Worst-Case Value at Risk

ORIE 5640
Spring 2023
September 28, 2023

Ashley Jiang, yj387
Nicolaj Hans Nielsen, nhn6

Contents

1 Introduction

One of the most widely used measures of risk in portfolios of financial assets is the Value at Risk, VaR. One of the paramount requirements in the standard form is complete information about the first and second moments of the returns, which is rarely possible in practice. Not to mention how overly optimistic estimates on historical data could be. When derivatives and options are added to the portfolio, the task becomes increasingly difficult because standard methods might wrongly assess the shared uncertainty between derivatives and the underlying asset. In this project, we introduce two methods that take a worst-case approach to all possible realizations of the returns to compute VaR. This alleviates the first problem of being too optimistic, given only the first and second moments. These allocation problems can be formulated as tractable SOCP that can be solved with readily available solvers. To handle non-linear portfolios with derivatives, we introduce Worst-case Polyhedral VaR, *WPVaR*, which can handle derivatives in a way that takes the shared uncertainty of bonds and underlying stocks into account. It can be shown that WPVaR is equivalent to the worst-case conditional value at risk and thus makes the WPVaR a coherent risk measure, unlike the standard VaR. These methods are from the field of distributionally robust optimization; however, we will also show the connection to robust optimization, which allows for graphical interpretations. We compare the methods on a fixed portfolio and an allocation problem with a simulated case where we make the assumptions that all assets follow the Black-Scholes framework. Then we investigate the calculated VaR when we let a standard Markowitz method allocate the portfolio. In the end, we will test the WPVaR with stock and options data from a portfolio of companies in the energy sector.

2 Decision Making Under Uncertainty

Portfolio optimization is an example of a decision problem subject to uncertainty. If we do not account for the uncertainty in an appropriate way, we end up with sub-optimal allocations and potentially devastating losses. However, there are different approaches to formulating decision problems under uncertainty. Thus the following section is a general introduction to the different programming methods that we will consider in this report. Initially, we will demonstrate why uncertainty makes optimization harder and introduce some basic notations. Let $\mathbf{x} \in \mathbb{R}^n$ be a decision variable of dimension m, and $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a convex function, then we can formulate a convex optimization problem as following:

$$\begin{aligned} &\underset{\boldsymbol{x}\in\mathbb{R}^n}{\text{minimize}} f(\boldsymbol{x}, \tilde{\boldsymbol{\xi}}) \\ &\text{subject to } g(\boldsymbol{x}, \tilde{\boldsymbol{\xi}}) \leq 0 \\ &\boldsymbol{x} \in \mathcal{X}, \end{aligned} \tag{2.1}$$

where $\tilde{\boldsymbol{\xi}}$ is some uncertainty in the data parameters e.g. the uncertainty in the first central moments. Notice, that in the equations above, both f, and g depend on $\tilde{\boldsymbol{\xi}}$. Since there might exist many different realizations of $\tilde{\boldsymbol{\xi}}$, the model above can be thought of as a large family of problems with a problem for each possible realization. As a result, it is unlikely that there exists one unique solution for the problem above which means we need a way to disambiguate the problem and formulate it as a tractable optimization problem. In the following, we will present several methods to do so.

2.1 Stochastic Programming

The idea in stochastic programming is that the decision maker has full and acute information about the probability distribution $\mathbb{Q}$ of the random vector $\tilde{\boldsymbol{\xi}}$. Thus we can formulate a version of the optimization problem as,

$$\begin{aligned} \underset{\boldsymbol{x}\in\mathbb{R}^n}{\text{minimize}} \quad & \mathbb{E}_{\mathbb{Q}}\left[f(\boldsymbol{x},\tilde{\boldsymbol{\xi}})\right] \\ \text{subject to } & \mathbb{Q}(g\left(\boldsymbol{x},\tilde{\boldsymbol{\xi}}\right) \leq 0) \geq 1-\epsilon \\ & \boldsymbol{x} \in \mathcal{X}, \end{aligned} \tag{2.2}$$

where $\mathbb{E}_{\mathbb{Q}}\left[\cdot\right]$ is the expectation with respect to the know distribution $\mathbb{Q}$. Further, the constraint $g\left(\boldsymbol{x},\tilde{\boldsymbol{\xi}}\right) \leq 0$ is satisfied with a high probability $1-\epsilon$ where $\epsilon \in (0,1)$ is a so-called *risk factor* in the *chance constraint.* With $\mathbb{Q}$ known and a chosen ϵ, we can formulate a chance constraint,

$$\mathbb{Q}(g(\boldsymbol{x},\tilde{\boldsymbol{\xi}}) \leq 0) \geq 1-\epsilon. \tag{2.3}$$

With a high probability $1-\epsilon$, the constraint will be satisfied. Unfortunately, these types of problems are notoriously hard or directly intractable to solve [1]. On top of that, because we need to estimate $\mathbb{Q}$ using historical data, the constraint itself can be very uncertain. This means we will have a biased $\hat{\mathbb{Q}}$ that leads to overly optimistic solutions. In other words, $\hat{\mathbb{Q}}$ could be far from satisfying the change constraint under the true $\mathbb{Q}$. Therefore, we will introduce a different approach.

2.2 Robust Optimization

Robust optimization takes a worst-case perspective on eq. (2.1) in order to disambiguate it; see [2] for a thorough introduction to the approach. The idea is that $\tilde{\boldsymbol{\xi}}$ is still random, but realizations happen in an *uncertainty set*, $\mathcal{U}$. And we choose to consider the worst possible realization of $\tilde{\boldsymbol{\xi}}$ in $\mathcal{U}$ which can be formulated into a min-max problem,

$$\begin{aligned} \underset{\boldsymbol{x}\in\mathbb{R}^n}{\text{minimize}} \quad & \max_{\boldsymbol{\xi}\in\mathcal{U}} f(\boldsymbol{x},\boldsymbol{\xi}) \\ \text{subject to} \quad & g(\boldsymbol{x},\boldsymbol{\xi}) \leq 0 \quad \forall\boldsymbol{\xi}\in\mathcal{U} \\ & \boldsymbol{x}\in\mathcal{X}. \end{aligned} \tag{2.4}$$

Thus, for any fixed $\boldsymbol{x}$, the function $\max_{\boldsymbol{\xi}\in\mathcal{U}} f(\boldsymbol{x},\boldsymbol{\xi})$ finds the worst case cost that can possibly happen when $\boldsymbol{\xi}$ can only take values in $\mathcal{U}$. Notice that our constraint, $g(\boldsymbol{x},\boldsymbol{\xi}) \leq 0$, needs to be satisfied for all possible realization of $\boldsymbol{\xi}\in\mathcal{U}$ which makes the constraint a semi-infinite constraint. The resulting problem eq. (2.4) is called the robust counterpart of eq. (2.1). We will explore different uncertainty sets when we move to portfolio optimization; however, the important thing to note is that the shape of $\mathcal{U}$ reflects our knowledge about the random vector $\boldsymbol{\xi}$. We might only have information about certain moments or some partial information about the domain, and we need to reflect that information in $\mathcal{U}$.

2.3 Distributionally Robust Optimization

In this framework, we also assume we have partial information about $\boldsymbol{\xi}$. However, this time we use this to constrain ourselves to a set of possible distributions $\mathcal{P}$ that satisfied the known properties $\mathbb{Q}$; thus, we consider distributions instead of uncertainty sets. We then take a worst-case approach and find the worst case over all probability distributions within $\mathcal{P}$,

$$\begin{aligned} \underset{\boldsymbol{x}\in\mathbb{R}^n}{\text{minimize}} \; & \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}(f(\boldsymbol{x},\tilde{\boldsymbol{\xi}})) \\ \text{subject to } \; & \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{P}(g(\boldsymbol{x},\tilde{\boldsymbol{\xi}}) \le 0) \ge 1-\epsilon \\ & \boldsymbol{x} \in \mathcal{X}. \end{aligned} \tag{2.5}$$

Under this setup, for any fixed $\boldsymbol{x}$, the function $\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}(f(\boldsymbol{x},\tilde{\boldsymbol{\xi}}))$ finds the worst case expected cost when evaluating all possible distributions $\mathbb{P}$ in $\mathcal{P}$. This time we have a distributionally robust change constraint where $g(\boldsymbol{x},\tilde{\boldsymbol{\xi}}) \le 0$ is satisfied with a high probability $1-\epsilon$ under all $\mathbb{P}$,

$$\mathbb{P}(g(\boldsymbol{x},\tilde{\boldsymbol{\xi}}) \le 0) \ge 1-\epsilon \quad \forall \mathbb{P}\in\mathcal{P} \iff \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{P}(g(\boldsymbol{x},\tilde{\boldsymbol{\xi}}) \le 0) \ge 1-\epsilon. \tag{2.6}$$

This approach turn the often intractable stochastic problem in eq. (2.2) into a tractable optimization problem with some assumptions and knowledge only about the first and second moment.

3 Worst-Case Value-at-Risk Optimization

In the following, we let $\mathbf{w} \in \mathbb{R}^m$ denote out allocation in terms of the percentage of invested wealth in m assets. We let $\tilde{\mathbf{r}} \in \mathbb{R}^m$ denote a random vector of asset returns over the investment horizon. Thus, the returns of a portfolio over the investment horizon are given by $\tilde{\mathbf{r}_p} = \mathbf{w}^\intercal \tilde{\mathbf{r}}$. One of the most widely used metrics for assessing the risk in a portfolio is *value at risk*(VaR). We let $\text{VaR}_\epsilon(\boldsymbol{w})$ denote the $(1-\epsilon)$-percentile of the portfolio **loss** distribution. Thus the VaR is the smallest $\gamma \in \mathbb{R}$ such that the loss $-\mathbf{w}^\intercal\tilde{\mathbf{r}}$ exceeds γ with probability no larger than ϵ, i.e.

$$\text{VaR}_\epsilon(\boldsymbol{w}) = \min\left\{\gamma : \mathbb{P}\left\{\gamma \le -\boldsymbol{w}^\top \tilde{\boldsymbol{r}}\right\} \le \epsilon\right\} \tag{3.1}$$

where $\mathbb{P}$ is the distribution of assets returns. We will consider the equivalent formulation,

$$\begin{aligned} \underset{\boldsymbol{w}\in\mathbb{R}^m,\gamma\in\mathbb{R}}{\text{minimize}} \quad & \gamma \\ \text{subject to} \quad & \mathbb{P}\left\{\gamma + \boldsymbol{w}^\top\tilde{\boldsymbol{r}} \ge 0\right\} \ge 1-\epsilon \\ & \boldsymbol{w} \in \mathcal{W}, \end{aligned} \tag{3.2}$$

where we introduce $\boldsymbol{w} \in \mathcal{W}$ with $\mathcal{W} \subseteq \mathbb{R}^m$ as the set of admissible portfolios e.g., that admits our budget constraint $\sum_{i=1}^m w_i = 1$. For all of the following optimization programs, we also only consider $\mathcal{W}$ to be a convex polyhedron.

3.1 Normal VaR

We will use the normal VaR as a benchmark to compare to. To calculate the normal VaR we assume $\tilde{\mathbf{r}} \in \mathbb{R}^m \sim \mathcal{N}(\mu_r, \boldsymbol{\Sigma}_r)$ where $\mu_r \in \mathbb{R}^m$ and $\boldsymbol{\Sigma}_r \in \mathbb{S}^m$ where $\mathbb{S}^m$ are all symmetric matrices of dimension $m \times m$. Under this assumption, we have the closed-form solution,

$$\text{VaR}_\epsilon(\boldsymbol{w}) = -\boldsymbol{\mu}_r^\top \boldsymbol{w} - \Phi^{-1}(\epsilon)\sqrt{\boldsymbol{w}^\top \boldsymbol{\Sigma}_r \boldsymbol{w}} \tag{3.3}$$

where $\Phi^{-1}(\cdot)$ is the inverse CDF of a standard normal distribution. The assumption of normality is known to greatly underestimate the VaR because financial returns are often highly skewed, [3]. Therefore, we need methods that produce more conservative estimates.

3.2 Worst-Case Value-at-Risk

The neat property of the Gaussian distribution is that it is fully determined by its mean and variance which makes the parameter estimation fairly simple. In contrast, [4] used ideas from distributionally robust optimization to come up with the ultra-conservative value at risk. Let $\mathcal{P}_r$ be the set of all possible distributions with mean and covariance matrix, $\mu_r, \mathbf{\Sigma}_r$. The set $\mathcal{P}_r$ might contain all kinds of distribution e.g., highly skewed and with different domains. In [4] they show the incredible result that even with a whole set of possible distributions, the VaR can be reduced to,

$$\begin{aligned} \text{WVaR}_\epsilon(\boldsymbol{w}) &= \min \left\{ \gamma : \sup_{\mathbb{P} \in \mathcal{P}_{\boldsymbol{r}}} \mathbb{P}\left\{ \gamma \leq -\boldsymbol{w}^\top \tilde{\boldsymbol{r}} \right\} \leq \epsilon \right\} \\ &= -\boldsymbol{\mu}_r^\top \boldsymbol{w} + \kappa(\epsilon)\sqrt{\boldsymbol{w}^\top \mathbf{\Sigma}_r \boldsymbol{w}} \end{aligned} \tag{3.4}$$

with $\kappa(\epsilon) = \sqrt{(1-\epsilon)/\epsilon}$. Notice the above is for some fixed $\mathbf{w}$; however, we can also formulate this as a portfolio allocation problem,

$$\begin{aligned} \underset{\boldsymbol{w} \in \mathbb{R}^m}{\text{minimize}} \quad & -\boldsymbol{\mu}_{\boldsymbol{r}}^\top \boldsymbol{w} + \kappa(\epsilon) \left\| \mathbf{\Sigma}_{\boldsymbol{r}}^{1/2} \boldsymbol{w} \right\|_2 \\ \text{subject to} \quad & \boldsymbol{w} \in \mathcal{W}, \end{aligned} \tag{3.5}$$

This is a tractable second-order cone program for which we have fast solvers. We will now show that the WVaR can also be formulated using a robust optimization program. Consider the robust counterpart of the standard VaR eq. (3.2) with an ellipsoidal set,

$$\mathcal{U} = \left\{ \boldsymbol{r} \in \mathbb{R}^m : (\boldsymbol{r} - \boldsymbol{\mu}_{\boldsymbol{r}})^\top \mathbf{\Sigma}_{\boldsymbol{r}}^{-1} (\boldsymbol{r} - \boldsymbol{\mu}_{\boldsymbol{r}}) \leq \delta^2 \right\} \tag{3.6}$$

where δ is a size parameter to control the size of the ellipsoid that encapsulates the set of possible returns. Using conic duality, it is shown in [4] that, $\gamma + \boldsymbol{w}^\top \boldsymbol{r} \geq 0 \quad \forall \boldsymbol{r} \in \mathcal{U} \iff -\boldsymbol{\mu}_{\boldsymbol{r}}^\top \boldsymbol{w} + \delta \left\| \mathbf{\Sigma}_{\boldsymbol{r}}^{1/2} \boldsymbol{w} \right\|_2 \leq \gamma$, and the robust program turns into the second-order cone program [5],

$$\begin{aligned} \underset{\boldsymbol{w} \in \mathbb{R}^m}{\text{minimize}} \quad & -\boldsymbol{\mu}_{\boldsymbol{r}}^\top \boldsymbol{w} + \delta \left\| \mathbf{\Sigma}_{\boldsymbol{r}}^{1/2} \boldsymbol{w} \right\|_2 \\ \text{subject to} \quad & \boldsymbol{w} \in \mathcal{W}. \end{aligned} \tag{3.7}$$

Note that when we compare the above equation eq. (3.7) with the distributionally robust program in eq. (3.5), they become equivalent if we set $\delta = \kappa(\epsilon)$.

3.2.1 Worst-Case Value-at-Risk with Derivatives

We now introduce derivatives into the portfolio such that we will have allocation w_i for $i = 1, 2, \ldots, n, n+1, \ldots, m$ where $w_1, w_2, ..., w_n$ denote weights on basic assets and $w_{n+1}, w_{n+2}, ..., w_m$ denote weights on derivatives. In total we have n basic assets and $m - n$ derivatives. We adapt to the notation used in [5] and let $\tilde{\boldsymbol{r}} = (\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\eta}})$ be the vector of random returns, $\tilde{\boldsymbol{\xi}}$ be the basic assets returns, and $\tilde{\boldsymbol{\eta}}$ be the derivative returns. With the mean and covariance, one could use WVaR directly, however, there are a couple of issues with that,

1. The first two moments of a portfolio with derivatives might be hard to estimate. Bonds have maturities, and their returns might first be meaningful close to the maturity; thus, we would need to do contract rolling to get long historical series of data.

2. WVaR would be a poor approximation as it disregards the perfectly known relationships between the basic assets and the derivatives through the pay-off function, $f : \mathbb{R}^n \mapsto \mathbb{R}^m$.

Since the derivatives are only uncertain because their underlying is uncertain; the returns on the derivatives can be written as $\tilde{\boldsymbol{r}}_\eta = f\left(\tilde{\boldsymbol{\xi}}\right)$

3. WVaR would overestimate the VaR drastically as the nonlinear f is only encapsulated through the linear matrix $\boldsymbol{\Sigma}_{\mathbf{r}}$. This can be understood geometrically using the robust optimization interpretation of the WVaR. If $\mathbf{mu_r}$ is highly symmetric but the option pay-off function f is skewed, then we would need to compensate and have an extra large symmetric ellipsoid just so that we can encapsulate the introduced skewness. As a result, the WVaR would produce overly pessimistic estimates. We will see this in a numerical example later.

To mitigate these issues, we will represent the returns for the derivatives using pay-off functions whose values depend on basic assets.

3.3 Worst-Case Polyhedral VaR Optimization

In the following, we will introduce European call and put options and assume they mature at the end of the investment horizon. Consider a partition of the allocation vector $\boldsymbol{w} = \left(\boldsymbol{w}^\xi, \boldsymbol{w}^\eta\right)$ where $\boldsymbol{w}^{\boldsymbol{\xi}} \in \mathbb{R}^n$ but we forbid short sales of the options; thus $\boldsymbol{w}^\eta \geq 0$. We now introduce the function f for both the basic assets and the derivatives s.t. f is just the identity map for the basic assets, $f_j\left(\tilde{\xi}\right)$.

3.3.1 Introducing Options

Let asset j be a call on basic asset i. Then let $\tilde{r}_j$, c_j, k_j denote the return, initial price, and strike price, respectively. If s_i is the initial price of the basic asset, then the price at the investment period is $s_i\left(1+\tilde{\xi}_i\right)$. We can now utilize directly the payoff function of a call which is a convex piecewise-linear function of $\tilde{\xi}_i$,

$$\begin{aligned} f_j(\tilde{\boldsymbol{\xi}}) &= \frac{1}{c_j}\max\left\{0, s_i\left(1+\tilde{\xi}_i\right) - k_j\right\} - 1 \\ &= \max\left\{-1, a_j + b_j\tilde{\xi}_i - 1\right\}, \quad \text{where} \quad a_j = \frac{s_i - k_j}{c_j} \quad \text{and} \quad b_j = \frac{s_i}{c_j} \end{aligned} \tag{3.8}$$

Likewise, if we have a put option on the basic asset i, then we can write,

$$f_j(\tilde{\boldsymbol{\xi}}) = \max\left\{-1, a_j + b_j\tilde{\xi}_i - 1\right\}, \quad \text{where} \quad a_j = \frac{k_j - s_i}{p_j} \quad \text{and} \quad b_j = -\frac{s_i}{p_j} \tag{3.9}$$

To write all of the assets more compactly, we introduce $\boldsymbol{a} \in \mathbb{R}^{m-n}, \mathbf{B} \in \mathbb{R}^{(m-n)\times n}$ such that we can write,

$$\tilde{\boldsymbol{r}} = f(\tilde{\boldsymbol{\xi}}) = \begin{pmatrix} \tilde{\boldsymbol{\xi}} \\ \max\{-\boldsymbol{e}, \boldsymbol{a} + \mathbf{B}\tilde{\boldsymbol{\xi}} - \boldsymbol{e}\} \end{pmatrix}. \tag{3.10}$$

This allows us to formulate the *worst-case polyhedral VaR* as

$$\begin{aligned} \mathrm{WCPVaR}_\epsilon(\boldsymbol{w}) &= \min\left\{\gamma : \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}\left\{\gamma \leq -\boldsymbol{w}^\top f(\tilde{\boldsymbol{\xi}})\right\} \leq \epsilon\right\} \\ &= \min_{0\leq g\leq \boldsymbol{w}^\eta} -\boldsymbol{\mu}^\top\left(\boldsymbol{w}^\xi + \mathbf{B}^\top\boldsymbol{g}\right) + \kappa(\epsilon)\left\|\boldsymbol{\Sigma}^{1/2}\left(\boldsymbol{w}^{\boldsymbol{\xi}} + \mathbf{B}^\top\boldsymbol{g}\right)\right\|_2 - \boldsymbol{a}^\top\boldsymbol{g} + \boldsymbol{e}^\top\boldsymbol{w}^\eta \end{aligned}$$

The second step above is highly non-trivial and involves many steps, but we refer to [5] for the outline and for all of the arguments [6]. Note again that the above model is a SOCP which we can easily solve, but it is the WPVaR for a given allocation, $\mathbf{w}$. In [7], it is shown that there exists an equivalent problem formulated as a robust optimization problem with a uncertainty set that is not symmetric:

$$\mathcal{U}_\epsilon^p = \left\{ \boldsymbol{r} \in \mathbb{R}^m : \exists \boldsymbol{\xi} \in \mathbb{R}^n, (\boldsymbol{\xi} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\boldsymbol{\xi} - \boldsymbol{\mu}) \leq \kappa(\epsilon)^2, \boldsymbol{r} = f(\boldsymbol{\xi}) \right\}$$

This formulation is useful because it enables us to construct a graphical illustration of the uncertainty set, as seen in fig. 1. We see how the uncertainty in the call option is just a piece-wise function of the elliptical uncertainty set of the stocks.

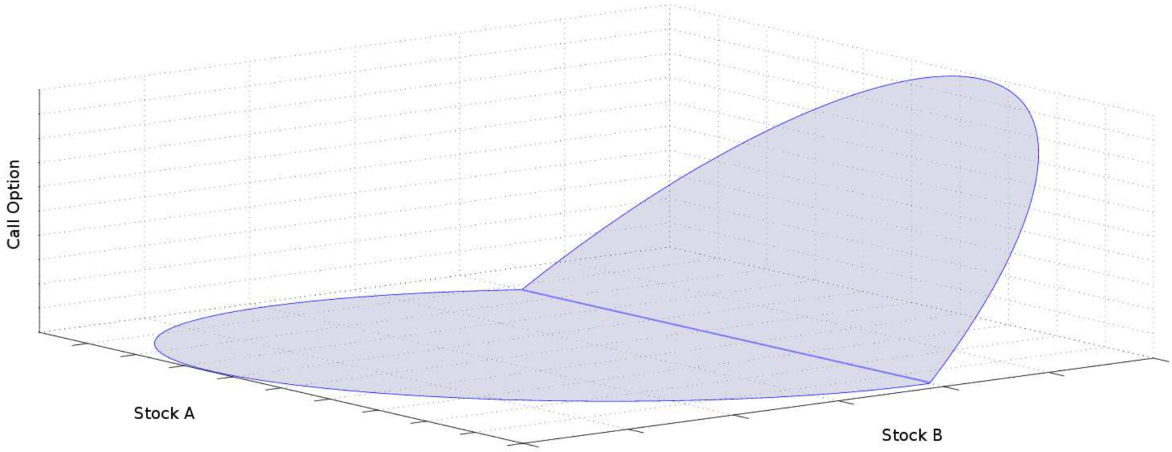


Figure 1 – A graphical illustration of $\mathcal{U}_\epsilon^p$ for a call option as a piece-wise linear transformation of the ellipsoidal uncertainty set of the stock. The figure directly replicates figure 4.1 in [6].

3.3.2 Relation to Worst-Case Conditional Value at Risk

The VaR is a useful measure but is often criticized for not having the subadditive property, which means that it might not promote diversification in a portfolio, and hence it cannot be considered a coherent risk measure [8]. Therefore, one can instead consider the conditional value at risk, *CVaR*. This is a conditional expectation of the portfolio loss if we have a period with a loss above VaR. The CVaR can be defined as [5],

$$\text{CVaR}_\epsilon(\mathbf{w}) = \inf_{\beta \in \mathbb{R}} \left\{ \beta + \frac{1}{\epsilon} \mathbb{E}_{\mathbb{P}} \left(\left(-\mathbf{w}^T \tilde{\mathbf{r}} - \beta \right)^+ \right) \right\}, \tag{3.11}$$

if we assume that $\tilde{\mathbf{r}}$ is from the distribution $\mathbb{P}$ and our tolerance is ϵ.Nevertheless, the problem with CVaR is that it is hard to estimate and cannot be determined only from the first and second central moments of $\tilde{\mathbf{r}}$ as VaR can. This is where our introduced frameworks have some useful properties. Consider the worst-case CVaR,

$$\text{WCVaR}_\epsilon(\mathbf{w}) = \sup_{\mathbb{P} \in \mathcal{P}_{\boldsymbol{r}}} \inf_{\beta \in \mathbb{R}} \left\{ \beta + \frac{1}{\epsilon} \mathbb{E}_{\mathbb{P}} \left(-\mathbf{w}^T \tilde{\mathbf{r}} - \beta \right)^+ \right\} \tag{3.12}$$

where $\mathcal{P}_{\boldsymbol{r}}$ is the set of possible distributions for $\boldsymbol{r}$. We especially care about the case where the portfolio also contains derivatives with expiration in the end of the investments horizon. These portfolios allow us to have a less conservative WPCVaR:

$$\text{WPCVaR}_\epsilon(\mathbf{w}) = \sup_{\mathbb{P} \in P} \inf_{\beta \in \mathbb{R}} \left\{ \beta + \frac{1}{\epsilon} \mathbb{E}_{\mathbb{P}} \left(- \left(\mathbf{w}^{\boldsymbol{\xi}} \right)^T \tilde{\boldsymbol{\xi}} - (\mathbf{w}^\eta)^T \max\{-\mathbf{e}, \mathbf{a} + \mathbf{B}\tilde{\boldsymbol{\xi}} - \mathbf{e}\} - \beta \right)^+ \right\}$$

Meanwhile WPVaR takes into account the relationship between basic assets and the options written on them. The surprising and non-trivial result is that is stated in theorem 6.1 (i) in [5],

$$\mathrm{WPVaR}_\epsilon(\mathbf{w}) = \mathrm{WPCVaR}_\epsilon(\mathbf{w}) \text{ for all } \mathbf{w} = \left(\mathbf{w}^\xi, \mathbf{w}^\eta\right) \in \mathbb{R}^n \times \mathbb{R}_+^{m-n} \tag{3.13}$$

This result is significant because it gives us a way to find the CVaR just through the WPVaR we introduced before. Notice that we only need first and second-order moments and the known relationships between basis assets and derivatives. The result is developed and shown in [6].
In the following, we will give a numerical example of the calculated VaR differs.

4 Application with Black-Scholes Economy

In the following, we will construct a simple example to showcase how a different fixed allocation would show different VaR under each method of calculating VaR. Our portfolio consists of four assets: two stocks A and B, and a call on stock A and a put on stock B. We then assume that the stock pair (S_A, S_B) follows a bivariate geometric Brownian motion with annual drift of 12% and 8%, and volatility of 30% and 20% with a contemporaneous correlation of 20%. The options mature within 21 days and have an initial price and strike price both of 100\$. We assume that we have a risk-free rate of 3% and 252 trading days per year, and as a result, we yield a price of \$3.58 and \$2.18 for the call and put options respectively, based on the Black-Scholes formula. We observe the following first two moments during this time horizon,

$$\boldsymbol{\mu_r} = \begin{bmatrix} 0.01 \\ 0.0067 \\ 0.1165 \\ -0.0856 \end{bmatrix}, \qquad \boldsymbol{\Sigma_r} = \begin{bmatrix} 0.0077 & 0.0010 & 0.1245 & -0.0204 \\ 0.0010 & 0.0034 & 0.0160 & -0.0670 \\ 0.1245 & 0.0160 & 2.5466 & -0.3028 \\ -0.0204 & -0.0670 & -0.3028 & 1.9580 \end{bmatrix} \tag{4.1}$$

Notice that in this example, we are able to use all of the methods because we can find the first two moments across all assets - even the options.

4.1 Fixed and Equal-weight Portfolio

Given this type of market, we first make the simple assumption that we have a portfolio of equal weights for each asset; thus, $\boldsymbol{w} = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}^\top$. Given this fixed portfolio, we will now examine how the VaR differs for each of the VaR methods introduced, see fig. 2.

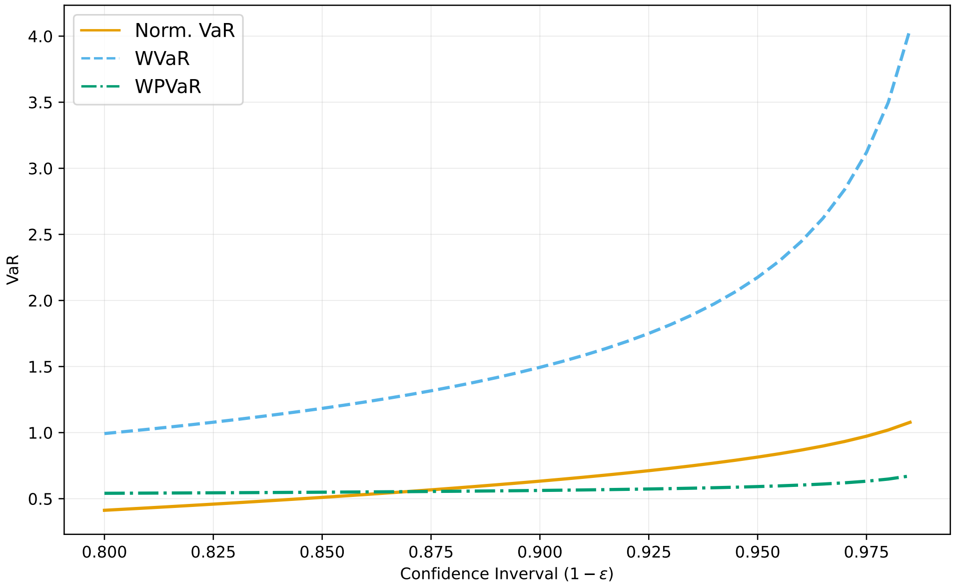


Figure 2 – A comparison of the VaR computed for an equal-weighted portfolio in the BS-economy with varying levels of confidence ϵ for each of the methods introduced.

We make the following observations:

1. **WVaR** worst-case value at risk seems overly conservative about the portfolio with derivatives. Recall with the robust optimization framework; we had a ellipsoidal set, $\mathcal{U}$. By introducing the derivatives, we will have a skewed distribution in the multidimensional space. However, with $WVaR$, we will only consider ellipsoids, making the results highly conservative.

2. **Gaussian** it appears that the estimates are slightly too optimistic for small $(1-\epsilon)$. However, even this method gradually becomes too conservative with derivatives present in the portfolio when $(1-\epsilon)$ becomes closer to one.

3. **WPVaR** the WPVaR remains low even as $\epsilon \to 0$. Recall that this is because it, by construction, only lets risk on the derivatives be calculated through the payoff and uncertainty of the underlying.

We now use this intuition as we move on to investigate what happens when we calculate the optimal allocations under each of the VaR models.

4.2 Calculating Optimal Allocation under VaR

In the following, we will investigate how the VaR changes when we change the allocation of wealth. In other words, we will find the allocation that minimizes the VaR under each method in the BS-economy. We cannot allow for short sales for the VaR framework to work for the normal VaR. Thus we impose the constraint $\mathbf{w} \geq 0$ for all of the methods. Additionally, we initially assume no transaction costs and no initial allocation. The VaR for different levels of confidence can be seen in fig. 3.

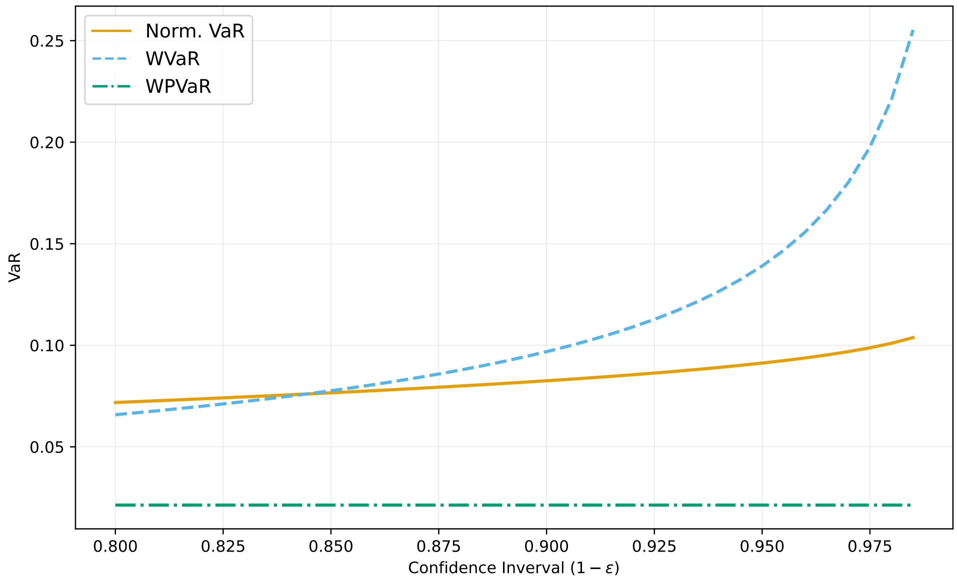


Figure 3 – A comparison of the VaR computed for the optimal portfolio with varying level of confidence ϵ for each of the methods introduced.

In figure fig. 3, we notice that the overall qualitative behavior is the same as that of a fixed $\mathbf{w}$. More specifically, the WVaR sill produces the most conservative estimate, with Gaussian Var being slightly more optimistic, and that WPVaR is far lower than the other methods. The allocation does not change much when the confidence level changes, so we will consider $\epsilon = 0.1$:

	$w^A_{\tilde{\boldsymbol{\xi}}}$	$w^B_{\tilde{\boldsymbol{\xi}}}$	$w^A_{\tilde{\boldsymbol{\eta}}}$	$w^B_{\tilde{\boldsymbol{\eta}}}$
Normal VaR	0.000	0.412	0.009	0.580
WVaR	0.087	0.881	0.000	0.032
WPVaR	0.000	0.979	0.000	0.021

Table 1 – The allocation for $\epsilon = 0.1$ in the BS-economy with two stocks A, B. Recall $w^A_{\tilde{\boldsymbol{\eta}}}$ is the allocation to call on stock A, and $w^B_{\tilde{\boldsymbol{\eta}}}$ is the allocation to a put on B.

In table 1, we notice that all of the methods allocate a substantial amount to the stock $w^B_{\tilde{\boldsymbol{\xi}}}$, which makes sense as it has the lowest variance which is shown in eq. (4.1). The objective for the Normal VaR is only to reduce the variance and loss. Since the put on B is negatively correlated with all other assets, it allocates a substantial amount in $w^B_{\tilde{\boldsymbol{\eta}}}$, but it also allocates weights to $w^A_{\tilde{\boldsymbol{\eta}}}$ which is the riskiest but it also promises a great return. On the other hand, the worst-case VaR, i.e., WVaR, tends to be very conservative and does not allocate anything to $w^A_{\tilde{\boldsymbol{\eta}}}$ because that would render the portfolio too risky. Instead, its optimal allocation is to go long in stock A, $w^A_{\tilde{\boldsymbol{\xi}}}$ but not its call option. WPVaR has the feature of understanding that the put option on stock B inherits the uncertainty in stock B and likewise for the call option on stock A. As a result, WPVaR allocates most of the weights to stock B, and the remaining amount to the put option on B. This way, the model effectively hedges the downside risk of stock B, making it a perfectly sensible strategy. The WaR and Normal VaR also followed this strategy partly but only from the variance-covariance information. In contrast, the WPVaR, by construction, knows how the stock B and put on B are interlinked.

4.2.1 Allow for Short Positions on the Basic Assets

We also conducted experiments with WVaR and WPVaR where we allowed for short positions on the stock; see the results in appendix A.1. The WVaR value did not change at all, while the WPVaR value sometimes changed slightly and started short-selling stock A and buying the associated call option on A. This can also be considered a reasonable strategy if one wants to hedge the risks. However, the short position is of order 10^{-9}; thus, one can dismiss the significance, but the allocation result can be seen in appendix A.1.

4.3 Calculating Optimal Allocation under Markowitz Model

We then changed our strategy from minimizing the risks to a mean-variance optimization approach using Markowitz Model still in the BS-economy setting. Then we would examine how VaR is different for each VaR method using allocations yielded from the Markowitz model, similar to what we did for a fixed-weight portfolio.
We first implement the simplest Markowitz model following the optimization framework below:

$$\begin{aligned} \max \quad & \mu^T x \\ s.t. \quad & x^T V x \leq \sigma^2 \\ & e^T x = 1 \end{aligned}$$

where μ is estimated with the historical returns for the four assets, V is the covariance matrix, and σ is our risk tolerance. Note that our model is slightly different because we did not include a risk-free asset. By choosing a risk tolerance of 0.05, we got a weight vector of $\boldsymbol{w} = \begin{bmatrix} -0.50 & 0.30 & 1 & 0.20 \end{bmatrix}^\top$. And the respective VaR value for each method is shown in fig. 4.

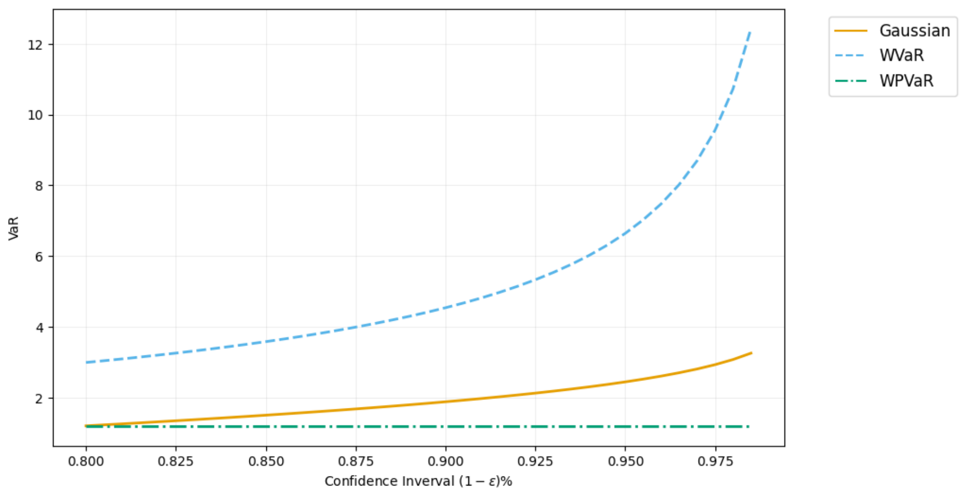


Figure 4 – A comparison of the VaR computed for the Markowitz portfolio with varying levels of confidence ϵ for each of the methods introduced.

Again, notice that the relative performance of each VaR method follows a similar pattern to the portfolios we constructed earlier. WVaR still produces the most conservative results, Gaussian still produces more optimistic results, and WPVaR again produces the most optimistic results. However, notice how much larger the values on y-axis have become. This is not surprising, considering our objective is to maximize the returns. Notice how the model allocates the most amount of asset to call option for stock A because it has the highest expected return even

though it is also highly risky. This result demonstrates how Markowitz is known to produce counterintuitive asset allocations when the portfolio return is skewed.
Next, we want to investigate how different the results are if we change our objective from maximizing the return to minimizing the risk. We change our model to the following

$$\begin{aligned} \min \quad & x^T V x \\ s.t. \quad & \mu^T x \geq R \\ & e^T x = 1 \end{aligned}$$

where R is a minimum return we want to yield.
The new weight vector therefore becomes $\boldsymbol{w} = \begin{bmatrix} -0.19 & 0.37 & 0.37 & 0.45 \end{bmatrix}^\top$. Note how the strategy has changed from that of the previous framework. Instead of heavily investing in an asset promising high return, we have a 'perfectly' hedged portfolio where we would short stock A and long the call option on it, then long stock B and short the put option on it. Moreover, we do so in a more balanced way. As a result, the VaR values are also different, as demonstrated below in fig. 5

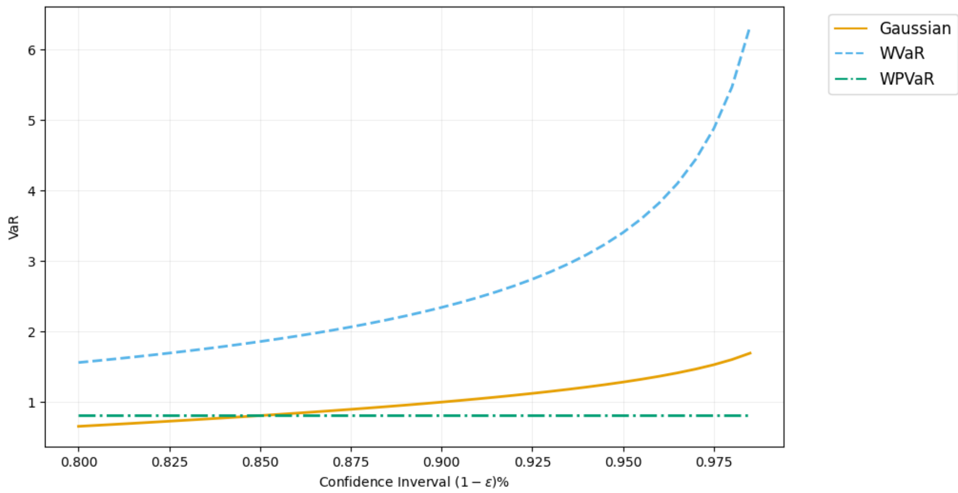


Figure 5 – A comparison of the VaR computed for the Markowitz portfolio with varying confidence levels ϵ for each of the methods introduced.

Compared to the previous graph, VaR values are lower on average, and the performance differences for each method are also smaller. Although pattern from previous portfolios also persists, compared to portfolios that minimize risk, i.e., the ones we calculated in part 4.3, the VaR values are much larger for the Markowitz portfolio.
Finally, we explored finding the optimal allocation by maximizing the difference between return and risks, i.e.

$$\begin{aligned} max \quad & \mu^T x - \frac{1}{2}\delta x^T V x \\ s.t. \quad & e^T x = 1 \end{aligned}$$

where delta is a risk aversion parameter that can only be positive. The larger δ is, the more risk-averse an investor is. By choosing a moderate δ value of 0.25, we got the following weight vector: $\boldsymbol{w} = \begin{bmatrix} -0.51 & 0.29 & 1.00 & 0.21 \end{bmatrix}^\top$. Notice how the strategy of this portfolio is similar to that of the previous one, where it is 'perfectly hedged' in a matter of speaking. We are again shorting stock A while longing the call option on it and longing stock B while shorting the put option on it. However, the difference is that we are allocating different weights to them. If, for the last strategy, we are relatively distributing our wealth more evenly, this time, we are more heavily invested in ones that promise a higher return. In other words, we are allocating more wealth to longing call option for stock A and less for a put option on stock B. And the respective VaR values can be seen in fig. 6

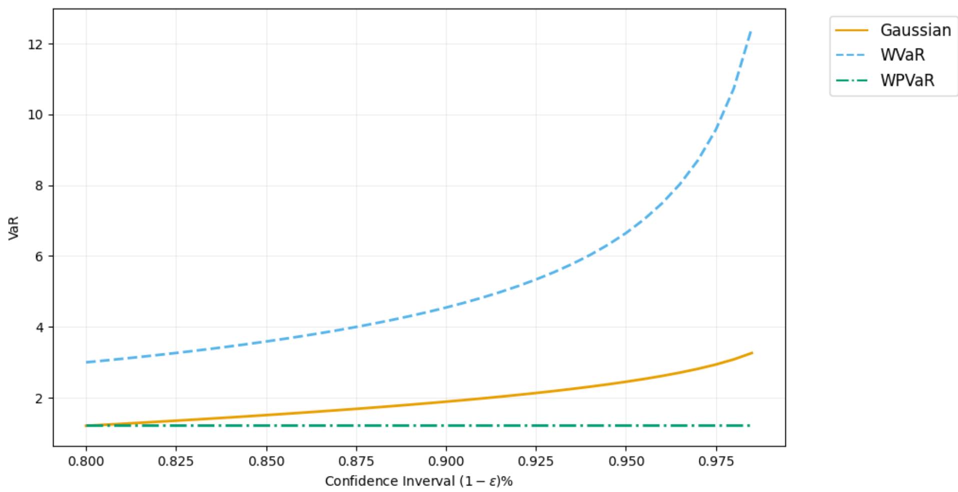


Figure 6 – A comparison of the VaR computed for the Markowitz portfolio with varying confidence levels ϵ for each of the methods introduced.

Notice how this graph is similar to the one we plotted from the first Markowitz portfolio, where we simply maximized the return. It is not surprising since the allocation of wealth is very similar despite having a different objective method. In a way, we can deduct from this result the Markowitz model would 'prioritize' getting higher expected returns even if that means taking on more risks.

5 Application with a Portfolio of Energy Companies

In the following, we will apply the VaR allocation method to a portfolio of stocks and options that are all in the energy sector. Specifically, we will consider the tickers, `'BP'`, `'CVX'` `'GE'`, `'XOM'`.

5.1 Data

We use the **Python** package, `yoptions` to fetch the option chain with the closest time to maturity. All of the companies had options with weekly maturity, and we fetched options data from EOD 5th of May, 2023, and thus they all had maturity a week later; the 12th of May 2023[1]. As the portfolio would expire in a week, we fetched the weekly prices on the stocks of the assets using the **Python** package `yfinance` and downloaded three years of data.

5.2 Prepossessing

For the stock data, we calculated the net-returns, $R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$, where P_t is the price in week t. We have 52 weeks of data to calculate the covariance matrix of the four stocks. Note that

[1] In the discussion, we will present several challenges we had when acquiring options data in general.

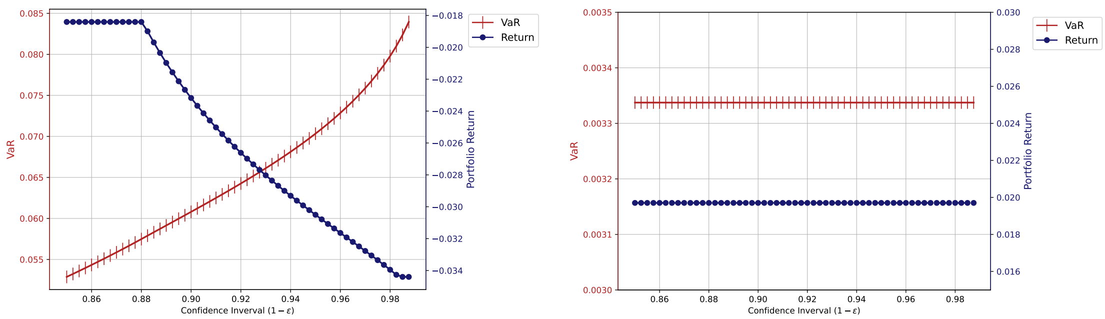
we cannot calculate the covariance matrix with the options. Because the options are prices in with current information in the market, we used a flat forward forecasting idea, and this used the returns from the week before for the $\boldsymbol{\mu}$. The moments are, thus,

$$\begin{bmatrix} \mu_{BP} \\ \mu_{CVX} \\ \mu_{GE} \\ \mu_{XOM} \end{bmatrix} = \begin{bmatrix} -0.078 \\ -0.050 \\ 0.013 \\ -0.082 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 0.00225 & 0.00174 & 0.00102 & 0.00201 \\ 0.00174 & 0.00226 & 0.00121 & 0.00221 \\ 0.00102 & 0.00121 & 0.00241 & 0.00114 \\ 0.00201 & 0.00221 & 0.00114 & 0.00252 \end{bmatrix} \tag{5.1}$$

Notice that this is a difficult market to invest in as most assets give a negative return.
For the options, we only wanted the most liquid and relatively near-the-money options. We used the rule of thump mentioned on [9]; options are liquid if the bid-ask spread is within 10% of the ask price. To filter the options deep in the money or out of the money, we used the condition $|\frac{V_t}{S_t}| < 0.03$, which is the current value of the option, V_t, divided by the stock price S_t. We divide by the stock price to make the condition account for the magnitude of the stock price. With these conditions, we reduce the number of options from 291 to 29. We took an additional conservative measure when we calculated eq. (3.8) and eq. (3.9). Instead of using the last price or mid-price, we used the ask price to ensure that the options were indeed traded in the market.

5.3 Allocation Results

We used the WPVaR and included the normal VaR as a reference, knowing that the normal VaR can only buy the stocks. We run the optimization for $\epsilon \in [0.15, 0.1525, 0.155, \ldots, 0.01]$. Given the size of portfolio, we allowed for at most 50% to one stock, $\max \mathbf{w}_{\tilde{\xi}} \leq 0.5$. This also forces the methods to be creative about the allocation as it would be trivial to just allocate 100% to the only asset with positive μ, μ_{GE}. In both cases, we also have the budget constraint, i.e., for the Normal VaR, $\mathbf{e}^\intercal \mathbf{w} = 1$, and for the WPVaR $\mathbf{e}^\intercal \mathbf{w}_{\tilde{\xi}} + \mathbf{e}^\intercal \mathbf{w}_{\tilde{\eta}} = 1$ For WPVaR we only allowed short-selling to up to 1/8, i.e., $\min \mathbf{w}_{\tilde{\xi}} \geq -\frac{1}{8}$. Additionally, we recall that WPVaR only allows long positions in the options, i.e., $\mathbf{w}_{\tilde{\eta}} \geq 0$. We show the VaR under each allocation and the estimated portfolio returns for each method in the graphs below



(a) The performance with the normal VaR **(b)** The performance with the WPVaR.

Figure 7 – The VaR and portfolio returns for the portfolio with energy companies. *Note the dual axis and that the axis changes for the two methods.*

In fig. 7a, we see that the normal VaR allocation balances the immediate returns with the potential loss. When we increase ϵ, the VaR increases, and the immediate return decreases because we are more concerned about the second order moment than the first, following the definition in eq. (3.3). In fig. 7b, we notice that the VaR is lower in general and remains low for all of ϵ. Likewise, the returns are higher and remain almost constant for all ϵ. In fact, the

VaR and returns vary slightly, which can be seen in fig. 8; however, it is on a scale of 10^{-10} which is very little. We observe that the result is like the ones we saw in previous sections, where the WPVaR remain unchanged for different ϵ. Notice, however, that the scale of return is much higher for the WPVaR than for the normal VaR. This can be due to both the allocation method but also because WPVaR can use options as well. We will now consider how the WPVaR allocated the wealth in the stocks:

$$\begin{bmatrix} w_{BP} & w_{CVX} & w_{GE} & w_{XOM} \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} -5.89 \cdot 10^{-11} & 0.500 & 0.469 & 2.03^{-10} \end{bmatrix} \tag{5.2}$$

We notice that it allocates a substantial amount to the company with the only positive projected return, `GE`. On top of that, it allocates a substantial amount to the stock with the lowest negative returns, `CVX`, and then some minimal amounts to the remaining assets. Because we have 29 options, we will only highlight the ones that the method allocated the most wealth. The WPVaR allocated 0.0162 and 0.0150 to a call on `BP` and a call `XOM`. Given that it is almost not investing in the underlying of the two and both of the stocks have a negative μ, it seems like quite a risky bet, and it surprised us, as we would have expected some investments in put-options to hedge the long positions. It is worth noticing that the WPVaR allocated a non-neglectable amount of the scale 10^{-10} to a series of different options that are predominately put options. These small allocations might be a way for the method to hedge against market movements as it did in the BS-economy. However, we would need to conduct further studies with more comprehensive options data to fully grasp the significance of these tiny allocations.

6 Discussion

This report introduced and considered different methods to calculate the worst-case value at risk. We also imposed some investment constraints on the allocation $\mathbf{W}$. However, we left out major ones like transaction costs which could change the effectiveness drastically, especially the WPVaR as transaction costs for options could be much larger than the ones for stocks. We also allowed for a perfectly new one-period allocation. In a real world situation, one would probably already have a portfolio that should be re-balanced or could take a bank loan. In an ideal world, we would have a multi-period setting with multiple re-balancing points. However, the availability of options data limited us greatly. It was hard and time-consuming to find options data with maturities and strikes that made sense to the current market conditions and with maturities that could match the investment horizon. For a multiperiod setting, one would also need a reliable method to deal with different maturities in the portfolio and consider doing contract rolling and take into account all possible related issues. With the current simple setup with WPVaR we only allowed for options with maturity at the end of the period. However, in [7], they mention that this could also be extended to include more general maturities. Additionally, they introduce the worst-case quadratic VaR, which can handle options that mature before or even after the investment horizon. The method builds on the *gamma-delta* approximation, *an approximation built on a Taylor-series expansion*, which is only adequate special settings and maturities. Although the method is interesting and relevant for further studies, we left it out to limit the scope.

The WPVaR sometimes found some surprising results when used to find the optimal allocation. If we did not remove options that were deeply out of the money or in the money, it would find very creative solutions that made the VaR negative, i.e., promised that we would never lose money. We mitigated this by removing the options as we found them unreal; however, it would be interesting to dive into how these allocations were put together and figure out if it was simply due to model deficiency.

We also noted that, as with any portfolio optimization models, the results depend greatly on the μ and $\mathbf{\Sigma}$. We used 52 weeks to estimate $\mathbf{\Sigma}$ to reflect the dependencies over a year, and if changed the number of points, it would change the allocation slightly. However, we often saw quite some changes when we changed μ. We took a flat forward scenario, i.e., the same return as we had in the week we would balance our portfolio. It was quite important that μ reflected the most recent market movements as all of the options were priced based on the latest changes. One could consider more advanced approaches to estimating the μ, e.g., with a Bayesian approach to incorporate one's beliefs and buying options in alignment with them, but we kept it simple here. Sometimes it was hard to thoroughly asses if an option allocation was good. In the BS-economy, the allocation seemed reasonable; however, when dealing with more than a couple of options, it quickly becomes hard to understand if it is reasonable or not. In the energy portfolio case, we saw many put options with the allocation of size 10^{-10}. They might have some effect, but it is hard to judge the significance of 15 immediate put options, each with an allocation of 10^{-10}.

7 Conclusion

In this project, we introduced and explored new methods of calculating value at risk, VaR. The methods take a worst-case perspective on the portfolio and ideas from distributionally robust optimization and robust optimization. However, with results from [7] [4], we showed how these could be formulated as simple SOCPs and even, in some cases, be interpreted as the conditional VaR. The Worst-case VaR, *WVaR*, was straightforward to implement and had desirable properties, but in practice, it seems to be too conservative and would significantly inflate the risk above the referenced normal Gaussian VaR. We also considered the worst-case polyhedral VaR, *WPVaR*. This method allows us to introduce options to the portfolio in a way that takes into account the payoff functions that links uncertainty between the underlying stock and the option. In contrast, the other methods ignore the fact that the uncertainty in the option is inherent from the uncertainty in the stock. We showed this with an example of a BS-economy where the normal VaR and WVaR were substantially higher than WPVaR at high confidence levels. We also experimented with Markowitz Portfolio optimization in several different ways, and calculated the respectively VaR as a comparison. We found that Markowitz model in general produces much higher VaR, especially in the case with maximizing high returns. This is not entirely surprising since the Markowitz model is known to perform poorly when the returns have a skewed distribution. Again it shows why WPVaR might be better for portfolios with derivative because it's capable of not only treating the derivatives just as a risky asset but also linking its expected return and risk to the underlying assets.

When we tested the allocation method on a real portfolio with stock from the energy sector, we saw that the WPVaR with options outperformed the referenced normal VaR without options. The WPVaR has great potential, and the allocations made sense qualitatively in the simple simulated BS-economy. However, we did ignore non-trivial factors such as transaction costs and one would probably need a more informative options data set with different market settings to fully understand the WPVaR allocation strategy and its performances under different settings.

References

[1] A. Nemirovski and A. Shapiro, "Convex approximations of chance constrained programs," *SIAM Journal on Optimization*, vol. 17, no. 4, pp. 969–996, 2007.

[2] A. Ben-Tal, L. El Ghaoui, and A. Nemirovski, *Robust optimization.* Princeton university press, 2009, vol. 28.

[3] D. Ruppert and D. S. Matteson, *Statistics and data analysis for financial engineering.* Springer, 2011, vol. 13.

[4] L. E. Ghaoui, M. Oks, and F. Oustry, "Worst-case value-at-risk and robust portfolio optimization: A conic programming approach," *Operations research*, vol. 51, no. 4, pp. 543–556, 2003.

[5] S. Zymler, D. Kuhn, and B. Rustem, "Worst-case value at risk of nonlinear portfolios," *Management Science*, vol. 59, no. 1, pp. 172–188, 2013.

[6] S. Zymler, "Distributionally robust optimization with applications to risk management," 2010.

[7] S. Zymler, B. Rustem, and D. Kuhn, "Robust portfolio optimization with derivative insurance guarantees," *European Journal of Operational Research*, vol. 210, no. 2, pp. 410–424, 2011.

[8] P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath, "Coherent measures of risk," *Mathematical finance*, vol. 9, no. 3, pp. 203–228, 1999.

[9] How is options liquidity determined? [Online]. Available: https://www.optiontradingpedia.com/options_liquidity.htm

A Numerical Results BS-Economy

A.1 Additional Experiment with Short-selling of Basic Assets

We also considered the optimization where we allowed for short-selling of the basic assets. In this case, we can only compare the allocation using WVaR and WPVaR. Here is the allocation for $\epsilon = 0.05$,

	$w^A_{\tilde{\boldsymbol{\xi}}}$	$w^B_{\tilde{\boldsymbol{\xi}}}$	$w^A_{\tilde{\boldsymbol{\eta}}}$	$w^B_{\tilde{\boldsymbol{\eta}}}$
WPVaR	-5.240807e-09	0.97869	6.287268e-10	0.021310
WVaR	8.966126e-02	0.87849	2.636404e-11	0.031848

Table 2 – The allocation to for $\epsilon = 0.05$ in the BS-economy with two stock A,B and a call option on A and a put option on B.

B Energy Portfolio

B.1 Covariance With Names

$$\begin{array}{c|cccc} & BP & CVX & GE & XOM \\ \hline BP & 0.00225 & 0.00174 & 0.00102 & 0.00201 \\ CVX & 0.00174 & 0.00226 & 0.00121 & 0.00221 \\ GE & 0.00102 & 0.00121 & 0.00241 & 0.00114 \\ XOM & 0.00201 & 0.00221 & 0.00114 & 0.00252 \end{array} \tag{B.1}$$

B.2 The VaR and Return Fluctuations WPVaR

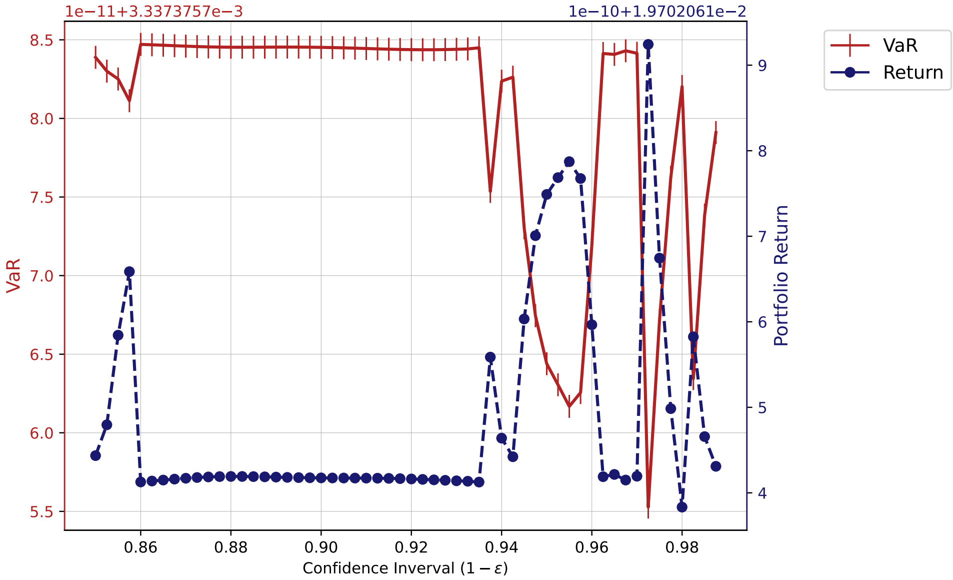


Figure 8 – The VaR and portfolio returns for the energy companies and the WPVaR. This is just to demonstate that the return and VaR varies but *notice* that the difference are on a scale of 10^{-10} and 10^{-11} thus the graphs look so drastic here but compare them with the more sensible scales in fig. 7b.