Proofs and Programs

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Contents

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Basis

- Lecture: Tue 8h-10h (Philippe Audebaud)
- Tutorial: We 8h-10h (Aurore)

10 Weeks of courses (3x3), which is really low.

$$Final\ mark = 50\% \cdot CC + 50\% \cdot Exam$$

No mid-time exam, but weekly homework.

Warning Presence at the courses and tutorial will have an impact on the marks.

Prerequisites

- L2.2 $\rightarrow$ Logical (Natacha P., Chapter 1 & 2):
  - Proof theory
  - Formal system for logic inference.
- λ-calculus
- Category theory

Part I
(Pure) λ-Calculus

1 Computing with functions ?

How do we do mathematics ?

A Having *structures:* numbers, spaces (points, vectors, functions) $\rightarrow$ Eilenberg-Mac Lane ($\sim$ 1942) Category theory

B Build, explore, transform structures $\rightarrow$ Church ($\sim$ 1930) *λ-Calculus*

C Compare "stuff": *equality* $\rightarrow$ Voevoski ($\sim$ 2006) Algebraic topology $\rightarrow$ search HoTT (Hight order Type Theory)

D Provide a framework (*rules*) to reasoning on all that! $\rightarrow$ 1st point

2 Church λ-calculus (informally)

$$\begin{aligned} f :& A \rightarrow B \\ & x \mapsto e \end{aligned}$$

Given $a \in A$, $f(a)$ is the "replacement of the occurrence of x in e by a"

$$f \stackrel{\text{def}}{=} \lambda x.e \qquad (\lambda\text{-abstraction})$$
$$f\ a = (\lambda a.e)\ a \qquad (\text{Application})$$

Notation

$$e\langle a/x\rangle$$

is the replacement in e of all the occurrences of a by x.

Example

1.

$$\lambda x.x$$
$$x \mapsto x$$

is the identity function

2.

$$\lambda x.y$$
$$x \mapsto y$$

Here x and y are variables, $x \neq y$. $(\lambda x.y)\ a$ leads to $y\langle a/x\rangle \equiv y$

$$(\lambda x.a)\ b \rightarrow_\beta a\langle b/x\rangle$$

$\rightarrow_\beta$ is a binary relation on lambda-terms $\Rightarrow$ idea of computation on terms.

Notion of α-equivalence

$$\lambda x.a \overset{?}{=}_\alpha \lambda y.b$$

Pick a *fresh* variable, let say z,

$$a\langle z/x\rangle =_\alpha b\langle z/y\rangle$$

All the results and proofs will be done up to α-equivalence (no difference made between $\lambda x.x$ and $\lambda y.y$).

3 A toolbox on λ-calculus

Let $\mathcal{X}$ be a measurable set of variables, ranged over by $x, y, z, ...$

Definition 1. *A λ-term e is generated by the grammar:*

$$a, b, e... ::= x \in \mathcal{X} \mid \lambda x.e \mid a\ b$$

The set of λ-terms is denoted Λ.

Definition 2 (Free variable)**.** *The set of free variables in e, denoted $FV(e)$ is defined inductively:*

- *if* $e \equiv x \in \mathcal{X},\ FV(x) \equiv \{x\}$
- *if* $e \equiv \lambda x.a_0,\ FV(\lambda x.a_0) \equiv FV(a_0) \setminus \{x\}$
- *if* $e \equiv a_1\ a_2,\ FV(a_1\ a_2) \equiv FV(a_1) \cup FV(a_2)$

A term e is closed *if* $FV(e) = \emptyset$

Definition 3 (Substitution)**.** *Given $x \in \mathcal{X}$, $a \in \Lambda$, the substitution of (all the) occurrences of a in $e \in \Lambda$, denoted $e\langle a/x\rangle$ is:*

- *if* $y \in \mathcal{X} \setminus \{x\},\ y\langle a/x\rangle \equiv y$ *and* $x\langle a/x\rangle \equiv a$

- $(\lambda y.e)\langle a/x\rangle = \lambda y.e\langle a/x\rangle$
- $(e\ f)\langle a/x\rangle = (e\langle a/x\rangle)\ f\langle a/x\rangle$

Definition 4 ($\to_\beta$ reduction)**.**

$$\begin{aligned}\to_\beta &\subseteq \Lambda \times \Lambda \\ &\Big\{\big(\underbrace{(\lambda x.a)\ b}_{redex},\ \underbrace{a\langle b/x\rangle}_{contraction}\ \big)\Big|x \in \mathcal{X},\ a, v \in \Lambda\Big\}\end{aligned}$$

Example

1. $$\begin{aligned}(\underbrace{\lambda x.(\lambda y.y)\ a}_{\twoheadrightarrow_\beta (\lambda y.y)\ b})\ b) \to_\beta ((\lambda y.y)\ a)\ \langle b/x\rangle &\equiv ((\lambda y.y)\ \langle b/x\rangle)a\langle b/x\rangle \\ &\equiv (\lambda y.y)\ a\langle b/x\rangle\end{aligned}$$

2. $$(\lambda x.y)\ a \to_\beta y$$

3. $$\begin{gathered}(\lambda x.x\ x)(\lambda x.x\ x) \to_\beta (x\ x)\langle \lambda x.x\ x/x\rangle \text{ or } (x\ x)\langle \lambda y.y\ y/x\rangle \\ (\lambda x.x\ x)(\lambda x.x\ x)\end{gathered}$$

   Russell paradox: we get an infinite β-reduction!

$$\to_\beta \subseteq\ \beta_0\ \subseteq \underbrace{\beta}_{\beta-\text{reduction}} = \beta_0^*$$

$$\to_\beta^* \text{ is the } \beta\text{-reduction, noted } \twoheadrightarrow_\beta$$

Definition 5 (β_0-contraction)**.** *Let $a, b \in \Lambda$. $a\ \beta_0\ b$ is defined by cases:*

- $x\ \beta_0\ x$
- $(\lambda x.u)v\ \beta_0\ u\langle v/x\rangle$
- $(\lambda x.u)\ \beta_0\ (\lambda x.v)$ *if* $u\ \beta_0\ v$
- $(u\ v)\ \beta_0\ (u'\ v)$ *if* $u\ \beta_0\ u'$
- $(u\ v)\ \beta_0\ (u\ v')$ *if* $v\ \beta_0\ v'$