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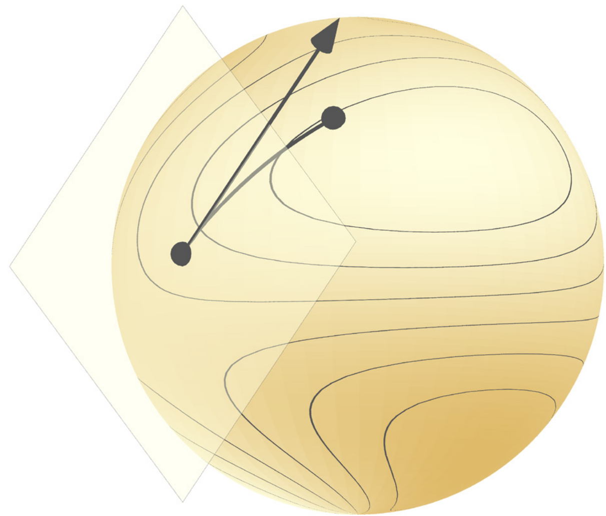
Connections



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Optimization on manifolds, MATH 512 @ EPFL

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Differentiating vector fields: let's try again

Say V is a smooth vector field on $\mathcal{M}$.

Unfortunately, we might have that $\mathrm{D}V(x)[u]$ is not tangent at x.

We need a new tool to differentiate vector fields.

This begs the question:

If $\mathrm{D}V$ is not good enough, then what *is* good enough?

A look back at linear spaces

If V is a smooth vector field on $\mathcal{E}$, then we differentiate "as usual":

$$\mathrm{D}V(x)[u] = \lim_{t \to 0} \frac{V(x+tu) - V(x)}{t}.$$

$\forall\, U, V, W : \mathbb{R}^d \to \mathbb{R}^d$ smooth

$\forall\, a, b \in \mathbb{R}$

$\forall\, u, w \in \mathbb{R}^d$

$\forall\, f : \mathbb{R}^d \to \mathbb{R}$ smooth

Here are four properties of that derivative:

0. $x \mapsto \mathrm{D}V(x)[U(x)]$ is a smooth vector field.

1. $\mathrm{D}V(x)[au + bw] = a\,\mathrm{D}V(x)[u] + b\,\mathrm{D}V(x)[w]$

2. $\mathrm{D}(aV + bW)(x)[u] = a\,\mathrm{D}V(x)[u] + b\,\mathrm{D}W(x)[u]$

3. $\mathrm{D}(fV)(x)[u] = \mathrm{D}f(x)[u] \cdot V(x) + f(x)\,\mathrm{D}V(x)[u]$

$(fV)(x)$
$= f(x)V(x).$

Connections: an axiomatic definition

Let $\mathcal{X}(\mathcal{M})$ denote the set of smooth vector fields on $\mathcal{M}$.

Def.: A connection on a manifold $\mathcal{M}$ is a map ∇, as follows:

$$\nabla : \mathrm{T}\mathcal{M} \times \mathcal{X}(\mathcal{M}) \to \mathrm{T}\mathcal{M} : \quad (u, V) \mapsto \nabla_u V$$

" $\nabla_{(x,u)} V$ "

such that $\nabla_u V$ is tangent at x if u is tangent at x, and:

0. Define $\nabla_U V$ by $(\nabla_U V)(x) = \nabla_{U(x)} V$: $\nabla_U V \in \mathcal{X}(M)$.
1. $\nabla_{au+bw} V = a \nabla_u V + b \nabla_w V$
2. $\nabla_u (aV + bW) = a \nabla_u V + b \nabla_u W$
3. $\nabla_u (fV) = Df(x)[u]\, V(x) + f(x) \nabla_u V$ (Leibniz rule)

$\forall a, b \in \mathbb{R}$
$\forall u, w \in T_x M$
$\forall U, V, W \in \mathcal{X}(M)$
$\forall f : M \to \mathbb{R}$ smooth.

Connections exist, and it's easy to pick one

Fact: For a linear space, $\nabla_u V = \mathrm{D}V(x)[u]$ defines a connection.

Fact: For an embedded submanifold $\mathcal{M}$ in a Euclidean space $\mathcal{E}$,

$$\nabla_u V = \mathrm{Proj}_x\left(\mathrm{D}\bar{V}(x)[u]\right)$$

defines a connection.

Fact: Every manifold $\mathcal{M}$ has infinitely many valid connections.