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Low-rank matrix completion: optimization on manifolds at work

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Recommender systems tell you which items you might like

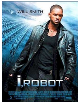
based on a huge database of ratings

$$\begin{array}{rcc} & \leftarrow n \text{ cols} \rightarrow & \\ X = & \begin{pmatrix} ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \end{pmatrix} & \begin{array}{c} \uparrow \\ m \text{ rows} \\ \downarrow \end{array} \end{array}$$

Ratings of items by the users are recorded

One row per item, one column per user

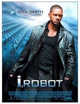
$$X = \begin{array}{cl} \begin{matrix} & & & & & & & & & & \text{user } j & & & & & \end{matrix} & \\ \begin{pmatrix} ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \end{pmatrix} & \begin{matrix} \\ \text{item } i \\ \\ \\ \\ \end{matrix} \end{array}$$



Ratings of items by the users are recorded

One row per item, one column per user

$$X = \begin{array}{cl}
\begin{array}{cccccccccccccccc} &&&&&&&&&& \text{user } j &&&&& \end{array} & \\
\begin{pmatrix}
? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? & ? & ? & 4 & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \\
? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ?
\end{pmatrix} & \begin{array}{l} \\ \text{item } i \\ \\ \\ \\ \end{array}
\end{array}$$



Most ratings are unknown. Our job is to complete X.

$$X = \begin{pmatrix} 1 & ? & 2 & ? & ? & 5 & ? & ? & ? & ? & 5 & ? & ? & ? & 2 & ? \\ 2 & ? & 2 & ? & ? & 4 & ? & ? & ? & 3 & 4 & ? & 5 & ? & ? & ? \\ 1 & ? & 5 & 2 & ? & 4 & ? & 4 & ? & ? & ? & 2 & ? & ? & ? & ? \\ ? & 1 & ? & 3 & ? & ? & ? & 3 & ? & ? & 3 & ? & 2 & ? & 5 & 5 \\ 4 & 4 & ? & ? & ? & ? & 5 & ? & ? & ? & 1 & ? & ? & 1 & ? & 4 \end{pmatrix}$$

We could exploit similarities between users and items

to complete the matrix

$$X = \begin{pmatrix} 1 & ? & 2 & ? & ? & 5 & ? & ? & ? & ? & 5 & ? & ? & ? & 2 & ? \\ 2 & ? & 2 & ? & ? & 4 & ? & ? & ? & 3 & 4 & ? & 5 & ? & ? & ? \\ 1 & ? & 5 & 2 & ? & 4 & ? & 4 & ? & ? & ? & 2 & ? & ? & ? & ? \\ ? & 1 & ? & 3 & ? & ? & ? & 3 & ? & ? & 3 & ? & 2 & ? & 5 & 5 \\ 4 & 4 & ? & ? & ? & ? & 5 & ? & ? & ? & 1 & ? & ? & 1 & ? & 4 \end{pmatrix}$$

In a global, automated, scalable way?

Scalability will guide the algorithm design

for both time and memory complexity

Netflix 1M$ prize:

- 17,700 movies;
- 480,000 users;
- 100,000,000 ratings (1%).

Scalability will guide the algorithm design

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Netflix 1M$ prize:

- 17,700 movies;
- 480,000 users;
- 100,000,000 ratings (1%).

$\Rightarrow$ The whole matrix won't fit into memory,

$\Rightarrow$ but the known ratings will.

We assume that X has low-rank r

Hence, that ratings are inner products in a small space $\mathbb{R}^r$

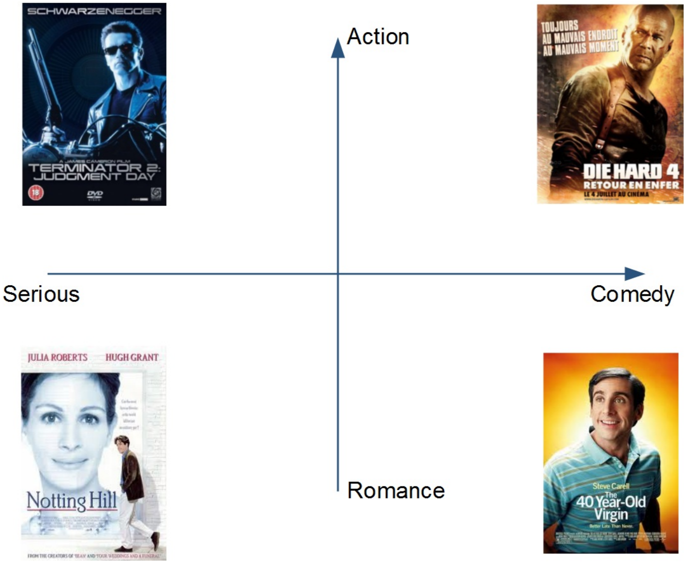
$$X = \begin{pmatrix} 1 & ? & 2 & ? & ? & 5 & ? & ? & ? & ? & 5 & ? & ? & ? & 2 & ? \\ 2 & ? & 2 & ? & ? & 4 & ? & ? & ? & 3 & 4 & ? & 5 & ? & ? & ? \\ 1 & ? & 5 & 2 & ? & 4 & ? & 4 & ? & ? & ? & 2 & ? & ? & ? & ? \\ ? & 1 & ? & 3 & ? & ? & ? & 3 & ? & ? & 3 & ? & 2 & ? & 5 & 5 \\ 4 & 4 & ? & ? & ? & ? & 5 & ? & ? & ? & 1 & ? & ? & 1 & ? & 4 \end{pmatrix}$$

$$\approx \begin{pmatrix} ? & ? \\ ? & ? \\ ? & ? \\ ? & ? \\ ? & ? \end{pmatrix} \cdot \begin{pmatrix} ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \end{pmatrix}$$

Rationale: only a few factors influence our preferences

We map items and users to this small dimensional space

without any human intervention



Toward a reasonable formulation

The optimal choice UW is an m-by-n matrix of rank r in best agreement with the k known entries of X

$$\min_{U \in \mathbb{R}^{m\times r}, W \in \mathbb{R}^{r\times n}} \sum_{\underbrace{(i,j)\in\Omega}_{\text{known entries}}} \overbrace{C_{ij}^2}^{\text{confidence}} \Big((UW)_{ij} - \overbrace{X_{ij}}^{\text{known ratings}}\Big)^2$$

This is reasonable

$$\min_{U\in\mathbb{R}^{m\times r}, W\in\mathbb{R}^{r\times n}} \boxed{\sum_{(i,j)\in\Omega} C_{ij}^2\big((UW)_{ij} - X_{ij}\big)^2}$$

- Very natural;
- Search space of dimension $r(m+n)$;
- Objective computable in $\mathcal{O}(rk)$ time;
- Ideal for Gaussian noise on ratings X_{ij}.

This is reasonable, *but*

$$\min_{U \in \mathbb{R}^{m \times r}, W \in \mathbb{R}^{r \times n}} \boxed{\sum_{(i,j) \in \Omega} C_{ij}^2 \big((UW)_{ij} - X_{ij} \big)^2}$$

Minimizers are not isolated.

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$$\min_{U \in \mathbb{R}^{m \times r}, W \in \mathbb{R}^{r \times n}} \boxed{\sum_{(i,j) \in \Omega} C_{ij}^2 \big((UW)_{ij} - X_{ij}\big)^2}$$

Minimizers are not isolated.

If (U, W) is a minimizer, $(UM, M^{-1}W)$ is too, for any r-by-r invertible matrix M.

The objective is invariant under invertible transformations

We don't want that. Why?

- The search space $\mathbb{R}^{m\times r} \times \mathbb{R}^{r\times n}$ is bigger than it ought to be;
- Most theoretical guarantees of convergence for iterative optimization methods assume isolated critical points;
- And it may prevent superlinear convergence rates altogether.

Partial solution: force U to be orthonormal

It's a kind of normalization

$$\min_{U \in \mathbb{O}(m,r), W \in \mathbb{R}^{r \times n}} \boxed{\sum_{(i,j) \in \Omega} C_{ij}^2 \big((UW)_{ij} - X_{ij} \big)^2}$$

$$\mathbb{O}(m,r) = \{ U \in \mathbb{R}^{m \times r} : U^\top U = I_r \}$$

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$$\mathbb{O}(m,r) = \{U \in \mathbb{R}^{m\times r} : U^\top U = I_r\}$$

Minimizers are still not isolated.

If (U, W) is a minimizer, $(UQ, Q^\top W)$ is too,
for any r-by-r orthogonal matrix Q.

The set of r-dimensional subspaces of $\mathbb{R}^m$ is the Grassmann manifold $\mathrm{Gr}(m, r)$

- The objective is well defined over the equivalence classes $[U] = \{UQ : Q \in \mathbb{R}^{r\times r}, Q^\top Q = I_r\}$;

- All matrices $UQ \in [U]$ share a common column space: $\mathrm{col}(UQ) = \mathscr{U} \in \mathrm{Gr}(m, r)$;

- We represent a point $\mathscr{U}$ on Grassmann with any orthonormal matrix U such that $\mathrm{col}(U) = \mathscr{U}$.

$$\min_{\mathscr{U} \in \mathrm{Gr}(m,r), W \in \mathbb{R}^{r \times n}} \boxed{\sum_{(i,j) \in \Omega} C_{ij}^2 \big((UW)_{ij} - X_{ij} \big)^2}$$

U is any m-by-r orthonormal matrix such that $\mathrm{col}(U) = \mathscr{U}$.

$$\min_{\mathscr{U} \in \mathrm{Gr}(m,r)} \min_{W \in \mathbb{R}^{r \times n}} \boxed{\sum_{(i,j) \in \Omega} C_{ij}^2 \big((UW)_{ij} - X_{ij} \big)^2}$$

U is any m-by-r orthonormal matrix such that $\mathrm{col}(U) = \mathscr{U}$.

W is the solution of a simple least squares problem.

The objective is a function of the column space $\mathscr{U}$

$$\min_{\mathscr{U} \in \mathrm{Gr}(m,r)} \underbrace{\boxed{\min_{W \in \mathbb{R}^{r \times n}} \sum_{(i,j) \in \Omega} C_{ij}^2 \big((UW)_{ij} - X_{ij}\big)^2}}_{f(\mathscr{U})}$$

U is any m-by-r orthonormal matrix such that $\mathrm{col}(U) = \mathscr{U}$.

W is the solution of a simple least squares problem.

f is not continuous :(

$$f(\mathscr{U}) = \boxed{\min_{W \in \mathbb{R}^{r \times n}} \sum_{(i,j) \in \Omega} C_{ij}^2 \big((UW)_{ij} - X_{ij}\big)^2}$$

Indeed, the least squares problem may not always have a unique solution.

This stems from unattended entries, see Dai, Milenkovic et al., 2010.

Regularization makes f smooth

$$f(\mathscr{U}) = \min_{W \in \mathbb{R}^{r \times n}} \sum_{(i,j) \in \Omega} C_{ij}^2 \big((UW)_{ij} - X_{ij}\big)^2$$

Regularization makes f smooth

$$f(\mathscr{U}) = \min_{W \in \mathbb{R}^{r \times n}} \sum_{(i,j) \in \Omega} C_{ij}^2 \big((UW)_{ij} - X_{ij}\big)^2 + \lambda \sum_{(i,j) \notin \Omega} (UW)_{ij}^2$$

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Computation of the inner objective looks like it costs $\mathcal{O}(mnr)$ time :(

A Matrix 101 trick to reduce computational costs

$$f(\mathscr{U}) = \boxed{\min_{W \in \mathbb{R}^{r \times n}} \sum_{(i,j) \in \Omega} C_{ij}^2 \big((UW)_{ij} - X_{ij}\big)^2 + \lambda \sum_{(i,j) \notin \Omega} (UW)_{ij}^2}$$

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$$\boxed{\sum_{(i,j) \notin \Omega} (UW)_{ij}^2} + \sum_{(i,j) \in \Omega} (UW)_{ij}^2 = \sum_{(i,j)} (UW)_{ij}^2$$

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$$\begin{aligned} \sum_{(i,j) \notin \Omega} (UW)_{ij}^2 + \sum_{(i,j) \in \Omega} (UW)_{ij}^2 &= \sum_{(i,j)} (UW)_{ij}^2 \\ &= \|UW\|_{\mathrm{F}}^2 \end{aligned}$$

A Matrix 101 trick to reduce computational costs

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$$\begin{aligned}
\sum_{(i,j) \notin \Omega} (UW)_{ij}^2 + \sum_{(i,j) \in \Omega} (UW)_{ij}^2 &= \sum_{(i,j)} (UW)_{ij}^2 \\
&= \|UW\|_{\mathrm{F}}^2 \\
&= \|W\|_{\mathrm{F}}^2
\end{aligned}$$

Complexity: $\mathcal{O}(r(k+n))$.

$$\|UW\|_{\mathrm{F}}^2 = \mathrm{trace}\left((UW)^\top UW\right) = \mathrm{trace}\left(W^\top U^\top UW\right) = \mathrm{trace}\left(W^\top W\right) = \|W\|_{\mathrm{F}}^2$$

This (final) objective has many good properties

for the low-rank matrix completion problem

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- It is natural and defined over the "right" space;

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- It is natural and defined over the "right" space;
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- It is efficiently computable, and so are $\operatorname{grad} f$ and $\operatorname{Hess} f$;

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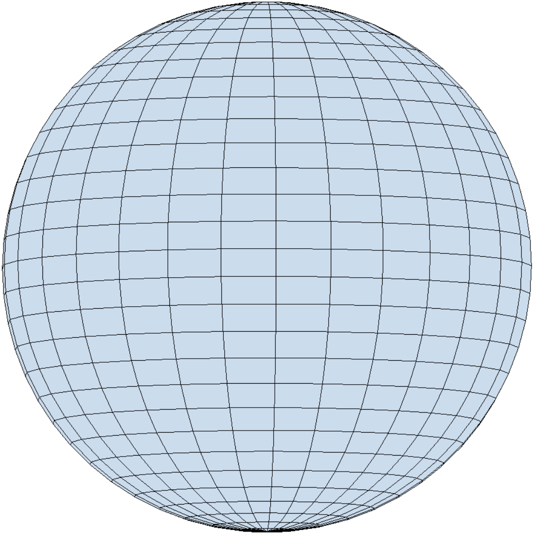
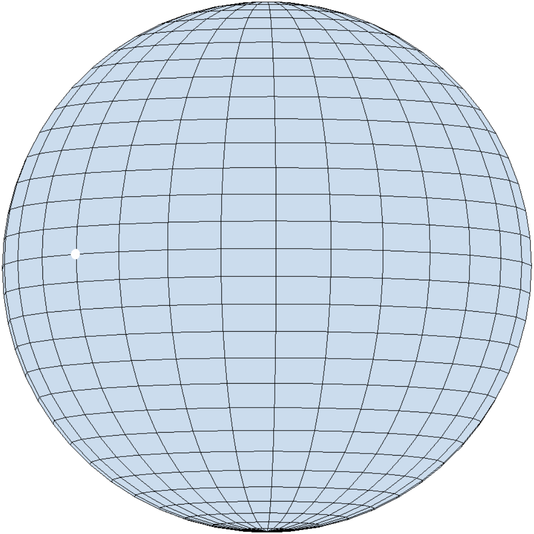
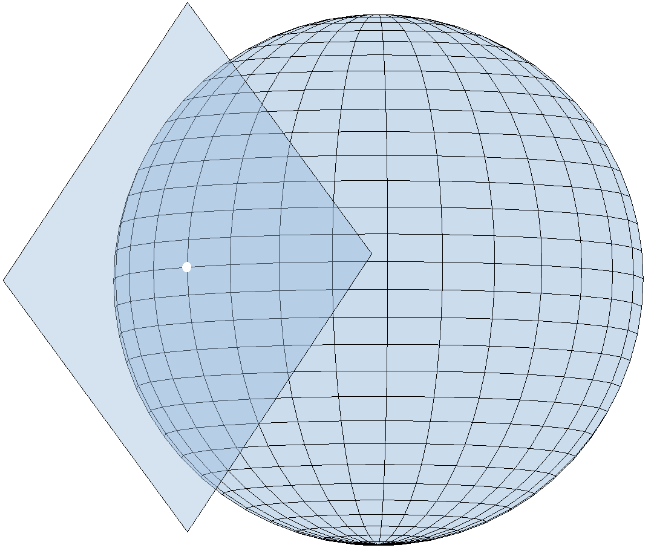
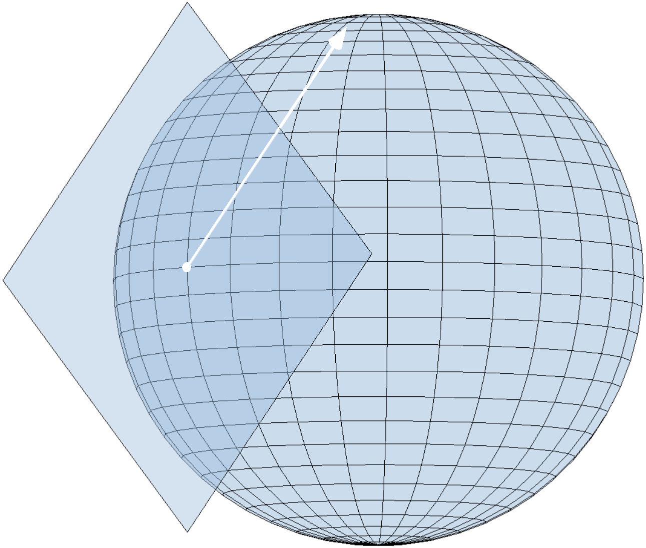
- It is natural and defined over the “right” space;
- It has isolated minimizers and it is smooth;
- It is efficiently computable, and so are $\operatorname{grad} f$ and $\operatorname{Hess} f$;
- It should be able to deal with noise.

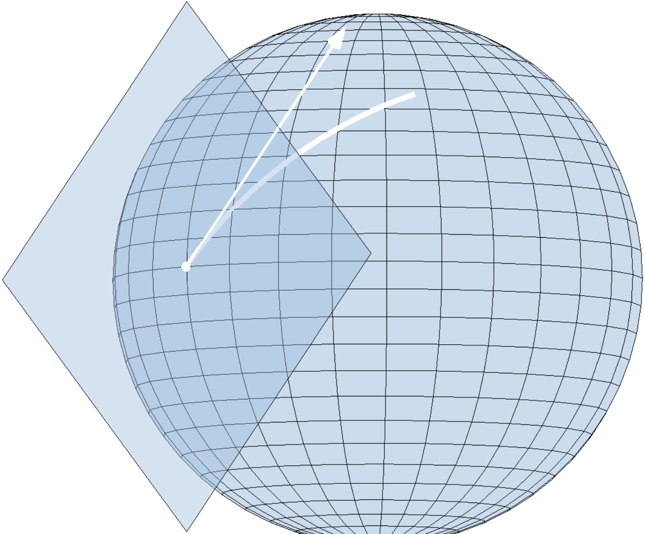
How do you minimize $f(\mathscr{U})$ over Grassmann?











We use a Riemannian trust-region method: GenRTR

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- build a quadratic model of f around the current iterate x,

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We use a Riemannian trust-region method: GenRTR

In $\mathbb{R}^n$, we would:

- build a quadratic model of f around the current iterate x,
- minimize the model in a *trust*-region around x,
- make the step if it decreases f and rescale the region accordingly.

Absil, Baker and Gallivan (2007) generalized this to manifolds.

Trust-region is much better than a Newton method

The trust-region approach:

- guarantees monotonic objective decrease, and
- can deal with approximate Hessians (even the identity)
- and with approximate solutions of the Newton equations.

GenRTR comes with proofs

- Our method is guaranteed to converge toward critical points,
- with second-order speed once near the limit point.
- Usually, the limit point is a local minimizer.

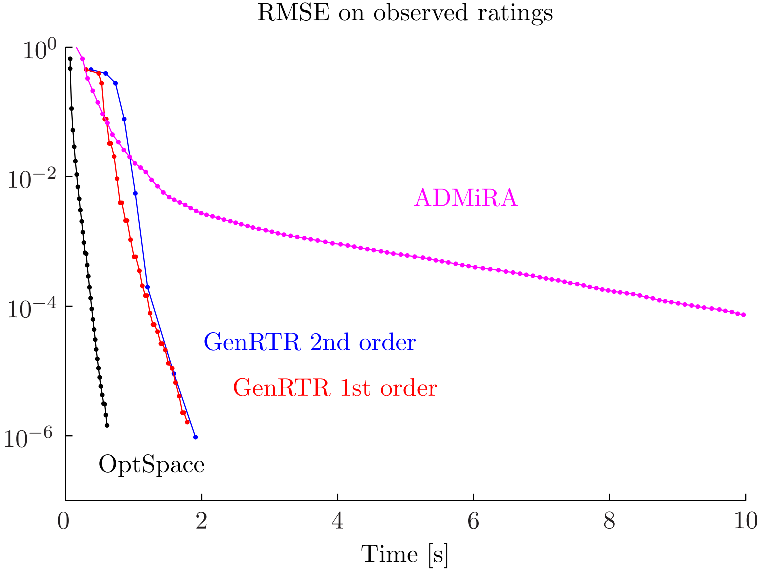
A few numerical tests

We compare four algorithms

- GenRTR with our objective function,
  with Hessian (2^{nd} order) and without Hessian (1^{st} order);
- OptSpace by Keshavan and Oh;
- ADMiRA by Lee and Bresler.

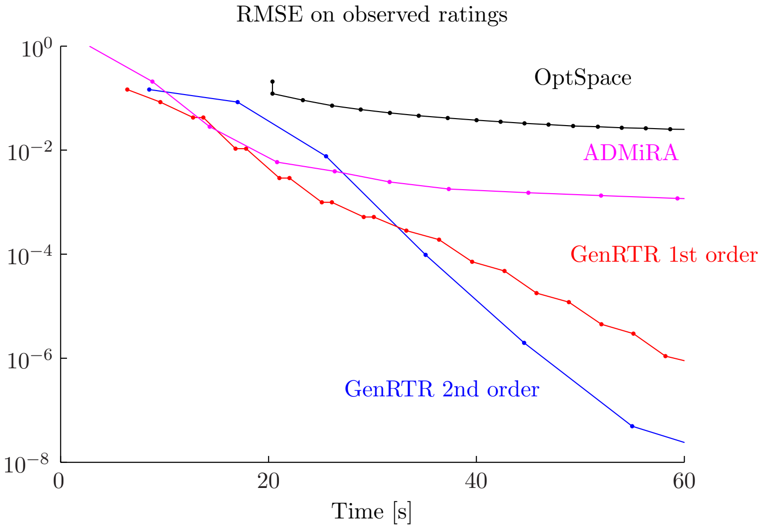
OptSpace is a serious contestant, ADMiRA less so.

Noiseless, $m = n = 1\,000$, $r = 4$, knowledge = 10%



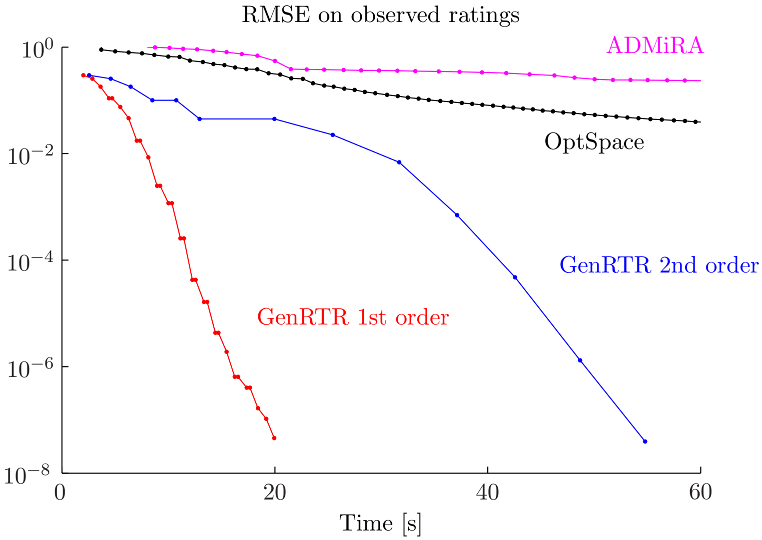
Our method seems to scale better.

Noiseless, $m = n = 10\,000$, $r = 4$, knowledge = 10%



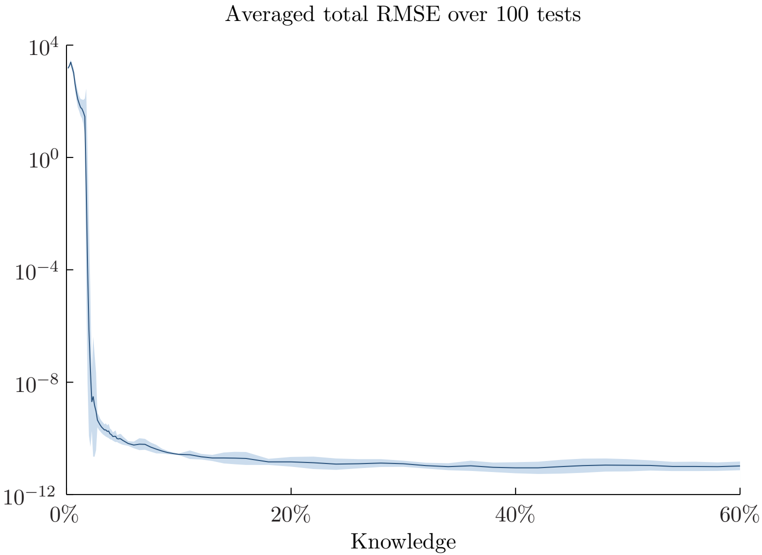
For rectangular matrices, we improve over OptSpace.

Noiseless, $m = 1\,000$, $n = 80\,000$, $r = 4$, knowledge = 2%



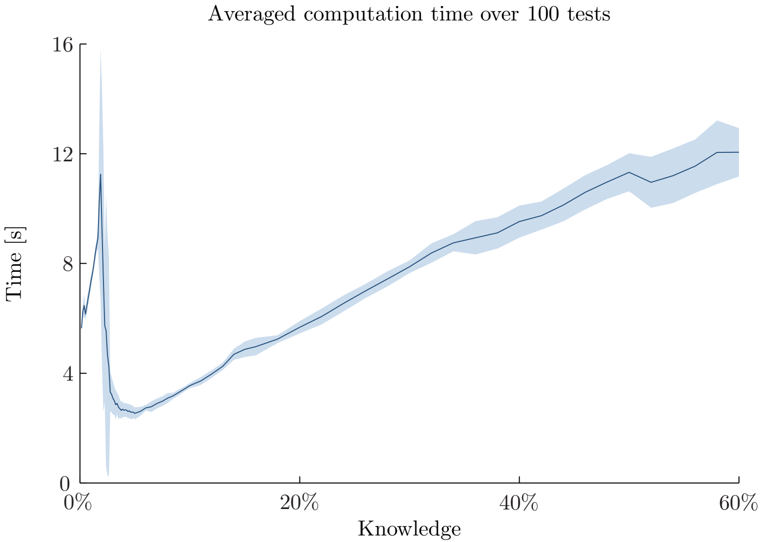
Given enough information, we consistently recover X.

Noiseless, $m = n = 1\,000$, $r = 5$



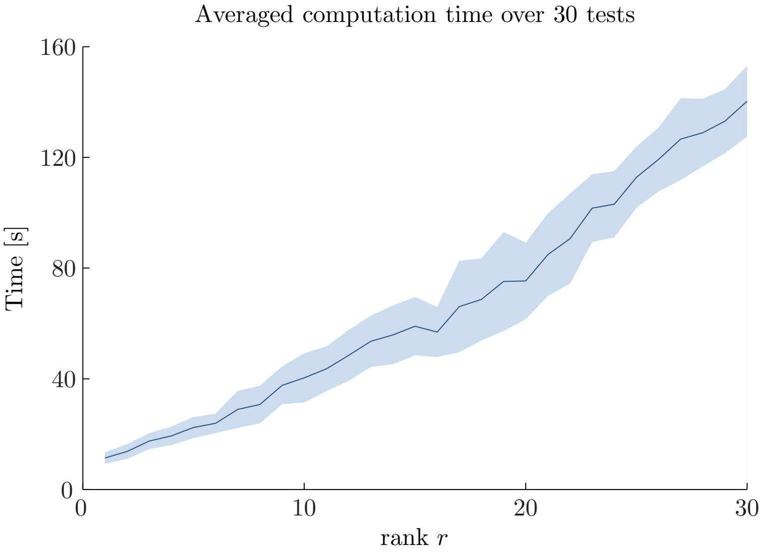
Computation time is proportional to ♯ known entries.

Noiseless, $m = n = 1\,000$, $r = 5$



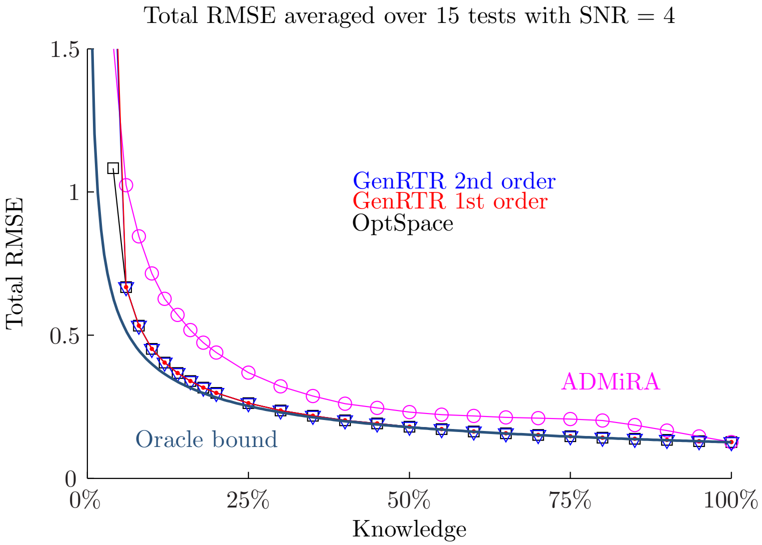
Computation time scales reasonably with the rank.

Noiseless, $m = n = 1\,000$, knowledge $= 25\%$



In the presence of noise, we are close to optimal.

Noisy, $m = n = 500$, $r = 4$



Final thoughts

Conclusions

Our contribution is a sensible objective function fed to a state-of-the-art theory-backed solver with the right complexity;

The algorithm scales well and is easily parallelizable;

In the noiseless case, we get exact completion if enough entries are known;

In the noisy case, we provide a close to optimal reconstruction.

Next step: application to real data sets.

Take home message

Many practical optimization problems are such that either the objective function presents some invariances, giving rise to a quotient manifold, or are constrained by nonlinear equalities describing a submanifold of $\mathbb{R}^n$.

Efficient tools are readily available to fully exploit the geometry of these problems such as, e.g., GenRTR.

These tools are backed up by a solid theory, described for example in the book by Absil, Mahony and Sepulchre: Optimization Algorithms on Matrix Manifolds, 2008.