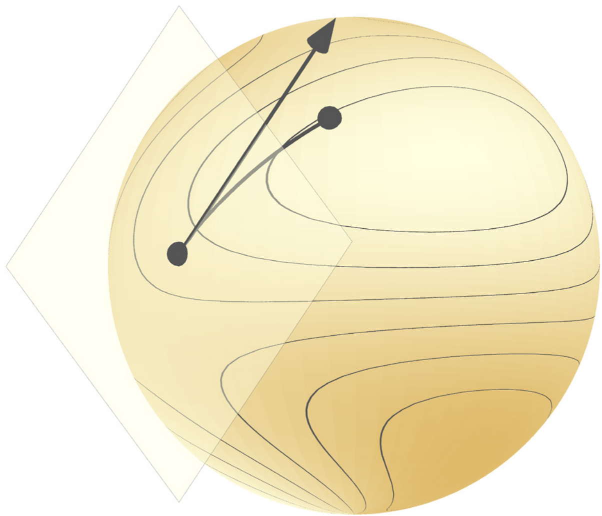
201

# Taylor expansions, first order



Spring 2023

Optimization on manifolds, MATH 512 @ EPFL

Instructor: Nicolas Boumal

# Function value along a curve

Our smooth cost function is $f : \mathcal{M} \to \mathbf{R}$.

Algorithms move from $x_k$ to $x_{k+1}$ along a (retraction) curve.

The value of $f$ changes along curves. How?

let $c : \mathbb{R} \to \mathcal{M}$, a smooth curve.

let $x = c(0)$, and let $v = c'(0)$.

$f : \mathcal{M} \to \mathbb{R}$

$c : \mathbb{R} \to \mathcal{M}$

$g = f \circ c : \mathbb{R} \to \mathbb{R}$

$g(t) = g(0) + t\, g'(0) + O(t^2)$

$g'(0) = (f \circ c)'(0)$

$= Df(x)[v]$

$$= \langle \operatorname{grad} f(x), v \rangle_x$$

$$f(c(t)) = f(x) + t\langle \operatorname{grad} f(x), v \rangle_x + O(t^2)$$

In particular, if $R$ is a retraction on $\mathcal{M}$, consider $c(t) = R_x(tv)$.

$$f(R_x(tv)) = f(x) + t\langle \operatorname{grad} f(x), v \rangle_x + O(t^2)$$

let $s = tv \in T_x\mathcal{M}$

$$f(R_x(s)) = f(x) + \langle \operatorname{grad} f(x), s \rangle_x + O(\|s\|_x^2)$$

# Take away

**Fact:** Consider $f : \mathcal{M} \to \mathbf{R}$ smooth.

If $c : \mathbf{R} \to \mathcal{M}$ is a smooth curve and $c(0) = x, c'(0) = v$, then

$$f\big(c(t)\big) = f(x) + t\langle \mathrm{grad} f(x), v\rangle_x + O(t^2).$$

Also, with $R$ a retraction on $\mathcal{M}$, we have:

$$f\big(R_x(s)\big) = f(x) + \langle \mathrm{grad} f(x), s\rangle_x + O(\|s\|_x^2).$$