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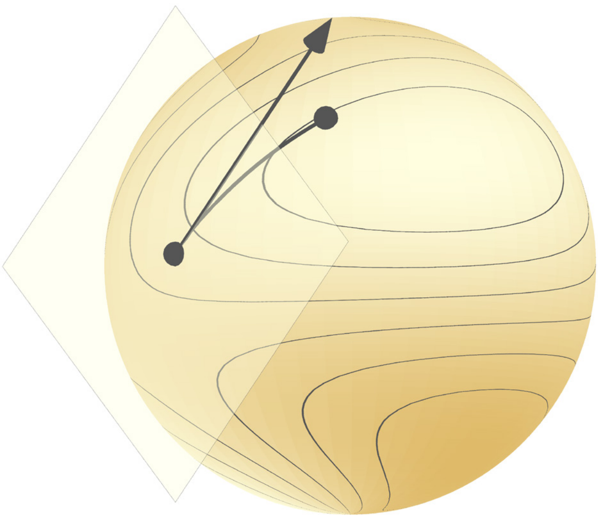
Tangent spaces to submanifolds



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Optimization on manifolds, MATH 512 @ EPFL

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$\mathcal{E}$ is a linear space of dimension d.

$\mathcal{M}$ is a submanifold of dimension n embedded in $\mathcal{E}$.

Pick $x \in \mathcal{M}$. We have the following, with U a neighborhood of x in $\mathcal{E}$:

1. A local defining function $h: U \to \mathbf{R}^k$ with $k = d - n$:

h smooth, $h(y) = 0 \iff y \in U \cap \mathcal{M}$; $Dh(x)$ has rank k.

2. A local diffeomorphism $F: U \to V$:

F, F^{-1} are smooth, and $F(U \cap \mathcal{M}) = V \cap E$

$E = \{y \in \mathbb{R}^d : y_{d-k+1} = \cdots = y_d = 0\}$.

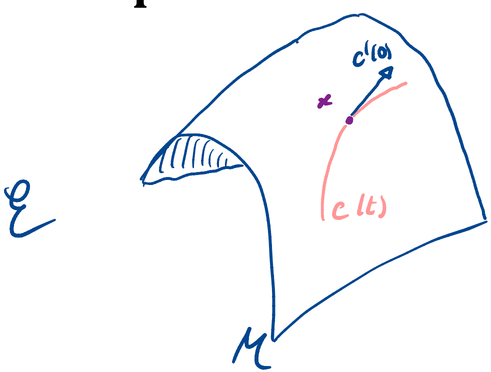
Two perspectives on linearizing $\mathcal{M}$ in $\mathcal{E}$

Perspective 1: based on local defining functions.

$x \in \mathcal{M} \cap \mathcal{U}$

$h(x+v) \simeq h(x) + Dh(x)[v] = 0 \iff v \in \ker Dh(x).$

Perspective 2: based on curves and "moving around".



$c : \mathbb{R} \to \mathcal{M} \subseteq \mathcal{E}$

$c(0) = x.$

$c'(0) = \lim_{t \to 0} \frac{c(t) - c(0)}{t} \in \mathcal{E}.$

Def.: A tangent vector at x is the velocity $c'(0)$ of a smooth curve $c\colon \mathbf{R} \to \mathcal{M}$ with $c(0) = x$.

The tangent space $\mathrm{T}_x\mathcal{M}$ is the set of all tangent vectors at x.

Theorem: $\mathrm{T}_x\mathcal{M} = \ker \mathrm{D}h(x)$ is linear, and $\dim \mathrm{T}_x\mathcal{M} = \dim \mathcal{M}$.

$$\underbrace{\dim \ker Dh(x)}_{\Rightarrow\, =\, n.} + \underbrace{\operatorname{rank} Dh(x)}_{= k = d - n} = d$$

Example 0: $\mathcal{M}$ open in $\mathcal{E}$.

Example 1: $\mathcal{M} = \mathrm{S}^{d-1}$.

Proof: (1) show $\mathrm{T}_x\mathcal{M} \subseteq \ker \mathrm{D}h(x)$, and

(2) show $\mathrm{T}_x\mathcal{M}$ contains a linear space of dimension n.

① Say $v \in T_x\mathcal{M}$: $\quad c : \mathbb{R} \to \mathcal{M}$, $\quad c(0) = x$, smooth

$c'(0) = v$

$h(c(t)) = 0$ for all t close to 0.

$c(t)$ $\in \mathcal{M}$ $\forall t$

$\in \mathcal{U}$ for all t close to 0

$h(y) = 0 \quad \forall y \in \mathcal{U} \cap \mathcal{M}$

$$\Rightarrow \underbrace{\frac{d}{dt} h(c(t))\Big|_{t=0}}_{} = 0$$

$$\| $$

$$\mathrm{D}h(c(t))[c'(t)]\Big|_{t=0}$$

$$\| $$

$$\mathrm{D}h(x)[v]$$

$\Longrightarrow \quad v \in \ker Dh(x).$

(2) $F(U \cap M) = V \cap E,$ $\quad E$ is a linear space of dim n.

$x \in U \cap M;$ $\quad F(x) \in V \cap E.$

$\gamma(t) = F(x) + tv,$ $\quad$ where $v \in E$ is arbitrary.

$c(t) = F^{-1}(\gamma(t)) = F^{-1}(\underbrace{F(x) + tv}_{\substack{\in E \text{ for all } t \\ \in V \text{ for } t \text{ close to } 0}}).$

$c(0) = F^{-1}(F(x)) = x;$

$\Longrightarrow c(t) \in U \cap M$ for t close to 0.

c is a smooth curve of M passing through x.

$$F^{-1}(F(x)) = x$$

$$DF^{-1}(F(x))[DF(x)[v]] = v$$

$$DF^{-1}(F(x)) \circ DF(x) = I$$

$$c'(0) = \frac{d}{dt} F^{-1}(F(x) + tv)\Big|_{t=0}$$

$$= \underbrace{DF^{-1}(F(x))}_{\text{invertible linear map}}[v]$$

arbitrary in E, which has dimension equal to $\dim M$.

$\square$