MA 576 Optimization for Data Science
Homework 4

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Problem 1

Consider the following sets; for each set (1) Prove or disprove it is convex, (2) Prove or disprove it is a cone, (3) Determine the extreme points (if convex), (4) Determine the recession cone (if convex).

1. $\mathbf{A} = \{(x_1, x_2) \in \mathbf{R}^2 : |x_1| + |x_2| \leq 1\}$

   1.) To prove a set is convex we have to show that for any two points in A, the line segment between them is also contained in A. Therefore, let (x_1, x_2) and (y_1, y_2) be two points in A. Then we need to show that for any $\alpha \in [0, 1]$, the point $((1-\alpha)x_1 + \alpha y_1, (1-\alpha)x_2 + \alpha y_2)$ is also in A. Therefore; by triangle inequality:

   $$|(1-\alpha)x_1 + \alpha y_1| \leq |(1-\alpha)x_1| + |\alpha y_1| \leq (1-\alpha)|x_1| + \alpha|y_1| \leq 1$$

   Because we have already defined α to be a point between [0,1]; we can remove it from $|x_1|$. Furthermore we can use the identical inequality above to prove the same relationship between x_2 and y_2 to show that set A is **convex**.

   2.) Because the set is bounded we can conclude that A is **not a convex cone** because although it contains rays through the origin; they do not stretch infinitely.

   3.) The set is described by a square with extreme points located at (-1,0), (1,0), (0,-1), (0,1).

2. $\mathbf{B} = \{x \in \mathbf{R}^3 : 0 \leq x_1 \leq x_2 \leq x_3\}$

   1.) Similar to the problem above, let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$; and both $x, y \in B$ Therefore we can conclude that $0 \leq x_1 \leq x_2 \leq x_3$ and $0 \leq y_1 \leq y_2 \leq y_3$. If $z = \alpha x + (\alpha - 1)y$. Then for each component in z :$\alpha x_1 + (\alpha - 1)y_1 = z_1$. This is also applicable to z_2 and z_3. Therefore we can conclude that $0 \leq z_1 \leq z_2 \leq z_3$ to show that set B is **convex**.

   2.) To prove that B is a cone we need to show that $\lambda x \in C$, $x \in C$ and $\lambda \geq 0$. Therefore; for any point in the convex set we can multiply it by a non-negative scalar; and the result will still be in the set B. If we multiply x by λ then we get $(\lambda x_1, \lambda x_2, \lambda x_3$. By cumulative property: $0 \leq \lambda x_1 \leq \lambda x_2 \leq \lambda x_3$ still holds, proving B **is a convex cone.**

   3.) Extreme points cannot be expressed as a convex combination of other points in B. Therefore; if $x_1 < x_2$ or $x_2 < x_3$ then these would not be extreme points because the point can just be rewritten as $(x_1, x_1, x_3$ or $(x_2, x_2, x_3$. Therefore the only way to find a unique point is to set $x_1 = x_2 = x_3$; since this point cannot be rewritten as a combination of other points.

   4.) To determine a recession cone we need to find a vector $v = (v_1, v_1, v_3)$ that can be added to any point in x such that it still remains in set B. This is straight forward since the only conndition that will keep this is $v_1 \leq v_2 \leq v_3$; so that $x + \lambda v \geq 0$. This way the set B still holds since $x_1 \leq x_2 \leq x_3$.

3. $\mathbf{C} = \{x \in \mathbf{R}^2 : x_2 \leq x_1^2\}$

   Let (x_1, x_2) and (y_1, y_2) be two points in C.To prove a set is not convex we need to find 2 points in the set C where the line between them is not fully included in the C. If we take two points in the set : (1, 1/2) and (-1, 1/2); we can see that they are in the set C:

$$\frac{1}{2} \leq (1)^2 \qquad\qquad \frac{1}{2} \leq (-1)^2$$

However the conditions for convexity are not met. The midpoint of these points is (0, 1/2); however this can be easily shown that it does not lie in C. Proof : $1/2 \leq (0)^2$. Therefore C **is not a convex set; and therefore not a cone**

4. $\mathbf{D} = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : \sqrt{x_1^2 + x_2^2} \leq x_3\}$

   1/2.) We will explore the conditions of convexity:

$$\sqrt{\lambda x_1^2 + \lambda x_2^2} \leq \lambda x_3$$
$$= \lambda\sqrt{x_1^2 + x_2^2} \leq \lambda x_3$$

   Set D proved to be a cone; therefore we can also conclude that D is convex.

   3.) The extreme points would be the boundary of the set located at point $(0, 0, x_3)$.

   4.) The recession cone would be the entire $\mathbf{R}^3$ sinjce there is no upper bound.

Problem 2

Consider the set of all non-decreasing functions $f : \mathbf{R} \to \mathbf{R}$. Prove or disprove it is a convex cone. To determine if its a cone we need to prove:

I.) If f is non-decreasing and $\lambda \geq 0$; then λf is also non decreasing.
To prove this we can show that for $x \leq y$, then $f(x) \leq f(y)$. Therefore for any scalar $\lambda \geq 0$, $\lambda f(x) \leq \lambda f(y)$.

II.) If f and g are non-decreasing, any combination would also be non-decreasing.
Let $\lambda \in [0, 1]$ and $x \leq y$.
Therefore:

$$h(x) = \alpha f(x) + (1 - \alpha f(x) \leq h(y) = \alpha f(y) + (1 - y)f(y)$$

Here we can see that both f and g can be expressed as a combination and are indeed increasing. Therefore we can conclude f **is a convex cone**

Problem 3

Let Ω be a set of n elements.

1. Describe the set $\mathbf{P}$ of all probability distributions (probability mass functions) supported on Ω. The probability mass function is the function that gives the probability that a random discrete variable is equal to some other value:

$$P = \{p \in \mathbf{R}^n : p_i \geq 0 \sum p_i = 1\},$$

   where p_i is the ith element in Ω

2. Show that $\mathbf{P}$ is convex.
   Let $x \in P$ and $y \in P$. Similar to before:

$$\lambda x + (1 - \lambda)y = \{\lambda x_1 + (1 - \lambda)y_1 + ... + \lambda x_n + (1 - \lambda)y_n\}$$

   However because $x_i, y_i \geq 0$, and $p_i \geq 0$ we can prove P is convex. We also know that because, by definition, the summation of probabilities is equal to 1; therefore P is convex.

3. Determine the extreme points of $\mathbf{P}$.
   In order for a point to be extreme, it cannot be rewritten as a combination of other points in P. For this to occur the probability mass function must be 1 at only one event; therefore this point is the only occurrence. Therefore Ω would only have the same point; meaning it's probability is 1.

Problem 4

$$K = \{x \in \mathbf{R}^n : 0 \leq x_1 \leq x_2 \leq ... \leq x_n\}$$

Compute the polar cone of $K\circ$
If C is a closed convex cone, in a vector space V; then the polar cone of C is denoted as $C\circ$
, is the set of functional's if on V such that $f(x)\ leq0$ for all $x \in C$.

Therefore, in order or the set K to have non-positive inner product of the set elements, $K\circ$ must be the set of all vectors that have non-positive components, given by:

$$K\circ = \{y \in \mathbf{R}^n :\leq x_n ... \leq x_2 \leq x_1 \leq 0\}$$

Problem 5

Consider the following optimization problems:

$$\begin{aligned} \textbf{minimize : } & -10x_1 - 2x_2 + x_3 + 5x_4 \\ \textbf{s.t. : } & 2x_1 - x_3 - x_4 = -5 \\ & 4x_1 + x_2 - 2x_4 = 3 \\ & x \geq 0 \end{aligned}$$

1. List all basic solutions indicating which ones are feasible.
   We first represent the constraints and the objective function in standard form so that we can easily identify the basic feasible solutions. This allows us toi see that x_3, x_4 are also non-basic and can be rewritten as

$$x_3 - -5_2 x_1 + x_3 \qquad\qquad x_2 = 3 - 4x_1 + 2x_4$$

   and we can substitute them into the objective function. Therefore; the basic solution that are feasible are subject to $-16x_1 + 2x_4 - 11$.

Problem 6

Consider the optimization problem:

$$\begin{aligned} \textbf{minimize : } & (2x - y)^2 - (y - z)^2 + (z - 1)^2 \\ \textbf{s.t. : } & x + 2y + 3z = 1 \\ & x, y, z \geq 0 \end{aligned}$$

1. What is the solution of the problem?
   All the basic solutions to this would be (1,0,0), (0,1/2,0) and (0,0,1/3). In order to find the solution to this function we can plug in these points into $f(x, y, z)$ to see which point is maximum.

$$f(1, 0, 0) = 5 \qquad\qquad f(0, 1/2, 0) = 3/2 \qquad\qquad f(0, 0, 1/3) = 5/9$$

   Therefore the solution would be f(1,0,0) = 5

2. What is the solution of the problem if the objective function is $f(x) = z + 1$ Solution would be maximized at z = 1/3; therefore the solution would be 4/3; since $1/3 + 1 = 4/3$

3. What is the solution of the problem if the objective function is $f(x) = x + z + 1$ Solution would be maximized at x = 1; therefore the solution would be 2; since $1 + 0 + 1 = 2$

Problem 7

Consider the following problem:

$$\textbf{minimize :}\ \log(y^2 - x^2)$$
$$\textbf{s.t. :}\ x^2 + y^2 \leq 1$$
$$2x - y \leq 0y \geq 1/2x \geq 0$$

The vortex of the region is shown above, where the intersection of the constraints are:

$$x = 0 \qquad\qquad y = 1/2$$

Therefore: $2x - y = 0$, since (0, -1) is not a feasible solution given our constraints. If we substitute both constraints, we find that $x = \frac{1}{4}$ and $y = \frac{1}{2}$. Another vortex can be found from $x^2 + y^2 \leq 1$. If $x^2 + (2x^2) = 1$; then $x = \frac{1}{\sqrt{5}}$ and $y = \frac{2}{\sqrt{5}}$. Again; both negative reciprocals are ignored since they don't satisfy our constraints. Therefore; in order to find the minimum of the function

$$f(0, 1/2) = -0.6 \qquad f(0, 1) = 0 \qquad f(1/4, 1/2) = -0.726 \qquad f(\frac{1}{\sqrt{5}}), \frac{2}{\sqrt{5}} = -0.22$$

Therefore the minimum is at point $(\frac{1}{4}, \frac{1}{2})$.