Written examination in the course

Optimisation Theory and Stochastic Processes

Thursday June 7th 2012

kl. 9 - 13 (4 hours)

All usual helping aids are allowed, i.e. books, notes, calculator, computer etc. All communication equipment and computer communication protocols must be turned off.

The questions should be answered in English.

REMEMBER to write your study number and page number on all sheets handed in.

The set consists of seven exercises. The total weighting for each of the exercises is stated in percentage. Sub-questions in each exercise have equal weight. You need 50 % in order to pass the exam.

It should be clear from the solution, which methods are used, and there should be a sufficient number of intermediate calculations, so the line of thought is clear.

Exercise 4: (12 %)

A multi-objective optimisation problem is formulated as:

$$\text{minimise} \qquad \begin{aligned} f_1(\mathbf{x}) &= (x_1-4)^2 + (x_2-2)^2 \\ f_2(\mathbf{x}) &= (x_1-4)^2 + (x_2-8)^2 \end{aligned} \tag{5}$$

Subject to the constraints:

$$\begin{aligned} g_1(\mathbf{x}) &= x_2 - 7 \leq 0 \\ g_2(\mathbf{x}) &= -x_1 - x_2 + 8 \leq 0 \end{aligned} \tag{6}$$

The two contour curves along with the constraints are shown in figure 1.

a) Illustrate the Pareto optimal points in figure 1 (the page should be handed in with the solution).

b) Sketch the Pareto front in the criterion space. The sketch should be based on function values from the contour plot. A coordinate system may be found in figure 2.

c) Indicate where the utopia point is located for the given problem.

*d) A weighting method is used in solving the problem (i.e. minimise $U = w_1 f_1(\mathbf{x}) + w_2 f_2(\mathbf{x})$), with $w_1 = w_2 = 1$. What is the solution to this problem - give both function value U, and design variables x_1 and x_2. **Hint:** Use the results from questions a) and b).*

Optimisation Theory - Exam 12

1 - Optimisation Problem:

Minimize $f(x) = (x_1 - 1)^2 + (x_2 - 1)^2$

Subject to $h(x) = x_1 + x_2 - 4 = 0$

a - Set up the Lagrangian function

$$L(x, v) = f(x) + \sum_{j=1}^{p} v_j h_j(x) = f(x) + v^T \bar{h}(x)$$

v_j, $j = 1, \dots, p$ are Lagrangian multiplier

L and f will have the same optimum,

$v^T h(x) = 0$

Thus:

$$L(x) = (x_1 - 1)^2 + (x_2 - 1)^2 + v \cdot (x_1 + x_2 - 4)$$

- Find the points that satisfy the KKT necessary conditions:

$$\frac{\partial L}{\partial x_k} = \underbrace{\frac{\partial f}{\partial x_k}}_{\nabla f} + \sum_{j=1}^{p} v_j \underbrace{\frac{\partial h_j}{\partial x_j}}_{\nabla h_j} + \sum_{i=1}^{m} \underbrace{\frac{\partial g_i}{\partial x_k}}_{\nabla g_i} = 0$$

$$\frac{\partial L}{\partial x_1} = 2x_1 - 2 + v = 0 \qquad 1$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - 2 + v = 0 \qquad 2$$

$$\frac{\partial L}{\partial v} = x_1 + x_2 - 4 = h(x) = 0 \qquad 3$$

, since eq. constraint recovered.

Using 1, 2 & 3 to solve for v:

$$\frac{(2 - v)}{2} + \frac{2 - v}{2} - 4 = 0$$

$\Downarrow$

$$v = -2 \qquad 4$$

Inserting $v = -2$ in 1 & 2 and solves for x_1 and x_2

$$\begin{cases} 2x_1 - 2 + (-2) = 0 \\ 2x_2 - 2 + (-2) = 0 \end{cases}$$

$\Downarrow$

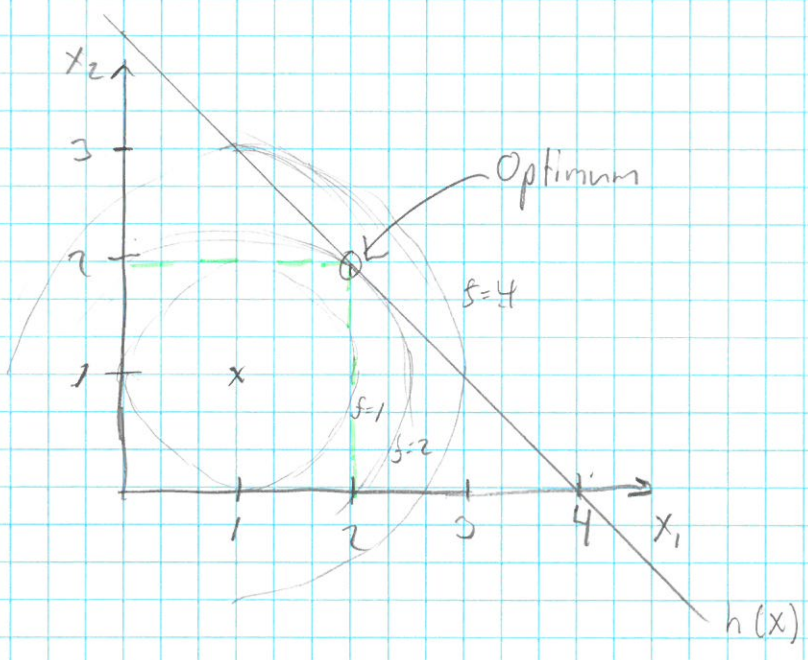
$$x_1 = +4/2 = +2$$

$$x_2 = +4/2 = +2$$

Thus the objective function is minimized to:

$$f(x) = (2-1)^2 + (2-1)^2 \Rightarrow f(x^*) = 2$$

- Check by graphical solution:



Optimisation Theory - Exam 12

2 - Gradient based Method

Minimization of:

$$f(x) = (1-x_1)^2 + (x_2-2)^2 + 2 \cdot x_1$$

The starting point is:

$$x^{(0)} = [3 \quad 1]^T$$

- 1. iteration of the steepest descent method:

$$c = \nabla f = \begin{bmatrix} 2x_1 - 2 + 2 \\ 2x_2 - 4 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 - 4 \end{bmatrix},$$

* The gradient is evaluated at the starting point, $x^{(0)}$:

$$c^{(0)} = \begin{bmatrix} 2 \cdot 3 \\ 2 \cdot 1 - 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

* Check the norm of the gradient:

$$\|c^{(0)}\| = \sqrt{6^2 + (-2)^2} = 6{,}3246$$

* Compared with the tolerance, ε:

$$6{,}3246 > \varepsilon, \qquad \varepsilon = 0{,}001$$

* Continue:

* The direction of steepest descent:

$$d^{(0)} = -c^{(0)}$$

$$d^{(0)} = [-6 \quad 2]$$

* Calculate the step size, α_0, the minimizes

$$f(\alpha) = f(x^{(0)} + \alpha_0 d^{(0)})$$

$$f(\alpha) = (1-(3+\alpha \cdot (-6)))^2 + ((1+\alpha \cdot 2) - 2)^2 + 2 \cdot (3 + \alpha \cdot (-6))$$

$$= 40\alpha^2 - 40\alpha + 11$$

Optimisation Theory - Exam 12

The 1D line search should be solved analytically.

Utilizing: $x^{(0)} + \alpha d^{(0)} = [\,3 + \alpha(-2) \quad 1 + \alpha \cdot 2\,]^T$

$$\frac{df(\alpha)}{d\alpha} = 2 \cdot 40\alpha - 40 = 80\alpha - 40$$

Setting $f'(\alpha) = 0$

$$80\alpha - 40 = 0$$
$$\Downarrow$$
$$\alpha = 1/2$$

* Update design

$$x^{(0+1)} = x^{(0)} + \alpha_0 d^{(0)}$$

$$x^{(1)} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 1/2 \begin{bmatrix} -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

- 1 iteration of Newton's methods:

Minimization of:

$$f(x) = (1 - x_1)^2 + (x_2 - 2)^2 + 2 \cdot x_1$$

The starting point is:

$$x^{(0)} = [\,3 \quad 1\,]^T$$

* The gradient vector:

$$c = \nabla f = \begin{bmatrix} 2x_1 - 2 + 2 \\ 2x_2 - 4 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 - 4 \end{bmatrix}$$

* The gradient vector evaluated at the starting point:

$$c^{(0)} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

* Check the norm of $c^{(0)}$:

$$\|c^{(0)}\| = \sqrt{6^2 + (-2)^2} = 6,3246$$

Optimisation Theory - Exam 12

* Compare w/ tolerance:
  $6{,}3246 > \varepsilon$
* Continue:
* Calculate the Hessian matrix, $H^{(0)}$

$$H^{(0)} = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

$$\frac{\partial^2 f}{\partial x_1^2} = 2$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = 0$$

$$\frac{\partial^2 f}{\partial x_2^2} = 2$$

Thus

$$H^{(0)} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$ - Constant for all iterations, since no variables.

Optimisation Theory - Exam 12

* Calculate the eigenvalues:

$$|A - \lambda \underline{I}| = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} = 0$$

$$= (2-\lambda)(2-\lambda) = 0$$

$$= 4 - 2\lambda + \lambda^2 - 2\lambda = 0$$

$$= \lambda^2 - 4\lambda + 4 = 0$$

$$= \lambda = \frac{4 \pm \sqrt{16 - 4 \cdot 4}}{2}$$

$$= \lambda_{1,2} = 2$$

$$= \lambda_1 = 2, \quad \lambda_2 = 2$$

Thus, the eigenvalues are positive, which means that the Hessian is positive definite, i.e. Newtons Methods can be applied.

* Calculate the new search direction:

$$d^{(0)} = -H^{-1} c^{(0)}$$

$$d^{(0)} = -\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

- Check the descent condition:

$$c^{(0)} d^{(0)} < 0$$

$$\begin{bmatrix} 6 \\ -2 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -18 \\ -2 \end{bmatrix} < 0$$

The descent condition is satisfied

Optimisation Theory - Exam 12

* Update the design

$$x^{(0+1)} = x^{(0)} + \alpha_0 d^{(0)}$$

- $\alpha_0 \rightarrow$ Step size

$\alpha_0 = 1/2$

$$x^{(1)} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 1/2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1,5 \\ 1,5 \end{bmatrix}$$

Optimisation Theory - Exam 12

3 - Linearised subproblem

Minimised $f(x) = -x_1^2 + 3x_2^2 + x_1 x_2 - 3$

Subject to $g_1(x) = 1/x_1 - 2x_2 \leq 0$

Lecture 6, Slide 11

$g_2(x) = x_1 - 2x_2^3 \leq 0$

Book, p. 502

$g_3(x) = x_1 \geq 0$

$g_4(x) = x_2 \geq 0$

- Linearise the problem at the point $(x_1, x_2) = (1, 1) = x^{(0)}$

Fcn Values

- Evaluating the cost function at $x^{(0)}$:

$f(1,1) = -1^2 + 3 \cdot 1^2 + 1 \cdot 1 - 3 = 0$

- Evaluating the constraints at $x^{(0)}$:

$g_1(1,1) = 1/1 - 2 \cdot 1 \leq 0 \Rightarrow -1 \leq 0$ (Inactive)

$g_2(1,1) = 1 - 2 \cdot 1^3 \leq 0 \Rightarrow -1 < 0$ (Inactive

$g_3(1,1) = -1 < 0$ - Sign change

$g_4(1,1) = -1 < 0$

- The gradient of the cost and constraints at $x^{(0)}$

Fcn gradients

$$c^{(0)} = \nabla f(1,1) = \begin{bmatrix} -2 \cdot 1 + 1 \\ 6 \cdot 1 + 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix} -4$$

$$\nabla g_1(1,1) = \begin{bmatrix} -1/1^2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\nabla g_2(1,1) = \begin{bmatrix} 1 \\ -6 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

$$\nabla g_3(1,1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\nabla g_4(1,1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Optimisation Theory - Exam 12

- Linearize using 1. order Taylor Approx.

$\bar{f} = f(x^{(0)}) + \nabla f(x^{(0)}) \cdot d$, $d = \Delta x^{(0)}$

$\bar{g} = g(x^{(0)}) + \nabla g^T(x^{(0)}) \cdot d$

Linearisation

$\bar{f} = 0 + \begin{bmatrix} -1 & 7 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = -d_1 - 7d_2$

$\bar{g}_1 = -1 + \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = -1 - d_1 + 2d_2 \leq 0$
$= -d_1 + 2d_2 \leq 1$

$g_2 = -1 + \begin{bmatrix} 1 & 6 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = -1 + d_1 + 6d_2 \leq 0$
$= d_1 + 6d_2 \leq 1$

$g_3 = -1 + \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = -1 - d_1 \leq 0$
$= -d_1 \leq 1$

$g_4 = -1 + \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = -1 - d_2 \leq 0$
$= -d_2 \leq 1$

Thus:

Minimise $\bar{f} = c^T d$

$= \begin{bmatrix} -1 & 7 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

Subject to $A^T d \leq b$

$A = \begin{bmatrix} -1 & 1 & -1 & 0 \\ 2 & 6 & 0 & -1 \end{bmatrix}$

$b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Optimisation Theory - Exam 12

- Solve the linearised sub-problem:

Minimise $f(x) = -x_1 + 7x_2$

Subject to $g_1(x) = -x_1 + 2x_2 \leq 1$

Book; p. 314

$g_2(x) = x_1 + 6x_2 \leq 1$

$x_1, x_2 \geq 0$

Lecture 5, Slide 4

Since $g_{1 \ldots 4} \leq b_{1 \ldots 4}$, the slack variables $x_{3 \ldots 6}$ are introduced:

$g_1(x) = -x_1 + 2x_2 + x_3 = 1$

$g_2(x) = x_1 + 6x_2 + x_4 = 1$

$x_1, x_2, x_3, x_4 \geq 0$

- Simplex Method:

Book, p. 314

- Setting up the Tableau:

Basic ↓	x_1	x_2	x_3	x_4	b	b/a
1 : x_3	-1	2	1	0	1	1/-1 = -1 = Negative
2 : x_4	1	6	0	1	1	1/1 = 1
3 : C	-1	7	0	0	f	

- Basic Variables: x_3, x_4 - Due to it only being in the identity matrix
- Non-basic variables: x_1, x_2

- Pivot Column: x_1, since C is lowest
- Pivot Row: x_3, since it has the lowest ratio

Optimisation Theory - Exam 12

Gauss-Jordan reduction:

- Pivot under pivot column

Basic ↓	x_1	x_2	x_3	x_4	b
x_1	0	8	1	2	2
x_3	1	6	0	1	1
c	0	13	0	1	$f+1$

- Add row 2 to 3
- Add row 2 to 1

Basic variable: x_1, x_3

Non-basic variable: x_2, x_4

Since the non-zero entries are non-negative in the cost row, the optimum is reached

Optimum: $f(x) = -1$.

Optimisation Theory - Exam 12

4 - Multi-objective optimisation problem

Minimise $f_1(x) = (x_1 - 4)^2 + (x_2 - 2)^2$

$f_2(x) = (x_1 - 4)^2 + (x_2 - 8)^2$

Subject to: $g_1(x) = x_2 - 7 \leq 0$

$g_2(x) = -x_1 - x_2 + 8 \leq 0$

- Pareto point

A point x^* in the feasible design space S is Pareto Optimal if there does not exist another point, x, in the set S, for which $f(x) \leq f(x^*)$

(Where the lowest value of each intersects)

- Criterion Space:

Point	f_1	f_2
1	25	1
2	16	4
3	9	9
4	4	16
5	2	25

- Utopia Point

A point where the minimum of each function intersects.

- Utopia point $\approx$ (2, 2)

Optimisation Theory - Exam 12

- Weighting Method:

Lecture ~~8~~, Slide 26

minimise $U = w_1 f_1(x) + w_2 f_2(x)$, with $w_1 = w_2 = 1$

Point	f_1	f_2	U
1	25	1	26
2	16	4	20
3	9	9	18
4	4	16	20
5	2	25	27

Smallest

Thus (9,9) is the point closets to the utopia point, hence comprise point ⇒ Pareto Optimal.

Or

$\frac{\partial U}{\partial x_1} = 0 \Rightarrow \qquad = x_1 =$

$\frac{\partial U}{\partial x_2} = 0 \Rightarrow \qquad = x_2 =$

Inserting in U yields:

$U =$

13