Proofs and Programs

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Contents

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Basis

- Lecture: Tue 8h-10h (Philippe Audebaud)
- Tutorial: We 8h-10h (Aurore)

10 Weeks of courses (3x3), which is really low.

$$Final\ mark = 50\% \cdot CC + 50\% \cdot Exam$$

No mid-time exam, but weekly homework.

Warning Presence at the courses and tutorial will have an impact on the marks.

Prerequisites

- L2.2 $\rightarrow$ Logical (Natacha P., Chapter 1 & 2):
  - Proof theory
  - Formal system for logic inference.
- λ-calculus
- Category theory

Part I
(Pure) λ-Calculus

1 Computing with functions ?

How do we do mathematics ?

A Having *structures:* numbers, spaces (points, vectors, functions) $\rightarrow$ Eilenberg-Mac Lane ($\sim$ 1942) Category theory

B Build, explore, transform structures $\rightarrow$ Church ($\sim$ 1930) *λ-Calculus*

C Compare "stuff": *equality* $\rightarrow$ Voevoski ($\sim$ 2006) Algebraic topology $\rightarrow$ search HoTT (Hight order Type Theory)

D Provide a framework (*rules*) to reasoning on all that! $\rightarrow$ 1st point

2 Church λ-calculus (informally)

$$\begin{aligned} f :& A \rightarrow B \\ & x \mapsto e \end{aligned}$$

Given $a \in A$, $f(a)$ is the "replacement of the occurrence of x in e by a"

$$f \stackrel{\text{def}}{=} \lambda x.e \qquad (\lambda\text{-abstraction})$$
$$f\ a = (\lambda a.e)\ a \qquad (\text{Application})$$

Notation

$$e < a/x >$$

is the replacement in e of all the occurrences of a by x.

Example

1. $$\lambda x.x$$
   $$x \mapsto x$$

   is the identity function

2. $$\lambda x.y$$
   $$x \mapsto y$$

   Here x and y are variables, $x \neq y$. $(\lambda x.y)\ a$ leads to $y < a/x > \equiv y$

   $$(\lambda x.a)\ b \rightarrow_\beta a < b/x >$$

   $\rightarrow_\beta$ is a binary relation on lambda-terms $\Rightarrow$ idea of computation on terms.

Notion of α-equivalence

$$\lambda x.a \stackrel{?}{=}_\alpha \lambda y.b$$

Pick a *fresh* variable, let say z,

$$a < z/x > =_\alpha b < z/y >$$

All the results and proofs will be done under the quotient implied by the α-equivalence

3 A toolbox on λ-calculus

Let $\mathcal{X}$ be a measurable set of variables, ranged over by $x, y, z, ...$

Definition 1. *A λ-term e is generated by the grammar:*

$$a, b, e ... ::= x \in \mathcal{X} \mid \lambda x.e \mid a\ b$$

The set of λ-terms is denoted Λ.

Definition 2 (Free variable)**.** *The set of free variables in e, denoted $FV(e)$ is defined inductively:*

- *if $e \equiv x \in \mathcal{X},\ FV(x) \equiv \{x\}$*
- *if $e \equiv \lambda x.a_0,\ FV(\lambda x.a_0) \equiv FV(a_0) \setminus \{x\}$*
- *if $e \equiv a_1\ a_2,\ FV(a_1\ a_2) \equiv FV(a_1) \cup FV(a_2)$*

A term e is closed *if $FV(e) = \emptyset$*

Definition 3 (Substitution)**.** *Given $x \in \mathcal{X}$, $a \in \Lambda$, the substitution of (all the) occurrences of a in $e \in \Lambda$, denoted $e < a/x >$ is:*

- *if $y \in \mathcal{X} \setminus,\ \{x\},\ y < a/x > \equiv y$ and $x < a/x > \equiv a$*

- $(\lambda y.e) < a/x >= \lambda y.e < a/x >$
- $(e\ f) < a/x >= (e < a/x >)\ f < a/x >$

Definition 4 ($\rightarrow_\beta$ reduction)**.**

$$\begin{aligned} \rightarrow_\beta &\subseteq \Lambda \times \Lambda \\ &\Big\{ \big(\underbrace{(\lambda x.a)\ b}_{redex}, \underbrace{a < b/x >}_{contraction} \big) \Big| x \in \mathcal{X},\ a, v \in \Lambda \Big\} \end{aligned}$$

Example

1.

$$\begin{aligned} (\ \underbrace{\lambda x.(\lambda y.y)\ a}_{\twoheadrightarrow_\beta (\lambda x.(\lambda y.y))\ b}\ )\ b) \rightarrow_\beta ((\lambda y.y)\ a)\ \ < b/x > &\equiv ((\lambda y.y)\ \ < b/x >)a < b/x > \\ &\equiv (\lambda y.y)\ a < b/x > \end{aligned}$$

2.

$$(\lambda x.y)\ a \rightarrow_\beta y$$

3.

$$\begin{gathered} (\lambda x.x\ x)(\lambda x.x\ x) \rightarrow_\beta (x\ x) < \lambda x.x\ x/x > \text{ or } (x\ x) < \lambda y.y\ y/x > \\ (\lambda x.x\ x)(\lambda x.x\ x) \end{gathered}$$

Russell paradox: we get an infinite β-reduction!

$$\rightarrow_\beta \subseteq\ \beta_0\ \subseteq \underbrace{\beta}_{\beta-\text{reduction}} = \beta_0^*$$

$$\rightarrow_\beta^* \text{ is the } \beta\text{-reduction, noted } \twoheadrightarrow_\beta$$

Definition 5 (β_0-contraction)**.** *Let $a, b \in \Lambda$. $a\ \beta_0\ b$ is defined by cases:*

- $x\ \beta_0\ x$
- $(\lambda x.u)v\ \beta_0\ u < v/x >$
- $(\lambda x.u)\ \beta_0\ (\lambda x.v)$ *if* $u\ \beta_0\ v$
- $(u\ v)\ \beta_0\ (u'\ v)$ *if* $u\ \beta_0\ u'$
- $(u\ v)\ \beta_0\ (u\ v')$ *if* $v\ \beta_0\ v'$