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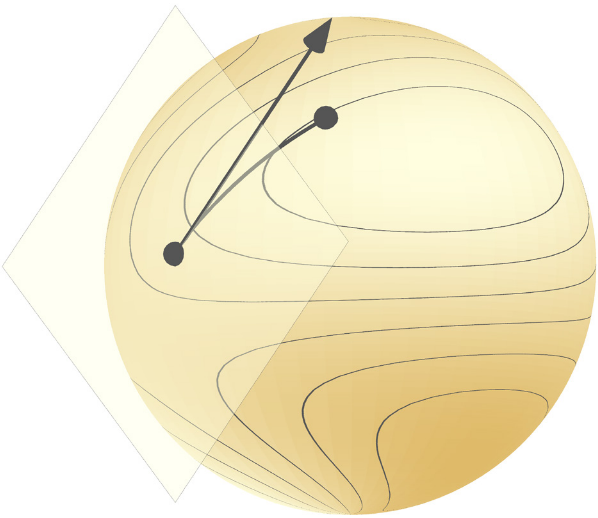
# Riemannian connections



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Optimization on manifolds, MATH 512 @ EPFL

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# Connections: from infinitely many, to one

We defined connections $\nabla$ by requiring *some* properties.

Since manifolds have infinitely many connections,
we can require even *more* properties.

Hopefully, exactly *one* connection will satisfy them.

This has two purposes:

1. We can pick convenient properties.
2. Uniqueness reduces arbitrariness.

# Where to from here? Let's imagine…

Let $\mathcal{M}$ be a Riemannian manifold.
The aim is to identify a special $\nabla$, called the Riemannian connection.

With *that* connection, given a smooth $f: \mathcal{M} \to \mathbf{R}$, we let

$$\mathrm{Hess}f(x): \mathrm{T}_x\mathcal{M} \to \mathrm{T}_x\mathcal{M}$$

denote the Riemannian Hessian of $f$ at $x$, defined by:

$$\mathrm{Hess}f(x)[u] = \nabla_u \mathrm{grad}f$$

We would like for Hess$f(x)$ to be symmetric. What does it take?

$\mathrm{Hess}f(x) : T_x\mathcal{M} \to T_x\mathcal{M}$ is symmetric or self-adjoint if :

$$\forall u, v \in T_x\mathcal{M} : \quad \langle \mathrm{Hess}f(x)[u], v \rangle_x = \langle \mathrm{Hess}f(x)[v], u \rangle_x$$

(in $\mathbb{R}^n$, if $M \in \mathbb{R}^{n \times n}$, $M^T = M$, $\langle u, Mv \rangle = u^T M v = u^T M^T v = (Mu)^T v = \langle Mu, v \rangle$)

$$\forall U, V \in \mathfrak{X}(\mathcal{M}) : \quad \langle \mathrm{Hess}f[U], V \rangle \overset{\text{want}}{=} \langle \mathrm{Hess}f[V], U \rangle$$

Notation: $\mathrm{Hess}f[U] \in \mathfrak{X}(\mathcal{M}) : (\mathrm{Hess}f[U])(x) = \mathrm{Hess}f(x)[U(x)]$

$\langle U, V \rangle : \mathcal{M} \to \mathbb{R} : \left(\langle U, V \rangle\right)(x) = \langle U(x), V(x) \rangle_x$.

What we want: $\langle \nabla_U \operatorname{grad} f, V \rangle = \langle \nabla_V \operatorname{grad} f, U \rangle$

Require: $U\langle W, V\rangle = \langle \nabla_U W, V\rangle + \langle W, \nabla_U V\rangle$

Notation: $U \in \mathfrak{X}(M)$, $f : M \to \mathbb{R}$ smooth

$Uf : M \to \mathbb{R}$ : $(Uf)(x) = Df(x)[U(x)]$

$Uf = \langle \text{grad} f, U\rangle$ $= \langle \text{grad} f(x), U(x)\rangle_x$

$$\langle \nabla_U \text{grad} f, V\rangle = U\langle \text{grad} f, V\rangle - \langle \text{grad} f, \nabla_U V\rangle.$$

$$\langle \nabla_V \text{grad} f, U\rangle = V\langle \text{grad} f, U\rangle - \langle \text{grad} f, \nabla_V U\rangle$$

Want: $U\langle \text{grad} f, V\rangle - V\langle \text{grad} f, U\rangle = \langle \text{grad} f, \nabla_U V - \nabla_V U\rangle$

Want: $U(Vf) - V(Uf) = (\nabla_U V - \nabla_V U) f$.

Define: $[U,V]$ (the Lie bracket of $U, V \in \mathfrak{X}(M)$)

acts on $f : M \to \mathbb{R}$ (smooth) as:

$$[U,V]f = U(Vf) - V(Uf).$$

Require: $[U,V]f = (\nabla_U V - \nabla_V U) f.$

Let's wrap up the imagination and clean up with proper definitions.

# Symmetry

Let $U, V$ be two smooth vector fields on a manifold $\mathcal{M}$.

Let $f : \mathcal{M} \to \mathbf{R}$ be a smooth function.

**Def.:** The action of $U$ on $f$ creates a new function $Uf$, defined by:

$$(Uf)(x) = \mathrm{D}f(x)[U(x)]$$

**Def.:** The Lie bracket $[U, V]$ acts on $f$ to create the new function:

$$[U, V]f = U(Vf) - V(Uf)$$

$\equiv$ torsion-free

**Def.:** A connection $\nabla$ is symmetric if

$$[U, V]f = (\nabla_U V - \nabla_V U)f$$

for all smooth vector fields $U, V$ and all smooth $f : \mathcal{M} \to \mathbf{R}$.

**Fact:** For a linear space, $\nabla_u V = \mathrm{D}V(x)[u]$ is a symmetric connection.

**Fact:** For an embedded submanifold of a Euclidean space,
$\nabla_u V = \mathrm{Proj}_x(\mathrm{D}\bar{V}(x)[u])$ is a symmetric connection.

# Compatibility with the metric

The symmetry property does *not* require a Riemannian metric.

From now on, assume $\mathcal{M}$ is a Riemannian manifold.

**Def.:** Given two vector fields $U, V$ on $\mathcal{M}$, define $\langle U, V\rangle : \mathcal{M} \to \mathbf{R}$ as:

$$\langle U, V\rangle(x) = \langle U(x), V(x)\rangle_x$$

**Def.:** A connection $\nabla$ is compatible with the metric if

$$W\langle U,V\rangle = \langle \nabla_W U, V\rangle + \langle U, \nabla_W V\rangle$$

for all smooth vector fields $U, V, W$ on $\mathcal{M}$.

**Fact:** For a Euclidean space, $\nabla_u V = \mathrm{D}V(x)[u]$ is compatible with the metric, *regardless* of which inner product we chose.

**Fact:** For a Riemannian submanifold of a Euclidean space, $\nabla_u V = \mathrm{Proj}_x(\mathrm{D}\bar{V}(x)[u])$ is compatible with the metric.

# The fundamental theorem of Riemannian geometry

**Theorem.** On a Riemannian manifold $\mathcal{M}$,
there exists a unique connection $\nabla$
which is symmetric and compatible with the metric.

It is called the Riemannian connection or Levi-Civita connection.

**Fact:** For a Euclidean space, $\nabla_u V = \mathrm{D}V(x)[u]$ is the LC connection.
**Fact:** For a Riemannian submanifold of a Euclidean space,
$\nabla_u V = \mathrm{Proj}_x(\mathrm{D}\bar{V}(x)[u])$ is the LC connection.

**Proof of uniqueness.**

Assume $\nabla$ is “a” Riemannian connection for $\mathcal{M}$.

Let $U, V, W$ be arbitrary smooth vector fields on $\mathcal{M}$.

Since $\nabla$ is compatible with the metric, we have:

$$
\begin{aligned}
U\langle V, W\rangle &= \langle \nabla_U V, W\rangle + \langle V, \nabla_U W\rangle \\
+\ V\langle W, U\rangle &= \langle \nabla_V W, U\rangle + \langle W, \nabla_V U\rangle \\
-\ W\langle U, V\rangle &= -\langle \nabla_W U, V\rangle - \langle U, \nabla_W V\rangle
\end{aligned}
$$

$+$

$$U\langle V, W\rangle + V\langle W, U\rangle - W\langle U, V\rangle$$

$$= \langle W, \nabla_U V + \nabla_V U\rangle + \langle V, \underbrace{\nabla_U W - \nabla_W U}_{[U,W]}\rangle + \langle U, \underbrace{\nabla_V W - \nabla_W V}_{[V,W]}\rangle$$

Since ∇ is symmetric, we know, for all $U, V, f$, that

$$[U,V]f = (\nabla_U V - \nabla_V U)f$$

$$\Rightarrow \quad [U,V] \text{ "="} \nabla_U V - \nabla_V U \quad : \quad \nabla_V U = \nabla_U V - [U,V]$$

This yields the *Koszul formula*:

$$2\langle W, \nabla_U V\rangle = \langle W, [U,V]\rangle - \langle V, [U,W]\rangle - \langle U, [V,W]\rangle$$
$$+ U\langle V,W\rangle + V\langle W,U\rangle - W\langle U,V\rangle$$