AALBORG UNIVERSITY
DENMARK

Engineering Optimisation - Concepts, Methods and Applications

Oral Re-Examination 2016 - Procedure and Questions

Rules *- please read these carefully!*

1. The total oral examination time is 20 minutes per student, ***including*** *time for voting. The actual examination time is therefore approximately 15-16 minutes. The students should therefore prepare so he/she is able to answer the drawn questions well within this time frame.*

2. Each student must answer two questions. The two questions will be drawn from two pools of questions (pool 1: questions 1-5 and pool 2: questions 6-8).

3. There is no time allocated for preparation; the examination starts immediately after the questions have been drawn by the student.

4. The questions included in this document are the foundation for the examination. The examiners may/will however ask small supplementary questions.

5. Notice that emphasis is on understanding the concepts/theory, not necessarily on completing all calculations.

6. The students are allowed to bring papers to the examination. It is however expected that the students will be able to answer the questions, without looking in the papers!

7. All 8 questions are included in this document, which is forwarded to the students at least two weeks before the day of the reexamination.

Exercise 1: (10 %)

The following optimisation problem is considered:

$$\begin{aligned} \text{Minimise} \quad f(\mathbf{x}) &= (3 - x_1)^2 + (x_2 + 2)^2 \\ \text{Subject to} \quad h(\mathbf{x}) &= -x_1 - x_2 + 2 = 0 \end{aligned} \tag{1}$$

a) Set up the Lagrangian function and find point(s) satisfying the KKT necessary conditions.

b) Check if the point(s) is an optimum point using the graphical method (make a simple sketch).

Exercise 2: (15 %)

We will consider gradient-based minimisation of the following unconstrained function:

$$f(\mathbf{x}) = \frac{3}{2}x_1^2 + x_1^2 - 6x_1 + x_1x_2 + \frac{1}{2}x_2^2 + 3 \tag{2}$$

The starting point is: $\mathbf{x}^{(0)} = [\ 1 \quad 1\ ]^T$.

a) Complete the first iteration of the steepest descent method for the function. The 1D line search problem should be solved analytically.

b) Can Newton's method be applied for determining the search direction in iteration 1?
If yes, then determine the search direction.
If no, then state an alternative robust method for determining the search direction.

Exercise 3: (9 %)

Solve the following problem by setting up the solution tree and using the *Local Minimization Branch & Bound Method*:

$$\text{minimise} \qquad f(\mathbf{x}) = -2x_1 - 3x_2$$

Subject to:

$$\begin{aligned} g_1(\mathbf{x}) &= 0.4x_1 + x_2 - 8 &\leq 0 \\ g_2(\mathbf{x}) &= x_1 + x_2 - 9.8 &\leq 0 \\ g_3(\mathbf{x}) &= 3x_1 - x_2 - 9 &\leq 0 \end{aligned}$$

Both x_1 and x_2 should be integer values

As a help for solving the problem the objective function contours and constraints are plotted in figure 1.

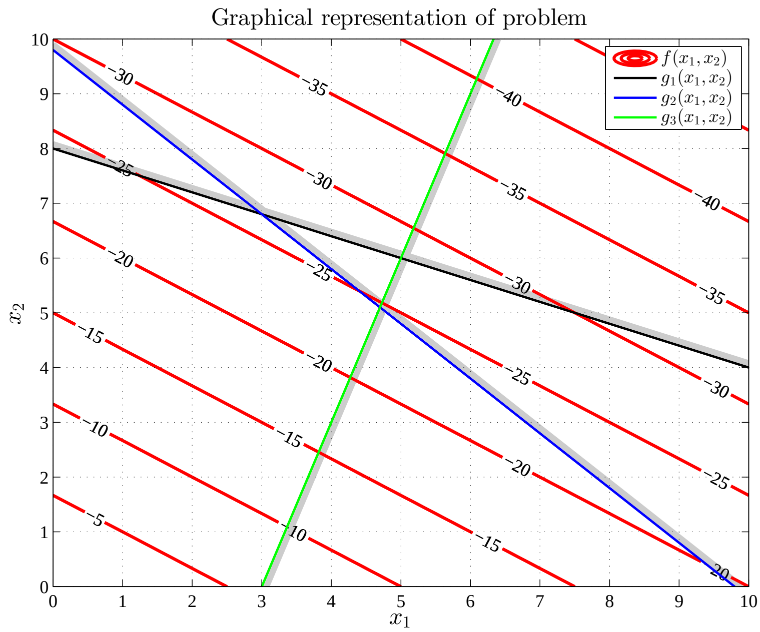

Figure 1: Graphical representation of problem in exercise 3.

Exercise 4: (8 %)

Solve the following linear optimisation problem using the basic steps of the Simplex method and tableau's:

$$\text{minimise} \qquad f(\mathbf{x}) = -5x_1 - 2x_2$$

Subject to the constraints:

$$\begin{aligned} g_1(\mathbf{x}) &= 4x_1 + 3x_2 &\leq& \quad 27 \\ g_2(\mathbf{x}) &= x_1 - 2x_2 &\leq& \quad 4 \\ & x_i \geq 0 &\forall& \quad x_i = \{1, 2\} \end{aligned}$$

Exercise 5: (8 %)

The following multi-objective optimisation problem is considered:

$$\text{minimise} \qquad \begin{aligned} f_1(\mathbf{x}) &= (x_1 - 2)^2 + (x_2 - 6)^2 + 5 \\ f_2(\mathbf{x}) &= (x_1 - 7)^2 + (x_2 - 1)^2 + 8 \end{aligned}$$

Figure 2 shows the Pareto optimal points in the design space and figure 3 shows the Pareto optimal set in the criterion space.

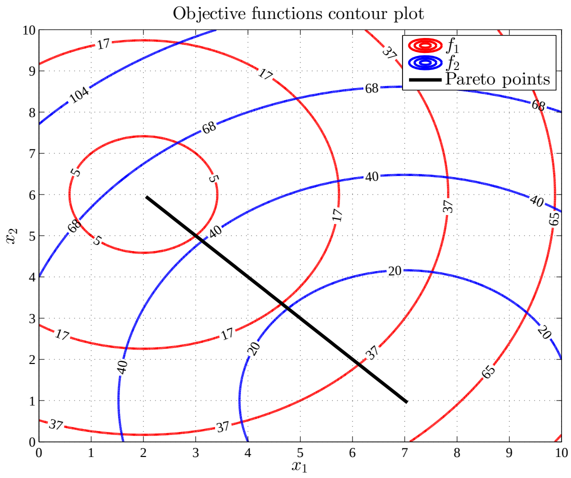

Figure 2: Contour curves and Pareto optimal points.

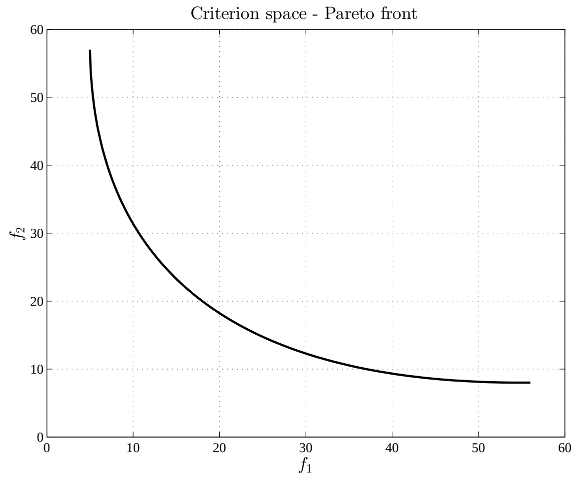

Figure 3: Pareto set in criterion space.

a) Determine the objective function values of the utopia point.

b) Assume that the multi-objective problem is solved as single objective problem, $U(\boldsymbol{x})$, using the weighting method with $w_1 = w_2 = 1$. Determine the minimum objective function value $U(\mathbf{x}^*)$, and the optimum set of design variables $\mathbf{x}^*$.

Exercise 6: (20 %)

The cost function $f(\boldsymbol{x})$ is defined as the sum of squares of the residual function $\boldsymbol{r}(\boldsymbol{x})$

$$f(\boldsymbol{x}) = \frac{1}{2}\sum_{i=1}^{m} r_i^2 \tag{3}$$

where

$$\boldsymbol{r}(\boldsymbol{x}) = \begin{bmatrix} \beta(x_2 - x_1^2) \\ 1 - x_1 \\ x_1 + sin(x_2) \end{bmatrix} \tag{4}$$

The initial cost function $f(\boldsymbol{x}_0) = 12.16$ for $\boldsymbol{x}_0 = [-1.2, 1]$ and the constant $\beta = 10$. The minimum value $f(\boldsymbol{x}^*) \approx 0.3195$ at $\boldsymbol{x}^* \approx [0.3190, 0.0976]$, see figure 4.

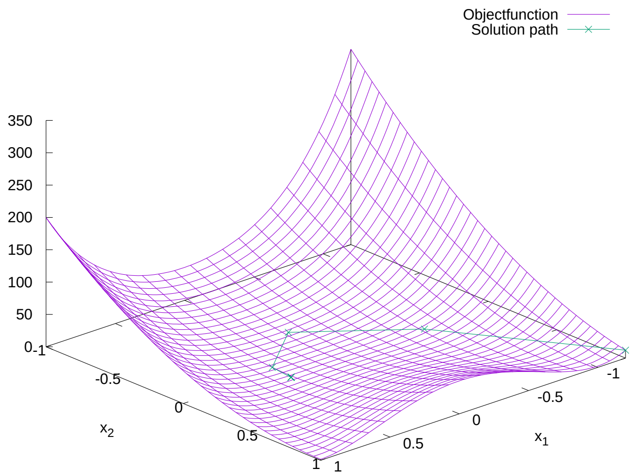

Figure 4: Surface plot of the cost function and the trajectory produced by an iterative solution scheme.

The following questions must be answered:

1. Derive an analytically expression for Jacobian matrix associated with equation 4.
2. Numerical solutions are expected for the following two sub-questions, where the initial parameters $\boldsymbol{x}_0 = [-1.2, 1]$ are inserted into the expressions derived above.
   (a) Using the initial parameters $\boldsymbol{x}_0 = [-1.2, 1]$; calculate the gradient and approximate the Hessian using only first order information from the Jacobian matrix derived above in question 1.
   (b) Complete the first 3 iterations using the Levenberg-Marquardt algorithm (where the step was defined as $\boldsymbol{s}_k = -(\boldsymbol{H} + \lambda \boldsymbol{I})^{-1}\nabla f(\boldsymbol{x}_k))$. Start the iterations using an initial damping parameter $\lambda = 10$ and report λ , object-function $f(\boldsymbol{x}_k)$ and $\boldsymbol{x}_k$ for the first 4 iterations.

iter.	x	f(x)	λ
0	$[-1.2\ ,\ 1]$	12.16	10
1			
2			
3			
4			

Exercise 7: (15 %)

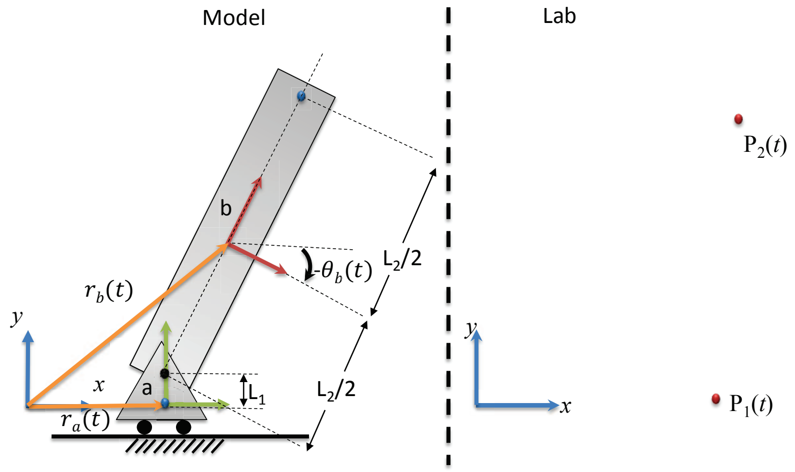

Figure 5: A mechanical system consiting of two rigid bodies.

Figure 5 shows two rigid bodies (a and b) in a two-dimensional space that are connected to each other with a revolute joint and body a is connected to ground with a translational joint. Two markers (the blue dots in the figure) are attached to the bodies: one on body a located at the origin of the local body reference frame (indicated with green) and one on body b located a distance of $L_2/2$ from the origin of the local body reference frame (indicated with red). The revolute joint is located a distance of L_1 and a distance of $L_2/2$ from the origin of the reference frame of body a and b respectively. The global reference frame is indicated in blue.

The position and orientation of the bodies are to be described in terms of a full cartesian formulation, i.e. position and orientation of the bodies relative to the global reference frame. The global position and rotation angles of the bodies are denoted, $r_a(t) = \begin{bmatrix} x_a(t) \\ y_a(t) \end{bmatrix}$, $r_b(t) = \begin{bmatrix} x_b(t) \\ y_b(t) \end{bmatrix}$, $\theta_a(t)$ and $\theta_b(t)$ as indicated on the figure. Please note that $\theta_a(t)$ is not indicated in the figure since it is assumed to always remain zero but a constraint equation to specify this much be included.

A motion capture experiment was performed, where the position of the markers were measured in the global reference frame for a duration of time and the measured coordinates are denoted $P_1(t)$ and $P_2(t)$.

From the measured trajectories of the markers, it is desired to set up a two-level nonlinear least-squares optimisation problem that can be used to compute the position and orientation of the rigid bodies as well as the distances L_1 and L_2. That is, the least-square difference between the measured markers and the points on the bodies at all time steps must be minimised while ensuring that the revolute and translational joint constraints are fulfilled.

The following questions must be answered:

1. Is the system of kinematic equations under-determinate, determinate or over-determinate when the motion of the system is driven by the measured markers? Justify the answer by counting the number of degrees-of-freedom of the mechanical system compared to the number of kinematic driver equations.

2. Formulate a two-level constrained nonlinear least-squares optimisation problem that can be used to compute both the positions, $r_a(t)$ and $r_b(t)$, and rotations, $\theta_a(t)$ and $\theta_b(t)$ as well as the lengths L_1 and L_2 simultaneously. For both the outer and inner optimisation problems, indicate the unknowns in the problem.

Exercise 8: (15 %)

We assume you are going to solve a design optimisation problem where the lowest eigenfrequency of a mechanical system is to be maximized while the mass must not be increased.

The mechanical system is discretized with a finite element model having more than 1000 degrees-of-freedom, and the design variables of the problem are denoted $x_i,\ i = 1, \ldots, n$. These have prescribed minimum values $\underline{x}_i,\ i = 1, \ldots, n$, and maximum values $\overline{x}_i,\ i = 1, \ldots, n$.
We assume that you have been given an analysis model that can compute the mass and eigenfrequencies ω_j of the mechanical system.
Gradients can also be computed by the analysis code, and it is assumed that all computed eigenfrequencies are distinct (simple), i.e. there are no multiple eigenfrequencies.

a) Explain the main mathematical difficulty of solving the optimization problem of maximizing the lowest eigenfrequency of the mechanical system.

b) State the mathematical formulation for three fundamentally different ways of formulating the optimization problem. Describe advantages and disadvantages of the three chosen optimisation problem formulations.