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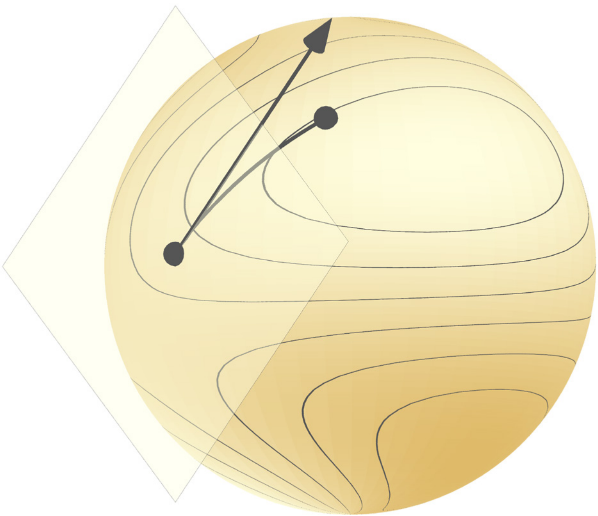
# Optimality conditions, second order



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Optimization on manifolds, MATH 512 @ EPFL

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# Necessary optimality conditions

Let $x$ be a point on a manifold $\mathcal{M}$.

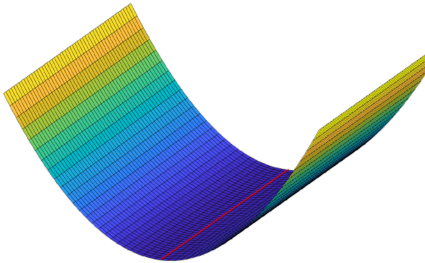
Consider a smooth function $f : \mathcal{M} \to \mathbf{R}$.

For any smooth curve $c : \mathbf{R} \to \mathcal{M}$ with $c(0) = x$, we have:

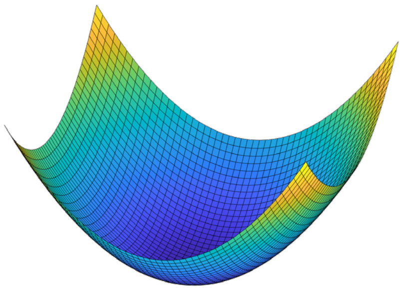
$$f\big(c(t)\big) = f(x) + t \cdot \underbrace{\boxed{(f \circ c)'(0)}}_{=0} + \frac{t^2}{2} \cdot \boxed{(f \circ c)''(0)} + O(t^3)$$

If $x$ is a local minimum, what do we learn about the boxed terms?

# Critical points, second order



**Def.:** A point $x \in \mathcal{M}$ is second-order critical or stationary for $f : \mathcal{M} \to \mathbf{R}$ if



$$(f \circ c)'(0) = 0 \quad \text{and} \quad (f \circ c)''(0) \geq 0$$

for all smooth curves $c$ on $\mathcal{M}$ such that $c(0) = x$.

**Fact:** If $x$ is a local minimum, then it is second-order critical.

**Fact:** On a Riemannian manifold, $x$ is second-order critical iff $\mathrm{grad} f(x) = 0$ and $\mathrm{Hess} f(x) \succcurlyeq 0$.

# Sufficient optimality conditions

**Fact:** On a Riemannian manifold,
if $\operatorname{grad} f(x) = 0$ and $\operatorname{Hess} f(x) \succ 0$, then $x$ is a local minimum.

**Proof sketch:**

$$f\big(R_x(s)\big) = f(x) + \frac{1}{2}\langle s, \operatorname{Hess} f(x)[s]\rangle_x + O(\|s\|_x^3)$$

Then argue $R_x$ can reach a ngbhd of $x$. See §6.1.