AALBORG UNIVERSITY
DENMARK

Written examination in the course

Optimisation Theory and

Modern Reliability from a Practical Approach

Wednesday June 11th 2014

kl. 9 - 13 (4 hours)

The examination set consist of two parts, one for »Optimisation Theory« and one for »Modern Reliability from a Practical Approach«.

For the part in Modern Reliability from a Practical Approach: No computers or other helping aids as slides, textbooks etc. may be used. This means that you should hand in your solution for this part, before starting up your computer or bringing out your notes if you wish to use these for the Optimisation Theory part. The pages in the reliability part are meant to be handed in.

For the Optimisation Theory part: All usual helping aids are allowed, i.e. books, notes, calculator, computer etc. All communication equipment and computer communication protocols must be turned off.

The questions should be answered in English.

REMEMBER to write your study number and page number on all sheets handed in.

The set consists of four exercises in the optimisation part (totalling 50% of the examination set) and 25 questions in the reliability part. The weighting for each exercise is stated in percentage. You need in total 50 % of the entire set (optimisation and reliability part) correct in order to pass the exam.

It should be clear from the solution, which methods are used, and there should be a sufficient number of intermediate calculations, so the line of thought is clear.

Exercise 1: (25 %)

A company produces two products: 1) a MP3 player and 2) a Watch-TV (a wristwatch TV).

The production process for each product is similar in that both require a certain number of hours of electronic work and a certain number of labor-hours in the assembly department. Each MP3 player takes 4 hours of electronic work and 2 hours in the assembly shop. Each Watch-TV requires 3 hours in electronics and 1 hour in assembly.

During the current production period, 240 hours of electronic time are available, and 100 hours of assembly department time are available.

Each MP3 player sold yields a profit of 7 Euro; each Watch-TV produced may be sold for a 5 Euro profit.

a) Formulate the problem of determining the best possible combination of MP3 players and Watch-TVs to manufacture to reach the maximum profit.

b) Solve the problem formulated in a) using graphical methods.

c) Set up the Lagrangian function for the problem formulated in a) and find point(s) satisfying the KKT necessary conditions. You may make use of results from b) when checking/solving the KKT necessary conditions.

Exercise 2: (7 %)

The following multi-objective optimisation problem is considered:

$$\text{minimise} \quad \begin{aligned} f_1(\mathbf{x}) &= (x_1-8)^2+(x_2-2)^2+3 \\ f_2(\mathbf{x}) &= (x_1-1)^2+(x_2-7)^2+10 \end{aligned}$$

Figure 1 shows the Pareto optimal points in the design space and figure 2 shows the Pareto optimal set in the criterion space.

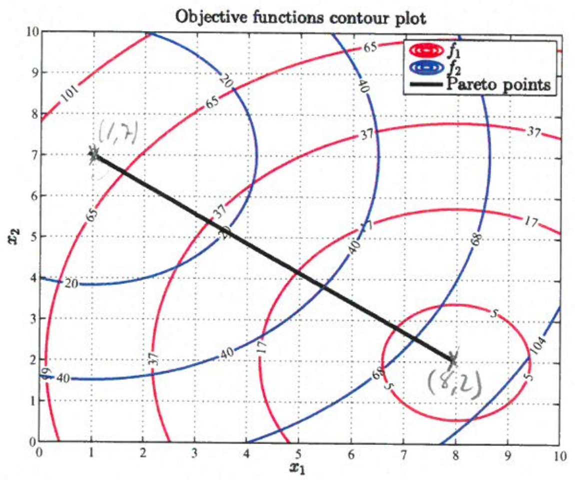


Figure 1: Contour curves and Pareto optimal points for the problem of exercise 3.

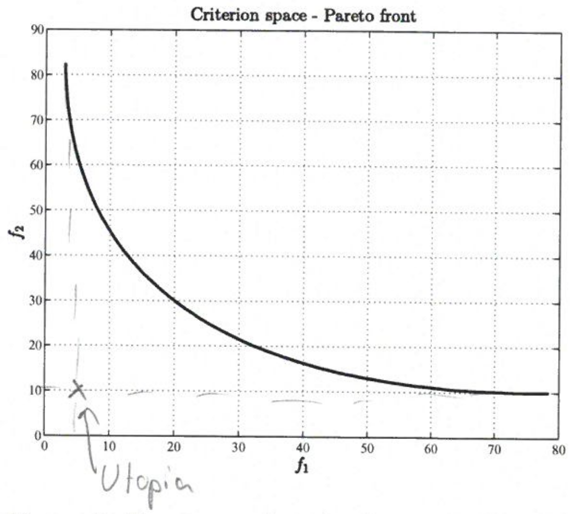


Figure 2: Pareto set in criterion space for the problem of exercise 3.

a) Determine the objective function values of the utopia point.

b) Assume that the multi-objective problem is solved as single objective problem using the weighting method (i.e. minimise $U(\mathbf{x}) = w_1 f_1(\mathbf{x}) + w_2 f_2(\mathbf{x})$), with $w_1 = w_2 = 1$. Determine the minimum objective function value $U(\mathbf{x}^*)$, and the optimum set of design variables $\mathbf{x}^*$.

Exercise 3: (10 %)

An optimisation problem is formulated as:

$$\text{minimise} \qquad f(\mathbf{x}) = 4x_1^2 + 2x_2^2 - 2x_1x_2 + 2x_1 - x_2$$

Subject to:

$$\begin{aligned} g_1(\mathbf{x}) &= x_1 - 12 &\leq 0 \\ g_2(\mathbf{x}) &= x_1^2 - x_1 - 2x_2 - 8 &\leq 0 \\ g_3(\mathbf{x}) &= x_1 + x_2 - 20 &\leq 0 \end{aligned}$$

The initial starting point for an optimisation is $\mathbf{x}^{(0)} = (-1, -2)$, for which the search direction is determined to $\mathbf{d}^{(0)} = [\ 2 \quad 7\ ]^T$ and the vector of Lagrange multipliers for the constraints is $\mathbf{u} = [\ 0 \quad 0 \quad 0\ ]^T$. Let $R_0 = 1$ and $\gamma = 0.5$. Choose the trial step according to the sequence $t_0 = 1$, $t_1 = \frac{1}{2}$, $t_2 = \frac{1}{4}$, $t_3 = \frac{1}{8}$...

Calculate the step size using the inexact line search procedure (approximate step size procedure) and determine the new design variables, $\mathbf{x}^{(1)}$, for the next iteration.

Exercise 4: (8 %)

Solve the following problem by setting up the solution tree and using the Branch & Bound Method with Local Minimization:

$$\text{minimise} \qquad f(\mathbf{x}) = -3x_1 - 8x_2$$

Subject to:

$$\begin{aligned} g_1(\mathbf{x}) &= -0.5x_1 + x_2 - 20 &\leq 0 \\ g_2(\mathbf{x}) &= 2x_1 + x_2 - 20 &\leq 0 \\ g_3(\mathbf{x}) &= 4x_1 - x_2 - 20 &\leq 0 \end{aligned}$$

Both x_1 and x_2 should be integer values

As a help for solving the problem the objective function contours and constraints are plotted in figure 3.

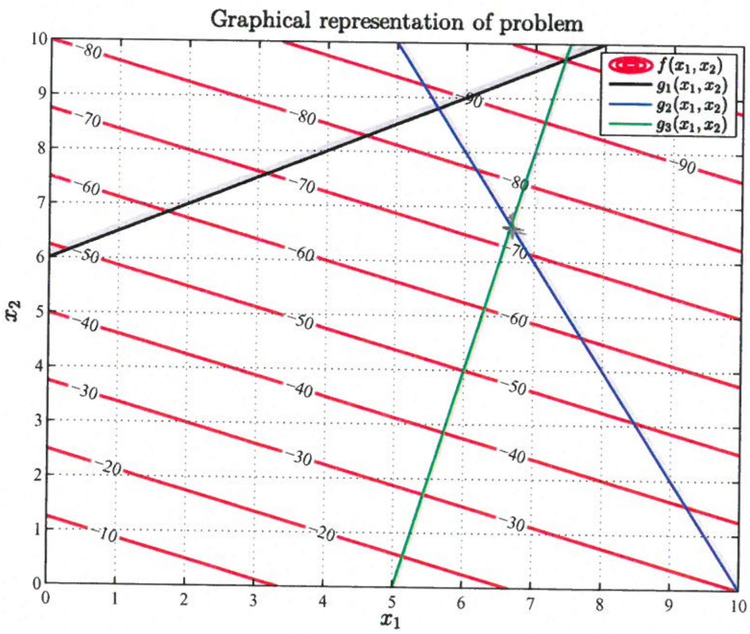


Figure 3: Graphical representation of problem in exercise 4.

Optimisation Theory - Exam 14

1

2 product:

	Work	Assembly	Profit
Product 1 :	4	2	7
Product 2 :	3	1	5

- Determine the best combination of 1 & 2 to reach max. profit:

Design Variables:

x_1 : Product 1

x_2 : Product 2

Profit fcn:

$f(x_1, x_2) = 7x_1 + 5x_2$

Constrains:

240 h of work and 100 h of assembly are available

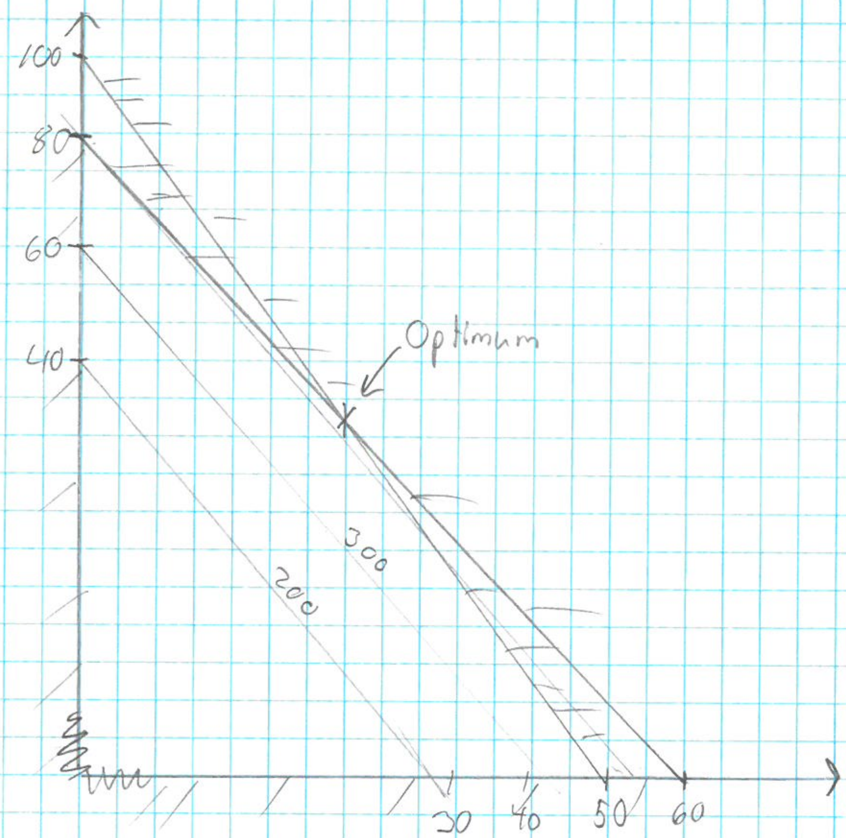
$4x_1 + 3x_2 \leq 240$

$2x_1 + x_2 \leq 100$

$x_1, x_2 \geq 0$

Optimisation Theory - Exam 14

- Graphical Solution



Constraints:

$x_2 = -\frac{4}{3}x_1 + 80$

$x_2 = -2x_1 + 100$

Optimum at intersection of the two constraint

$$\begin{cases} 4x_1 + 3x_2 = 240 \\ 2x_1 + x_2 = 100 \end{cases} \quad + 7 \cdot 40 = 157,5$$

$\Downarrow$

$x_1 = 30$

$x_2 = 40$

$f = 5 \cdot 30 + 7 \cdot 40 = 430$

Optimisation Theory - Exam 14

- Set up the Lagrangian function:

Lecture 6, slide 3

Book, p. 190

minimise $f(x_1, x_2) = -(7x_1 + 5x_2)$

Subject to $g_1(x) = 4x_1 + 3x_2 \leq 240$

$g_2(x) = 2x_1 + x_2 \leq 100$

$x_1, x_2 \geq 0$

- $L(x,u) = f(x) + u^T g(x)$

$\Rightarrow f(x) + u_1 g_1(x) + u_2 g_2(x)$

Thus:

$L(x,u) = -7x_1 - 5x_2 + u_1(4x_1 + 3x_2 - 240) + u_2(2x_1 + x_2 - 100)$

- Gradient:

$\frac{\partial L}{\partial x_1} = -7 + 4u_1^* + 2u_2^*$ 1

$\frac{\partial L}{\partial x_2} = -5 + 3u_1^* + u_2^*$ 2

- Feasiability Check:

$g_1(x^*) \leq 0 \Rightarrow (4$

$g_2(x^*) \leq 0$

- Switch Condition

$u_1 g_1 = 0 \Rightarrow u_1(4x_1^* + 3x_2^* - 240) = 0$ 3

$u_2 g_2 = 0 \Rightarrow u_2(2x_1^* + x_2^* - 100) = 0$ 4

- Non-negativity check:

$u_1^* \geq 0$

$u_2^* \geq 0$

Optimisation Theory - Exam 14

- Guess from Graphical Solution:

$$\begin{cases} g_1(x^*) = 0 & \rightarrow \text{Active} \\ g_2(x^* = 0 & \rightarrow \text{Active} \end{cases}$$

$$\Downarrow$$

$$\begin{cases} 4x_1^* + 3x_2^* - 240 = 0 \\ 2x_1^* + x_2^* - 100 = 0 \end{cases}$$

$$\Downarrow$$

$$x_1^* = 30$$

$$x_2^* = 40$$

- Non-negative

$$\begin{cases} -7 + 4u_1^* + 2u_2^* = 0 \\ -5 + 3u_1^* + u_2^* = 0 \end{cases}$$

$$\Downarrow$$

$$u_1 = 3/2$$

$$u_2 = 1/2$$

Since $u_{1,2} > 0$, the necessary condition of KKT is satisfied.

Optimisation Theory - Exam 14

2 - Multi-Objective Optimisation Problem:

minimise $f_1(x) = (x_1 - 8)^2 + (x_2 - 2)^2 + 3$

$f_2(x) = (x_1 - 1)^2 + (x_2 - 7)^2 + 10$

- The objective function values of the utopia point:

$f_1(8,2) = 3$

$f_2(1,7) = 10$

- The minimum objective function value:

Minimise $U(x) = w_1 f_1(x) + w_2 f_2(x)$, $w_1 = w_2 = 0$

f_1	f_2	U
65	5	70
35	20	65
15	40	55
3	65	78

Smallest.

$$\begin{cases} 15 = (x_1 - 8)^2 + (x^2 - 2)^2 + 3 \\ 40 = (x_1 - 1)^2 + (x_2 - 7)^2 + 10 \end{cases}$$

$\Downarrow$

$x_1 = x_2 = 4{,}5$

$U = 55$

Or

$\frac{\partial U}{\partial x_1} = 0 \Rightarrow \qquad = x_1 =$

$\frac{\partial U}{\partial x_2} = 0 \Rightarrow \qquad = x_2 =$

Inserting in U yields:

$U =$

Optimisation Theory - Exam 14

3 - Optimisation problem

minimise $f(x) = 4x_1^2 + 2x_2^2 - 2x_1x_2 + 2x_1 - x_2$

Subject to $g_1(x) = x_1 - 12 \leq 0$

$g_2(x) = x_1^2 - x_1 - 2x_2 - 8 \leq 0$

$g_3(x) = x_1 + x_2 - 20 \leq 0$

- Initial Starting point: $x^{(0)} = [-1, -2]$
- Search direction: $d^{(0)} = [2 \quad 7]^T$
- Vector of Lagrange multiplier $u = [0 \quad 0 \quad 0]^T$

- Let $R_0 = 1$ and $\gamma = 0{,}5$,

- Choose a trial step according to the

Lecture 7 Sequence, $t_0 = 1$, $t_1 = 1/2$, $t_2 = 1/4$, $t_3 = 1/8$

Book, p. 542 - Calculate the step size using the inexact line search procedure (Approximate Step Size procedure):

$$r_0 = \sum_{i=0}^{m} u_i^{(0)} = 0$$

$$R = \max\{R_0, r_0\} \Rightarrow \max\{1, 0\} = 1$$

$$V_0 = \max\{g_1^{(0)}, g_2^{(0)}, g_3^{(0)}\}$$

$$\max\{-13, -4, -23\} \Rightarrow \text{Set to } 0$$

- Value of the descent function:

$$\Phi_0 = f_0 + RV_0 \Rightarrow$$

$$\Rightarrow 4\cdot(-1)^2 + 2\cdot(-2)^2 - 2\cdot(-1)(-2) + 2\cdot(-1) - (-2) + 1\cdot 0$$

$$= 8 + 1\cdot 0 = 8$$

Optimisation Theory - Exam 14

- The constant:

$$\beta_0 = \gamma \|d^{(0)}\|^2 \Rightarrow 0{,}5 \cdot (\sqrt{2^2+7^2})^2 = 26{,}5$$

- Evaluate the new value of the descent function to check the descent conditions:

$$x^{(1,0)} = x^{(0)} + t_{0,3} d^{(0)} \quad , \quad t_{0\ldots3} \text{ - Trial Steps}$$

$$x^{(1,0)} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 & 1/2 & 1/4 & 1/8 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

$$x^{(1,0)} = \begin{array}{cccc} \begin{bmatrix} 1 & 0 & -0{,}5 & -0{,}75 \\ (5) & 1{,}5 & -0{,}25 & -1{,}1250 \end{bmatrix} \\ t_0 \quad\quad t_1 \quad\quad t_2 \quad\quad t_3 \end{array}$$

- Evaluate the cost and constraint function:

$$f^{(1,0)} = \begin{bmatrix} 41 & 5 & 0{,}125 & 2{,}71988 \end{bmatrix}$$

$$V_{1,0} = V(t_0) = \max\{-11 \quad -18 \quad -14\} \Rightarrow 0$$

$$V_{1,0} = V(t_1) = \max\{-12 \quad -11 \quad -18{,}5\} \Rightarrow 0$$

$$V_{1,0} = V(t_2) = \max\{-11{,}5 \quad -7{,}75 \quad -19{,}25\} \Rightarrow 0$$

$$V_{1,0} = V(t_3) = \max\{-12{,}75 \quad -5{,}93 \quad -13\} \Rightarrow 0$$

- Evaluate the maximum constraint violation to calculate the descent function:

$$\Phi_{1,0} = f_{1,0} + R V_{1,0} \Rightarrow \begin{bmatrix} 41 & 5 & 0{,}125 & 2{,}7188 \end{bmatrix}$$

$$\Phi_0 - t_{0\ldots3}\beta_0 = 9 - \begin{bmatrix} 1 & 1/2 & 1/4 & 1/8 \end{bmatrix} \cdot 26{,}5$$

$$= \begin{bmatrix} -17{,}5 & -4{,}25 & 2{,}375 & 5{,}6875 \end{bmatrix}$$

The descent condition, $\Phi_{1,0} \leq \Phi_0 - t_0\beta_0$, is at $t_2 = 1/4$, since $0{,}125 \leq 2{,}375$

Optimisation Theory - Exam 14

The design is updated:

$$x^{(1,1)} = x^{(0)} + t_2 d^{(0)}$$

$$= \begin{bmatrix} -1 \\ -2 \end{bmatrix} + 1/4 \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} -0{,}5 \\ -0{,}25 \end{bmatrix}$$

4 - The solution tree / Branch & Bound Method

Book, p. 622 — with Local Minisation

Minimise $f(x) = -3x_1 - 8x_2$

Subject to:

$g_1(x) = -0{,}5x_1 + x_2 - 20 \le 0$

$g_2(x) = 2x_1 + x_2 - 20 \le 0$

$g_3(x) = 4x_1 - x_2 - 20 \le 0$

x_1 & x_2 are integers

- Start at the optimal continuous solution:
  - From graphical solution:

$x_1 = 6{,}7$, $x_2 = 6{,}7$, $f(6{,}7, 6{,}7) = -73{,}7$

- Solution Tree: - Local Minimisation - Ignore integer constraits for one variable during.

Node 1: $x_1 = 7{,}4$, $x_2 = 10{,}6$, $f = -107$

$x_1 \le 7$: $x_1 = 7$, $x_2 = 8$, $f = -85$ — Stop - Integer feasiable Solution

$x_1 \ge 8$: $x_1 = 8$, x_2, $f =$ — Stop - Infeasiable Solution

Restriction on x_1
↳ 7 is the largest value below 7,4 that is an integer, 8 the smallest above.
Since no integer solution between 6 & 7.
So here the algorithm eliminate the space in between 7 and 8 from the feasiable region.

Since all constraits are "less than or equal"
↳ Round down
At Node 1, the best known solution is: $x_1 = 7$, $x_2 = 10$, $f = -101$.

Thus, the optimum integer solution is:

$x_1 = 7$, $x_2 = 8$, for which $f = -85$