(x_1, x_2) = \begin{bmatrix} 1 \\ -6x_2^2 \end{bmatrix}_{x^0} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}_{x^0} \rightarrow \bar{g}_2 = 1(x_1 - 1) - 6 \cdot (x_2 - 1) = x_1 - 6x_2 + 5$$

$$\nabla g_3(x_1, x_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}_{x^0} \rightarrow -1 \cdot (x_1 - 1) = -x_1 + 1$$

$$\nabla g_4(x_1, x_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}_{x^0} \rightarrow -1 \cdot (x_2 - 1) = -x_2 + 1$$

Add them to the right-side of new inequality $\bar{g}$ and $d_1 = -1$ and $d_2 = -7$.

which yields matrix A and vector b:

$$A = \begin{bmatrix} \overset{-1}{\cancel{-1/x_1^2}} & 1 & -1 & 0 \\ -2 & \underset{-6}{\cancel{-6x_2^2}} & 0 & -1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -1 + 2\cdot 1 \\ -1 + 2\cdot 1^3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The linearised subproblem may now be rewritten as: $\quad d_1 = x_1 - 1$, $d_2 = x_2 - 1$

$$\text{Minimise} \quad \bar{f} = [-1 \quad 7]\cdot\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \boxed{-d_1 + 7\cdot d_2} \qquad \bar{f} = c^T\cdot d$$

Apply simplex directly. $\quad$ Subject $A^T\cdot d \le b$

Subject to the constraints:

$$\begin{bmatrix} -1/x_1^2 & -2 \\ 1 & -6x_2^2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}\cdot\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} \overset{-d_1}{\cancel{-d_1/x_1^2}} & -2d_2 \\ d_1 & \overset{-d_2}{\cancel{-6x_2^2\cdot d_2}} \\ -d_1 & 0 \\ 0 & -d_2 \end{bmatrix} \le \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

We can finish here.

Written in terms of the original variables may be formulated as $(d = x - x^{(0)})$

$$\text{Minimise} \quad f(x_1,x_2) = f(x^{(0)}) + c^T\cdot d = 0 + [-1 \quad 7]\cdot\begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}$$

$$f(x_1,x_2) = -1\cdot(x_1 - 1) + 7\cdot(x_2 - 1) = 1 - x_1 + 7x_2 - 7$$

$$\boxed{\bar{f}(x_1,x_2) = -x_1 + 7x_2 - 6}$$

Subject to the constraints:

$$\bar{g}_1(x_1,x_2) = \frac{1}{x_1} - 2x_2 \le 0$$

$$g_2(x_1,x_2) = x_1 - 2x_2^3 \le 0$$

$$g_3(x_1,x_2) = -x_1 \le 0$$

$$g_4(x_1,x_2) = -x_2 \le 0.$$

EXAM 2012

Exercise 3 (Continuation)

b) Solve the linearised subproblem using tableaus and the basic steps of the simplex method.

Minimise: $f(x_1, x_2) = -x_1 + 7x_2 - 6$

Subject to:

$g_1 = 1/x_1 - 2x_2 \leq 0$

$g_2 = x_1 - 2x_2^3 \leq 0$

$g_3 = -x_1 \leq 0$

$g_4 = -x_2 \leq 0.$

Written as a standard linear programming problem:

Minimise $f(x_1, x_2) = -x_1 + 7 \cdot x_2 - 6$

Subject to:

$1/x_1 - 2x_2 + x_3 = 0$

$x_1 - 2x_2^3 + x_4 = 0$

$-x_1 \leq 0$

$-x_2 \leq 0$

$-x_3 \leq 0$

$-x_4 \leq 0.$

Or written in a matrix form.

Minimise $f(x) = c^T \cdot x$

Subject to: $A \cdot x = b$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad c = \begin{bmatrix} -1 \\ 7 \\ 0 \\ 0 \end{bmatrix} \qquad A = \begin{bmatrix} & -2 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The initial tableau may now be set up as:

Basic var. ↓	x_1	x_2	x_3	x_4	b	Ratio b_i/a_{ij}

EXAM 2012

Exercise 4:

A multi-objective optimisation problem formulated as:

$$\text{minimize} \quad f_1(x) = (x_1-4)^2 + (x_2-2)^2$$
$$f_2(x) = (x_1-4)^2 + (x_2-8)^2$$

Subject to the constraints:

$$g_1(x) = x_2 - 7 \leq 0$$
$$g_2(x) = -x_1 - x_2 + 8 \leq 0.$$

The 2 contour curves along with the constraints are shown in Fig 1.

a) Illustrate the Pareto optimal points in Figure 1.

b) Sketch the Pareto front in the criterion space. The sketch should be based on function values from the contour plot. A coordinate system is given.

c) Indicate where the utopia point is located for the given problem. $(4,1)$ f_1, f_2

minimo de f_1 y de f_2

d) A weighting method is used in solving the problem ($U = w_1 \cdot f_1(x) + w_2 \cdot f_2(x)$ with $w_1 = w_2 = 1$. What is the solution to this problem? Give both function value U, and design variables x_1 and x_2.

---

a) Drawn in Figure 1. It will goe inside the design space (constraints)

b) $g_1 \to \quad x_2 \leq 7$
$g_2 \to \quad x_1 \geq -x_2 + 8$

PLOT in Fig 2.

Pareto points	f_1	f_2
1 (4,4)	4	16
2 (4,4.5)	6.25	12.25
3 (4,5)	9	9
4 (4,5.5)	12.25	6.25
5 (4,6)	16	4
6 (4,6.5)	20.25	2.25
7 (4,7)	25	1

c) Utopia point → When f_1 and f_2 are minimum.

EXAM 2012

Exercise 5: A system with an input "u" and an output "y" and zero initial conditions is described by the following differential equation.

$$\frac{d^2y}{dt^2} + 2\cdot\frac{dy}{dt} + 3y = 5\cdot\frac{du}{dt} + u.$$

a) Derive the impulse response of the system.

Rewriting the differential equation:

$$\ddot{y} + 2\cdot\dot{y} + 3y = 5\cdot\dot{u} + u$$

Applying LAPLACE,

$$y\cdot(s^2 + 2s + 3) = u\cdot(5s + 1)$$

$$H(s) = \frac{Y(s)}{U(s)} = \frac{\text{output}}{\text{input}} = \frac{5s+1}{s^2+2s+3} \rightarrow H(s) = \frac{5\cdot s + 1}{s^2 + 2\cdot s + 3}$$

check in MAPLE.

In time domain:

$$\boxed{h(t) = \mathcal{L}^{-1}[H(s)] = 5\cdot e^{-t}\left(\cos(\sqrt{2}\cdot t) - \frac{2\cdot\sqrt{2}\cdot\sin(\sqrt{2}\cdot t)}{5}\right)}$$

From MATLAB → $5\cdot e^{-t}\cdot\left(\cos(\sqrt{2}\cdot t) - \frac{2\cdot\sqrt{2}\cdot\sin(\sqrt{2}t)}{5}\right) =$

Use ILAPLACE(F)!!

b) u is WSS, zero-mean Gaussian random process with σ_u^2.

Autocorrelation R_{yy}:

$$R_{yy} = \mathcal{L}^{-1}\{S_{yy}(f)\} \rightarrow S_{yy}(f) = S_{uu}\cdot|H(s)|^2 \qquad S_{uu} = \sigma_u^2$$

since u is WSS.
LECTURE 4 - Slide 30

$$S_{yy} = \sigma_u^2\cdot\frac{(5s+1)^2}{(s^2+2\cdot s+3)^2} \rightarrow R_{yy} = \mathcal{L}^{-1}\left\{\sigma_u^2\cdot\frac{(5s+1)^2}{(s^2+2s+3)^2}\right\}$$

$$\boxed{R_{yy} = \frac{1}{4}\cdot\sigma_u^2\cdot e^{-t}\cdot(34\cdot t\cdot\cos(\sqrt{2}\cdot t) - \sin(\sqrt{2}\cdot t))\cdot\sqrt{2}\cdot(-33+40t)}$$

c) Determine the power spectrum S_{yy}.

From the previous section:

$$\boxed{S_{yy} = \sigma_u^2\cdot\frac{(5s+1)^2}{(s^2+2s+3)^2}}$$

Now the system is replaced by a square wave detector, that is a nonlinear system without memory. Our system is described by:

$$y = u^2$$

d) Verify that the output of the system is no longer Gaussian.

e) Determine the autocorrelation function r_{yy} of the output and its variance.

EXAM 2012

Exercise 6: In a digital communication system, consider a source whose output under hypothesis H_1 is a constant voltage of value m, while its output under H_0 is zero.
The received signal is corrupted by N, an additive white Gaussian noise of zero mean and variance σ^2.

a) Find the probability density function (pdf) of the output under both hypotheses.

b) Calculate the log-likelihood function

c) Find the MAP decision rule for the following a priori probability distributions: $P[H_0] = P[H_1] = 0.5$.

---

a)
$$\begin{cases} H_1: & Y = m + N \\ H_0: & Y = N. \end{cases}$$

$$f(N) = \frac{1}{\sqrt{2\pi}\cdot\sigma}\cdot e^{-\frac{1}{2\cdot\sigma^2}\cdot N^2}$$

$$\boxed{\begin{aligned} f(y|H_1) &= f(N)\Big|_{N=y-m} = \frac{1}{\sqrt{2\pi}\cdot\sigma_y}\cdot e^{-\frac{1}{2\cdot\sigma_y^2}\cdot(y-m)^2} \\ f(y|H_0) &= f(N)\Big|_{N=y} = \frac{1}{\sqrt{2\pi}\cdot\sigma_y}\cdot e^{-\frac{1}{2\cdot\sigma_y^2}\cdot y^2} \end{aligned}}$$

b) log-likelihood:

$$L(y) = \frac{f(y|H_1)}{f(y|H_0)} = \frac{\frac{1}{\cancel{\sqrt{2\pi}}\cdot\sigma_y}\cdot e^{\frac{-1}{2\cdot\sigma_y^2}\cdot(y-m)^2}}{\frac{1}{\cancel{\sqrt{2\pi}}\cdot\sigma_y}\cdot e^{\frac{-1}{2\cdot\sigma_y^2}\cdot y^2}}$$

$$\frac{e^x}{e^y} = e^{x-y}$$

$$e^{\frac{-1}{2\cdot\sigma_y^2}\cdot(y-m)^2+\left(+\frac{1}{2\sigma_y^2}\cdot y^2\right)}$$

$$L(y) = e^{\frac{-1}{2\cdot\sigma_w^2}\cdot[y^2-(y-m)^2]}$$

$$\boxed{l(y) = \ln(L(y)) = \frac{-1}{2\cdot\sigma_w^2}\cdot[y^2-(y-m)^2]}$$

c) Find the MAP decision rule ($P(H_0)=P(H_1)=0.5$).

$$l(y) \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{P(H_0)}{P(H_1)}\right) = 0 \qquad (\ln(1)=0)$$

$$\frac{-1}{2\cdot\sigma_y^2}\cdot[\cancel{y^2}-(y-m)^2] \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$(\cancel{y^2}+m^2-2my) = -m^2+2my$$

$$+m^2 \pm 2my$$

$$+m^2+2my \underset{H_0}{\overset{H_1}{\gtrless}} 0 \rightarrow \cancel{\boxed{y \underset{H_0}{\overset{H_1}{\gtrless}} \frac{m^2}{2m}}} = \frac{m}{2}$$

$$\frac{m}{2\cdot\sigma^2}\cdot(m-2y)$$

$$\boxed{\begin{array}{l} y \overset{H_0}{>} \frac{m}{2} \\ y \underset{H_1}{<} \frac{m}{2} \end{array}}$$

$$V_2^2-V_1^2-2V_2\cdot y+2V_1\cdot y \qquad m^2 \underset{H_0}{\overset{H_1}{\gtrless}}$$

$$\cancel{m}(m-2y) \underset{}{\overset{H_1}{\gtrless}} 0$$

$$m-2y \underset{H_0}{\overset{H_1\ H_0}{\gtrless}} 0$$

$$(-V_2+V_1)\cdot(V_1-2y+V_2)$$

$$V_1^2-V_2^2+2y\cdot V_2-2y\cdot V_1$$

$$V_1^2-V_2^2$$

$$\boxed{y \overset{H_0}{>} \frac{m}{2} \qquad y \overset{H_1}{<} \frac{m}{2}}$$

$$(-V_2^2+V_1)\cdot(V_1-2y+V_2)$$

$$(y-V_2)^2 = \cancel{y^2}+V_2^2-2V_2\cdot y$$

$$(y-V_1^2) = \cancel{y^2}+V_1^2-2\cdot V_1\cdot y$$

$$V_2^2-V_1^2-2V_2y+2V_1\cdot y$$

$$V_2\cdot(-2y+V_2)+V_1\cdot(V_1+2y)$$

Exercise 7

EXAM 2012

We wish to observe a variable Y of a system. But the observation X is actually:

$$X = 0.9 \cdot Y + W,$$

where W is a normal random variable which is independent of Y.

The variances $\sigma_W = \sigma_Y = 1$ and the mean values $\mu_W = \mu_Y = 0.2$ are given.

Find the best linear estimator of Y:

$$\hat{Y} = h_0 + h^- \cdot X$$

$$\boxed{\hat{Y} = 0{,}011 + 0{,}4972 \cdot X}$$

$$\hat{Y} = a + bX$$

$$\hat{Y} = h_0 + \sum_{m=1}^{M} h_m \cdot X(m)$$

$$h_0 = a = \mu_Y - (h^-)^T \cdot \mu_X$$

$$h^- = b = (\Sigma_{XX})^{-1} \cdot \Sigma_{XY}$$

LECTURE 7 Slide 16.

- $\Sigma_{XX} = E\left[(X - \mu_X) \cdot (X - \mu_X)^T \right] = \sigma_X^2 = 0.9^2 \cdot \sigma_Y^2 + \sigma_W^2$
- $\Sigma_{XY} = E\left[(X - \mu_X) \cdot (Y - \mu_Y) \right] =$ where $X = 0.9 \cdot Y + W$ and $\mu_X = 0.9 \cdot \mu_Y + \mu_W$

$$= E\left[\left((0.9 \cdot Y + W) - (0.9 \cdot \mu_Y + \mu_W) \right) \cdot (Y - \mu_Y) \right]$$

$$= E\left[\left(0.9 \cdot (Y - \mu_Y) + W - \mu_W\right) \cdot (Y - \mu_Y) \right]$$

$$= 0{,}9 \cdot E\left[(Y - \mu_Y)^2 \right] + \underbrace{E\left[(W - \mu_W) \cdot (Y - \mu_Y) \right]}_{= 0}$$

From solution lecture 7.

$$\Sigma_{XY} = 0{,}9 \cdot \sigma_Y^2$$

- $h^- = (\Sigma_{XX})^{-1} \cdot \Sigma_{XY} = \dfrac{0.9 \cdot \sigma_Y^2}{\sigma_X^2} = \dfrac{0{,}9 \cdot \sigma_Y^2}{0.9^2 \cdot \sigma_Y^2 + \sigma_W^2} = \dfrac{0.9}{0{,}9^2 + 1} = 0{,}4972$

$$\boxed{h^- = 0{,}4972}$$

- $h^0 = \mu_Y - (h^-)^T \cdot \mu_X = \mu_Y - (h^-)^T \cdot (0{,}9 \mu_Y + \mu_W) = 0.2 - 0.4972 \cdot (0.9 \cdot 0{,}2 + 0{,}2)$

$$h^0 = 0{,}2 - 0{,}188936 = 0{,}011064 \qquad \boxed{h^0 = 0{,}011}$$

Possible exam questions:

- Modify Example_Quad1(2)_SteepestDescent to introduce the Conjugate Gradient Method (made with Steepest descent)

$$V = \left(\sum_{i=1}^{k} \left[w_i \left(f_i(x) - f_i^{(0)} \right) \right]^p \right)^{1/p}$$

$$\boxed{V = \sum_{i=1}^{k} w_i \cdot f_i(x)}$$

$$w_1 \cdot f_1(x) + w_2 f_2(x)$$

$$\left. \begin{aligned} 2x_1 - v - 4 &= 0 \\ 2x_2 - v + 2 &= 0 \\ -x_1 - x_2 + 4 &= 0 \end{aligned} \right\} \rightarrow$$

$$v = 2x_1 - 4$$

$$2x_2 - 2x_1 + 4 + 2 = 0$$

$$2x_2 - 2x_1 + 6 = 0$$

$$2x_2 = \frac{2x_1 - 6}{2}$$

$$x_2 = x_1 - 3$$

$$-x_1 - x_1 + 3 + 4 = 0$$

$$-2x_1 = -7$$

$$\boxed{x_1 = -7/2}$$

$$v = 2 \cdot \left(\frac{-7}{2} \right) - 4$$

$$v = \frac{-14}{2} - 4$$

$$v = -7 - 4 = -11$$

$$\boxed{v = -11}$$

$$x_2 = \frac{-7}{2} - 3$$

$$\boxed{x_2 = -6{,}5}$$

IMPORTANT:

- Lecture 4 → Slide 12

$$\frac{1}{x_1}$$

$$(2 - x_1)^2 = 2^2 + x_1^2 - 2 \cdot 2x_1 = x_1^2 - 4x_1 + 4$$

$$(x_1 - 2)^2 = x_1^2 + 4 + 4x_1$$

LECTURE 1 (7 ex): 2.1 | 2.3 | 2.5 | 2.14 | 2.20 | 3.1 | 3.2

LECTURE 2: (5 ex) 4.8 | 4.10 | 4.22 | 4.45 | 4.67

LECTURE 3 (5 exercises): 10.4 | 10.22 | 10.32 (Ying) | 10.42 (Ying) | 10.52

LECTURE 4 (5 ex): 10.67 | 11.9 | 11.10 | 11.21 | 11.22 (extra 11.1)

LECTURE 5: 8.4 | 8.38 | 8.44. (MATLAB: 8.55 | 8.57 | 8.59)

LECTURE 6: 5.14 | 12.2 | 12.15 | 12.26 ⇝ cylinder volume = 150 m^3

LECTURE 7: 12.51 (cylinder vol. = 150 m^3), 12.52 | 13.11. / Simulated Annealing ex.

LECTURE 8: 17.9 | 17.1 | 17.2 | 17.4.

400 + 200

Stochastic:

LECTURE 1:

LECTURE 2:

LECTURE 3:

LECTURE 4:

LECTURE 5:

LECTURE 6:

LECTURE 7:

LECTURE 8:

AALBORG UNIVERSITET

Written examination in the course

Optimisation Theory and Stochastic Processes

Tuesday June 11th 2013

kl. 9 - 13 (4 hours)

All usual helping aids are allowed, i.e. books, notes, calculator, computer etc. All communication equipment and computer communication protocols must be turned off.

The questions should be answered in English.

REMEMBER to write your study number and page number on all sheets handed in.

The set consists of seven exercises. The total weighting for each of the exercises is stated in percentage. You need 50 % in order to pass the exam.

It should be clear from the solution, which methods are used, and there should be a sufficient number of intermediate calculations, so the line of thought is clear.

Exercise 1: (10 %)

The following optimisation problem is considered:

$$\begin{aligned} \text{Minimise} \quad f(\mathbf{x}) &= (2-x_1)^2 + (x_2+1)^2 \\ \text{Subject to} \quad h(\mathbf{x}) &= -x_1 - x_2 + 4 = 0 \end{aligned} \tag{1}$$

a) Set up the Lagrangian function and find point(s) satisfying the KKT necessary conditions.

b) Check if the point(s) is an optimum point using the graphical method (make a simple sketch).

Exercise 2: (15 %)

We will consider gradient-based minimisation of the following unconstrained function:

$$f(\mathbf{x}) = 10(x_1^2 - x_2) + x_1^2 - 2x_1 + 5 \tag{2}$$

The starting point is: $\mathbf{x}^{(0)} = [\ -1 \quad 3\ ]^T$.

a) Complete the first iteration of the steepest descent method for the function. The 1D line search problem should be solved analytically.

b) Can Newton's method be applied for determining the search direction in iteration 1?
If yes, then determine the search direction.
If no, then state an alternative robust method for determining the search direction.

Exercise 3: (18 %)

An optimisation problem is given as:

$$\text{minimise} \qquad f(\mathbf{x}) = (x_1-1)^2 + 2\,(x_2-1)^2 - x_1 x_2 \tag{3}$$

Subject to the constraints:

$$\begin{aligned} g_1(\mathbf{x}) &= 5 - x_1 - x_2 &\leq 0 \\ g_2(\mathbf{x}) &= x_1^2 - x_2 - 36 &\leq 0 \\ g_3(\mathbf{x}) &= \tfrac{x_1^2}{4} - x_1 + x_2 - 8 &\leq 0 \\ g_4(\mathbf{x}) &= -x_1 &\leq 0 \\ g_5(\mathbf{x}) &= -x_2 &\leq 0 \end{aligned} \tag{4}$$

a) Complete one iteration of the Sequential Linear Programming (SLP) method for the above problem, where you solve step 4 graphically. Use a starting point of $\mathbf{x}^{(0)} = (5,5)$ and 20% move limits. As a help the contour plot of the linearised objective function for $\mathbf{x}^{(0)} = (5,5)$ is shown in figure . The page may be handed in with the solution.

b) Describe in words, if there are any limitations in using the SLP-method and/or if there are any type of problems, for which the SLP-method is unsuited.

Question 5: (points 25%)

A system with an **input** u and an **output** y and zero initial conditions is described by the following differential equation:

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = 2\frac{du(t)}{dt} + 5u(t), \tag{6}$$

a) Derive the impulse response of the system.

From now on, we assume: u is a wide-sense stationary, white zero-mean Gaussian random process with $\sigma_u^2 = 0.5$.

b) Determine the autocorrelation function R_{yy}.

c) Derive the power spectrum S_{yy}.

Assume that the system is replaced by a continuous-time integrator. In other words, from now on our system is described by:

$$u(t) = \frac{dy(t)}{dt} \tag{7}$$

d) Find the mean value of the output(μ_y).

e) Determine the autocorrelation function of the output (R_{yy}).

Question 6: (points 15%)

In a communication system, consider a source whose output under hypothesis H_0 is a constant voltage of value v_1, while its output under H_1 is a constant voltage of value v_2. The received signal is corrupted by N, an additive white Gaussian noise of zero mean, and variance $\sigma^2 = 1$.

a) Find the probability density function of the output under both hypotheses.

b) Calculate the log-likelihood function.

c) Find the MAP decision rule for the following a priori probability distributions: $P[H_0] = 0.4$ and $P[H_1] = 0.6$

Question ~~3~~ 7:(points 10%)

We wish to observe a variable Y of a system. But the observation X is actually $X = k_1Y + k_2W$ where W is a normal random variable which is independent of Y. Note that k_1 and k_2 are some known constants. The variances $\sigma_W = \sigma_Y = 1$ and the mean values $\mu_W = \mu_Y = 0.2$ are given.

Find the best linear estimator of Y

$$\hat{Y} = a + bX \tag{8}$$

in terms of k_1 and k_2.

AALBORG UNIVERSITET

Written examination in the course

Optimisation Theory and Stochastic Processes

Wednesday June 1th 2011

kl. 8.30 - 11.30 (3 hours)

All usual helping aids are allowed, i.e. books, notes, calculator, computer etc. All communication equipment and computer communication protocols must be turned off.

The questions should be answered in English.

REMEMBER to write your study number and page number on all sheets handed in.

The set consists of 6 exercises. The total weighting for each of the exercises is stated in percentage. Sub-questions in each exercise have equal weight.

It should be clear from the solution, which methods are used, and there should be a sufficient number of intermediate calculations, so the line of thought is clear.

Exercise 1: (10 %)

a) Find stationary points for the following function:

$$f(\mathbf{x}) = -2 \cdot x_1^2 + 3 \cdot x_1 \cdot x_2 - 2 \cdot x_2^2 + 2 \tag{1}$$

b) Determine the local minimum, local maximum, or inflection (saddle) points for the function.

↳ LECTURE 2 (4.22...)

Exercise 2: (20 %)

We will consider gradient based minimization of the following unconstrained function: ↗ LECTURE 3 (10.52..)

$$f(\mathbf{x}) = 2 \cdot x_1^2 + x_2^2 + 2 \cdot x_1 \cdot x_2 - 4 \cdot x_2 \tag{2}$$

The starting point is: $\mathbf{x}^{(0)} = [\ 1 \quad 1\ ]^T$.

a) Complete the first iteration of the steepest descent method for the function. The 1D line search problem should be solved analytically.

b) Determine the search direction for the first iteration of the modified Newton's method for the function.

Exercise 3: (10 %)

Solve the following linear optimisation problem using the basic steps of the Simplex method and tableau's:

$$\text{minimise} \qquad f(\mathbf{x}) = -3x_1 + 2x_2 \tag{3}$$

Subject to the constraints:

$$\begin{array}{rcccc} g_1(\mathbf{x}) & = & -\frac{x_1}{2} + x_2 & \leq & 2 \\ g_2(\mathbf{x}) & = & x_1 + x_2 & \leq & 3 \\ & & x_i \geq 0 & \forall & x_i = \{1, 2\} \end{array} \tag{4}$$

Exercise 4: (10 %)

A multi-objective optimisation problem is formulated as:

$$\text{minimise} \qquad \begin{array}{l} f_1(\mathbf{x}) = (x_1 - 3)^2 + (x_2 - 2)^2 \\ f_2(\mathbf{x}) = (x_1 - 6)^2 + (x_2 - 5)^2 \end{array} \tag{5}$$

The two contour curves are shown in figure 1.

a) Illustrate the Pareto optimal points in figure 1 (the page should be handed in with the solution).

b) Sketch the Pareto front in the criterion space. The sketch may be based on function values from the contour plot. A coordinate system may be found in figure 2.

Page to be handed in with the solution!

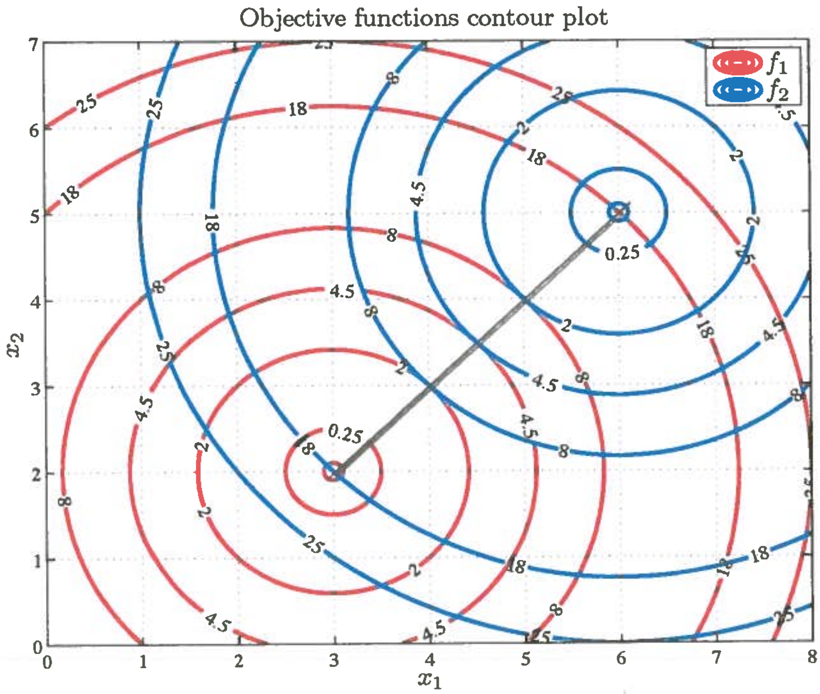

Figure 1: Contour curves for the problem of exercise 4.

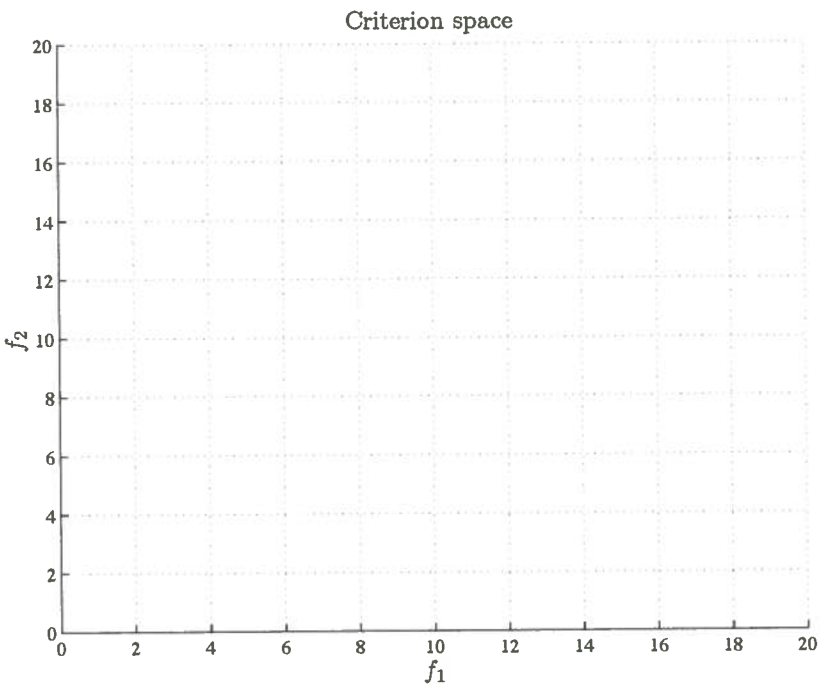

Figure 2: Coordinate system for plotting the criterion space Pareto front in exercise 4.

Exercise 5: Linear systems and ARMA(1,1) process (30 %)

Let us consider the following linear system:

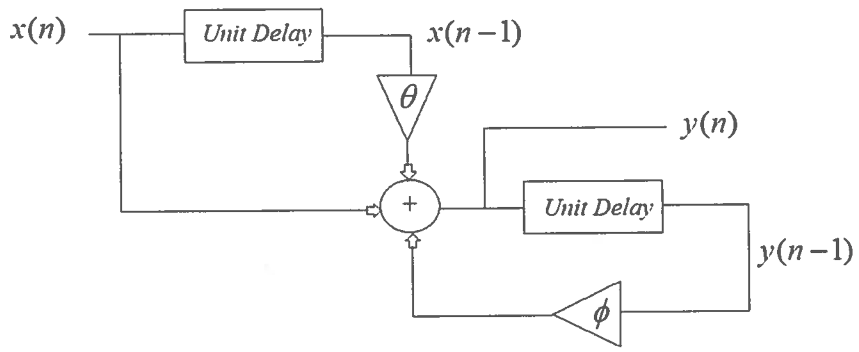

Figure 3: System for exercise 5.

the input-output relationship of which is given by:

$$y(n) = \phi y(n-1) + x(n) + \theta x(n-1) \tag{6}$$

a) Derive the impulse response $h(n)$ of the linear system for initialization $x(n) = y(n) = 0,\ n < 0$.

b) Determine the range of values of ϕ and θ for which the linear system is stable.

From now on, we assume that $\phi = \theta = 0.5$.

c) Derive the transfer function $H(f)$ of the linear system.

Let us now assume that the input sequence is a white noise sequence $\{X(n)\}$ with unit variance, i.e. $E[X(n)] = 0, R_{XX}(k) = [X(n)X(n+k)] = \sigma_X^2\delta(k)$ with $\sigma_X^2 = 1$.

d) Show that $\{Y(n)\}$ is wide-sense stationary only if $E[Y(n)] = 0$ for any n.

e) Derive the power spectrum $S_{YY}(f)$ of $\{Y(n)\}$.

f) What is the interpretation of surface under graph of $S_{YY}(f)$, i. e. the quantity $\int_{-0.5}^{+0.5} S_{YY}(f)df$.

Notice: you do not have to calculate this integral, but only specify the manner it can be interpreted.

Exercise 6: Detection of a Gaussian random signal in background noise (20 %)

Detection of a Gaussian random signal in background noise:

H_0: Only Gaussian noise W is present

H_1: A random Gaussian signal X plus Gaussian noise W is present.

More specifically the received signal under both hypotheses reads:

$$H_0 \, : \, Y = W \tag{7}$$

$$H_1 \, : \, Y = X + W \tag{8}$$

Where

A. W is a zero-mean Gaussian random variable with variance $\sigma_W^2 = 1$, i.e. $W \sim N(0,1)$.

B. X is a zero-mean Gaussian random variable with σ_X^2, i.e. $X \sim N(0, \sigma_X^2)$.

We further assume that X and W are independent. The signal to noise ratio is:

$$\eta = \left(\frac{\sigma_X}{\sigma_W}\right)^2 = 3 \tag{9}$$

a) Find the probability density function of Y under both hypotheses, i. e. $f(y|H_0)$ and $f(y|H_1)$.

Hint: Notice that the variance of the sum of two independent random variables equals to the sum of their individual variances.

b) Show that $L(y) = \frac{f(y|H_1)}{f(y|H_0)}$ is given by:

$$L(y) = \frac{1}{\sqrt{1+\eta}} \exp\left(\frac{1}{2}\frac{\eta}{(1+\eta)}\left|\frac{y}{\sigma_W}\right|^2\right) = \frac{1}{2}\exp\left(\frac{3}{8}|y|^2\right) \tag{10}$$

c) Calculate the log-likelihood function: $l(y) = \ln\left(\frac{f(Y|H_1)}{f(Y|H_0)}\right)$

d) Find the MAP decision rule for the following a priori probability distributions: $P[H_0] = P[H_1] = \frac{1}{2}$.

EXAM 2011

Exercise 1:

a) Find stationary points for the following function:

$$f(x) = -2 \cdot x_1^2 + 3 \cdot x_1 \cdot x_2 - 2 \cdot x_2^2 + 2.$$

b) Determine the local minimum, local maximum or inflection (saddle) points.

a) We need to compute the gradient:

$$\nabla f(x) = c(x) = \begin{bmatrix} -4x_1 + 3x_2 \\ -4x_2 + 3x_1 \end{bmatrix}$$

Stationary points $\rightarrow$ $c(x) = 0$ $\rightarrow$ $\begin{bmatrix} -4x_1 + 3x_2 \\ -4x_2 + 3x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

This yields: $x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad x^* = (0,0)$

The function has 1 stationary point.

To determine the local minimum/maximum and inflection, the Hessian is needed:

$$H(x) = \begin{bmatrix} -4 & 3 \\ 3 & -4 \end{bmatrix}$$

Hessian is constant!

And the eigen-value should be calculated:

$$\begin{vmatrix} -4-\lambda & 3 \\ 3 & -4-\lambda \end{vmatrix} = 0 \rightarrow \left[(-4-\lambda)\cdot(-4-\lambda) - (3\cdot 3)\right] = \left[16 + 4\lambda + 4\lambda + \lambda^2 - 9\right]$$
$$= \lambda^2 + 8\lambda + 7 = 0$$

$$\lambda = \frac{-8 \pm \sqrt{64-28}}{2} = \frac{-8 \pm \sqrt{36}}{2}$$

$$\lambda = \begin{cases} -1 \\ -7 \end{cases}$$

Both eigen values are negative, which means the Hessian is Negative Definite ✓✓

↳ So then, we have a LOCAL MAXIMUM. How to know if it's global maximum?

Exercise 2:

We will consider gradient based minimization of the following:

$$f(x) = 2x_1^2 + x_2^2 + 2 \cdot x_1 \cdot x_2 - 4x_2$$

Start point is $x^{(0)} = [1 \quad 1]^T$.

a) Complete the 1st iteration of the steepest descent method for the function. The 1D line search problem should be solved analytically.

b) Determine the search direction for the 1st iteration of the modified Newton's method.

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a) Function value at $x^{(0)} \rightarrow f^{(0)} = 1$.

The gradient: $\nabla f(x) = c(x) = \begin{bmatrix} 4x_1 + 2x_2 \\ 2x_2 + 2x_1 - 4 \end{bmatrix}$

Evaluated at the starting point: $c(x^0) = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

The direction of the steepest descent can be computed as:

$$d^{(0)} = -c(x^0) = -c^{(0)} = \begin{bmatrix} -6 \\ 0 \end{bmatrix} \rightarrow d^{(0)} = (-6, 0)$$

Next step is to compute the step size, having $x^{(1)} = x^0 + \alpha \cdot d^{(0)}$

$f(\alpha) = f(x^{(0)} + \alpha \cdot d^{(0)})$ = where $x^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $d^{(0)} = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$ $\quad \begin{matrix} 1 - 6\alpha \\ 1 \end{matrix}$

$$f(\alpha) = 2 \cdot (1 - 6\alpha)^2 + 1^2 + 2 \cdot 1 \cdot (1 - 6\alpha) - 4 \cdot 1$$

$$f(\alpha) = 2 \cdot (1 + 36\alpha^2 - 12\alpha) + 1 + 2 - 12\alpha - 4 = 72\alpha^2 - 36\alpha + 1$$

$$f(\alpha) = 72\alpha^2 - 36\alpha + 1 \rightarrow f'(\alpha) = 144\alpha - 36 = 0$$

$$\boxed{\alpha = 0{,}25} \leftarrow \alpha = 36/144 = 1/4 = 0{,}25$$

New design,

$$x^{(1)} = x^{(0)} + \alpha \cdot d^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0.25 \begin{bmatrix} -6 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} \quad x^{(1)} = (-0{,}5, 1)$$

which gives the following cost function value: $\boxed{f^{(1)} = -3{,}5}$

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Exercise 2 (continuation):

b) Search direction → Modified Newton's method.

To determine search direction with modified Newton's method:

$$d^{(k)} = -\left[H(x)^k\right]^{-1} \cdot \nabla f\left(x^{(k)}\right) \rightarrow d^{(k)} = -H(x)^{k^{-1}} \cdot c(x^k)$$

Using the previous gradient, $c(x) = \begin{bmatrix} 4x_1 + 2x_2 \\ 2x_2 + 2x_1 - 4 \end{bmatrix}$

And at x^0, $\quad c^0 = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \quad f^0 = 1. \quad x^0 = (1, 1)$

We need to calculate the Hessian:

$$H(x) = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \rightarrow H^{-1} = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

And the eigen values of $H(x)$ are:

$\lambda_1 = 0{,}7639 \qquad \lambda_2 = 5{,}2361$ both positive.

The search direction is then,

$$d^{(0)} = -H^{-1} \cdot c^{(0)} = -\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} \quad \boxed{d^{(0)} = (-3, 3)}$$

The descent condition is checked,

$$c^{(0)} \cdot d^{(0)} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} -18 \\ 0 \end{bmatrix} = -18 < 0 \quad \text{Descent!}$$

Exercise 3:

Solve the following linear optimisation problem using the basic steps of the Simplex method and tableau's:

$$\text{Minimise} \quad f(x) = -3x_1 + 2x_2$$

Subject to.

$$g_1(x) = -\frac{x_1}{2} + x_2 \leq 2 \qquad + x_3$$

$$g_2(x) = x_1 + x_2 \leq 3 \qquad + x_4$$

$$g_3(x) = -x_2 \leq 0$$

$$g_4(x) = -x_2 \leq 0.$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad c = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} \; \end{bmatrix} \qquad b =$$

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Exercise 4: A multi objective optimisation problem is formulated as:

Minimise $f_1(x_1,x_2) = (x_1-3)^2 + (x_2-2)^2$

$f_2(x_1,x_2) = (x_1-6)^2 + (x_2-5)^2$

The 2 contour curves are shown in Figure 1 (attached paper)

a) Illustrate the Pareto optimal points in Figure 1.

b) Sketch the Pareto front in the criterion space. The sketch may be based on function values from the contour plot.