MA 576 Optimization for Data Science
Homework 5

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Problem 1

Let Q be pd. State and prove a generalization of the projection theorem where the objective function is $(y-x)^T Q(y-x)$ instead of $(y-x)^T(y-x)$.

For a given vector x and subspace S, there exists a unique vector y that minimizes the distance between x and y. A vector y is the orthogonal projection of x and S, and the difference between x and y is orthogonal to the subspace S.

To generalise the projection theorem for the given **objective function** - $(y-x)^T Q(y-x)$. We can calculate the gradient of the objective function:

$$\nabla f = 2Q(y-x)$$

For a point to be a **stationary point** the gradient of the objective function should be orthogonal to the subspace S.:

$$\nabla f(y) = 2Q(y-x) \perp S$$

If y is the orthogonal projection of x and S, the vector $y-x$ is orthogonal to S. Using the properties of positive definite matrices:

$$(y-x) \perp S \Longleftrightarrow Q(y-x) \perp S$$

Therefore the optimality condition for our generalized problem is $Q(y-x) \perp S$. The generalized projection theorem states that, given a vector x and a subspace S, there exists a unique vector y in S that minimizes the distance between x and y, given by the objective function $(y-x)^T Q(y-x)$.

Problem 2

Find the optimal solution of the problem

$$\begin{aligned} \text{minimize } & -xyz \\ \text{s.t. } & x+2y+4x \leq 12 \\ & x,y,z \geq 0. \end{aligned}$$

We can solve this problem using the **Lagrangian Method**. We will start by re-writing the equation:

$$L(x,y,z\lambda) = -xyz + \lambda(x+2y+4z-12)$$

To solve this optimization problem we set the partial derivatives and set them equal to 0:

$$\begin{aligned}
\frac{\partial L}{\partial x} &= -yz + \lambda = 0 \\
\frac{\partial L}{\partial y} &= -xz + 2\lambda = 0 \\
\frac{\partial L}{\partial z} &= -xy + 4\lambda = 0 \\
\frac{\partial L}{\partial \lambda} &= x + 2y + 4z - 12 = 0
\end{aligned}$$

If we solve this equations, we find that:

$$x = 4 \qquad y = 2 \qquad z = 1$$

Therefore we find that the minimum is going to be $f(x, y, z) = -xyz = -(4)(2)(1) = -8$; and all the conditions of this optimization problem are met.

Problem 3

Find the optimal solution of the problem

$$\begin{aligned}
\text{maximize } & x^2 + 2xy + 2y^2 - 3x + y \\
\text{s.t. } & x + y = 1 \\
& x, y \geq 0.
\end{aligned}$$

We can solve this problem using the **Lagrangian Method**. We will start by re-writing the equation:

$$L(x, y, z\lambda) = x^2 + 2xy + 2y^2 - 3x + y + \lambda(1 - x - y)$$

To solve this optimization problem we set the partial derivatives and set them equal to 0:

$$\begin{aligned}
\frac{\partial L}{\partial x} &= 2x + 2y - 3 - \lambda = 0 \\
\frac{\partial L}{\partial y} &= 2x + 4y + 1 - \lambda = 0 \\
\frac{\partial L}{\partial \lambda} &= 1 - x - y = 0
\end{aligned}$$

If we solve this equations, we find that:

$$x = 3 \qquad y = -2$$

We can solve for y and get that $y = -2$; however that solution is not feasible. Therefore we need to check the non-negativity constraints as well; such as where $x = 0$ or $y = 0$. For the first case we an objective function value of $f(0, 1) = 3$; while the second case gives us $f(1, 0) = -2$. Therefore we can conclude that the second case will give us the maximum of 3, with $x = 0$ and $y = 1$.

Problem 4

$$\begin{aligned}
\text{minimize } & x^2 - y^2 - z^2 \\
\text{s.t. } & x^4 + y^4 + z^4 \geq 2.
\end{aligned}$$

1. Is it a convex problem?

   The constraint function $g(x, y, z)$ is convex since x^4, y^4, and z^4 are all convex functions (with non-negative second-order derivatives) and the sum of convex functions is also convex. However the objective function is not convex since the Hessian Matrix is not positive definite. Therefore the problem is not a convex optimization problem.

2. Find all KKT points.

   We can solve this problem using the **Lagrangian Method**. We will start by re-writing the equation:

   $$L(x, y, z\lambda) = x^2 - y^2 - z^2 + \lambda(2 - x^4 - y^4 - z^4)$$

   The KKT conditions for this problem are therefore:

   $$\begin{aligned} 2x - 4\lambda x^3 &= 0 \\ -2y - 4\lambda y^3 &= 0 \\ -2z - 4\lambda z^3 &= 0 \\ x^4 + y^4 + z^4 &= 2 \\ \lambda &\geq 0 \\ \lambda(x^4 + y^4 + z^4 - 2) &= 0 \end{aligned}$$

   By solving these conditions we can solve and find two cases:

   $$\begin{aligned} x = y = z &= 0 \\ x = y = z &= \pm\frac{1}{\sqrt{2}} \end{aligned}$$

3. Find the optimal solution of the problem.

   To find the optimal solution of the problem, we will evaluate the objective function at the KKT points found above.

   Therefore; the optimal solution would be $f(x, y, z) = \frac{-1}{\sqrt{2}}^2 + \frac{-1}{\sqrt{2}}^2 + \frac{-1}{\sqrt{2}}^2 = \frac{-3}{2}$

Problem 5

Consider

$$\begin{aligned} \text{minimize} \quad & -y \\ \text{s.t.} \quad 2x - (5 - y)^3 &\leq 0 \\ x, y &\geq 0. \end{aligned}$$

The minimizer is $x^* = (0, 5)$. Check if KKT optimality conditions are satisfied at the point. A plot may be useful.

1. Primal Feasibility - The point must satisfy the constraints of the problem.$2x-(5-y)^3 \leq 0, x, y \geq 0$.

2. Dual Feasibility - The dual variables corresponding to the inequality constraints are non-negative.Similar to previous problems, use the lagrangian method:

   $$L(x, y, \lambda) = -y + \lambda(2x - (5 - y)^3)$$

   Since $x^* = (0, 5)$ are the constraints given, it implies that $\lambda = 0$.

3. The gradient of the Lagrangian with respect to x is 2λ, which implies that it is also zero at x^*.

4. The gradient of the Lagrangian with respect to y is $3\lambda(5 - y)^2 - 1$, which implies that it is not 0 at x^*.

Therefore we can conclude that it does not imply the KKT optimality conditions since the complementary slackness condition is not satisfied.