

Proofs and Programs

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*<https://perso.ens-lyon.fr/philippe.audebaud/PnP/>

Basis

- Lecture: Tue 8h-10h (Philippe Audebaud)
- Tutorial: We 8h-10h (Aurore)

10 Weeks of courses (3x3), which is really low.

$$Final\ mark = 50\% \cdot CC + 50\% \cdot Exam$$

No mid-time exam, but weekly homework.

Warning Presence at the courses and tutorial will have an impact on the marks.

Prerequisites

- L2.2 \rightarrow Logical (Natacha P., Chapter 1 & 2):
 - Proof theory
 - Formal system for logic inference.
- λ -calculus
- Category theory

Part I

(Pure) λ -Calculus

1 Computing with functions ?

How do we do mathematics ?

- A Having *structures*: numbers, spaces (points, vectors, functions) \rightarrow Eilenberg-Mac Lane (~ 1942) Category theory
- B Build, explore, transform structures \rightarrow Church (~ 1930) λ -Calculus
- C Compare "stuff": *equality* \rightarrow Voevodski (~ 2006) Algebraic topology \rightarrow search HoTT (Hight order Type Theory)
- D Provide a framework (*rules*) to reasoning on all that! \rightarrow 1st point

2 Church λ -calculus (informally)

$$\begin{array}{l} f : A \rightarrow B \\ x \mapsto e \end{array}$$

Given $a \in A$, $f(a)$ is the "replacement of the occurrence of x in e by a "

$$\begin{array}{ll} f \stackrel{\text{def}}{=} \lambda x. e & (\lambda\text{-abstraction}) \\ f\ a = (\lambda a. e)\ a & (\text{Application}) \end{array}$$

Notation

$$e\langle a/x \rangle$$

is the replacement in e of all the occurrences of a by x .

Example

1.

$$\begin{aligned} \lambda x.x \\ x \mapsto x \end{aligned}$$

is the identity function

2.

$$\begin{aligned} \lambda x.y \\ x \mapsto y \end{aligned}$$

Here x and y are variables, $x \neq y$. $(\lambda x.y) a$ leads to $y\langle a/x \rangle \equiv y$

$$(\lambda x.a) b \rightarrow_{\beta} a\langle b/x \rangle$$

\rightarrow_{β} is a binary relation on lambda-terms \Rightarrow idea of computation on terms.

Notion of α -equivalence

$$\lambda x.a \stackrel{?}{=}_{\alpha} \lambda y.b$$

Pick a *fresh* variable, let say z ,

$$a\langle z/x \rangle =_{\alpha} b\langle z/y \rangle$$

All the results and proofs will be done under the quotient implied by the α -equivalence

3 A toolbox on λ -calculus

Let \mathcal{X} be a measurable set of variables, ranged over by x, y, z, \dots

Definition 1. A λ -term e is generated by the grammar:

$$a, b, e \dots ::= x \in \mathcal{X} \mid \lambda x.e \mid a b$$

The set of λ -terms is denoted Λ .

Definition 2 (Free variable). The set of free variables in e , denoted $FV(e)$ is defined inductively:

- if $e \equiv x \in \mathcal{X}$, $FV(x) \equiv \{x\}$
- if $e \equiv \lambda x.a_0$, $FV(\lambda x.a_0) \equiv FV(a_0) \setminus \{x\}$
- if $e \equiv a_1 a_2$, $FV(a_1 a_2) \equiv FV(a_1) \cup FV(a_2)$

A term e is closed if $FV(e) = \emptyset$

Definition 3 (Substitution). Given $x \in \mathcal{X}$, $a \in \Lambda$, the substitution of (all the) occurrences of a in $e \in \Lambda$, denoted $e\langle a/x \rangle$ is:

- if $y \in \mathcal{X} \setminus \{x\}$, $y\langle a/x \rangle \equiv y$ and $x\langle a/x \rangle \equiv a$

- $(\lambda y.e)\langle a/x \rangle = \lambda y.e\langle a/x \rangle$
- $(e f)\langle a/x \rangle = (e\langle a/x \rangle) f\langle a/x \rangle$

Definition 4 (\rightarrow_β reduction).

$$\rightarrow_\beta \subseteq \Lambda \times \Lambda$$

$$\left\{ \left(\underbrace{(\lambda x.a) b}_{\text{redex}}, \underbrace{a\langle b/x \rangle}_{\text{contraction}} \right) \mid x \in \mathcal{X}, a, v \in \Lambda \right\}$$

Example

1.

$$\begin{aligned} (\underbrace{\lambda x.(\lambda y.y) a}_{\rightarrow_\beta (\lambda x.(\lambda y.y))} b) &\rightarrow_\beta ((\lambda y.y) a) \langle b/x \rangle \equiv ((\lambda y.y) \langle b/x \rangle) a \langle b/x \rangle \\ &\equiv (\lambda y.y) a \langle b/x \rangle \end{aligned}$$

2.

$$(\lambda x.y) a \rightarrow_\beta y$$

3.

$$\begin{aligned} (\lambda x.x x)(\lambda x.x x) &\rightarrow_\beta (x x) \langle \lambda x.x x/x \rangle \text{ or } (x x) \langle \lambda y.y y/x \rangle \\ &(\lambda x.x x)(\lambda x.x x) \end{aligned}$$

Russell paradox: we get an infinite β -reduction!

$$\rightarrow_\beta \subseteq \beta_0 \subseteq \underbrace{\beta}_{\beta\text{-reduction}} = \beta_0^*$$

\rightarrow_β^* is the β -reduction, noted \rightarrow_β

Definition 5 (β_0 -contraction). Let $a, b \in \Lambda$. $a \beta_0 b$ is defined by cases:

- $x \beta_0 x$
- $(\lambda x.u)v \beta_0 u\langle v/x \rangle$
- $(\lambda x.u) \beta_0 (\lambda x.v)$ if $u \beta_0 v$
- $(u v) \beta_0 (u' v)$ if $u \beta_0 u'$
- $(u v) \beta_0 (u v')$ if $v \beta_0 v'$