AALBORG UNIVERSITY
DENMARK

Written examination in the course

Optimisation Theory and

Modern Reliability from a Practical Approach

Friday June 10th 2016

kl. 9 - 13 (4 hours)

All usual helping aids are allowed, i.e. books, notes, calculator, computer etc. All communication equipment and computer communication protocols must be turned off.

The questions should be answered in English.

REMEMBER to write your study number and page number on all sheets handed in.

The set consists of eight exercises. The total weighting for each of the exercises is stated in percentage. You need 50 % in order to pass the exam.

It should be clear from the solution, which methods are used, and there should be a sufficient number of intermediate calculations, so the line of thought is clear.

Exercise 1: (10 %)

The following optimisation problem is considered:

$$\begin{aligned} \text{Minimise} \quad f(\mathbf{x}) &= (3 - x_1)^2 + (x_2 + 2)^2 \\ \text{Subject to} \quad h(\mathbf{x}) &= -x_1 - x_2 + 2 = 0 \quad \Rightarrow X_2 = -X_1 + 2 \end{aligned} \tag{1}$$

a) Set up the Lagrangian function and find point(s) satisfying the KKT necessary conditions.

b) Check if the point(s) is an optimum point using the graphical method (make a simple sketch).

Exercise 2: (15 %)

We will consider gradient-based minimisation of the following unconstrained function:

$$f(\mathbf{x}) = \frac{3}{2}x_1^2 + x_1^2 - 6x_1 + x_1x_2 + \frac{1}{2}x_2^2 + 3 \tag{2}$$

The starting point is: $\mathbf{x}^{(0)} = [\ 1 \quad 1\ ]^T$.

a) Complete the first iteration of the steepest descent method for the function. The 1D line search problem should be solved analytically.

b) Can Newton's method be applied for determining the search direction in iteration 1?
If yes, then determine the search direction.
If no, then state an alternative robust method for determining the search direction.

Exercise 3: (9 %)

Solve the following problem by setting up the solution tree and using the *Local Minimization Branch & Bound Method*:

$$\text{minimise} \qquad f(\mathbf{x}) = -2x_1 - 3x_2$$

Subject to:

$$\begin{aligned} g_1(\mathbf{x}) &= 0.4x_1 + x_2 - 8 \leq 0 \\ g_2(\mathbf{x}) &= x_1 + x_2 - 9.8 \leq 0 \\ g_3(\mathbf{x}) &= 3x_1 - x_2 - 9 \leq 0 \end{aligned}$$

Both x_1 and x_2 should be integer values

As a help for solving the problem the objective function contours and constraints are plotted in figure 1.

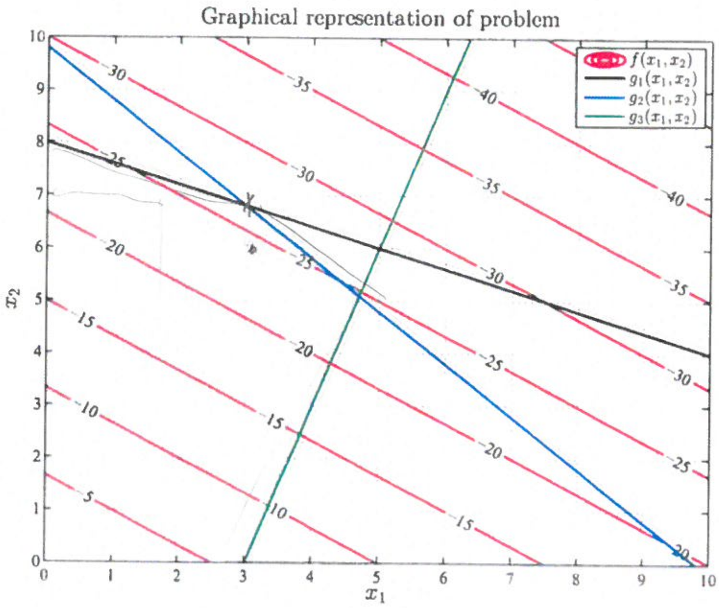


Figure 1: Graphical representation of problem in exercise 3.

Exercise 4: (8 %)

Solve the following linear optimisation problem using the basic steps of the Simplex method and tableau's:

$$\text{minimise} \qquad f(\mathbf{x}) = -5x_1 - 2x_2$$

Subject to the constraints:

$$\begin{aligned} g_1(\mathbf{x}) &= 4x_1 + 3x_2 \leq 27 \\ g_2(\mathbf{x}) &= x_1 - 2x_2 \leq 4 \\ & x_i \geq 0 \quad \forall \quad x_i = \{1, 2\} \end{aligned}$$

Exercise 5: (8 %)

The following multi-objective optimisation problem is considered:

$$\text{minimise} \qquad \begin{aligned} f_1(\mathbf{x}) &= (x_1 - 2)^2 + (x_2 - 6)^2 + 5 \\ f_2(\mathbf{x}) &= (x_1 - 7)^2 + (x_2 - 1)^2 + 8 \end{aligned}$$

Figure 2 shows the Pareto optimal points in the design space and figure 3 shows the Pareto optimal set in the criterion space.

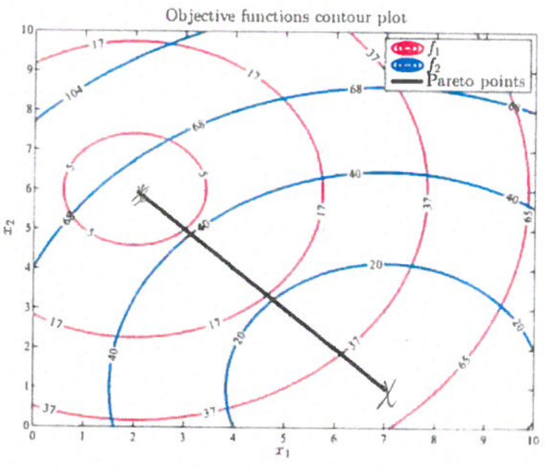


Figure 2: Contour curves and Pareto optimal points.

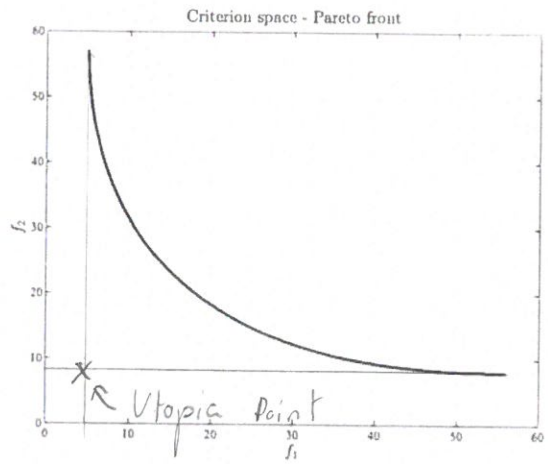


Figure 3: Pareto set in criterion space.

a) Determine the objective function values of the utopia point.

b) Assume that the multi-objective problem is solved as single objective problem, $U(\boldsymbol{x})$, using the weighting method with $w_1 = w_2 = 1$. Determine the minimum objective function value $U(\mathbf{x}^*)$, and the optimum set of design variables $\mathbf{x}^*$.

Reliability

Exercise 6 (16%)

8 of paper clips are tested with 180 º bending angle, and the cycles to failure for each clip are recorded as: 20, 50, 40, 10, 30, 80, 70, and 60.

1) *State the relationship between accumulated failure and time in Weibull distribution.*
2) *Arrange these time-to-failure numbers by using median ranking method.*
3) *Plot the ranking number as listed in (b) and the cycles to failure in the attached Weibull paper below. Please find out the values for β and η, respectively.*
4) *Explain how to identify the values of β and η in the Weibull plot.*

Appendix I – Median rank table

sample size $= n$
failure rank $= i$

i \ n	1	2	3	4	5	6	7	8	9	10
1	.5000	.2929	.2063	.1591	.1294	.1091	.0943	.0830	.0741	.0670
2		.7071	.5000	.3864	.3147	.2655	.2295	.2021	.1806	.1632
3			.7937	.6136	.5000	.4218	.3648	.3213	.2871	.2594
4				.8409	.6853	.5782	.5000	.4404	.3935	.3557
5					.8706	.7345	.6352	.5596	.5000	.4519
6						.8906	.7705	.6787	.6065	.5481
7							.9057	.7979	.7129	.6443
8								.9170	.8194	.7406
9									.9259	.8368
10										.9330

Exercise 7 (17%)

A mining drill head was tested in the laboratory to determine its expected lifetime. The following data were collected:

Stress level ($\times 10^6$ N)	1.0	2.0	3.0	3.5
Mean cycles to failure ($\times 10^3$)	20.3	7.5	1.8	0.35

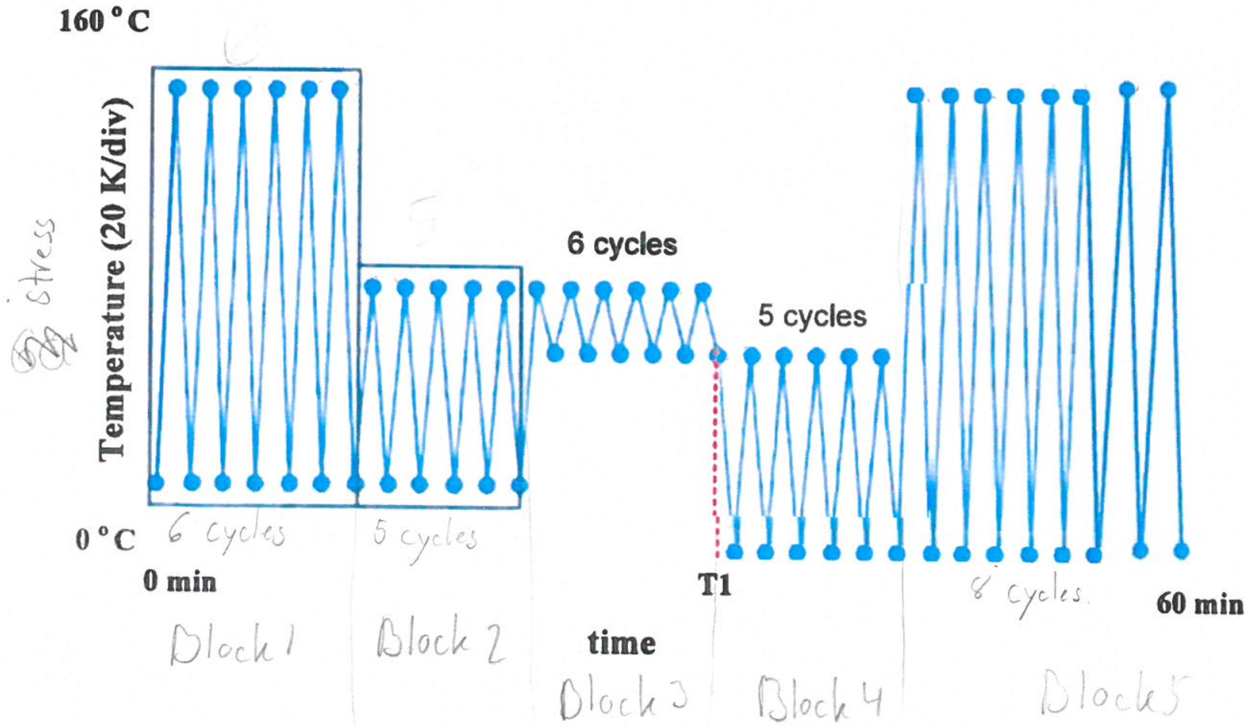
The head will operate in the drill with the following most stressful levels, respectively:

Proportion of cycles	0.8	0.15	0.04	0.01

a) In case of 100 cycles per hour, what will be the expected time to failure in service?
b) what will it be in case of 10 cycles per hour?

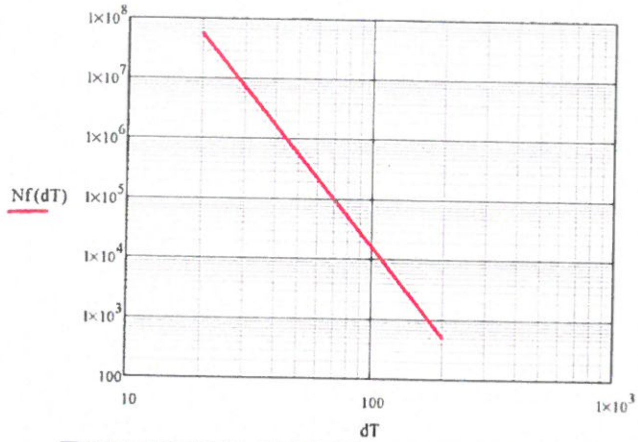
Exercise 8 (17%)

It has been found that the thermal cycle or temperature excursion is the main cause of failure for a component. The 1 hour thermal profiles of this component in the real-field operation has been recorded and shown in the following figure, where many thermal cycles at different levels of ΔT can be identified. The y-axis is the temperature of the component with the scale of 20 degree per div, and x-axis is the time with scale of 10 minute per div.



The lifetime of this component has been tested and modeled as Coffin-Masson lifetime model, as shown in the following function and table.

Coffin-Masson lifetime model of component



Coffin-Masson Model:

$$N_f = A \cdot \Delta T^{\alpha}$$

where:

$A = 1.99 \times 10^{14}$, $\alpha = -5.039$

ΔT	Nf	ΔT	Nf	ΔT	Nf
10	NA	70	$1.00*10^5$	130	$4.43*10^3$
20	$5.53*10^7$	80	$5.12*10^4$	140	$3.05*10^3$
30	$7.17*10^6$	90	$2.83*10^4$	150	$2.16*10^3$
40	$1.68*10^6$	100	$1.66*10^4$	160	$1.56*10^3$
50	$5.47*10^5$	110	$1.03*10^4$	170	$1.15*10^3$
60	$2.18*10^5$	120	$6.63*10^3$	180	860

Nf: the number of thermal cycles to failure.

1) *List the Rainflow counting table which summarizes the number of thermal cycles at different stress levels.*
2) *Calculate the damage (in %) of component at time T1.*
3) *Calculate the damage (in %) of component within this 60 minutes or 1 hour.*
4) *Assuming the component will experience the repeated thermal profiles of this 1 hour until the failure, calculate how many hours the component can survive.*

Optimisation Theory - Exam 16

1 - Optimisation Problem:

minimise $f(x) = (3-x_1)^2 + (x_2+2)^2$

Subject to $h(x) = -x_1 - x_2 + 2 = 0$

- Set up the Lagrangian:

$$L(x) = f(x) + v h(x)$$
$$= (3-x_1)^2 + (x_2+2)^2 + v(-x_1 - x_2 + 2)$$
$$= (3-x_1)^2 + (x_2+2)^2 - vx_1 - x_2 v + 2v$$

- The KKT necessary condition:

$$\frac{\partial L}{\partial x_1} = -6 + 2x_1 - v = 0$$

$$\frac{\partial L}{\partial x_2} = 2x_2 + 4 - v = 0$$

$$\frac{\partial L}{\partial v} = -x_1 - x_2 + 2 = 0$$

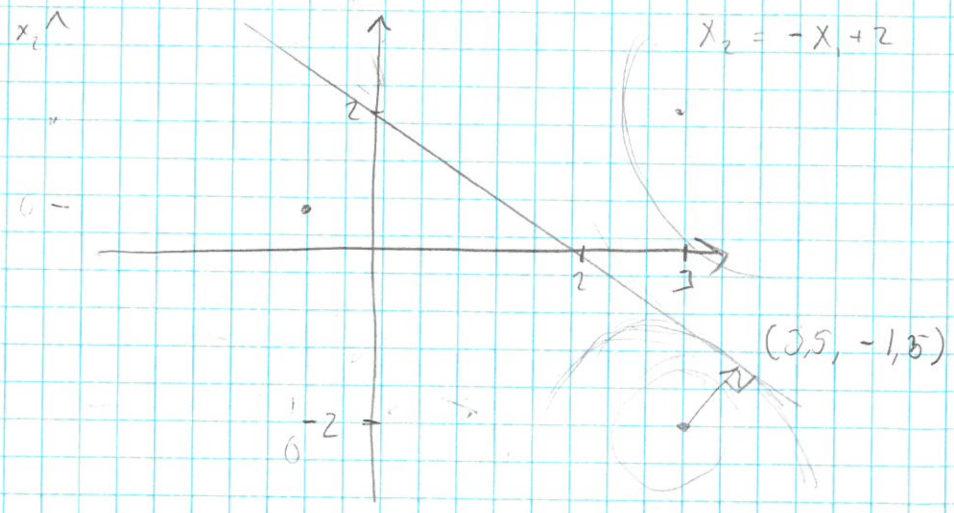
$\Downarrow$

$x_1 = 7/2$

$x_2 = -3/2$

$v = 1$

- Graphical solution



Optimisation Theory - Exam 16

2 - Gradient-based minimisation

$$f(x) = \tfrac{3}{2} x_1^2 + x_1^2 - 6x_1 + x_1 x_2 + \tfrac{1}{2} x_2^2 + 3$$

Starting point : $x^{(0)} = [1 \quad 1]^T$

- First iteration of the steepest descent method:
  - Gradient

$$c = \nabla f = \begin{bmatrix} 5x_1 + x_2 - 6 \\ x_1 + x_2 \end{bmatrix}$$

$$c^{(0)} = \begin{bmatrix} 5 \cdot 1 + 1 - 6 \\ 1 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

  - Check for convergence:

$$\| c^{(0)} \| = \sqrt{(0)^2 + 2^2} = 2 > \varepsilon \rightarrow \text{Continue}$$

  - Search direction:

$$d^{(0)} = -c^{(0)}$$

$$= [0 \quad -2]$$

- Step size (Calculated anallytically)

$$f(\alpha) = f(x^{(0)} + \alpha_0 d^{(0)}), \quad x^{(0)} + \alpha_0 d^{(0)} = \begin{matrix} 1 + 0 \\ 1 + \alpha_0(-2) \end{matrix}$$

$$= \tfrac{3}{2}(1+0)^2 + (1+0)^2 + (1+0)(1+\alpha_0(-2))$$

$$+ \tfrac{1}{2}(1 + \alpha_0(-2))^2 + 3$$

$$= 2\alpha_0^2 + 4\alpha + 7$$

$$\frac{df(\alpha)}{d\alpha} = 4\alpha - 4 = 0$$

$$\Rightarrow \alpha = 1$$

$$x = \quad 2$$

Optimisation Theory - Exam 16

- Update design:

$$x^{(1)} = x^{(0)} + \alpha_0 d^{(0)}$$

$$x^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

- Newton's Method

$$d^{(0)} = -\left[H^{(0)} \right]^{-1}$$

$$H^{(0)} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

$$\frac{\partial^2 f}{\partial x_1^2} = 5$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 1$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = 1$$

$$\frac{\partial^2 f}{\partial x^2} = 1$$

$$H^{(0)} = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$$

Book, P. 463

- Since the Hessian is positive definite, Newton's method can be applied.

Optimisation Theory - Exam 16

$$C + H\,\Delta X = 0$$

$$\Downarrow$$

$$\Delta X = -H^{-1} C$$

$$\Delta X = -\begin{bmatrix} 0{,}25 & -0{,}25 \\ -0{,}25 & 1{,}25 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -0{,}5 \\ 2{,}5 \end{bmatrix}$$

Thus:

$$X^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0{,}5 \\ 2{,}5 \end{bmatrix} = \begin{bmatrix} -0{,}5 \\ 3{,}5 \end{bmatrix}$$

Optimisation Theory - Exam 16

3 - Branch & Bound Method

minimise $f(x) = -2x_1 - 3x_2$

Subject to: $g_1(x) = 0{,}4x_1 + x_2 - 8 \leq 0$

$g_2(x) = x_1 + x_2 - 9{,}8 \leq 0$

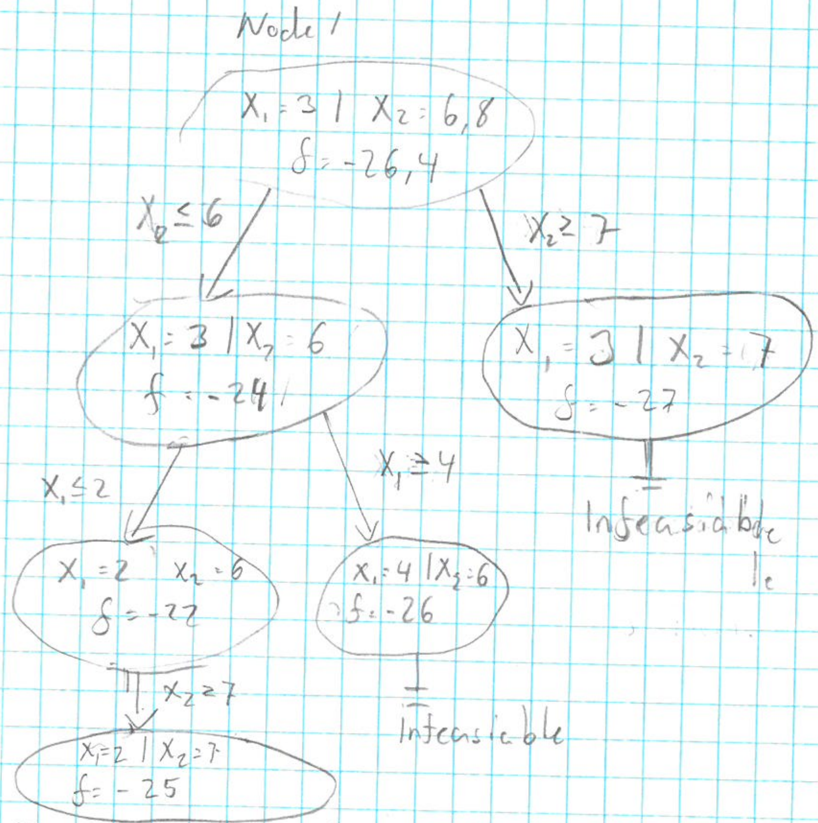
$g_3(x) = 3x_1 - x_2 - 9 \leq 0$

x_1 & x_2 are integers

Continuous solution:

$x_1 = 3, \quad x_2 = 6{,}8$

$f = -26{,}4$



Thus the optimal integer solution is

$x_1 = 2, \quad x_2 = 7$

$f(2,7) = -25$

Optimisation Theory - Exam 16

4 - Simplex Method

Minimise $f(x) = -5x_1 - 2x_2$

Subject to: $g_1(x) = 4x_1 + 3x_2 \leq 27$

$g_2(x) = x_1 - 2x_2 \leq 4$

$x_i > 0, \; x_i = \{1, 2\}$

- Since $g_{1 \dots 4} \leq b_{1 \dots 4}$, the slack variables $x_{3 \dots 4}$ are introduced:

$$g_1(x) = 4x_1 + 3x_2 + x_3 = 27$$

$$g_2(x) = x_1 - 2x_2 + x_4 = 4$$

- Tableau:

Basic ↓	x_1	x_2	x_3	x_4	b	Ratio
x_3	4	3	1	0	27	$27/4 = 6{,}75$
x_4	1	-2	0	1	4	$4/1 = 0{,}75$
C	-5	-2	0	0	δ	

Basic Variable: x_3, x_4

Non-basic Variable: x_1, x_2

Pivot Coloum: x_1, since C is lowest

Pivot Row: x_4, lowest ration

-

Optimisation Theory - Exam 16

- Gauss - Jordan reduction

Basic ↓	x_1	x_2	x_3	x_4	b	Ratios
x_1	0	11	1	-4	11	1
x_3	1	-2	0	1	4	$-2/4 = -1/2$
C	0	-12	0	5	$f+20$	

- Multiply row 2 with 5, add to row 3
- Multiply row 2 with -4, add to row 1

Since the non-zero entries of c is of opposite sign, → Continue:

Basic variables: x_1, x_3

Pivot Column: x_2

row: x_1

- Tableau:

Basic ↓	x_1	x_2	x_3	x_4	b
x_1	0	1	$1/11$	$-4/11$	1
x_2	1	0	$2/11$	$3/11$	6
C	0	0	$12/11$	$-7/11$	$f+32$

- Multiply row 1 with $12/11$ add to 3
- Multiply row 1 with $7/11$
- Multiply new 1 with 2, add to 2

Since the non-zero entries are non-negative in the cost row, the optimum is reached.

Optimisation Theory - Exam 16

Optimum: $f(x) = -32$

$x_1 = 1, \quad x_2 = 6$

5 - Multi-objective optimisation:

minimise $f_1(x) = (x_1 - 2)^2 + (x_2 - 6)^2 + 5$

$f_2(x) = (x_1 - 7)^2 + (x_2 - 1)^2 + 8$

- Objective fcn value of the utopia point:

$f_1(x) = 5, \quad x_1 = 2, \; x_2 = 6$

$f_2(x) = 8 \quad x_1 = 7, \; x_2 = 1$

- The minimum objective fcn value $U(x^*)$

Minimise $U(x) = w_1 f_1(x) + w_2 f_2(x)$, $w_1 = w_2 = 1$

f_1	f_2	U
5	43	48
17	23	40 — Smallest
37	10	47

$$\begin{cases} 17 = (x_1 - 2)^2 + (x_2 - 6)^2 + 5 \\ 23 = (x_1 - 7)^2 + (x_2 - 1)^2 + 8 \end{cases}$$

$\Downarrow$

$x_1 = 4{,}5$ - By fmincon in Matlab.

$x_2 = 3{,}5$

Or

$\frac{\partial U}{\partial x_1} = 0 \Rightarrow \quad = x_1 =$

$\frac{\partial U}{\partial x_2} = 0 \Rightarrow \quad = x_2 =$

Insert in U yields:

$U =$

Modern Reliability - Exam 16

6 -

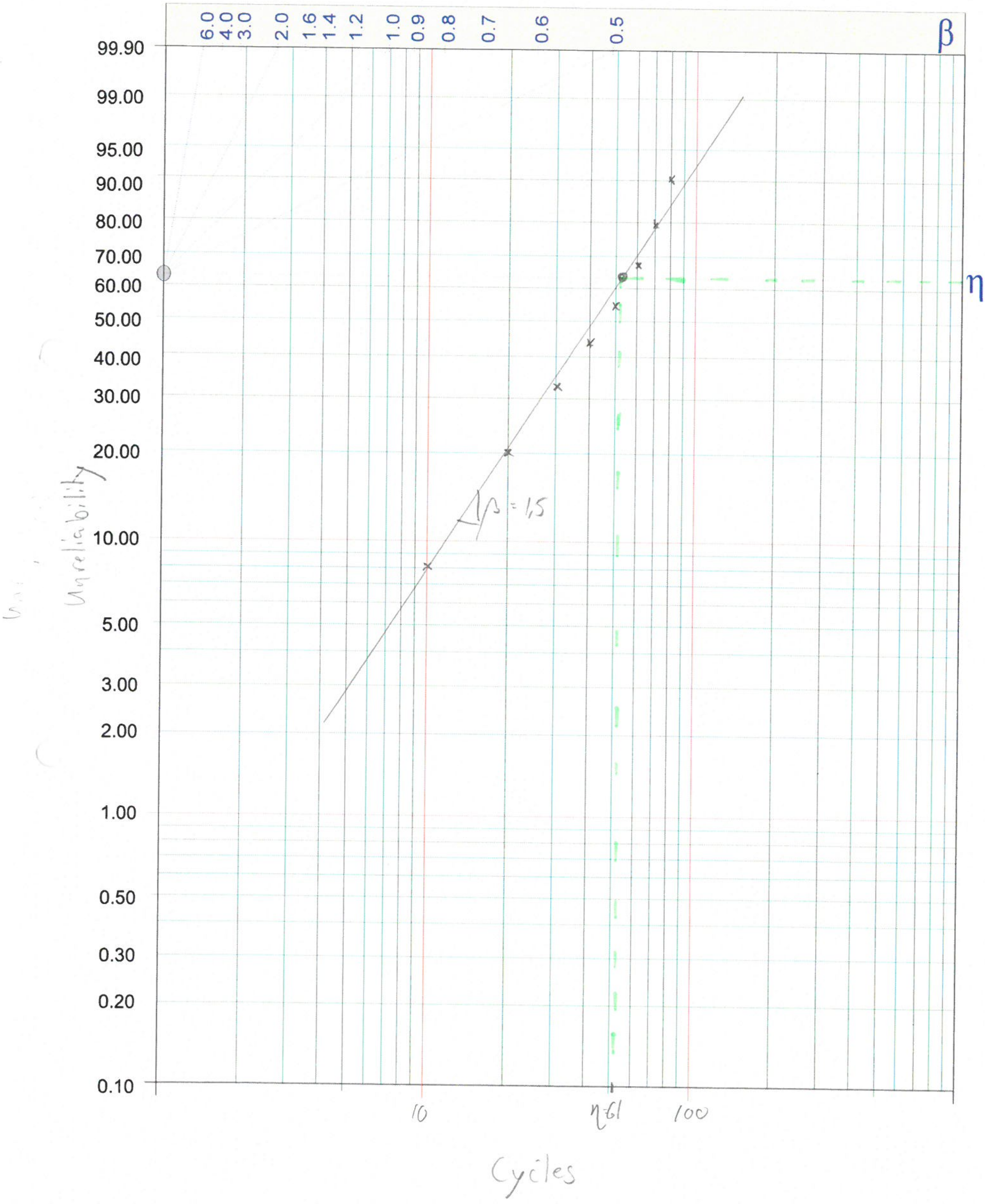
- Relationship between accumulated failure and time in Weibull Distribution: $1-R = 1-\exp\left(-\left(\frac{t}{\eta}\right)^{\beta}\right)$

$$1 - R(t) = 1 - \exp\left[-\left(\frac{t-\gamma}{\eta}\right)^{\beta}\right]$$

- Median Rank Method:

Cycles	MR
10	0,083
20	0,2021
30	0,3213
40	0,4404
50	0,5596
60	0,6787
70	0,7979
80	0,9170

- Plotted on the weibull paper (next page)
  - $\beta = 1,5$ - Slope of the Weibull-Distribution
  - $\eta = 61$ - The value of cycles at which the unreliability is 63,2%.


β
6.0
4.0
3.0
2.0
1.6
1.4
1.2
1.0
0.9
0.8
0.7
0.6
0.5
99.90
99.00
95.00
90.00
80.00
70.00
60.00
50.00
40.00
30.00
20.00
10.00
5.00
3.00
2.00
1.00
0.50
0.30
0.20
0.10
η
Unreliability
β = 1,5
10
η 61
100
Cycles

Modern Reliability

7

Stress Level ($\cdot 10^6$ N)	1.0	2.0	3.0	3.5
Mean cycles to fail ($\cdot 10^3$)	20.3	7,5	1,8	0,35

Proportion of cycles	0.8	0,15	0,04	0,01

- For 100 cycles/h, calculate the expected time to failure in service:

- Total consumed Lifetime:

$$TCL = \sum_{i=1}^{k} \frac{n_i}{N_i}$$

$$= \frac{0,8}{20,3 \cdot 10^3} + \frac{0,15}{7,5 \cdot 10^3} + \frac{0,04}{1,8 \cdot 10^3} + \frac{0,01}{0,35 \cdot 10^3} = 1,102 \cdot 10^{-4}$$

- Cycles to fail:

$$CTF = \frac{100\%}{1,102 \cdot 10^{-4}} \simeq 9074 \text{ cycles to fail}$$

- Thus:

$$TTF\Big|_{100 \text{ cycles}} = \frac{CTF}{\text{cycles/h}} \Rightarrow \frac{9074}{100} = 90,74 h$$

$$TTF\Big|_{10} = \frac{9074}{10} = 907,4 h \simeq 37,8 \text{ days.}$$

Modern Reliability - Exam 16

8 - Rainflow

- Rain flow Table

	Stress lvl	# of cycles	N_F
Block 1	$\Delta T = 140°C$	6	3050
Block 2	$\Delta T = 70°C$	5	$1{,}00 \cdot 10^5$
Block 3	$\Delta T = 20°C$	6	$5{,}53 \cdot 10^7$
Block 4	$\Delta T = 70°C$	5	$1{,}00 \cdot 10^5$
Block 5	$\Delta T = 160°C$	8	860

- The damage of the component at T1:
  - Proportion of cycles:
    - Block 1 $\quad 6/30 = 0{,}2$
    - 2 $\quad 5/30 = 1/6$
    - 3 $\quad 6/30 = 0{,}2$
    - 4 $\quad 5/30 = 1/6$
    - 5 $\quad 8/30 = 0{,}2667$

  - Total consumed life time

$$TCL = \sum_{i=1}^{k} \frac{n_i}{N_i}$$

$$= \frac{0{,}2}{3050} + \frac{1/6}{1{,}00 \cdot 10^5} + \frac{0{,}2}{5{,}53 \cdot 10^7} = 6{,}724 \cdot 10^{-5}$$