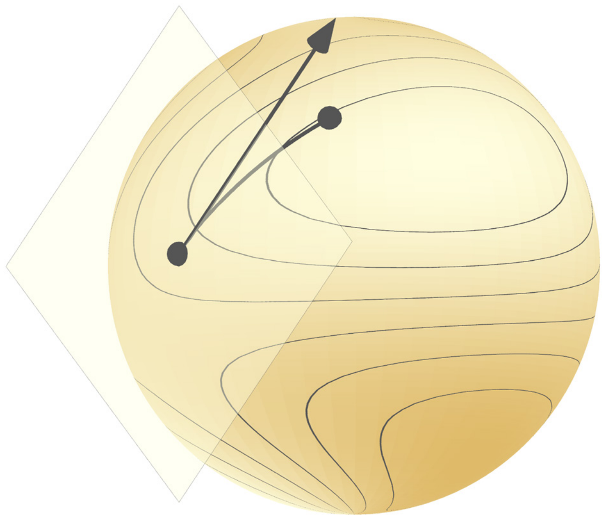
106

Local frames



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Optimization on manifolds, MATH 512 @ EPFL

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Expanding vectors and vector fields

Given a basis $w_1, \ldots, w_n$ for a tangent space $\mathrm{T}_x\mathcal{M}$, for all $v \in \mathrm{T}_x\mathcal{M}$,

$$v = a_1 w_1 + \cdots + a_n w_n$$

with unique real coefficients $a_1, \ldots, a_n$.

If we have a vector field V, then we can expand $V(x)$ in that basis.

Given a basis for *each* tangent space $\mathrm{T}_y\mathcal{M}$, we can expand *each* $V(y)$.

It is convenient if these expansions are somehow related…

On occasions, for theory, we resort to the following:

Smoothly varying bases of tangent spaces

Def.: A (smooth) local frame around x on a manifold $\mathcal{M}$ of dimension n is a collection of smooth vector fields $W_1, \ldots, W_n$ defined on a ngbhd $\mathcal{U} \subseteq \mathcal{M}$ of x such that, for all $y \in \mathcal{U}$,

$$W_1(y), \ldots, W_n(y) \text{ form a basis of } \mathrm{T}_y\mathcal{M}.$$

Claim: There exists a local frame around each x on a manifold $\mathcal{M}$.

Claim: If $\mathcal{M}$ is Riemannian, we can make the bases orthonormal.

We can use local frames to check that a vector field V is smooth:

Say $W_1, \dots, W_n$ form a local frame on $\mathcal{U} \subseteq \mathcal{M}$.

If V is a vector field on $\mathcal{M}$, then for each $x \in \mathcal{U}$ we have:

$$V(x) = f_1(x)W_1(x) + \cdots + f_n(x)W_n(x)$$

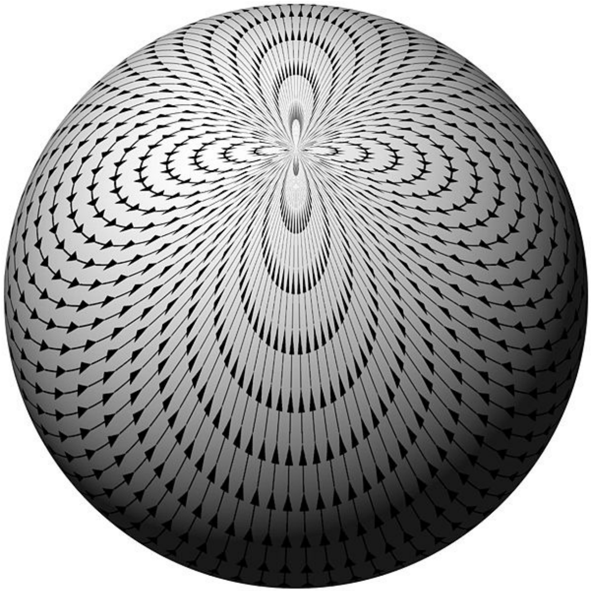
for some unique real numbers $f_1(x), \dots, f_n(x)$.

Claim: V is smooth on $\mathcal{U}$ if and only if $f_1, \dots, f_n \colon \mathcal{U} \to \mathbf{R}$ are smooth.

Why not *global* frames?

The Hairy Ball Theorem implies that, if W is a smooth vector field on the sphere S^2, then $W(x) = 0$ for some x.

Now imagine that we have a global frame on S^2, that is, smooth vector fields W_1, W_2 on S^2 such that $W_1(x), W_2(x)$ are linearly independent for all x. That can't be!



Manifolds which admit global frames are called parallelizable. Examples: $\mathbf{R}^d$, S^1.

https://en.wikipedia.org/wiki/Hairy_ball_theorem