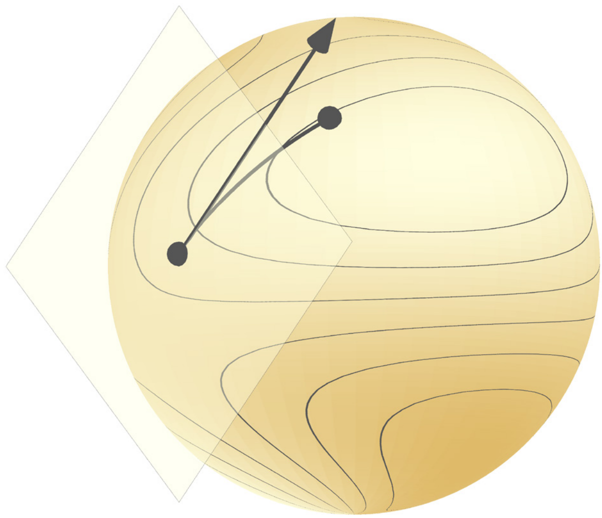
502

Smooth sets and functions: now with charts



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Optimization on manifolds, MATH 512 @ EPFL

Instructor: Nicolas Boumal

It all starts with a simple question

Say M is a set. *Just* a set.

Let $f : M \to \mathbf{R}$ be some function on that set.

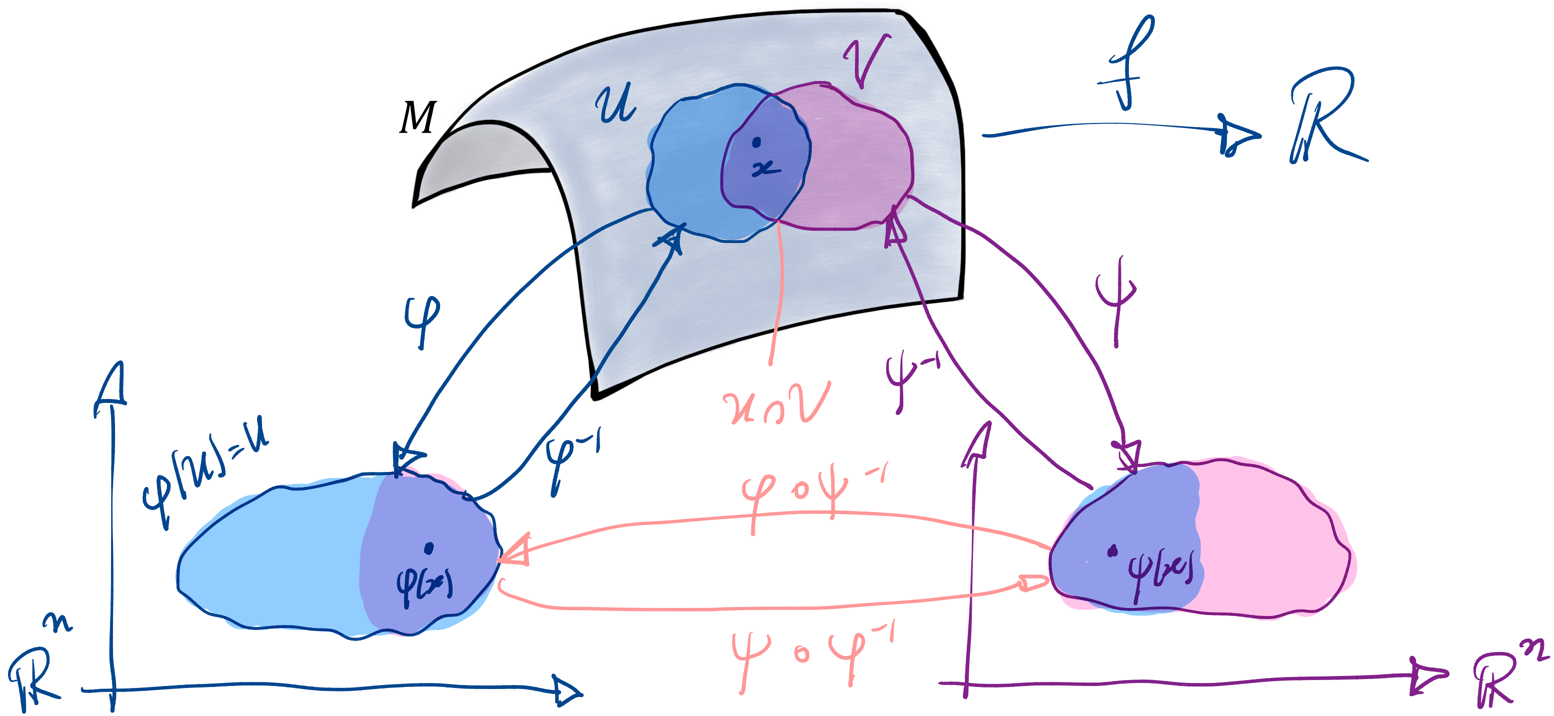
What should it mean for f to be "smooth" at $x \in M$?

$$\begin{tikzcd} \mathcal{U} \subseteq M \arrow[r, "f"] \arrow[d, leftrightarrow] & \mathbb{R} \\ U \subseteq \mathbb{R}^n \arrow[ur] & \end{tikzcd}$$

Def.: An n-dimensional chart for M is a pair $(\mathcal{U}, \varphi)$ such that:

$\mathcal{U}$ is a subset of M and

φ is an invertible map from $\mathcal{U}$ to $\varphi(\mathcal{U})$, open in $\mathbf{R}^n$.



We'll say f is smooth at x as judged through (U, φ) if $f \circ \varphi^{-1}$ is smooth at $\varphi(x)$.

We also say that f is smooth at x as judged by (V, ψ) if $f \circ \psi^{-1}$ is smooth at $\psi(x)$.

$$f \circ \psi^{-1} = f \circ \varphi^{-1} \circ \varphi \circ \psi^{-1}$$

$f : M \to \mathbb{R}$, $\varphi^{-1} : U \subseteq \mathbb{R}^n \to M$

$f \circ \varphi^{-1} : U \to \mathbb{R}$

Def.: Two charts $(\mathcal{U}, \varphi)$ and $(\mathcal{V}, \psi)$ for M are compatible if they have the same dimension n and, if $\mathcal{U} \cap \mathcal{V} \neq \emptyset$, then

$\varphi(\mathcal{U} \cap \mathcal{V})$ is open in $\mathbf{R}^n$,

$\psi(\mathcal{U} \cap \mathcal{V})$ is open in $\mathbf{R}^n$, and

$\psi \circ \varphi^{-1}: \varphi(\mathcal{U} \cap \mathcal{V}) \to \psi(\mathcal{U} \cap \mathcal{V})$ is a diffeomorphism (C^∞).

Def.: An atlas for M is a collection of compatible charts whose domains cover M.

Def.: A manifold* is a pair $\mathcal{M} = (M, \mathcal{A})$: a set with an atlas. The dimension of $\mathcal{M}$, $\dim \mathcal{M}$, is that of any chart.

Example 0: $\mathbf{R}^n$, with the single chart $\varphi(x) = x$ (identity on $\mathbf{R}^n$).

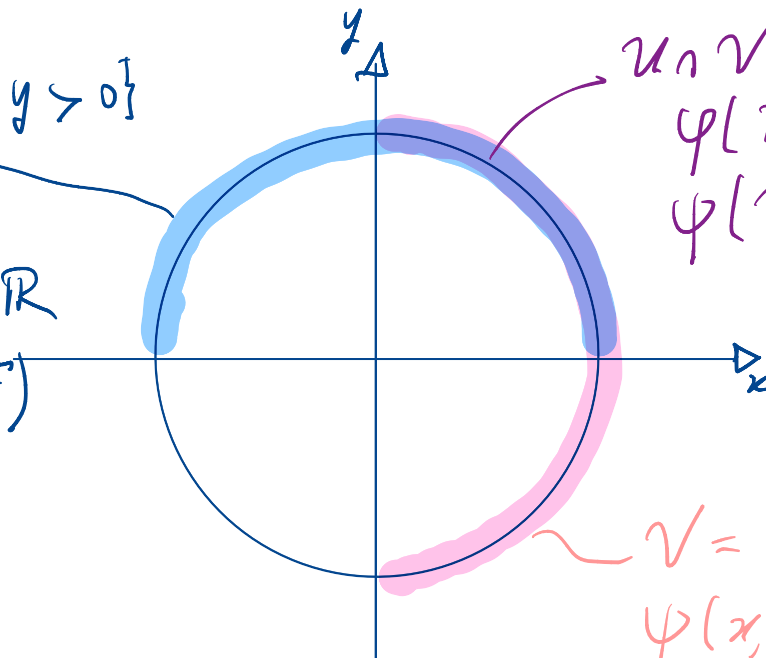
Example 1: $M = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$, with atlas as follows:

$\mathcal{U} = \{(x,y) \in M : y > 0\}$

$\varphi(x,y) = x$

$\varphi : \mathcal{U} \to (-1,1) \subset \mathbb{R}$

$\varphi^{-1}(t) = (t, \sqrt{1-t^2})$



$\mathcal{U} \cap \mathcal{V}$

$\left.\begin{array}{l} \varphi(\mathcal{U} \cap \mathcal{V}) = (0,1) \subset \mathbb{R} \\ \psi(\mathcal{U} \cap \mathcal{V}) = (0,1) \subset \mathbb{R} \end{array}\right\}$ open

$(\varphi \circ \psi^{-1})(t) = \sqrt{1-t^2}$.

$\mathcal{V} = \{(x,y) \in M : x > 0\}$

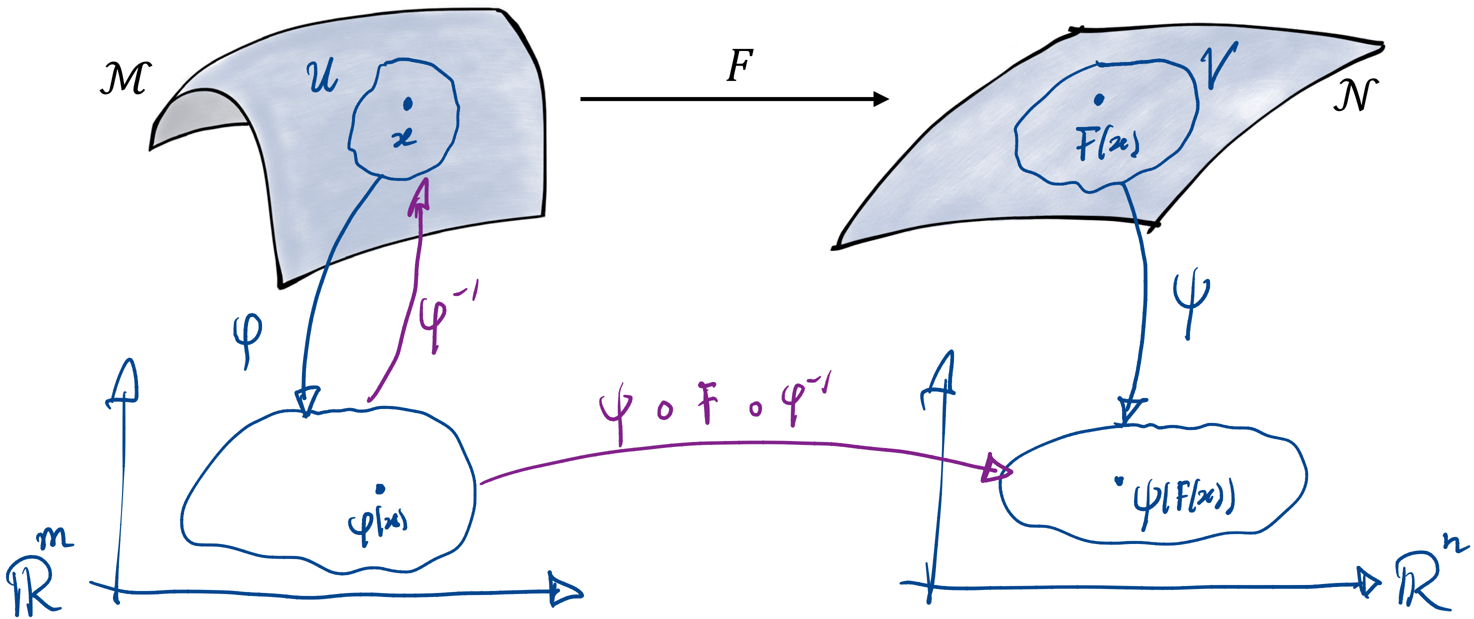
$\psi(x,y) = y$, $\psi : \mathcal{V} \to (-1,1)$

$\psi^{-1}(t) = (\sqrt{1-t^2}, t)$

Smooth functions

Let $\mathcal{M} = (M, \mathcal{A})$ and $\mathcal{N} = (N, \mathcal{B})$ be two manifolds*.

What should it mean for $F: \mathcal{M} \to \mathcal{N}$ to be smooth at $x \in \mathcal{M}$?



Def.: $F\colon \mathcal{M} \to \mathcal{N}$ is smooth at x if, with some chart $(\mathcal{U}, \varphi)$ of $\mathcal{M}$ and some chart $(\mathcal{V}, \psi)$ of $\mathcal{N}$ such that $x \in \mathcal{U}$ and $F(x) \in \mathcal{V}$, the map

$$\tilde{F} = \psi \circ F \circ \varphi^{-1}$$

is smooth at $\varphi(x)$.

Fact: This definition does not depend on the choice of charts.

Surely, we want a topology on $\mathcal{M}$

A topology on $\mathcal{M}$ is a suitable collection of subsets deemed open.

Useful to define local minima, to define convergence…

A topology also defines continuity of functions:

f is continuous if the pre-image of each open set is open.

We better make sure that smooth functions are continuous!

Charts are smooth, so they should be continuous.

$\varphi : \mathcal{U} \to \mathbb{R}^n$ and $\varphi(\mathcal{U})$ is open in $\mathbb{R}^n$,

so $\varphi^{-1}(\varphi(\mathcal{U}))$ must be open in the topology (still TBD) of M.

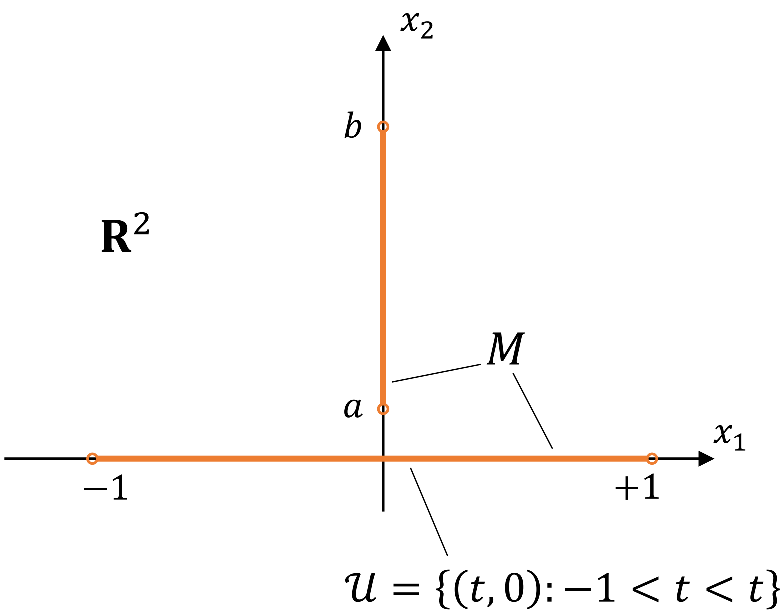
But $\varphi^{-1}(\varphi(\mathcal{U})) = \mathcal{U}$, so the domain of each chart must be deemed open.

Now add to the atlas all the charts that could have been in there without breaking compatibility: those chart domains form a basis for our topology.

Def.: Given an atlas $\mathcal{A}$ for a set M, the maximal atlas $\mathcal{A}^+$ is the collection of all charts that are compatible with $\mathcal{A}$.

Def.: The atlas topology associated to $\mathcal{A}^+$ states that a subset of M is open iff it is the union of a collection of chart domains.

The atlas topology can be uncomfortable...



It's possible to pick an atlas for M such that, in the atlas topology, the sequence

$$(1/2,0), (1/3,0), (1/4,0), (1/5,0) \dots$$

converges to $(0,\alpha)$ for *all* $\alpha \in \{0\} \cup (a,b)$: the limit is not unique! (and just plain odd)

$\mathcal{U}_\alpha = \mathcal{U} \setminus \{(0,0)\} \cup \{(0,\alpha)\}$

$\varphi_\alpha: \mathcal{U}_\alpha \to (-1,1): x \mapsto \varphi_\alpha(x) = x_1$

Atlas: $\{(\mathcal{U}_\alpha, \varphi_\alpha): \alpha = 0 \text{ or } \alpha \in (a,b)\}$

Atlas topology is neither Hausdorff nor second-countable.

Example 3.2.2 in Brickell & Clark, 1970, illustrating how an atlas can induce an uncomfortable topology.

Def.: A (smooth) manifold is a pair $\mathcal{M} = (M, \mathcal{A}^+)$ whose atlas topology is Hausdorff and second-countable.

Fact: Let $\mathcal{A}$ be an atlas. The atlas topology (of $\mathcal{A}^+$) is

Hausdorff if two distinct points of M belong either to the same chart domain or to two disjoint chart domains of $\mathcal{A}$; and

Second-countable if countably many chart domains cover M.

Embedded geometry fits nicely in here.

A subset M of a linear space $\mathcal{E}$ may admit many genuinely different maximal atlases, yielding different manifolds (for the same set!).

However,
If M is what we called an embedded submanifold of $\mathcal{E}$, then:

- There exists a unique maximal atlas that turns M into a manifold such that atlas topology $\equiv$ subspace topology (as we've been using!)
- With that atlas, a function on M is smooth (as seen through charts) if and only if it admits a smooth extension.