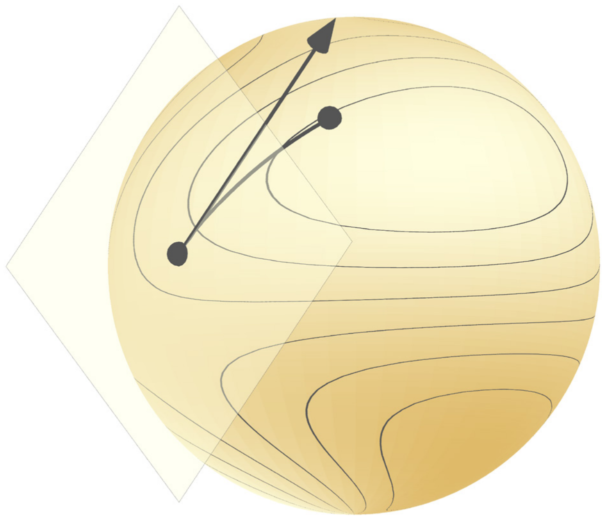
104

Retractions, vector fields and tangent bundles

Spring 2023

Optimization on manifolds, MATH 512 @ EPFL

Instructor: Nicolas Boumal



Moving on manifolds: towards retractions

To move around on $\mathcal{M}$, we want retractions—still to be defined.

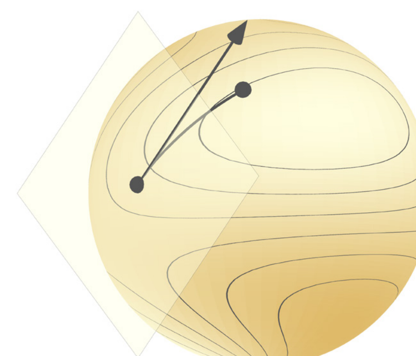
A retraction is a map R which takes as input a point x and a tangent vector v at x, and outputs a new point on $\mathcal{M}$, denoted by $R_x(v)$.

$= R(x, v)$

Thus, the *domain* of R as a map $(x, v) \mapsto R_x(v)$ is:

$R : T\mathcal{M} \to \mathcal{M}$

$$T\mathcal{M} = \{(x, v) : x \in \mathcal{M} \text{ and } v \in \mathrm{T}_x\mathcal{M}\}$$



We will want R to be smooth. Meaning?

Tangent bundles

$\mathcal{M} \subseteq \mathcal{E}$: $x \in \mathcal{M} \Rightarrow x \in \mathcal{E}$

$v \in T_x\mathcal{M} \subseteq \mathcal{E} \Rightarrow v \in \mathcal{E}$

Def.: The tangent bundle of a manifold $\mathcal{M}$ is the set

$$\mathrm{T}\mathcal{M} = \{(x, v): x \in \mathcal{M} \text{ and } v \in \mathrm{T}_x\mathcal{M}\}.$$

Theorem: If $\mathcal{M}$ is an embedded submanifold of $\mathcal{E}$, then $\mathrm{T}\mathcal{M}$ is an embedded submanifold of $\mathcal{E} \times \mathcal{E}$. Moreover, $\dim \mathrm{T}\mathcal{M} = 2 \dim \mathcal{M}$.

Proof. Pick $(\bar{x}, \bar{v}) \in T\mathcal{M}$: $\bar{x} \in \mathcal{M}$, $\bar{v} \in T_{\bar{x}}\mathcal{M}$.

Pick a local defining function $h: U \to \mathbb{R}^k$ for $\mathcal{M}$ around $\bar{x}$:

U is a nghbhd of $\bar{x}$ in $\mathcal{E}$; $h^{-1}(0) = U \cap \mathcal{M}$; h is smooth;

$\operatorname{rank}(Dh(\bar{x})) = k$. $\quad T_{\bar{x}}\mathcal{M} = \ker Dh(\bar{x})$; so: $Dh(\bar{x})[\bar{v}] = 0$.

$x \in \mathcal{U}$: $\quad h(x) = 0 \iff x \in \mathcal{M}$

$Dh(\bar{x}) : \underset{\mathbb{R}^d}{\underset{\wr\wr}{\mathcal{E}}} \to \mathbb{R}^k$: $Dh(\bar{x}) \sim$ matrix of size $k \times d$.

$Dh(\bar{x})\, Dh(\bar{x})^T$: matrix of size $k \times k$.

$x \mapsto \det(Dh(x)\, Dh(x)^T)$ is a smooth function on $\mathcal{U}$,

and it is nonzero at $\bar{x} \in \mathcal{U}$.

If need be, make $\mathcal{U}$ smaller so that the determinant

is nonzero for all $x \in \mathcal{U}$; Then: $\operatorname{rank}(Dh(x)) = k$

for all $x \in \mathcal{U}$.

We now have that $v \in T_x \mathcal{M} \iff Dh(x)[v] = 0.$

for all x in $\mathcal{U}$

Let $H : \mathcal{U} \times \mathcal{E} \to \mathbb{R}^{2k}$:

$$H(x, v) \triangleq \begin{bmatrix} h(x) \\ Dh(x)[v] \end{bmatrix} = 0 \iff \begin{cases} h(x) = 0 \iff x \in \mathcal{M} \\ Dh(x)[v] = 0 \iff v \in T_x\mathcal{M}. \end{cases}$$

$$\Updownarrow$$

$$(x, v) \in T\mathcal{M}.$$

$$DH(x, v)[(\underbrace{\dot{x}}_{\in \mathcal{E}}, \underbrace{\dot{v}}_{\in \mathcal{E}})] = \begin{bmatrix} Dh(x)[\dot{x}] \\ \mathcal{L}(x, v)[\dot{x}] + Dh(x)[\dot{v}] \end{bmatrix}$$

$$= \begin{bmatrix} Dh(x) & 0 \\ \ell(x,v) & Dh(x) \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix}$$

$$\operatorname{rank}(DH(x,v)) = \operatorname{rank}(Dh(x)) + \operatorname{rank}(Dh(x)) = 2k = \dim \mathbb{R}^{2k} \;\checkmark$$

So H is a local defining function for TM around $(\bar{x},\bar{v}) \in TM$.

and
$$\begin{aligned} \dim TM &= \dim(\mathcal{E} \times \mathcal{E}) - 2k \\ &= 2\dim \mathcal{E} - 2k \\ &= 2(\dim \mathcal{E} - k) = 2\dim M. \end{aligned}$$

$\square$

Retractions

$R_x : T_x\mathcal{M} \to \mathcal{M}$

Def.: A retraction is a smooth map

$$R : \mathrm{T}\mathcal{M} \to \mathcal{M} : (x, v) \mapsto R_x(v)$$

such that each curve

$$c(t) = R_x(tv)$$

c is a smooth curve on $\mathcal{M}$.

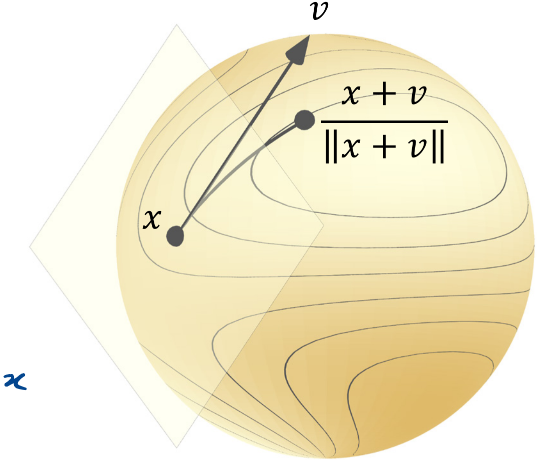
satisfies $c(0) = x$ and $c'(0) = v$.

Example 0: On $\mathcal{M} = \mathcal{E}$, $\qquad R_x(v) = x + v$

$c(t) = R_x(tv) = x + tv.$

$\bar{R}: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $\bar{R}(x, v) = \frac{x+v}{\sqrt{1+\|v\|^2}}$

Example 1: On $\mathcal{M} = S^{d-1}$, $\quad R_x(v) = \frac{x+v}{\|x+v\|}$



$c(t) = R_x(tv) = \frac{x+tv}{\|x+tv\|}$

$\|x+tv\|^2$

$= (x+tv)^T(x+tv)$

$= 1 + t(v^Tx + x^Tv) + t^2 v^Tv$

$= 1 + t^2\|v\|^2$

$= \frac{x+tv}{\sqrt{\|x+tv\|^2}}$

$= \frac{x+tv}{\sqrt{1+t^2\|v\|^2}}$

$c(0) = x$

$c'(0) = v$.

Example 2: On $\mathcal{M} = S^{d-1}$, $\quad R_x(v) = \cos(\|v\|)x + \frac{\sin(\|v\|)}{\|v\|}v$

$TM = \{(y,v) : y \in M \text{ and } v \in T_y M\}$

Vector fields

Def.: A vector field V on a manifold $\mathcal{M}$ is a map $V: \mathcal{M} \to \mathrm{T}\mathcal{M}$ such that each $V(x)$ is tangent at x.

"$V(x) \in T_x M$"

$V(x) = (x, v)$ for some $v \in T_x M$

It is a smooth vector field if it is also a smooth map.

A vector field is smooth iff it can be smoothly extended:

Claim: If $\mathcal{M}$ is embedded in $\mathcal{E}$, then V is a smooth vector field iff there exists a smooth vector field $\bar{V}$ on a ngbhd of $\mathcal{M}$ s.t. $V = \bar{V}|_{\mathcal{M}}$.