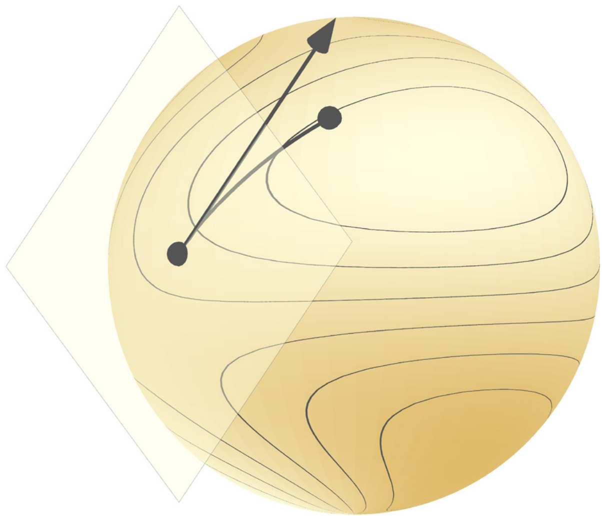
101

What is a smooth manifold?



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Optimization on manifolds, MATH 512 @ EPFL

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Smooth manifolds: a word about words

For us, smooth means C^∞ (infinitely differentiable).

We work with *smooth* manifolds only, so I'll just say "manifold".

At first, we focus entirely on submanifolds of linear spaces:

Submanifolds are manifolds: I'll say "manifold" for general claims.

All algorithms we discuss work for general manifolds.

Other, related notions: topological manifolds, algebraic varieties, …

Submanifolds, informally

Consider a linear space $\mathcal{E}$ (say, $\mathbf{R}^d$).

A subset of $\mathcal{E}$ is a submanifold if, locally around each point,

$\sim$ “the set can be linearized.”

$\sim$ “the set looks linear.”

We make this precise, with two equivalent definitions.

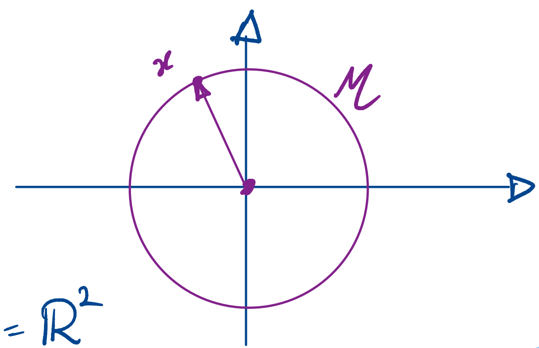
Submanifolds via local defining functions

Consider a set $\mathcal{M} = \{x \in \mathcal{E} : h(x) = 0\}$ with $h : \mathcal{E} \to \mathbf{R}^k$.

Maybe if h is smooth, then $\mathcal{M}$ is smooth?

Example 1: $h(x) = x_1^2 + x_2^2 - 1 = 0$

$$\underbrace{h : \mathbb{R}^2}_{\mathcal{E}} \to \mathbb{R}$$



$\mathcal{E} = \mathbb{R}^2$

$$h(x) = x^T x - 1$$

$$\underbrace{h(x+v)}_{h(x)=0} \simeq \underbrace{h(x)}_{=0} + \underbrace{Dh(x)[v] = 0}_{\Leftrightarrow\ v \in \ker Dh(x)}$$

$$Dh(x)[v] = \lim_{t \to 0} \frac{h(x+tv) - h(x)}{t} = 2x^T v = 0 \Leftrightarrow x^T v = 0$$

Example 2: $h(x) = x_1^2 - x_2^2$

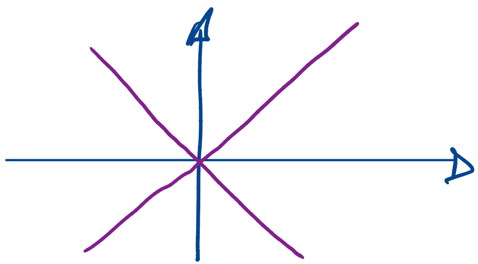
$h: \mathbb{R}^2 \to \mathbb{R}$

$h(x) = 0 \iff x_1^2 = x_2^2 \iff x_1 = \pm x_2.$

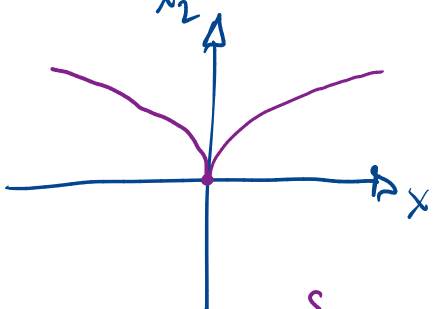
$Dh(x)[v] = \frac{d}{dt} h(x+tv)\Big|_{t=0}$

$= \frac{d}{dt}\left((x_1 + t v_1)^2 - (x_2 + t v_2)^2 \right)\Big|_{t=0}$

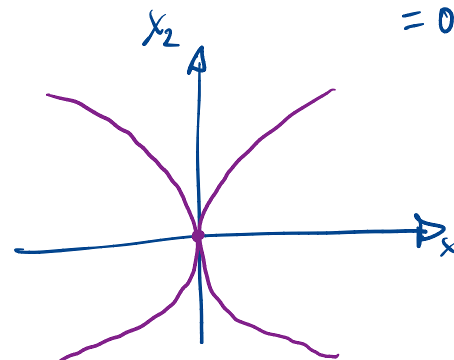
$= 2x_1 v_1 - 2x_2 v_2 = \begin{bmatrix} 2x_1 & -2x_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$



Example 3: $h(x) = x_1^2 - x_2^3$ $= 0 \iff x_1 = \pm x_2^{3/2}$



Example 4: $h(x) = x_1^2 - x_2^4$ $= 0 \iff x_1 = \pm x_2^2$



! For all <u>closed</u> sets S in $\mathcal{E}$, there exists a C^∞ fct $h: \mathcal{E} \to \mathbb{R}$ s.t. $S = h^{-1}(0)$.

The embedding space $\mathcal{E}$ is a linear space of dimension d.

non empty

Def.: $\mathcal{M}$ open in $\mathcal{E}$ is an embedded submanifold of dimension $n = d$.

non empty

$\mathcal{M} \subset \mathcal{E}$ is an embedded submanifold of dimension $n < d$ if:

for all $x \in \mathcal{M}$, there exists a neighborhood U of x in $\mathcal{E}$ such that

$$U \cap \mathcal{M} = \{y \in U : h(y) = 0\}, = h^{-1}(0)$$

where $h: U \to \mathbf{R}^{d-n}$ is smooth and $\mathrm{D}h(x)$ has rank $d - n$.

local defining function for $\mathcal{M}$ around x.

Equivalently: linear up to diffeomorphism

open set in a linear space

Def.: A diffeomorphism is an invertible map $F: U \to V$, where U, V are open and both F and F^{-1} are smooth.

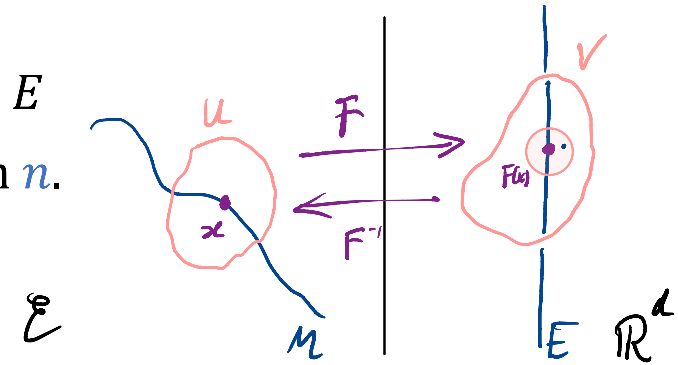
open set in a linear space of the same dimension

Theorem: $\mathcal{M}$ is an embedded submanifold of dimension n iff:

For all $x \in \mathcal{M}$, there exists a ngbhd U of x in $\mathcal{E}$, an open set $V \subseteq \mathbf{R}^d$ and a diffeomorphism $F: U \to V$ such that

$$F(U \cap \mathcal{M}) = V \cap E$$

where E is a linear subspace of dimension n.

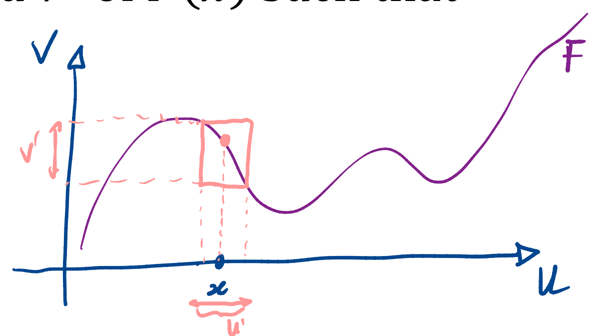


$$F^{-1}(F(x)+v) \cong F^{-1}(F(x)) + DF^{-1}(F(x))[v]$$
$$= x + [\text{some linear map}][v].$$

Proof.

We will use the **inverse function theorem (IFT):**

> If $F: U \to V$ is smooth and if $\mathrm{D}F(x)$ is invertible,
> then we can reduce U to a smaller neighborhood U' of x
> and V to a smaller neighborhood V' of $F(x)$ such that
> $F: U' \to V'$ is a diffeomorphism.



Assume we have a local defining function h in a ngbhd U of $x \in \mathcal{M}$.
We need to build a diffeomorphism F that linearizes $\mathcal{M}$ around x.
(The other direction is left as an exercise.)

$h : U \to \mathbb{R}^k$, smooth; $x \in U$, U open in $\mathcal{E}$. $\dim \mathcal{E} = d$

$h(y) = 0 \Leftrightarrow y \in U \cap \mathcal{M}$; $\operatorname{rank}(Dh(x)) = k = d - n$, $\dim \mathcal{M} = n$.

$Dh(x) : \underset{\underset{\mathbb{R}^d}{\shortparallel}}{\mathcal{E}} \to \mathbb{R}^k$; $\mathcal{E} \equiv \mathbb{R}^d$ after choosing a basis.

$Dh(x)$ "is" a matrix of dimension $k \times d$. Order the basis

such that $Dh(x) = [\underbrace{A}_{k\times(d-k)} \quad \underbrace{B}_{k\times k}] \in \mathbb{R}^{k\times d}$, B invertible

Define $F : U \to \mathbb{R}^d$: $F(y) = \begin{bmatrix} y_1 \\ \vdots \\ y_{d-k} \\ h(y)_1 \\ \vdots \\ h(y)_k \end{bmatrix}$; F is smooth.

$DF(x) : \mathbb{R}^d \to \mathbb{R}^d$; $DF(x) = \begin{bmatrix} I_{d-k} & 0_{(d-k)\times k} \\ A & B \end{bmatrix}$

$\left(DF(x)^{-1} = \begin{bmatrix} I_{d-k} & 0 \\ -B^{-1}A & B^{-1} \end{bmatrix} \right)$

$\Rightarrow$ $DF(x)$ is invertible, and therefore IFT applies:

Reduce U so that F is a diffeomorphism.

$V = F(U)$

? $F(U \cap M) \overset{?}{=} V \cap E$, $E = \{ y \in \mathbb{R}^d : y_{d-k+1} = \dots = y_d = 0 \}$.

$$F(y) = \begin{bmatrix} y_1 \\ \vdots \\ y_{d-k} \\ h(y)_1 \\ \vdots \\ h(y)_k \end{bmatrix}$$

$\underline{y \in U}$: $y \in U \cap M$

$$\Updownarrow$$

$$h(y) = 0$$

$$\Updownarrow$$

$$F(y) \in V \cap E$$

$$\Longrightarrow \quad F(U \cap M) = V \cap E.$$

$\square$

Topology on embedded submanifolds

Let $\mathcal{M}$ be an embedded submanifold of $\mathcal{E}$.

Def.: $\mathcal{U} \subseteq \mathcal{M}$ is open if $\mathcal{U} = U \cap \mathcal{M}$ for some $U \subseteq \mathcal{E}$ open.

Def.: A neighborhood of x on $\mathcal{M}$ is an open set $\mathcal{U} \subseteq \mathcal{M}$ with $x \in \mathcal{U}$.

Fact: If $\mathcal{U}$ is open in $\mathcal{M}$, then $\mathcal{U}$ is an embedded submanifold of $\mathcal{E}$.