Bayesian Parametric Spatial *Split-Population* Survival Models for Clustered Political Event Processes

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Abstract

Scholars frequently employ Split-Population survival models when data on political event processes include "immune" cases that never experience the event of interest and "at-risk" cases, which eventually will. Although theories of political processes often posit that underlying risk factors among spatially adjacent units affect their survival time and probability of becoming immune, existing Split-Population survival models assume that these baseline risks are spatially independent. We develop a time-varying Bayesian Parametric Spatial Split-Population survival model that statistically addresses how spatial autocorrelation influences the units' survival time and their probability of becoming immune. We illustrate the model's advantages via Monte Carlo simulations and demonstrate in replication analyses of democratic survival and post-civil war peace duration that our approach significantly alters previous findings.

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Political Scientists often employ survival models to address questions about the duration of political processes. The widespread use of these models has driven methodologists to statistically account for different forms of heterogeneity in the data generation process (d.g.p.) of censored survival data. One such type of heterogeneity is inflation in the number of non-failure cases, which takes the form of an "immune" fraction that will never experience the political event of interest. The presence of this immune fraction violates the assumption of conventional survival models that all censored cases will eventually experience the failure event (Box-Steffensmeier and Zorn 1999; Box-Steffensmeier, Radcliffe and Bartels 2005; Beger, Dorff and Ward 2016). The frequent need to relax this assumption has led to the development of split-population (SP) survival models, which first estimate the probability of being immune and then the survival rate of at-risk cases. In addition to their widespread applications to substantive research questions, these models have since been extended to address misclassified failures (Bagozzi et al. 2019) and "triadic duration" interdependence (Chiba, Metternich and Ward 2015).

A second key form of heterogeneity in survival data involves spatial clustering (spatial autocorrelation) among individual units (Darmofal 2009; Hays, Schilling and Boehmke 2015). While spatial autocorrelation in binary and continuous dependent variables has been studied extensively (Beck, Gleditsch and Beardsley 2006; Franzese and Hays 2007; Williams, Seki and Whitten 2016; Betz, Cook and Hollenbach 2019), spatial clustering in underlying unobserved risk factors can also influence the survival of political processes (Darmofal 2009, 2015). Failing to properly account for spatial autocorrelation with spatially-weighted random effects (frailties) leads to faulty inferences (Franzese and Hays 2007; Darmofal 2015). Scholars have thus developed conventional spatial survival models to address the possibility that adjacent units share unobservable risk factors that affect *when* they experience a failure event (Banerjee, Wall and Carlin 2003; Darmofal 2009, 2015).

Existing research on SP and spatial survival models has provided a diverse toolbox for analyzing event-history data, but these disparate sets of models have not been drawn together in a unified statistical framework. Consequently, current applications of SP survival models

assume that the occurrence of political events across adjacent units are spatially independent, while conventional spatial survival models only account for spatial dependence in the duration of political processes. This poses a serious methodological challenge since numerous theories of political event processes suggest that spatially clustered units share common risk factors that influence *not just* the duration of a given political process, *but also* their probability of being immune to the process-terminating event.

Consider, for example, democratic regime survival. Scholars have long argued that although some democracies are at risk of authoritarian reversal, other "consolidated" democracies are immune from such institutional change (Przeworski et al. 2000; Boix and Stokes 2003; Svolik 2008). Although recent work has utilized parametric SP frailty survival models to account for these two subpopulations (Svolik 2008), even these models assume that the frailties themselves are spatially independent. Accordingly, existing SP survival models cannot statistically evaluate the widespread contention that spatially clustered democracies share common unobservable features (norms) that influence the survival of their institutions *and* the probability that these institutions consolidate (Gasiorowski and Power 1998; Brinks and Coppedge 2006). Extant spatial survival models can account for spatial autocorrelation in the duration of democracy, but they cannot address the presence of at-risk and consolidated democracies in the data nor evaluate how theoretically-identified covariates and spatial dependence affect the probability that a democracy is immune from authoritarian reversal.

This example is hardly unique: political scientists have used nonspatial SP survival models to analyze irregular leadership changes (Beger, Dorff and Ward 2016), PAC contributions to U.S. House members (Box-Steffensmeier, Radcliffe and Bartels 2005), interstate war (Clark and Regan 2003), and foreign intervention in civil conflict (Findley and Teo 2006). Using nonspatial models in these research areas may be insufficient given that coup-proofing, partisan influences, and conflict tend to cluster geographically or politically (Darmofal 2009, 2015; Böhmelt, Ruggeri and Pilster 2017; Braithwaite 2005). If spatial autocorrelation matters for process survival or the probability of being immune to an event, then ignoring it can lead to inconsistent and biased parameter estimates (Beck, Gleditsch and Beardsley 2006; Franzese and Hays 2007; Darmofal

2015).

To address these statistical challenges, we develop a parametric Bayesian Spatial SP survival model with time-varying covariates, which builds on but also departs from extant spatial survival and non-mixture cure models (e.g., Banerjee and Carlin 2004; Darmofal 2009). Similar to a regular SP survival model, our model consists of a split-stage equation that estimates the probability of a unit being immune from a failure event and a second-stage equation that estimates the survival probability conditional upon being at risk of failure. In our approach, however, *both* equations can include spatially autocorrelated frailties whose joint distribution is interpretable in a spatial context. This allows us to explicitly model spatial autocorrelation in each equation of the model rather than assuming spatially independent frailties. Hence, in contrast to nonspatial SP survival models, our model permits researchers to evaluate whether common underlying risk factors among spatially adjacent units affect the survival time of interest *and* odds of being immune. Further, unlike conventional spatial and non-mixture cure models, our model incorporates time-varying covariates and spatial frailties in *both* the survival and split stages. This is important for political scientists who often work with time-varying covariates in panel survival data that also exhibit spatial dependence.

We estimate the Spatial SP survival model via Bayesian Markov-Chain-Monte-Carlo (MCMC) methods, which we also provide in our user-friendly R-package (see Supplemental File). Similar to Banerjee and Carlin (2004) and Darmofal's (2009) model, this spatial dependence is incorporated by specifying in each equation a conditionally autoregressive (CAR) prior for the spatially autocorrelated frailties across neighboring units that the researcher defines in an adjacency matrix. After presenting the Bayesian Spatial SP survival model, we apply it to prominent studies of democratic survival (Svolik 2008) and post-civil war peace duration (Walter 2015, presented in Supplemental Appendix). Our applications reveal that fully modeling theoretically relevant spatial autocorrelation leads to some key inferential differences from the studies' original results. For instance, unlike results from Svolik's (2008) nonspatial SP survival models (and also Gasiorowski and Power 1998; Treisman 2018), we find a nonmonotonic (weak U-shaped) relationship between economic growth and the probability of democratic consolida-

tion, but an unreliable association between growth and democratic survival. Our Spatial SP survival model also suggests that GDP per capita's influence on democracies being immune from authoritarian reversals is weaker than those reported in previous studies, and that the association between previous military regimes and democratic breakdown is unreliable. In our replication of Walter's (2015) analysis of post-war peace duration, we find that accounting for spatial autocorrelation in each stage leads to a reversal in her primary finding that increased political freedoms can help peace endure after civil war.

We next describe our time-varying Bayesian Spatial SP survival model and the MCMC algorithm for its estimation. We then illustrate our model's advantages relative to nonspatial SP survival models via Monte Carlo simulations. We present our main empirical application and conclude with a brief discussion of future extensions.

1 Bayesian Spatial *Split-Population* Survival Model

We first define a general parametric SP survival model with unit-specific, spatially independent frailties. We then develop our Spatial SP survival model with time-varying covariates that incorporates spatial frailties in the model's split and survival stages. Next, we describe the Bayesian modeling approach employed to account for such spatially dependent frailties between adjacent units. This provides the foundation for our Bayesian Spatial SP *Weibull* model.

Suppose we have continuous time duration data in which units $i = \{1, 2, ...N\}$ may experience an event of interest. Some subjects that survive until the end of the sampling period may not experience the event, all of which are "censored" ($\widetilde{C}_i = 0$ if censored; $\widetilde{C}_i = 1$ otherwise). The duration of interest t has the probability density function (PDF) $f(t) = \Pr(T_i = t)$, where T is an observation's duration of time until experiencing the event or censoring. The cumulative distribution function (CDF) for the probability of the event on or before t is $\Pr(T_i \leq t) \equiv F(t) = \int_0^t f(t)dt$. The probability of survival is $\Pr(T_i \geq t) \equiv S(t) = 1 - F(t)$. Thus, the hazard of an event at t given that the event has not yet occurred is $h(t) = \frac{f(t)}{S(t)}$.

Consider an SP survival model for the duration t that splits the sample into an "at-risk"

population ($Y_i = 0$) and an "immune" population ($Y_i = 1$) that will not experience the event of interest. Define the probability with which units enter the immune fraction as $\alpha_i = Pr(Y_i = 1)$. Let $V_i \sim N(0, \sigma^2)$ be the unit-specific frailty that accounts for unobserved heterogeneity among units that influence probability α_i. Probit or logit can be used to estimate α_i, and is defined for the logit case as:

$$\alpha_i = \frac{\exp(\mathbf{Z}_i\boldsymbol{\gamma} + V_i)}{1 + \exp(\mathbf{Z}_i\boldsymbol{\gamma} + V_i)} \tag{1}$$

where $\mathbf{Z}_i$ are p2-dimensional covariates. $\boldsymbol{\gamma}$ is the corresponding parameter vector in $\mathbb{R}^{p_2}$. Equation (1) is the split-stage equation of the parametric SP survival model in which each unit's frailty is *independent* of other individual random effects. Let $W_i = \log \omega_i$ (where $W_i \sim N(0, \sigma^2)$) denote the unit-specific frailty term that captures the possibility that some units are at greater risk of experiencing the event of interest in the survival stage due to unobserved factors. Suppose that W_i is also independent of other individual random effects. The proportional hazards of the parametric SP survival model with unit-specific frailties is

$$h(t_i|\mathbf{X}_i\boldsymbol{\beta}, W_i) = h_0(t_i)\omega_i \exp(\mathbf{X}_i\boldsymbol{\beta}) = h_0(t_i) \exp(\mathbf{X}_i\boldsymbol{\beta} + W_i) \tag{2}$$

where $h_0(t_i)$ is the baseline hazard (e.g., Weibull, log-logistic, log-normal), $\mathbf{X}_i$ are the p1-dimensional covariates, $\boldsymbol{\beta}$ is the corresponding parameter vector in $\mathbb{R}^{p_1}$. Given the aforementioned density and survival functions and equations (1)-(2), the general parametric SP survival model's log-likelihood *without* time-varying covariates but with nonspatial i.i.d. frailties (V_i, W_i) is

$$\ln L = \sum_{i=1}^{N} \{\widetilde{C}_i \ln[(1 - \alpha_i) f(t_i|\mathbf{X}_i\boldsymbol{\beta}, W_i)] + (1 - \widetilde{C}_i) \ln[\alpha_i + (1 - \alpha_i) S(t_i|\mathbf{X}_i, \boldsymbol{\beta}), W_i]\}. \tag{3}$$

We follow the steps in Bagozzi et al. (2019) to incorporate time-varying covariates in the log-likelihood in (3). To do so, we re-define our survival data with unique "entry time" duration $t0$ and "exit time" duration t for each period. Let $t0_{ij}$ denote observation i's elapsed time since inception until the beginning of time period j and t_{ij} the elapsed time since that observation's inception until the end of period j. An observation is then coded as censored ($\tilde{C}_{ij} = 0$) or failed ($\tilde{C}_{ij} = 1$) at time t_{ij}. The PDF, CDF, survival probability, and hazard of an event remain as defined earlier, but the probability of survival up until period j is now $S(t0) = 1 - F(t0)$ where

$F(t0) = \int_0^{t0} f(t0)$.

As described in the Supplemental Appendix, once $S(t0)$ is defined, we can extend the log-likelihood in (3) to accommodate time-varying covariates $\mathbf{X}_{ij}$ and associated parameter vectors $\boldsymbol{\beta}$ by conditioning an observation's hazard and survival probability for time t upon its probability of survival until $t0$. This leads to the parametric SP survival model's log-likelihood with *time-varying* covariates and nonspatial i.i.d. frailties (V_i, W_i):

$$\ln L = \sum_{i=1}^{N} \left\{ \widetilde{C}_{ij} \ln \left[(1-\alpha_{ij}) \frac{f(t_{ij}|\mathbf{X}_{ij}\boldsymbol{\beta}, W_i)}{S(t0_{ij}|\mathbf{X}_{ij}\boldsymbol{\beta}, W_i)} \right] + (1-\widetilde{C}_{ij}) \ln \left[\alpha_i + (1-\alpha_i) \frac{S(t_{ij}|\mathbf{X}_{ij}\boldsymbol{\beta}, W_i)}{S(t0_{ij}|\mathbf{X}_{ij}\boldsymbol{\beta}, W_i)} \right] \right\} \tag{4}$$

where $\alpha_{ij} = \frac{\exp(\mathbf{Z}_{ij}\boldsymbol{\gamma} + V_i)}{1+\exp(\mathbf{Z}_{ij}\boldsymbol{\gamma} + V_i)}$ is the model's split first-stage equation with time-varying covariates.

Note that (4) reduces to the parametric SP survival model's log-likelihood *without* unit-specific frailties when $V_i = W_i = 0$. We denote this non-frailty model as the "Pooled" SP survival model (e.g., Banerjee, Wall and Carlin 2003) whose properties include parameter identification even when identical covariates are included in $\mathbf{X}_{ij}$ and $\mathbf{Z}_{ij}$ as well as reduction to a standard parametric survival model when $\alpha_{ij} = 0$ (Box-Steffensmeier and Zorn 1999). If each unit's unique frailty is independent from those of other units in the survival data, then researchers estimate the SP survival model with nonspatial unit-specific frailties in the split-stage (V_i) and survival-stage (W_i). In a Bayesian SP survival framework, the influence of unobserved heterogeneity among units (given by their unique frailty) on the units' probability of entering the immune fraction *and* their hazard is statistically accounted for in the split and survival stages via the Exchangeable normal prior $W_i \sim N(0, 1/\tau)$ and $V_i \sim N(0, 1/\tau)$, where τ is the precision parameter (Banerjee and Carlin 2004). Bayesian estimation of this "Exchangeable" SP survival model proceeds by assigning a prior for the model's parameters and V_i and W_i (Banerjee, Wall and Carlin 2003; Banerjee and Carlin 2004).

The Exchangeable SP survival model is appropriate if each unit's unique frailty is spatially *independent* from other individual random effects, but it is insufficient for evaluating theories which posit that spatial dependence between neighboring units influences their propensity for being immune and the time until they experience the event of interest. As Banerjee and Carlin

(2004) and Darmofal (2009) note, we can account for such spatial dependence by allowing V_i and W_i to be spatially autocorrelated across adjacent units. A common Bayesian approach for modeling spatially autocorrelated frailties in survival data involves specifying a CAR prior, which relaxes the assumption of exchangeability (Besag, York and Mollié 1991; Banerjee and Carlin 2004; Darmofal 2009). The CAR prior approach is used to model spatial dependence in conventional survival models or *just* the survival stage of non-mixture cure models *without* time-varying split-stage covariates (Banerjee and Carlin 2004; Darmofal 2009). We *extend* this approach by incorporating separate CAR priors for both the split-stage and survival-stage frailties (V_i, W_i) in our SP survival model, which also incorporates time-varying split and survival-stage covariates. This permits us to model spatially dependent frailties between neighboring units whose spatial relationships are defined via an adjacency matrix $\mathbf{A}$.

To incorporate spatially dependent frailties in our SP survival model, we first collect the frailty terms V_i into the vector $\mathbf{V} = \{V_1, ..., V_N\}$, and W_i into $\mathbf{W} = \{W_1,....,W_N\}$. We employ separate CAR priors on $\mathbf{V}$ and $\mathbf{W}$, which implies the following CAR model structure for our Spatial SP survival model: $\mathbf{V}|\lambda \sim \text{CAR}(\lambda)$ in the split-stage and $\mathbf{W}|\lambda \sim \text{CAR}(\lambda)$ in the survival-stage. CAR denotes conditionally autoregressive structure (Besag, York and Mollié 1991), while λ is the precision parameter. The CAR(λ) prior for both $\mathbf{V}$ and $\mathbf{W}$ has a joint distribution in the Spatial SP survival model (Banerjee, Wall and Carlin 2003), and is formally characterized in equations A.9-A.10 (Supplemental Appendix). As defined in Darmofal (2009, 246), the resulting conditional distribution of the spatial frailties for $\mathbf{V}$ and $\mathbf{W}$, respectively is,

$$V_i|V_{i' \neq i} \sim N(\overline{V_i}, 1/(\lambda m_i)), \quad W_i|W_{i' \neq i} \sim N(\overline{W_i}, 1/(\lambda m_i)) \tag{5}$$

where $\overline{W_i} = m_i^{-1} \sum_{i' \ adj \ i} W_{i'}$ and $\overline{V_i} = m_i^{-1} \sum_{i' \ adj \ i} V_{i'}$ in (5). $\overline{W}_i$ is the average of the neighboring $W_{i' \neq i}$, where i' *adj* i denotes that i' is a *neighbor* of i defined in terms of spatial contiguity between units in the adjacency matrix $\mathbf{A}$, and m_i is the number of these adjacencies. The same definitions apply to $\overline{V_i}$. In matrix $\mathbf{A}$ with elements $a_{ii'}$, we let $a_{ii'} = 1$ if units i and i' are spatially adjacent, and $a_{ii'} = 0$ otherwise to account for the possibility that spatially proximate units—unlike spatially distant units—share common unmodeled factors that influence these

adjacent units' probability of entering the immune fraction and their risk of experiencing the event of interest. Matrix **A** can also incorporate other non-spatial forms of dependence such as trade flows and institutional similarities between countries (Beck, Gleditsch and Beardsley 2006). We extend equation (4) in the Supplemental Appendix to define the *general parametric* Spatial SP survival model's log-likelihood (equation A.15) that incorporates time-varying covariates and spatially dependent frailties in the model's split and survival stages. This model provides numerous advantages.

First, the Spatial SP survival model directly evaluates whether spatially proximate units share common unobserved factors that make them distinct from spatially distant units in split-population survival data. This facilitates assessment of theories about political event processes wherein spatial clustering between units influences not just event occurrence but also the probability of units being immune to the event. Second, unlike extant spatial survival models (Banerjee and Carlin 2004; Darmofal 2009), our Spatial SP model incorporates time-varying covariates in both the split and survival stages. This is necessary for analyzing panel survival data that exhibits spatial dependence. Third, Bayesian estimation of our model via MCMC methods (described below) incorporates Gibbs sampling, Metropolis-Hastings algorithm, and slice-sampling. This enables flexible modeling of spatial associations in survival data with an immune fraction.

The log-likelihood of our Spatial SP survival model can be used in conjunction with any commonly used parametric survival distribution. Since political scientists often assume a Weibull distribution (Box-Steffensmeier and Zorn 1999; Svolik 2008; Findley and Teo 2006), we focus on the Spatial SP *Weibull* model here (*Log-Logistic* version is presented in the Supplemental Appendix). The density and survival functions of a Weibull distributed survival time t is:

$$f(t_{ij}|\rho, \mathbf{X}_{ij}, \boldsymbol{\beta}, W_i) = \rho(\exp(\mathbf{X}_{ij}\boldsymbol{\beta} + W_i))(\exp(\mathbf{X}_{ij}\boldsymbol{\beta} + W_i)t_{ij})^{\rho-1} \exp(-(\exp(\mathbf{X}_{ij}\boldsymbol{\beta} + W_i)t_{ij})^{\rho}$$

$$S(t_{ij}|\rho, \mathbf{X}_{ij}, \boldsymbol{\beta}, W_i) = \exp(-(\exp(\mathbf{X}_{ij}\boldsymbol{\beta} + W_i)t_{ij})^{\rho}. \quad (6)$$

In the Supplemental Appendix, we derive the following log-likelihood of our Spatial SP *Weibull*

model with time-varying covariates and spatial frailties in the model's split and survival stages:

$$\ln L(\rho, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{W}, \mathbf{V}) = \sum_{i=1}^{N} \left\{ \begin{array}{c} \widetilde{C}_{ij} \ln \left[(1-\alpha_{ij}) \frac{\rho(\exp(\mathbf{X}_{ij}\boldsymbol{\beta}+W_i))(\exp(\mathbf{X}_{ij}\boldsymbol{\beta}+W_i)t_{ij})^{\rho-1} \exp(-(\exp(\mathbf{X}_{ij}\boldsymbol{\beta}+W_i)t_{ij})^{\rho}}{\exp(-(\exp(\mathbf{X}_{ij}\boldsymbol{\beta}+W_i)t0_{ij})^{\rho}} \right] \\ + \ (1-\widetilde{C}_{ij}) \ln \left[\alpha_{ij} + (1-\alpha_{ij}) \frac{\exp(-(\exp(\mathbf{X}_{ij}\boldsymbol{\beta}+W_i)t_{ij})^{\rho}}{\exp(-(\exp(\mathbf{X}_{ij}\boldsymbol{\beta}+W_i)t0_{ij})^{\rho}} \right] \end{array} \right\} \tag{7}$$

where $\alpha_{ij} = \frac{\exp(\mathbf{Z}_{ij}\boldsymbol{\gamma}+V_i)}{1+\exp(\mathbf{Z}_{ij}\boldsymbol{\gamma}+V_i)}$ is the split first stage equation. We employ the CAR prior for $\mathbf{V} = \{V_i\}$ in the split stage and $\mathbf{W} = \{W_i\}$ in the survival stage to account for spatially autocorrelated frailties across neighboring units defined in the adjacency matrix.

To estimate our time-varying Spatial SP Weibull model via MCMC methods, we assign a prior for each of its three parameters (ρ, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$), a hyperprior $p(\lambda)$ to λ, and then define the parameters' conditional posterior distributions. We assign the weakly-informative multivariate Normal (MVN) prior to $\boldsymbol{\beta} = \{\beta_1, ..., \beta_{p_1}\}$ and $\boldsymbol{\gamma} = \{\gamma_1, ..., \gamma_{p_2}\}$, and the Gamma prior for ρ with shape and scale parameters a_ρ and b_ρ:

$$\begin{gathered} \rho \ \sim \ \text{Gamma}(a_\rho, b_\rho), \quad \boldsymbol{\beta} \sim \text{MVN}_{p_1}(\mu_\beta, \Sigma_\beta), \quad \boldsymbol{\gamma} \sim \text{MVN}_{p_2}(\mu_\gamma, \Sigma_\gamma) \\ \Sigma_\beta \ \sim \text{IW}(S_\beta, \nu_\beta); \ \ \Sigma_\gamma \ \sim \text{IW}(S_\gamma, \nu_\gamma) \end{gathered} \tag{8}$$

where a_ρ, b_ρ, S_β, ν_β, S_γ, ν_γ are the hyperparameters.[5] For robustness checks, we also estimate our model by assigning the multivariate Cauchy prior while retaining the Gamma prior for ρ. Following Banerjee and Carlin (2004) and Darmofal (2009), we assign the Gamma hyperprior $\lambda \sim \text{Gamma}(a_\lambda, b_\lambda)$ for λ.[6] We use Bayesian hierarchical modeling to estimate Σ_β and Σ_γ employing the Inverse-Wishart (IW) distribution when using the MVN or Cauchy prior and the Gamma hyperprior for λ. To identify the intercept, we impose the constraint that the frailties sum to zero ($\sum_i W_i = 0$ and $\sum_i V_i = 0$). The joint posterior distribution for the time-varying Bayesian SP Weibull model with spatially dependent frailties in the split and survival stages is:

[5]The MVN prior minimizes the possibly undue influence of the prior on the posterior estimates.
[6]We specify the vague prior $(a_\lambda, b_\lambda) = (0.001, 1/0.001) = (0.001, 1000)$ as done for ρ.

$$\pi(\boldsymbol{\beta}, \boldsymbol{\gamma}, \rho, \mathbf{W}, \mathbf{V}, \lambda, \Sigma_\beta, \Sigma_\gamma | \mathbf{C}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \mathbf{t0}, \boldsymbol{\gamma}) \quad \propto \quad L(\boldsymbol{\beta}, \boldsymbol{\gamma}, \rho, \mathbf{W}, \mathbf{V} | \mathbf{C}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \mathbf{t0})$$
$$\pi(\mathbf{W}|\lambda)\pi(\mathbf{V}|\lambda)\pi(\beta|\mu_\beta, \Sigma_\beta)\pi(\gamma|\mu_\gamma, \Sigma_\gamma)\pi(\rho)\pi(\lambda)\pi(\Sigma_\beta)\pi(\Sigma_\gamma) \quad (9)$$

where $L(\boldsymbol{\beta}, \boldsymbol{\gamma}, \rho, \mathbf{W}, \mathbf{V} | \mathbf{C}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \mathbf{t0})$ is given by equation (7). $\pi(\mathbf{W}|\lambda)$ and $\pi(\mathbf{V}|\lambda)$ are defined via their respective conditional distributions. $\pi(\beta|\mu_\beta, \Sigma_\beta)$, $\pi(\gamma|\mu_\gamma, \Sigma_\gamma)$, $\pi(\rho), \pi(\Sigma_\beta)$ and $\pi(\Sigma_\gamma)$ are from equation (8). $\pi(\lambda)$ is the Gamma hyperprior.

We estimate our Spatial SP Weibull model with an MCMC algorithm for Bayesian inference. Since closed form distributions for the posterior distributions of $\boldsymbol{\beta}, \boldsymbol{\gamma}$, $\rho, \lambda, \mathbf{W}$, $\mathbf{V}$ are not available, our MCMC method's update scheme incorporates Gibbs Sampling (for λ), Metropolis-Hastings algorithm (for $\mathbf{W}$ and $\mathbf{V}$ given λ), and slice-sampling with step-out and shrinkage (Neal 2003; Bagozzi et al. 2019) for updating $\boldsymbol{\beta}, \boldsymbol{\gamma}$, ρ. This MCMC algorithm (described in the Supplemental Appendix) proceeds in five steps:

1. Choose starting points β_0, γ_0, ρ_0, λ_0, and corresponding $\mathbf{W}_0 = \{W_1, ..., W_N\}$ and $\mathbf{V}_0 = \{V_1, ..., V_N\}$; then set $i = 0$.
2. Update $\Sigma_\beta \sim \pi(\Sigma_\beta|\beta)$, $\Sigma_\gamma \sim \pi(\Sigma_\gamma|\gamma)$, and $\lambda \sim \pi(\lambda|\mathbf{W}, \mathbf{V})$.[7]
3. Update $\beta \sim \pi(\beta|\mathbf{C}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \mathbf{W}, \mathbf{V}, \gamma, \rho, \mu_\beta, \Sigma_\beta)$, $\gamma \sim \pi(\gamma|\mathbf{C}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \mathbf{W}, \mathbf{V}, \beta, \rho, \mu_\gamma, \Sigma_\gamma)$, $\mathbf{W} \sim \pi(\mathbf{W}|\mathbf{C}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \mathbf{V}, \beta, \gamma, \rho, \lambda)$ and $\mathbf{V} \sim \pi(\mathbf{V}|\mathbf{C}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \mathbf{W}, \beta, \gamma, \rho, \lambda)$.
4. Update $\rho \sim \pi(\rho|\mathbf{C}, \boldsymbol{\alpha}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \mathbf{W}, \mathbf{V}, \beta, \gamma, a_\rho, b_\rho)$.
5. Set $i = i + 1$, go to Step 2, repeat N iterations.

2 Monte Carlo Simulations

We conduct 8 Monte Carlo (MC) experiments to evaluate the performance of our Bayesian SP Weibull models (i) *without* any unit-specific frailties ("Pooled" SP model), (ii) with i.i.d. frailties in the model's split and survival stages ("Exchangeable" SP model), and (iii) with spatially autocorrelated frailties in both stages ("Spatial" SP model). Our primary MC experiments simulate an SP Weibull distributed outcome variable (with immune fraction α) with

[7]Conditional distributions of $\Sigma_\beta \sim \pi(\Sigma_\beta|\beta)$, $\Sigma_\gamma \sim \pi(\Sigma_\gamma|\gamma)$, and $\lambda \sim \pi(\lambda|\mathbf{W}, \mathbf{V})$ are defined in the Supplemental Appendix.

spatially dependent frailties in both stages of the model. We explore the performance of all three models when the fraction of the immune population is $\alpha = 25\%$ and the proportion of units with spatially dependent frailties is held at 40% (Experiment 1); when the immune fraction is varied while the share of units with spatially dependent frailties remains at 40% (Experiment 2); when the immune fraction is $\alpha = 25\%$ but the proportion of units that exhibit spatially dependent frailties is varied (Experiment 3); and when *both* the immune fraction and share of units with spatially dependent frailties varies (Experiment 4). We also re-evaluate our primary MC results using an alternative prior (Experiment 5), assess a Bayesian Spatial SP *Log-Logistic* model (Experiment 6), and evaluate our model's performance (i) compared to a Bayesian Spatial SP model that incorporates spatial frailties in just the survival stage (Experiment 7) and (ii) when the SP Weibull d.g.p. does *not* include i.i.d. or spatially dependent frailties.

For each experiment listed above, we set $sims = 100$ and evaluate the models' performance at different sample sizes: $N = 1,000$, $N = 1,500$, and $N = 2,000$. We assign our survival stage covariates ($\mathbf{x}$) as $\mathbf{x} = (\mathbf{1}, \mathbf{x_1})'$ where $\mathbf{x_1}$ is drawn from $Uniform[-3,\ 3]$. Parameter values are assigned as $(\beta_0,\ \beta_1)' = (1,\ 2)'$ for our survival-stage predictors. The simulated SP Weibull outcome variable for Experiments 1, 3, 5-8 have a moderate immune fraction level of $\alpha = 25\%$, whereas in Experiments 2 and 4, we increase the immune population as: $\alpha = 25\%,\ 33\%, 48\%, 60\%$. To generate these immune fraction levels, we first define the split-stage covariates ($\mathbf{z}$) as $\mathbf{z} = (\mathbf{1}, \mathbf{z_1}, \mathbf{z_2})'$, where $\mathbf{z_1} = Uniform[0,\ 5]$ and $\mathbf{z_2} \equiv \mathbf{x_1}$. Our split-stage parameters are then defined as $(\gamma_0,\ \gamma_1,\ \gamma_2)' = (-3,\ -1,\ 3.5)'$ for our baseline immune fraction level of $\alpha = 25\%$ (Experiments 1, 3, 5-8), or as follows in Experiments 2 and 4: $(\gamma_0,\ \gamma_1,\ \gamma_2)' = (4,\ -3,\ 3)'$ for $\alpha = 33\%$, $(\gamma_0,\ \gamma_1,\ \gamma_2)' = (-4,\ 1,\ 2.5)'$ for $\alpha = 40\%$, $(\gamma_0,\ \gamma_1,\ \gamma_2)' = (-3,\ 1,\ 3)'$ for $\alpha = 48\%$, and $(\gamma_0,\ \gamma_1,\ \gamma_2)' = (-4,\ 2,\ 1)'$ for $\alpha = 60\%$.

As described in the Supplemental Appendix, we incorporate information about the spatial relationship between units (with unit-specific frailties V_i, W_i) in our simulated data via an adjacency matrix $\mathbf{A} = \{a_{ii'}\}$. To generate $\mathbf{A}$, we consider a hypothetical space with five areal units (e.g., countries), where each has at least one adjacent "neighbor." $\mathbf{A} = \{a_{ii'}\}$ is thus a 5

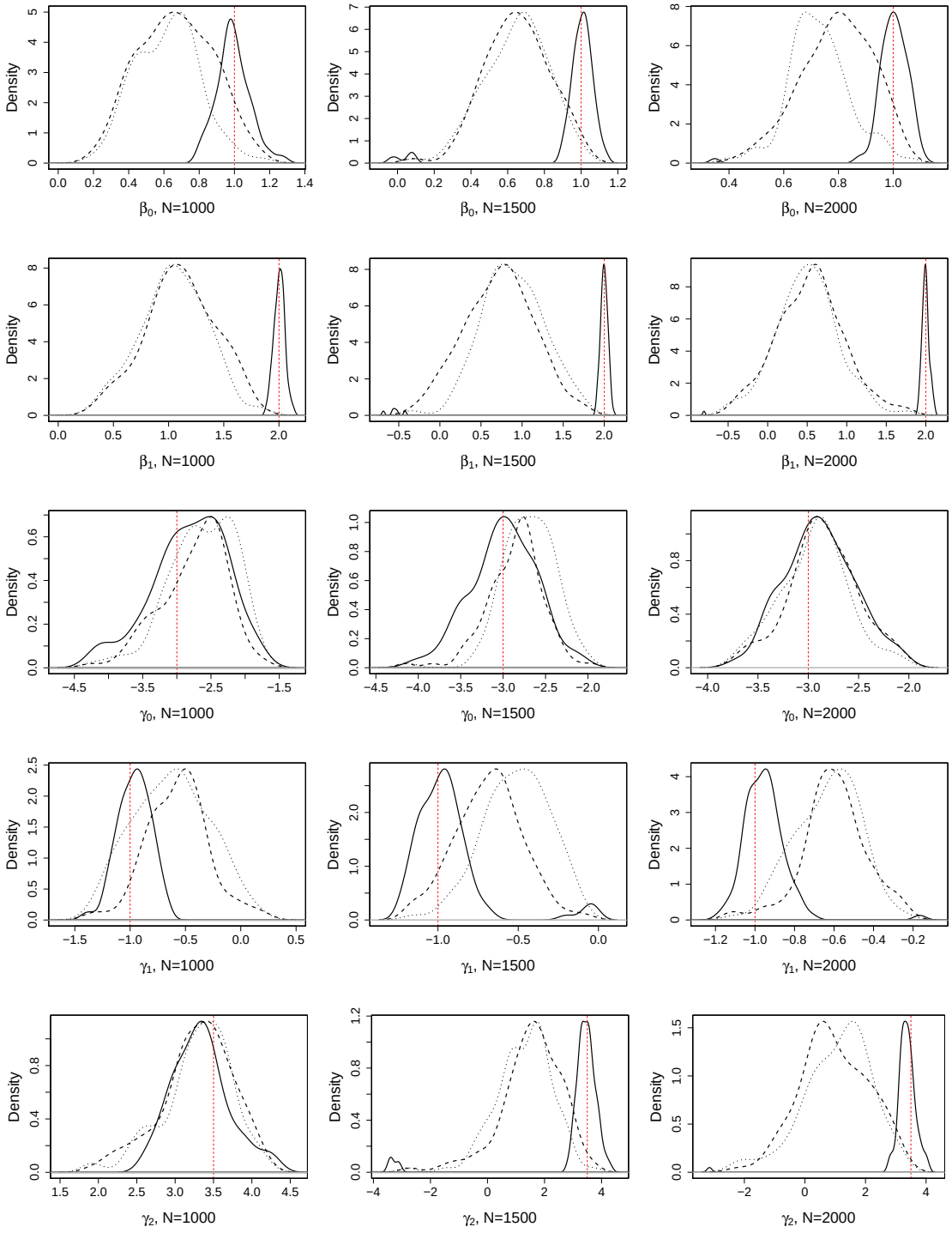
$\times$ 5 symmetric and positive definite matrix where $a_{ii'} = 1$ if units i and i' are adjacent and $a_{ii'} = 0$ if i and i' are not adjacent. Each unit's frailty is $\mathbf{V} = \{V_1 ... V_5\}$ in the split-stage and $\mathbf{W} = \{W_1 ... W_5\}$ in the survival-stage. We use matrix $\mathbf{A}$ to vary the proportion of units in each stage that exhibit spatially dependent frailties in the simulated data. In Experiments 1-2, 5-6, 8, we set the share of units with spatially dependent frailties to 40% by assigning a 1 to the following elements in $\mathbf{A}$: $a_{12} = a_{21}$, $a_{23} = a_{32}$, $a_{35} = a_{53}$, and $a_{14} = a_{41}$. All other cells are assigned 0. We use a similar procedure (see Supplemental Appendix) to vary this proportion in Experiments 3 and 4. After constructing $\mathbf{A}$ for each MC experiment, we assign the true theoretical values for the unit-specific frailties $\mathbf{V} = \{V_1 ... V_5\}$ and $\mathbf{W} = \{W_1 ... W_5\}$, which are drawn from the CAR prior's conditional distribution described in equation (5): $N(\overline{W_i}, 1/(\lambda m_i))$ and $N(\overline{V_i}, 1/(\lambda m_i))$.[8] Experiments 1-5 and 7-8 use an MVN prior, while Experiment 6 uses a multivariate Cauchy prior. In each experiment, we evaluate the MCMC-simulated mean parameter estimates and standard errors (MCSEs), root mean square error (RMSE), and 95% empirical coverage probabilities (CPs). To save space, the results from our MC experiments are described in detail in the Supplemental Appendix. We summarize our findings here.

The results from Experiment 1 presented in Figure 1 and Tables A.1-A.2 (Supplemental Appendix) show that the $\hat{\beta}'$s (β_0, β_1) and $\hat{\gamma}'$s ($\widehat{\gamma}_0$, $\widehat{\gamma}_1$, $\widehat{\gamma}_2$) from the Bayesian Spatial SP Weibull model converge to their true theoretical values with negligible RMSEs and high 95% empirical CPs in the 84%-92% range. The nonspatial Bayesian Pooled and Exchangeable SP Weibull models' $\hat{\beta}'$s and $\hat{\gamma}'$s, however, substantially deviate from their true values no matter the size of N. Results also reveal that *unlike* the (nonspatial) Exchangeable SP Weibull model, the Spatial SP Weibull model retrieves the true values of *each* $\{\widehat{w}_1 ... \widehat{w}_5\}$ in $\mathbf{W}$ and *each* $\{\widehat{v}_1 ... \widehat{v}_5\}$ in $\mathbf{V}$ with negligible RMSEs and high 72-94% CPs (Table A.3, Figure A.1, Supplemental Appendix).

The $\hat{\beta}$'s and $\hat{\gamma}$'s from Experiment 2—where the immune fraction's size is increased from $\alpha = 25\%$ to $\alpha = 60\%$—also favor the Bayesian Spatial SP Weibull model over the other two nonspatial Bayesian SP Weibull models at all immune fraction levels and sample sizes (Tables A.4-A.5, Supplemental Appendix). Further, as indicated in Figure 2a-b, the averaged

[8] We set $\lambda = 0.1$; m_i denotes the number of adjacencies for i.

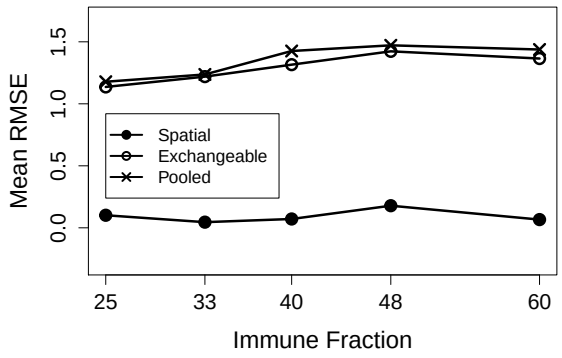
Figure 1: MC Experiment 1 β,γ Densities



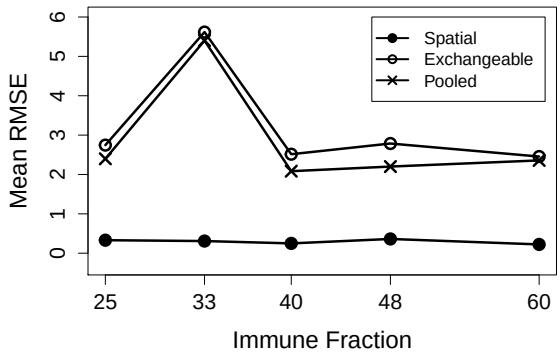
$N = 1,000$, $N = 1,500$, and $N = 2,000$ $\hat{\beta}$ and $\hat{\gamma}$ RMSEs from the Bayesian Spatial SP Weibull model are close to 0 no matter the size of the immune fraction, whereas the averaged $\hat{\beta}$ and $\hat{\gamma}$ RMSEs from the other two models are consistently larger. The Bayesian Spatial SP Weibull model's $\widehat{w}_i$'s and $\widehat{v}_i$'s rapidly converge to their true values (with 79-97% CPs) for all immune fraction levels and all sample sizes, while the nonspatial Pooled and Exchangeable SP Weibull models fail in this regard (Tables A.6-A.7).

Figure 2: MC Experiment 2 β,γ Mean RMSEs

(a) $\hat{\beta}$ coefficients



(b) $\hat{\gamma}$ coefficients



Experiment 3 applies Experiment 1's d.g.p. (where $\alpha = 25\%$) but varies the level of spatial dependence in the frailties (30%, 40%, 60%, 80%). The $\hat{\beta}$'s and $\hat{\gamma}$'s from both the nonspatial SP Weibull models—as well as $\widehat{w}_i$'s and $\widehat{v}_i$'s from the Exchangeable SP Weibull model—are severely biased at all levels of spatial dependence, and become increasingly biased as the degree of spatial dependence in the d.g.p. grows (Tables A.8-A.11, Figure A.2, Supplemental Appendix). The Bayesian Spatial SP Weibull model, however, retrieves the true $\hat{\beta}, \hat{\gamma}$, $\widehat{w}_i$, $\widehat{v}_i$ values with low RMSEs and high CPs when the proportion of adjacent units in $\mathbf{A}$ increases.

Experiment 4 compares the three Bayesian SP Weibull models' performance when the true d.g.p. is SP Weibull with spatial frailties while varying both the immune fraction size *and* the proportion of units with spatially dependent frailties. This exercise is conducted for the following four conditions: (1) small immune fraction ($\alpha = 25\%$) with mild spatial dependence in the frailties (30%), (2) small immune fraction ($\alpha = 25\%$) with highly spatially dependent frailties (80%), (3) large immune fraction ($\alpha = 65\%$) with mild spatial dependence in the

frailties (30%), and (4) large immune fraction ($\alpha = 65\%$) with severe spatial dependence in the frailties (80%). Results show that the Bayesian Spatial SP Weibull model successfully retrieves—while the two nonspatial Bayesian SP Weibull models $\hat{\beta}$'s and $\hat{\gamma}$'s diverge from—the true values of $\hat{\beta}$ and $\hat{\gamma}$ across all four conditions above (Tables A.12-A.13, Figure A.3a-b, Supplemental Appendix). Additionally, *in contrast* to the nonspatial Exchangeable SP model, the $\widehat{w}_i$'s and $\widehat{v}_i$'s from the Bayesian Spatial SP Weibull model always converge to their true value with negligible RMSEs and high CPs across all four conditions, no matter the size of N (Tables A.14-A.15). The results from Experiments 1-4 thus confirm that the Bayesian Spatial SP Weibull model outperforms the nonspatial Pooled and Exchangeable SP Weibull model when the true SP Weibull d.g.p. exhibits spatial dependence between units in both stages. Moreover, the Bayesian Spatial SP Weibull model outperforms the nonspatial SP Weibull models even when the immune fraction is small and the degree of spatial dependence in the frailties is low.

We provide the results from Experiments 5-8 in the Supplemental Appendix. In brief, the Bayesian Spatial SP Weibull model, unlike the nonspatial models, retrieves the true values of β, γ, $\mathbf{W}$, $\mathbf{V}$ when using the multivariate Cauchy prior (instead of the MVN prior) in Experiment 5. Experiment 6 shows that the $\hat{\beta}$'s, $\hat{\gamma}$'s, $\widehat{w}_i$'s and $\widehat{v}_i$'s from the Bayesian Spatial SP Log-Logistic model converge to their true theoretical values when this model is applied to an SP Log-logistic d.g.p. ($\alpha = 25\%$) that exhibits spatial dependence. Experiment 7 reveals that our Bayesian SP Weibull model with spatially autocorrelated frailties in the split and survival stages outperforms the Bayesian SP Weibull model with spatially autocorrelated frailties in only the survival stage when using the d.g.p. from Experiment 1. In Experiment 8, we find that the Bayesian Spatial SP Weibull's $\hat{\beta}$'s and $\hat{\gamma}$'s tend to be slightly biased toward zero, but their RMSEs remain relatively small, and their CPs fall between 76%-93% when the true SP Weibull d.g.p. does *not* have spatially dependent or i.i.d. random effects. Thus, misapplying the Bayesian Spatial SP Weibull model to an SP survival d.g.p. where the unit-specific frailties do not exhibit spatial autocorrelation does not lead to substantial bias.

3 Empirical Applications

To illustrate the applicability of our model, we estimate our Bayesian Spatial SP Weibull model using survival data from two published studies in Political Science. The second application is Walter's (2015) study of post-civil war peace survival presented in the Supplemental Appendix. In this section, we employ our Bayesian SP Weibull models to replicate Svolik's (2008) analysis of democratic consolidation and survival published in the APSR.

Most democratization scholars distinguish conceptually between *transitional* democracies that are "at-risk" of reverting to authoritarianism, and *consolidated* democracies, in which the "democratic regime becomes sufficiently durable that democratic breakdown... is no longer likely" (Gasiorowski and Power 1998, 743; Przeworski et al. 2000). Traditionally, a consolidated democracy formally meets the following criteria: a second election for the national executive has been held following an inaugural democratic election, alternation of executive power through constitutional means, and democratic institutions have been sustained for at least 12 years (Gasiorowski and Power 1998, 746-747; Przeworski et al. 2000; Gassebner, Lamla and Vreeland 2013).

Empirical analyses of democratic *consolidation*, which typically employ discrete choice models, consistently find that GDP per capita has a positive and highly significant effect on the probability of democratic consolidation, while the effect of economic growth and parliamentary systems on democratic consolidation is insignificant (Gasiorowski and Power 1998; Gassebner, Lamla and Vreeland 2013). Presidential regimes have a negative but weakly significant or insignificant impact on democratic consolidation, and there are mixed results for the association between past authoritarian institutions and democratic consolidation (Gasiorowski and Power 1998; Cheibub 2007). The democratic *survival* literature (which typically uses conventional survival models) reports that unlike Presidential systems, parliamentary systems help democracies endure (Przeworski et al. 2000). Economic growth also has a positive and highly significant influence on democratic regime survival (Treisman 2018). Democratic states preceded by military rule are less likely to survive as democracies, though other previous authoritarian regime-types

(civilian dictatorship, monarchy) have no tangible effect (Boix and Stokes 2003; Cheibub 2007). The most robust finding in this literature, however, is that GDP per capita has a strong positive influence on survival of democracy (Przeworski et al. 2000; Treisman 2018; Gassebner, Lamla and Vreeland 2013). This has led Przeworski and Limongi (1997) to infer that democratic survival "increases monotonically with per capita income," and then endures *indefinitely* once GDP per capita reaches approximately US $6,000 (p.165).

These insights are undoubtedly important, but Svolik (2008) rightly points out that by employing standard duration or discrete choice models, these studies assume that all democracies face the same baseline risk of reversal to authoritarian rule. This assumption is unjustified since the population of democracies includes an "at-risk" set of transitional democracies and an "immune fraction" of fully consolidated democracies whose risk of authoritarian reversal is negligible (Svolik 2008). The existence of these two populations implies that the observed survival of democracy results from two separate processes: "democracies that survive because they are consolidated and democracies that are not consolidated but survive because of some favorable circumstances" (Svolik 2008, 153). Given these two subpopulations, Svolik (2008) estimates nonspatial parametric SP survival models with and without i.i.d. frailties via MLE to re-assess existing findings about democratic consolidation and survival using his data on democratic spells across 133 countries during 1789–2001. Rather than defining consolidation arbitrarily, he argues that all right-censored observations in his data are either consolidated or transitional democracies that have not yet reverted to authoritarian rule. The split-stage in his SP survival model estimates the probability of democratic consolidation (62% of his cases are right-censored), and the survival stage estimates the duration of democracy. Svolik (2008) incorporates seven covariates in both stages of the model: *GDP per capita*, *GDP growth*, *Presidential* system, *Parliamentary* system, previous *Military* dictatorship, previous *Civilian* dictatorship, and previous *Monarchy*.

Svolik (2008) presents several results that are robust to the exclusion and inclusion of nonspatial frailties. First, he finds that *GDP per capita* has a positive and statistically significant effect on the probability of democratic consolidation in the split stage but is not significant

in the survival stage. His split-stage finding corroborates previous research (Gasiorowski and Power 1998; Gassebner, Lamla and Vreeland 2013), but his insignificant survival-stage result for *GDP per capita* departs from the otherwise robust finding in the literature that increased income per capita helps democracy endure longer (Przeworski et al. 2000; Treisman 2018). Second, consistent with other studies, Svolik (2008) finds that *GDP growth* has an insignificant association with democratic consolidation (Gasiorowski and Power 1998), but a positive and highly significant impact on democratic regime survival (Treisman 2018). Third, he finds that *Presidential* systems are significantly less likely to become consolidated democracies, but have a positive albeit insignificant influence on survival of democracy. Fourth, previous military regimes have a negative but insignificant effect on the duration of democracy, but are significantly less likely to consolidate (Svolik 2008). Previous monarchies do not appear to systematically influence democratic consolidation, but are associated with extended democratic survival. Finally, Svolik (2008) finds no significant effect of *Parliamentary* systems or previous *Civilian* dictatorships in either stage of his models.

Svolik's (2008) analysis is an important contribution to an extensive literature on durability of democracies, but his SP models assume that neither the likelihood of democratic consolidation nor the prospects for democratic survival exhibits spatial autocorrelation. We find this assumption to be untenable because democracies tend to cluster in space. In fact, "since 1815, the probability that a randomly chosen country will be a democracy is about 0.75 if the majority of its neighbors are democracies, but only 0.14 if the majority of its neighbors are nondemocracies" (Gleditsch and Ward 2006, 916). To better understand this empirical phenomenon, note that studies have found that autocracies in close proximity to democracies are more likely to experience a democratic transition (Gleditsch and Ward 2006; Brinks and Coppedge 2006). Once these democracies emerge, they form regional clusters of institutions that reinforce one another through various regional political and normative channels. These neighborhood effects are important for democratic regime survival over and above any independent effects of how long neighboring democracies have endured (Gleditsch and Ward 2006; Gates et al. 2006). More specifically, even nascent democracies tend to survive when they are geographically proximate

to regimes with similar institutional portfolios (Gates et al. 2006; Gleditsch and Ward 2006), and geographic proximity between consolidated democracies and new democracies also increases the odds of consolidation and survival of the latter's democratic institutions (Gasiorowski and Power 1998; Kopstein and Reilly 2000; Treisman 2018). This is because stable democracies transmit ideas, norms and political pressures that are conducive to democracy (Gasiorowski and Power 1998, 745) leading to "regional production chains" of democratic institutions and practices (Kopstein and Reilly 2000, 25; Brinks and Coppedge 2006). This incentivizes leaders in nearby democracies to maintain their institutions, since failing to do so can lead to political isolation and costly diplomatic repercussions. Moreover, the clustering of these norms increases the odds that elites and citizens in newer democracies will internalize them and ultimately sustain a democratic rule of law (Gasiorowski and Power 1998; Brinks and Coppedge 2006).

Thus, democratic clustering increases the regional legitimacy of democratic institutions, which reinforces the "spatially dependent nature" of democratic norms (Kopstein and Reilly 2000, 1) that make it costly for elites to engage in democratic backsliding (Brinks and Coppedge 2006). This reduces the risk of authoritarian reversal, thereby increasing the prospects for consolidation and survival of democracies in predominantly democratic neighborhoods. Although these mechanisms are often intangible and difficult to measure, democratic neighborhood effects clearly have important latent influences on both democratic consolidation and institutions and survival. For this reason, it is essential to account for spatially dependent frailties in both the split and survival stages of an SP model of democratic survival and consolidation.

As a preliminary verification exercise for these claims, we conduct join-count and Moran's I tests on Svolik's (2008) data. A detailed discussion of these tests is reported in the Supplemental Appendix, but the results indicate the presence of spatial clustering among potentially consolidated democracies as well as more durable democracies. The join-count statistics in Figure A.4a depict significant spatial clustering in the probability of democratic consolidation, especially in the twentieth century. The Moran's I values in Figure A.4b also suggest spatial autocorrelation in the survival of democracies throughout the observed period.

These theoretical claims and statistical tests suggest that it is insufficient to estimate SP

survival models that do not account for the spatial clustering of democracy when assessing the determinants of democratic consolidation and survival. We thus apply our Bayesian Spatial SP Weibull model to Svolik's (2008) data to evaluate whether spatial autocorrelation matters for consolidation and duration of democracy. We focus on replicating Svolik's (2008) main specification of interest: nonspatial MLE SP Weibull model with i.i.d. frailties (this model's results are very similar to those from his MLE SP Weibull model without frailties). We estimate and compare Svolik's (2008) original MLE SP Weibull model with i.i.d. frailties, a nonspatial Bayesian SP Weibull model without frailties ("Pooled" SP), a nonspatial Bayesian SP Weibull model with i.i.d. frailties in each stage ("Exchangeable" SP), and our Bayesian SP Weibull model with spatially autocorrelated frailties between democratic neighbors in the model's split and survival stages ("Spatial" SP). The split-stage estimates the probability of *democratic consolidation*, and the survival stage models *democratic duration*. We include the exact same set of covariates in each stage as Svolik's (2008) original analysis (see above).

Unlike Svolik's (2008) MLE and the nonspatial Bayesian Pooled and Exchangeable SP models, our Bayesian Spatial SP Weibull model incorporates spatially-weighted frailties across neighboring democracies via the adjacency matrix $\mathbf{A}$. We construct matrix $\mathbf{A}$ with elements $a_{ii'}$ such that for each year $a_{ii'} = 1$ if the capital of country i is less than 800km from the capital of country i', and $a_{ii'} = 0$ if countries i and i' are greater than 800km from each other. Using geographic proximity as the spatial relationship of interest is appropriate for our purposes because it allows the frailties to correlate with those of neighboring democracies. Indeed, defining adjacency in $\mathbf{A}$ in this way means that we explicitly account for our contention that the underlying risk propensity of a democracy to survive and consolidate is correlated with the risk propensities of neighboring democracies rather than assuming spatial independence even within the same regions. We incorporate the spatial information in matrix $\mathbf{A}$ by employing separate CAR priors for the frailty terms vector $\mathbf{V}$ (split-stage) and $\mathbf{W}$ (survival-stage), where: $\mathbf{V}|\lambda \sim \text{CAR}(\lambda)$ and $\mathbf{W}|\lambda \sim \text{CAR}(\lambda)$. The resulting conditional distribution of the spatial frailties between the geographically neighboring democracies is given by equation (5) above. The Bayesian Pooled, Exchangeable, and Spatial SP Weibull models are each estimated on Svolik's

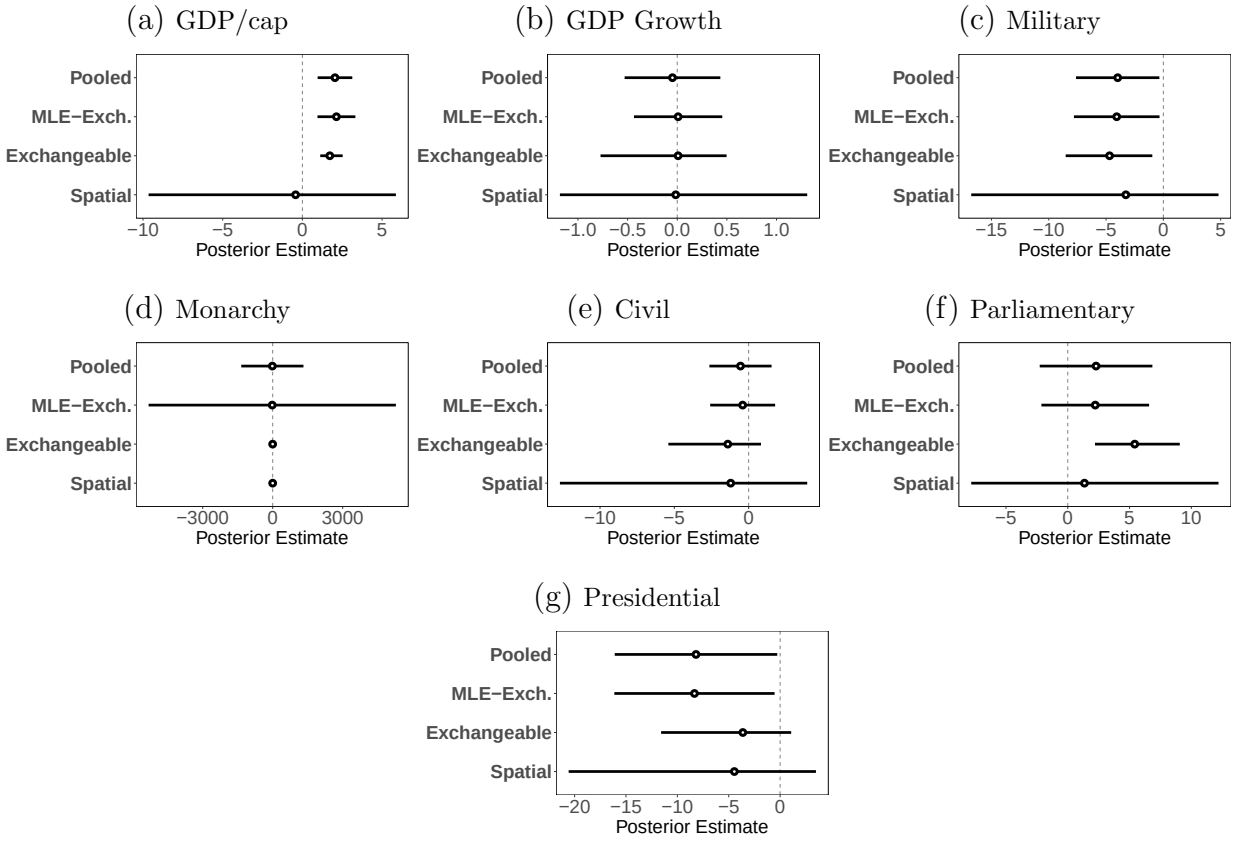
sample using the MVN prior and our MCMC algorithm with the following hyperparameters: $a = 1$, $b = 1$, $S_\beta = I_{p1}$, $S_\gamma = I_{p2}$, $\nu_\beta = p1$ and $\nu_\gamma = p2$.[9] We also assign the Gamma hyperprior $\lambda \sim \text{Gamma}(a_\lambda, b_\lambda)$ for λ (with vague prior $(a_\lambda, b_\lambda) = (0.001, 1/0.001)$) for Bayesian inference in the Spatial SP model.

We first discuss the split-stage and then the survival-stage results from our SP Weibull models. To begin, we use three choropleth maps to illustrate the posterior mean of the spatial frailties obtained from the Spatial SP Weibull model's split-stage (Figures A.5a-c, Supplemental Appendix). These maps illustrate whether spatially clustered democracies—unlike spatially distant democracies—share common underlying risk factors for democratic consolidation, and are divided into three temporal "waves" of democratization (Huntington 1991). The first wave occurred from 1828-1922, the second from 1948-1962, and the third from 1978-2001. Due to space constraints, we only present the split-stage posterior spatial frailty estimates for the third wave here and analyze the first and second wave results in the Supplemental Appendix. In Figure A.5a, note that there are distinct spatial bands in the frailties, which range from -0.014 to 0.027 with corresponding standard deviation of 0.01 to 0.09. The map reveals strong spatial clustering in the underlying factors linked to democratic consolidation, as states with a higher baseline risk for democratic consolidation are in similar geographic neighborhoods, whereas those with lower propensities cluster in *separate* regions. Moreover, the posterior mean of the variance of the frailties in the Spatial SP Weibull's split stage is approximately twice as large as the posterior mean of the nonspatial random effects from the Exchangeable SP model's split stage. These spatial patterns also hold for the first and second waves of democracy (Figures A.5b-c). Thus, as per theoretical expectations, neighboring democracies indeed appear to share common unobserved factors that influence the probability of democratic consolidation.

The dot-whisker plots in Figure 3 reveal mixed results for the split-stage covariates' posterior mean estimates from the nonspatial Pooled and Exchangeable SP and Spatial SP models. *Monarchy* and *Civilian* are each negative and unreliable (statistically insignificant) in all three

[9]Our Bayesian SP Weibull models' results are based on a set of 50,000 iterations after 4,000 burn-in scans and thinning of 10.

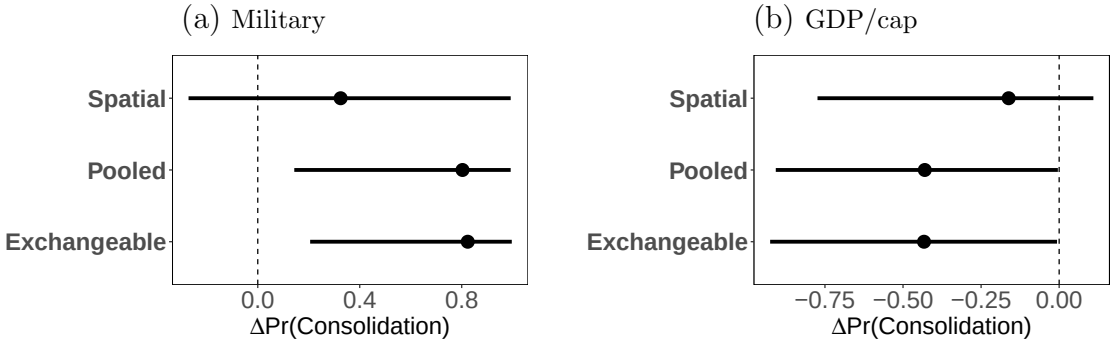
Figure 3: Split-Stage Covariates



models' split stages, as the 95% Bayesian Credible Intervals [BCI] (confidence intervals in the MLE model) always include zero. *Presidential* is negative and unreliable in the Bayesian Exchangeable and Spatial SP models, but reliable (statistically significant) in the MLE and Pooled SP models. The positive split-stage estimate of *Parliamentary* is unreliable in all the models, barring the Bayesian Exchangeable SP model. In general, our Bayesian nonspatial models produce similar results to Svolik's (2008) original split-stage results obtained via MLE. However, the split-stage posterior estimates for previous *Military* regimes, *GDP per capita* and *GDP growth* from our Spatial SP Weibull model vary considerably from their corresponding results in the nonspatial models.

To see this, first note that similar to Svolik's (2008) findings, Figure 3c shows that the split-stage estimate of *Military* is negative and highly reliable in the nonspatial SP Weibull models. In Figure 4a, increasing the *Military* dummy from 0 to 1 reliably decreases the probability of *democratic consolidation* by almost 80% in the Exchangeable and Pooled SP models, as the

Figure 4: Predicted Probability of Democratic Consolidation



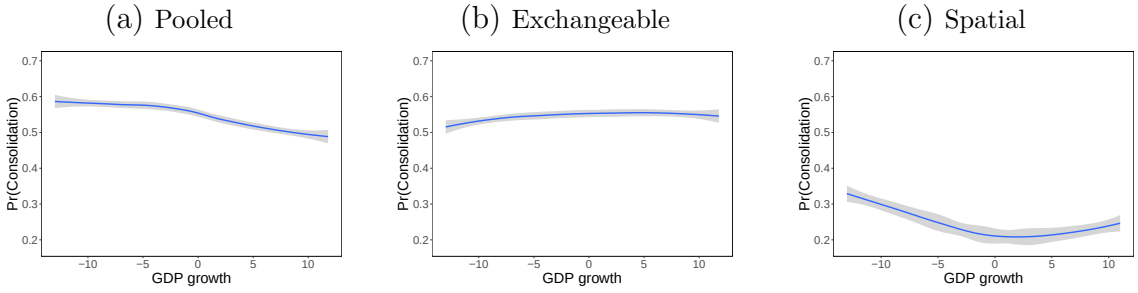
95% BCI of these substantive effects exclude zero. By comparison, the negative posterior mean estimate and substantive influence of *Military* on democratic consolidation in the Spatial SP model's split stage is unreliable, since its 95% BCI includes zero.

The split stage estimate for *GDP per capita* is positive and highly reliable in all the nonspatial Pooled and Exchangeable SP models (Figure 3a), but the corresponding posterior estimate in the Spatial SP Weibull model is not only inconsistent with the other nonspatial SP models but also unreliable. Increasing *GDP per capita* from 1 SD below to 1 SD above its mean in the split stage reliably increases the odds of democratic consolidation in the nonspatial Pooled and Exchangeable SP models. However, income per capita has a negligible and unreliable substantive effect in the Spatial SP Weibull model's split stage (Figure 4b). This result is reinforced when we assess the marginal effect of *GDP per capita*—calculated for its entire observed sample range—on the likelihood of democratic consolidation from the nonspatial and Spatial SP Weibull models (Figures A.6a-c, Supplemental Appendix). These figures show that once a country achieves the "threshold" income level of approximately US$6,000, the probability of democratic consolidation increases from 0.2 to 0.8 and then monotonically increases to 0.9 for all per capita income levels above this threshold in the Pooled and Exchangeable SP models. This result is not only highly reliable but also consistent with earlier studies that report a similar positive and significant association between per capita income and democratic consolidation (Gasiorowski and Power 1998; Gassebner, Lamla and Vreeland 2013). In contrast, the effect of *GDP per capita* on the likelihood of democratic consolidation in our Spatial SP Weibull model increases to *just* 0.55 and is unreliable. Thus, once we account for spatial autocorrela-

tion, the association between *GDP per capita* and democratic consolidation is simply not as substantively strong and robust as widely reported in the democratic consolidation literature.

Next, consider the other key economic covariate: *GDP growth*. The negative split-stage posterior mean estimate of economic growth seems unreliable in all the models in 4b, but a more careful and detailed assessment of the marginal effect of *GDP growth*—calculated for this covariate's entire observed sample range—on the probability of democratic consolidation illustrated in Figures 5a-c reveal an intriguing result. As consistently reported in previous research (Gasiorowski and Power 1998; Svolik 2008), these figures first show that there is a monotonically decreasing (or weakly-increasing) relationship between *GDP growth* and the odds of democratic consolidation across the nonspatial SP Weibull models. By contrast, *GDP growth* has a *nonmonotonic* and reliable association with the probability of democratic consolidation in the Spatial SP Weibull model. Unlike the nonspatial SP Weibull models, the aforementioned nonmonotonic relationship suggests that while democracies tend to become fragile during recessionary conditions, their odds of institutional consolidation rises significantly in periods of positive (and increasing) economic growth. Modeling spatial autocorrelation in our SP survival model thus unpacks a substantively interesting association between GDP growth and democratic consolidation that would be overlooked in nonspatial SP survival models.

Figure 5: GDP Growth and Democratic Consolidation

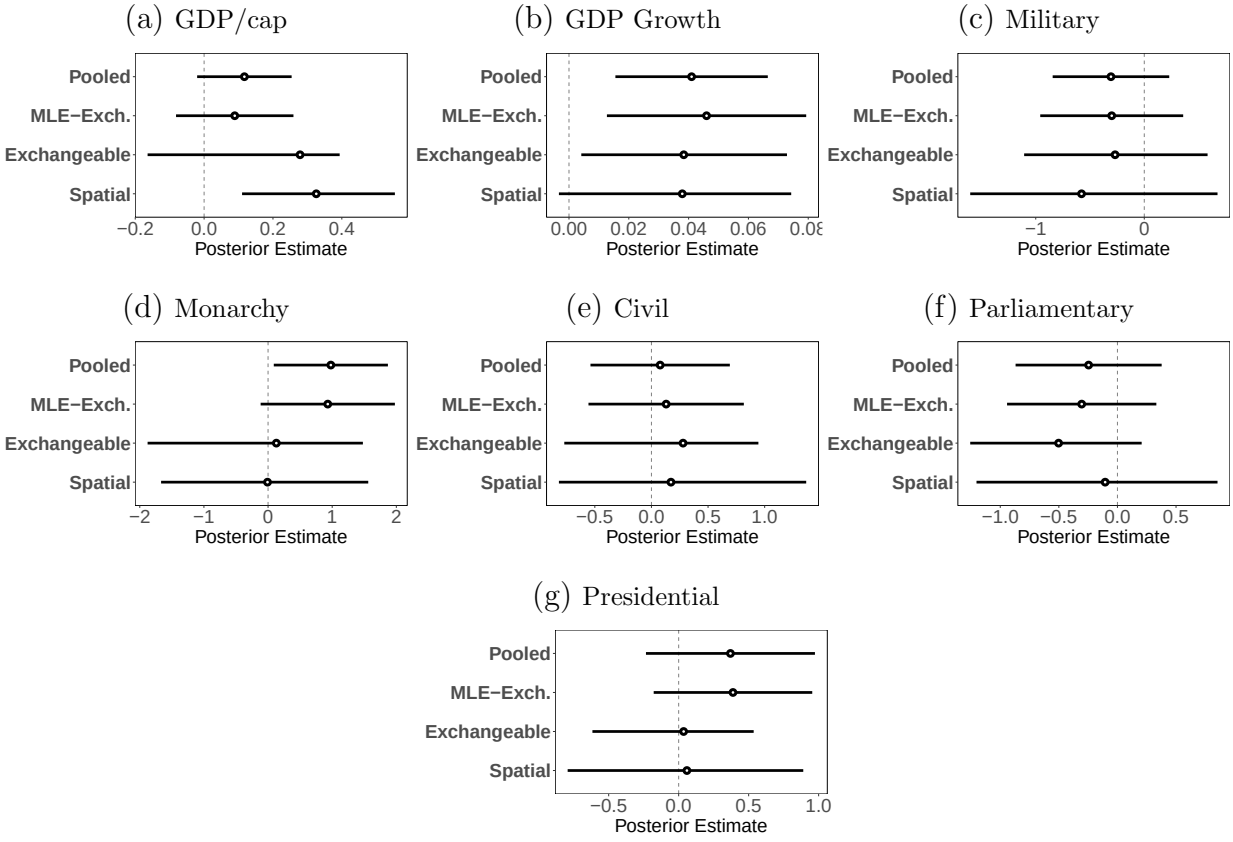


We now turn to discuss the survival stage posterior estimates from the Pooled, Exchangeable, and Spatial SP models. Owing to space constraints, we discuss Figure A.7a here, which maps the posterior mean of the survival-stage spatial frailties for the third wave of democracy (1978-2001) in Svolik's sample, and discuss the other two maps in the Supplemental Appendix.

The distinct spatial bands in this map's frailties vary from -0.028 to 0.027 with corresponding standard deviation of 0.02 to 0.11. These spatial bands again reveal strong geographic clustering associated with democratic regime survival: those democracies with greater underlying propensity for democratic survival tend to be located near countries with similarly propensities for democratic survival, and those with a lower propensity for democratic survival are located in disparate geographic areas. Furthermore, the posterior mean of the variance of the frailties from the Spatial SP Weibull's survival stage is about twice as large as those obtained from the frailties in the Exchangeable SP model's survival stage. The prominent spatial clustering of democratic survival rates also holds for the earlier waves of democracy (Figures A.7b-c).

The survival stage estimates illustrated in Figure 6's dot-whisker plot reveals that estimates for *Parliamentary* and previous *Military* regimes are each negative and always unreliable. *Civilian* dictatorships and *Presidential* systems are each positive but also unreliable across all three Bayesian SP Weibull models. This mirrors Svolik's (2008) finding for these covariates, but the survival-stage results for *GDP per capita*, *GDP growth*, and *Monarchy* differ substantially between the nonspatial SP and Spatial SP Weibull models. First, note that Figure 6 and the acceleration factor plot (AFP) in Figure A.8a (Supplemental Appendix) show that the effect of *Monarchy* on democratic survival is positive but almost always unreliable—as also shown by Svolik (2008)—in the Pooled and Exchangeable SP models' survival stage. However, the corresponding posterior mean estimate (and AFP in Figure A.8a) from the Bayesian Spatial SP model suggests that the influence of *Monarchy* in the survival stage is *negative* albeit unreliable. Thus, once we account for spatial autocorrelation between geographically clustered democracies in the Bayesian SP model's survival stage, we find no support for Svolik's (2008) finding that democracies previously ruled by a monarch survive longer. Moreover, consistent with Svolik's (2008) and Treisman's (2018) findings, the survival stage estimate of *GDP growth* is positive and highly reliable in the nonspatial Pooled and Exchangeable SP models. However, the 95% BCI of the effect of *GDP growth* on democratic survival in the Bayesian Spatial SP model includes zero and is thus unreliable (Figure 6, Figure A.8b). Our Spatial SP model's results therefore suggest that shared unobservable risk factors among spatially clustered democracies

Figure 6: Survival-Stage Covariates



likely exert a more powerful influence on democratic survival than economic growth.

Finally, we find in our nonspatial Bayesian SP models that *GDP per capita* is positively associated with democratic survival, but this effect is unreliable given that the 95% credible intervals include zero. This corroborates Svolik's (2008) finding for income per capita in the survival stage of his MLE nonspatial SP survival models which, in turn, allows him to challenge other research that emphasizes the positive influence of higher income on democratic survival (Przeworski et al. 2000; Treisman 2018; Gassebner, Lamla and Vreeland 2013). However, the posterior estimate of the effect of per capita income on democratic survival in our Bayesian Spatial SP model is positive and highly reliable (Figure 6 and Figure A.8c). Thus, even though our SP survival model accommodates the mixture of at-risk and fully consolidated democracies like Svolik's (2008) SP Weibull model, the positive association between higher income levels and survival of democracy remains—contrary to Svolik's (2008) finding—robust when we model spatial autocorrelation in a SP survival framework.

Our results remain consistent when using the Cauchy prior. The Bayesian Spatial SP Weibull model's overall fit outperforms the Bayesian Pooled and Exchangeable SP survival models as per Deviance Information Criterion values (see Supplemental Appendix). Autocorrelation plots, convergence tests and stationarity tests show that all parameter estimates have properly converged in our Bayesian Spatial SP Weibull model (see Figures A.9-A.10, Tables A.22-A.23 in Supplemental Appendix).

4 Conclusion

Scholars have increasingly employed nonspatial SP survival models to study a variety of political phenomena (Box-Steffensmeier, Radcliffe and Bartels 2005; Beger, Dorff and Ward 2016; Svolik 2008). Others have focused on modeling spatial autocorrelation in binary and continuous dependent variables (Beck, Gleditsch and Beardsley 2006; Franzese and Hays 2007; Beck, Gleditsch and Beardsley 2006; Betz, Cook and Hollenbach 2019). These two research areas provide rich insights but also talk past each other. This is surprising as many theories of political event processes suggest that spatially clustered units share common factors that influence their probability of being immune from a failure event as well as the time until the event. We take an important step forward by developing a parametric Bayesian Spatial SP survival model that is estimated via MCMC methods. While we build on previous work (Banerjee and Carlin 2004; Darmofal 2009), our primary innovation is that our statistical framework allows scholars to model spatial autocorrelation in the model's split *and* survival stages. Additionally, unlike non-mixture cure models, our model incorporates covariates in not merely the survival-stage but also the split-stage. These features permit researchers to evaluate theoretical claims about how shared experiences among spatially clustered units influence not just the duration of a given political process in these states, but also their odds of being immune to the process-terminating event.

Modeling spatial autocorrelation in a split-population survival framework can lead to dramatically different substantive inferences when survival data that include "immune" and "at-

risk" populations exhibit spatial dependence. Indeed, when we account for spatial autocorrelation in Svolik's (2008) data, we not only find previously overlooked nonmonotonic relationships between key covariates and democratic consolidation, but also numerous split-stage and survival-stage results that lead us to question widely-accepted findings in the democratic survival and consolidation literature. We also reverse Walter's (2015) main finding in her study of post-civil war peace: after accounting for the immune fraction and spatial autocorrelation in her data, increased civil liberties in the aftermath of violent conflict leads to a *quicker* return to internal war.

Apart from statistically accounting for spatial autocorrelation in SP survival data, our model also incorporates time-varying covariates in the split *and* survival stages. This is particularly useful for analyzing panel survival data that exhibit spatial dependence and diverges from extant spatial models that only allow for time-invariant (if any) covariates in the split-stage. Moreover, our extensive MC experiments reveal that the Bayesian Spatial SP survival model provides accurate estimates in the presence of such spatial dependence compared to nonspatial Pooled and Exchangeable SP survival models. We also provide a R-package with functions to estimate all three of the Bayesian SP models examined here. We hope these models will be extended in future work to address spatial autocorrelation in a semi-parametric SP survival setting, though this will be computationally and technically challenging. Empirically, the Bayesian Spatial SP model is widely applicable to a variety of additional substantive research areas, including coup-proofing, alliance formation, and the survival of power-sharing institutions, all of which may involve spatial clustering in the probability of the event occurring and the duration of each process.

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SUPPLEMENTAL APPENDIX for
Bayesian Parametric Spatial *Split-Population* Survival Models for Clustered Political Event Processes

September 22, 2020

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I Full Conditional Distributions and Slice Sampling

As listed in the Table of Contents, we first derive the closed form of the full conditional distributions of $\pi(\Sigma_\beta|\boldsymbol{\beta_i})$, $\pi(\Sigma_\gamma|\boldsymbol{\gamma_i})$ and $\lambda \sim \pi(\lambda|\mathbf{W},\mathbf{V})$ required for Gibbs sampling in Step 1 of the MCMC estimation of our Bayesian Spatial SP (Weibull) model:

1. Σ_β :
$$\begin{aligned}\pi(\Sigma_\beta|\boldsymbol{\beta}) &\propto \pi(\boldsymbol{\beta}|\boldsymbol{\mu}_\beta=\mathbf{0},\Sigma_\beta)\times\pi(\Sigma_\beta)\\ &\propto |\Sigma_\beta|^{-\frac{1}{2}}\exp\Big\{-\frac{1}{2}\Big(\boldsymbol{\beta}'\Sigma_\beta^{-1}\boldsymbol{\beta}\Big)\Big\}\times|\Sigma_\beta|^{-\frac{\nu_1+p_1+1}{2}}\exp\Big\{-\frac{1}{2}\mathrm{tr}\Big(\mathbf{I}_{p_1}\Sigma_\beta^{-1}\Big)\Big\}\\ &= |\Sigma_\beta|^{-\frac{1+\nu_1+p_1+1}{2}}\exp\Big\{-\frac{1}{2}\mathrm{tr}\Big(\boldsymbol{\beta}\boldsymbol{\beta}'\Sigma_\beta^{-1}\Big)\Big\}\times\exp\Big\{-\frac{1}{2}\mathrm{tr}\Big(\mathbf{I}_{p_1}\Sigma_\beta^{-1}\Big)\Big\}\\ &= |\Sigma_\beta|^{-\frac{1+\nu_1+p_1+1}{2}}\exp\Big\{-\frac{1}{2}\mathrm{tr}\Big((\boldsymbol{\beta}\boldsymbol{\beta}'+\mathbf{I}_{p_1})\Sigma_\beta^{-1}\Big)\Big\}\\ &\sim \mathrm{IW}(\boldsymbol{\beta}\boldsymbol{\beta}'+\mathbf{I}_{p_1},\quad 1+\nu_1)\end{aligned}\tag{A.1}$$

2. Σ_γ :
$$\begin{aligned}\pi(\Sigma_\gamma|\boldsymbol{\gamma}) &\propto \pi(\boldsymbol{\gamma}|\boldsymbol{\mu}_\gamma=\mathbf{0},\Sigma_\gamma)\times\pi(\Sigma_\gamma)\\ &\propto |\Sigma_\gamma|^{-\frac{1}{2}}\exp\Big\{-\frac{1}{2}\Big(\boldsymbol{\gamma}'\Sigma_\gamma^{-1}\boldsymbol{\gamma}\Big)\Big\}\times|\Sigma_\gamma|^{-\frac{\nu_1+p_2+1}{2}}\exp\Big\{-\frac{1}{2}\mathrm{tr}\Big(\mathbf{I}_{p_2}\Sigma_\gamma^{-1}\Big)\Big\}\\ &= |\Sigma_\gamma|^{-\frac{1+\nu_2+p_2+1}{2}}\exp\Big\{-\frac{1}{2}\mathrm{tr}\Big(\boldsymbol{\gamma}\boldsymbol{\gamma}'\Sigma_\gamma^{-1}\Big)\Big\}\times\exp\Big\{-\frac{1}{2}\mathrm{tr}\Big(\mathbf{I}_{p_2}\Sigma_\gamma^{-1}\Big)\Big\}\\ &= |\Sigma_\gamma|^{-\frac{1+\nu_2+p_2+1}{2}}\exp\Big\{-\frac{1}{2}\mathrm{tr}\Big((\boldsymbol{\gamma}\boldsymbol{\gamma}'+\mathbf{I}_{p_2})\Sigma_\gamma^{-1}\Big)\Big\}\\ &\sim \mathrm{IW}(\boldsymbol{\gamma}\boldsymbol{\gamma}'+\mathbf{I}_{p_2},\quad 1+\nu_2)\end{aligned}\tag{A.2}$$

3. The full conditional distribution of $\lambda \sim \pi(\lambda|\mathbf{W},\mathbf{V})$ in our model that incorporates spatial individual frailty terms is:

$$\begin{aligned}\pi(\lambda|\mathbf{W},\mathbf{V}) &\propto \pi(\mathbf{W}|\lambda)\times\pi(\mathbf{V}|\lambda)\times\pi(\lambda)\\ &\propto \Big(\prod_{i=1}^{N}\pi(W_i|\lambda)\Big)\times\Big(\prod_{i=1}^{N}\pi(V_i|\lambda)\Big)\times\pi(\lambda)\\ &\propto \Big(\prod_{i=1}^{N}N(\overline{W_i},1/(\lambda m_i))\times N(\overline{V_i},1/(\lambda m_i))\Big)\times\mathrm{Gamma}(a_\lambda,b_\lambda)\\ &\propto \lambda^{\frac{N}{2}}\lambda^{\frac{N}{2}}\exp\{-\sum_{i=1}^{N}\frac{\lambda m_i}{2}\Big((W_i-\overline{W_i})^2+(V_i-\overline{V_i})^2\Big)\}\times\lambda^{a_\lambda-1}\exp\{-b_\lambda\lambda\}\\ &\propto \lambda^{N+a_\lambda-1}\exp\{-(\sum_{i=1}^{N}\frac{m_i}{2}\Big((W_i-\overline{W_i})^2+(V_i-\overline{V_i})^2\Big)+b_\lambda)\lambda\}\\ &\sim \mathrm{Gamma}(N+a_\lambda,\quad\sum_{i=1}^{N}\frac{m_i}{2}\Big((W_i-\overline{W_i})^2+(V_i-\overline{V_i})^2\Big)+b_\lambda)\end{aligned}\tag{A.3}$$

We next describe the slice sampling algorithm for estimating $\boldsymbol{\beta}$,$\boldsymbol{\gamma}, \mathbf{W}, \mathbf{V}$, ρ. Following current practice in Bayesian mixture survival models, we use the univariate slice sampler with step-out and shrinkage in Step 2 (Neal 2003; Mahani, Sharabiani and Mahani (2016)), where the closed form of the full conditional distribution is intractable. Below are the steps to perform slice sampling for $\boldsymbol{\beta}$ (slice sampling for $\boldsymbol{\gamma}$, $\mathbf{W}, \mathbf{V}$, ρ are done in exactly the same manner and is hence not described here to avoid repetition). For $\boldsymbol{\beta}_p$, $p = \{1, ..., P\}$,

- **Step 0.** Choose an arbitrary starting point β_{p_0} and size of the slice w, and set $i = 0$.
- **Step 1.** Draw y from Uniform$(0, f(\beta_{p_0}))$ defining slice $S = \{\beta_p : y < f(\beta_p)\}$, where

$$\begin{aligned} f(\beta_p) &\propto \pi(\beta_p|\boldsymbol{\beta}_{-p}, \mathbf{C}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \mathbf{t0}, \boldsymbol{\gamma}) \qquad \text{if exponential} \\ &\propto \pi(\beta_p|\boldsymbol{\beta}_{-p}, \mathbf{C}, \mathbf{X}, \mathbf{Z}, \mathbf{t}, \mathbf{t0}, \boldsymbol{\gamma}, \rho) \qquad \text{if Weibull.} \end{aligned} \tag{A.4}$$

- **Step 2.** Find an interval, $I = (L, R)$, around β_{p_0} that contains all, or much, of the slice, where the initial interval is determined as:

$$u \sim \text{Uniform}(0, w), \qquad L = \beta_{p_0} - u, \qquad R = \beta_{p_0} + (w - u) \tag{A.5}$$

  and expand the interval until its ends are outside the slice or until the limit on steps (limit on steps $= m$) is reached ("stepping-out" procedure), by comparing y and $(f(L), f(R))$:

$$\begin{aligned} &J = \text{Floor}(\text{Uniform}(0, m)) \\ &K = (m - 1) - J \\ &\text{Repeat while } J > 0 \text{ and } y < f(L) : L = L - w, J = J - 1 \\ &\text{Repeat while } K > 0 \text{ and } y < f(R) : R = R + w, K = K - 1 \end{aligned} \tag{A.6}$$

- **Step 3.** Draw a new point β_{p_1} from the part of the slice within this interval I, and shrink the interval on each rejection ("shrinkage" procedure):

$$\begin{aligned} \text{Repeat:} \qquad & \beta_{p1} \sim \text{Uniform}(L, R) \\ & \text{if } y < f(\beta_{p_1}), \text{ accept } \beta_{p_1} \text{ and exit loop} \\ & \text{if } \beta_{p_1} < \beta_{p_0}, \text{ then } L = \beta_{p_1} \\ & \qquad\qquad \text{else } R = \beta_{p_1}. \end{aligned} \tag{A.7}$$

- **Step 4.** Set $i = i + 1$, $\beta_{p_0} = \beta_{p_1}$, and go to Step 1.
- **Step 5.** After N iterations, summarize the parameter estimates using all sampled values (via, e.g., credible intervals or posterior means).

II CAR Prior and Probability Distributions

Recall the split stage frailties vector $\mathbf{V} = \{V_1, ..., V_N\}$ and the survival stage frailties vector $\mathbf{W} = \{W_1, ..., W_N\}$. We employ separate CAR priors on $\mathbf{V}$ and $\mathbf{W}$, which implies the following CAR model structure for our Bayesian Spatial SP framework:

$$\begin{aligned} \mathbf{W}|\lambda &\sim \text{CAR}(\lambda), \\ \mathbf{V}|\lambda &\sim \text{CAR}(\lambda), \end{aligned} \tag{A.8}$$

where CAR denotes the conditional autoregressive structure (Besag, York and Mollié 1991), while λ refers to the precision of the random effects distribution. By using λ, we reflect a conditionally autoregressive prior that incorporates neighbor (and non-neighbor) definitions between units in the model's split and survival stages via the aforementioned adjacency matrix. The most commonly used form of the CAR (λ) prior (Bernardinelli and Montomoli 1992; Banerjee, Wall and Carlin 2003; Darmofal 2009) for $\mathbf{W}$ has joint distribution proportional to

$$\lambda^{1/2} \exp\left[-\frac{\lambda}{2} \sum_{i \ adj \ i'} (W_i - W_{i'})^2\right] \propto \lambda^{1/2} \exp\left[-\frac{\lambda}{2} \sum_{i=1}^{I} m_i W_i (W_i - \overline{W}_i)\right] \tag{A.9}$$

The CAR (λ) prior for $\mathbf{V}$ has a joint distribution proportional to

$$\lambda^{1/2} \exp\left[-\frac{\lambda}{2} \sum_{i \ adj \ i'} (V_i - V_{i'})^2\right] \propto \lambda^{1/2} \exp\left[-\frac{\lambda}{2} \sum_{i=1}^{I} m_i V_i (V_i - \overline{V}_i)\right] \tag{A.10}$$

In (A.9), I is the number of units in the data, $i \ adj \ i'$ indicates that units i and i' are adjacent (where information about i and i' being neighbors or non-neighbors is reflected in the adjacency matrix $\mathbf{A} = \{a_{ii'}\}$), $\overline{W}_i$ is the average of the neighboring $W_{i' \neq i}$,[1] and m_i is the number of these adjacencies (Banerjee *et al*, 2003; Darmofal 2009, 2015). The exact same definitions apply for equation (A.10) associated with V_i. Given (A.9)-(A.10), the conditional distribution of the spatial frailties that results from the CAR prior in the Spatial SP survival model is,

$$\begin{aligned} W_i|W_{i' \neq i} &\sim N(\overline{W}_i, 1/(\lambda m_i)), \\ V_i|V_{i' \neq i} &\sim N(\overline{V}_i, 1/(\lambda m_i)), \end{aligned} \tag{A.11}$$

[1] This can be stated more formally as $\overline{W}_i = m_i^{-1} \sum_{j \text{ adj } i} W_j$.

The expression in (A.11) in each case produces "a conditional distribution for the random effects that is normally distributed with a conditional mean equal to the average of the random effects for neighbors of i and a conditional variance that is inversely proportional to the number of units neighboring i" (Darmofal 2009: 246; Thomas et al. 2004).

III Time-Varying (Spatial) SP Survival Model

Log-Likelihood and Properties

To develop the log likelihood function of our general parametric SP survival model with time-varying covariates and unit-specific frailties that are spatially *independent*, we first re-define our survival data with unique "entry time" duration as $t0$ and "exit time" as duration t for each period at which an observation is observed. As described in the text, $t0_{ij}$ denotes observation i's elapsed time since inception until the beginning of time period j and t_{ij} denotes the elapsed time since that observation's inception until the end of period j. An observation's status at time t_{ij} is then coded as censored $\tilde{C}_{ij} = 0$ or as having failed (i.e., having ended) at time t_{ij} $\tilde{C}_i = 1$. For t, the PDF ($f(t)$), CDF ($\Pr(T_i \leq t) \equiv F(t)$), probability of survival ($S(t)$), and hazard of an event ($h(t) = \frac{f(t)}{S(t)}$) remain as defined above. However, we must now also define the probability of survival up until period j as $S(t0) = 1 - F(t0)$, where $F(t0) = \int_0^{t0} f(t0)$. With $S(t0)$ defined, we can extend the log-likelihood function of the standard general parametric survival model to accommodate time varying covariates $\mathbf{X}$ and associated parameter vectors of β by conditioning an observation's hazard and survival probability for time t upon its probability of survival until $t0$. Doing so leads to,

$$lnL = \sum_{i=1}^{N} \left\{ \tilde{C}_i \ln \left[\frac{f(t_{ij}|\mathbf{X}_{ij}, \beta)}{S(t0_{ij}|\mathbf{X}_{ij}, \beta)} \right] + (1 - \tilde{C}_i) \ln \left[\frac{S(t_{ij}|\mathbf{X}_{ij}, \beta)}{S(t0_{ij}|\mathbf{X}_{ij}, \beta)} \right] \right\} \tag{A.12}$$

which can be re-defined as

$$lnL = \sum_{i=1}^{N} \left\{ \tilde{C}_i \ln \left[f(t_{ij}|\mathbf{X}_{ij}, \beta) \right] + (1 - \tilde{C}_i) \ln \left[S(t_{ij}|\mathbf{X}_{ij}, \beta) \right] - ln(S(t0_{ij}|\mathbf{X}_{ij}, \beta)) \right\} \tag{A.13}$$

Recall that in the SP "exchangeable" survival model in the text, the nonspatial i.i.d. random effects are: $V_i \sim N(0, \sigma^2)$ in the split-stage in equation (1) and $W_i \sim N(0, \sigma^2)$ in the survival

stage in equation (2). Using equations (1)-(2) from the text and the procedure leading to equations (A.12)-(A.13), we can extend the log-likelihood function in equation (3) in the text to define the log-likelihood of the general parametric SP survival model with time-varying covariates and nonspatial i.i.d. random effects (frailties) as,

$$\ln L = \sum_{i=1}^{N} \left\{ \widetilde{C}_{ij} \ln \left[(1-\alpha_{ij}) \frac{f(t_{ij}|\mathbf{X}_{ij}, \boldsymbol{\beta}, W_i)}{S(t0_{ij}|\mathbf{X}_{ij}\boldsymbol{\beta}, W_i)} \right] + (1-\widetilde{C}_{ij}) \ln \left[\alpha_{ij} + (1-\alpha_{ij}) \frac{S(t_{ij}|\mathbf{X}_{ij}, \boldsymbol{\beta}, W_i)}{S(t0_{ij}|\mathbf{X}_{ij}\boldsymbol{\beta}, W_i)} \right] \right\}, \tag{A.14}$$

where $\alpha_{ij} = \dfrac{\exp(\mathbf{Z}_{ij}\boldsymbol{\gamma} + V_i)}{1 + \exp(\mathbf{Z}_{ij}\boldsymbol{\gamma} + V_i)}$. Expression (A.14) is stated in equation (4) in the text. Consider the vectors $\mathbf{V} = \{V_1, ..., V_N\}$ and $\mathbf{W} = \{W_1, ..., W_N\}$ defined earlier. Then, following the steps described above and from (A.14), the log-likelihood of the general parametric SP survival model with spatially dependent random effects in the split and survival stage is:

$$\ln L = \sum_{i=1}^{N} \left\{ \widetilde{C}_{ij} \ln \left[(1-\alpha_{ij}) \frac{f(t_{ij}|\mathbf{X}_{ij}, \boldsymbol{\beta}, \mathbf{W})}{S(t0_{ij}|\mathbf{X}_{ij}\boldsymbol{\beta}, \mathbf{W})} \right] + (1-\widetilde{C}_{ij}) \ln \left[\alpha_{ij} + (1-\alpha_{ij}) \frac{S(t_{ij}|\mathbf{X}_{ij}, \boldsymbol{\beta}, \mathbf{W})}{S(t0_{ij}|\mathbf{X}_{ij}\boldsymbol{\beta}, \mathbf{W})} \right] \right\} \tag{A.15}$$

where $\alpha_{ij} = \dfrac{\exp(\mathbf{Z}_{ij}\boldsymbol{\gamma} + \mathbf{V})}{1 + \exp(\mathbf{Z}_{ij}\boldsymbol{\gamma} + \mathbf{V})}$. For the Spatial SP Weibull model, we use

$$\begin{aligned}
f(t_{ij}|\rho, \mathbf{X}_{ij}, \boldsymbol{\beta}, \mathbf{W}) &= \rho(\exp(\mathbf{X}_{ij}\boldsymbol{\beta} + \mathbf{W}))(\exp(\mathbf{X}_{ij}\boldsymbol{\beta} + \mathbf{W})t_{ij})^{\rho-1} \exp(-(\exp(\mathbf{X}_{ij}\boldsymbol{\beta} + \mathbf{W})t_{ij})^{\rho} \\
S(t_{ij}|\rho, \mathbf{X}_{ij}, \boldsymbol{\beta}, \mathbf{W}) &= \exp(-(\exp(\mathbf{X}_{ij}\boldsymbol{\beta} + \mathbf{W})t_{ij})^{\rho} \\
S(t0_{ij}|\rho, \mathbf{X}_{ij}, \boldsymbol{\beta}, \mathbf{W}) &= \exp(-(\exp(\mathbf{X}_{ij}\boldsymbol{\beta} + \mathbf{W})t0_{ij})^{\rho} \qquad (A.16)
\end{aligned}$$

Suppose that the survival time t has a Log-Logistic distribution. Then the density function and the survival function for the log-likelihood in (A.15) are:

$$f(t_{ij}|\rho, \mathbf{X}_{ij}, \boldsymbol{\beta}, \mathbf{W}) = \frac{\rho(\exp(\mathbf{X}_{ij}\boldsymbol{\beta} + \mathbf{W}))(\exp(\mathbf{X}_{ij}\boldsymbol{\beta} + \mathbf{W})t_{ij})^{\rho-1}}{(1 + \exp(-(\exp(\mathbf{X}_{ij}\boldsymbol{\beta} + \mathbf{W})t_{ij})^{\rho})^2}$$

$$S(t_{ij}|\rho, \mathbf{X}_{ij}, \boldsymbol{\beta}, \mathbf{W}) = \frac{1}{1 + (\exp(\mathbf{X}_{ij}\boldsymbol{\beta} + \mathbf{W})t_{ij})^{\rho}} \tag{A.17}$$

In the Log-Logistic case, $S(t0_{ij}|\mathbf{X}_{ij}\boldsymbol{\beta}, \mathbf{W}) = \dfrac{1}{1 + (\exp(\mathbf{X}_{ij}\boldsymbol{\beta} + \mathbf{W})t0_{ij})^{\rho}}$ and $\alpha_{ij} = \dfrac{\exp(\mathbf{Z}_{ij}\boldsymbol{\gamma} + \mathbf{V})}{1 + \exp(\mathbf{Z}_{ij}\boldsymbol{\gamma} + \mathbf{V})}$. We adopt the CAR prior $\mathbf{V}|\lambda \sim \text{CAR}(\lambda)$ for $\mathbf{V}$ in the split stage and $\mathbf{W}|\lambda \sim \text{CAR}(\lambda)$ for $\mathbf{W}$ in the survival stage of the Bayesian Spatial SP Weibull and Log-Logistic models. In the time-varying Bayesian Spatial SP survival models, the CAR(λ) prior for $\mathbf{V}$ and $\mathbf{W}$ has a joint distribution stated in (A.10) and (A.9) respectively.

IV Monte Carlo Results

We first describe below how we generate the adjacency matrix $\mathbf{A} = \{a_{ii'}\}$ for our Monte Carlo (MC) experiments which was briefly discussed in our paper. To start, recollect that the adjacency matrix $\mathbf{A} = \{a_{ii'}\}$ in the simulated data used for our MC experiments incorporates information about *spatial adjacency*—specifically, geographic contiguity—between units in the simulated data whose unit-specific frailty is given by V_i in $\mathbf{V}$ for the split stage and W_i in $\mathbf{W}$ for the survival stage. To generate $\mathbf{A}$, we consider a hypothetical space where there are five areal (i.e., spatial) units (e.g., countries), where each spatial unit has at least one "neighbor" to which it is adjacent (e.g., shares a border). This results in a 5×5 symmetric and positive definite $\boldsymbol{A}$ matrix in which,

$$a_{ii'} = \begin{cases} 1 & \text{if the } i\text{th and the } i'\text{th units are adjacent; and} \\ 0 & \text{if the } i\text{th and the } i'\text{th units are not adjacent.} \end{cases} \tag{A.18}$$

The frailty for each unit in this 5×5 adjacency matrix $\boldsymbol{A}$ is given by $\mathbf{V} = \{V_1 ... \ V_5\}$ in the split stage and $\mathbf{W} = \{W_1 ... \ W_5\}$ in the survival stage. As discussed in the paper, we use the information about neighboring versus non-neighboring units in this adjacency matrix to vary the proportion of units that exhibit spatially dependent frailties in the simulated data. For instance, in MC experiments 1, 2, 5 and 6, we assume that the share of units that have spatially dependent frailties is 40%. We set this 40% level for the experiments by assigning a value of 1 to each of the following elements $a_{ii'}$ in matrix $\boldsymbol{A}$: $a_{12} = a_{21}$, $a_{23} = a_{32}$, $a_{35} = a_{53}$ and $a_{14} = a_{41}$. All other cells (including diagonal cell entries) are assigned a value of 0. Hence the 5×5 symmetric and positive definite adjacency matrix in this case is given by (A.19):

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{A.19}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \tag{A.20}$$

Suppose that we want to increase the proportion of units with spatially dependent frailties

to 60%. Building on the previous scenario, we can set this 60% level in the 5×5 adjacency matrix by assigning a value of 1 to each of the following elements $a_{ii'}$ in $\mathbf{A}$: $a_{12} = a_{21}$, $a_{23} = a_{32}$, $a_{35} = a_{53}$, $a_{14} = a_{41}$, $a_{42} = a_{24}$ and $a_{54} = a_{45}$. All other cells are assigned a value of 0. The 5×5 symmetric and positive definite adjacency matrix in this latter case is expressed in (A.20). The adjacency matrices in (A.19)–(A.20) can be adjusted to account for greater or lower proportion of units with spatially dependent frailties in the simulated data for MC experiments.

We turn to present the tables and figures obtained from our 8 MC experiments discussed in our paper's "Monte Carlo Simulations" section. This list of tables and figures are: Tables A.1 to A.3 and Figure A.1 from Experiment 1; Tables A.4 to A.7 from Experiment 2;[2] Tables A.8 to A.11 and Figure A.2 from Experiment 3; Tables A.12 to A.15 and Figure A.3 from Experiment 4; Tables A.16 and A.17 from Experiment 5; Tables A.18 and A.19 from Experiment 6; Table A.20 from Experiment 7; and Table A.21 from Experiment 8.

Each of these eight MC experiments evaluate the performance of the following three models: the Bayesian Pooled SP, Exchangeable SP and Spatial SP Weibull model. Recall that our primary MC experiments simulate a SP Weibull distributed outcome variable (with immune fraction α) that exhibits spatially dependent frailties across neighboring units in both the split and survival stage. Experiment 1 evaluates the performance of these three models when the immune fraction is $\alpha = 25\%$ and the proportion of units with spatially dependent frailties is 40%. The results for $\hat{\beta}$ and $\widehat{\gamma}$ from Experiment 1 are presented in Tables 1 and 2 respectively. The results in Table A.1 show that as N increases, the $\hat{\beta}'$s from the Bayesian Spatial SP Weibull model converge to their true values with negligible RMSEs and high 95% empirical CPs in the 85%-92% range. In contrast, the Bayesian Pooled and Exchangeable SP Weibull models severely underestimate β_0 and β_1 no matter the size of N. Further, the two nonspatial models produce RMSEs for their $\hat{\beta}'$s that are 3-30 times larger than the Bayesian Spatial SP Weibull model as well as poor CPs (0-20%). The $\hat{\gamma}$ results from Experiment 1 in Table A.2 reveal that the Bayesian Spatial SP Weibull model retrieves the true parameter values of $\widehat{\gamma}_0$, $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ with high 95% CPs and negligible RMSEs for all sizes of N. While the $\hat{\gamma}_0$'s from the

[2]The main results from this MC experiment are illustrated in Figure 1 in the paper.

nonspatial Bayesian Pooled and Exchangeable SP Weibull models perform well, both models substantially underestimate or overestimate $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ that also exhibit high RMSEs and CPs of 0.

Table A.1: Markov Chain Monte Carlo (MCMC) β Estimates for Experiment 1

	Experiment 1: Spatial SP Weibull D.G.P. – β Estimates							
#Obs.	Model	$\hat{\beta}_0$	*MCSE*($\hat{\beta}_0$)	*RMSE*($\hat{\beta}_0$)	*CP*($\hat{\beta}_0$)	$\hat{\beta}_1$	*MCSE*($\hat{\beta}_1$)	*RMSE*($\hat{\beta}_1$)	*CP*($\hat{\beta}_1$)
1000	Bayes Pooled SP	0.546	0.016	0.454	0.010	0.075	0.008	1.925	0
	Bayes Exch. SP	0.589	0.014	0.411	0.030	0.140	0.008	1.860	0
	Bayes Spatial SP	0.936	0.012	0.114	0.880	1.912	0.005	0.135	0.880
1500	Bayes Pooled SP	0.634	0.010	0.366	0	0.029	0.005	1.971	0
	Bayes Exch. SP	0.816	0.009	0.184	0.360	-0.031	0.004	2.031	0
	Bayes Spatial SP	0.958	0.005	0.079	0.860	1.868	0.006	0.175	0.850
2000	Bayes Pooled SP	0.685	0.011	0.315	0.060	-0.037	0.005	2.037	0
	Bayes Exch. SP	0.729	0.026	0.271	0.200	-0.059	0.003	2.059	0
	Bayes Spatial SP	0.998	0.006	0.043	0.910	1.970	0.008	0.062	0.920

Note: True parameter values are $\beta_0 = 1$ and $\beta_1 = 2$.

Table A.2: Markov Chain Monte Carlo (MCMC) γ Estimates for Experiment 1

	Experiment 1: Spatial SP Weibull D.G.P. – γ Estimates											
#Obs.	Model	$\hat{\gamma}_0$	*MCSE*($\hat{\gamma}_0$)	*RMSE*($\hat{\gamma}_0$)	*CP*($\hat{\gamma}_0$)	$\hat{\gamma}_1$	*MCSE*($\hat{\gamma}_1$)	*RMSE*($\hat{\gamma}_1$)	*CP*($\hat{\gamma}_1$)	$\hat{\gamma}_2$	*MCSE*($\hat{\gamma}_2$)	*RMSE*($\hat{\gamma}_2$)	*CP*($\hat{\gamma}_2$)
1000	Bayes Pooled SP	-3.026	0.153	0.399	0.970	-0.038	0.022	0.962	0	-2.433	0.052	5.933	0
	Bayes Exch. SP	-3.738	0.208	0.795	0.770	-0.016	0.030	0.984	0	-3.391	0.113	6.891	0
	Bayes Spatial SP	-2.812	0.133	0.512	0.820	-0.925	0.031	0.154	0.860	3.071	0.100	0.598	0.770
1500	Bayes Pooled SP	-3.130	0.103	0.281	0.980	0.002	0.016	1.002	0	-2.503	0.038	6.003	0
	Bayes Exch. SP	-3.200	0.099	0.328	0.950	-0.032	0.018	0.968	0	-3.254	0.063	6.754	0
	Bayes Spatial SP	-2.947	0.098	0.338	0.930	-0.945	0.022	0.136	0.890	3.124	0.078	0.567	0.830
2000	Bayes Pooled SP	-3.256	0.102	0.330	0.900	-0.028	0.012	0.972	0	-2.202	0.039	5.702	0
	Bayes Exch. SP	-3.255	0.102	0.340	0.920	-0.019	0.015	0.981	0	-3.169	0.086	6.669	0
	Bayes Spatial SP	-2.886	0.077	0.290	0.880	-0.950	0.017	0.085	0.910	3.312	0.071	0.294	0.840

Note: True parameter values are $\gamma_0 = -3$, $\gamma_1 = -1$, and $\gamma_2 = 3.5$.

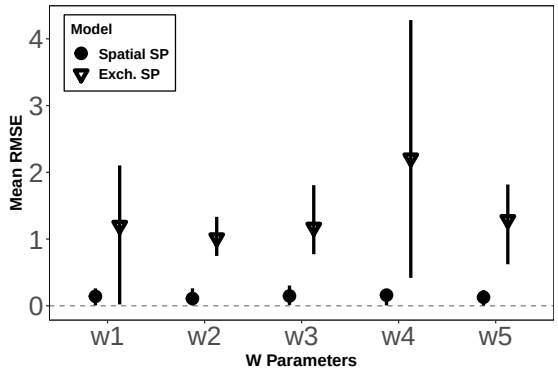
Next, the Experiment 1 results for $\{\widehat{w}_1 ... \widehat{w}_5\}$ and $\{\widehat{v}_1 ... \widehat{v}_5\}$ presented in Table A.3 and Figure A.1 show that the Bayesian Spatial SP Weibull model retrieves the true values of *each* $\{\widehat{w}_1 ... \widehat{w}_5\}$ in $\mathbf{W}$ and *each* $\{\widehat{v}_1 ... \widehat{v}_5\}$ in $\mathbf{V}$ with negligible RMSEs and high CPs in the 72-94% range. However, the values of $\{\widehat{w}_1 ... \widehat{w}_5\}$ and $\{\widehat{v}_1 ... \widehat{v}_5\}$ from the Bayesian Exchangeable SP model deviate substantially from their true values and have high RMSEs and poor CPs. Experiment 1's results thus confirm that the Bayesian Spatial SP Weibull model outperforms the nonspatial Pooled and Exchangeable SP Weibull model when the true d.g.p exhibits spatial dependence between units in both stages.

Table A.3: Markov Chain Monte Carlo (MCMC) $\boldsymbol{W}$ and $\boldsymbol{V}$ Estimates for Experiment 1

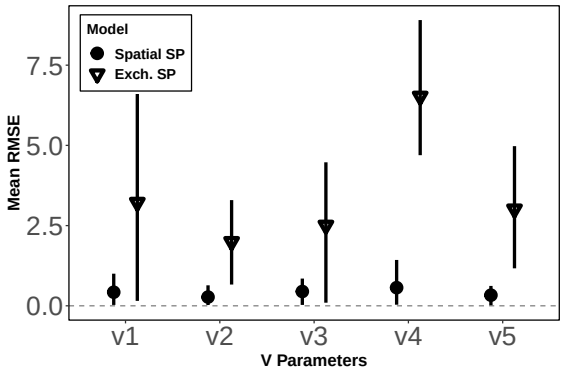
#Obs.	Experiment 1: Spatial SP Weibull D.G.P. – $\boldsymbol{W}$ estimates																			
	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate		$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_5$	$\hat{w}_5$	$\hat{w}_5$	$\hat{w}_5$
1000	True Value	2.07	–	–	–	-1.804	–	–	–	-1.62	–	–	–	0.65	–	–	–	0.704	–	–	–
	Bayes Exch. SP	0.086	0.027	1.984	0	-0.626	0.037	1.178	0	-0.703	0.04	0.917	0	1.303	0.04	0.652	0.08	-0.06	0.032	0.763	0
	Bayes Spatial SP	1.989	0.03	0.158	0.83	-1.733	0.041	0.136	0.85	-1.655	0.059	0.154	0.84	0.736	0.028	0.172	0.83	0.663	0.032	0.112	0.84
1500	True Value	-1.803	–	–	–	1.159	0	0	0	0.579	–	–	–	-1.714	–	–	–	1.779	–	–	–
	Bayes Exch. SP	-0.276	0.036	1.527	0	0.313	0.028	0.846	0	-0.351	0.031	0.93	0	0.243	0.038	1.957	0	0.07	0.029	1.709	0
	Bayes Spatial SP	-1.736	0.038	0.159	0.79	1.109	0.027	0.093	0.78	0.506	0.037	0.178	0.79	-1.568	0.031	0.222	0.76	1.688	0.03	0.173	0.77
2000	True Value	0.188	–	–	–	1.399	–	–	–	2.667	—	–	–	-6.116	–	–	–	1.862	–	–	–
	Bayes Exch. SP	0.26	0.042	0.098	0.93	0.376	0.04	1.024	0	0.988	0.04	1.678	0	-2.089	0.147	4.027	0	0.466	0.037	1.397	0
	True Value	-2.58	–	–	–	-1.762	–	–	–	2.073	–	–	–	-0.919	–	–	–	3.188	–	–	–
	Bayes Spatial SP	-2.526	0.038	0.109	0.72	-1.75	0.034	0.099	0.72	2.021	0.034	0.111	0.77	-0.901	0.03	0.085	0.82	3.156	0.029	0.097	0.75
#Obs.	Experiment 1: Spatial SP Weibull D.G.P. – $\boldsymbol{V}$ estimates																			
	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$
1000	True Value	2.407	–	–	–	0.499	–	–	–	1.983	–	–	–	-5.315	–	–	–	0.426	–	–	–
	Bayes Exch. SP	-3.58	0.179	5.987	0	2.676	0.173	2.177	0.06	2.356	0.175	0.574	0.85	-0.191	0.084	5.124	0	-1.261	0.126	1.687	0.08
	Bayes Spatial SP	1.861	0.089	0.606	0.75	0.639	0.07	0.297	0.89	1.773	0.092	0.391	0.89	-4.662	0.195	0.833	0.80	0.389	0.059	0.281	0.92
1500	True Value	-0.621	–	–	–	-0.558	–	–	–	2.908	–	–	–	-3.485	–	–	–	1.757	–	–	–
	Bayes Exch. SP	2.664	0.097	3.285	0	-1.624	0.071	1.067	0.08	-1.148	0.076	4.056	0	2.915	0.098	6.401	0	-2.807	0.087	4.564	0
	Bayes Spatial SP	-0.598	0.051	0.372	0.88	-0.495	0.047	0.245	0.91	2.531	0.076	0.515	0.81	-3.006	0.096	0.612	0.87	1.568	0.062	0.425	0.90
2000	True Value	-0.457	–	–	–	0.76	–	–	–	-0.591	0	–	–	-0.005	–	–	–	0.293	–	–	–
	Bayes Exch. SP	-0.09	0.062	0.39	0.77	-1.989	0.089	2.749	0	-3.483	0.112	2.892	0	8.054	0.413	8.059	0	-2.492	0.091	2.785	0
	True Value	-2.088	–	–	–	-1.829	–	–	–	4.147	–	–	–	-2.225	–	–	–	1.995	–	–	–
	Bayes Spatial SP	-1.975	0.063	0.294	0.94	-1.615	0.057	0.277	0.85	3.805	0.094	0.433	0.76	-2.122	0.057	0.253	0.91	1.907	0.059	0.291	0.89

Figure A.1: Average RMSE for $\boldsymbol{W}$ and $\boldsymbol{V}$ Estimates in Experiment 1

(a) Average RMSE for $\boldsymbol{W}$



(b) Average RMSE for $\boldsymbol{V}$



In Experiment 2, we first assess the β and γ MC results from the three above mentioned Bayesian SP Weibull models using the same d.g.p. as Experiment 1 but with one key difference—we increase the immune fraction from $\alpha = 25\%$ to $33\%, 40\%, 48\%$ and 60%. The $\hat{\beta}$ and $\widehat{\gamma}$ from Experiment 2, which are reported in Tables A.4 and A.5, strongly favor the Bayesian Spatial SP Weibull model over the Bayesian Pooled SP and Exchangeable SP Weibull models when α is increased to above 25% for all sample sizes. Figures 1a-1b further illustrate that the averaged $\hat{\beta}$ and $\hat{\gamma}$ RMSE values (averaged across $N = 1,000$, $N = 1,500$, and $N = 2,000$) from the Bayesian Spatial SP Weibull model remain not just consistently close to 0 but are about 15-20 times smaller than those from the Bayesian Pooled and Exchangeable SP Weibull models when the immune fraction increases from 25% to 60%. The MC Experiment 2 results for $\{\widehat{w}_1 ... \widehat{w}_5\}$ in $\mathbf{W}$ and $\{\widehat{v}_1 ... \widehat{v}_5\}$ in $\mathbf{V}$ that are discussed in the text are presented here in Tables A.6 and A.7 respectively.

Experiment 3 begins by comparing the β and γ parameters retrieved from each of the three Bayesian SP Weibull models using the same d.g.p. from Experiment 1 (where $\alpha = 25\%$) but under circumstances in which we increase the proportion of *adjacent* units with spatially dependent split and survival stage frailties from 30% to 40%, 60% and then 80%. The MC results from this experiment for $N = 1,000$, $N = 1,500$, and $N = 2,000$ reveals that the $\hat{\beta}'$s and $\widehat{\gamma}$'s from the Bayesian Spatial SP Weibull converge to their exact true theoretical value with high CPs (range from 85%–93%) as the share of units that exhibit spatially dependent split and survival stage frailties grow from 30% to 60% and then to 80%. By contrast, the $\hat{\beta}'$s and $\widehat{\gamma}$'s from the Bayesian Pooled SP and Exchangeable SP Weibull models deviate substantially from their true theoretical values and exhibits CPs that range from 0% to 12% when the share of contiguous areal units in the simulated data increases from 30% to 80% (Table A.8-A.9, Figure 2.A). Further, as the proportion of contiguous units increases, we find in Figures A.2 and A.11 that the Bayesian Spatial SP Weibull model's $\hat{\beta}$ and $\widehat{\gamma}$ RMSEs remain effectively flat and close to 0, whereas those of the two nonspatial Bayesian SP Weibull models increase dramatically.

Table A.4: Markov Chain Monte Carlo (MCMC) β Estimates for Experiment 2 (Varying Immune Fraction, Spatial SP Weibull D.G.P.)

	Experiment 2: Immune Fraction (α) = 33%							
#Obs.	Model	$\hat{\beta}_0$	$MCSE(\hat{\beta}_0)$	$RMSE(\hat{\beta}_0)$	$CP(\hat{\beta}_0)$	$\hat{\beta}_1$	$MCSE(\hat{\beta}_1)$	$RMSE(\hat{\beta}_1)$	$CP(\hat{\beta}_1)$
1000	Bayes Pooled SP	0.196	0.017	0.804	0	0.274	0.009	1.726	0
	Bayes Exch. SP	0.852	0.028	0.161	0.90	-0.116	0.009	2.116	0
	Bayes Spatial SP	0.999	0.011	0.058	0.91	2.002	0.005	0.042	0.93
1500	Bayes Pooled SP	0.225	0.016	0.775	0	0.256	0.009	1.744	0
	Bayes Exch. SP	0.389	0.01	0.611	0	0.135	0.005	1.865	0
	Bayes Spatial SP	0.998	0.006	0.054	0.91	1.997	0.006	0.039	0.93
2000	Bayes Pooled SP	0.352	0.012	0.648	0	0.276	0.007	1.724	0
	Bayes Exch. SP	0.571	0.047	0.429	0.01	-0.133	0.004	2.133	0
	Bayes Spatial SP	0.996	0.007	0.045	0.91	2.005	0.007	0.036	0.92
	Experiment 2: Immune Fraction (α) = 40%							
#Obs.	Model	$\hat{\beta}_0$	$MCSE(\hat{\beta}_0)$	$RMSE(\hat{\beta}_0)$	$CP(\hat{\beta}_0)$	$\hat{\beta}_1$	$MCSE(\hat{\beta}_1)$	$RMSE(\hat{\beta}_1)$	$CP(\hat{\beta}_1)$
1000	Bayes Pooled SP	-0.002	0.054	1.002	0.02	0.071	0.027	1.929	0
	Bayes Exch. SP	0.172	0.01	0.828	0	0.156	0.006	1.844	0
	Bayes Spatial SP	1.039	0.03	0.103	0.90	1.992	0.007	0.054	0.95
1500	Bayes Pooled SP	0.09	0	0.91	0	0.089	0	1.911	0
	Bayes Exch. SP	0.505	0.011	0.495	0	-0.012	0.004	2.012	0
	Bayes Spatial SP	0.969	0.01	0.091	0.90	1.967	0.01	0.082	0.93
2000	Bayes Pooled SP	0.26	0	0.74	0	-0.063	0	2.063	0
	Bayes Exch. SP	0.306	0.01	0.694	0	-0.022	0.004	2.022	0
	Bayes Spatial SP	0.991	0.008	0.049	0.93	1.995	0.015	0.048	0.86
	Experiment 2: Immune Fraction (α) = 48%							
#Obs.	Model	$\hat{\beta}_0$	$MCSE(\hat{\beta}_0)$	$RMSE(\hat{\beta}_0)$	$CP(\hat{\beta}_0)$	$\hat{\beta}_1$	$MCSE(\hat{\beta}_1)$	$RMSE(\hat{\beta}_1)$	$CP(\hat{\beta}_1)$
1000	Bayes Pooled SP	-0.054	0.043	1.054	0	0.006	0.023	1.994	0
	Bayes Exch. SP	0.615	0.016	0.385	0.04	-0.307	0.005	2.307	0
	Bayes Spatial SP	0.974	0.046	0.174	0.91	1.949	0.009	0.093	0.95
1500	Bayes Pooled SP	0.191	0.001	0.809	0	-0.011	0	2.011	0
	Bayes Exch. SP	0.075	0.007	0.925	0	0.012	0.003	1.988	0
	Bayes Spatial SP	0.907	0.015	0.147	0.93	1.894	0.014	0.158	0.94
2000	Bayes Pooled SP	0.094	0	0.906	0	-0.056	0	2.056	0
	Bayes Exch. SP	0.079	0.008	0.921	0	-0.014	0.003	2.014	0
	Bayes Spatial SP	0.823	0.011	0.233	0.90	1.787	0.017	0.264	0.86
	Experiment 2: Immune Fraction (α) = 60%							
#Obs.	Model	$\hat{\beta}_0$	$MCSE(\hat{\beta}_0)$	$RMSE(\hat{\beta}_0)$	$CP(\hat{\beta}_0)$	$\hat{\beta}_1$	$MCSE(\hat{\beta}_1)$	$RMSE(\hat{\beta}_1)$	$CP(\hat{\beta}_1)$
1000	Bayes Pooled SP	-0.242	0.011	1.242	0	0.336	0.006	1.664	0
	Bayes Exch. SP	-0.09	0.012	1.09	0	0.319	0.006	1.681	0
	Bayes Spatial SP	1.005	0.023	0.1	0.96	1.993	0.007	0.047	0.96
1500	Bayes Pooled SP	-0.224	0.013	1.224	0	0.47	0.008	1.53	0
	Bayes Exch. SP	-0.004	0.014	1.004	0	0.373	0.006	1.627	0
	Bayes Spatial SP	1.023	0.013	0.084	0.91	1.992	0.007	0.056	0.88
2000	Bayes Pooled SP	-0.206	0.001	1.206	0	0.24	0	1.76	0
	Bayes Exch. SP	-0.003	0.01	1.003	0	0.214	0.004	1.786	0
	Bayes Spatial SP	0.973	0.018	0.071	0.93	1.997	0.01	0.041	0.95

Note: True parameter values are $\beta_0 = 1$, $\beta_1 = 2$.

Table A.5: Markov Chain Monte Carlo (MCMC) γ Estimates for Experiment 2 (Varying Immune Fraction, Spatial SP Weibull D.G.P.)

	Experiment 2: Immune Fraction (α) = 33%											
#Obs.	Model	$\hat{\gamma}_0$	*MCSE*($\hat{\gamma}_0$)	*RMSE*($\hat{\gamma}_0$)	*CP*($\hat{\gamma}_0$)	$\hat{\gamma}_1$	*MCSE*($\hat{\gamma}_1$)	*RMSE*($\hat{\gamma}_1$)	*CP*($\hat{\gamma}_1$)	$\hat{\gamma}_2$	*MCSE*($\hat{\gamma}_2$)	*RMSE*($\hat{\gamma}_2$)	*CP*($\hat{\gamma}_2$)
	Bayes Pooled SP	-3.907	0.239	7.907	0	-0.024	0.052	2.976	0.01	-2.661	0.082	5.661	0
1000	Bayes Exch. SP	-3.347	0.214	7.347	0	-0.037	0.057	2.963	0.01	-3.184	0.179	6.184	0
	Bayes Spatial SP	3.682	0.119	0.492	0.83	-2.771	0.097	0.323	0.73	2.78	0.083	0.311	0.80
	Bayes Pooled SP	-3.983	0.209	7.983	0	-0.027	0.018	2.973	0	-2.321	0.07	5.321	0
1500	Bayes Exch. SP	-3.939	0.152	7.939	0	-0.047	0.022	2.953	0	-3.53	0.09	6.53	0
	Bayes Spatial SP	3.744	0.092	0.395	0.83	-2.846	0.08	0.248	0.80	2.882	0.071	0.246	0.87
	Bayes Pooled SP	-3.409	0.111	7.409	0	-0.036	0.015	2.964	0	-2.419	0.039	5.419	0
2000	Bayes Exch. SP	-3.525	0.166	7.525	0	-0.012	0.019	2.988	0	-3.129	0.108	6.129	0
	Bayes Spatial SP	3.847	0.082	0.319	0.86	-2.892	0.074	0.206	0.84	2.886	0.07	0.238	0.83
	Experiment 2: Immune Fraction (α) = 40%												
#Obs.	Model	$\hat{\gamma}_0$	*MCSE*($\hat{\gamma}_0$)	*RMSE*($\hat{\gamma}_0$)	*CP*($\hat{\gamma}_0$)	$\hat{\gamma}_1$	*MCSE*($\hat{\gamma}_1$)	*RMSE*($\hat{\gamma}_1$)	*CP*($\hat{\gamma}_1$)	$\hat{\gamma}_2$	*MCSE*($\hat{\gamma}_2$)	*RMSE*($\hat{\gamma}_2$)	*CP*($\hat{\gamma}_2$)
	Bayes Pooled SP	-3.631	0.407	1.192	0.73	-0.088	0.115	1.088	0.02	-1.183	0.073	3.683	0.01
1000	Bayes Exch. SP	-4.546	0.27	0.746	0.90	-0.008	0.035	1.008	0.02	-3.697	0.147	6.197	0
	Bayes Spatial SP	-3.573	0.157	0.557	0.81	0.929	0.032	0.135	0.85	2.316	0.076	0.251	0.87
	Bayes Pooled SP	-3.791	0.015	0.38	0.190	-0.012	0.002	1.012	0	-2.675	0.006	5.175	0
1500	Bayes Exch. SP	-3.676	0.141	0.468	0.90	-0.022	0.02	1.022	0	-3.388	0.115	5.888	0
	Bayes Spatial SP	-4.027	0.105	0.389	0.90	0.991	0.023	0.1	0.92	2.442	0.053	0.26	0.89
	Bayes Pooled SP	-3.64	0.009	0.407	0.06	-0.015	0.001	1.015	0	-2.3	0.004	4.8	0
2000	Bayes Exch. SP	-3.666	0.086	0.445	0.49	-0.022	0.005	1.022	0	-3.335	0.067	5.835	0
	Bayes Spatial SP	-3.859	0.094	0.31	0.87	0.967	0.019	0.073	0.91	2.403	0.058	0.165	0.83
	Experiment 2: Immune Fraction (α) = 48%												
#Obs.	Model	$\hat{\gamma}_0$	*MCSE*($\hat{\gamma}_0$)	*RMSE*($\hat{\gamma}_0$)	*CP*($\hat{\gamma}_0$)	$\hat{\gamma}_1$	*MCSE*($\hat{\gamma}_1$)	*RMSE*($\hat{\gamma}_1$)	*CP*($\hat{\gamma}_1$)	$\hat{\gamma}_2$	*MCSE*($\hat{\gamma}_2$)	*RMSE*($\hat{\gamma}_2$)	*CP*($\hat{\gamma}_2$)
	Bayes Spatial SP	-3.631	0.407	1.192	0.73	-0.088	0.115	1.088	0.02	-1.183	0.073	3.683	0.01
1000	Bayes Exch. SP	-4.546	0.27	0.746	0.90	-0.008	0.035	1.008	0.02	-3.697	0.147	6.197	0
	Bayes Spatial SP	-3.573	0.157	0.557	0.81	0.929	0.032	0.135	0.85	2.316	0.076	0.251	0.87
	Bayes Pooled SP	-3.791	0.015	0.38	0.190	-0.012	0.002	1.012	0	-2.675	0.006	5.175	0
1500	Bayes Exch. SP	-3.676	0.141	0.468	0.90	-0.022	0.02	1.022	0	-3.388	0.115	5.888	0
	Bayes Spatial SP	-4.027	0.105	0.389	0.90	0.991	0.023	0.1	0.92	2.442	0.053	0.26	0.89
	Bayes Pooled SP	-3.64	0.009	0.407	0.06	-0.015	0.001	1.015	0	-2.3	0.004	4.8	0
2000	Bayes Exch. SP	-3.666	0.086	0.445	0.49	-0.022	0.005	1.022	0	-3.335	0.067	5.835	0
	Bayes Spatial SP	-3.859	0.094	0.31	0.87	0.967	0.019	0.073	0.91	2.403	0.058	0.165	0.83
	Experiment 2: Immune Fraction (α) = 60%												
#Obs.	Model	$\hat{\gamma}_0$	*MCSE*($\hat{\gamma}_0$)	*RMSE*($\hat{\gamma}_0$)	*CP*($\hat{\gamma}_0$)	$\hat{\gamma}_1$	*MCSE*($\hat{\gamma}_1$)	*RMSE*($\hat{\gamma}_1$)	*CP*($\hat{\gamma}_1$)	$\hat{\gamma}_2$	*MCSE*($\hat{\gamma}_2$)	*RMSE*($\hat{\gamma}_2$)	*CP*($\hat{\gamma}_2$)
	Bayes Pooled SP	-5.188	0.423	1.249	0.81	-0.026	0.037	2.026	0	-3.07	0.145	4.07	0
1000	Bayes Exch. SP	-5.049	0.348	1.117	0.79	-0.086	0.047	2.086	0	-3.8	0.153	4.8	0
	Bayes Spatial SP	-3.675	0.143	0.482	0.83	1.902	0.061	0.199	0.85	0.915	0.031	0.145	0.89
	Bayes Pooled SP	-5.336	0.346	1.343	0.68	-0.042	0.038	2.042	0	-2.667	0.112	3.667	0
1500	Bayes Exch. SP	-4.516	0.211	0.681	0.89	-0.057	0.027	2.057	0	-3.587	0.14	4.587	0
	Bayes Spatial SP	-3.887	0.112	0.373	0.88	1.954	0.041	0.15	0.88	0.969	0.025	0.118	0.92
	Bayes Pooled SP	-5.107	0.023	1.107	0	-0.041	0.002	2.041	0	-2.668	0.009	3.668	0
2000	Bayes Exch. SP	-4.22	0.139	0.437	0.69	-0.025	0.006	2.025	0	-3.31	0.08	4.31	0
	Bayes Spatial SP	-3.945	0.095	0.306	0.90	1.973	0.034	0.133	0.86	0.977	0.027	0.101	0.91

Note: True parameter values are $\gamma_0 = 4$, $\gamma_1 = -3$, and $\gamma_2 = 3$ for $\alpha = 33\%$;
$\gamma_0 = -4$, $\gamma_1 = 1$, and $\gamma_2 = 2.5$ for $\alpha = 40\%$; $\gamma_0 = -3$, $\gamma_1 = 1$, and $\gamma_2 = 3$ for $\alpha = 48\%$;
$\gamma_0 = -4$, $\gamma_1 = 2$, and $\gamma_2 = 1$ for $\alpha = 60\%$.

Table A.6: Markov Chain Monte Carlo (MCMC) $\boldsymbol{W}$ Estimates for Experiment 2

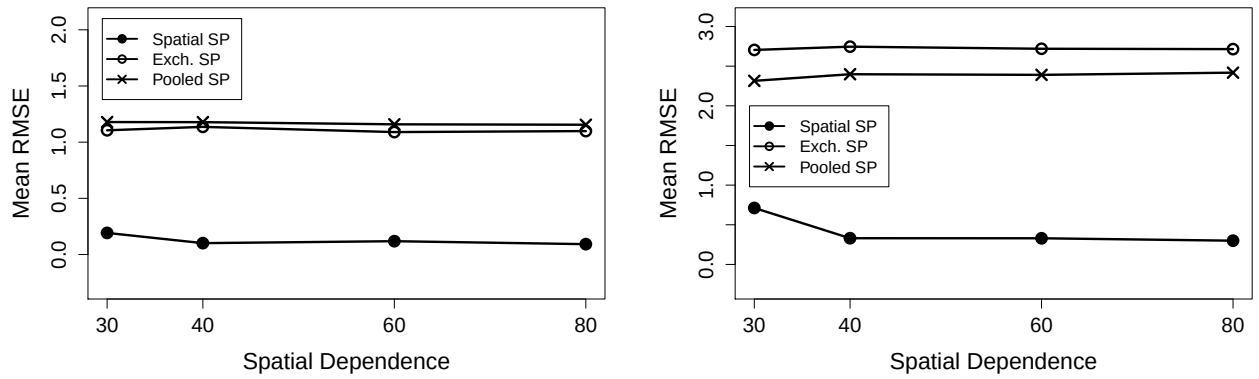
#Obs.	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_5$	$\hat{w}_5$	$\hat{w}_5$	$\hat{w}_5$
	Experiment 2: Immune Fraction (α) = 33%																				
1000	True Value	-3.368	–	–	–	-4.014	–	–	–	-1.999	–	–	–	6.061	–	–	–	3.321	–	–	–
	Bayes Exch. SP	-0.463	0.07	2.905	0	-1.267	0.069	2.747	–	-0.296	0.05	1.704	–	1.603	0.041	4.457	0	0.423	0.04	2.898	0
	True Value	2.07	–	–	–	-1.804	–	–	0	-1.62	–	–	0	0.65	–	–	–	0.704	–	–	–
	Bayes Spatial SP	2.085	0.03	0.092	0.87	-1.802	0.04	0.109	0.87	-1.632	0.054	0.155	0.89	0.657	0.026	0.085	0.90	0.693	0.032	0.092	0.92
1500	True Value	-1.803	–	–	–	1.159	–	–	–	0.579	–	–	–	-1.714	–	–	–	1.779	–	–	–
	Bayes Exch. SP	-0.188	0.035	1.615	0	0.193	0.028	0.966	0	-0.251	0.029	0.829	0	0.108	0.037	1.822	0	0.137	0.026	1.642	0
	True Value	-1.803	–	–	–	1.159	–	–	–	0.579	–	–	–	-1.714	–	–	–	1.779	–	–	–
	Bayes Spatial SP	-1.763	0.034	0.106	0.79	1.134	0.03	0.09	0.84	0.577	0.036	0.099	0.88	-1.723	0.033	0.085	0.90	1.776	0.026	0.083	0.87
2000	True Value	-0.106	–	–	–	-1.953	–	–	–	2.433	–	–	–	6.078	–	–	–	-6.451	–	–	–
	Bayes Exch. SP	-0.09	0.054	0.104	0.98	-0.186	0.063	1.767	0	0.731	0.052	1.702	0	1.629	0.052	4.448	0	-2.084	0.25	4.367	0
	True Value	-2.58	–	–	–	-1.762	–	–	–	2.073	–	–	–	-0.919	–	–	–	3.188	–	–	–
	Bayes Spatial SP	-2.585	0.036	0.085	0.81	-1.743	0.035	0.095	0.79	2.074	0.036	0.086	0.84	-0.942	0.027	0.068	0.84	3.197	0.031	0.079	0.81
	Experiment 2: Immune Fraction (α) = 40%																				
1000	True Value	2.07	–	–	–	-1.804	–	–	–	-1.62	–	–	–	0.65	–	–	–	0.704	–	–	–
	Bayes Exch. SP	-0.064	0.027	2.134	0	-0.525	0.033	1.279	0	-0.524	0.038	1.097	0	1.218	0.033	0.567	0.02	-0.105	0.028	0.809	0
	True Value	2.07	–	–	–	-1.804	–	–	–	-1.62	–	–	–	0.65	–	–	–	0.704	–	–	–
	Bayes Spatial SP	2.012	0.044	0.155	0.85	-1.753	0.072	0.197	0.89	-1.553	0.109	0.282	0.87	0.618	0.039	0.123	0.91	0.676	0.039	0.134	0.92
1500	True Value	1.141	–	–	–	2.611	–	–	–	-4.585	–	–	–	2.227	–	–	–	-1.394	–	–	–
	Bayes Exch. SP	0.564	0.031	0.577	0	0.748	0.03	1.863	0	-1.128	0.059	3.457	0	0.624	0.028	1.603	0	-0.808	0.037	0.586	0.02
	True Value	-1.803	–	–	–	1.159	–	–	–	0.579	–	–	–	-1.714	–	–	–	1.779	–	–	–
	Bayes Spatial SP	-1.772	0.046	0.146	0.88	1.142	0.027	0.091	0.85	0.545	0.053	0.163	0.85	-1.672	0.037	0.128	0.88	1.757	0.035	0.134	0.85
2000	True Value	-2.58	–	–	–	-1.762	–	–	–	2.073	–	–	–	-0.919	–	–	–	3.188	–	–	–
	Bayes Exch. SP	-0.182	0.038	2.397	0	0.01	0.032	1.772	0	-0.214	0.027	2.287	0	0.218	0.03	1.137	0	0.168	0.025	3.019	0
	True Value	-2.58	–	–	–	-1.762	–	–	–	2.073	–	–	–	-0.919	–	–	–	3.188	–	–	–
	Bayes Spatial SP	-2.558	0.044	0.113	0.82	-1.715	0.039	0.104	0.89	2.008	0.055	0.136	0.83	-0.926	0.03	0.088	0.88	3.191	0.04	0.094	0.88
	Experiment 2: Immune Fraction (α) = 48%																				
10000	True Value	-3.368	–	–	–	-4.014	–	–	–	-1.999	–	–	–	6.061	–	–	–	3.321	–	–	–
	Bayes Exch. SP	-0.283	0.055	3.085	0	-0.622	0.056	3.392	0	-0.205	0.041	1.795	0	1.03	0.031	5.031	0	0.079	0.032	3.241	0
	True Value	2.07	–	–	–	-1.804	–	–	–	-1.62	–	–	–	0.65	–	–	–	0.704	–	–	–
	Bayes Spatial SP	1.949	0.055	0.246	0.82	-1.622	0.101	0.277	0.89	-1.651	0.157	0.384	0.87	0.682	0.053	0.186	0.93	0.643	0.052	0.189	0.91
1500	True Value	-1.803	–	–	–	1.159	–	–	–	0.579	–	–	–	-1.714	–	–	–	1.779	–	–	–
	Bayes Exch. SP	-0.208	0.033	1.595	0	0.203	0.027	0.956	0	-0.148	0.029	0.727	0	0.114	0.034	1.827	0	0.041	0.027	1.739	0
	True Value	-1.803	–	–	–	1.159	–	–	–	0.579	–	–	–	-1.714	–	–	–	1.779	–	–	–
	Bayes Spatial SP	-1.764	0.062	0.197	0.88	1.132	0.029	0.103	0.93	0.503	0.062	0.21	0.92	-1.571	0.043	0.233	0.88	1.7	0.037	0.185	0.85
2000	True Value	-2.58	–	–	–	-1.762	–	–	–	2.073	–	–	–	-0.919	–	–	–	3.188	–	–	–
	Bayes Exch. SP	-0.196	0.034	2.384	0	-0.059	0.03	1.703	0	-0.064	0.025	2.137	0	0.092	0.028	1.01	0	0.227	0.024	2.961	0
	True Value	-2.58	–	–	–	-1.762	–	–	–	2.073	–	–	–	-0.919	–	–	–	3.188	–	–	–
	Bayes Spatial SP	-2.376	0.052	0.311	0.84	-1.58	0.049	0.254	0.84	1.756	0.059	0.399	0.78	-0.785	0.035	0.229	0.85	2.984	0.044	0.306	0.83
	Experiment 2: Immune Fraction (α) = 60%																				
10000	True Value	2.07	–	–	–	-1.804	–	–	–	-1.62	–	–	–	0.65	–	–	–	0.704	–	–	–
	Bayes Exch. SP	-0.074	0.027	2.143	0	-0.46	0.038	1.344	0	-0.516	0.038	1.104	0	1.128	0.035	0.478	0.06	-0.078	0.031	0.782	0
	True Value	2.07	–	–	–	-1.804	–	–	–	-1.62	–	–	–	0.65	–	–	–	0.704	–	–	–
	Bayes Spatial SP	1.981	0.052	0.184	0.91	-1.776	0.056	0.185	0.93	-1.552	0.083	0.259	0.87	0.64	0.032	0.1	0.93	0.707	0.037	0.14	0.92
1500	True Value	1.141	–	–	–	2.611	–	–	–	-4.585	–	–	–	2.227	–	–	–	-1.394	–	–	–
	Bayes Exch. SP	0.75	0.033	0.391	0.08	0.881	0.032	1.73	0	-1.49	0.065	3.096	0	0.668	0.029	1.559	0	-0.809	0.039	0.584	0.01
	True Value	-1.803	–	–	–	1.159	–	–	–	0.579	–	–	–	-1.714	–	–	–	1.779	–	–	–
	Bayes Spatial SP	-1.805	0.038	0.137	0.87	1.117	0.027	0.117	0.84	0.625	0.072	0.209	0.85	-1.71	0.031	0.116	0.86	1.772	0.039	0.117	0.97
2000	True Value	-2.58	–	–	–	-1.762	–	–	–	2.073	–	–	–	-0.919	–	–	–	3.188	–	–	–
	Bayes Exch. SP	-0.154	0.039	2.426	0	0.046	0.034	1.808	0	-0.179	0.028	2.252	0	0.190	0.032	1.109	0	0.097	0.026	3.09	0
	True Value	-2.58	–	–	–	-1.762	–	–	–	2.073	–	–	–	-0.919	–	–	–	3.188	–	–	–
	Bayes Spatial SP	-2.55	0.048	0.122	0.90	-1.71	0.041	0.108	0.84	1.949	0.127	0.248	0.83	-0.895	0.037	0.1	0.85	3.206	0.043	0.11	0.87

Table A.7: Markov Chain Monte Carlo (MCMC) $\boldsymbol{V}$ Estimates for Experiment 2

#Obs.	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Experiment 2: Immune Fraction (α) = 33%																				
	Estimate	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$
1000	True Value	-2.029	–	–	–	3.393	–	–	–	-1.239	–	–	–	-1.16	–	–	–	1.035	–	–	–
	Bayes Exch. SP	5.132	0.328	7.161	0	5.683	0.412	2.31	0.16	3.034	0.212	4.273	0	-9.161	0.905	8.001	0.01	-4.687	0.33	5.722	0
	True Value	2.407	–	–	–	0.499	–	–	–	1.983	–	–	–	-5.315	–	–	–	0.426	–	–	–
	Bayes Spatial SP	2.081	0.088	0.447	0.76	0.516	0.073	0.301	0.96	1.699	0.102	0.4	0.88	-4.738	0.17	0.691	0.74	0.442	0.059	0.249	0.97
1500	True Value	-0.621	–	–	–	-0.558	–	–	–	2.908	–	–	–	-3.485	–	–	–	1.757	–	–	–
	Bayes Exch. SP	3.025	0.129	3.646	0	-1.874	0.098	1.318	0.09	-1.254	0.094	4.161	0	3.158	0.138	6.643	0	-3.056	0.122	4.812	0
	True Value	-0.621	–	–	–	-0.558	–	–	–	2.908	–	–	–	-3.485	–	–	–	1.757	–	–	–
	Bayes Spatial SP	-0.771	0.059	0.258	0.90	-0.379	0.049	0.263	0.89	2.583	0.071	0.407	0.72	-3.192	0.101	0.447	0.78	1.759	0.07	0.253	0.95
2000	True Value	2.878	–	–	–	0.433	–	–	–	-0.419	–	–	–	-0.07	–	–	–	-2.822	–	–	–
	Bayes Exch. SP	0.46	0.114	2.418	0	3.213	0.144	2.78	0	-2.954	0.181	2.535	0.01	-9.384	0.81	9.315	0	8.665	0.635	11.487	0
	True Value	-2.088	–	–	–	-1.829	–	–	–	4.147	–	–	–	-2.225	–	–	–	1.995	–	–	–
	Bayes Spatial SP	-2.083	0.074	0.25	0.93	-1.605	0.065	0.326	0.81	3.842	0.102	0.431	0.76	-2.171	0.064	0.223	0.95	2.018	0.059	0.211	0.95
	Experiment 2: Immune Fraction (α) = 40%																				
#Obs.	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$
1000	True Value	2.407	–	–	–	0.499	–	–	–	1.983	–	–	–	-5.315	–	–	–	0.426	–	–	–
	Bayes Exch. SP	-3.891	0.253	6.298	0	2.862	0.26	2.365	0.1	2.461	0.208	0.743	0.88	-0.095	0.098	5.22	0	-1.337	0.172	1.763	0.11
	True Value	2.407	–	–	–	0.499	–	–	–	1.983	–	–	–	-5.315	–	–	–	0.426	–	–	–
	Bayes Spatial SP	2.047	0.095	0.445	0.80	0.744	0.088	0.368	0.87	1.819	0.129	0.432	0.89	-4.988	0.160	0.510	0.80	0.379	0.057	0.242	0.95
1500	True Value	-2.597	–	–	–	-1.999	–	–	–	0.98	–	–	–	-1.783	–	–	–	5.399	–	–	–
	Bayes Exch. SP	-1.466	0.088	1.131	0.240	-3.757	0.136	1.758	0.010	6.804	0.334	5.825	0	-3.307	0.131	1.524	0.04	1.726	0.116	3.673	0
	True Value	-0.621	–	–	–	-0.558	–	–	–	2.908	–	–	–	-3.485	–	–	–	1.757	–	–	–
	Bayes Spatial SP	-0.742	0.061	0.294	0.90	-0.442	0.041	0.244	0.80	2.706	0.082	0.351	0.87	-3.34	0.087	0.394	0.85	1.818	0.058	0.3	0.88
2000	True Value	-2.088	–	–	–	-1.829	–	–	–	4.147	–	–	–	-2.225	–	–	–	1.995	–	–	–
	Bayes Exch. SP	4.047	0.092	6.135	0	2.845	0.082	4.673	0	-3.557	0.089	7.704	0	1.630	0.059	3.855	0	-4.964	0.101	6.959	0
	True Value	-2.088	–	–	–	-1.829	–	–	–	4.147	–	–	–	-2.225	–	–	–	1.995	–	–	–
	Bayes Spatial SP	-2	0.069	0.238	0.91	-1.641	0.055	0.262	0.81	3.83	0.102	0.39	0.73	-2.155	0.054	0.217	0.92	1.965	0.065	0.216	0.89
	Experiment 2: Immune Fraction (α) = 48%																				
#Obs.	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$
1000	True Value	-2.029	–	–	–	3.393	–	–	–	-1.239	–	–	–	-1.16	–	–	–	1.035	–	–	–
	Bayes Exch. SP	5.156	0.329	7.185	0	5.807	0.418	2.414	0.26	2.969	0.213	4.207	0	-9.262	0.945	8.101	0	-4.671	0.318	5.705	0
	True Value	2.407	–	–	–	0.499	–	–	–	1.983	–	–	–	-5.315	–	–	–	0.426	–	–	–
	Bayes Spatial SP	1.907	0.127	0.588	0.74	0.785	0.132	0.424	0.86	1.719	0.206	0.534	0.87	-4.836	0.18	0.698	0.79	0.424	0.081	0.288	0.92
1500	True Value	-0.621	–	–	–	-0.558	–	–	–	2.908	–	–	–	-3.485	–	–	–	1.757	–	–	–
	Bayes Exch. SP	2.901	0.11	3.522	0	-1.747	0.088	1.188	0.15	-1.143	0.085	4.051	0	3.113	0.119	6.599	0	-3.124	0.112	4.881	0
	True Value	-0.621	–	–	–	-0.558	–	–	–	2.908	–	–	–	-3.485	–	–	–	1.757	–	–	–
	Bayes Spatial SP	-0.755	0.087	0.391	0.84	-0.446	0.046	0.243	0.90	2.595	0.101	0.464	0.81	-3.109	0.109	0.557	0.83	1.716	0.077	0.41	0.89
2000	True Value	-2.088	–	–	–	-1.829	–	–	–	4.147	–	–	–	-2.225	–	–	–	1.995	–	–	–
	Bayes Exch. SP	4.171	0.11	6.26	0	2.858	0.092	4.687	0	-3.499	0.097	7.646	0	1.546	0.067	3.77	0	-5.076	0.122	7.071	0
	True Value	-2.088	–	–	–	-1.829	–	–	–	4.147	–	–	–	-2.225	–	–	–	1.995	–	–	–
	Bayes Spatial SP	-1.612	0.094	0.648	0.81	-1.303	0.072	0.564	0.76	3.301	0.119	0.902	0.73	-1.865	0.064	0.478	0.85	1.479	0.076	0.646	0.84
	Experiment 2: Immune Fraction (α) = 60%																				
#Obs.	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$
1000	True Value	2.407	–	–	–	0.499	–	–	–	1.983	–	–	–	-5.315	–	–	–	0.426	–	–	–
	Bayes Exch. SP	-4.406	0.371	6.812	0	3.186	0.298	2.687	0.13	2.467	0.262	0.856	0.79	0.079	0.134	5.394	0	-1.326	0.24	1.752	0.28
	True Value	2.407	–	–	–	0.499	–	–	–	1.983	–	–	–	-5.315	–	–	–	0.426	–	–	–
	Bayes Spatial SP	2.087	0.08	0.397	0.76	0.616	0.068	0.311	0.86	1.866	0.097	0.351	0.89	-4.991	0.165	0.545	0.82	0.422	0.052	0.246	0.96
1500	True Value	-2.597	–	–	–	-1.999	–	–	–	0.98	–	–	–	-1.783	–	–	–	5.399	–	–	–
	Bayes Exch. SP	-1.477	0.121	1.119	0.46	-4.077	0.187	2.078	0.04	7.342	0.433	6.362	0	-3.633	0.21	1.85	0.08	1.846	0.177	3.553	0
	True Value	-0.621	–	–	–	-0.558	–	–	–	2.908	–	–	–	-3.485	–	–	–	1.757	–	–	–
	Bayes Spatial SP	-0.745	0.059	0.252	0.90	-0.428	0.037	0.192	0.91	2.725	0.078	0.313	0.85	-3.359	0.093	0.319	0.87	1.807	0.055	0.214	0.90
2000	True Value	-2.088	–	–	–	-1.829	–	–	–	4.147	–	–	–	-2.225	–	–	–	1.995	–	–	–
	Bayes Exch. SP	4.416	0.124	6.505	0	3.078	0.106	4.907	0	-3.704	0.137	7.851	0	1.803	0.082	4.028	0	-5.594	0.25	7.589	0
	True Value	-2.088	–	–	–	-1.829	–	–	–	4.147	–	–	–	-2.225	–	–	–	1.995	–	–	–
	Bayes Spatial SP	-2.021	0.068	0.248	0.89	-1.701	0.055	0.251	0.86	3.896	0.111	0.411	0.81	-2.179	0.053	0.226	0.86	2.004	0.057	0.194	0.94

Figure A.2: Average RMSE for β and γ Estimates in Experiment 3

(a) Average RMSE for $\hat{\beta}$ (b) Average RMSE for $\hat{\gamma}$



The difference between the performance of the Bayesian Exchangeable SP and Spatial SP Weibull models in Experiment 3 is even more stark in the case of **W** and **V**. Tables A.10 and A.11 reveal that as the extent of adjacent units with spatially dependent split and survival stage frailties increases, the Bayesian Spatial SP—*but not* the Bayesian Exchangeable SP—Weibull model retrieves true theoretical values for $\{\widehat{w}_1...\widehat{w}_5\}$ and $\{\widehat{v}_1...\widehat{v}_5\}$ with negligible RMSEs but very high CPs. The nonspatial Bayesian Exchangeable SP Weibull's RMSEs and CPs for **W** and **V** also display higher bias and poor coverage as the share of units with spatially dependent frailties increases from 30% to 80%.

In Experiment 4, we also assume a SP Weibull distributed outcome variable d.g.p. with spatially dependent split and survival stage frailties. However, in this MC Experiment, we vary both the share of immune fraction α and the proportion of units with spatially dependent frailties in the following four conditions. For the first condition, we allow a low share of the immune fraction ($\alpha = 25\%$) and a low proportion (30%) of units that exhibit spatially dependent split and survival stage frailties. The next condition is also $\alpha = 25\%$ but high proportion of units (80%) with spatially dependent frailties in each stage. The third condition captures a high immune fraction share of $\alpha = 65\%$ but low proportion of units with(30%) spatially dependent split and survival stage frailties. The fourth condition is high immune fraction $\alpha = 65\%$ and high proportion of units (80%) with spatially dependent frailties in each stage.

In Experiment 4, the $\hat{\beta}'$s and $\hat{\gamma}'$s from the Bayesian Spatial SP Weibull model are largely

Table A.8: Markov Chain Monte Carlo (MCMC) β Estimates for Experiment 3 (Varying Spatial Dependence, Spatial SP Weibull D.G.P.)

	Experiment 3: Spatial Dependence = 30%							
#Obs.	Model	$\hat{\beta}_0$	$MCSE(\hat{\beta}_0)$	$RMSE(\hat{\beta}_0)$	$CP(\hat{\beta}_0)$	$\hat{\beta}_1$	$MCSE(\hat{\beta}_1)$	$RMSE(\hat{\beta}_1)$	$CP(\hat{\beta}_1)$
1000	Bayes Pooled SP	0.539	0.019	0.461	0.03	0.076	0.01	1.924	0
	Bayes Exch. SP	1.065	0.021	0.099	0.98	-0.055	0.006	2.055	0
	Bayes Spatial SP	0.381	0.215	0.645	0.95	2.054	0.042	0.116	0.99
1500	Bayes Pooled SP	0.504	0.02	0.496	0.01	0.179	0.01	1.821	0
	Bayes Exch. SP	0.817	0.009	0.183	0.37	-0.018	0.004	2.018	0
	Bayes Spatial SP	0.956	0.007	0.086	0.91	1.876	0.007	0.165	0.90
2000	Bayes Pooled SP	0.638	0.012	0.362	0	-0.01	0.006	2.01	0
	Bayes Exch. SP	0.78	0.014	0.22	0.24	-0.059	0.004	2.059	0
	Bayes Spatial SP	0.982	0.008	0.05	0.90	1.945	0.009	0.092	0.90
	Experiment 3: Spatial Dependence = 60%							
#Obs.	Model	$\hat{\beta}_0$	$MCSE(\hat{\beta}_0)$	$RMSE(\hat{\beta}_0)$	$CP(\hat{\beta}_0)$	$\hat{\beta}_1$	$MCSE(\hat{\beta}_1)$	$RMSE(\hat{\beta}_1)$	$CP(\hat{\beta}_1)$
1000	Bayes Pooled SP	0.589	0.015	0.411	0.03	0.064	0.007	1.936	0
	Bayes Exch. SP	0.891	0.014	0.121	0.86	-0.022	0.005	2.022	0
	Bayes Spatial SP	0.968	0.023	0.114	0.85	1.949	0.006	0.089	0.91
1500	Bayes Pooled SP	0.614	0.015	0.386	0.01	0.091	0.008	1.909	0
	Bayes Exch. SP	0.813	0.009	0.188	0.27	-0.027	0.004	2.027	0
	Bayes Spatial SP	0.96	0.005	0.08	0.88	1.84	0.006	0.195	0.87
2000	Bayes Pooled SP	0.699	0.011	0.301	0.02	-0.011	0.005	2.011	0
	Bayes Exch. SP	0.867	0.008	0.134	0.51	-0.047	0.004	2.047	0
	Bayes Spatial SP	0.961	0.006	0.066	0.88	1.864	0.007	0.167	0.87
	Experiment 3: Spatial Dependence = 80%							
#Obs.	Model	$\hat{\beta}_0$	$MCSE(\hat{\beta}_0)$	$RMSE(\hat{\beta}_0)$	$CP(\hat{\beta}_0)$	$\hat{\beta}_1$	$MCSE(\hat{\beta}_1)$	$RMSE(\hat{\beta}_1)$	$CP(\hat{\beta}_1)$
1000	Bayes Pooled SP	0.656	0.013	0.344	0.09	0.028	0.007	1.972	0
	Bayes Exch. SP	0.851	0.014	0.155	0.75	-0.013	0.006	2.013	0
	Bayes Spatial SP	1.001	0.017	0.072	0.92	1.98	0.006	0.07	0.96
1500	Bayes Pooled SP	0.649	0.014	0.351	0.03	0.061	0.007	1.939	0
	Bayes Exch. SP	0.807	0.009	0.193	0.33	-0.029	0.005	2.029	0
	Bayes Spatial SP	0.973	0.004	0.068	0.86	1.856	0.005	0.171	0.87
2000	Bayes Pooled SP	0.609	0.01	0.391	0	0.066	0.005	1.934	0
	Bayes Exch. SP	0.829	0.009	0.171	0.29	-0.03	0.004	2.03	0
	Bayes Spatial SP	0.98	0.005	0.056	0.94	1.921	0.006	0.115	0.86

Note: True parameter values are $\beta_0 = 1$, $\beta_1 = 2$.

equal to their true theoretical values, while the $\hat{\beta}'$s and $\hat{\gamma}'$s from the two nonspatial Bayesian Pooled and Exchangeable SP Weibull models deviate substantially from their true values for *each* of the four aforementioned conditions regardless of the sample size (Tables A.12–A.13). We also average and plot the $N = 1,000$, $N = 1,500$, and $N = 2,000$ RMSE results obtained for $\hat{\beta}$ and $\hat{\gamma}$ from Experiment 4 for each of the four conditions outlined above. Figure A.3a (for $\hat{\beta}$) and Figure A.3b (for $\widehat{\gamma}$) reveal that the Bayesian Spatial SP Weibull's $\hat{\beta}$ and $\hat{\gamma}$ RMSEs remain consistently close to 0 and are also substantially smaller than those obtained from the Bayesian Pooled SP and Exchangeable SP Weibull models under each of the four conditions

Table A.9: Markov Chain Monte Carlo (MCMC) γ Estimates for Experiment 3 (Varying Spatial Dependence, Spatial SP Weibull D.G.P.)

	Experiment 3: Spatial Dependence = 30%											
#Obs.	Model	$\hat{\gamma}_0$	$MCSE(\hat{\gamma}_0)$	$RMSE(\hat{\gamma}_0)$	$CP(\hat{\gamma}_0)$	$\hat{\gamma}_1$	$MCSE(\hat{\gamma}_1)$	$RMSE(\hat{\gamma}_1)$	$CP(\hat{\gamma}_1)$	$\hat{\gamma}_2$	$MCSE(\hat{\gamma}_2)$	$RMSE(\hat{\gamma}_2)$	$CP(\hat{\gamma}_2)$
1000	Bayes Pooled SP	-2.588	0.122	0.441	0.88	-0.049	0.02	0.951	0	-1.947	0.035	5.447	0
	Bayes Exch. SP	-3.051	0.149	0.408	0.97	-0.07	0.029	0.93	0.01	-3.25	0.179	6.75	0
	Bayes Spatial SP	-2.379	1.187	2.326	0.80	-1.816	0.371	0.874	0.930	4.469	0.75	1.471	0.94
1500	Bayes Pooled SP	-3.711	0.194	0.721	0.82	0.032	0.017	1.032	0.01	-1.939	0.058	5.439	0
	Bayes Exch. SP	-3.302	0.107	0.415	0.90	-0.021	0.018	0.979	0	-3.273	0.07	6.773	0
	Bayes Spatial SP	-2.951	0.092	0.300	0.94	-0.947	0.022	0.128	0.920	3.137	0.081	0.526	0.89
2000	Bayes Pooled SP	-2.738	0.081	0.323	0.90	-0.035	0.012	0.965	0	-2.016	0.027	5.516	0
	Bayes Exch. SP	-3.292	0.097	0.341	0.91	-0.017	0.015	0.983	0	-3.263	0.085	6.763	0
	Bayes Spatial SP	-2.854	0.074	0.324	0.84	-0.983	0.017	0.094	0.95	3.288	0.073	0.365	0.82
	Experiment 3: Spatial Dependence = 60%												
#Obs.	Model	$\hat{\gamma}_0$	$MCSE(\hat{\gamma}_0)$	$RMSE(\hat{\gamma}_0)$	$CP(\hat{\gamma}_0)$	$\hat{\gamma}_1$	$MCSE(\hat{\gamma}_1)$	$RMSE(\hat{\gamma}_1)$	$CP(\hat{\gamma}_1)$	$\hat{\gamma}_2$	$MCSE(\hat{\gamma}_2)$	$RMSE(\hat{\gamma}_2)$	$CP(\hat{\gamma}_2)$
1000	Bayes Pooled SP	-2.824	0.129	0.383	0.95	-0.041	0.021	0.959	0	-2.331	0.043	5.831	0
	Bayes Exch. SP	-3.320	0.143	0.491	0.91	-0.018	0.027	0.982	0	-3.334	0.121	6.834	0
	Bayes Spatial SP	-2.928	0.135	0.425	0.92	-1.006	0.032	0.136	0.96	3.344	0.108	0.428	0.90
1500	Bayes Pooled SP	-3.356	0.141	0.493	0.88	-0.018	0.016	0.982	0	-2.323	0.049	5.823	0
	Bayes Exch. SP	-3.327	0.105	0.466	0.87	-0.009	0.018	0.991	0	-3.299	0.063	6.799	0
	Bayes Spatial SP	-3.032	0.093	0.296	0.95	-0.938	0.022	0.152	0.90	3.105	0.08	0.609	0.83
2000	Bayes Pooled SP	-2.787	0.075	0.284	0.93	-0.025	0.012	0.975	0	-2.282	0.028	5.782	0
	Bayes Exch. SP	-3.071	0.089	0.295	0.96	-0.03	0.015	0.97	0	-3.148	0.061	6.648	0
	Bayes Spatial SP	-2.945	0.061	0.261	0.93	-0.931	0.014	0.13	0.86	3.092	0.057	0.532	0.87
	Experiment 3: Spatial Dependence = 80%												
#Obs.	Model	$\hat{\gamma}_0$	$MCSE(\hat{\gamma}_0)$	$RMSE(\hat{\gamma}_0)$	$CP(\hat{\gamma}_0)$	$\hat{\gamma}_1$	$MCSE(\hat{\gamma}_1)$	$RMSE(\hat{\gamma}_1)$	$CP(\hat{\gamma}_1)$	$\hat{\gamma}_2$	$MCSE(\hat{\gamma}_2)$	$RMSE(\hat{\gamma}_2)$	$CP(\hat{\gamma}_2)$
1000	Bayes Pooled SP	-2.958	0.143	0.379	0.96	-0.027	0.022	0.973	0	-2.566	0.05	6.066	0
	Bayes Exch. SP	-3.394	0.147	0.559	0.88	-0.032	0.026	0.968	0	-3.357	0.112	6.857	0
	Bayes Spatial SP	-2.872	0.13	0.412	0.93	-0.99	0.029	0.13	0.92	3.385	0.099	0.373	0.86
1500	Bayes Pooled SP	-3.206	0.13	0.355	0.97	-0.016	0.016	0.984	0	-2.418	0.047	5.918	0
	Bayes Exch. SP	-3.235	0.104	0.334	0.93	-0.027	0.019	0.973	0	-3.26	0.057	6.76	0
	Bayes Spatial SP	-2.997	0.089	0.301	0.94	-0.946	0.02	0.155	0.81	3.159	0.072	0.56	0.84
2000	Bayes Pooled SP	-2.945	0.086	0.257	0.96	-0.031	0.013	0.969	0	-2.362	0.032	5.862	0
	Bayes Exch. SP	-3.148	0.092	0.354	0.91	-0.022	0.014	0.978	0	-3.151	0.056	6.651	0
	Bayes Spatial SP	-2.881	0.055	0.281	0.89	-0.969	0.014	0.092	0.91	3.234	0.05	0.391	0.88

Note: True parameter values are $\gamma_0 = -3$, $\gamma_1 = -1$, and $\gamma_2 = 3.5$.

that we focus on here. The MC results for the spatially dependent split and survival stage frailty terms from Experiment 4 in Tables A.14–A.15 show that the values for $\{\widehat{w}_1 ... \widehat{w}_5\}$ in $\mathbf{W}$

Table A.10: Markov Chain Monte Carlo (MCMC) $\boldsymbol{W}$ Estimates for Experiment 3

#Obs.	Experiment 3: Spatial Dependence = 30%																			
	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_5$	$\hat{w}_5$	$\hat{w}_5$	$\hat{w}_5$
1000	True Value	-3.314	–	–	–	-4.227	–	–	–	-1.379	–	–	–	4.736	–	–	–	4.185	–	–	–
	Bayes Exch. SP	-0.619	0.071	2.695	0	-1.767	0.076	2.460	0	-0.153	0.046	1.226	0	1.633	0.038	3.103	0	0.905	0.035	3.279	0
	True Value	-3.314	–	–	–	-4.227	–	–	–	-1.379	–	–	–	4.736	–	–	–	4.185	–	–	–
	Bayes Spatial SP	-3.488	0.256	0.483	0.95	-5.14	1.226	1.398	0.89	-1.273	0.238	0.368	0.99	5.216	0.267	0.563	0.89	4.685	0.276	0.599	0.87
1500	True Value	-2.759	–	–	–	1.430	–	–	–	0.610	–	–	–	-1.061	–	–	–	1.781	–	–	–
	Bayes Exch. SP	-0.426	0.041	2.333	0	0.524	0.029	0.906	0	-0.456	0.029	1.065	0	0.272	0.035	1.333	0	0.086	0.029	1.695	0
	True Value	-2.759	–	–	–	1.430	–	–	–	0.610	–	–	–	-1.061	–	–	–	1.781	–	–	–
	Bayes Spatial SP	-2.633	0.047	0.207	0.80	1.403	0.026	0.098	0.79	0.488	0.04	0.213	0.76	-0.964	0.033	0.183	0.80	1.706	0.031	0.157	0.74
2000	True Value	-0.429	–	–	–	1.284	–	–	–	3.076	–	–	–	-5.069	–	–	–	1.138	–	–	–
	Bayes Exch. SP	-0.04	0.034	0.389	0.07	0.118	0.031	1.166	0	0.878	0.031	2.198	0	-1.282	0.077	3.788	0	0.326	0.032	0.813	0
	True Value	-3.745	–	–	–	-2.589	–	–	–	2.835	–	–	–	-0.019	–	–	–	3.517	–	–	–
	Bayes Spatial SP	-3.651	0.035	0.203	0.80	-2.535	0.031	0.146	0.81	2.725	0.032	0.190	0.83	0.007	0.019	0.079	0.88	3.454	0.024	0.131	0.79
#Obs.	Experiment 3: Spatial Dependence = 60%																			
	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_5$	$\hat{w}_5$	$\hat{w}_5$	$\hat{w}_5$
1000	True Value	-2.555	–	–	–	-2.723	–	–	–	-1.186	–	–	–	3.716	–	–	–	2.748	–	–	–
	Bayes Exch. SP	-0.44	0.049	2.115	0	-1.096	0.044	1.627	0	-0.115	0.038	1.071	0	1.107	0.031	2.609	0	0.545	0.031	2.203	0
	True Value	-2.555	–	–	–	-2.723	–	–	–	-1.186	–	–	–	3.716	–	–	–	2.748	–	–	–
	Bayes Spatial SP	-2.505	0.046	0.178	0.87	-2.71	0.108	0.26	0.80	-1.16	0.04	0.131	0.88	3.665	0.036	0.124	0.83	2.710	0.038	0.133	0.85
1500	True Value	-1.893	–	–	–	0.95	–	–	–	0.489	–	–	–	-0.864	–	–	–	1.317	–	–	–
	Bayes Exch. SP	-0.265	0.037	1.628	0	0.298	0.032	0.652	0	-0.335	0.030	0.825	0	0.259	0.034	1.123	0	0.043	0.027	1.275	0
	True Value	-1.893	–	–	–	0.95	–	–	–	0.489	–	–	–	-0.864	–	–	–	1.317	–	–	–
	Bayes Spatial SP	-1.765	0.038	0.197	0.72	0.91	0.027	0.092	0.82	0.392	0.037	0.185	0.80	-0.77	0.030	0.165	0.85	1.233	0.029	0.142	0.78
2000	True Value	-2.72	–	–	–	-1.391	–	–	–	1.932	–	–	–	-0.236	–	–	–	2.415	–	–	–
	Bayes Exch. SP	-0.296	0.044	2.425	0	-0.069	0.034	1.322	0	-0.227	0.027	2.159	0	0.268	0.033	0.504	0.020	0.324	0.027	2.091	0
	True Value	-2.72	–	–	–	-1.391	–	–	–	1.932	–	–	–	-0.236	–	–	–	2.415	–	–	–
	Bayes Spatial SP	-2.572	0.030	0.222	0.82	-1.318	0.025	0.152	0.79	1.770	0.026	0.235	0.82	-0.188	0.018	0.11	0.84	2.307	0.021	0.177	0.77
#Obs.	Experiment 3: Spatial Dependence = 80%																			
	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_5$	$\hat{w}_5$	$\hat{w}_5$	$\hat{w}_5$
1000	True Value	-2.223	–	–	–	-2.466	–	–	–	-1.106	–	–	–	3.688	–	–	–	2.106	–	–	–
	Bayes Exch. SP	-0.38	0.048	1.843	0	-0.987	0.044	1.479	0	-0.101	0.039	1.004	0	1.054	0.032	2.635	0	0.415	0.03	1.691	0
	True Value	-2.223	–	–	–	-2.466	–	–	–	-1.106	–	–	–	3.688	–	–	–	2.106	–	–	–
	Bayes Spatial SP	-2.2	0.044	0.142	0.88	-2.423	0.079	0.214	0.82	-1.117	0.037	0.138	0.82	3.647	0.033	0.115	0.88	2.094	0.032	0.103	0.89
1500	True Value	-1.504	–	–	–	0.844	–	–	–	0.441	–	–	–	-0.899	–	–	–	1.118	–	–	–
	Bayes Exch. SP	-0.199	0.033	1.304	0	0.236	0.029	0.608	0	-0.293	0.031	0.735	0	0.221	0.034	1.12	0	0.036	0.029	1.082	0
	True Value	-1.504	–	–	–	0.844	–	–	–	0.441	–	–	–	-0.899	–	–	–	1.118	–	–	–
	Bayes Spatial SP	-1.405	0.037	0.155	0.78	0.811	0.029	0.088	0.83	0.349	0.034	0.168	0.70	-0.796	0.031	0.163	0.78	1.041	0.029	0.132	0.80
2000	True Value	-2.157	–	–	–	-1.184	–	–	–	1.642	–	–	–	-0.335	–	–	–	2.036	–	–	–
	Bayes Exch. SP	-0.248	0.039	1.909	0	-0.071	0.034	1.113	0	-0.154	0.028	1.796	0	0.198	0.029	0.533	0	0.276	0.029	1.76	0
	True Value	-2.157	–	–	–	-1.184	–	–	–	1.642	–	–	–	-0.335	–	–	–	2.036	–	–	–
	Bayes Spatial SP	-2.086	0.027	0.146	0.83	-1.142	0.024	0.11	0.84	1.557	0.025	0.150	0.79	-0.313	0.019	0.081	0.90	1.985	0.02	0.114	0.82

Table A.11: Markov Chain Monte Carlo (MCMC) $\boldsymbol{V}$ Estimates for Experiment 3

#Obs.	Experiment 3: Spatial Dependence = 30%																			
	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$
1000	True Value	-2.583	–	–	–	5.085	–	–	–	-1.465	–	–	–	-1.684	–	–	–	0.647	–	–	–
	Bayes Exch. SP	5.106	0.277	7.689	0	5.833	0.366	0.891	0.95	2.033	0.128	3.499	0	-6.861	0.436	5.176	0	-6.112	0.374	6.759	0
	True Value	-2.583	–	–	–	5.085	–	–	–	-1.465	–	–	–	-1.684	–	–	–	0.647	–	–	–
	Bayes Spatial SP	-3.457	0.611	1.204	0.89	7.722	1.356	3.195	0.63	-2.293	0.317	1.026	0.78	-2.558	0.356	1.098	0.76	0.587	0.487	1.054	0.87
1500	True Value	-1.299	–	–	–	-1.211	–	–	–	3.691	–	–	–	-2.653	–	–	–	1.472	–	–	–
	Bayes Exch. SP	4.239	0.137	5.537	0	-1.913	0.075	0.706	0.46	-1.267	0.078	4.958	0	1.831	0.084	4.485	0	-2.89	0.098	4.362	0
	True Value	-1.299	–	–	–	-1.211	–	–	–	3.691	–	–	–	-2.653	–	–	–	1.472	–	–	–
	Bayes Spatial SP	-1.003	0.061	0.473	0.84	-1.262	0.051	0.212	0.90	3.371	0.102	0.499	0.90	-2.3	0.078	0.491	0.86	1.194	0.054	0.421	0.87
2000	True Value	-0.022	–	–	–	1.699	–	–	–	-0.212	–	–	–	-1.273	–	–	–	-0.192	–	–	–
	Bayes Exch. SP	0.624	0.056	0.645	0.43	-1.994	0.085	3.693	0	-4.264	0.114	4.052	0	7.227	0.288	8.501	0	-1.593	0.067	1.401	0
	True Value	-2.98	–	–	–	-2.613	–	–	–	5.838	–	–	–	-1.987	–	–	–	1.742	–	–	–
	Bayes Spatial SP	-2.738	0.072	0.505	0.85	-2.453	0.059	0.363	0.91	5.465	0.129	0.63	0.83	-1.797	0.044	0.265	0.90	1.523	0.046	0.332	0.91
#Obs.	**Experiment 3: Spatial Dependence = 60%**																			
	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$
1000	True Value	-1.649	–	–	–	2.852	–	–	–	-0.859	–	–	–	-0.978	–	–	–	0.635	–	–	–
	Bayes Exch. SP	3.843	0.172	5.492	0	3.856	0.188	1.04	0.51	1.861	0.115	2.72	0	-5.439	0.245	4.461	0	-4.121	0.168	4.756	0
	True Value	-1.649	–	–	–	2.852	–	–	–	-0.859	–	–	–	-0.978	–	–	–	0.635	–	–	–
	Bayes Spatial SP	-1.46	0.085	0.376	0.90	2.69	0.133	0.429	0.94	-0.742	0.061	0.327	0.88	-0.999	0.063	0.335	0.89	0.511	0.071	0.321	0.91
1500	True Value	-0.892	–	–	–	-0.667	–	–	–	2.637	–	–	–	-2.145	–	–	–	1.067	–	–	–
	Bayes Exch. SP	2.905	0.107	3.797	0	-1.362	0.074	0.702	0.44	-1.033	0.08	3.669	0	1.63	0.084	3.775	0	-2.141	0.077	3.208	0
	True Value	-0.892	–	–	–	-0.667	–	–	–	2.637	–	–	–	-2.145	–	–	–	1.067	–	–	–
	Bayes Spatial SP	-0.705	0.055	0.411	0.83	-0.646	0.045	0.209	0.91	2.279	0.076	0.52	0.82	-1.802	0.068	0.496	0.83	0.874	0.05	0.375	0.85
2000	True Value	-2.128	–	–	–	-1.633	–	–	–	4.107	–	–	–	-1.556	–	–	–	1.211	–	–	–
	Bayes Exch. SP	4.009	0.101	6.138	0	2.06	0.077	3.694	0	-3.05	0.082	7.157	0	0.512	0.047	2.068	0	-3.532	0.086	4.742	0
	True Value	-2.128	–	–	–	-1.633	–	–	–	4.107	–	–	–	-1.556	–	–	–	1.211	–	–	–
	Bayes Spatial SP	-1.78	0.051	0.523	0.88	-1.329	0.037	0.367	0.87	3.545	0.078	0.671	0.85	-1.391	0.034	0.297	0.90	0.954	0.035	0.388	0.92
#Obs.	**Experiment 3: Spatial Dependence = 80%**																			
	Model	$\hat{v}_1$	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$
1000	True Value	-1.397	–	–	–	2.481	–	–	–	-0.752	–	–	–	-0.800	–	–	–	0.468	–	–	–
	Bayes Exch. SP	3.504	0.17	4.901	0	3.271	0.182	1.093	0.55	1.794	0.116	2.546	0	-5.383	0.224	4.583	0	-3.186	0.141	3.654	0.01
	True Value	-1.397	–	–	–	2.481	–	–	–	-0.752	–	–	–	-0.80	–	–	–	0.468	–	–	–
	Bayes Spatial SP	-1.329	0.079	0.328	0.92	2.389	0.109	0.359	0.91	-0.653	0.061	0.318	0.90	-0.846	0.063	0.269	0.95	0.438	0.067	0.286	0.93
1500	True Value	-0.64	–	–	–	-0.492	–	–	–	2.241	–	–	–	-2.067	–	–	–	0.959	–	–	–
	Bayes Exch. SP	2.282	0.089	2.922	0	-1.091	0.067	0.611	0.52	-0.961	0.071	3.202	0	1.525	0.078	3.592	0	-1.755	0.075	2.714	0
	True Value	-0.64	–	–	–	-0.492	–	–	–	2.241	–	–	–	-2.067	–	–	–	0.959	–	–	–
	Bayes Spatial SP	-0.464	0.048	0.337	0.84	-0.458	0.045	0.195	0.95	1.939	0.064	0.423	0.81	-1.757	0.064	0.424	0.84	0.741	0.049	0.35	0.84
2000	True Value	-1.692	–	–	–	-1.34	–	–	–	3.399	–	–	–	-1.402	–	–	–	1.034	–	–	–
	Bayes Exch. SP	3.237	0.09	4.928	0	1.675	0.072	3.015	0	-2.594	0.076	5.993	0	0.612	0.051	2.014	0	-2.93	0.077	3.964	0
	True Value	-1.692	–	–	–	-1.34	–	–	–	3.399	–	–	–	-1.402	–	–	–	1.034	–	–	–
	Bayes Spatial SP	-1.487	0.04	0.319	0.95	-1.198	0.034	0.271	0.91	3.066	0.062	0.478	0.84	-1.302	0.031	0.233	0.92	0.921	0.03	0.279	0.93

Table A.12: Markov Chain Monte Carlo (MCMC) β Estimates for Experiment 4 (Varying Immune Fraction and Spatial Dependence, Spatial SP Weibull D.G.P.)

	Experiment 4: Immune Fraction = 25%, Spatial Dependence = 30%							
#Obs.	Model	$\hat{\beta}_0$	*MCSE*($\hat{\beta}_0$)	*RMSE*($\hat{\beta}_0$)	*CP*($\hat{\beta}_0$)	$\hat{\beta}_1$	*MCSE*($\hat{\beta}_1$)	*RMSE*($\hat{\beta}_1$)	*CP*($\hat{\beta}_1$)
1000	Bayes Pooled SP	0.546	0.016	0.454	0.01	0.075	0.008	1.925	0
	Bayes Exch. SP	0.589	0.014	0.411	0.03	0.14	0.008	1.86	0
	Bayes Spatial SP	0.936	0.012	0.114	0.88	1.912	0.005	0.135	0.88
2000	Bayes Pooled SP	0.685	0.011	0.315	0.06	-0.037	0.005	2.037	0
	Bayes Exch. SP	0.729	0.026	0.271	0.2	-0.059	0.003	2.059	0
	Bayes Spatial SP	0.998	0.006	0.043	0.91	1.97	0.008	0.062	0.92
	Experiment 4: Immune Fraction = 25% Spatial Dependence = 80%							
#Obs.	Model	$\hat{\beta}_0$	*MCSE*($\hat{\beta}_0$)	*RMSE*($\hat{\beta}_0$)	*CP*($\hat{\beta}_0$)	$\hat{\beta}_1$	*MCSE*($\hat{\beta}_1$)	*RMSE*($\hat{\beta}_1$)	*CP*($\hat{\beta}_1$)
1000	Bayes Pooled SP	0.656	0.013	0.344	0.09	0.028	0.007	1.972	0
	Bayes Exch. SP	0.851	0.014	0.155	0.75	-0.013	0.006	2.013	0
	Bayes Spatial SP	1.001	0.017	0.072	0.92	1.98	0.006	0.07	0.96
2000	Bayes Pooled SP	0.609	0.01	0.391	0	0.066	0.005	1.934	0
	Bayes Exch. SP	0.829	0.009	0.171	0.29	-0.03	0.004	2.03	0
	Bayes Spatial SP	0.98	0.005	0.056	0.94	1.921	0.006	0.115	0.86
	Experiment 4: Immune Fraction = 60%, Spatial Dependence = 30%							
#Obs.	Model	$\hat{\beta}_0$	*MCSE*($\hat{\beta}_0$)	*RMSE*($\hat{\beta}_0$)	*CP*($\hat{\beta}_0$)	$\hat{\beta}_1$	*MCSE*($\hat{\beta}_1$)	*RMSE*($\hat{\beta}_1$)	*CP*($\hat{\beta}_1$)
1000	Bayes Pooled SP	-0.348	0.012	1.348	0	0.375	0.007	1.625	0
	Bayes Exch. SP	-0.076	0.012	1.076	0	0.273	0.006	1.727	0
	Bayes Spatial SP	1.006	0.023	0.088	0.96	1.999	0.008	0.048	0.95
2000	Bayes Pooled SP	-0.281	0.001	1.281	0	0.283	0	1.717	0
	Bayes Exch. SP	-0.097	0.015	1.097	0	0.190	0.003	1.81	0
	Bayes Spatial SP	0.667	0.085	0.344	0.53	1.985	0.01	0.049	0.94
	Experiment 4: Immune Fraction = 60%, Spatial Dependence = 80%							
#Obs.	Model	$\hat{\beta}_0$	*MCSE*($\hat{\beta}_0$)	*RMSE*($\hat{\beta}_0$)	*CP*($\hat{\beta}_0$)	$\hat{\beta}_1$	*MCSE*($\hat{\beta}_1$)	*RMSE*($\hat{\beta}_1$)	*CP*($\hat{\beta}_1$)
1000	Bayes Pooled SP	-0.231	0.011	1.231	0	0.339	0.006	1.661	0
	Bayes Exch. SP	-0.002	0.012	1.002	0	0.225	0.006	1.775	0
	Bayes Spatial SP	1.036	0.03	0.103	0.91	1.990	0.007	0.054	0.92
2000	Bayes Pooled SP	-0.2	0.001	1.2	0	0.285	0	1.715	0
	Bayes Exch. SP	-0.056	0.009	1.056	0	0.214	0.003	1.786	0
	Bayes Spatial SP	0.97	0.012	0.076	0.88	2	0.007	0.043	0.97

Note: True parameter values are $\beta_0 = 1$, $\beta_1 = 2$.

and $\{\widehat{v}_1 ... \widehat{v}_5\}$ in $\mathbf{V}$ from the Bayesian Spatial SP Weibull model converge to their true values for each sample size with negligible RMSEs and almost always extremely high CPs. This holds for each of the four scenarios being examined here with respect to the level of the immune fraction and the proportion of units with spatially dependent frailties. By contrast, the values of $\{\widehat{w}_1 ... \widehat{w}_5\}$ and $\{\widehat{v}_1 ... \widehat{v}_5\}$ from the Bayesian Exchangeable SP Weibull model deviate sharply from their true values and exhibit high RMSE but low CPs that are frequently 0 for each of the four conditions listed above.

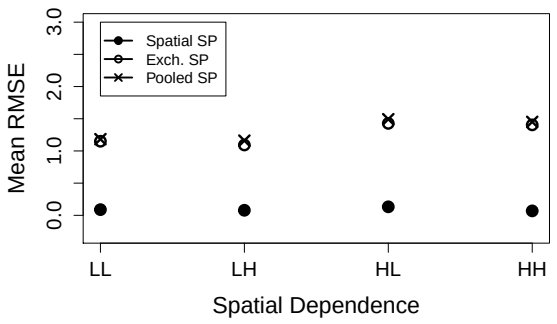
Table A.13: Markov Chain Monte Carlo (MCMC) γ Estimates for Experiment 4 (Varying Immune Fraction and Spatial Dependence, Spatial SP Weibull D.G.P.)

	Experiment 4: Immune Fraction = 25%, Spatial Dependence = 30%											
#Obs.	Model	$\hat{\gamma}_0$	$MCSE(\hat{\gamma}_0)$	$RMSE(\hat{\gamma}_0)$	$CP(\hat{\gamma}_0)$	$\hat{\gamma}_1$	$MCSE(\hat{\gamma}_1)$	$RMSE(\hat{\gamma}_1)$	$CP(\hat{\gamma}_1)$	$\hat{\gamma}_2$	$MCSE(\hat{\gamma}_2)$	$RMSE(\hat{\gamma}_2)$	$CP(\hat{\gamma}_2)$
1000	Bayes Pooled SP	-3.026	0.153	0.399	0.97	-0.038	0.022	0.962	0	-2.433	0.052	5.933	0
	Bayes Exch. SP	-3.738	0.208	0.795	0.77	-0.016	0.03	0.984	0	-3.391	0.113	6.891	0
	Bayes Spatial SP	-2.812	0.133	0.512	0.82	-0.925	0.031	0.154	0.86	3.071	0.1	0.598	0.77
2000	Bayes Pooled SP	-3.256	0.102	0.33	0.90	-0.028	0.012	0.972	0	-2.202	0.039	5.702	0
	Bayes Exch. SP	-3.255	0.102	0.34	0.92	-0.019	0.015	0.981	0	-3.169	0.086	6.669	0
	Bayes Spatial SP	-2.886	0.077	0.29	0.88	-0.95	0.017	0.085	0.91	3.312	0.071	0.294	0.84
	Experiment 4: Immune Fraction = 25%, Spatial Dependence = 80%												
#Obs.	Model	$\hat{\gamma}_0$	$MCSE(\hat{\gamma}_0)$	$RMSE(\hat{\gamma}_0)$	$CP(\hat{\gamma}_0)$	$\hat{\gamma}_1$	$MCSE(\hat{\gamma}_1)$	$RMSE(\hat{\gamma}_1)$	$CP(\hat{\gamma}_1)$	$\hat{\gamma}_2$	$MCSE(\hat{\gamma}_2)$	$RMSE(\hat{\gamma}_2)$	$CP(\hat{\gamma}_2)$
1000	Bayes Pooled SP	-2.958	0.143	0.379	0.96	-0.027	0.022	0.973	0	-2.566	0.05	6.066	0
	Bayes Exch. SP	-3.394	0.147	0.559	0.88	-0.032	0.026	0.968	0	-3.357	0.112	6.857	0
	Bayes Spatial SP	-2.872	0.13	0.412	0.93	-0.99	0.029	0.13	0.92	3.385	0.099	0.373	0.86
2000	Bayes Pooled SP	-2.945	0.086	0.257	0.96	-0.031	0.013	0.969	0	-2.362	0.032	5.862	0
	Bayes Exch. SP	-3.148	0.092	0.354	0.91	-0.022	0.014	0.978	0	-3.151	0.056	6.651	0
	Bayes Spatial SP	-2.881	0.055	0.281	0.89	-0.969	0.014	0.092	0.91	3.234	0.05	0.391	0.88
	Experiment 4: Immune Fraction = 60%, Spatial Dependence = 30%												
#Obs.	Model	$\hat{\gamma}_0$	$MCSE(\hat{\gamma}_0)$	$RMSE(\hat{\gamma}_0)$	$CP(\hat{\gamma}_0)$	$\hat{\gamma}_1$	$MCSE(\hat{\gamma}_1)$	$RMSE(\hat{\gamma}_1)$	$CP(\hat{\gamma}_1)$	$\hat{\gamma}_2$	$MCSE(\hat{\gamma}_2)$	$RMSE(\hat{\gamma}_2)$	$CP(\hat{\gamma}_2)$
1000	Bayes Pooled SP	-5.009	0.44	1.057	0.88	-0.24	0.116	2.24	0	-2.438	0.133	3.438	0.02
	Bayes Exch. SP	-4.951	0.343	1.087	0.83	-0.046	0.052	2.046	0	-3.652	0.166	4.652	0
	Bayes Spatial SP	-3.664	0.15	0.553	0.78	1.891	0.059	0.2	0.84	0.933	0.035	0.165	0.86
2000	Bayes Pooled SP	-4.552	0.02	0.576	0.14	-0.052	0.002	2.052	0	-2.399	0.007	3.399	0
	Bayes Exch. SP	-4.387	0.168	0.446	0.72	-0.016	0.007	2.016	0	-3.446	0.124	4.446	0
	Bayes Spatial SP	-3.993	0.109	0.284	0.94	1.948	0.035	0.114	0.96	0.946	0.028	0.1	0.91
	Experiment 4: Immune Fraction = 60%, Spatial Dependence = 80%												
#Obs.	Model	$\hat{\gamma}_0$	$MCSE(\hat{\gamma}_0)$	$RMSE(\hat{\gamma}_0)$	$CP(\hat{\gamma}_0)$	$\hat{\gamma}_1$	$MCSE(\hat{\gamma}_1)$	$RMSE(\hat{\gamma}_1)$	$CP(\hat{\gamma}_1)$	$\hat{\gamma}_2$	$MCSE(\hat{\gamma}_2)$	$RMSE(\hat{\gamma}_2)$	$CP(\hat{\gamma}_2)$
1000	Bayes Pooled SP	-4.797	0.334	0.872	0.93	-0.059	0.039	2.059	0	-3.036	0.119	4.036	0
	Bayes Exch. SP	-4.779	0.305	0.898	0.86	0.016	0.04	1.984	0	-3.62	0.169	4.62	0
	Bayes Spatial SP	-3.94	0.136	0.411	0.89	2.005	0.047	0.158	0.94	1.009	0.034	0.126	0.92
2000	Bayes Pooled SP	-4.395	0.017	0.475	0.22	-0.058	0.002	2.058	0	-2.73	0.006	3.73	0
	Bayes Exch. SP	-4.417	0.125	0.537	0.49	0.001	0.006	1.999	0	-3.466	0.086	4.466	0
	Bayes Spatial SP	-3.936	0.094	0.299	0.92	1.977	0.033	0.114	0.94	0.977	0.025	0.089	0.94

Note: True parameter values are $\gamma_0 = -3$, $\gamma_1 = -1$, and $\gamma_2 = 3.5$ for Immune Fraction = 25%; $\gamma_0 = -4$, $\gamma_1 = 2$, and $\gamma_2 = 1$ for Immune Fraction = 60%.

Figure A.3: Average RMSE for β and γ Estimates in Experiment 4

(a) Average RMSE for $\hat{\beta}$



(b) Average RMSE for $\hat{\gamma}$

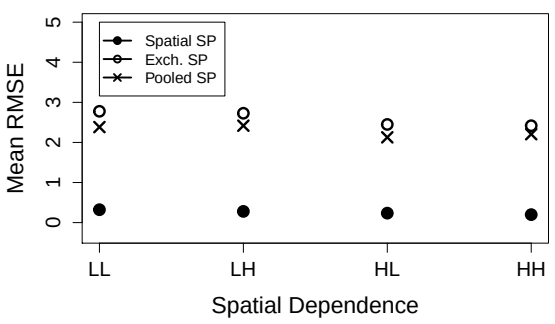


Table A.14: Markov Chain Monte Carlo (MCMC) $\boldsymbol{W}$ Estimates for Experiment 4

#Obs.	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_5$	$\hat{w}_5$	$\hat{w}_5$	$\hat{w}_5$
	Experiment 4: Immune Fraction = 25%, Spatial Dependence = 30%																				
1000	True Value	2.07	–	–	–	-1.804	–	–	–	-1.62	–	–	–	0.65	–	–	–	0.704	–	–	–
	Bayes Exch. SP	0.086	0.027	1.984	0	-0.626	0.037	1.178	0	-0.703	0.04	0.917	0	1.303	0.04	0.652	0.08	-0.06	0.032	0.763	0
	True Value	2.07	–	–	–	-1.804	–	–	–	-1.62	–	–	–	0.65	–	–	–	0.704	–	–	–
	Bayes Spatial SP	1.989	0.03	0.158	0.83	-1.733	0.041	0.136	0.85	-1.655	0.059	0.154	0.84	0.736	0.028	0.172	0.83	0.663	0.032	0.112	0.84
2000	True Value	0.188	–	–	–	1.399	–	–	–	2.667	–	–	–	-6.116	–	–	–	1.862	–	–	–
	Bayes Exch. SP	0.26	0.042	0.098	0.93	0.376	0.04	1.024	0	0.988	0.04	1.678	0	-2.089	0.147	4.027	0	0.466	0.037	1.397	0
	True Value	-2.58	–	–	–	-1.762	–	–	–	2.073	–	–	–	-0.919	–	–	–	3.188	–	–	–
	Bayes Spatial SP	-2.526	0.038	0.109	0.72	-1.75	0.034	0.099	0.72	2.021	0.034	0.111	0.77	-0.901	0.03	0.085	0.82	3.156	0.029	0.097	0.75
	Experiment 4: Immune Fraction = 25%, Spatial Dependence = 80%																				
#Obs.	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_5$	$\hat{w}_5$	$\hat{w}_5$	$\hat{w}_5$
1000	True Value	-2.223	–	–	–	-2.466	–	–	–	-1.106	–	–	–	3.688	–	–	–	2.106	–	–	–
	Bayes Exch. SP	-0.38	0.048	1.843	0	-0.987	0.044	1.479	0	-0.101	0.039	1.004	0	1.054	0.032	2.635	–	0.415	0.03	1.691	0
	True Value	-2.223	–	–	–	-2.466	–	–	–	-1.106	–	–	–	3.688	–	–	–	2.106	–	–	–
	Bayes Spatial SP	-2.2	0.044	0.142	0.88	-2.423	0.079	0.214	0.82	-1.117	0.037	0.138	0.82	3.647	0.033	0.115	0.88	2.094	0.032	0.103	0.89
2000	True Value	-2.157	–	–	–	-1.184	–	–	–	1.642	–	–	–	-0.335	–	–	–	2.036	–	–	–
	Bayes Exch. SP	-0.248	0.039	1.909	0	-0.071	0.034	1.113	0	-0.154	0.028	1.796	0	0.198	0.029	0.533	0	0.276	0.029	1.76	0
	True Value	-2.157	–	–	–	-1.184	–	–	–	1.642	–	–	–	-0.335	–	–	–	2.036	–	–	–
	Bayes Spatial SP	-2.086	0.027	0.146	0.83	-1.142	0.024	0.11	0.84	1.557	0.025	0.15	0.79	-0.313	0.019	0.081	0.9	1.985	0.02	0.114	0.82
	Experiment 4: Immune Fraction = 60%, Spatial Dependence = 30%																				
#Obs.	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_5$	$\hat{w}_5$	$\hat{w}_5$	$\hat{w}_5$
1000	True Value	3.295	–	–	–	-2.183	–	–	–	-1.923	–	–	–	0.141	–	–	–	0.67	–	–	–
	Bayes Exch. SP	0.059	0.026	3.236	0	-0.423	0.039	1.759	0	-0.512	0.041	1.412	0	0.792	0.034	0.651	0.01	0.084	0.03	0.586	0.01
	True Value	3.295	–	–	–	-2.183	–	–	–	-1.923	–	–	–	0.141	–	–	–	0.67	–	–	–
	Bayes Spatial SP	3.235	0.052	0.175	0.90	-2.161	0.052	0.187	0.84	-1.906	0.097	0.264	0.89	0.148	0.03	0.124	0.91	0.684	0.035	0.123	0.94
2000	True Value	-0.429	–	–	–	1.284	–	–	–	3.076	–	–	–	-5.069	–	–	–	1.138	–	–	–
	Bayes Exch. SP	0.027	0.035	0.456	0	0.118	0.027	1.166	0	0.482	0.028	2.594	0	-0.867	0.066	4.202	–	0.241	0.03	0.898	0
	True Value	-3.745	–	–	–	-2.589	–	–	–	2.835	–	–	–	-0.019	–	–	–	3.517	–	–	–
	Bayes Spatial SP	-3.372	0.09	0.395	0.52	-2.226	0.09	0.388	0.49	1.418	0.45	1.445	0.34	0.346	0.089	0.374	0.5	3.835	0.093	0.335	0.66
	Experiment 4: Immune Fraction = 60%, Spatial Dependence = 80%																				
#Obs.	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_5$	$\hat{w}_5$	$\hat{w}_5$	$\hat{w}_5$
1000	True Value	-2.223	–	–	–	-2.466	–	–	–	-1.106	–	–	–	3.688	–	–	–	2.106	–	–	–
	Bayes Exch. SP	-0.255	0.041	1.968	0	-0.629	0.039	1.837	0	-0.055	0.036	1.051	0	0.737	0.03	2.951	0	0.202	0.028	1.904	0
	True Value	-2.223	–	–	–	-2.466	–	–	–	-1.106	–	–	–	3.688	–	–	–	2.106	–	–	–
	Bayes Spatial SP	-2.205	0.047	0.164	0.89	-2.323	0.134	0.345	0.89	-1.137	0.045	0.172	0.90	3.613	0.04	0.154	0.87	2.052	0.041	0.148	0.89
2000	True Value	-0.011	–	–	–	0.817	–	–	–	2.013	–	–	–	-3.713	–	–	–	0.894	–	–	–
	Bayes Exch. SP	0.004	0.031	0.066	0.86	0.12	0.028	0.697	0	0.295	0.027	1.718	0	-0.547	0.043	3.165	0	0.129	0.029	0.765	0
	True Value	-2.157	–	–	–	-1.184	–	–	–	1.642	–	–	–	-0.335	–	–	–	2.036	–	–	–
	Bayes Spatial SP	-2.119	0.035	0.114	0.83	-1.146	0.034	0.098	0.90	1.535	0.082	0.183	0.89	-0.304	0.028	0.088	0.92	2.033	0.035	0.112	0.87

Table A.15: Markov Chain Monte Carlo (MCMC) $\boldsymbol{V}$ Estimates for Experiment 4

#Obs.	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$
	Experiment 4: Immune Fraction =25%, Spatial Dependence = 30%																				
1000	True Value	2.407	–	–	–	0.499	–	–	–	1.983	–	–	–	-5.315	–	–	–	0.426	–	–	–
	Bayes Exch. SP	-3.58	0.179	5.987	0	2.676	0.173	2.177	0.06	2.356	0.175	0.574	0.85	-0.191	0.084	5.124	0	-1.261	0.126	1.687	0.08
	True Value	2.407	–	–	–	0.499	–	–	–	1.983	–	–	–	-5.315	–	–	–	0.426	–	–	–
	Bayes Spatial SP	1.861	0.089	0.606	0.75	0.639	0.07	0.297	0.89	1.773	0.092	0.391	0.89	-4.662	0.195	0.833	0.80	0.389	0.059	0.281	0.92
2000	True Value	-0.457	–	–	–	0.76	–	–	–	-0.591	–	–	–	-0.005	–	–	–	0.293	–	–	–
	Bayes Exch. SP	-0.09	0.062	0.39	0.77	-1.989	0.089	2.749	0	-3.483	0.112	2.892	0	8.054	0.413	8.059	0	-2.492	0.091	2.785	0
	True Value	-2.088	–	–	–	-1.829	–	–	–	4.147	–	–	–	-2.225	–	–	–	1.995	–	–	–
	Bayes Spatial SP	-1.975	0.063	0.294	0.94	-1.615	0.057	0.277	0.85	3.805	0.094	0.433	0.76	-2.122	0.057	0.253	0.91	1.907	0.059	0.291	0.89
	Experiment 4: Immune Fraction =25%, Spatial Dependence = 80%																				
#Obs.	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$
1000	True Value	-1.397	–	–	–	2.481	–	–	–	-0.752	–	–	–	-0.800	–	–	–	0.468	–	–	–
	Bayes Exch. SP	3.504	0.17	4.901	0	3.271	0.182	1.093	0.55	1.794	0.116	2.546	0	-5.383	0.224	4.583	0	-3.186	0.141	3.654	0.01
	True Value	-1.397	–	–	–	2.481	–	–	–	-0.752	–	–	–	-0.800	–	–	–	0.468	–	–	–
	Bayes Spatial SP	-1.329	0.079	0.328	0.92	2.389	0.109	0.359	0.91	-0.653	0.061	0.318	0.90	-0.846	0.063	0.269	0.95	0.438	0.067	0.286	0.93
2000	True Value	-1.692	–	–	–	-1.34	–	–	–	3.399	-	–	–	-1.402	–	–	–	1.034	–	–	–
	Bayes Exch. SP	3.237	0.09	4.928	0	1.675	0.072	3.015	0	-2.594	0.076	5.993	0	0.612	0.051	2.014	0	-2.93	0.077	3.964	0
	True Value	-1.692	–	–	–	-1.34	–	–	–	3.399	–	–	–	-1.402	–	–	–	1.034	–	–	–
	Bayes Spatial SP	-1.487	0.04	0.319	0.95	-1.198	0.034	0.271	0.91	3.066	0.062	0.478	0.84	-1.302	0.031	0.233	0.92	0.921	0.03	0.279	0.93
	Experiment 4: Immune Fraction =60%, Spatial Dependence = 30%																				
#Obs.	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$
1000	True Value	2.652	–	–	–	-0.046	–	–	–	2.052	–	–	–	-4.398	–	–	–	-0.26	–	–	-
	Bayes Exch. SP	-6.905	0.672	9.557	0	3.912	0.331	3.958	0.02	3.333	0.278	1.357	0.55	0.686	0.168	5.083	0	-1.026	0.232	0.911	0.72
	True Value	2.652	–	–	–	-0.046	–	–	–	2.052	–	–	–	-4.398	–	–	–	-0.26	–	–	–
	Bayes Spatial SP	2.484	0.097	0.371	0.89	0.095	0.072	0.272	0.94	1.837	0.106	0.418	0.86	-4.097	0.149	0.472	0.85	-0.319	0.05	0.231	0.92
2000	True Value	-0.022	–	–	–	1.699	–	–	–	-0.212	–	–	–	-1.273	–	–	–	-0.192	–	–	–
	Bayes Exch. SP	0.752	0.076	0.78	0.42	-2.219	0.113	3.918	0	-4.939	0.176	4.727	0	8.14	0.331	9.413	0	-1.735	0.085	1.542	0.03
	True Value	-2.98	–	–	–	-2.613	–	–	–	5.838	–	–	–	-1.987	–	–	–	1.742	–	–	–
	Bayes Spatial SP	-2.714	0.104	0.36	0.87	-2.399	0.076	0.311	0.81	4.987	0.215	0.862	0.57	-1.695	0.053	0.319	0.74	1.821	0.062	0.216	0.93
	Experiment 4: Immune Fraction =60%, Spatial Dependence = 80%																				
#Obs.	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$
1000	True Value	2.652	–	–	–	-0.046	–	–	–	2.052	–	–	–	-4.398	–	–	–	-0.26	–	–	–
	Bayes Exch. SP	-6.905	0.672	9.557	0	3.912	0.331	3.958	0.02	3.333	0.278	1.357	0.55	0.686	0.168	5.083	0	-1.026	0.232	0.911	0.72
	True Value	2.652	–	–	–	-0.046	–	–	–	2.052	–	–	–	-4.398	–	–	–	-0.26	–	–	–
	Bayes Spatial SP	2.484	0.097	0.371	0.89	0.095	0.072	0.272	0.94	1.837	0.106	0.418	0.86	-4.097	0.149	0.472	0.85	-0.319	0.05	0.231	0.92
2000	True Value	-0.022	–	–	–	1.699	–	–	–	-0.212	–	–	–	-1.273	–	–	–	-0.192	–	–	–
	Bayes Exch. SP	0.752	0.076	0.78	0.42	-2.219	0.113	3.918	0	-4.939	0.176	4.727	0	8.14	0.331	9.413	0	-1.735	0.085	1.542	0.03
	True Value	-2.98	–	–	–	-2.613	–	–	–	5.838	–	–	–	-1.987	–	–	–	1.742	–	–	–
	Bayes Spatial SP	-2.714	0.104	0.36	0.87	-2.399	0.076	0.311	0.81	4.987	0.215	0.862	0.57	-1.695	0.053	0.319	0.74	1.821	0.062	0.216	0.93

Finally, we evaluated the robustness of our MC findings to four additional scenarios. First, Experiment 5 re-evaluates our primary MC findings when each of the three Bayesian SP Weibull models are specified with a multivariate Cauchy prior as opposed to our favored multivariate normal prior. The MC results from this exercise—reported below in Tables A.16 and A.17—reveal that the Bayesian Spatial SP Weibull model outperform the Bayesian Pooled SP and Exchangeable SP Weibull models in the context of a multivariate Cauchy prior when using the d.g.p. employed for Experiments 1-4. Second, in Experiment 6, we analyze the Bayesian Spatial SP Log-Logistic model with spatially dependent split and survival stage frailties applied to a SP Log-logistic d.g.p (where $\alpha = 25\%$ and the share of adjacent units with spatially dependent frailties is 40%). The MC results of the obtained β, γ, $\mathbf{W}$ and $\mathbf{V}$ from the Bayesian Spatial SP *Log-Logistic* model substantially outperform those obtained from the remaining two nonspatial Bayesian SP Log-logistic models in this experiment (see Tables A.18 – A.19).

Table A.16: Markov Chain Monte Carlo (MCMC) β, γ Estimates for Experiment 5

	Experiment 5: Cauchy Prior, Spatial SP Weibull D.G.P. – β Estimates							
#Obs.	Model	$\hat{\beta}_0$	$MCSE(\hat{\beta}_0)$	$RMSE(\hat{\beta}_0)$	$CP(\hat{\beta}_0)$	$\hat{\beta}_1$	$MCSE(\hat{\beta}_1)$	$RMSE(\hat{\beta}_1)$	$CP(\hat{\beta}_1)$
1000	Bayes Pooled SP	0.538	0.016	0.462	0.040	0.077	0.008	1.923	0
	Bayes Exch. SP	0.586	0.014	0.414	0.020	0.142	0.007	1.858	0
	Bayes Spatial SP	0.976	0.011	0.086	0.910	1.947	0.005	0.093	0.900
1500	Bayes Pooled SP	0.631	0.011	0.369	0	0.0360	0.006	1.964	0
	Bayes Exch. SP	0.765	0.015	0.235	0.310	0.112	0.005	1.888	0
	Bayes Spatial SP	0.971	0.005	0.069	0.890	1.888	0.007	0.148	0.860
2000	Bayes Pooled SP	0.684	0.011	0.316	0.050	-0.035	0.005	2.035	0
	Bayes Exch. SP	0.922	0.009	0.089	0.790	-0.079	0.004	2.079	0
	Bayes Spatial SP	0.997	0.007	0.044	0.900	1.974	0.009	0.065	0.900

	Experiment 5: Cauchy Prior, Spatial SP Weibull D.G.P. – γ Estimates											
#Obs.	Model	$\hat{\gamma}_0$	$MCSE(\hat{\gamma}_0)$	$RMSE(\hat{\gamma}_0)$	$CP(\hat{\gamma}_0)$	$\hat{\gamma}_1$	$MCSE(\hat{\gamma}_1)$	$RMSE(\hat{\gamma}_1)$	$CP(\hat{\gamma}_1)$	$\hat{\gamma}_2$	$MCSE(\hat{\gamma}_2)$	$RMSE(\hat{\gamma}_2)$	$CP(\hat{\gamma}_2)$
1000	Bayes Pooled SP	-3.131	0.159	0.407	0.960	-0.015	0.022	0.985	0	-2.449	0.054	5.949	0
	Bayes Exch. SP	-3.741	0.189	0.815	0.750	-0.002	0.03	0.998	0	-3.350	0.101	6.85	0
	Bayes Spatial SP	-2.774	0.124	0.478	0.830	-0.930	0.029	0.136	0.910	3.190	0.094	0.445	0.830
1500	Bayes Pooled SP	-3.054	0.100	0.255	0.99	-0.012	0.016	0.988	0	-2.476	0.038	5.976	0
	Bayes Exch. SP	-3.496	0.138	0.573	0.820	-0.010	0.02	0.99	0	-3.290	0.103	6.79	0
	Bayes Spatial SP	-2.867	0.092	0.341	0.880	-0.954	0.022	0.121	0.93	3.143	0.079	0.51	0.850
2000	Bayes Pooled SP	-3.264	0.104	0.345	0.91	-0.022	0.013	0.978	0	-2.208	0.040	5.708	0
	Bayes Exch. SP	-3.040	0.090	0.306	0.900	-0.019	0.015	0.981	0	-3.190	0.075	6.690	0
	Bayes Spatial SP	-2.920	0.081	0.305	0.870	-0.982	0.020	0.086	0.940	3.370	0.082	0.301	0.800

Note: True parameter values are $\beta_0 = 1$, $\beta_1 = 2$, $\gamma_0 = -3$, $\gamma_1 = -1$, and $\gamma_2 = 3.5$.

Third, in Experiment 7, we assess the performance of the Bayesian Spatial SP Weibull model in which we include spatially dependent (i.e. autocorrelated) frailties between adjacent units in just the model's survival stage but *not* the split stage (the model's split stage neither includes spatial nor nonspatial i.i.d random effects). This model is evaluated using the d.g.p

Table A.17: Markov Chain Monte Carlo (MCMC) $\boldsymbol{W}$ and $\boldsymbol{V}$ Estimates for Experiment 5

#Obs.	Experiment 5: Cauchy Prior and Spatial SP Weibull D.G.P. – $\boldsymbol{W}$ estimates																			
	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_5$	$\hat{w}_5$	$\hat{w}_5$	$\hat{w}_5$
1000	True Value	2.07	–	–	–	-1.804	–	–	–	-1.62	–	–	–	0.65	–	–	–	0.704	–	–	–
	Bayes Exch. SP	0.113	0.029	1.957	0	-0.644	0.037	1.16	0	-0.691	0.042	0.929	0	1.269	0.039	0.618	0.06	-0.046	0.031	0.75	0
	True Value	2.07	–	–	–	-1.804	–	–	–	-1.62	–	–		0.65	–	–	–	0.704	–	–	–
	Bayes Spatial SP	2.014	0.033	0.129	0.87	-1.754	0.044	0.132	0.82	-1.648	0.056	0.169	0.86	0.702	0.026	0.125	0.86	0.686	0.032	0.098	0.88
1500	True Value	1.141	–	–	–	2.611	–	–	–	-4.585	–	–	–	2.227	–	–	–	-1.394	–	–	–
	Bayes Exch. SP	0.672	0.033	0.469	0.1	1.076	0.033	1.535	0	-1.647	0.071	2.938	0	1.003	0.03	1.224	0	-1.104	0.039	0.29	0.57
	True Value	-1.803	–	–	–	1.159	–	–	–	0.579	–	–	–	-1.714	–	–	–	1.779	–	–	–
	Bayes Spatial SP	-1.711	0.036	0.142	0.78	1.11	0.026	0.082	0.88	0.521	0.036	0.147	0.83	-1.612	0.032	0.181	0.84	1.692	0.029	0.144	0.86
2000	True Value	-2.58	–	–	–	-1.762	–	–	–	2.073	–	–	–	-0.919	–	–	–	3.188	–	–	–
	Bayes Exch. SP	-0.256	0.04	2.324	0	-0.107	0.035	1.655	0	-0.237	0.029	2.31	0	0.275	0.031	1.194	0	0.325	0.028	2.863	0
	True Value	-2.58	–	–	–	-1.762	–	–	–	2.073	–	–	–	-0.919	–	–	–	3.188	–	–	–
	Bayes Spatial SP	-2.554	0.041	0.113	0.83	-1.731	0.032	0.103	0.73	2.045	0.034	0.117	0.74	-0.904	0.031	0.085	0.80	3.144	0.033	0.104	0.78
#Obs.	Experiment 5: Cauchy Prior and Spatial SP Weibull D.G.P. – $\boldsymbol{V}$ estimates																			
	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$
1000	True Value	2.407	–	–	–	0.499	–	–	–	1.983	–	–	–	-5.315	–	–	–	0.426	–	–	–
	Bayes Exch. SP	-3.482	0.187	5.889	0	2.542	0.163	2.043	0.05	2.322	0.158	0.597	0.82	-0.235	0.084	5.08	0	-1.147	0.127	1.573	0.13
	True Value	2.407	–	–	–	0.499	–	–	–	1.983	–	–	–	-5.315	–	–	–	0.426	–	–	–
	Bayes Spatial SP	1.991	0.087	0.506	0.75	0.638	0.071	0.294	0.90	1.772	0.09	0.39	0.88	-4.767	0.187	0.70	0.83	0.366	0.057	0.264	0.93
1500	True Value	-2.597	–	–	–	-1.999	–	–	–	0.98	–	–	–	-1.783	–	–	–	5.399	–	–	–
	Bayes Exch. SP	-1.441	0.082	1.156	0.22	-3.625	0.128	1.626	0.06	6.487	0.303	5.507	0	-3.12	0.126	1.337	0.07	1.698	0.117	3.701	0
	True Value	-0.621	–	–	–	-0.558	–	–	–	2.908	–	–	–	-3.485	–	–	–	1.757	–	–	–
	Bayes Spatial SP	-0.666	0.05	0.374	0.83	-0.458	0.045	0.231	0.91	2.51	0.079	0.464	0.85	-3.006	0.1	0.562	0.84	1.62	0.06	0.444	0.84
2000	True Value	-2.088	–	–	–	-1.829	–	–	–	4.147	–	–	–	-2.225	–	–	–	1.995	–	–	–
	Bayes Exch. SP	3.902	0.109	5.991	0	2.684	0.086	4.513	0	-3.352	0.102	7.499	0	1.558	0.06	3.783	0	-4.792	0.122	6.787	0
	True Value	-2.088	–	–	–	-1.829	–	–	–	4.147	–	–	–	-2.225	–	–	–	1.995	–	–	–
	Bayes Spatial SP	-1.986	0.064	0.29	0.93	-1.664	0.057	0.27	0.89	3.877	0.107	0.407	0.83	-2.161	0.06	0.264	0.89	1.934	0.06	0.295	0.90

employed for Experiment 1 where the immune fraction is $\alpha = 25\%$, while the proportion of units with spatially dependent frailties is 40%. The $\hat{\beta}'$s and $\hat{\gamma}'$s as well as each $\{\widehat{w}_1 ... \widehat{w}_5\}$ in **W** obtained from the aforementioned model—that incorporates only survival-stage spatially autocorrelated frailties—deviates from their true theoretical values and exhibit high RMSEs but low CPs (Table A.20). We also find (but do not report to save space) that the Bayesian Spatial SP Weibull model with spatially dependent (i.e. autocorrelated) frailties in only the split stage—but not the survival stage—performs poorly when using the d.g.p from Experiment 1. Thus, our Bayesian Spatial SP Weibull model with spatially autocorrelated frailties in the split and survival stage outperforms the Bayesian Spatial SP Weibull model with spatially dependent frailties in only the split or just the survival stage.

Fourth, Experiment 8 evaluates the MC results from just the Bayesian Spatial SP Weibull model when we simulate a SP Weibull distributed outcome variable (where $\alpha = 25\%$) *without* any frailties (that is, without spatially dependent frailties and without nonspatial i.i.d. random effects) whatsoever. The results from this experiment are presented in Table A.20. We find across all obtained $\hat{\beta}$ and $\widehat{\gamma}$ parameters of interest that the Bayesian Spatial SP Weibull model recovered average parameter estimates that are largely identical to their true values with low

RMSEs as well as high coverage probabilities when the d.g.p. is SP Weibull distributed (for $\alpha = 25\%$) *without* any frailties. Thus, in instances where researchers encounter a SP Weibull distributed outcome variable that does not exhibit spatial dependence among units in the survival data, they can still use the Bayesian Spatial SP Weibull model to obtain reliable estimates for the split and survival stage.

Table A.18: Markov Chain Monte Carlo (MCMC) β, γ Estimates for Experiment 6 (Log-logistic Likelihood, Spatial SP Weibull D.G.P.)

	Experiment 6: Log-logistic Likelihood, Spatial SP Weibull D.G.P. – β Estimates								
#Obs.	Model	$\hat{\beta}_0$	$MCSE(\hat{\beta}_0)$	$RMSE(\hat{\beta}_0)$	$CP(\hat{\beta}_0)$	$\hat{\beta}_1$	$MCSE(\hat{\beta}_1)$	$RMSE(\hat{\beta}_1)$	$CP(\hat{\beta}_1)$
1000	Bayes Spatial SP	0.956	0.144	0.376	0.98	1.987	0.037	0.127	0.97
1500	Bayes Spatial SP	1.4	0.004	0.413	0.08	1.785	0.006	0.246	0.84
2000	Bayes Spatial SP	1.43	0.004	0.437	0.04	1.885	0.007	0.143	0.85

	Experiment 6: Log-logistic Likelihood, Spatial SP Weibull D.G.P. – γ Estimates												
#Obs.	Model	$\hat{\gamma}_0$	$MCSE(\hat{\gamma}_0)$	$RMSE(\hat{\gamma}_0)$	$CP(\hat{\gamma}_0)$	$\hat{\gamma}_1$	$MCSE(\hat{\gamma}_1)$	$RMSE(\hat{\gamma}_1)$	$CP(\hat{\gamma}_1)$	$\hat{\gamma}_2$	$MCSE(\hat{\gamma}_2)$	$RMSE(\hat{\gamma}_2)$	$CP(\hat{\gamma}_2)$
1000	Bayes Spatial SP	-4.073	1.537	2.154	0.90	-2.190	0.449	1.246	0.93	5.676	1.085	2.621	0.82
1500	Bayes Spatial SP	-3.247	0.088	0.363	0.93	-0.954	0.017	0.165	0.88	3.117	0.073	0.784	0.85
2000	Bayes Spatial SP	-3.208	0.068	0.314	0.91	-0.983	0.016	0.12	0.91	3.349	0.07	0.497	0.90

Note: True parameter values are $\beta_0 = 1$, $\beta_1 = 2$, $\gamma_0 = -3$, $\gamma_1 = -1$, and $\gamma_2 = 3.5$.

Table A.19: Markov Chain Monte Carlo (MCMC) $\boldsymbol{W}$ and $\boldsymbol{V}$ Estimates for Experiment 6

	Experiment 6: Log-logistic Likelihood and Spatial SP Weibull D.G.P. – $\boldsymbol{W}$ estimates																			
#Obs.	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_1$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_2$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_3$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_4$	$\hat{w}_5$	$\hat{w}_5$	$\hat{w}_5$	$\hat{w}_5$
1000	True Value	-3.368	–	–	–	-4.014	–	–	–	-1.999	–	–	–	6.061	–	–	–	3.321	–	–	–
	Bayes Spatial SP	-3.273	0.17	0.377	0.96	-5.15	0.885	1.479	0.88	-1.816	0.143	0.342	0.97	6.527	0.179	0.579	0.91	3.711	0.173	0.53	0.94
1500	True Value	-1.803	–	–	–	1.159	–	–	–	0.579	–	–	–	-1.714	–	–	–	1.779	–	–	–
	Bayes Spatial SP	-1.681	0.028	0.203	0.81	1.128	0.02	0.105	0.88	0.433	0.028	0.233	0.85	-1.51	0.025	0.291	0.84	1.631	0.023	0.22	0.87
2000	True Value	-2.58	–	–	–	-1.762	–	–	–	2.073	–	–	–	-0.919	–	–	–	3.188	–	–	–
	Bayes Spatial SP	-2.476	0.029	0.185	0.81	-1.674	0.027	0.166	0.81	1.913	0.028	0.212	0.83	-0.841	0.024	0.133	0.87	3.078	0.024	0.185	0.85
	Experiment 6: Log-logistic Likelihood and Spatial SP Weibull D.G.P. – $\boldsymbol{V}$ estimates																			
#Obs.	Model	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*	*MEAN*	*MCSE*	*RMSE*	*CP*
	Estimate	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_1$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_2$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_3$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_4$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$	$\hat{v}_5$
1000	True Value	-2.029	–	–	–	3.393	–	–	–	-1.239	–	–	–	-1.16	–	–	–	1.035	–	–	–
	Bayes Spatial SP	-3.306	0.535	1.655	0.76	5.637	1.689	2.91	0.72	-1.942	0.381	1.097	0.88	-2.085	0.437	1.336	0.83	1.697	0.674	1.31	0.88
1500	True Value	-0.621	–	–	–	-0.558	–	–	–	2.908	–	–	–	-3.485	–	–	–	1.757	–	–	–
	Bayes Spatial SP	-0.611	0.042	0.477	0.86	-0.457	0.035	0.259	0.88	2.517	0.068	0.596	0.89	-3.036	0.082	0.784	0.87	1.587	0.049	0.602	0.84
2000	True Value	-2.088	–	–	–	-1.829	–	–	–	4.147	–	–	–	-2.225	–	–	–	1.995	–	–	–
	Bayes Spatial SP	-2.014	0.054	0.497	0.87	-1.604	0.045	0.395	0.86	3.836	0.09	0.616	0.89	-2.121	0.046	0.382	0.89	1.903	0.049	0.507	0.88

Table A.20: Markov Chain Monte Carlo (MCMC) β, γ, $\boldsymbol{W}$ Estimates for Experiment 7

Model	**Experiment 7: Spatial SP Weibull D.G.P. – β Estimates**							
	$\hat{\beta}_0$	$MCSE(\hat{\beta}_0)$	$RMSE(\hat{\beta}_0)$	$CP(\hat{\beta}_0)$	$\hat{\beta}_1$	$MCSE(\hat{\beta}_1)$	$RMSE(\hat{\beta}_1)$	$CP(\hat{\beta}_1)$
Bayes Spat Surv 1000	-0.01	0.17	1.01	0.59	3.063	0.141	0.477	0.51
Bayes Spat Surv 1500	2.023	0.105	0.873	0.52	2.919	0.107	0.397	0.55
Bayes Spat Surv 2000	2.07	0.106	0.902	0.49	2.939	0.109	0.373	0.65

Model	**Experiment 7: Spatial SP Weibull D.G.P. – γ Estimates**											
	$\hat{\gamma}_0$	$MCSE(\hat{\gamma}_0)$	$RMSE(\hat{\gamma}_0)$	$CP(\hat{\gamma}_0)$	$\hat{\gamma}_1$	$MCSE(\hat{\gamma}_1)$	$RMSE(\hat{\gamma}_1)$	$CP(\hat{\gamma}_1)$	$\hat{\gamma}_2$	$MCSE(\hat{\gamma}_2)$	$RMSE(\hat{\gamma}_2)$	$CP(\hat{\gamma}_2)$
Bayes Spat Surv 1000	-3.587	1.159	1.556	0.95	-1.499	0.319	0.55	0.97	4.111	0.677	1.017	0.96
Bayes Spat Surv 1500	-1.554	0.048	1.487	0	-0.638	0.013	0.362	0.02	1.969	0.029	1.531	0
Bayes Spat Surv 2000	-0.902	0.036	2.098	0.01	-0.523	0.01	0.477	0	1.463	0.016	2.037	0

Model	**Experiment 7: Spatial SP Weibull D.G.P. – $\boldsymbol{W}$ Estimates**																			
	$\hat{w}_1$	$MCSE(\hat{w}_1)$	$RMSE(\hat{w}_1)$	$CP(\hat{w}_1)$	$\hat{w}_2$	$MCSE(\hat{w}_2)$	$RMSE(\hat{w}_2)$	$CP(\hat{w}_2)$	$\hat{w}_3$	$MCSE(\hat{w}_3)$	$RMSE(\hat{w}_3)$	$CP(\hat{W}_3)$	$\hat{w}_4$	$MCSE(\hat{w}_4)$	$RMSE(\hat{w}_4)$	$CP(\hat{w}_4)$	$\hat{w}_5$	$MCSE(\hat{w}_5)$	$RMSE(\hat{w}_5)$	$CP(\hat{w}_5)$
True Value	-3.368	–	–	–	-4.014	–	–	–	-1.999	–	–	–	6.061	–	–	–	3.321	–	–	–
Bayes Spat Surv 1000	-2.909	0.224	0.516	0.94	-6.828	0.947	2.814	0.49	-1.501	0.194	0.518	0.89	7.016	0.23	0.955	0.67	4.222	0.231	0.903	0.7
True Value	-1.803	–	–	–	1.159	–	–	–	0.579	-	–	–	-1.714	–	–	–	1.779	–	–	–
Bayes Spat Surv 1500	-1.666	0.035	0.156	0.59	1.09	0.027	0.096	0.71	0.504	0.037	0.133	0.72	-1.598	0.034	0.143	0.69	1.669	0.029	0.129	0.69
True Value	-2.58	–	–	–	-1.762	–	–	–	2.073	–	–	–	-0.919	–	–	–	3.188	–	–	–
Bayes Spat Surv 2000	-2.417	0.034	0.168	0.5	-1.677	0.034	0.112	0.68	1.929	0.034	0.156	0.48	-0.907	0.028	0.077	0.80	3.071	0.031	0.132	0.51

Note: True parameter values are $\beta_0 = 1$, $\beta_1 = 2$, $\gamma_0 = -3$, $\gamma_1 = -1$, and $\gamma_2 = 3.5$.

Table A.21: Markov Chain Monte Carlo (MCMC) β, γ, W Estimates for Experiment 8 (Non-Spatial SP Weibull D.G.P.)

#Obs.	Model	**Experiment 8: Non-Spatial SP Weibull D.G.P. – β Estimates**							
		$\hat{\beta}_0$	$MCSE(\hat{\beta}_0)$	$RMSE(\hat{\beta}_0)$	$CP(\hat{\beta}_0)$	$\hat{\beta}_1$	$MCSE(\hat{\beta}_1)$	$RMSE(\hat{\beta}_1)$	$CP(\hat{\beta}_1)$
1000	Bayes Spatial SP	0.924	0.005	0.110	0.88	1.726	0.005	0.319	0.84
1500	Bayes Spatial SP	0.958	0.004	0.070	0.90	1.904	0.004	0.145	0.93
2000	Bayes Spatial SP	0.958	0.003	0.073	0.88	1.864	0.003	0.168	0.90

#Obs.	Model	**Experiment 8: Non-Spatial SP Weibull D.G.P. – γ Estimates**											
		$\hat{\gamma}_0$	$MCSE(\hat{\gamma}_0)$	$RMSE(\hat{\gamma}_0)$	$CP(\hat{\gamma}_0)$	$\hat{\gamma}_1$	$MCSE(\hat{\gamma}_1)$	$RMSE(\hat{\gamma}_1)$	$CP(\hat{\gamma}_1)$	$\hat{\gamma}_2$	$MCSE(\hat{\gamma}_2)$	$RMSE(\hat{\gamma}_2)$	$CP(\hat{\gamma}_2)$
1000	Bayes Spatial SP	-3.059	0.118	0.415	0.83	-0.959	0.025	0.188	0.79	2.904	0.093	0.998	0.76
1500	Bayes Spatial SP	-3.026	0.091	0.334	0.92	-0.978	0.022	0.135	0.86	3.256	0.077	0.515	0.81
2000	Bayes Spatial SP	-3.05	0.079	0.279	0.92	-0.948	0.017	0.123	0.89	3.168	0.062	0.534	0.86

Note: True parameter values are $\beta_0 = 1$, $\beta_1 = 2$, $\gamma_0 = -3$, $\gamma_1 = -1$, and $\gamma_2 = 3.5$.

V Application I: Democratic Survival (Svolik, 2008)

We present below details about the operationalization of the adjacency matrix employed for the Svolik (2008) application as well as several results from this application that were briefly mentioned in the paper. We start with a more detailed description of the adjacency matrix used for the Svolik (2008) application. Next, we discuss in depth the results from the Joint Count and Moran I test statistics. This is followed by a presentation of the (i) choropleth maps that illustrate the posterior mean of the spatial frailties that influences democratic consolidation for each of the three waves of democracy and (ii) marginal effect of per capita income on the probability of democratic consolidation. We then present choropleth maps of the posterior mean of the spatial frailties that influences democratic survival for each of the three waves of democracy. We also illustrate the AFP figures for each key survival stage covariate discussed in the text. Finally, we present autocorrelation plots and report the Geweke as well as Heidelberg-Welch test results for the split and survival stage covariates from our main Bayesian SP models of interest estimated on the Svolik (2008) sample.

Spatial Weights Matrix

Before conducting preliminary spatial tests and estimating our Bayesian Spatial SP Weibull model using Svolik's (2008) data, we first specify the spatial relationship between each country in the sample, all of which are democracies, via the adjacency (a.k.a spatial weights) matrix **A**. This matrix is then used to estimate our Bayesian SP models. Recall that because Svolik's (2008) study examines the survival of democratic institutions, only nominally democratic countries are included in the sample. Our task is to incorporate the spatial relationship of these countries into the estimation procedure so that our unit-level frailty terms can exhibit spatial clustering rather than requiring the untenable assumption that the country-level underlying risk propensities associated with democratic survival and consolidation are randomly distributed across space. We capture the spatial relationships between each country in Svolik's (2008) sample in the symmetric adjacency matrix **A** with elements $a_{ii'}$. We define geographi-

cally proximate democracies as those whose capitals are within 800km of one another. Thus, for each year in Svolik's (2008) data, $a_{ii'} = 1$ for pairs of countries that meet this geographic criterion, and $a_{ii'} = 0$ for pairs of democracies whose capitals are further than 800km from each other. Thus, if an entire row in the matrix is comprised of only 0s, either that country is not proximate to any other country or it is only proximate (according to our definition) to non-democracies. By defining matrix $\mathbf{A}$ in this way, we effectively capture the geographic proximity of the political centers of each democracy in the sample, which is theoretically consistent with the argument we make in the main text (drawn from a vast body of literature on democratic survival) that democratic norms and political incentives to refrain from backsliding internationally reinforce one another within clusters of proximate democratic states.

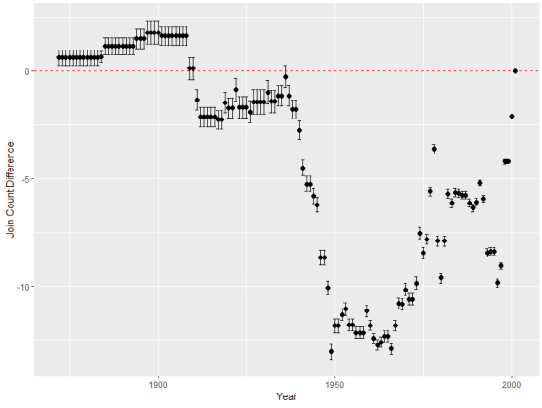
Join Count Test and Moran's I Statistics

As we noted in the main text, we employed two spatial autocorrelation tests—the join count and Moran's I statistics—to detect spatial clustering among possibly consolidated democracies as well as in the survival of democracy. The join count test is a common spatial statistic for binary or categorical outcomes used to assess whether the number of pairs of incongruent neighbors is lower than what is expected by chance (Cliff and Ord 1981). Thus, negative values of this statistic indicate positive spatial clustering. We use this test to evaluate whether potentially consolidated democracies tend to cluster in geographic neighborhoods. The global Moran's I statistic measures overall spatial autocorrelation in a dataset by evaluating how similar a unit is to those surrounding it (Cliff and Ord 1981). Positive values indicate spatial clustering of similar values. We use this statistic to test whether the number of years democracies have survived in $time = t$ clusters in space. To conduct these tests, we construct a separate cross-sectional adjacency matrix with elements $a_{ii'}$ for each year in Svolik's (2008) data, wherein once again proximate pairs of democratic countries (within 800 km of each other) are assigned a weight of 1 ($a_{ii'} = 1$). Our outcome of interest for the joint count analysis is whether a country is right-censored (potentially consolidated) in Svolik's (2008) data, and the outcome for the Moran's I

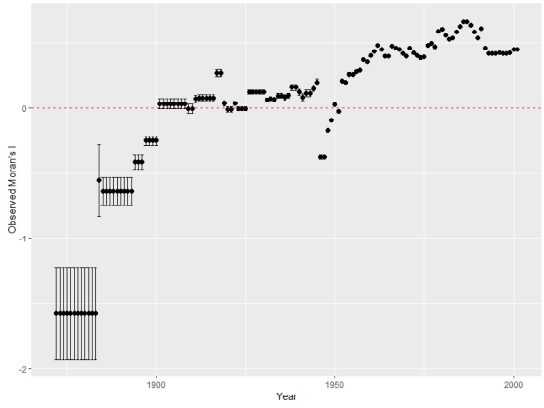
test is the number of years each democracy has survived. Figure A.4a plots the results of the join count tests, displaying the difference between the observed and expected join counts with 95% confidence intervals. The figure shows significant spatial clustering of possibly consolidated democracies, particularly during the twentieth century. This time-period fully includes both the second and third wave of democracy and several years from the first wave of democracy. For our purposes, this preliminary test suggests that countries in the fraction of consolidated democracies may exhibit spatial clustering. Figure A.4b reports the resulting Moran's I statistics for each year. Clearly, a large proportion of the sampling period, most notably between 1925-1945 and from 1952-2001, exhibits positive spatial clustering in the endurance of democracy. This suggests that geographic clustering of transitional and consolidated democracies and also clustering among consolidated democracies may have positively influenced democratic survival, particularly in the second and third waves of democratization.

Figure A.4: Pre-Estimation Plots for Svolik (2008) Application

(a) Joint Counts



(b) Moran's I



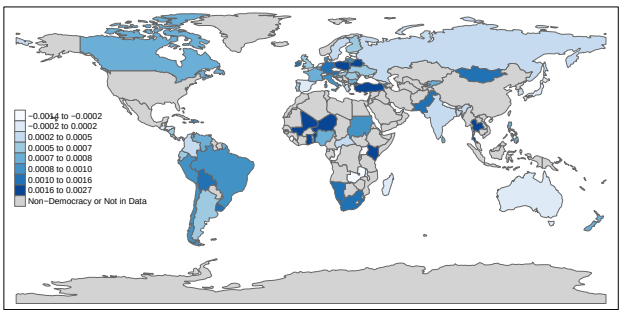
Democratic *Consolidation* Results

We discuss the results from the choropleth maps for the split stage frailties in Figures A.5a–A.5c. While we presented our discussion of the spatial frailties associated with democratic consolidation during the third wave of democracy (1978-2001) in the main manuscript, we visually illustrate the distinct spatial bands among these split-stage frailties in Figure A.5a. Like, the posterior means of the split-stage spatial frailties illustrated in Figure A.5b show similar

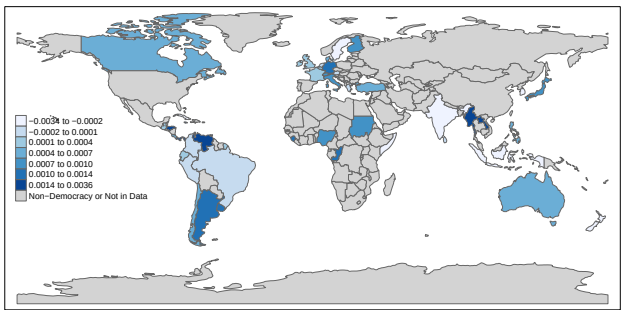
results for the second wave of democracy that includes the 1948-62 period. The spatial frailty terms represented in Figure A.5b range from −0.034 to 0.038 with corresponding standard deviation of 0.024. This figure reveals that during the second wave of democracy, the underlying factors associated with democratic consolidation again appear to cluster along distinct spatial bands, and that distant frailty values also appear to be spatially distant from one another. This suggests that failing to account for the clustering of heterogeneous influences on the probability of democratic consolidation will lead to inaccurate inferences during the second wave of democracy as well. These spatial patterns are less pronounced during the first wave of democracy (1828-1922), as shown in Figure A.5c, though this is consistent with the fact that democracy in this period was much rarer and far fewer countries became democracies during this period relative to the other two waves.

Figure A.5: Democratic Consolidation Maps

(a) Third Wave



(b) Second Wave



(c) First Wave

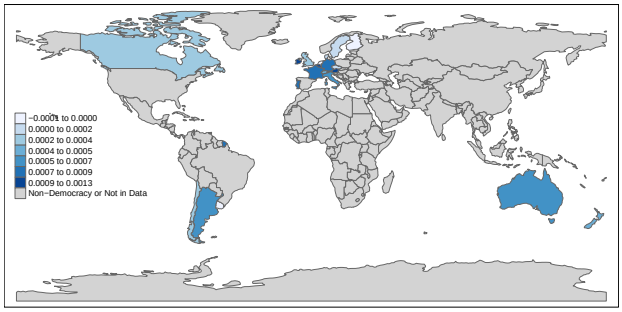
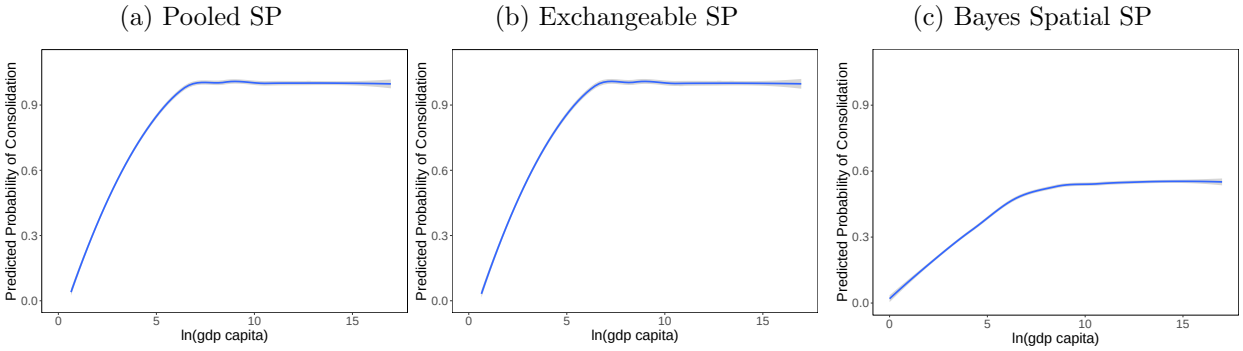


Figure A.6 illustrates the marginal effect of each value of *GDP per capita*—within its observed range in the sample—on the probability of democratic consolidation in the nonspatial pooled and exchangeable SP models and the Spatial SP model. As described in our paper, the

marginal effect of income per capita on the probability of *democratic consolidation* across its observed sample range increases sharply to over 0.9 in the nonspatial pooled and exchangeable SP Weibull (Figures A.6a–A.6b) models when it reaches the threshold of slightly above $ 6,300/- level. But this is not the case in the Spatial SP Weibull model (Figure A.6c) in which the effect of per capita income on the probability of *democratic consolidation* increases to just 0.5 (and is thus much weaker) when it reaches the same threshold of slightly above $6,300/-.

Figure A.6: Predicted Probability of Consolidation by Level of GDP/capita



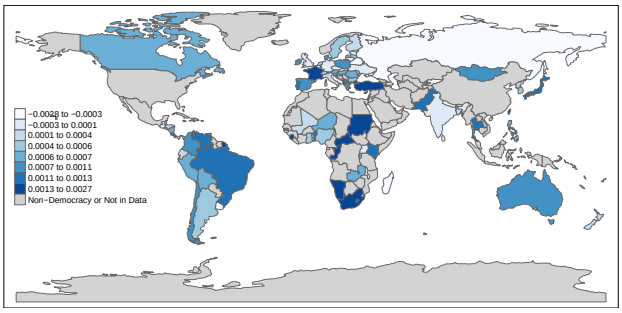
Democratic *Survival* Analysis

The posterior means of the survival-stage spatial frailties discussed in the main article are presented in Figure A.7a below. Again, the map clearly shows that during the third wave of democracy (1978-2001), the underlying and unobserved factors associated with democratic survival among at-risk democracies appear to have similar influences among spatially proximate states, whereas spatially distant democracies tend to have more variable frailty terms. We observe similar results for the first and second waves of democracy in Figures A.7b–A.7c, respectively. The spatial frailty terms in Figure A.7b for the second wave of democracy (1948-62) range from -0.028 to 0.065 with corresponding standard deviation of 0.024, and the spatial frailty terms in Figure A.7c for the first wave (1828-1922) range from -0.001 to 0.013 with standard deviation of 0.023. These frailty values indeed tend to cluster along distinct spatial bands in both maps, though much like the split-stage (democratic consolidation) frailties, this clustering is much more pronounced in the second and third wave of democracy than the first.

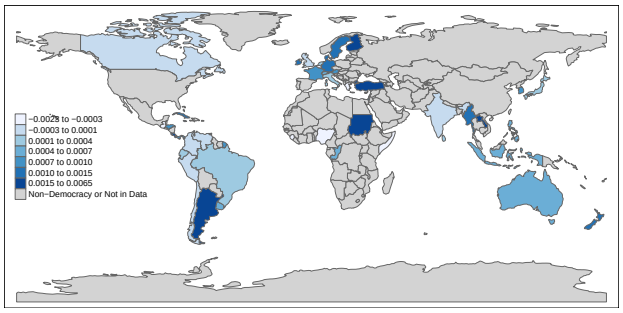
Taken together, however, for most of the temporal period in Svolik's (2008) sample, the underlying unit-level factors that contribute to the survival rates of at-risk or transitional democracies do appear to manifest in geographic neighborhoods. The persistence of these spatial patterns is not only substantively interesting, but also clearly indicates that assuming that the random effects in Svolik's (2008) analysis are i.i.d. with respect to space is untenable.

Figure A.7: Democratic Survival Maps

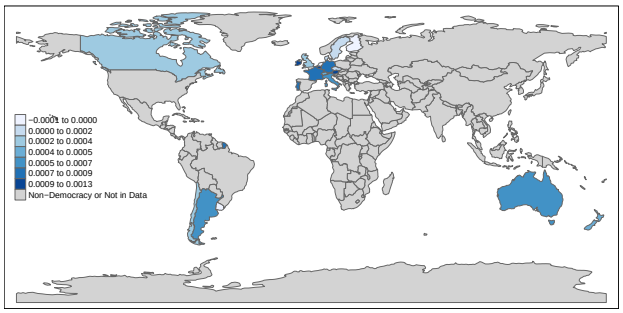
(a) Third Wave



(b) Second Wave



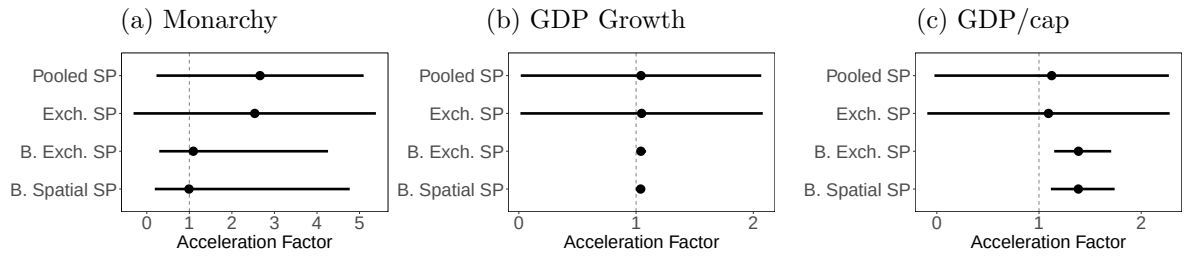
(c) First Wave



In our paper, we discussed the marginal effect of the following three survival stage covariates on democratic survival obtained from the nonspatial pooled and exchangeable SP Weibull models as well as the Spatial SP Weibull model: *Monarchy*, *GDP growth*, *GDP per capita*. These reported marginal effects are (as noted in our paper) illustrated below in the AFPs in Figure A.8a (*Monarchy*), Figure A.8b (*GDP growth*) and Figure A.8c (*GDP per capita*).

We next turn to compare the model fit based on the Deviance Information Criterion (DIC) results obtained from each of the three Bayesian SP Weibull models estimated for the Svolik sample: the pooled SP, exchangeable SP, and spatial SP model. We calculated the Deviance Information Criterion (DIC) of our Bayesian SP Weibull models using the following algorithm.

Figure A.8: Acceleration Factor Plots



First, the formula of DIC in this case is,

$$DIC = -2 * (L - P), \tag{A.21}$$

where L is the log likelihood of the data given the posterior means of the parameters ($\hat{\theta}$):

$$L = logp(\beta, \gamma, \lambda|\hat{\theta}), \tag{A.22}$$

and P is an estimate of the effective number of parameters in the model:

$$P = 2 * \Big[L - \frac{1}{S}\sum_{s=1}^{S} logp(\beta, \gamma, \lambda|\theta_s)\Big], \tag{A.23}$$

where S is the number of posterior samples and θ_s is the parameter vector for the s^{th} sample.

Based on the information above, the steps of calculating the DIC are as follows:

- **Step 1.** Obtain $\hat{\theta}$, a vector of posterior means of the parameters.
- **Step 2.** Calculate the log-likelihood of the data given $\hat{\theta}$ (i.e., L) using the respective Bayesian (MF) Weibull log-likelihood function(s).
- **Step 3.** Compute the log-likelihood of the data given the first posterior sample, θ_1. Repeat S times (for θ_2, θ_3, θ_4, ..., θ_S).
- **Step 4.** Obtain $\sum_{s=1}^{S} logp(\beta, \gamma, \lambda|\theta_s)$ by summing up the results from Step 3.
- **Step 5.** Calculate P using equation (A.23).
- **Step 6.** Calculate DIC using equation (A.21).

We calculate the DIC for each of three Bayesian SP Weibull models estimated on Svolik's sample using Steps 1-6. The Bayesian Spatial SP Weibull model's DIC value (8118.3) is lower

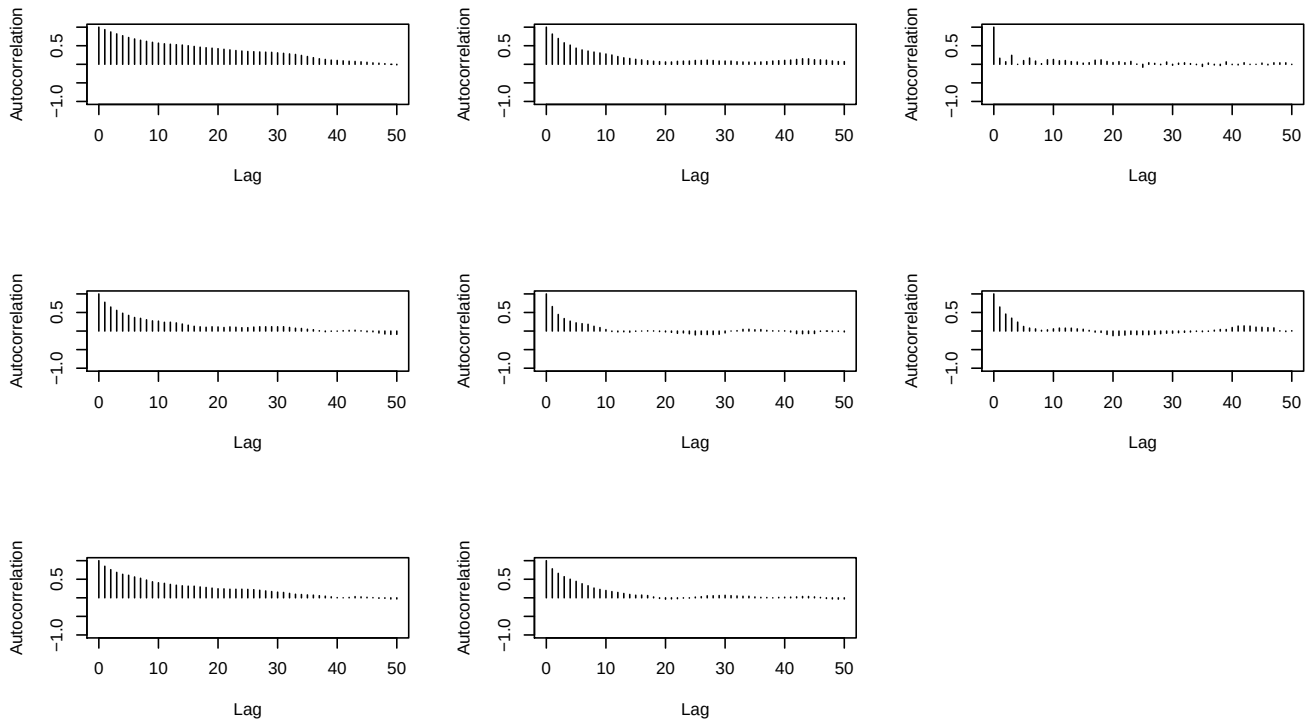
than that obtained from the Bayesian pooled (8217.5) and exchangeable (8193.7) SP models respectively. Thus, in terms of overall fit, the Bayesian Spatial SP Weibull model's performance is superior (with respect to DIC performance) compared to the pooled and exchangeable SP Weibull models.

Convergence Tests and Diagnostics

This subsection first presents the autocorrelation plots (illustrated for lags 1-50) for each split and survival stage covariate from our main model of interest estimated on Svolik's (2008) sample: the Bayesian Spatial SP Weibull model. The autocorrelation plots in Figure A.9 includes from left to right—in the top, middle, and bottom row respectively—the following split stage covariates: *GDP per capita*, *GDP growth*, *Military* (top row); *Civilian*, *Monarchy*, *Parliamentary* (middle row); *Presidential* (bottom row). The autocorrelation plots in Figure A.10 includes from left to right (again from the top, middle and bottom row) the same set of survival stage covariates in exactly the same order as listed above for the split stage covariates. As briefly mentioned in the paper, autocorrelation plots of each parameter in the Bayesian Spatial SP Weibull model split and survival stage covariates in the figures listed above reveals not only good mixing and rapid convergence, but also indicates that there is no high degree of autocorrelation for the respective posterior samples.

Next, consider the results from the Geweke (1992) convergence diagnostic tests for each split and survival stage covariate from the Bayesian Spatial SP Weibull model. The Geweke convergence diagnostic (Geweke, 1992) test that we employ compares the location of the sampled parameter on two different time intervals of the Markov chain in order to assess convergence. If the mean values of the parameter in the two time intervals are approximately close to each other, then it is safe to assume that the two different parts of the Markov chain have similar locations in the state space, and hence that the two samples come from the same distribution. Because many of the estimated split and survival stage parameters in the Spatial SP model exhibits "slow mixing," we compare the first 10% of the Markov chain and the last 50% of

Figure A.9: Autocorrelation Plot for γ Covariates

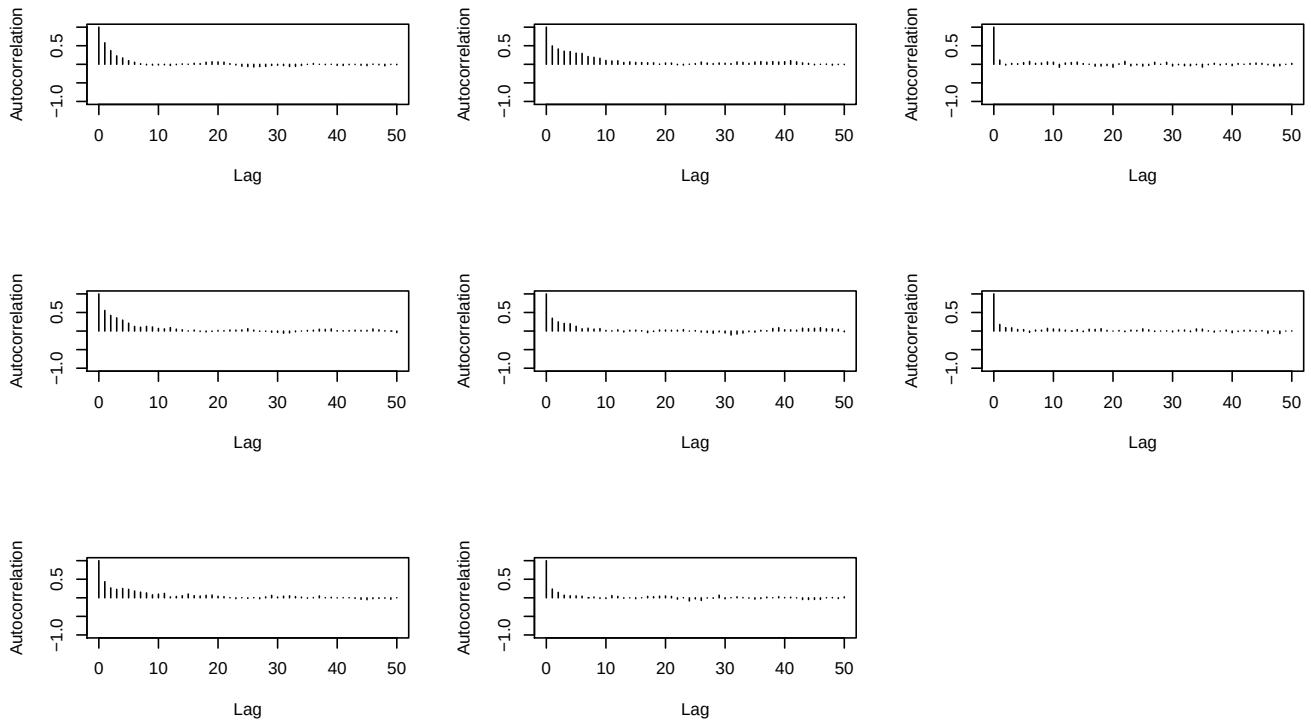


Note: Variable names are (from left to right, top to bottom): GDP/capita, GDP growth, Military, Civil, Monarchy, Parliamentary, Presidential

the chain. The Geweke (1992) convergence diagnostic method summarized here computes a z-statistic where the difference in the two sample means is divided by the asymptotic standard error of their difference. z-scores from the Geweke convergence diagnostics for the (i) split stage covariates is reported in Table A.22 and (ii) survival stage covariates in Table A.23. The Geweke diagnostics in Table A.22–A.23 show that none of the split- and survival-stage parameters in the Bayeisan Spatial SP model produced significant z-scores. This indicates that there is no evidence against convergence for each of these parameters.

Finally, we use the Heidelberger and Welch (1983) test (hereafter, HW test) of stationarity which is a convergence diagnostic that determines whether or not the last part of a Markov chain of each parameter has stabilized. That is, this test of stationarity determines whether the trace of simulated values arises from a stationary stochastic process. This test uses the Cramer-von-Mises statistic to assess evidence of non-stationarity for each parameter in the model. As indicated in Tables A.22–A.23, the HW test of stationarity fails to reject the null hypothesis

Figure A.10: Autocorrelation Plot for β Covariates



Note: Variable names are (from left to right, top to bottom): GDP/capita, GDP growth, Military, Civil, Monarchy, Parliamentary, Presidential

Table A.22: Convergence Diagnostics for γ Covariates

	Geweke	Heidelberger & Welch		
	Z-score	stationary test	start iteration	p-value
GDP/capita	1.120	pass	1	0.193
GDP growth	0.400	pass	335	0.718
Military	0.817	pass	1	0.442
Civil	0.589	pass	1	0.390
Monarchy	-1.152	pass	1	0.387
Parliamentary	0.256	pass	1	0.390
Presidential	-0.201	pass	1	0.316

that the Markov chain of *each* split stage parameter (column 2, Table A.22) and *each* survival stage parameter (column 2, Table A.23) in the Bayesian Spatial SP Weibull model is from a stationary distribution. The HW test thus shows that all the split and survival stage parameters have passed the test of stationarity, which implies that they have converged properly.

Table A.23: Convergence Diagnostics for β Covariates

	Geweke	Heidelberger & Welch		
	Z-score	stationary test	start iteration	p-value
GDP/capita	-0.849	pass	1	0.476
GDP growth	-0.171	pass	1	0.388
Military	-1.009	pass	1	0.579
Civil	-1.335	pass	335	0.214
Monarchy	1.384	pass	1	0.430
Parliamentary	-0.596	pass	1	0.783
Presidential	0.547	pass	1	0.263

VI Application II: Post Civil War Peace (Walter, 2015)

In this section, we use our Bayesian Spatial SP model to re-examine Walter's (2015) widely-cited claim that political freedoms and civil liberties can increase the probability of post-war peace survival. The importance of consolidating peace after civil wars has motivated a wide body of empirical research on why some civil wars recur and others do not. Some of this work uses the civil war itself as the unit of analysis (i.e. Doyle and Sambanis 2000; Walter 2004; Toft 2010; Quinn, Mason and Gurses 2007) and estimates the probability of recurrence with a logit model. Other scholars are more concerned with the duration of post-war peace (i.e. Hartzell, Hoddie and Rothchild 2001; Fortna 2004; Flores and Nooruddin 2012; Mason and Greig 2017) and use conventional parametric or semi-parametric survival models. Regardless, most of the findings in this literature are related to the characteristics of the initial war or the structure of the post-war environment. Most fundamentally, long wars are often followed by a lower risk of conflict recurrence (Doyle and Sambanis 2000; Walter 2004) and longer peace periods (Hartzell et al. 2001; Mason and Greig 2017), but particularly intense wars tend to quickly recur (Doyle and Sambanis 2000; Hartzell et al. 2001; Toft 2010). The effect of how wars end on peace survival is still debated, with some arguing that outright victories lead to a lower risk of recurrence (Licklider 1995; Toft 2010), whereas others contend that peace agreements can lead to lasting peace if they include certain institutional provisions (Hartzell et al. 2001; Hartzell and Hoddie 2003; Matanock 2017) or are supported by international actors (Hartzell, et al. 2001; Quinn, et al. 2007; Walter 1999). Other scholars have emphasized the structure

of the post-war environment over the characteristics of the initial war (Walter 2004). These studies have found that weak economies lead to war renewal (Collier et al. 2003; Walter 2004; Doyle and Sambanis 2000; Quinn et al. 2007), whereas peacekeepers (Fortna 2004; Doyle and Sambanis 2000; Quinn et al. 2007) and democratic power-sharing institutions (Gates et al. 2016) can lead to enduring peace as long as elections are delayed (Flores and Nooruddin 2012).

Walter (2015) contributes to this literature by arguing that political accountability in the aftermath of a civil war can reduce the probability that war recurs. Political participation by the public and freedom of the press, for instance, can all increase the costs of reneging on a settlement, thereby allowing the government to credibly commit to not resume violence. Walter uses conventional survival models to demonstrate that large-scale civil wars are less likely to recur in countries where political freedoms and civil liberties are stronger.[3] None of her control variables reach standard levels of statistical significance after accounting for political freedoms.

Walter's (2015) analysis is thorough, but her theory and empirical method conflate the survival and consolidation of post-war peace. Her primary testable expectation is that "Civil wars that are fought against governments with limited accountability should be more likely to repeat themselves than civil wars in countries with highly accountable governments" (1248). Her use of a Cox/Weibull models, however, means that she is actually evaluating the effect of political freedoms and accountability on *the time until a given war occurs, assuming that all wars recur.* We interpret her argument to mean that political accountability affects the probability that a government-rebel dyad will be *at risk* of renewed war, which implies that some wars will not be at risk of renewal. This is a perfectly testable claim with a split-population model, since the probability of being in the subset of cases that does not experience conflict recurrence is estimated in the first stage. Estimating a second stage—in this context, the survival of peace after war among at-risk cases—can add empirical nuance to the argument by testing whether political and civic freedoms in post-war countries can help peace survive longer.

Still, in order to justify the use of a split-population model we must establish that there is a structural split in the data between the wars that may recur and those that never will.

[3]Walter's (2015) sample consists of 77 post-war spells between 1945-2009, with 24 ending in war renewal.

Beyond what the varied research designs (e.g. logit versus Weibull) in the literature imply, we suspect this structural split exists in Walter's (2015) data for two reasons. First, Walter (2015) only codes a war as recurring if violence resumes between the government and the *same rebel group*. In doing so, she defines conflict recurrence *dyadically* rather than conflating the onset of new wars with the recurrence of old ones (Walter 2004). The consequence of this coding rule is that actors that are completely eliminated via military victory or groups that are demobilized entirely after a war has ended most likely will not re-emerge in the same form. Renewed war within these dyads is therefore structurally improbable and empirically distinct from the subset of rebel groups that retain some capabilities in the post-war period.

Our second reason to suspect that there is a structural split between cases of temporary and consolidated peace in Walter's (2015) data is based on a visual inspection of the Kaplan-Meier curve (available upon request), which shows the proportion of cases that survive (war does not recur) as a function of time. Consistent with our expectations, even the bottom 95% confidence interval never drops below 40%, suggesting that a large portion of cases never recur in the sample. Given these empirical patterns, and since split-population models reduce to standard survival models if no cured fraction exists, we see no reason to favor a Cox/Weibull regression over a split-population model. We also find it far more plausible that some dyadic conflicts will never recur than that *all* conflicts recur if given enough time. Thus, we believe that a split-population model is far more appropriate for testing Walter's (2015) argument.

A split-population model can account for the aforementioned d.g.p., but we also have theoretical and empirical reasons to suspect that spatially autocorrelated heterogeneity exists in Walter's (2015) data. It is widely accepted in the literature that civil wars (and peaceful states) tend to cluster in space (Braithwaite 2010;Buhaug and Gleditsch 2008; Darmofal 2015), but the nature of this spatial autocorrelation is still debated (Forsberg 2014). The dominant view in the literature is that civil wars spread to other states via direct *diffusion* processes, such as transborder ethnic kin (Buhaug and Gleditsch 2008), refugee flows (Salehyan and Gleditsch 2006), or transnational rebel bases (Salehyan 2007), but many have argued that the country-

level factors that influence the risk of conflict onset are what cluster in space,[4] which accounts for much of the spatial dependence in civil wars as well (e.g., Elbadawi and Sambanis 2000). Moreover, there are good reasons to question the magnitude of the effect of conflict diffusion processes, since very few cases of contiguous conflicts are actually causally linked (Black 2013).

We suspect that country-level propensities for post-war peace consolidation and survival may exhibit *spatially clustered heterogeneity*. While some diffusion processes may lead to *conflict contagion*, *peace stability* is more likely to be regionally clustered due to the clustering of observable and unobservable political attributes. In the split stage, post-war countries may never return to violence if they are surrounded by similarly peaceful countries that have demonstrated the institutional capability and political interest to prevent the resurgence of violence in the region (Braithwaite 2010), or if latent localized interests among elites or civilian populations to contain violence—if even to prevent war recurrence in their own contiguous countries—are clustered in space (Doyle and Sambanis 2000). In the survival stage, indirect regional peacekeeping efforts (e.g. Beardsley 2011), stable institutions (Gates et al. 2006), or other latent geographic factors that transcend the borders of a single state may influence the time that rebels take to remobilize or governments take to re-engage with dormant dissident movements. If any of these unobserved factors are not *randomly* distributed in space, then failing to account for this heterogeneity, or improperly accounting for it with a spatial lag that is better suited for capturing diffusion processes, could lead to faulty inferences (Darmofal 2015).

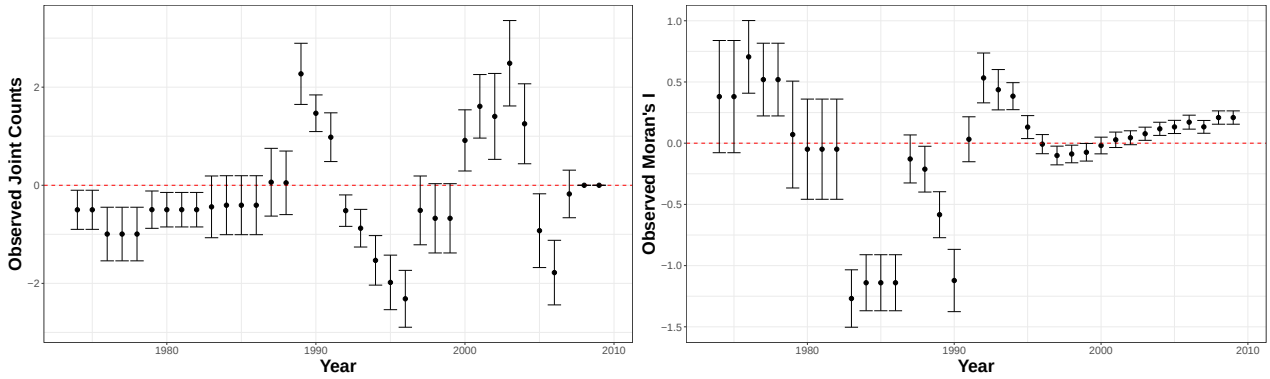
The first step in determining whether to account for either type of spatial dependence is to assess whether it exists in the absence of any covariates. We employ two basic statistical diagnostic tests—join count analysis and Moran's I—to evaluate the possibility of spatial dependence in the split and survival stages, respectively. Figure A.11a plots the observed-expected join counts of potentially "consolidated" peace-spells in Walter's (2015) dataset for each year, along with their 90% confidence intervals. Negative values indicate clustering (positive spatial autocorrelation), and positive values indicate spatial dispersion. The figure depicts clear spatially correlated patterns of peace consolidation, though the direction of the autocorrelation

[4]For example, political institutions and stability often cluster (Gates et al. 2006; Gleditsch and Ward 2006).

Figure A.11: Pre-estimation Spatial Autocorrelation Diagnostics

(a) Join Count (Peace Consolidation)

(b) Moran's I (Peace Survival)



varies over time. Countries in which dyadic post-war peace was consolidated seemed to cluster through the 1970s, early 1980s, 1990s, and late 2000s, whereas the early 1990s and early 2000s witnessed a dispersion of consolidated peace beyond what would be expected by chance. Figure A.11b depicts the Moran's I values for spatial autocorrelation in post-war peace duration. Once again, we observe changes in the direction of spatial autocorrelation over time, but nevertheless significant degrees of spatial autocorrelation of some type in the vast majority of years between 1975 and 2010. Taken together with our theoretical reasons to suspect some form of spatial autocorrelation in the data, these diagnostic tests suggest that it is entirely plausible that unobserved geographic factors that transcend the borders of a single state may lead to spatially autocorrelated heterogeneity in the consolidation and endurance of peace after war.

In sum, Walter's sample of post-civil war peace spells consists of two populations: the "at risk" fraction of observations in which civil conflicts can recur and the "immune" fraction in which the possibility of civil war recurrence is implausible. Extant research on the spatial clustering of peace and war as well as our spatial autocorrelation tests also reveal that spatial proximity may affect the probability of post-war peace survival and consolidation. Given these characteristics in Walter's sample of post-civil war states, we use her replication data on post-war peace spells to estimate each of our three Bayesian SP Weibull models: the *pooled*, non-frailty model, the *exchangeable* model with i.i.d. random effects incorporated in each stage, and the SP Weibull model with spatially weighted frailties in both stages. The split stage

estimates the probability that dyadic peace is "consolidated," and the survival stage estimates the duration of post-war peace among the wars at risk of recurrence.

It should be noted that our goal here is not to make novel or comprehensive substantive claims about the causes of conflict recurrence, but rather to illustrate the applicability of our model and to assess whether and to what degree accounting for spatial autocorrelation might affect the results of previous analyses that do not account for it. Our econometric approach is also different from what Walter (2015) uses to fit her data. Our model specifications are therefore slightly different from the original specifications in Walter's (2015) article, and we do not re-estimate models for all of her disaggregated measures of political accountability and civic freedoms.[5] Rather, for the sake of brevity, we focus on her use of the Freedom House Politics and Freedom index, which is a composite measure of political rights and civil liberties indicators and most generally encompasses her variable of interest compared to her other independent variables. Since this represents the primary variable of interest in her analysis, and because her argument seems to imply that political rights and civil liberties should affect the probability of *not* being at risk of conflict recurrence, we include the Freedom House index in both stages.

In the split stage, we also include GDP/capita (logged), which may reflect economic grievances or state capacity, both of which likely affect the ability of a rebel group to remobilize after a war has ended. Our third covariate in the split stage is whether the war ended in an outright victory by either side. Wars that terminate with an outright victory have been shown to decrease the probability of conflict recurrence in extant studies (e.g. Toft 2010), but it also may be a strong predictor of an observation entering the cured fraction because the rebel group in these conflicts is less likely to independently endure relative to conflicts that end in a peace agreement, ceasefire, or low activity. In the survival stage, we include GDP/capita, how the previous war ended (*Victory*, *Comprehensive Peace Agreement*), *Political Instability* (whether there was a change in the country's Polity score in the previous year), the *Intensity* of the previous conflict (logged total battle deaths), *Ethnic Fractionalization*, and the presence of *UN Peacekeepers*.

[5]We also fill in missing data with a single iteration of "hot deck" data imputation. This is a rudimentary approach, but as this is simply an illustration of our model, we did not want to overcomplicate the data analysis.

Some of these variables affect the ability of either warring party to re-mobilize (Walter 2004, 2015). For instance, intense wars may lead to war weariness, making it difficult for rebels or governments to encourage the population to rally for renewed war; ethnic fractionalization may lead to sustained grievances or difficulty mobilizing for rebels (Gubler and Selway 2012); and GDP/capita may capture either economic grievances or stronger incentives to capture control of the state. Finally, political instability may present a window of opportunity for dissidents to violently demand change, and UN peacekeepers have been shown to improve the survivability of peace by deterring and containing potential rebellions (Fortna 2004;Beardsley 2011).

The pooled Bayesian SP model evaluates the effects of our covariates on the probability of peace consolidation and survival of post-war peace, while the exchangeable SP model accounts for heterogeneous country-level effects but assumes those effects are spatially i.i.d. As we establish above, the consolidation and survival of peace after civil war is likely influenced by spatially clustered heterogeneous effects. The Bayesian spatial SP Weibull model accounts for this type of spatial autocorrelation by employing the CAR prior approach and via the time-invariant binary adjacency matrix $\boldsymbol{A} = \{a_{ii'}\}$ that captures the spatial relationship of each state in the data. We define spatial proximity as $a_{ii'} = 1$ if the distance between the capitals of state i and i' is less than 800 km and $a_{ii'} = 0$ otherwise (e.g. Murdoch and Sandler 2003).

We do not overcomplicate the construction of our spatial weights matrix $\boldsymbol{A}$ with any content beyond geographic proximity because we do not have specific predictions for what specific country-level factor may be spatially clustered while also influencing the baseline risk propensity for peace stability.[6] As we discussed previously, any number of regional characteristics, including transnational ethnic kin (Buhaug and Gleditsch 2008), transnational rebel safe-havens (Salehyan 2007), regional political institutions, economic and political histories, indirect regional effects of neighboring peacekeeping operations (Beardsley 2011), security capabilities (Braithwaite 2010), etc., many of which are difficult to accurately observe or measure, could be influencing these risk propensities, as could spatially heterogeneous clusters of different combinations of these. We subsume all of these possible factors under the umbrella of spatial

[6]See King (1996) and Beck, Gleditsch and Beardsley (2006).

proximity to other countries that have terminated civil wars with the general expectation that, consistent with the literature on neighborhoods of war and peace, these shared experiences and associated unobserved risk factors for post-war stability occur in geographic clusters.

We allow the frailties between neighboring units in Walter's data to be spatially correlated by employing separate CAR priors for the frailty vectors $\mathbf{V}$ and $\mathbf{W}$, which implies the following CAR model structure for the Bayesian Spatial SP Weibull model: $\mathbf{V}|\lambda \sim \text{CAR}(\lambda)$ and $\mathbf{W}|\lambda \sim \text{CAR}(\lambda)$. The resulting conditional distribution of the spatial frailties between the geographically proximate cases is given by equation (5) in the main text. The Bayesian pooled, exchangeable, and Spatial SP models are each estimated using the multivariate Normal prior and our slice-sampling (MCMC) algorithm for which we specify the hyperparameters as: $a = 1$, $b = 1$, $S_\beta = I_{p1}$, $S_\gamma = I_{p2}$, $\nu_\beta = p1$ and $\nu_\gamma = p2$.[7] Given the CAR prior for the spatially dependent random effects in each stage of the Bayesian Spatial SP Weibull model, we also assign the Gamma hyperprior $\lambda \sim \text{Gamma}(a_\lambda, b_\lambda)$ for λ (with vague prior $(a_\lambda, b_\lambda) = (0.001, 1/0.001)$).

Results

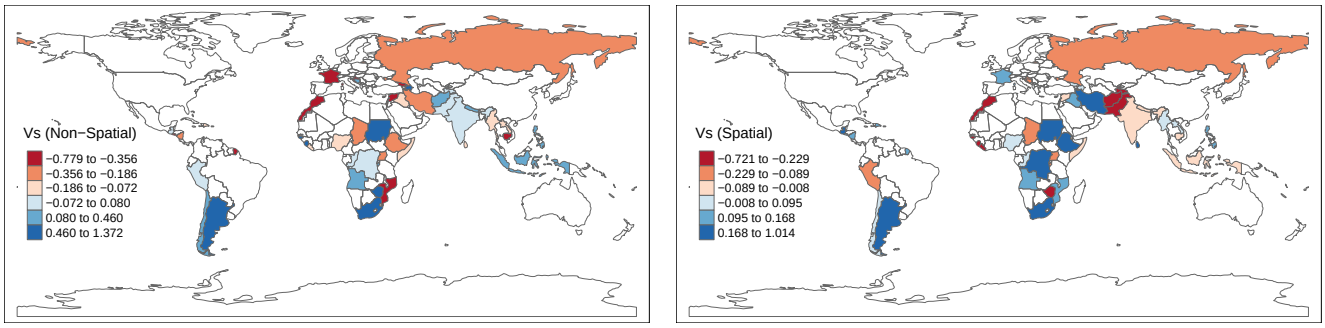
We test Walter's (2015) theory that civil liberties improve the prospects for post-war peace by fitting her data with three Bayesian SP models. We proceed with a discussion of the split-stage nonspatial and spatial frailties before analyzing the posterior mean γ estimates. We then turn to discuss the survival-stage results in the same manner. We display the posterior mean estimates of the nonspatial i.i.d. and spatial frailties from the exchangeable and spatial models in the choropleth maps in Figures A.12a-A.12b $v_i \in \mathbf{V}$ and A.14a-A.14b $w_i \in \mathbf{W}$. In all four maps, countries in blue represent those with greater "risk factors" for peace consolidation or survival, whereas red countries are those with few unobserved factors associated with post-war peace.

Turning first to the split-stage frailties ($v_i \in \mathbf{V}$) in Figures A.12a–A.12b, we see that the distinct spatial bands in the random effects range from -0.779 to 1.372 in the exchangeable model and -0.721 to 1.014 in the spatial model, with the corresponding standard deviation

[7]The Bayesian SP Weibull specification results are based on a set of X iterations after X burn-in scans.

Figure A.12: Split Stage Frailty Maps

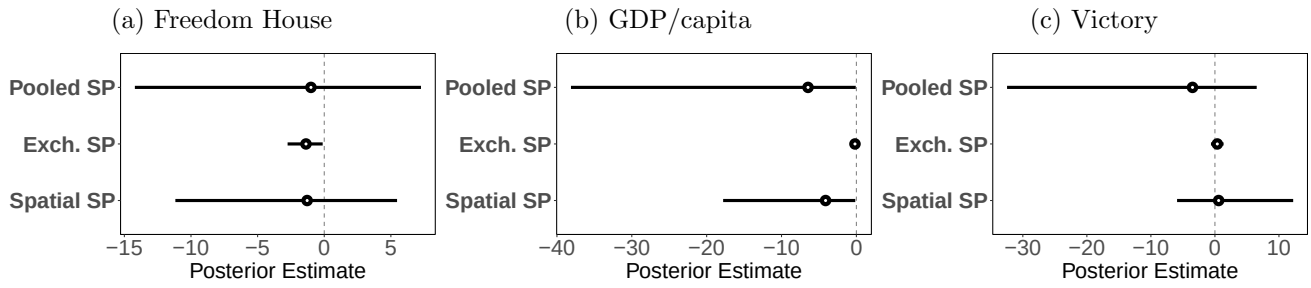
(a) Non-spatial (b) Spatial



of the posterior means being 0.301. The maps reveal some notable changes in underlying propensities for the consolidation of peace after accounting for the spatial clustering of these heterogeneous effects. For instance, Africa, Central America, and the Middle-east/Caucasus regions exhibit a checkerboard pattern with relatively little spatial clustering in Figure A.12a when assuming that the frailty terms in the split stage are i.i.d. However, in Figure A.12b the frailties become markedly more pronounced and clustered, and in some cases like Iran, Iraq, Nicaragua, and El Salvador, this clustering corresponds to a reversal in the direction of the frailties. Moreover, the posterior mean of the variance of the random effects from the Spatial SP Weibull's split stage is about two times larger than those of the non-spatial exchangeable SP model's survival stage. This further suggests that spatially proximate states in Walter's (2015) sample share common unobserved factors that influence the likelihood of peace consolidation *and* makes them distinct from states that are not in the same geographic neighborhood.

Although we do observe some interesting spatial patterns in the split-stage random effects, our *Freedom House* and *Victory* γ covariates are generally unreliable predictors of post-war peace consolidation, particularly in the pooled and spatial Bayesian SP models. The posterior mean estimates in Figure A.13b reveal that the most consistent predictor of consolidated peace is actually *GDP/capita*, which is negative in all three models and statistically reliable in the exchangeable and spatial models. This result contradicts the conventional wisdom that higher levels of economic development should be associated with a lower risk of conflict recurrence (Collier, et al. 2003), though it may be a reflection of the construction of the sample, which only

Figure A.13: Split Stage Posterior Means

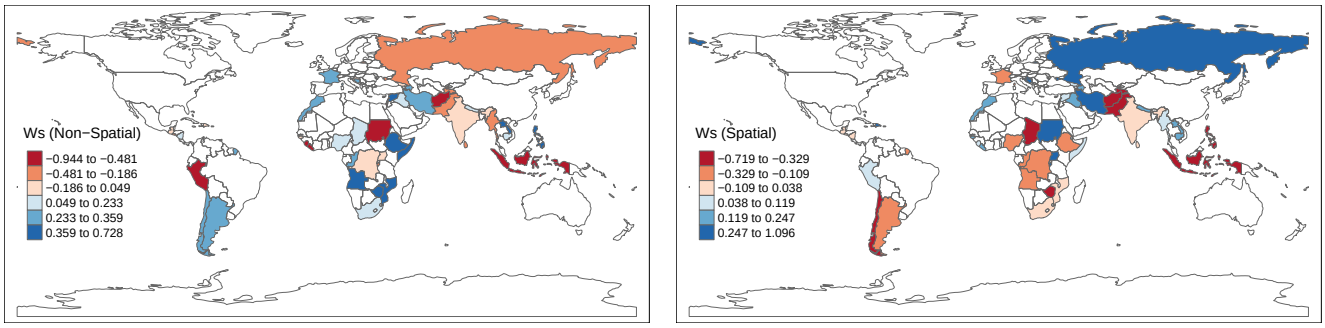


includes countries that have experienced a civil war. In contrast, we cannot reliably infer any systematic relationship between *Freedom House* or *Victory* and the probability of consolidated peace, though the posterior mean γ estimate for *Freedom House* in the non-spatial frailty model is negative and weakly reliable. Thus, we find no empirical evidence to support extant claims in the literature that outright military victories (e.g., Licklider 1995; Toft 2010) and increased political freedoms and civil liberties can reduce the risk of resurgent violence. If anything, Walter's (2015) political freedoms index is *negatively* associated with consolidated peace.

Turning to the survival stage results, we first examine whether the underlying risk factors for peace survival tend to cluster in space by mapping the survival-stage frailty terms in $w_i \in \mathbf{W}$ in the Bayesian exchangeable (Figure A.14a) and spatial (Figure A.14b) SP models. The values in the non-spatial $\mathbf{W}$ range from -0.944 to 0.728 with corresponding posterior standard deviation of 0.416, whereas the values in the spatial $\mathbf{W}$ range from -0.719 to 1.096 with standard deviation of 0.326. In comparing the two figures, we again see that the map of i.i.d random effects exhibits a fairly random distribution of underlying risk factors for the survival of peace, but the map of spatially weighted frailty terms shows distinct spatial bands in these effects across multiple regions. For instance, if we assume i.i.d. random effects, we might infer from Figure A.14a that the Democratic Republic of the Congo and Chad, both of which have had multiple recurring conflicts after brief periods of peace, have moderate to reasonably high levels of risk factors for the survival of peace. After accounting for spatial autocorrelation in these effects, many of the Central African countries that experienced a civil war appear to have unobserved risk factors for a quick return to conflict, which is more consistent with what we see in the data. Moreover,

Figure A.14: Survival Stage Frailty Maps

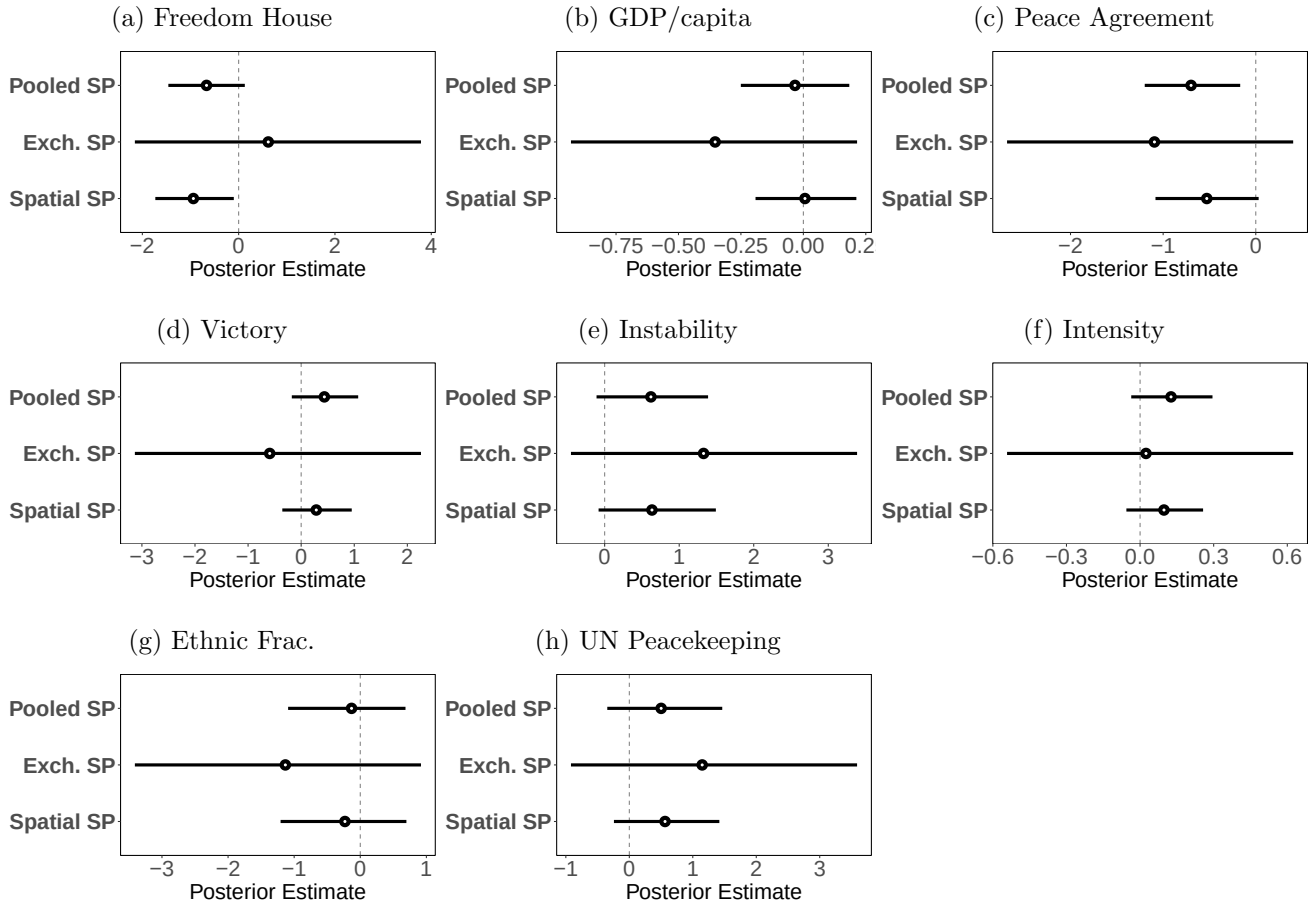
(a) Non-spatial (b) Spatial



Afghanistan and Pakistan seem to have some of the lowest underlying characteristics that could contribute to lasting peace, Central America seems to have a cluster of moderate levels of these characteristics, and Iran, the Caucasus, and Russia exhibit a band of factors that contribute to longer dyadic peace spells. Thus, the unobserved risk factors that influence the durability of post-war peace appear to manifest in geographic neighborhoods. The posterior mean of the variance of the random effects from the spatial SP Weibull's survival stage is approximately twice as large than that of the exchangeable SP model's survival stage, further indicating that geographically proximate post-conflict states in Walter's sample share common latent factors relative to states that are not in the same neighborhoods.

Our posterior estimates for the βs across our pooled, exchangeable, and spatial models reveal that the spatial clustering of heterogeneous effects have important substantive implications as well. Much like Walter's (2015) analysis, almost all of the control variables have little systematic effect on the survival of post war peace. The survival-stage posterior mean estimates for *GDP/capita* and *Ethnic Fractionalization* (Figures A.15b, A.15g) suggest a negative relationship with peace duration in each model, but these relationships are always statistically unreliable. *Victory*, *Instability*, *Intensity*, and *Peacekeeping* all have a positive, but again unreliable, relationship with the survival of post-war peace (Figures A.15d, A.15e, A.15f, A.15h). We find some evidence in the pooled Bayesian SP model that *Peace Agreements* (Figure A.15c) may have a reliable and deleterious effect on post-war peace survival (Licklider 1995), but the effect is negligible and unreliable after accounting for i.i.d. or spatially autocorrelated frailties.

Figure A.15: Survival Stage Posterior Means



More importantly for our purposes, however, Figure A.15a reveals dramatically different results for the effect of political rights and civil liberties on the survival of post-war peace across the three Bayesian SP Weibull models. The survival-state posterior mean estimate for *Freedom House* is negative in the nonspatial pooled model and positive in the nonspatial exchangeable model, but in both cases the association is statistically unreliable. Thus, even though the direction of the relationship between the *Freedom House* index and the survival of post-war peace is positive in our i.i.d. random effects model, consistent with the results from Walter's (2015) conventional survival model, the effect is not robust to our Bayesian SP framework. In fact, the posterior mean estimate for *Freedom House* in the survival stage of our Bayesian spatial SP Weibull model is actually *negative* and highly reliable, suggesting that in direct contrast to Walter's (2015) argument, increased political rights and civil liberties *reduces* the duration of post-war peace. Taken together, these results not only contradict Walter's (2015)

findings but also imply that the risk propensities for peace survival tend to cluster in geographic neighborhoods. This spatial autocorrelation goes unaccounted for when using conventional i.i.d. random effects, which can clearly lead to severe inferential mistakes (Darmofal 2015).

We verify that our main findings are not a result of convergence failures in the Markov chains by conducting a few convergence diagnostic tests. First, we visually inspect autocorrelation plots for all parameters in each stage of all three Bayesian SP models (available upon request). Across all three models, the level of autocorrelation quickly converges to almost 0, indicating that there is no severe autocorrelation in the posterior samples for any of the parameters in either stage. Second, we employ the Geweke (1992) diagnostic test for each parameter, which compares the location of the sampled parameter at two different points in the Markov chain. If the mean values of the parameter at both points are statistically similar (indicated by small Z-scores), then the two samples are likely drawn from the same distribution. Lastly, we use the Heidelberger and Welch (1983, HW) stationarity test, which evaluates whether the Markov chain has stabilized. Statistically insignificant p-values indicate relative parameter stability. Table A.24 provides the results for the Geweke and HW convergence tests. For each of our three models, we report the Z-score of the Geweke test in the first column, and then the results from the HW stationarity tests in the subsequent three columns. Barring our control for military *Victory* in the survival stage of the Bayesian SP spatial frailty model, the p-values from the HW tests lead us to fail to reject the null hypothesis that the Markov chain for each parameter is from a stationary distribution. Similarly, the Geweke tests for most of our variables–with the exception of *Victory* in the pooled and spatial frailty models–produce statistically insignificant Z-scores in all three Bayesian SP models. We can therefore be confident that the Markov chains for all of the other variables, including our variable of interest, have properly converged.

Table A.24: Convergence Diagnostics

	Pooled SP				Exchangeable SP				Spatial SP			
	Geweke	Heidelberger & Welch			Geweke	Heidelberger & Welch			Geweke	Heidelberger & Welch		
	Z-score	Start	P-value	Stationarity	Z-score	Start	P-value	Stationarity	Z-score	Start	P-value	Stationarity
X *Covariates*												
Freedom House	-0.17	1	0.41	pass	0.29	1	0.55	pass	0.6	1	0.24	pass
GDP/Capita	-0.42	1	0.55	pass	-1.53	1	0.86	pass	-0.35	1	0.76	pass
Peace Agreement	-0.4	1	0.98	pass	-0.88	117	0.07	pass	-1.63	1	0.41	pass
Victory	-3.55	40	0.13	pass	-1.77	1	0.07	pass	3.48	-	0.002	fail
Instability	0.15	1	0.45	pass	0.44	1	0.44	pass	1.33	1	0.2	pass
Intensity	-0.54	1	0.97	pass	0.89	1	0.61	pass	1.01	1	0.86	pass
Ethnic Frac.	-0.53	1	0.23	pass	-1.39	1	0.31	pass	-1.71	78	0.07	pass
UN Peacekeeping	0.13	1	0.93	pass	-0.21	1	0.16	pass	-0.15	1	0.54	pass
Z *Covariates*												
Freedom House	-0.99	1	0.17	pass	0.5	1	0.44	pass	-0.94	1	0.69	pass
GDP/Capita	-1.39	78	0.054	pass	-1.15	40	0.19	pass	0.73	1	0.53	pass
Victory	1.51	1	0.19	pass	1.7	40	0.26	pass	-0.17	1	0.055	pass

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