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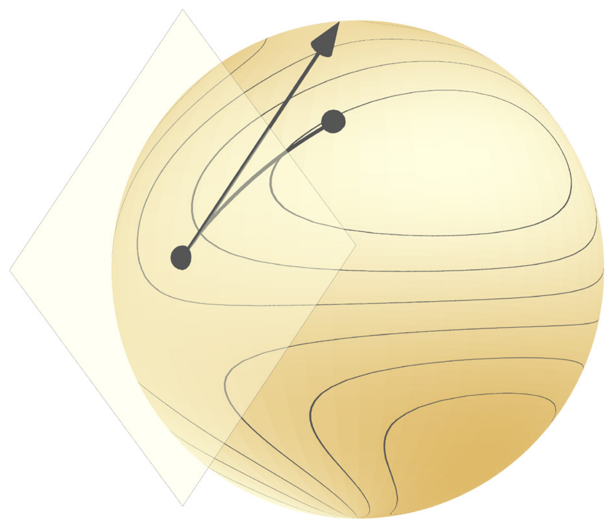
Riemannian Hessians



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Optimization on manifolds, MATH 512 @ EPFL

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Let $\mathcal{M}$ be a Riemannian manifold.

Let ∇ denote the (unique) Riemannian connection on $\mathcal{M}$.

Def.: Let $f: \mathcal{M} \to \mathbf{R}$ be smooth.

Its Riemannian Hessian at x is the linear map

$$\mathrm{Hess} f(x): \mathrm{T}_x\mathcal{M} \to \mathrm{T}_x\mathcal{M}$$

defined by

$$\mathrm{Hess} f(x)[u] = \nabla_u \mathrm{grad} f.$$

Fact: $\mathrm{Hess} f(x)$ is symmetric.

$\forall u, v \in T_x\mathcal{M}:$

$\langle \mathrm{Hess} f(x)[u], v \rangle_x = \langle \mathrm{Hess} f(x)[v], u \rangle_x$

$\Longrightarrow$ spectral theorem applies.

$\{x \in \mathbb{R}^d : x^T x = 1\}$ $\langle u, v \rangle = u^T v$

Example. Consider S^{d-1} as a Riemannian submanifold of $\mathbf{R}^d$.
Let $f : \mathcal{M} \to \mathbf{R}$ be smooth on S^{d-1}, with extension $\bar{f}$.
Compute $\mathrm{Hess} f(x)$ in terms of $\bar{f}$.

$$T_x S^{d-1} = \{ v \in \mathbb{R}^d : x^T v = 0 \} ; \quad \mathrm{Proj}_x(u) = u - (x^T u) x$$

$$\mathrm{grad} f(x) = \mathrm{Proj}_x(\mathrm{grad} \bar{f}(x)) = \mathrm{grad} \bar{f}(x) - (x^T \mathrm{grad} \bar{f}(x)) x$$

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$$\bar{G}(x) = \mathrm{grad} \bar{f}(x) - (x^T \mathrm{grad} \bar{f}(x)) x$$

$$D\bar{G}(x)[u] = \mathrm{Hess} \bar{f}(x)[u] - (x^T \mathrm{grad} \bar{f}(x)) u - (u^T \mathrm{grad} \bar{f}(x) + x^T \mathrm{Hess} \bar{f}(x)[u]) x$$

$$\operatorname{Hess} f(x)[u] = \operatorname{Proj}_x\left(D\overline{G}(x)[u] \right)$$

$$= \operatorname{Proj}_x\left(\operatorname{Hess} \bar{f}(x)[u] \right) - \left(x^T \operatorname{grad} \bar{f}(x) \right) u .$$

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