Performance Evaluation in Networks

Eric Thierry*
ENS de Lyon

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*eric.thierry@ens-lyon.fr

0 Introduction

0.1 Generalities

Performance of computer and communication systems/networks. Objectives:

- Observation
- Prediction
- Control & Optimization

Background needed:

- Not much, try to give always the specification of the system we will use

Systems can be architecture/hardware, code/software , communication network (included distributed systems) and logistic/industrial processes, etc.

The metrics will be:

- Speed, bandwidth, delay, load, losses.
- Worst case, average, ..

Use of either a mathematical analysis of an abstract model or a simulation (math model, scale model). Comparisons with experiments/measures on a real system (statistics).

Objectives:

- Designing and analysing mathematical models (probability assumptions)
- Improve knowledge in probability/statistical tool
- Improve knowledge about communication networks
- practising simulation/statistic with a simulation/statistic software

Why are there so many probabilities?

- Sometimes intrinsic in the system (i.e. noise in the core of the system
- The users themselves introduce probabilities (Ethernet protocol and randomize algorithms)
- Probabilistic is a good way to have a good "rough estimation"
- Usually works well in practice

Use of statistics: Do I need to do more experiments? Do I need to run them longer (asymptotical behaviour)?

Classical questions: Overload? Average waiting time?

Bibliography:

- Performance Evaluation of Computer and Communication systems
- Introduction aux probabilités et à la statistique
- The Art of Computer Systems Performance Analysis
- http://perfeval.epfl.ch/
- http://proweb.

0.2 Refresher about Probability

Cf first year

Usual working hypothesis:

- A list Ω of the outcomes and a measure of their occurrences via the measures of events.

Definition 1. *Ω is the universe, $\mathcal{F}$ a set of subsets of Ω, $\mathbb{P}$ function from Ω to $\mathbb{R}$.*

Vocabulary:

- $\omega \in \Omega$ is an outcome
- $A \in \mathcal{F}$ is an event
- ω realise A if $\omega \in A$

Recalls:

- Conditional probabilities
- Independence of events
- Law of total probabilities
- General/Real Random Variable
- Cumulative distribution function: $F_x(x) = \mathbb{P}(X \leq x)$ ($\lim\limits_{x \to -\infty} F(x) = 0$ and $\lim\limits_{x \to +\infty} F(x) = 1$, F non decreasing, F right continuous)
- Discrete/Continuous r.v.[1]
- Random vectors & Joint distribution
- Expectation
- Composition of discrete/continuous r.v. (if g is Lebesgue integrable, then $\mathbb{E}(g(X)) = \int_x g(x)f(x)dx$)
- Same for vectors
- Moments of a r.v. (generalisation of expectation and standard deviation)

[1] Random Variable

- Generating functions (Probabilities $G_X(s) = \mathbb{E}(s^x)$, moments $M_X(t) = \mathbb{E}(e^{tX})$, characteristic $\Phi_X(t) = \mathbb{E}(e^{itX})$)
- Markov, Bienaymé-Tchebychev, Jensen (convex function), Hölder, Minkowski
- Chernoff and Hoeffding
- Convergence (in law, in proba, almost sure)
- Central limit theorem
- Law of large numbers

1 Random number generation

cf Webpage

2 Markov chains

Recalls

3 Martingales and stuff

4 Continuous time Markov chains

Definition 2. *A process $(X_t)_{t\in\mathbb{R}_+}$ over a space of states E is a homogeneous continuous Markov chain if:*

$$\begin{aligned}\forall t_1 < ... < t_n \in \mathbb{R}_+, \mathbb{P}(X_{t_n} = i_n | X_{t_{n-1}} = i_{n-1}, ..., X_{t_1} = i_1) &= \mathbb{P}(X_{t_n} = i_n | X_{t_{n-1}} = i_{n-1} && \text{(Markov)}\\ &= \mathbb{P}(X_{t_n - t_{n-1}} = i_n | X_0 = i_{n-1}) && \text{(Homogeneous)}\end{aligned}$$

Notation: $p_{ij}(t) = \mathbb{P}(X_t = j | X_0 = i) \to P_t = (p_{ij}(t))_{i,j\in E}$

Kolmogorov equations: $P_{t+s} = P_s P_t, \forall s, t \in \mathbb{R}_+$ (same proof)

Discrete case: $P_n = P^n$ with $P = P_1 = \mathbb{P}(X_1 = j | X_0 = i)$ ("kernel")

Continuous case: $P_t \underset{t\to 0}{\to} ?$

Classical assumption: $P_t \underset{t\to 0}{\to} Id$ i.e $\forall i, j, p_{ij}(t) \underset{t\to 0}{\to} 0$ if $i \neq j$ and $p_{ii}(t) \underset{t\to 0}{\to} 1$
("standard") ("continuity") ("differentiability at 0")

$$\begin{aligned}\exists Q, \frac{P_t - Id}{t} &\underset{t\to 0}{\to} Q\\ \text{i.e.}\forall i \neq j, \frac{p_{ij}(t)}{t} &\to q_{ij}\\ \forall i, \frac{p_{ii}(t) - 1}{t} &\to q_{ii}\end{aligned}$$

Q = infinitesimal generator = "kernel"

Remarks about coeff of Q:

- $\forall i \neq j,\ q_{ij} \geq 0$
-

$$\begin{aligned}
\forall t \in \mathbb{R}_+, \sum_{j\in E} p_{ij}(t) &= 1 \\
\Rightarrow \forall t > 0, \frac{\sum_{j\in E} P_{ij}(t) - 1}{t} &= 0 \\
(t \to 0) \Rightarrow \sum_{j \in E} q_{ij} &= 0 \\
\Rightarrow q_{ii} &\leq 0
\end{aligned}$$

Vocabulary: q_{ij} = transition *rate* from i to j

Theorem 1 (Kolmogorov)**.** *With previous assumptions,*

$$\forall t \in \mathbb{R}, \frac{\mathrm{d}P_t}{\mathrm{d}t} = QP_t$$

Proof. Use $P_{t+s} = P_t P_s$, more precisely:

$$\begin{aligned}
p_{ij}(t+h) - p_{ij}(t) &= \sum_k p_{ik}(h)p_{kj}(t) - p_{ij}(t) \\
&= \sum_{k\neq i} p_{ik}(h)p_{kj}(t) - (p_{ii}(h) - 1)p_{ij}(t)
\end{aligned}$$

Divide by h and make h tend to 0:

$$\begin{aligned}
\lim_{h\to 0} \frac{p_{ij}(t+h) - p_{ij}(t)}{h} &= \sum_{i \neq k} q_{ik}p_{kj}(t) + q_{ii}p_{ij}(t) \\
&= \text{coeff } (i,j) \text{ of } QP_t
\end{aligned}$$

□

Theorem 2 (Kolmogorov bis)**.**

$$\frac{\mathrm{d}P_t}{\mathrm{d}t} = P_t Q$$

Corollary 1. *Since we know that $P_0 = I$ the identity matrix:*

$$P_t = e^{tQ} \underset{def}{=} \sum_{n\in\mathbb{N}} \frac{(tQ)^n}{n!}$$

Remark: all the work on finite matrices apply to those infinite matrices up to defining/using matrix norms (many norms defined by combinations of sup, $\sum$)

Alternate description: "jump process". 1 Trajectory of HMC = alternation of *waiting times* and jumps.

Waiting times: suppose you are at state i at time 0.
Consider

$$\begin{aligned}
W &= \text{the time when you leave } i \\
&= \inf\{t \geq 0 | X_t \neq i\} \\
\mathbb{P}(W > s+t | W > s) &= ? \\
\{W > s\} &= \{\forall 0 \leq u \leq s, X_u = i\} \\
&= \mathbb{P}(\forall \underbrace{0 \leq u \leq s+t}_{s \leq u \leq s+t}, X_u = i | \forall \underbrace{0 \leq u \leq s, X_u = i}_{X_s = i}) \\
&= \mathbb{P}(W > t) \ \rightarrow \ \text{Memoryless law} \\
&= exp(\underbrace{\lambda}_{?}) \\
\mathbb{P}(W > t) &= e^{-\lambda t} \underset{t \text{ small}}{\simeq} p_{ii}(t) = \mathbb{P}(X_t = i | X_0 = i) \\
&\overset{\text{def}}{=} 1 + Q_{ii} t + o(t)
\end{aligned}$$

Jump probability: supp. that you are at state i at time 0.
Consider an an interval $[s, s+t[$ such that $s \leq W < s+t$, and suppose that the chain jumps only in this interval,

$$\mathbb{P}(\text{jump to } i | \text{it jumps}) \underset{t \text{ small}}{\simeq} \frac{p_{ij}(t)}{1 - p_{ii}(t)} = \frac{q_{ij} t + o(t)}{-q_{ii} t + o(t)} \overset{t \to 0}{\rightarrow} \frac{q_{ij}}{-q_{ii}}$$

Evolution/asymptotic behaviour of HMC

Invariant measure/distribution: $\pi \in \mathbb{R}_+^E$ satisfying $\forall t \leq 0, \pi P_t = \pi$ (and proba dist, i.e. $\sum_{i \in E} \pi_i = 1$)

Theorem 3.

$$\forall t \geq 0, \pi P_t = \pi \Leftrightarrow \pi Q = 0$$

Proof.

$$\begin{aligned}
\pi Q = 0 &\Leftrightarrow \forall n \in \mathbb{N}^*, \pi Q^n = 0 \\
&\Leftrightarrow \forall t \in \mathbb{R}_+, \pi \sum_{n=0}^{+\infty} \frac{t^n Q^n}{n!} = \pi
\end{aligned}$$

□

Irreducible HMC: rate transition graph

$$= \begin{cases} \text{vertices} & \text{states } E \\ \text{directed edges } i \to j & \text{if } q_{ij} > 0 \end{cases}$$

irreducible $\overset{\text{def}}{=}$ graph strongly connected.

Theorem 4 (Convergence result)**.** *Suppose that an HMC is irreducible and admits an invariant proba distribution π.*

$$\lim_{t \to \infty} P_{ij}(t) = \pi_j$$

Moreover, let $f : E \to \mathbb{R}$ such that $\sum_{i \in E} \pi_i |f(i)| < \infty$.
Then

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t f(X_s) \mathrm{d}s = \sum_{i \in E} \pi_i f(i)$$

5 Queues

Queue = buffer + server

Classical questions:

- Average waiting time for a client
- Proportion of time the server is busy
- Distribution of nb of clients waiting in the queue

Kendall notation (for single queue)

$$\underbrace{A}_{\substack{\text{law of interarrival}\\\text{of clients}}} \;/\; \underbrace{B}_{\substack{\text{law of service}\\\text{time (time to}\\\text{serve one client)}}} \;/\; \underbrace{S}_{\substack{\text{nb of}\\\text{process}}} \;(/\; \underbrace{N}_{\substack{\text{nb of}\\\text{buffers}}}\;)$$

All are usually assumed independent.

Classical laws:

- M (Markov) = exponential law
- D (Deterministic) = constant time
- E (Erlang) = Erlang law
- G (General) = arbitrary law

Example: analysis of the $M(\lambda)/M(\mu)/1$ queue.

$$X_t = \text{ size of the queue at time } t$$

Different behaviour:

- $X_t = 0 \to$ new client will arrive within time $\sim exp(\lambda)$
- $X_t \geq 1 \to$ one client is being saved within time $\sim exp(\mu)$ and one new client will arrive within time $\sim exp(\lambda)$

5.1 M/M/1 queue

$(N_t)_{t\in\mathbb{R}_+}$ is a continuous time HMC.

Search for invariant proba distribution π $\pi Q = \pi \Rightarrow \forall n \in \mathbb{N}, \pi_n = \left(\frac{\mu}{\lambda}\right)^n \pi_0$ An invariant proba measure exists iff $\sum_{n=0}^{+\infty} \pi_n = 1 \Leftrightarrow \frac{\mu}{\lambda} < 1$. Then $\pi_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$ (geometric law).
under the assumption that the system is at the permanent/stationary state/regime, i.e. N_0 follows the invariant distribution π, let's study some performance parameters.

1. Average nb of packets in the system (we assume that we have the distribution π at time 0)

$$\mathbb{E}_\pi(N_t) = \sum_{n=0}^{+\infty} n\pi_n = \frac{\frac{\mu}{\lambda}}{1 - \frac{\mu}{\lambda}} \leftarrow \text{ mean for geometric law}$$

2. Let W = waiting time before being served

   Two cases:

   - arriving in an empty queue $\Leftrightarrow$ being served immediately
     $\mathbb{P}(W=0) = \mathbb{P}_\pi(N_t = 0) = \pi_0 = 1 - \frac{\mu}{\lambda}$
   - arriving in a non-empty queue: if there is a packet (with probability π_n) you must wait or sum of n independent exponential laws $Exp(\lambda)$
     $\rightarrow$ density $f(t)$ for W:

$$\begin{aligned} f(t) &= \sum_{n=1}^{+\infty} \underbrace{\pi_n}_{\left(\frac{\mu}{\lambda}\right)^n\left(1-\frac{\mu}{\lambda}\right)} \times \underbrace{\text{density of a sum of independent } Exp(\lambda)}_{e^{-\lambda t}\frac{\lambda^n t^{n-1}}{(n-1)!}} \\ &= e^{-\lambda t}\left(1-\frac{\mu}{\lambda}\right)\mu \underbrace{\sum_{n=1}^{+\infty}\frac{(\mu t)^{n-1}}{(n-1)!}}_{e^{\mu t}} \\ &= e^{-(\lambda-\mu t}(\lambda-\mu)\frac{\mu}{\lambda} \end{aligned}$$

   We have the law of W, let's compute $\mathbb{E}_\pi(W)$

$$\begin{aligned} \mathbb{E}_\pi(W) = \int_{t=0}^{+\infty} t f(t)\mathrm{d}t &= \int_0^{+\infty} t e^{-(\lambda-\mu)t}(\lambda-\mu)\frac{\lambda}{\mu}\mathrm{d}t \\ &= \frac{\mu}{\lambda}\int_0^{+\infty} t \underbrace{(\lambda-\mu)e^{-(\lambda-\mu)t}}_{\text{density for } Exp(\lambda-\mu} \mathrm{d}t \\ &= \frac{\mu}{\lambda}\frac{1}{\lambda-\mu} \end{aligned}$$

   Average time before leaving the system $\overline{T}$ i.e. spent in the system $= \mathbb{E}_\pi(W) + \underbrace{\frac{1}{\lambda}}_{\substack{\text{average}\\\text{waiting time}}} = \frac{\mu}{\lambda}\frac{1}{\lambda-\mu} + \frac{1}{\lambda} = \frac{1}{\lambda-\mu}$

   Remark

$$\underbrace{\overline{N}}_{\substack{\text{average size}\\\text{of the queue}}} = \underbrace{\mu}_{\text{input rate}} \underbrace{\overline{T}}_{\substack{\text{average time}\\\text{spent in the queue}}} \qquad \text{(Little's law)}$$

5.2 M/M/∞ queue

Behaviour of N_t = number of packet in the system?

Search for an invariant distribution π Substitution:

$$\pi_n = \frac{1}{n!}\left(\frac{\mu}{\lambda}\right)^n \pi_0$$

There is always an invariant distribution because $\sum_{n=0}^{+\infty}\pi_n = \pi_0 \sum_{n=0}^{+\infty}\frac{1}{n!}\left(\frac{\mu}{\lambda}\right)^n = \pi_0 e^{\frac{\mu}{\lambda}}$

$$\Rightarrow \forall n \in \mathbb{N}, \pi_n = e^{-\frac{\mu}{\lambda}} \frac{1}{n!} \left(\frac{\mu}{\lambda}\right)^n$$

Under the stationary assumption

1. $\mathbb{E}_\pi(N_t) = \overline{N} =$ mean value for Poisson laws of parameter $\frac{\lambda}{\mu} = \frac{\lambda}{\mu}$
2. $\mathbb{E}_\pi(\underbrace{\text{time spent in the system}}_{\text{service time}}) =$ mean value for $Exp(\lambda) = \frac{1}{\lambda}$.

Remark

$$\overline{N} = \mu \overline{T}$$

A useful relation: Little's law

In many cases,

$$\text{average nb of guys in the system} = \text{average arrival rate} \times \text{average time spend in the system}$$
$$\overline{N} = \mu \overline{T}$$

Examples

1. M/M/1 queue in the stationary state
2. M/M/∞ in the stationary state
3. General deterministic case with finite horizon

Definition 3 (Pseudo-inverse)**.**

$$f^{(-1)}(n) = \inf\{t \mid f(t) \geq n\}$$

Definition 4 (Definitions)**.**

- $A(t) =$ *number* $n \in \mathbb{N}$ *of packets entering between time 0 and time* $t \in \mathbb{R}_+$
- $B(t) =$ *number* $n \in \mathbb{N}$ *of packets leaving between time 0 and time* $t \in \mathbb{R}_+$
- $N(t) = A(t) - B(t) =$ *number of packet* in *the system at time* t
- $T(n) = B^{(-1)}(n) - A^{(-1)}(n) =$ *time spent in the system by the n-th packet*

"Averages:"

- *"finite horizons"* $=$ *on* $[0, t]$
- $\mu = \frac{A(t)}{t}$ *(arrival rate)*
- $\overline{T} = \frac{1}{A(t)} \sum_{i=1}^{A(t)} T(i)$ *(average time spent in the system)*
- $\overline{N} = \frac{1}{t} \int_{x=0}^{t} N(x)\mathrm{d}x$ *(average number of packets).*

Theorem 5 (Little's theorem (1961))**.** *If* $A(t) = B(t)$

$$\overline{N} = \mu \overline{T}$$

Proof. Graphical proof, interpreting integrals as areas.

$$\underbrace{\frac{1}{t} \int_{x=0}^{t} N(x)\mathrm{d}x}_{\overline{N}} = \underbrace{\frac{A(t)}{t}}_{\mu} \underbrace{\frac{1}{A(t)} \sum_{i=1}^{A(t)} T(i)}_{\overline{T}}$$

□