

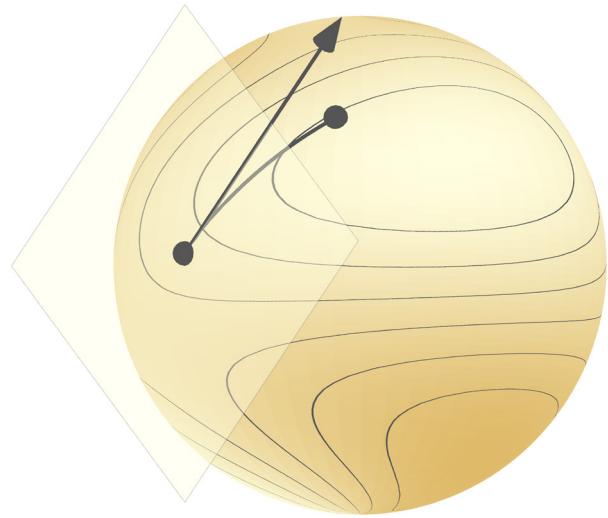
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# Smooth sets and functions: now with charts

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Optimization on manifolds, MATH 512 @ EPFL

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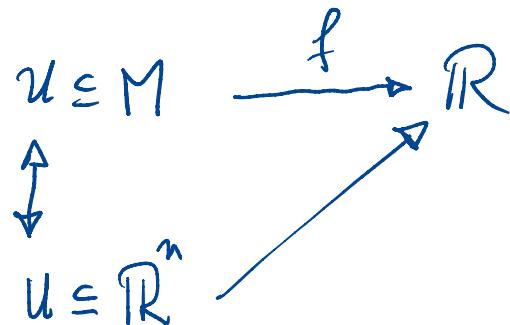


# It all starts with a simple question

Say  $M$  is a **set**. Just a set.

Let  $f: M \rightarrow \mathbf{R}$  be some **function** on that set.

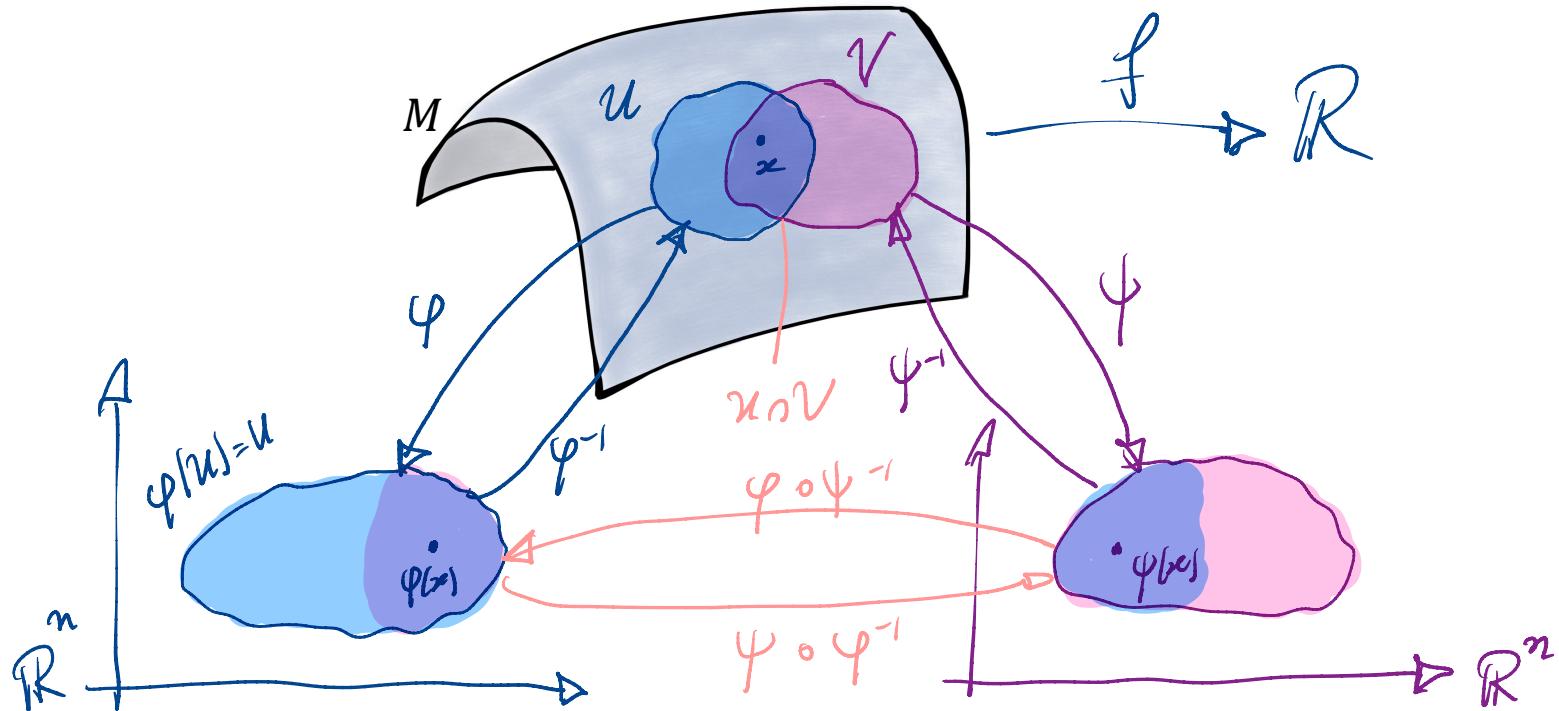
What should it mean for  $f$  to be “**smooth**” at  $x \in M$ ?



**Def.:** An  $n$ -dimensional chart for  $M$  is a pair  $(U, \varphi)$  such that:

$U$  is a subset of  $M$  and

$\varphi$  is an invertible map from  $U$  to  $\varphi(U)$ , open in  $\mathbb{R}^n$ .



We'll say  $f$  is smooth at  $x$  as judged through  $(U, \varphi)$

if  $f \circ \varphi^{-1}$  is smooth at  $\varphi(x)$ .

We also say that  $f$  is smooth at  $x$  as judged by  $(V, \psi)$

if  $f \circ \psi^{-1}$  is smooth at  $\psi(x)$ .

$$f \circ \varphi^{-1} = f \circ \varphi^{-1} \circ \varphi \circ \psi^{-1}$$



$$f: M \rightarrow \mathbb{R}, \quad \varphi: U \subseteq \mathbb{R}^n \rightarrow M$$

$$f \circ \varphi^{-1}: U \rightarrow \mathbb{R}$$

**Def.:** Two **charts**  $(U, \varphi)$  and  $(V, \psi)$  for  $M$  are **compatible** if they have the same dimension  $n$  and, if  $U \cap V \neq \emptyset$ , then

$\varphi(U \cap V)$  is open in  $\mathbf{R}^n$ ,

$\psi(U \cap V)$  is open in  $\mathbf{R}^n$ , and

$\psi \circ \varphi^{-1}: \varphi(U \cap V) \rightarrow \psi(U \cap V)$  is a diffeomorphism ( $C^\infty$ ).

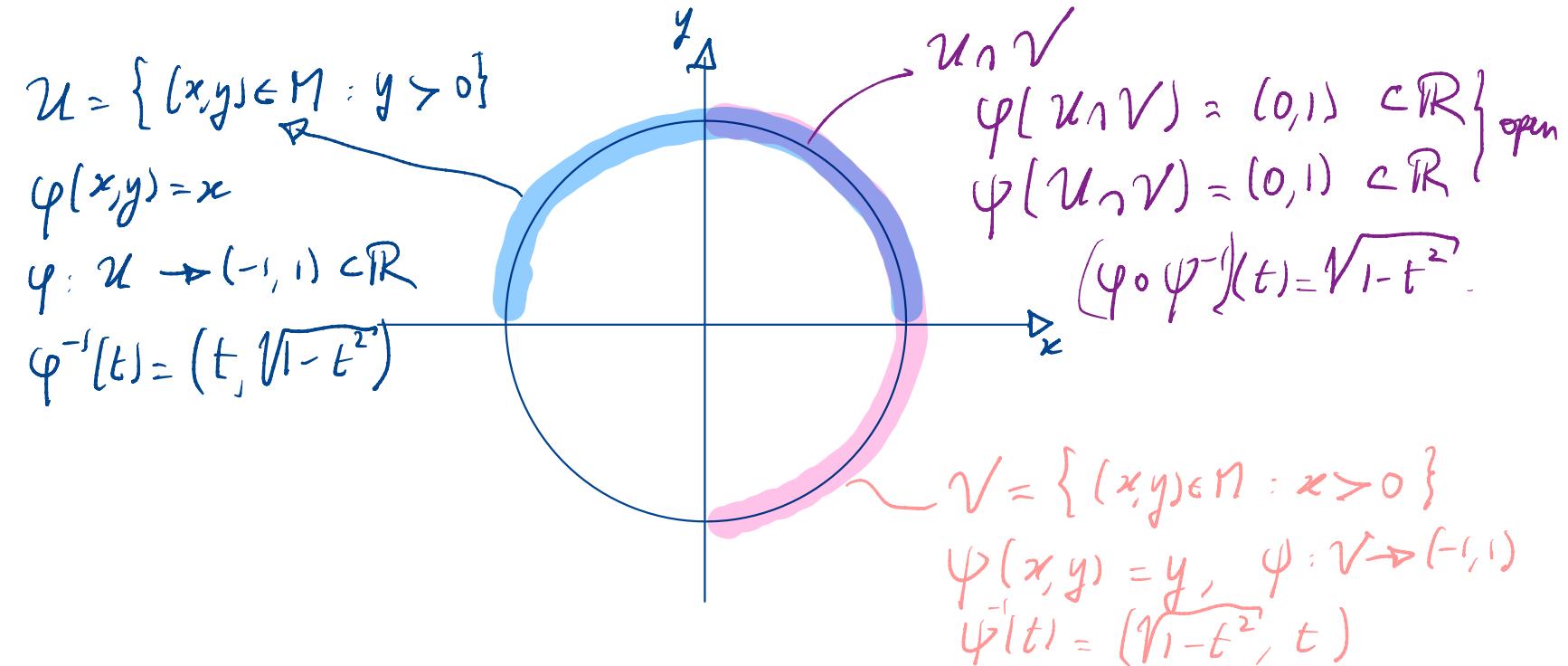
**Def.:** An **atlas** for  $M$  is a collection of compatible charts whose domains cover  $M$ .

**Def.:** A **manifold\*** is a pair  $\mathcal{M} = (M, \mathcal{A})$ : a set with an atlas.

The **dimension** of  $\mathcal{M}$ ,  $\dim \mathcal{M}$ , is that of any chart.

**Example 0:**  $\mathbf{R}^n$ , with the single chart  $\varphi(x) = x$  (identity on  $\mathbf{R}^n$ ).

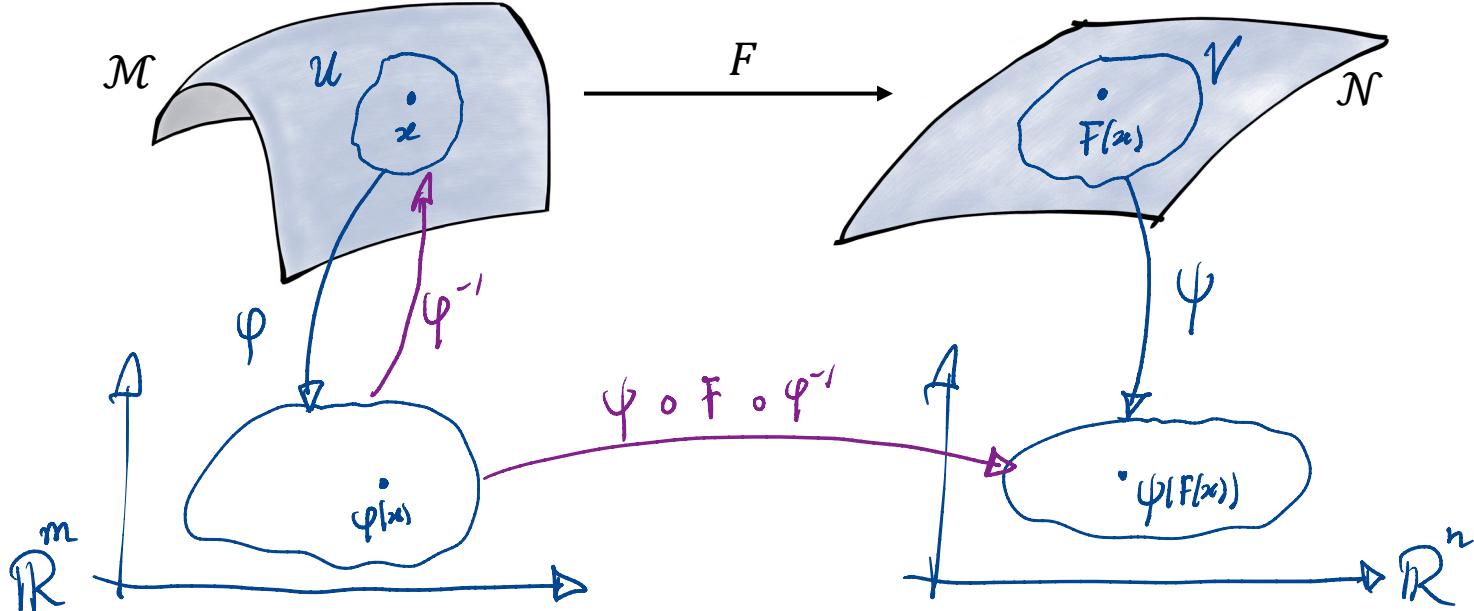
**Example 1:**  $M = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$ , with atlas as follows:



# Smooth functions

Let  $\mathcal{M} = (M, \mathcal{A})$  and  $\mathcal{N} = (N, \mathcal{B})$  be two manifolds\*.

What should it mean for  $F: \mathcal{M} \rightarrow \mathcal{N}$  to be smooth at  $x \in \mathcal{M}$ ?



**Def.:**  $F: \mathcal{M} \rightarrow \mathcal{N}$  is smooth at  $x$  if, with some chart  $(U, \varphi)$  of  $\mathcal{M}$  and some chart  $(V, \psi)$  of  $\mathcal{N}$  such that  $x \in U$  and  $F(x) \in V$ , the map

$$\tilde{F} = \psi \circ F \circ \varphi^{-1}$$

is smooth at  $\varphi(x)$ .

**Fact:** This definition does not depend on the choice of charts.

# Surely, we want a topology on $\mathcal{M}$

A **topology** on  $\mathcal{M}$  is a suitable collection of subsets deemed open.

Useful to define local minima, to define convergence...

A topology also defines continuity of functions:

$f$  is continuous if the pre-image of each open set is open.

We better make sure that smooth functions are continuous!

Charts are smooth, so they should be continuous.

$\varphi: U \rightarrow \mathbb{R}^n$  and  $\varphi(U)$  is open in  $\mathbb{R}^n$ ,

so  $\varphi^{-1}(\varphi(U))$  must be open in the topology (still TBD) of  $U$ .

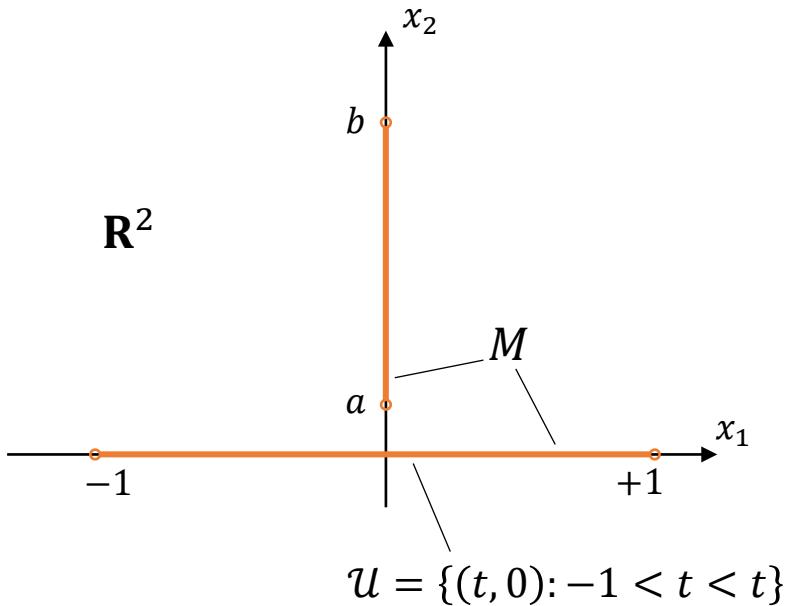
But  $\varphi^{-1}(\varphi(U)) = U$ , so the domain of each chart must be deemed open.

Now add to the atlas all the charts that could have been in there without breaking compatibility: those chart domains form a basis for our topology.

**Def.:** Given an atlas  $\mathcal{A}$  for a set  $M$ , the **maximal atlas**  $\mathcal{A}^+$  is the collection of all charts that are compatible with  $\mathcal{A}$ .

**Def.:** The **atlas topology** associated to  $\mathcal{A}^+$  states that a subset of  $M$  is open iff it is the union of a collection of chart domains.

# The atlas topology can be uncomfortable...



It's possible to pick an atlas for  $M$  such that, in the atlas topology, the sequence

$$(1/2, 0), (1/3, 0), (1/4, 0), (1/5, 0) \dots$$

*converges to  $(0, \alpha)$  for all  $\alpha \in \{0\} \cup (a, b)$ :*  
the limit is not unique! (and just plain odd)

$$\mathcal{U}_\alpha = \mathcal{U} \setminus \{(0, 0)\} \cup \{(0, \alpha)\}$$

$$\varphi_\alpha: \mathcal{U}_\alpha \rightarrow (-1, 1): x \mapsto \varphi_\alpha(x) = x_1$$

$$\text{Atlas: } \{(\mathcal{U}_\alpha, \varphi_\alpha): \alpha = 0 \text{ or } \alpha \in (a, b)\}$$

Atlas topology is neither **Hausdorff**  
nor **second-countable**.

**Def.:** A **(smooth) manifold** is a pair  $\mathcal{M} = (M, \mathcal{A}^+)$  whose atlas topology is Hausdorff and second-countable.

**Fact:** Let  $\mathcal{A}$  be an atlas. The atlas topology (of  $\mathcal{A}^+$ ) is **Hausdorff** if two distinct points of  $M$  belong either to the same chart domain or to two disjoint chart domains of  $\mathcal{A}$ ; and **Second-countable** if countably many chart domains cover  $M$ .

# Embedded geometry fits nicely in here.

A subset  $M$  of a linear space  $\mathcal{E}$  may admit many genuinely different maximal atlases, yielding **different manifolds** (for the **same set!**).

E.g.:  $(\mathbf{R}, x \mapsto x)$  and  $(\mathbf{R}, x \mapsto x^3)$  yield different maximal atlases on  $\mathbf{R}$ .

However,

-  If  $M$  is what we called an **embedded** submanifold of  $\mathcal{E}$ , then there exists a **unique** maximal **atlas** that turns  $M$  into a manifold s.t.
  1. The **atlas topology  $\equiv$  subspace topology** (as we've been using!) and
  2. A function on  $M$  is smooth (as seen through charts) if and only if it admits a smooth extension (as we've been using!).