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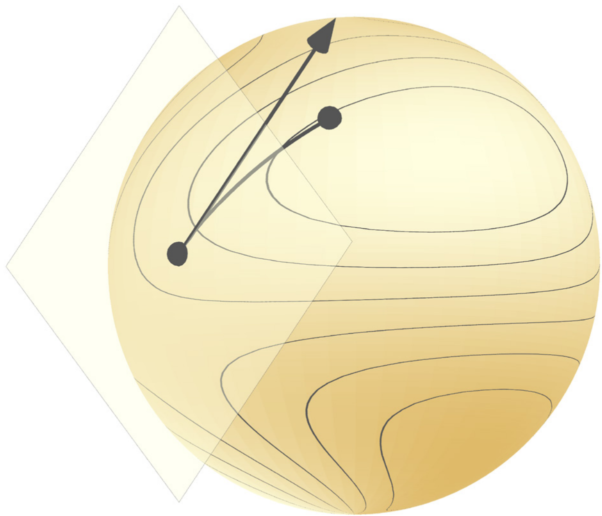
Differentiating vector fields: why do it, and how not to do it



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Optimization on manifolds, MATH 512 @ EPFL

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Why we want to differentiate vector fields

Let $f: \mathcal{M} \to \mathbf{R}$ be smooth. Consider a smooth curve $c: \mathbf{R} \to \mathcal{M}$.

Taylor expand $g = f \circ c: \mathbf{R} \to \mathbf{R}$ beyond first order?

$c(0) = x$

$c'(0) = v \in T_x\mathcal{M}$

$$g(t) = g(0) + t\, g'(0) + \left[\frac{t^2}{2} g''(0)\right] + \cdots$$

$g(0) = f(c(0)) = f(x)$

$$g'(t) = (f \circ c)'(t) = Df(c(t))[c'(t)] = \langle \operatorname{grad} f(c(t)), c'(t) \rangle_{c(t)}$$

$$g(t) = f(x) + t\langle \operatorname{grad} f(x), v \rangle_x + \left[\frac{t^2}{2} \cdots\right] + \cdots$$

$g'(0) = \langle \operatorname{grad} f(x), v \rangle_x$

$$\boxed{g''(0) = \frac{d}{dt} \langle \operatorname{grad} f(c(t)), c'(t) \rangle_{c(t)} \Big|_{t=0}}$$

We already know how to differentiate vector fields… right?

Let $\mathcal{M}$ be an embedded submanifold of $\mathcal{E}$.

Say V is a smooth vector field on $\mathcal{M}$. That is:

$$V : \mathcal{M} \to \underbrace{\mathrm{T}\mathcal{M}}_{} \text{ is smooth, and } V(x) \in \mathrm{T}_x\mathcal{M} \text{ for all } x \in \mathcal{M}$$

$\mathrm{T}\mathcal{M} = \{(x, v) : x \in \mathcal{M} \text{ and } v \in \mathrm{T}_x\mathcal{M}\}$

How could we differentiate V?

$DG(x): T_x\mathcal{M} \to T_{G(x)}(T\mathcal{M})$

Recall: If $G: \mathcal{M} \to \mathcal{N}$ is smooth, and $\bar{G}$ is a smooth extension, then

$$DG(x)[v] = D\bar{G}(x)[v] \text{ for all } x \in \mathcal{M} \text{ and } v \in T_x\mathcal{M}.$$

Example: Let $S^{d-1} = \{x \in \mathbf{R}^d : x^\top x = 1\}$,

as a Riemannian submanifold of $\mathbf{R}^d$ with $\langle u, v\rangle = u^\top v$.

Let $f: S^{d-1} \to \mathbf{R}$ be defined by $f(x) = \frac{1}{2}x^\top Ax$.

$\hookrightarrow A = A^T \in \mathbb{R}^{d \times d}$

Task: compute $D(\operatorname{grad} f)(x)[v]$.

$\bar{f}: \mathbb{R}^d \to \mathbb{R}$, $\quad \bar{f}(x) = \frac{1}{2}x^T A x$

$\operatorname{grad} \bar{f}(x) = Ax$

$D\bar{f}(x)[v] = x^T A v$

$= \langle Ax, v \rangle$

$\operatorname{grad} f(x) = \operatorname{Proj}_x(\operatorname{grad} \bar{f}(x))$

$$T_x S^{d-1} = \{ v \in \mathbb{R}^d : v^T x = 0 \}$$

$$\mathrm{Proj}_x(u) = u - (x^T u)x = (I - xx^T)u.$$

$$\vdots$$

$$= (I - xx^T)Ax$$

$$= Ax - (x^T A x)x$$

Define $\bar{G} : \mathbb{R}^d \to \mathbb{R}^d$: $\bar{G}(x) = Ax - (x^T A x)x$.

this is a smooth extension of grad f.

$v \in T_x S^{d-1}$

$$D\bar{G}(x)[v] = Av - (x^T A x)v - (v^T A x + x^T A v)x$$

$$= \underbrace{\mathrm{Proj}_x(Av)}_{\in T_x S^{d-1}} \underbrace{- (x^T A x)v}_{\in T_x S^{d-1}} \underbrace{- (v^T A x)x}_{\notin T_x S^{d-1}}.$$

Fine. Let's be pragmatic then… no?

The issue in the example is that $\mathrm{D}V(x)[u]$ might not be in $\mathrm{T}_x\mathcal{M}$.

… Can't we just orthogonally project $\mathrm{D}V(x)[u]$ to $\mathrm{T}_x\mathcal{M}$?

It's reasonable, but *kind of arbitrary*. Still:

1. We'll argue it's always a good option (among many).
2. It's the right option for Riemannian submanifolds.
3. For other Riemannian manifolds, there is another right option.