Manopt: a brand new toolbox for optimization on manifolds

Nicolas Boumal
Université catholique de Louvain
nicolas.boumal@uclouvain.be

Bamdev Mishra
Université de Liège
b.mishra@ulg.ac.be

Optimization on manifolds is a powerful paradigm to address nonlinear (and possibly nonconvex) optimization problems of the general form

$$\min_{x \in \mathcal{M}} f(x), \tag{1}$$

where $\mathcal{M}$ is a Riemannian manifold [1]. We propose a Matlab toolbox called Manopt to solve problems of this class using known algorithms on known geometries. In the presentation, we will motivate the importance of this class of optimization problems in applications and illustrate how the toolbox can be used to address them.

One practically important class of manifolds consists in Riemannian submanifolds of $\mathbb{R}^n$ endowed with its usual Frobenius inner product, that is, smooth surfaces embedded in space. Examples include the sphere, the set of orthonormal matrices, the set of rotation matrices, the set of fixed-rank matrices, etc. As a consequence, algorithms designed to solve optimization problems on manifolds can address all the following (nonconvex) constrained problems naturally ($\|\cdot\|_\mathrm{F}$ denotes the Frobenius norm):

$$\min_{X \in \mathbb{R}^{m\times n}} f(X) \text{ s.t. } \|X\|_\mathrm{F} = 1, \tag{2}$$

$$\min_{X \in \mathbb{R}^{m\times n}} f(X) \text{ s.t. } X^\top X = I_n, \tag{3}$$

$$\min_{X \in \mathbb{R}^{n\times n}} f(X) \text{ s.t. } X^\top X = I_n,\ \det(X) = 1, \tag{4}$$

$$\min_{X \in \mathbb{R}^{m\times n}} f(X) \text{ s.t. } \operatorname{rank}(X) = k. \tag{5}$$

The paradigm of optimization on manifolds is even more flexible, as it also includes the more abstract class of Riemannian quotient manifolds. In optimization problems, these typically arise when the objective function is invariant under some group action. For example, if $f(X)$ is merely a function of the space spanned by the columns of the matrix X, noted $\operatorname{span}(X)$, one can partition $\mathbb{R}^{m\times n}$ into equivalence classes, where X and Y are deemed equivalent if $\operatorname{span}(X) = \operatorname{span}(Y)$. The set of all such equivalence classes equipped with a proper metric is a Riemannian manifold known as the Grassmann manifold. This means that still the same algorithms may be used (with the same theoretical guarantees) on problems of the form

$$\min_{X \in \mathbb{R}^{m\times n}} f(\operatorname{span}(X)). \tag{6}$$

Increasingly many algorithms from nonlinear optimization are generalized to the realm of optimization on manifolds. The particularization of these to a specific manifold $\mathcal{M}$ typically requires little effort. One usually just needs a few tools to represent points and tangent vectors and to move along crude approximations of geodesics. Such toolsets have been described for many geometries.

The toolbox we propose regroups known algorithms and known geometries under a common framework. Manopt is designed such that a commonly-skilled Matlab user can obtain working code based on state-of-the-art algorithms with minimal effort. For basic use, one only needs to pick a manifold from the library, describe the cost function (and possible derivatives) and pass it on to a solver. Accompanying tools help the user in common tasks such as numerically checking whether the cost function agrees with its derivatives up to the appropriate order, etc.

As an example, for a given symmetric matrix A of size n, the eigenvalue problem of minimizing the Rayleigh "quotient" $f(x) = -x^\top Ax$ under the constraint $x^\top x = 1$ is written in Manopt as follows:

```
% Pick the manifold (the constraint).
manifold = spherefactory(n);
problem.M = manifold;

% Define the cost function and its gradient.
problem.cost = @(x) -x'*(A*x);
problem.grad = @(x) manifold.proj(x, -2*A*x);

% Solve.
x = trustregions(problem);
```

Advanced use gives the user fine control over stopping criteria, solver parameters, which (problem-specific) statistics to log at each iteration, etc. Furthermore, a caching system is in place to prevent redundant computations once the early prototyping stage is over.

Code and documentation are freely available from the web: `http://www.manopt.org`.

We acknowledge the RANSO group, led by P.-A. Absil, Y. Nesterov and R. Sepulchre, for advice and support. Both authors are research fellows with the FNRS (aspirants).

References

[1] P.-A. Absil, R. Mahony, and R. Sepulchre. *Optimization Algorithms on Matrix Manifolds*. Princeton University Press, Princeton, NJ, 2008.