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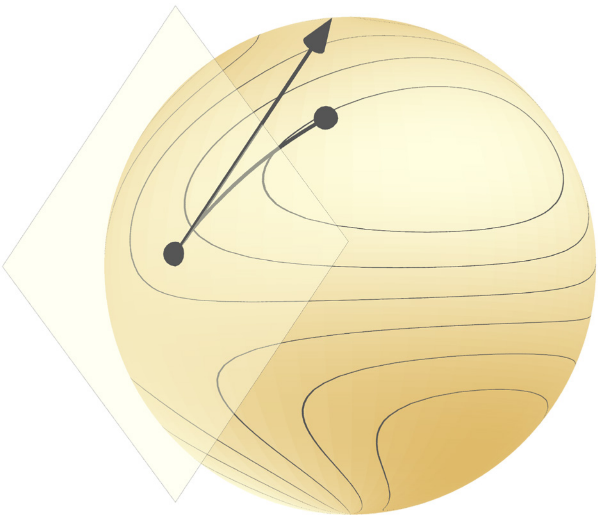
# Retractions, vector fields and tangent bundles



Spring 2023

Optimization on manifolds, MATH 512 @ EPFL

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# Moving on manifolds: towards retractions

To move around on $\mathcal{M}$, we want retractions—still to be defined.

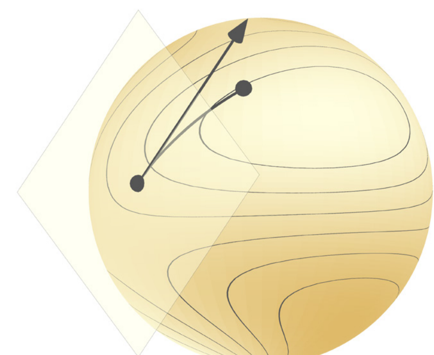
A retraction is a map $R$ which takes as input a point $x$ and a tangent vector $v$ at $x$, and outputs a new point on $\mathcal{M}$, denoted by $R_x(v)$.

$= R(x, v)$

Thus, the *domain* of $R$ as a map $(x, v) \mapsto R_x(v)$ is:

$R : T\mathcal{M} \to \mathcal{M}$

$$T\mathcal{M} = \{(x, v) : x \in \mathcal{M} \text{ and } v \in \mathrm{T}_x\mathcal{M}\}$$



We will want $R$ to be smooth. Meaning?

# Tangent bundles

$\mathcal{M} \subseteq \mathcal{E}$ : $x \in \mathcal{M} \Rightarrow x \in \mathcal{E}$

$v \in T_x\mathcal{M} \subseteq \mathcal{E} \Rightarrow v \in \mathcal{E}$

**Def.:** The tangent bundle of a manifold $\mathcal{M}$ is the set

$$\mathrm{T}\mathcal{M} = \{(x, v) : x \in \mathcal{M} \text{ and } v \in \mathrm{T}_x\mathcal{M}\}.$$

**Theorem**: If $\mathcal{M}$ is an embedded submanifold of $\mathcal{E}$, then $\mathrm{T}\mathcal{M}$ is an embedded submanifold of $\mathcal{E} \times \mathcal{E}$. Moreover, $\dim \mathrm{T}\mathcal{M} = 2 \dim \mathcal{M}$.

**Proof.** Pick $(\bar{x}, \bar{v}) \in T\mathcal{M}$ : $\bar{x} \in \mathcal{M}$, $\bar{v} \in T_{\bar{x}}\mathcal{M}$.

Pick a local defining function $h : U \to \mathbb{R}^k$ for $\mathcal{M}$ around $\bar{x}$:

$U$ is a nghbhd of $\bar{x}$ in $\mathcal{E}$; $h^{-1}(0) = U \cap \mathcal{M}$; $h$ is smooth;

$\operatorname{rank}(Dh(\bar{x})) = k$. $\quad T_{\bar{x}}M = \ker Dh(\bar{x})$; so: $Dh(\bar{x})[\bar{v}] = 0$.

$x \in U$: $\quad h(x) = 0 \iff x \in M$

$$Dh(\bar{x}) : \underset{\substack{\wr\wr \\ \mathbb{R}^d}}{\mathcal{E}} \to \mathbb{R}^k \quad : \quad Dh(\bar{x}) \sim \text{matrix of size } k \times d.$$

$Dh(\bar{x})\, Dh(\bar{x})^T$ : matrix of size $k \times k$.

$x \mapsto \det\left( Dh(x)\, Dh(x)^T \right)$ is a smooth function on $U$,

and it is nonzero at $\bar{x} \in U$.

If need be, make $U$ smaller so that the determinant

is nonzero for all $x \in U$; Then: $\operatorname{rank}(Dh(x)) = k$

for all $x \in U$.

We now have that $v \in T_x\mathcal{M} \iff Dh(x)[v] = 0$.

for all $x$ in $\mathcal{U}$

Let $H : \mathcal{U} \times \mathcal{E} \to \mathbb{R}^{2k}$:

$$H(x, v) \triangleq \begin{bmatrix} h(x) \\ Dh(x)[v] \end{bmatrix} = 0 \iff \begin{cases} h(x) = 0 \iff x \in \mathcal{M} \\ Dh(x)[v] = 0 \iff v \in T_x\mathcal{M} \end{cases}$$

$$\Updownarrow$$

$$(x, v) \in T\mathcal{M}.$$

$$DH(x, v)[(\underbrace{\dot{x}}_{\in \mathcal{E}}, \underbrace{\dot{v}}_{\in \mathcal{E}})] = \begin{bmatrix} Dh(x)[\dot{x}] \\ L(x, v)[\dot{x}] + Dh(x)[\dot{v}] \end{bmatrix}$$

$$= \begin{bmatrix} Dh(x) & 0 \\ \mathcal{L}(x,v) & Dh(x) \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix}$$

$$\operatorname{rank}(DH(x,v)) = \operatorname{rank}(Dh(x)) + \operatorname{rank}(Dh(x)) = 2k = \dim \mathbb{R}^{2k} \checkmark$$

So $H$ is a local defining function for $TM$ around $(\bar{x}, \bar{v}) \in TM$.

and
$$\begin{aligned} \dim TM &= \dim(\mathcal{E} \times \mathcal{E}) - 2k \\ &= 2 \dim \mathcal{E} - 2k \\ &= 2(\dim \mathcal{E} - k) = 2 \dim M. \end{aligned}$$

$\square$

# Retractions

$R_x : T_x \mathcal{M} \to \mathcal{M}$

**Def.:** A retraction is a smooth map

$$R: \mathrm{T}\mathcal{M} \to \mathcal{M}: (x, v) \mapsto R_x(v)$$

such that each curve

$$c(t) = R_x(tv)$$

c is a smooth curve on $\mathcal{M}$.

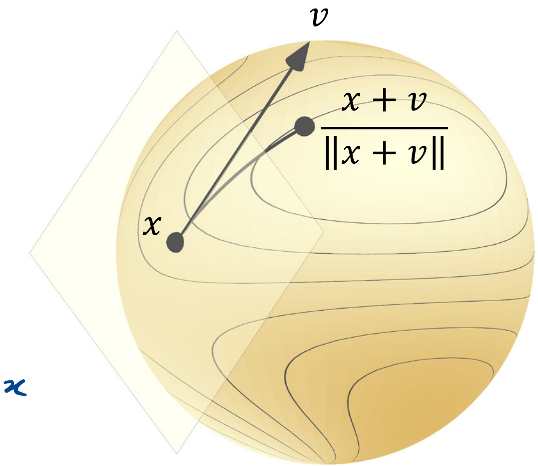
satisfies $c(0) = x$ and $c'(0) = v$.

**Example 0:** On $\mathcal{M} = \mathcal{E}$, $\quad R_x(v) = x + v$

$c(t) = R_x(tv) = x + tv.$

$\bar{R}: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d, \quad \bar{R}(x, v) = \frac{x+v}{\sqrt{1+\|v\|^2}}$

**Example 1:** On $\mathcal{M} = \mathrm{S}^{d-1}$, $\quad R_x(v) = \dfrac{x+v}{\|x+v\|}$




$$c(t) = R_x(tv) = \frac{x+tv}{\|x+tv\|}$$

$$\|x+tv\|^2$$
$$= (x+tv)^T(x+tv)$$
$$= 1 + t(v^T x + x^T v) + t^2 v^T v$$
$$= 1 + t^2\|v\|^2$$

$$= \frac{x+tv}{\sqrt{\|x+tv\|^2}}$$

$$= \frac{x+tv}{\sqrt{1+t^2\|v\|^2}}$$

$\to c(0) = x$

$\to c'(0) = v.$

**Example 2:** On $\mathcal{M} = \mathrm{S}^{d-1}$, $\quad R_x(v) = \cos(\|v\|)x + \dfrac{\sin(\|v\|)}{\|v\|}v$

$T\mathcal{M} = \{(y,v) : y \in \mathcal{M} \text{ and } v \in T_y\mathcal{M}\}$

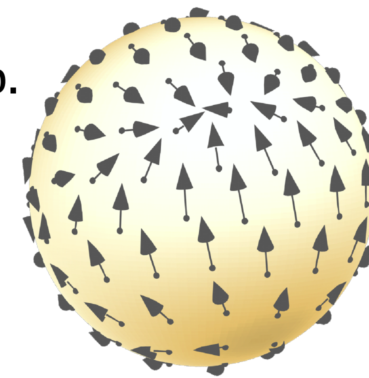
# Vector fields

**Def.:** A vector field $V$ on a manifold $\mathcal{M}$ is a map $V: \mathcal{M} \to \mathrm{T}\mathcal{M}$ such that each $V(x)$ is tangent at $x$.

" $V(x) \in T_x\mathcal{M}$ "

$V(x) = (x, v)$ for some $v \in T_x\mathcal{M}$

It is a smooth vector field if it is also a smooth map.



A vector field is smooth iff it can be smoothly extended:

**Claim:** If $\mathcal{M}$ is embedded in $\mathcal{E}$, then $V$ is a smooth vector field iff there exists a smooth vector field $\bar{V}$ on a ngbhd of $\mathcal{M}$ s.t. $V = \bar{V}|_{\mathcal{M}}$.