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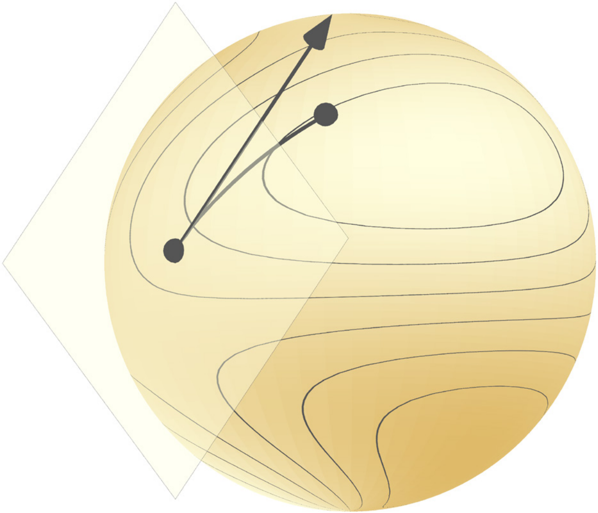
# Geodesic convexity: the basics



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Optimization on manifolds, MATH 512 @ EPFL

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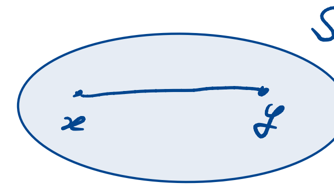
# Convexity on a Riemannian manifold $\mathcal{M}$




**Def.:** A set $S \subseteq \mathbf{R}^n$ is convex if

$$x, y \in S \Rightarrow (1-t)x + ty \in S \text{ for all } t \in [0,1].$$

if $\mathcal{M} = \mathbb{R}^n$ $|||$

**Def.:** A set $S \subseteq \mathcal{M}$ is geodesically convex if

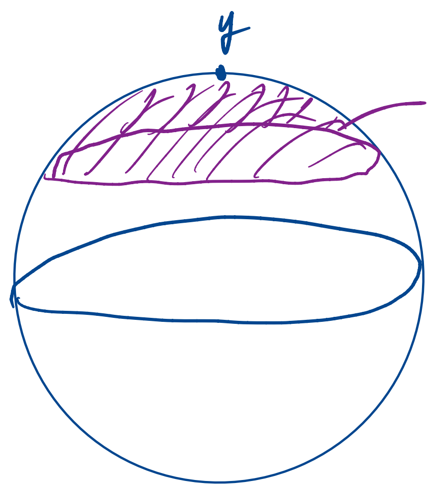
for all $x, y \in S$ there exists a geodesic segment $c : [0,1] \to \mathcal{M}$

s.t. $c(0) = x$, $c(1) = y$,

$c(t) \in S \quad \forall t \in [0,1]$.

Ex 1: If $M$ is complete and connected, then $S = M$ is g-convex.

Ex 2:




$\{x \in S^2 : \mathrm{dist}(x, y) \le r\}$

is g-convex.

**Def.:** A function on a subset of $\mathbf{R}^n$ is convex if its domain is convex and

$$f\big((1-t)x + ty\big) \leq (1-t)f(x) + tf(y)$$

for all $x, y$ in the domain and $t \in [0,1]$.

**Def.:** A function $f: S \to \mathbf{R}$ is geodesically convex if

S is g-convex and for all geodesic segments $c: [0,1] \to M$

s.t. $c(0) = x$, $c(1) = y$ and $c(t) \in S$ $\forall t \in [0,1]$,

We have: $f(c(t)) \leq (1-t) f(x) + t f(y)$ $\forall t \in [0,1]$.

Ex: $f(x) = \frac{1}{2} \operatorname{dist}(x, y)^2$ is g-convex on the domain

$\{x \in \mathcal{M} : \operatorname{dist}(x, y) \leq r\}$,

provided $r$ is small enough.

**Properties:**

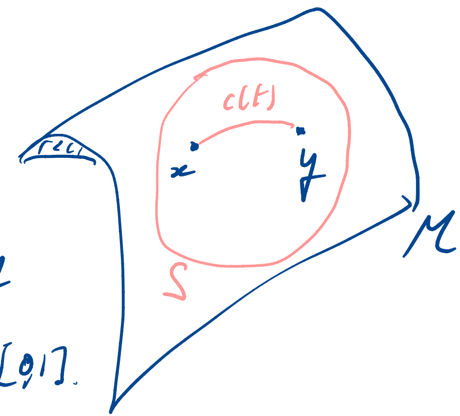
1. Sublevel sets of g-convex functions are g-convex sets.
2. Intersections of such sublevel sets are g-convex sets.
3. Sums of nonnegatively scaled g-convex functions are g-convex.
4. The pointwise maximum of g-convex functions is g-convex.

Let $f: S \to \mathbf{R}$ be geodesically convex on the Riemannian manifold $\mathcal{M}$.

We say $\min_{x \in S} f(x)$ is a geodesically convex problem.

**Fact:** If $x$ is a local minimum, then it is a global minimum.

Pf: For contradiction, say there exists $y \in S$ s.t. $f(y) < f(x)$.




Since $S$ is g-convex, $\exists\ c: [0,1] \to \mathcal{M}$ s.t. $c(0) = x$, $c(1) = y$, $c(t) \in S\ \forall t \in [0,1]$.

Since $f$ is g-convex, we know:

$$f(c(t)) \leq (1-t) f(x) + t f(y) \quad \forall\, t \in [0,1].$$

$$< f(x) \cancel{- t f(x)} \cancel{+ t f(x)} \quad \forall\, t \in (0,1]$$

This contradicts the fact that $x$ is a local min. $\square$

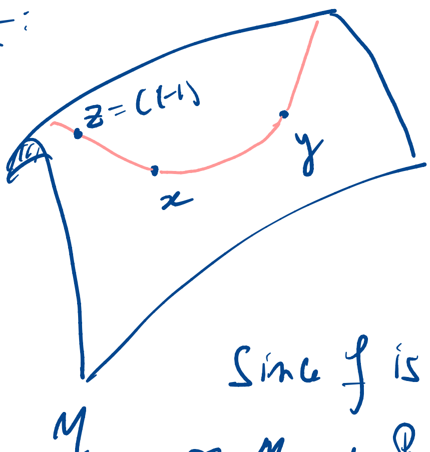
Say $\mathcal{M}$ is a complete Riemannian manifold. It is g-convex as a whole.

Let $f: \mathcal{M} \to \mathbf{R}$ be geodesically convex on the whole manifold $\mathcal{M}$.

**Fact:** If $f$ attains its maximum value on $\mathcal{M}$, then $f$ is constant.

**Cor.:** If $\mathcal{M}$ is compact, then $f$ is constant.

Pf:




Let $x \in \mathcal{M}$ be a max. of $f$.

For contradiction, say $\exists\, y \in \mathcal{M}$ s.t. $f(y) < f(x)$.

Let $c : \mathbb{R} \to \mathcal{M}$ be a geodesic s.t. $c(0) = x$ and $c(1) = y$.

Let $z = c(-1)$.

Since $f$ is g-convex and $c : [-1, 1] \to \mathcal{M}$ is a geodesic segment on $\mathcal{M}$, we have:

$$f(x) \leq \frac{1}{2} \underbrace{f(z)}_{\leq f(x)} + \frac{1}{2} \underbrace{f(y)}_{< f(x)} < f(x).$$

$\square$

# More definitions of g-convex functions

**Def.:** A function $f : S \to \mathbf{R}$ is geodesically convex if $S$ is g-convex and

$$f\big(c(t)\big) \leq (1-t)f\big(c(0)\big) + tf\big(c(1)\big)$$

for all geodesic segments $c: [0,1] \to \mathcal{M}$ that stay in $S$ and all $t \in [0,1]$.

We say that $f$ is geodesically strictly convex if moreover

$$f\big(c(t)\big) < (1-t)f\big(c(0)\big) + tf\big(c(1)\big)$$

for all $c$ as above with $c(0) \neq c(1)$ and all $t \in (0,1)$.

We say $f$ is geodesically $\mu$-strongly convex with $\mu > 0$ if moreover

$$f\big(c(t)\big) \leq (1-t)f\big(c(0)\big) + tf\big(c(1)\big) - \frac{t(1-t)\mu}{2}\mathrm{Length}(c)^2$$

for all $c$ as above and all $t \in [0,1]$.

**Properties of the set of minimizers:**

1. If $f : S \to \mathbf{R}$ is g-convex, they form a g-convex set (may be empty).
2. If $f : S \to \mathbf{R}$ is strictly g-convex, it has at most one minimizer.
3. If $f : S \to \mathbf{R}$ is strongly g-convex and differentiable, and $S$ is closed and nonempty, then $f$ has exactly one minimizer.

# Competing definitions of g-convex sets

There is some leeway in how we define g-convex sets.

The one here is permissive, yet still fruitful for optimization.

It has some downsides though (e.g., not closed under intersection).

See §11.3 for two common (and more restrictive) definitions.

All three coincide if $\mathcal{M}$ is complete and each pair $x, y$ is connected by a unique geodesic (e.g., Cartan–Hadamard manifolds).