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Distance, geodesics and complete manifolds

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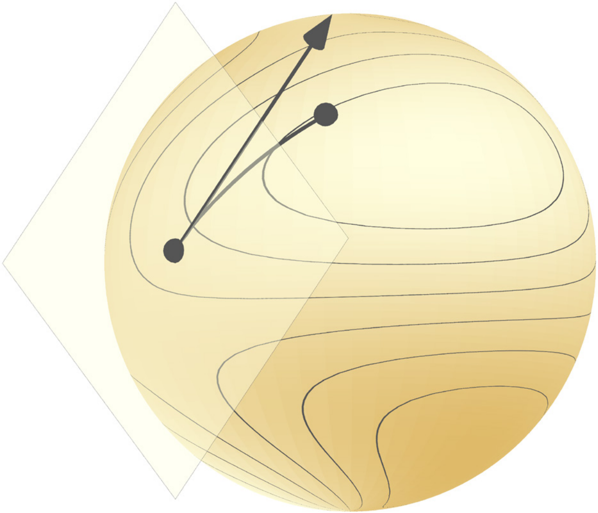
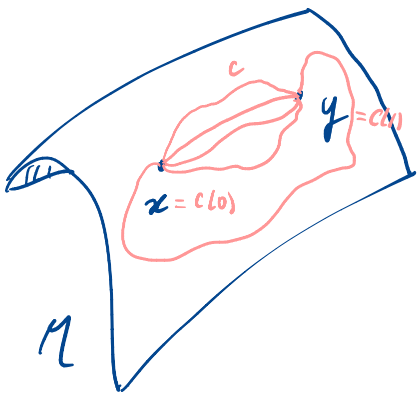
Optimization on manifolds, MATH 512 @ EPFL

Instructor: Nicolas Boumal



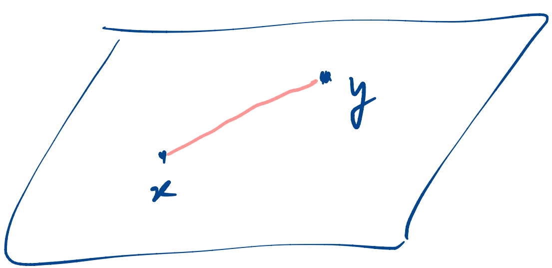
The Riemannian metric induces a distance

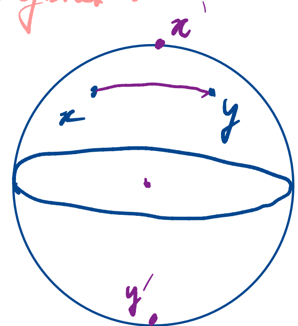


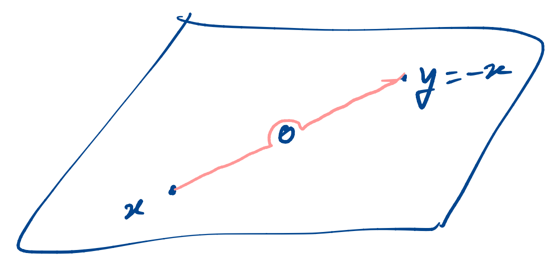
$$\mathrm{Length}(c) = \int_0^1 \| c'(t) \|_{c(t)} \, dt$$

$$\mathrm{dist}(x,y) = \inf_{\substack{c:[0,1]\to M \\ c(0)=x,\ c(1)=y}} \mathrm{Length}(c)$$

This defines a distance if M is connected.







Def: The length of a smooth curve segment $c: [0,1] \to \mathcal{M}$ is

$$\text{Length}(c) = \int_0^1 \|c'(t)\|_{c(t)} \mathrm{d}t$$

Fact: If $\mathcal{M}$ is connected, the function dist: $\mathcal{M} \times \mathcal{M} \to \mathbf{R}$,

$$\text{dist}(x, y) = \inf_{\substack{c:[0,1]\to\mathcal{M} \\ c(0)=x, c(1)=y}} \text{Length}(c),$$

defines a distance on $\mathcal{M}$. We call it the Riemannian distance.

With that distance, $\mathcal{M}$ is a metric space.

The metric topology and the manifold topology are the same.

Continuous: $x \mapsto \text{dist}(x, y)$ | $\{x \in \mathcal{M} : \text{dist}(x, y) < r\}$ open with time parametrization s.t. $\|c'(t)\|_{c(t)}$ is constant.

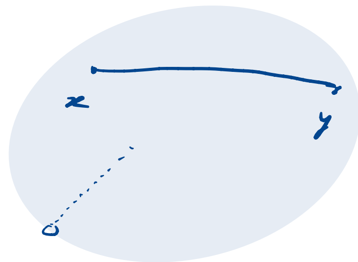
Fact: If there exists a curve segment $c: [0,1] \to \mathcal{M}$ from x to y such that $\text{Length}(c) = \text{dist}(x, y)$, then c is a geodesic.
We call such c a minimizing geodesic.

Def.: $x_0, x_1, x_2, \ldots$ is a Cauchy sequence if its points eventually get arbitrarily close: $\forall \varepsilon > 0, \exists K$ s.t. $\operatorname{dist}(x_k, x_\ell) < \varepsilon$ for all $k, \ell \geq K$.

Def.: $\mathcal{M}$ is metrically complete if all Cauchy sequences converge.

Fact: On a complete and connected manifold, each pair x, y is connected by a minimizing geodesic.

Completeness is sufficient but not necessary.



Metric completeness captures more than existence of minimizing geodesics: it captures the fact that geodesics “keep going forever”.

Recall: a curve c is a geodesic if $\frac{\mathrm{D}}{\mathrm{d}t} c'(t) = 0$ for all t.

This is a smooth differential equation:

> The solution c is uniquely determined by initial conditions $c(0) = x$ and $c'(0) = v$.
>
> But $c(t)$ is not necessarily defined for all t: what can stop it?

Def.: $\mathcal{M}$ is geodesically complete if all geodesics exist for all $t \in \mathbf{R}$.

Theorem (Hopf–Rinow). For $\mathcal{M}$ connected, these are equivalent:

- $\mathcal{M}$ is geodesically complete;
- $\mathcal{M}$ is metrically complete;
- A subset of $\mathcal{M}$ is compact iff it is closed and bounded.

Example: If $\mathcal{M}$ is embedded in $\mathcal{E}$ and closed in $\mathcal{E}$, it is complete.