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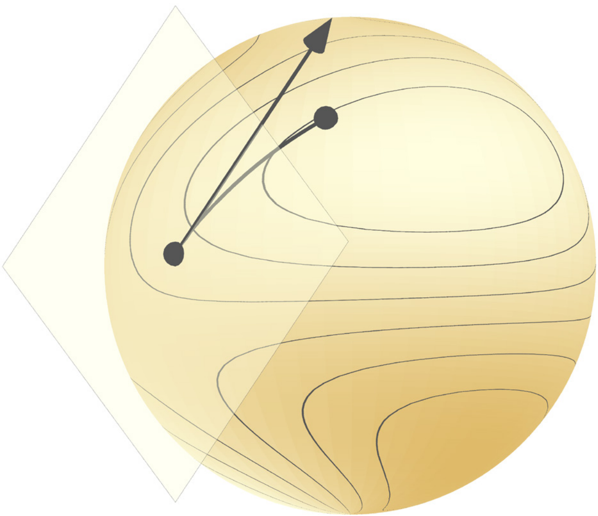
Differentiating vector fields along curves



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Optimization on manifolds, MATH 512 @ EPFL

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Recall why we introduced connections

$f : \mathcal{M} \to \mathbb{R}, \qquad c : \mathbb{R} \to \mathcal{M}, \qquad g = f \circ c : \mathbb{R} \to \mathbb{R}.$

$$g(t) = g(0) + t\, g'(0) + \frac{t^2}{2} g''(0) + O(t^3)$$

$$g'(t) = Df(c(t))[c'(t)] = \langle \operatorname{grad} f(c(t)), c'(t) \rangle_{c(t)}.$$

$$g''(0) = \frac{d}{dt} \langle \operatorname{grad} f(c(t)), c'(t) \rangle_{c(t)} \Big|_{t=0} = ???$$

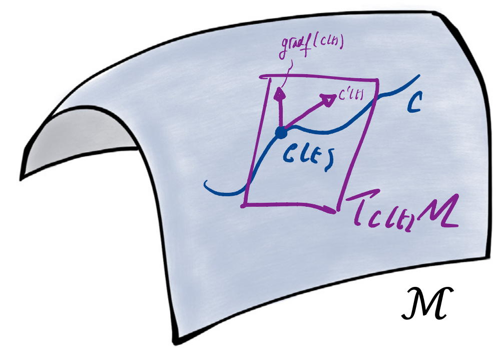
Each connection ∇ induces a way to differentiate vector fields along curves.

Let $c\colon \mathbf{R} \to \mathcal{M}$ be a smooth curve on a manifold with connection ∇.

Def.: A (smooth) vector field along c is a (smooth) map $Z\colon \mathbf{R} \to \mathrm{T}\mathcal{M}$ such that $Z(t)$ is in $\mathrm{T}_{c(t)}\mathcal{M}$ for all t.

The set of smooth vector fields along c is denoted $\mathcal{X}(c)$.

Examples: c', $\underbrace{\operatorname{grad} f}_{} \circ \underbrace{c}_{}$, $U \circ c$



Theorem: There exists a unique operator $\frac{\mathrm{D}}{\mathrm{d}t} : \mathcal{X}(c) \to \mathcal{X}(c)$ satisfying the following three properties:

linearity 1. $\frac{D}{dt}(aY + bZ)(t) = a\frac{D}{dt}Y(t) + b\frac{D}{dt}Z(t)$, $\quad \forall a, b \in \mathbb{R}$

Leibniz 2. $\frac{D}{dt}(gZ)(t) = g'(t)Z(t) + g(t)\frac{D}{dt}Z(t)$ $\quad \forall Y, Z \in \mathcal{X}(c)$, $\forall g : \mathbb{R} \to \mathbb{R}$ smooth

Chain rule 3. $\frac{D}{dt}(U \circ c)(t) = \nabla_{c'(t)}U$ $\quad \forall U \in \mathcal{X}(M)$

Theorem: If ∇ is compatible with the metric, then $\frac{\mathrm{D}}{\mathrm{d}t}$ also satisfies:

product rule 4. $\frac{d}{dt}\langle Y(t), Z(t)\rangle_{c(t)}\Big|_t = \langle \frac{D}{dt}Y(t), Z(t)\rangle_{c(t)} + \langle Y(t), \frac{D}{dt}Z(t)\rangle_{c(t)}$

We call $\frac{\mathrm{D}}{\mathrm{d}t}$ the covariant derivative induced by ∇.

Proof sketch (uniqueness)

Assume $\frac{D}{dt}$ has properties 1, 2, 3. Let $Z \in \mathcal{X}(c)$.

Let $W_1, \ldots, W_n$ form a local frame around some point on U.

$$Z(t) = \sum_{i=1}^{n} a_i(t)\, W_i(c(t))$$

smooth smooth smooth

$$\frac{D}{dt} Z(t) \overset{1}{=} \sum_{i=1}^{n} \frac{D}{dt}\left(a_i (W_i \circ c)\right)(t)$$

$$\overset{2}{=} \sum_{i=1}^{n} \left[a_i'(t) \cdot W_i(c(t)) + a_i(t) \frac{D}{dt}(W_i \circ c)(t) \right]$$

$$\overset{3}{=} \sum_{i=1}^{n} \left[a_i'(t) \cdot W_i(c(t)) + a_i(t)\, \nabla_{c'(t)} W_i \right].$$

Fact: For a Euclidean space, $\frac{\mathrm{D}}{\mathrm{d}t}Z(t) = \frac{\mathrm{d}}{\mathrm{d}t}Z(t) = \lim_{\delta\to 0}\frac{Z(t+\delta)-Z(t)}{\delta}$.

Fact: For a Riemannian submanifold of a Euclidean space,

$$\frac{\mathrm{D}}{\mathrm{d}t}Z(t) = \mathrm{Proj}_{c(t)}\left(\frac{d}{dt} Z(t)\right)$$

Finite difference approximation of Hess$f(x)$

$$\mathrm{Hess} f(x)[u] = \nabla_u \,\mathrm{grad} f = \frac{D}{dt}\big(\mathrm{grad} f \circ \underbrace{c}_{c(0)=x,\; c'(0)=u}\big)(0)$$

If $\mathcal{M}$ is a Riemannian submanifold of a Euclidean space, then:

$$\mathrm{Hess} f(x)[u] = \mathrm{Proj}_x\left(\frac{d}{dt}(\mathrm{grad} f \circ c)(0)\right)$$

$$= \mathrm{Proj}_x\left(\lim_{t \to 0} \frac{\mathrm{grad} f(c(t)) - \mathrm{grad} f(c(0))}{t}\right)$$

$$= \lim_{t \to 0} \frac{\mathrm{Proj}_x(\mathrm{grad} f(c(t))) - \mathrm{grad} f(x)}{t}$$

$$\simeq \frac{\mathrm{Proj}_x\left(\mathrm{grad} f\left(c(\bar{t})\right)\right) - \mathrm{grad} f(x)}{\bar{t}}$$

, for $\bar{t} > 0$ small (e.g., 10^{-4})

$$\boxed{\mathrm{Hess} f(x)[u] \simeq \frac{\mathrm{Proj}_x\left(\mathrm{grad} f\left(R_x(\bar{t}u)\right)\right) - \mathrm{grad} f(x)}{\bar{t}}}$$

Acceleration along a curve

Let $\mathcal{M}$ be a manifold with a connection ∇ and the induced $\frac{\mathrm{D}}{\mathrm{d}t}$.

Def.: The acceleration along c is $c''(t) = \frac{\mathrm{D}}{\mathrm{d}t} c'(t)$.

Def.: A geodesic is a curve c such that $c''(t) = 0$ for all t.

Example: On S^{d-1}, pick a point x and a unit vector v tangent at x.
Let $c(t) = \cos(t)\, x + \sin(t)\, v$. Acceleration?