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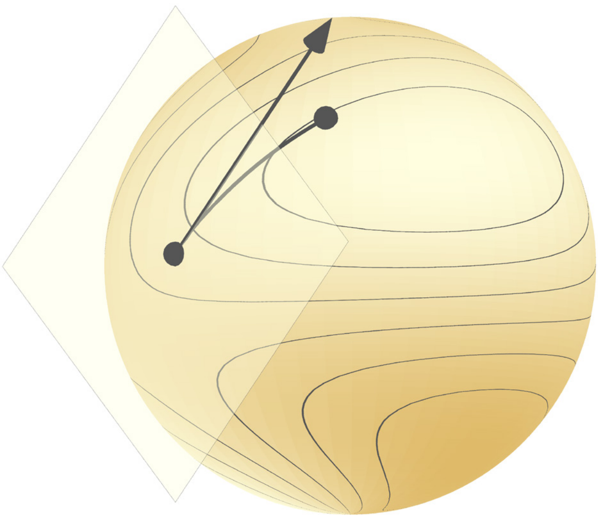
Smooth maps and differentials



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Optimization on manifolds, MATH 512 @ EPFL

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Smooth maps on/to manifolds

$\mathcal{M}$ is an embedded submanifold of $\mathcal{E}$. Same for $\mathcal{M}'$ in $\mathcal{E}'$.

What does it mean for $F: \mathcal{M} \to \mathcal{M}'$ to be smooth?

$S^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$

Example: $f: \mathrm{S}^{d-1} \to \mathbf{R}$ defined by $f(x) = x^\top A x$

$\bar{f}: \mathbb{R}^d \to \mathbb{R}$, $\bar{f}(x) = x^\top A x$ is smooth (C^∞).

$f = \bar{f}|_{S^{d-1}}$

restriction of $\bar{F}$

Def.: $F: \mathcal{M} \to \mathcal{M}'$ is smooth if there exists a ngbhd U of $\mathcal{M}$ in $\mathcal{E}$ and a smooth map $\bar{F}: U \to \mathcal{E}'$ such that $\underbrace{F = \bar{F}|_{\mathcal{M}}}_{F(y) = \bar{F}(y) \ \forall y \in \mathcal{M}.}$

smooth extension of F

Fact: Composition preserves smoothness.

$F: \mathcal{M} \to \mathcal{M}'$, $\quad G: \mathcal{M}' \to \mathcal{M}''$, $\quad G \circ F: \mathcal{M} \to \mathcal{M}''$

Claim: if F and G are smooth, then $G \circ F$ is smooth.

To prove F is smooth, it's enough to check locally (proof omitted):

Def.: $F: \mathcal{M} \to \mathcal{M}'$ is smooth at $x \in \mathcal{M}$ if it is smooth on a ngbhd of x.

Fact: If F is smooth at all $x \in \mathcal{M}$, then it is smooth on $\mathcal{M}$.

Differentials of maps on/to manifolds

For a smooth map $\bar{F}: \mathcal{E} \to \mathcal{E}'$, the differential at x is the linear map:

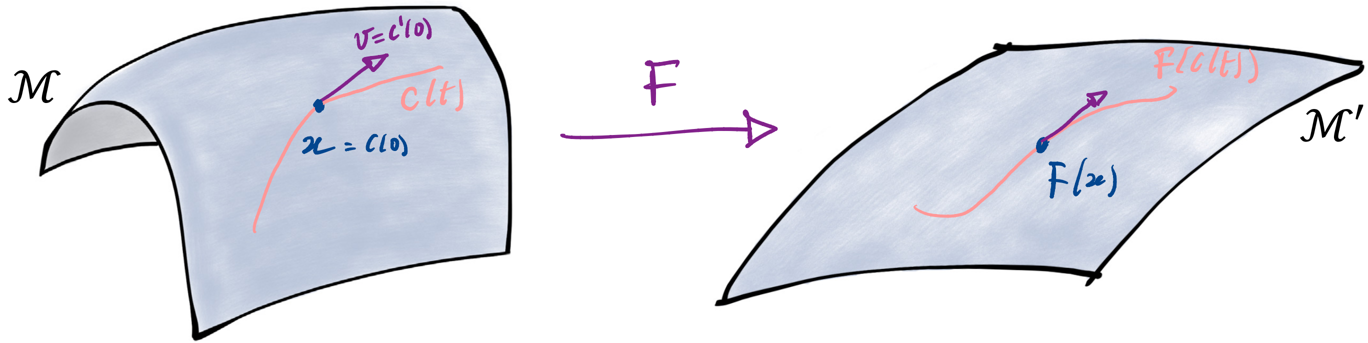
$D\bar{F}(x): \mathcal{E} \to \mathcal{E}'$
linear

$$\mathrm{D}\bar{F}(x)[v] = \lim_{t \to 0} \frac{\bar{F}(x + tv) - \bar{F}(x)}{t}$$

Quid for a smooth $F: \mathcal{M} \to \mathcal{M}'$? Two perspectives:

Perspective 1: If $v \in T_x\mathcal{M}$, there exists a smooth curve $c: \mathbb{R} \to \mathcal{M}$ s.t. $c(0) = x$ and $c'(0) = v$.

$F \circ c : \mathbb{R} \to \mathcal{M}'$



Def.: The differential of F at x is the map $\mathrm{D}F(x)\colon \mathrm{T}_x\mathcal{M} \to \mathrm{T}_{F(x)}\mathcal{M}'$:

$$\mathrm{D}F(x)[v] = \frac{\mathrm{d}}{\mathrm{d}t}(F \circ c)\bigg|_{t=0} = (F \circ c)'(0) = \lim_{t \to 0} \frac{F\big(c(t)\big) - F(c(0))}{t}$$

where c is a smooth curve on $\mathcal{M}$ satisfying $c(0) = x$ and $c'(0) = v$.

a) Is this well defined?
(no dependence on choice of c?)

b) Is $\mathrm{D}F(x)$ linear?

Perspective 2: Since $F : \mathcal{M} \to \mathcal{M}'$ is smooth, there exist $\mathcal{U}$ (a nghbhd of $\mathcal{M}$ in $\mathcal{E}$) and $\bar{F} : \mathcal{U} \to \mathcal{E}'$ s.t. a) $\bar{F}$ is smooth,
b) $F = \bar{F}|_{\mathcal{M}}$.

$$\mathrm{D}F(x)[v] = (F \circ c)'(0) = (\bar{F} \circ c)'(0)$$

$$= \mathrm{D}\bar{F}(c(0))[c'(0)] = \mathrm{D}\bar{F}(x)[v].$$

chain rule

Fact: $\mathrm{D}F(x) = \mathrm{D}\bar{F}(x)|_{\mathrm{T}_x\mathcal{M}}$,

where $\bar{F}$ is any smooth extension of F around x.

In particular, $\mathrm{D}F(x)$ is a linear map from $\mathrm{T}_x\mathcal{M}$ to $\mathrm{T}_{F(x)}\mathcal{M}'$.

Properties of smooth maps and differentials

Chain rule: If $F: \mathcal{M} \to \mathcal{M}'$ and $G: \mathcal{M}' \to \mathcal{M}''$ are smooth,

then $G \circ F: \mathcal{M} \to \mathcal{M}''$ is smooth,

and $\mathrm{D}(G \circ F)(x)[v] = \mathrm{D}G\big(F(x)\big)\big[\mathrm{D}F(x)[v]\big]$.

Linearity: If $F_1, F_2: \mathcal{M} \to \mathcal{E}'$ are smooth, for $a_1, a_2 \in \mathbf{R}$,

then $F(x) = a_1 F_1(x) + a_2 F_2(x)$ is smooth,

and $\mathrm{D}F(x)[v] = a_1 \mathrm{D}F_1(x)[v] + a_2 \mathrm{D}F_2(x)[v]$.

Product rule: If $f: \mathcal{M} \to \mathbf{R}$ and $F: \mathcal{M} \to \mathcal{E}'$ are smooth,

then $G(x) = f(x) F(x)$ is smooth,

and $\mathrm{D}G(x)[v] = \mathrm{D}f(x)[v]\, F(x) + f(x)\, \mathrm{D}F(x)[v]$.

Caveat about smooth extensions

Let $F: \mathcal{M} \to \mathcal{M}'$ with $\mathcal{M}$ embedded in $\mathcal{E}$.

If F is smooth, we can smoothly extend it to a ngbhd of $\mathcal{M}$ in $\mathcal{E}$.
But it is *not* necessarily possible to smoothly extend F to all of $\mathcal{E}$.

Example: $\mathcal{E} = \mathbf{R}$, $\mathcal{M} = \mathbf{R}\backslash\{0\}$, $f: \mathcal{M} \to \mathbf{R}: x \mapsto f(x) = \mathrm{sign}(x)$.

It is possible if $\mathcal{M}$ is closed though (proof omitted):
Fact: If $\mathcal{M}$ is closed in $\mathcal{E}$, $F: \mathcal{M} \to \mathcal{M}'$ is smooth if and only if there exists a smooth $\bar{F}: \mathcal{E} \to \mathcal{E}'$ such that $F = \bar{F}|_{\mathcal{M}}$.