Barycentric method for parametric space computation of a spline surface with "scalar parametric field"

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1 Definitions

Let S a discrete surface of $\mathbb{R}$ with four edges and defined by a set of k points.

Let

$$P = (P_i)_{0 \leq i \leq m-1} = \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix}_{0 \leq i \leq m-1}$$

The set of m points belonging to the given inside surface.

Let

$$E = (E_j)_{0 \leq j \leq n-1} = \begin{pmatrix} X_j \\ Y_j \\ Z_j \end{pmatrix}_{0 \leq j \leq n-1}$$

The set of n points belonging to the edges (data).

$$k = n + m$$

Let $N = N_j(u_j, v_j)_{0 \leq j \leq n-1}$,

The parameters set of the edges points (secondary data), $N \in [0,1]^{2n}$.

Let $M = M_i(u_i, v_i)_{0 \leq i \leq m-1}$,

The parameters set of the inside surface points (unknown to compute). $M \in [0,1]^{2m}$.

Let φ the application "parameteric scalar field" such that $\varphi(P) = M$ and defined by :

$$\forall i \in [0, m-1], u_i = \frac{\sum\limits_{j=0}^{n-1} \frac{u_j}{4\pi r_j^2}}{\sum\limits_{j=0}^{n-1} \frac{1}{4\pi r_j^2}} \tag{1}$$

With $\forall i \in [0, m-1], \forall j \in [0, n-1], r_j = \|P_i - E_j\| = P_iE_j$

$$\forall i \in [0, m-1], u_i = \frac{\sum_{j=0}^{n-1} \frac{u_j}{S_{P_iE_j}}}{\sum_{j=0}^{n-1} \frac{1}{S_{P_iE_j}}} = \frac{\sum_{j=0}^{n-1} \alpha_j u_j}{\sum_{j=0}^{n-1} \alpha_j} \tag{2}$$

The product $\alpha_j u_j$ corresponds to the "surfacic density of the parameter u_j at the distance r_j" (see part 3).

2 Demonstrations

2.1 φ is a parameterization of P (as a bijection from $\mathbb{R}^3$ to $[0,1]^2$)

Proof (reductio ad absurdum) :

Let suppose

$$\begin{aligned} &\exists P_{i'} \neq P_i, \varphi(P_{i'}) = \varphi(P_i) = M_i \\ &\Leftrightarrow r_{j'} = r_j \\ &\Leftrightarrow P_{i'} = P_i \end{aligned}$$

Hence the contradiction.

Reciprocal way proof is idem.

2.2 φ is isotropic (obvious)

2.3 φ is continuous on the edges, by extension (figure 2)

Proof :

Let

$$\epsilon = r_{j'} = \|P_i - E_{j'}\|$$

$$P_i \rightarrow E_{j'} \Leftrightarrow \epsilon \rightarrow 0$$

$$\lim_{\epsilon \to 0} u_i = \lim_{\epsilon \to 0} \frac{\sum_{j=0}^{n-1} \frac{u_j}{r_{j'}^2}}{\sum_{j=0}^{n-1} \frac{1}{r_{j'}^2}} = \frac{\frac{u_j}{\epsilon^2}}{\frac{1}{\epsilon^2}} = u_j$$

Likewise we have

$$\lim_{\epsilon \to 0} v_i = v_j$$

Hence the continuity on the edges.

2.4 φ is continuous inside

Proof :

Let $P_{i'}$ such that $r_{j'} - r_j = \epsilon$

$$\lim_{\epsilon \to 0} u_{i'} = \lim_{\epsilon \to 0} \frac{\sum\limits_{j=0}^{n-1} \frac{u_j}{r_{j'}^2}}{\sum\limits_{j=0}^{n-1} \frac{1}{r_{j'}^2}} = \lim_{\epsilon \to 0} \frac{\sum\limits_{j=0}^{n-1} \frac{u_j}{(r_j+\epsilon)^2}}{\sum\limits_{j=0}^{n-1} \frac{1}{(r_j+\epsilon)^2}} = u_i$$

According to definition 1.

2.5 The isobarycenter G of E (edges points), if it belongs to P, has for parameterization -by φ- their parameters isobarycenter (equation 3).

Proof :

Let

$$G = \begin{pmatrix} X_G \\ Y_G \\ Z_G \end{pmatrix} = \frac{1}{n} \begin{pmatrix} \sum\limits_{j=0}^{n-1} X_j \\ \sum\limits_{j=0}^{n-1} Y_j \\ \sum\limits_{j=0}^{n-1} Z_j \end{pmatrix} \quad ; \quad E_{j'} = \begin{pmatrix} X_{j'} \\ Y_{j'} \\ Z_{j'} \end{pmatrix}$$

$$\forall j' \in [0, n-1], r_{j'} = \|G - E_{j'}\| = \left\| \frac{1}{n} \begin{pmatrix} \sum\limits_{j=0}^{n-1} X_j - nX_{j'} \\ \sum\limits_{j=0}^{n-1} Y_j - nY_{j'} \\ \sum\limits_{j=0}^{n-1} Z_j - nZ_{j'} \end{pmatrix} \right\| = \left\| \frac{1}{n} \begin{pmatrix} \sum\limits_{j=0}^{n-1} (X_j - X_{j'}) \\ \sum\limits_{j=0}^{n-1} (Y_j - Y_{j'}) \\ \sum\limits_{j=0}^{n-1} (Z_j - Z_{j'}) \end{pmatrix} \right\|$$

Hence

$\forall j' \in [0, n-1]$,

$$r_{j'}^2 = \left[(X_G - X_{j'})^2 + (Y_G - Y_{j'})^2 + (Z_G - Z_{j'})^2\right]$$
$$r_{j'}^2 = \frac{1}{n^2} \left[\left(\sum_{j=0}^{n-1} X_j - X_{j'} \right)^2 + \left(\sum_{j=0}^{n-1} Y_j - Y_{j'} \right)^2 + \left(\sum_{j=0}^{n-1} Z_j - Z_{j'} \right)^2 \right]$$

In this last expression, for the specific case of the barycenter, we notice : $r_j = r_{j'}$.

Hence

$$u_G = \frac{\sum_{j'=0}^{n-1} \frac{u_{j'}}{r_{j'}^2}}{\sum_{j'=0}^{n-1} \frac{1}{r_{j'}^2}}$$

$$u_G = \frac{\frac{1}{r_j^2} \sum_{j'=0}^{n-1} u_{j'}}{\frac{n}{r_j^2}}$$

$$u_G = \frac{1}{n} \sum_{j=0}^{n-1} u_j \tag{3}$$

2.6 Any H isobarycenter of p points has for parameterization by φ the isobarycenter of their parameters

Let

$$H = \begin{pmatrix} X_H \\ Y_H \\ Z_H \end{pmatrix} = \frac{1}{p} \begin{pmatrix} \sum_{i'=0}^{p-1} X_{i'} \\ \sum_{i'=0}^{p-1} Y_{i'} \\ \sum_{i'=0}^{p-1} Z_{i'} \end{pmatrix} \quad ; \quad E_j = \begin{pmatrix} X_j \\ Y_j \\ Z_j \end{pmatrix}$$

A barycenter of p inside points.

Proof (from end to begining)

$$u_H = \frac{1}{p} \sum_{i'=0}^{p-1} u_{i'} \tag{4}$$

$$u_H = \frac{1}{p} \sum_{i'=0}^{p-1} \left(\frac{\sum_{j=0}^{n-1} \frac{u_j}{r_j^2}}{\sum_{j=0}^{n-1} \frac{1}{r_j^2}} \right)$$

$$u_H = \left(\frac{\sum_{j=0}^{n-1} \frac{u_j}{r_j^2}}{\sum_{j=0}^{n-1} \frac{1}{r_j^2}} \right) \sum_{i'=0}^{p-1} \frac{1}{p}$$

$$u_H = \frac{\sum_{j=0}^{n-1} \frac{u_j}{r_j^2}}{\sum_{j=0}^{n-1} \frac{1}{r_j^2}}$$

Which is by definition (equation 1) the parameterization of H by φ. Hence the result (equation 4) by reordering the steps of the proof in the reverse order.

3 Explanations and recalls

We consider a parameter u_j (here belonging to the surface edges) like having a remote influence on the inside points parameters. Hence the idea of a "parametric scalar field". Just like for other fields (electromagnetic, gravitational), this intensity on a point located at a r_j distance of the parameter is written like :

$$\rho_j = \frac{u_j}{4\pi r_j^2}$$

Additional observation : despite it doesn't take part into the parameters computation, the 4π coefficient actually allows to link with field computation via the solid angle notion (tridimensional space is seen under a 4π steradians angle -just like the sphere seen from its center for instance-). Because of this, the barycentric parameterization technique applies well to volumic surfaces in particular.

4 Figures

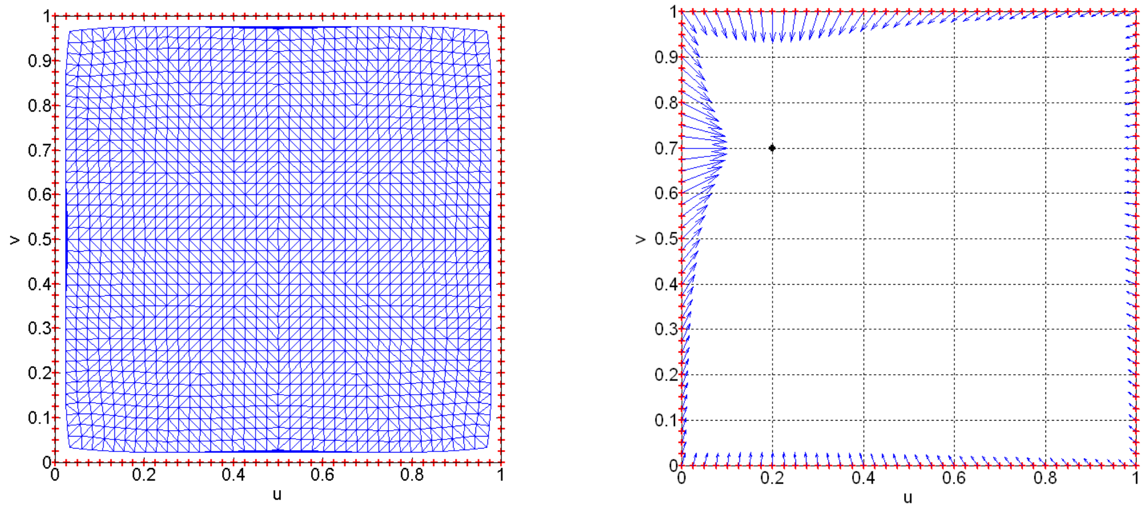

Figure 1: Left : parameters mesh created from a regular 41×41 data mesh (square flat surface) with barycentric method. Red crosses are the edges parameters (data) computed with curvi linear abscissa. Right : vectorized weights of this contour points parameters on an inside point, $(0.2, 0.7)$ here (black cross).

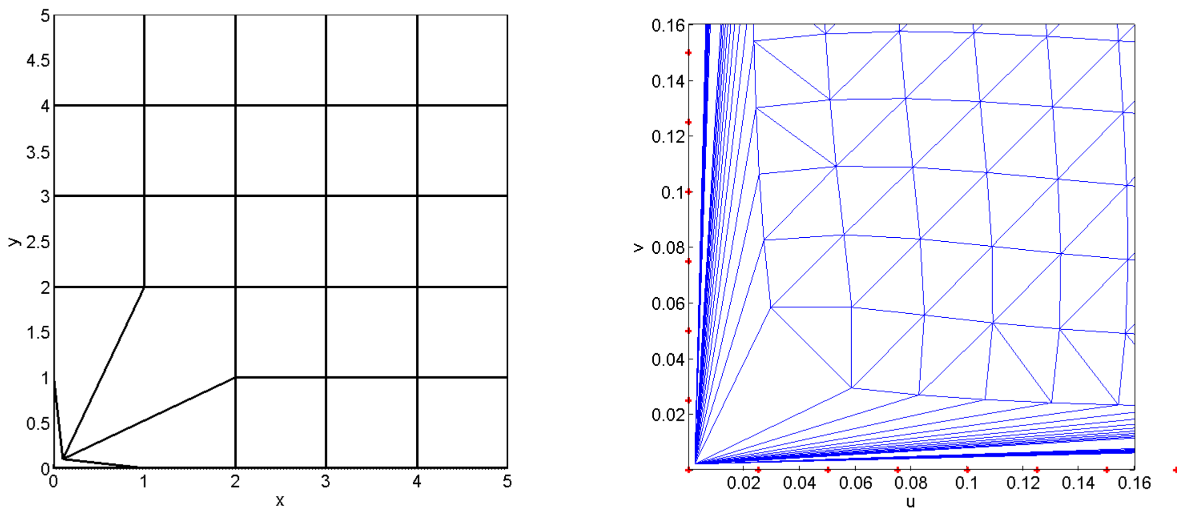

Figure 2: Left : a streching test of the corner of the previous data mesh, described in 1. Right : its effects on the parameters mesh, showing evidence of continuity.