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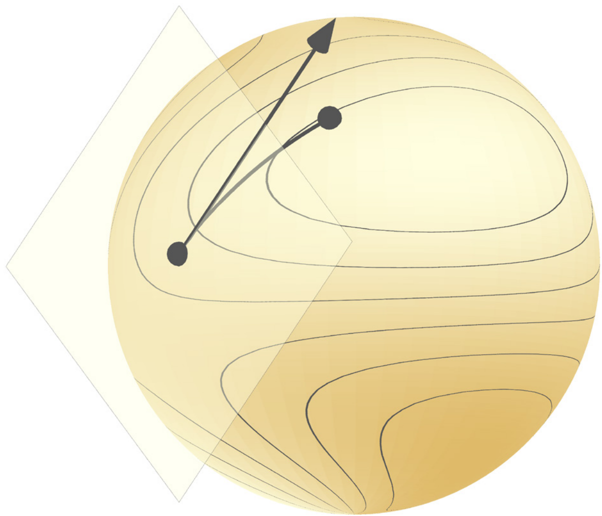
Tangent vectors without embedding space



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Optimization on manifolds, MATH 512 @ EPFL

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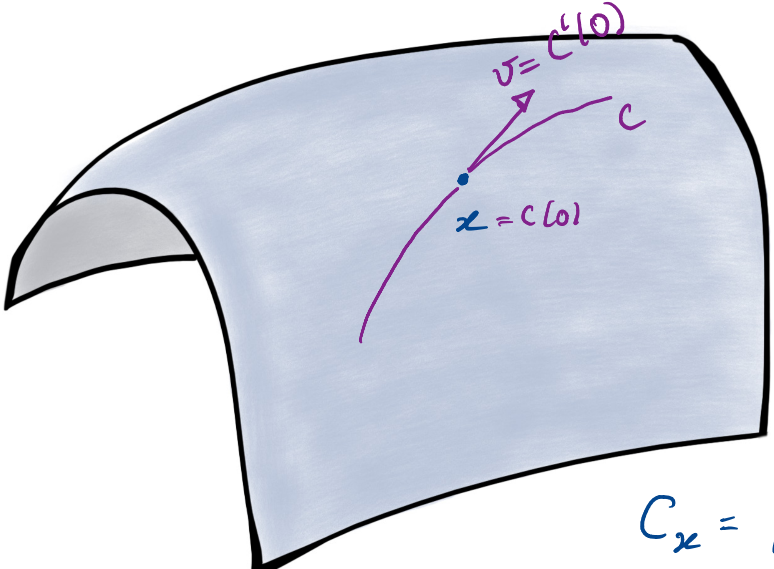
An abstract look at the embedded case…



$\mathcal{M} \subset \mathbf{R}^3$

A tangent vector is any element of $\mathcal{E}$ I can get as $c'(0)$, w/ $c : \mathbb{R} \to \mathcal{M}$ a smooth curve, $c(0) = x$.

$$T_x\mathcal{M} = \left\{ c'(0) \;\middle|\; \begin{array}{l} c : \mathbb{R} \to \mathcal{M} \text{ smooth} \\ c(0) = x \end{array} \right\}$$

$$C_x = \left\{ c \;\middle|\; \begin{array}{l} c : \mathbb{R} \to \mathcal{M} \\ \text{smooth, } c(0) = x \end{array} \right\}$$

Define an equivalence relation $\sim$ on C_x:

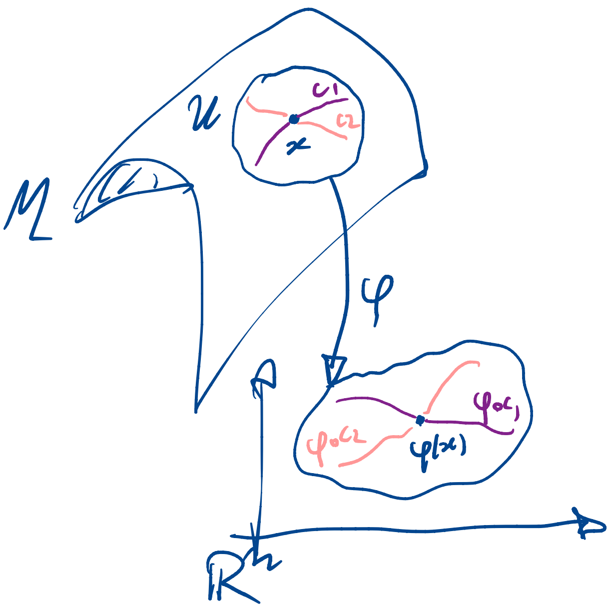
$$c_1 \sim c_2 \iff c_1'(0) = c_2'(0).$$

$\Rightarrow$ $C_x/\sim$ is one-to-one with T_xM.

... to serve as inspiration for general manifolds.

Consider a point x on a manifold $\mathcal{M} = (M, \mathcal{A}^+)$.

Let $C_x = \{c: \mathbf{R} \to \mathcal{M} \mid c \text{ is smooth and } c(0) = x\}$.



Define an equivalence relation $\sim$ on C_x:

$$c_1 \sim c_2 \iff (\varphi \circ c_1)'(0) = (\varphi \circ c_2)'(0)$$

Then, $C_x/\sim$ "is" the tangent space at x.

Fact: The equivalence relation on C_x defined by

$$c_1 \sim c_2 \quad \Leftrightarrow \quad (\varphi \circ c_1)'(0) = (\varphi \circ c_2)'(0)$$

is independent of the choice of chart $(\mathcal{U}, \varphi)$ around x.

Let (V, ψ) be another chart for M around x.

Let $c \in C_x$; Compute:

$$(\psi \circ c)'(0) = (\underbrace{\psi \circ \varphi^{-1}}_{\mathbb{R}^n \leftarrow M \leftarrow \mathbb{R}^n} \circ \underbrace{\varphi \circ c}_{\mathbb{R}^n \leftarrow M \leftarrow \mathbb{R}})'(0)$$

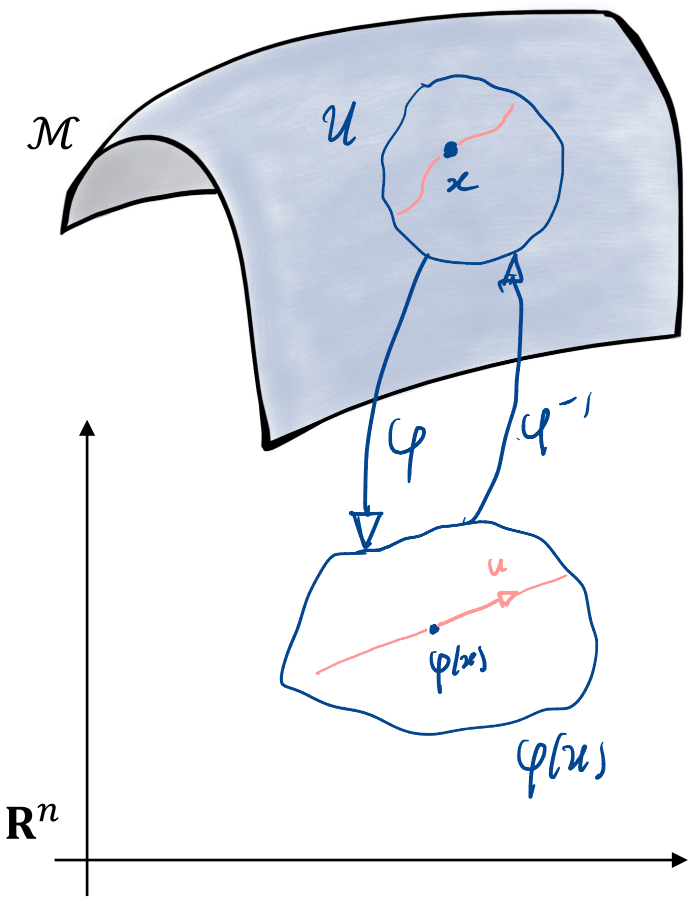
$$= D(\psi \circ \varphi^{-1})(\varphi(x))\,[(\varphi \circ c)'(0)]$$

Now, make the tangent space linear.

$$C_x = \{c\colon \mathbf{R} \to \mathcal{M} \mid c \text{ is smooth and } c(0) = x\}$$

Def.: The tangent space $\mathrm{T}_x\mathcal{M}$ is the quotient set $C_x/\sim$.
A tangent vector $v \in \mathrm{T}_x\mathcal{M}$ is an equivalence class of curves.

How do we "add" two tangent vectors, $v_1 + v_2$? Or "scale", αv_1?



$$\Theta_x^\varphi : C_x/\sim \;\longrightarrow\; \mathbb{R}^n$$

$$\Theta_x^\varphi([c]) = (\varphi \circ c)'(0)$$

Θ_x^φ is injective b/c

if $\Theta_x^\varphi([c_1]) = \Theta_x^\varphi([c_2])$

then $(\varphi \circ c_1)'(0) = (\varphi \circ c_2)'(0)$

$$\equiv \quad c_1 \sim c_2 \quad \equiv \quad [c_1] = [c_2].$$

For surjective, notice that $\forall u \in \mathbb{R}^n$,

we have: $\Theta_x^\varphi\Big(\big[t \mapsto \varphi^{-1}(\varphi(x) + tu)\big]\Big) = u.$

$$\Theta_x^{\varphi}([c]) = (\varphi \circ c)'(0);$$

Define: $[c_1] + [c_2] \triangleq (\Theta_x^{\varphi})^{-1}\left(\Theta_x^{\varphi}([c_1]) + \Theta_x^{\varphi}([c_2])\right).$

$$\alpha[c] \triangleq (\Theta_x^{\varphi})^{-1}(\alpha\,\Theta_x^{\varphi}([c])).$$

Remember: $(\psi \circ c)'(0) = D(\psi \circ \varphi^{-1})(\varphi(x))\left[(\varphi \circ c)'(0)\right]$

$$\Theta_x^{\psi}([c]) = D(\psi \circ \varphi^{-1})(\varphi(x))\left[\Theta_x^{\varphi}([c])\right]$$

If $\mathcal{M}$ is embedded, we now have two notions of tangent spaces.
They are equivalent:

Fact: If $\mathcal{M}$ is an embedded submanifold of a linear space $\mathcal{E}$, then $[c] \mapsto c'(0)$ is a linear bijection from $C_x/\sim$ to $\ker \mathrm{D}h(x)$ where h is a local defining function for $\mathcal{M}$ around x.

Thus, both formalisms yield the same conclusions, always.