Technical University of Denmark

Financial Risk Management

42106

Project 2: *Credit Value Adjustments*

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1 Introduction

In this project, we wish to investigate a portfolio consisting of 3 interest rate swaps. The contractual details of the swaps are listed below.

- The first swap is a 10-year interest swap on a notional of \$2 million, where we pay a fixed rate of 1% and receive the floating rate.
- The second swap is a 7-year interest swap on a notional of \$10 million, where we pay a fixed rate of 1.5% and receive the floating rate.
- The third and last swap is 4-year interest swap on a notional of \$3 million, where we pay the floating rate and receive a fixed rate of 2%.

The payments of all 3 swaps are made on a quarterly basis, and the swaps are entered with Morgan Stanley as counterpart. When valuing financial derivatives we discount the cash-flow with an appropriate risk-free rate. The choice might depend on whether it is a collateralized derivative or not. The present value is then known as the *no-default* value as it is assumed that neither side will default [1, p. 229]. This introduces the need to adjust the valuation such that it accounts for the associated *credit risk*. For this reason, we want to estimate our *Expected Exposure* (EE) and *Credit Value Adjustments* (CVA) for each of the swaps. Furthermore, we also want to estimate these metrics under the use of *Netting*, *Collateral*, and the mix of the two in order to assess these tools in relation to mitigating credit risk.

2 Calculation of CVA

The CVA is the amount of a party must reduce the value of a contract due to counterparty default risk.

Let $V(t,T)$ be the risk-free value of a derivative at time $0 \leq t \leq T$ where T is the time of maturity. With the assumption that the counterparty can default, we let $V^D(t,T)$ be the value of the defaultable value of the same derivative. The CVA for that contract can be calculated as

$$\mathrm{CVA}\,(t,T) = V\,(t,T) - V^D\,(t,T)\,. \tag{1}$$

If we assume that $t=0$ then $\mathrm{CVA}\,(0,T) = V\,(0,T) - V^D\,(0,T)$. It is proven in [2] that we can write the CVA at time $t=0$ as

$$\mathrm{CVA}(0,T) = \mathbb{E}\left[(1-R)\cdot 1_{\{\tau\leq T\}}\mathrm{DF}(0,\tau)(\mathrm{NPV}(\tau,T))^{+}\right]. \tag{2}$$

Here $t \leq \tau \leq T$ is the time of default of the counterpart, $1 - R$ is the loss given default, $\mathrm{DF}(0, \tau)$ is the discount factor, and $\mathrm{NPV}(\tau, T))^{+} = \max\{\mathrm{NPV}(\tau, T),\ 0\}$ is the expected exposure. The net present value of the derivative is the expected future cash flow under some market filtration $\mathcal{F}_t$:

$$\mathrm{NPV}(\tau, T) = \mathbb{E}\left[\Pi(t,T) \mid \mathcal{F}_t\right]. \tag{3}$$

$\mathcal{F}_t$ should just be seen as the available information we have at time t, and $\Pi(t, T)$ is the discounted net cash flow between us and the counterpart. In essence $\mathrm{NPV}(\tau, T)$ is a condensed way of stating that we should price the value of the derivative today only with the information we have. At a future time step we will know something else and we should update the value of the expected cash flow.

In this assignment, we will make the following assumptions:

- Loss given default is constant over time i.e. R is constant.
- τ is independent of $\mathrm{NPV}(\tau, T)$. If market participants knew τ, they would react to the information and the valuation after and before τ would different.
- We divide the time to maturity into N intervals $0 < t_0 < t_1 < \ldots, < t_N$.
- τ is replaced by t_j such that if $t_{j-1} < \tau < t_j$, then $\mathrm{NPV}(\tau) \approx \mathrm{NPV}(t_j)$.

With these assumption, we can simply equation 2 to be:

$$\mathrm{CVA}_0 \approx (1-R)\sum_{j=1}^{n}\left(\mathbb{P}\left[\tau\leq t_j\right]-\mathbb{P}\left[\tau\leq t_{j-1}\right]\right)\cdot\mathrm{DF}_j\cdot\mathrm{EE}_j \tag{4}$$

where $\mathbb{P}[\tau \leq t_j] - \mathbb{P}[\tau \leq t_{j-1}]$ is the probability of default between the time points and we have introduced EE_j for the expected exposure. This can be written equivalently to match the equation from the slide for week 7:

$$\mathrm{CVA}_0 \approx (1-R)\sum_{j=1}^{n}\mathrm{DF}(t_j)\mathrm{EE}(t_j)\mathrm{PD}(t_{j-1},t_j) \tag{5}$$

In the following, we will consider each component separately and show explicitly how they can be calculated.

- $\mathrm{DF}(t_j)$: will be calculated using a Vasicek model for the short-term rates.
- $\mathrm{EE}(t_j)$: will be covered in section 5.
- $\mathrm{PD}(t_{j-1}, t_j)$: in section 6 we will cover the probability of default.

3 Model for Short-Term Rates

In order to simulate the exposure of the portfolio over time, we introduce the Vasicek model. It is an Ornstein-Uhlenbeck stochastic process which incorporates mean reversion.

$$\mathrm{d}r(t) = k(\theta - r(t))\mathrm{d}t + \sigma \mathrm{d}W(t) \tag{6}$$

We discretize the time steps t such that the N'th step is denoted t_N. The short-term interest rate at time t_N is denoted r_N. Furthermore, the time steps are assumed to be fixed Δt. Then the numerical solution can be found as

$$\begin{aligned} r_{N+1} &= r_N + k(\theta - r_N)(t_{N+1} - t_N) + \sigma\sqrt{t_{N+1} - t_N} Z \\ &= r_N + k(\theta - r_N)\Delta t + \sigma\sqrt{\Delta t} Z \end{aligned}$$

where $Z \sim \mathcal{N}(0, 1)$. For the interest simulations in the following provided constants[1] are used:

- Initial spot rate of $r_0 = 0.005$
- Mean reversion rate of $k = 0.3$
- Mean reversion level of $\theta = 0.02$
- Interest rate volatility of $\sigma = 0.01$

A total 50,000 simulations are carried out.

[1] Financial Risk Management – Project 2 2022

4 Valuation of Interest Rate Swaps

A plain vanilla interest rate swap is an agreement between two parties to exchange coupon payments on a particular notional. In this project, we only concern ourselves with fixed-for-floating. The floating payments will be referred to as the *floating leg*, and the fixed payments will be referred to as the *fixed leg*. Depending on whether the owner receives or pays the fixed interest rate, the swap agreement can be seen as a long position in one bond and a short position in another.

Let r_f denote the fixed interest rate and $r_v(t_i)$ the variable interest rate at time t_i. Let the notional from which the payments are calculated be denoted L_0. Let $ZBC(t_i, t_j)$ where $t_i \leq t_j$ be the value of a zero-coupon bond at time t_i maturing at t_j. Notice that $ZBC(t_i, t_i)$ trades at face value. The prices of the zero-coupon bonds will be used to discount the future values of the cash-flows to their present values. If we receive the floating coupon payments, the *Mark-to-Market* (f_{t_i}) of a swap will be

$$f_{t_i} = \sum_{i=1}^{N} L_0 \cdot (r_f - r_v(t_{i-1})) \cdot ZCB(t_i, t_j) \tag{7}$$

The ZCB from time t_i to maturity t_j follows from a no-arbitrage argument of the simulated short-term rates. These were calculated with the Vasicek model, i.e:

$$ZCB(t_i, t_j) = A(t_i, t_j)e^{-r(t_i)B(t_i,t_j)} \tag{8}$$

Where r(t_i) is the simulated interest rate at time t_i, and

$$B(t_i, t_j) = \frac{1 - e^{-k(t_j - t_i)}}{k} \tag{9}$$

and

$$A(t_i, t_j) = \exp\left\{ \left(\theta - \frac{\sigma^2}{k^2}\right)(B(t_i, t_j) - t_j + t_i) - \frac{\sigma^2}{4k} B^2(t_i, t_j) \right\} \tag{10}$$

5 Expected Exposure

The expected exposure is the expected amount of capital we would risk loosing if our counterparty defaults. In the following, we will go through ways to calculate the expected exposure where we gradually increase contract specifications.

Consider a plain vanilla interest rate swap contract with no netting or posted collateral:

$$\mathrm{EE}(t) = \mathbb{E}\left[\max\left\{f(t), 0\right\}\right]. \tag{11}$$

where $f(t)$ is the mark-to-market value of the swap contract seen from our perspective such that $f(t) > 0$, if the contract has value in our favor. We will first partition the time into time steps $0 < t_0 < t_1, \ldots, < t_N$ and hence for the i'th time point we have $\mathrm{EE}(t_i) = \mathbb{E}\left[\max\left\{f(t_i), 0\right\}\right]$. Using a Monte Carlo framework we will now make a simulation based approximation of the expected exposure. When we have the fixed and instance of the floating interest rate, it is straight forward value the forward contract using the method introduced in section 4. Given a set of model parameter, we can easily simulate many instances of floating rates using the the Vasicek model introduced in section 3.
Therefore, with N_s simulations and the partitioned time, we will use the following approximation:

$$\mathrm{EE}(t_i) = \frac{1}{N_s}\sum_{k=1}^{N_s} \max\left\{f^k(t_i), 0\right\}. \tag{12}$$

This is the exposure of one contract. However, we might have multiple contracts in our portfolio. Therefore, introduce $f_c(t_i)$ as the MtM value of the c'th swap contract in a portfolio of N_c contracts. We can write the generic formula as:

$$\mathrm{EE}(t) = \mathbb{E}\left[\sum_{c=1}^{N_c} \max\left\{f_c(t), 0\right\}\right]. \tag{13}$$

In our version where time is partitioned that would be

$$\mathrm{EE}(t_i) = \frac{1}{N_s}\sum_{k=1}^{N_s}\sum_{c=1}^{N_c} \max\left\{f_c^k(t_i), 0\right\}. \tag{14}$$

5.1 Netting and Collateral

In this report, the effects of netting and collateral are considered with regards to default risk.

In broad terms, netting is an agreement[2] between two parties that their derivative contracts are counted as one such that opposite value contracts cancel each other out and defaulting on one defaults all of them.

Netting is divided into two categories, payment netting and closeout netting. In the provided case, bilateral clearing is used and calculations only account for close-out netting.
A netting contract enters close-out, when a counter party files for bankruptcy, or fails to make payments or margin calls on one derivative contract covered in the netting agreement. In practice, bilateral netting is used far less today, as vast regulations has enforced central clearing and thus multilateral netting. The advantage of netting derivatives are great, as the risk on the contracts is reduced, thus the prices are likewise reduced. Multilateral netting has a few important advantages as it is much easier to regulate and monitor the simple company structure of a central counter party, rather than the complex financial institutions. The downside is the increase in collateral that is necessary to post. The involved banks needs better liquidity to handle the same contracts as previously, further the restrictions on rehypothecation[3] intensifies this. As these types of trades represented great source of income to the largest banks prior to the financial crisis, they are developing new instruments which falls into less regulated areas, mean while the regulators are trying to keep up.[3, p. 399-413]

5.1.1 Exposure under Netting

The exposure is changed slightly when we introduce netting. In the most simple setting, we can consider two contracts. When we find the value of the portfolio of swap contracts, we would net their value at each time point to calculate the exposure:

$$\mathrm{EE}(t)_\nu = \mathbb{E}\left[\max\left\{f_1(t) + f_2(t), 0\right\}\right]. \tag{15}$$

In the general case, we might have contracts with and without netting. Therefore, introduce NA as the set of contracts with a netting agreement then expected exposure on a portfolio level is now:

[2] usually an International Swap and Derivatives master agreement (ISDA)
[3] reusing collateral provided in one contract to be posted in the next

$$\mathrm{EE}(t) = \mathbb{E}\left[\max\left\{\sum_{c\in\mathrm{NA}} f_c(t), 0\right\} + \sum_{c\notin\{\mathrm{NA}\}} \max\left\{f_c(t), 0\right\}\right]. \tag{16}$$

Note: there could be different sets of netting agreements and then we would just introduce multiple NA_k's; one for each set.

In our case we will only either have netting for all or for none, hence the expected exposure reduces to:

$$\mathrm{EE}(t) = \mathbb{E}\left[\max\left\{\sum_{c=1}^{N_c} f_c(t), 0\right\}\right]. \tag{17}$$

5.2 Collateral

Consider again the simple instance with only one contract and let $C(t)$ denote the posted collateral at time t. Then the expected exposure is:

$$\mathrm{EE}(t) = \mathbb{E}\left[\max\left\{f(t) - C(t), 0\right\}\right] \tag{18}$$

Note that if $C(t) < 0$, then we have posted collateral and hence if the counterparty defaults, we would also expect this amount to be lost. On the other hand, if we have received collateral then $C(t) > 0$. In this case, it is an asset to us and will reduce our exposure to the counterparty. If the counterparty defaults, the value will not be fully paid and the exposure equal the present value which is $f(t) - C(t)$ when $f(t) > C(t)$; otherwise, it is 0.

Collateral can be posted for each contract separately or as a total amount to the counterparty. If it for each contract then we would calculate the expected exposure using the simulated interest rates as:

$$\mathrm{EE}(t_j) = \frac{1}{N_s}\sum_{k=1}^{N_s}\sum_{c=1}^{N_c} \max\left\{f_c^k(t_j) - C_c(t_j), 0\right\}. \tag{19}$$

If just one large amount of capital is posted for the all contracts, the expected exposure is:

$$\mathrm{EE}(t) = \frac{1}{N_s}\sum_{k=1}^{N_s} \max\left[\sum_{s}^{N_c} \max\left\{f_c^k(t_j), 0\right\} - C(t_j), 0\right]. \tag{20}$$

In our case, we are also to calculate the expected exposure with both netting and collateral and this can be calculated with the formula:

$$\mathrm{EE}(t) = \mathbb{E}\left[\max\left\{\sum_{c=1}^{N_c} f_c(t) - C(t), 0\right\}\right]. \tag{21}$$

6 Default intensity

In the following, we will introduce how we calculate the probability of default PD. To do that, we first introduce the framework to consider the distribution of the default intensity. Let τ denote the time of default for our counterparty and introduce t as the current time such that $\tau < t$ where market participants have the information $\mathcal{F}_t$. The probability of default of our counterparty in a time interval $(t, t+\Delta t)$ can be approximated by

$$\mathbb{P}\left[\tau \in (t, t+\Delta t) | \mathcal{F}_t\right] \approx \lambda_t \Delta t, \qquad \tau > t \tag{22}$$

This is only a valid approximation for small Δt. In the above equation λ_t is the hazard rate which is the probability that the company will default in an infinitesimal time after t [3, p. 296]:

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{P(T \in (t, t+\Delta t) \mid T > t)}{\Delta t}. \tag{23}$$

Using the notation introduced in this course and property (7) on [3, p. 297] with subsequent derivation on [3, p. 300], the survival function can be defined as:

$$V(t) = \exp\left(-\int_0^t \lambda\left(u\right) \mathrm{d}u\right). \tag{24}$$

We can equivalently define the probability of default up to τ as

$$Q(t) = 1 - V(t) \tag{25}$$

where $Q(t)$ is the cumulative distribution function for τ.

In the following, we will introduce a framework to calibrate the default intensities, $\lambda(t)$, to market data. When we have calibrated $\lambda(t)$, we can easily compute the instances of $Q(t)$ we need to calculate the CVA.

In our case, it is convenient to introduce the average intensities over some time period $(0, t)$:

$$\bar{\lambda}(t) = \frac{1}{t} \int_0^t \lambda(u) \mathrm{d}u, \tag{26}$$

hence subsequently we have $V(t) = \exp\left[-\bar{\lambda}(t)t\right]$ and $Q(t) = 1 - \exp\left[-\bar{\lambda}(t)t\right]$. The next step is to recall the assumption that we can partition the time into $0 < t_0 < t_1 < \ldots, < t_N$. We can now define a picewise constant $\lambda(t)$ from average intensities. Using the introduced $\bar{\lambda}(t)$ of equation 26, we define:

$$\lambda(t) = \begin{cases} \bar{\lambda}\left(t_1\right) & t \leq t_1 \\ \frac{\bar{\lambda}(t_2)t_2 - \bar{\lambda}(t_1)t_1}{t_2 - t_1} & t_1 < t \leq t_2 \\ \ldots & \ldots \end{cases} \tag{27}$$

This is a convenient way to define the intensities as we can find average default intensities directly from CDS spreads. For a CDS spread of a company with fixed maturity t_m, we define

$$\bar{\lambda}(t_m) = \frac{s(t_m)}{1 - R}, \tag{28}$$

where R is the coverage rate. The above result is actually only an approximation, however, we will not cover how it is derived but only use the result directly.

Therefore, the steps to obtain the default intensities is to use the CDS spreads from Morgan Stanley to calculate a set of $\bar{\lambda}(t_m)$ for all $t_m \leq t_N$. Then we construct the piece-wise $\lambda(t)$ to be used to calculate $Q(t_i)$. Using this parameterization, we can find the probability of default as $PD\left(t_{t_{i-1}}, t_i\right) = Q(t_i) - Q(t_{i-1})$ for all time steps $t_0, t_1, \ldots, t_N$.

The results we obtain from the CDS spreads from morgan Stanley are as follows.

time period (years)	λ
(0, 0.5)	0.00501053
(0, 1)	0.00731316
(0, 2)	0.01016316
(0, 3)	0.01370263
(0, 4)	0.01677105
(0, 5)	0.02060526
(0, 7)	0.02645
(0, 10)	0.03133684
(0, 20)	0.03802105
(0, 30)	0.03998158

Table 1: Table showing the average Hazard rates used to calculate the stepwise probability of default

time period	PD
(0, 0.5)	0.002502
(0.5, 1.0)	0.004784
(1.0, 2.0)	0.012835
(2.0, 3.0)	0.020153
(3.0, 4.0)	0.024609
(4.0, 5.0)	0.033013
(5.0, 7.0)	0.071124
(7.0, 10.0)	0.099999

Table 2: stepwise probability of default up until year 10

Using the found values in Table 2 we can now calculate the between half yearly probability of default that will be used in the calculation of CVA. And they are as follows:

time	PD
0.25	0.002502
0.75	0.004784
1.25	0.006417
1.75	0.006417
2.25	0.010077
2.75	0.010077
3.25	0.012305
3.75	0.012305
4.25	0.016507
4.75	0.016507
5.25	0.017781
5.75	0.017781
6.25	0.017781
6.75	0.017781
7.25	0.016667
7.75	0.016667
8.25	0.016667
8.75	0.016667
9.25	0.016667
9.75	0.016667

Table 3: probability of default at the between half yearly points

7 Estimation of CVA

With all of the knowledge we have from the previous sections, we can now make our simulations for the interest rates, and from these find the EE for the 3 swaps. Together with the EE, we will also find the 97.5% Peak Exposure (PE) and the maximum Peak Exposure. The EE and PE metrics will be plotted together with the mark-to-market simulations. In all the following figures and tables all the values presented will be in million USD.

For the 10-year interest swap we gain the following results of EE and PE can be seen on following plot:

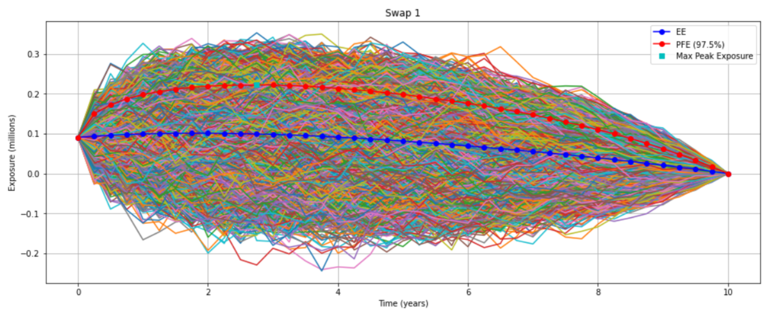

Figure 1: Figure showing 50.000 simulations of the Mark-to-Market value, the Expected Exposure, Peak Exposure and Max Peak Exposure for the 10-year swap

The result for the 7-year interest swap can be seen in the following plot:

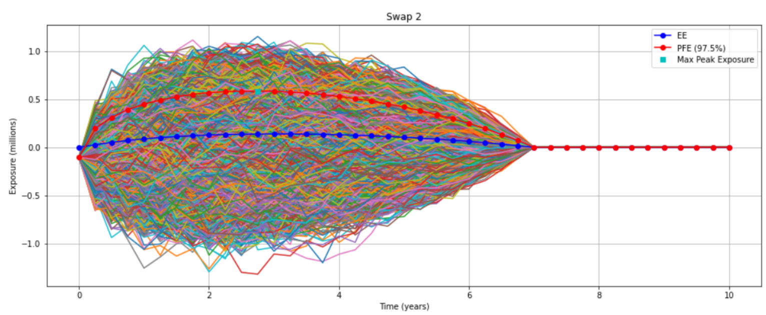

Figure 2: Figure showing 50.000 simulations of the Mark-to-Market value, the Expected Exposure, Peak Exposure and Max Peak Exposure for the 7-year swap

The result for the 4-year interest swap can be seen in the following plot:

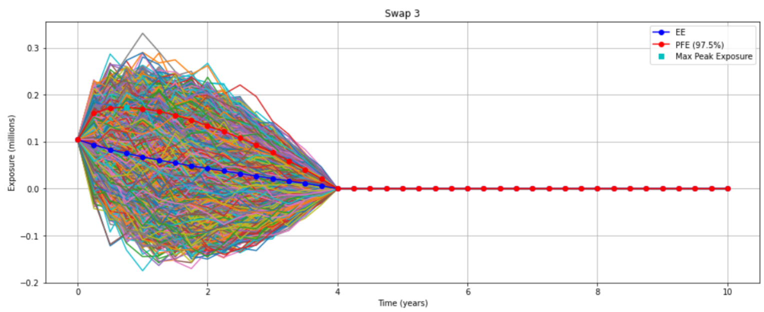

Figure 3: Figure showing 50.000 simulations of the Mark-to-Market value, the Expected Exposure, Peak Exposure and Max Peak Exposure for the 4-year swap

Based on these simulations, we calculate the CVA and get the results displayed in Table 4 together with the maximum value of the PE.

	Max Peak Exposure (97.5%)	CVA
10-year swap	0.2218	0.006095
7-year swap	0.5833	0.005760
4-year swap	0.1736	0.000827

Table 4: Table of Max Peak Exposure (97.5%) and CVA charge in millions

8 CVA under Netting and Collateral

When we are use netting and collateral, we obtain the maximum PE and calculated CVA that are shown in Table 5. In this Table we see that we gain the lowest CVA when we have collateral of $100,000 for each of the swaps, but if we net the swaps and get collateral of $100,000 in total we get all most the same CVA charge.

	Max Peak Exposure (97.5%)	CVA
Netting	0.7576	0.011502
Collateral	0.6229	0.004195
Netting and Collateral	0.6575	0.006715

Table 5: Table of Max Peak Exposure (97.5%) and CVA charge in millions

8.1 Exposure of Netted Swaps

To show the EE and PE for the netted swaps, they have been plotted together with simulated mark-to-market values under netting. The Resulting plot can be seen here in Figure 4.

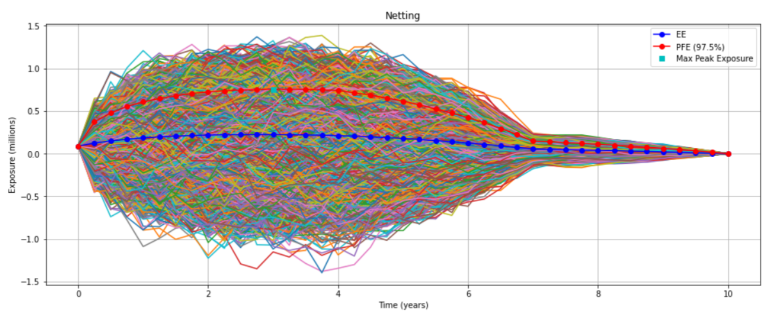

Figure 4: Figure showing 50.000 simulations of the Mark-to-Market value, the Expected Exposure, Peak Exposure and Max Peak Exposure for netted portfolio

8.2 Exposure with Collateral Posted per Swap

For the Swaps where Collateral has been posted for each of the swaps, the EE and PE have been plotted together with simulated mark-to-market values under netting. The Resulting plot can be seen here in Figure 5.

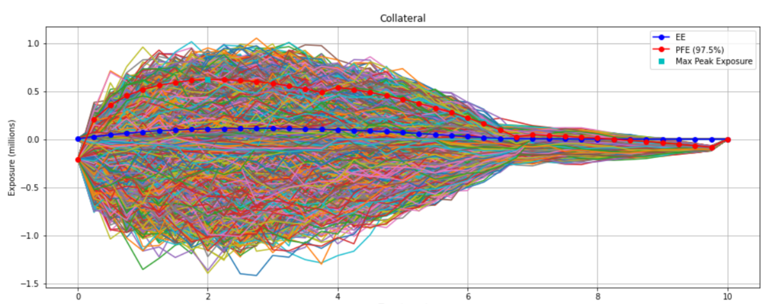

Figure 5: Figure showing 50.000 simulations of the Mark-to-Market value, the Expected Exposure, Peak Exposure and Max Peak Exposure for per swap collateral

8.3 Exposure with Netting and Portfolio Collateral

Lastly we have also plotted the PE and EE for the netted swaps, but this time where there has been posted collateral for the netted portfolio. The mark-to-market for this portfolie together with the EE and PE can be seen in Figure 6.

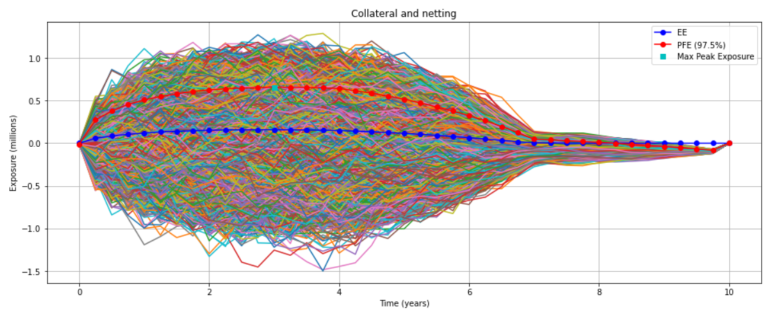

Figure 6: Figure showing 50.000 simulations of the Mark-to-Market value, the Expected Exposure, Peak Exposure and Max Peak Exposure for netting and portfolio collateral

9 Conclusion

Discounting cash-flows to value financial contracts result in a *no-default* value. The *Credit Value Adjustment* (CVA) estimates can be used to adjust the price of our financial contracts such that we account for credit risk. In other words, this is the price we should be willing to pay when trading these contracts in the market. It was demonstrated with the simulations that *netting* and posting of *collateral*, separately and in conjunction, can substantially reduce the CVA as the *Expected Exposure* (EE) in decreased.

The risk metrics are based on several assumptions with drawbacks. First and foremost, it is assumed that the market prices the *Credit Default Swaps* (CDS) correctly such that the *Probability of Default* (PD) of the counterpart is reflected. This is most likely not the case. Furthermore, a verification of the probabilities is practically impossible. This makes the quality of the estimates questionable. Another assumption worth mentioning is the independence between PD and EE. This is most likely not the case as large claims on an institution with potential large margin calls might lead to *wrong-way risk* meaning there is a positive correlation between the two quantities. Lastly, the parameter estimate in the modelling of the *Short-Term Rates* are assumed to be constant. It is more plausible that volatility and mean-level varies with over time.

References

[1] J. C. Hull, *Options, Futures, and Other Derivatives*, 2015.

[2] D. Brigo, M. Morini, and A. Pallavicini, *Counterparty credit risk, collateral and funding: with pricing cases for all asset classes.* John Wiley & Sons, 2013, vol. 478.

[3] J. Pitman, *Probability.* Springer Science & Business Media, 1999.