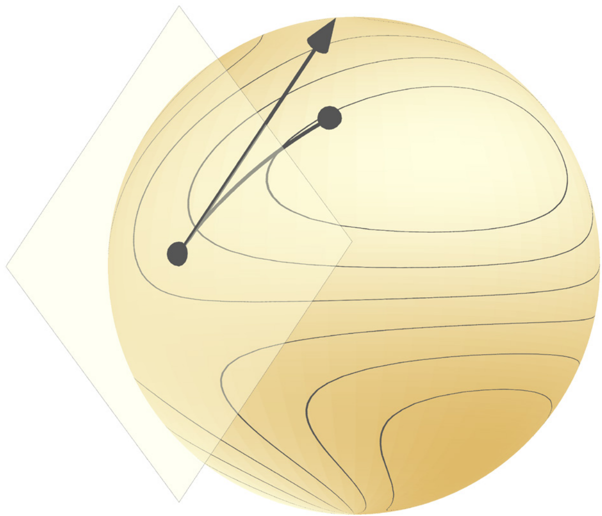
501

From embedded to general manifolds: upgrading our foundations



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Optimization on manifolds, MATH 512 @ EPFL

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Some sets are smooth (in a sense to be defined) yet are not explicitly embedded.

Example: some quotient sets in optimization with symmetries.

PCA : input is a cloud of points in $\mathbb{R}^d$

output is a linear subspace that "fits" the cloud.

We built everything on just three concepts

What is
a smooth function?

What is
a tangent vector?

What is
a smooth set?

Those three foundational definitions rely on an embedding $\mathcal{M} \subseteq \mathcal{E}$:

- The set $\mathcal{M}$ is smooth if it admits local defining functions.
  We equip it with the subspace topology inherited from $\mathcal{E}$.

- A function $F: \mathcal{M} \to \mathcal{N}$ is smooth if it has a smooth extension.

- A tangent vector is the velocity of a curve on $\mathcal{M}$ viewed in $\mathcal{E}$.

What is
a smooth function?

What is
a tangent vector?

What is
a smooth set?

All other concepts, tools and algorithms rely on those three.

$\mathrm{Hess}f$

$\nabla, \frac{\mathrm{D}}{\mathrm{d}t}$

$\mathrm{grad}f$

$\langle u, v \rangle_x$

$\mathfrak{X}(\mathcal{M})$

R

$\mathrm{D}F(x)[v]$

$\mathrm{T}\mathcal{M}$

$T\mathcal{M} \triangleq \{(x,v) : x \in \mathcal{M},\ v \in T_x\mathcal{M}\}$

What is a smooth function?

What is a tangent vector?

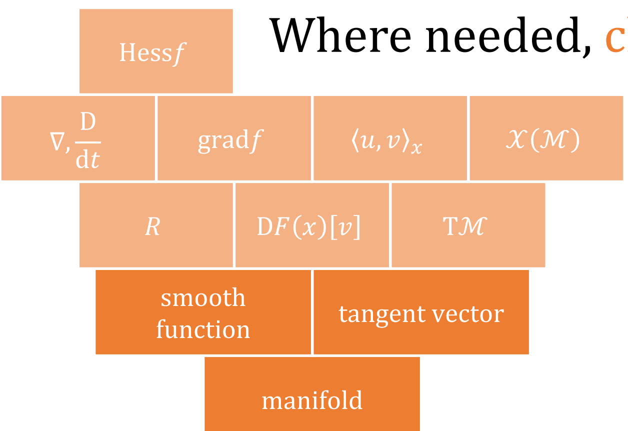
What is a smooth set?

The plan is clear:

Replace the three foundational definitions to remove any references to a (possibly inexistent) embedding space.

(Mostly) copy-paste definitions of the derived concepts.

Where needed, change proofs to cater to the new definitions.



Check that both perspectives are fully compatible (they are).

Wait but why?

Good mathematical reasons not to go general:

Most applications are on embedded submanifolds.

Whitney's and Nash's embedding theorems say:

"Every (Riemannian) manifold can be (isometrically) embedded into some Euclidean space."

Good mathematical reasons to go general:

Some applications are not embedded, e.g., quotient manifolds.

Mere existence of an embedding is useless for computation.

Everyone out there speaks the general language.