Exam : Statistical modelling and its applications

Mines Saint-Étienne – 5th January 2016

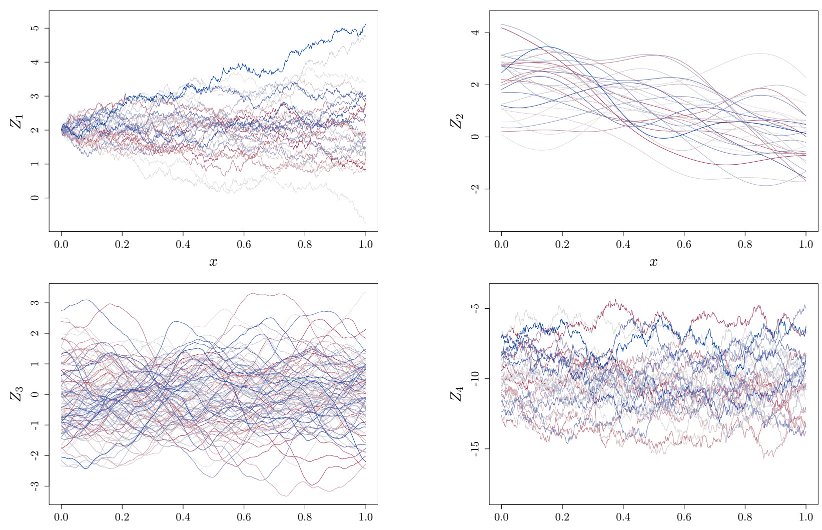
No document allowed except the Gaussian process regression handout and an A4 sheet with hand written notes.

Exercise 1 (5 pts)

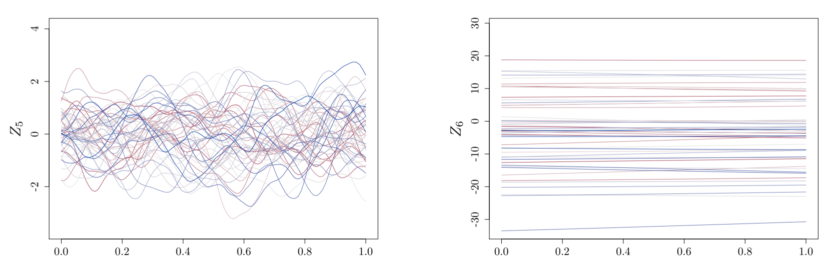
1. [1 pt] Detail how to generate samples from a one-dimensional Gaussian process with mean $\mu(.)$ and kernel $k(.,.)$.

2. [1.5 pts] The following figures show samples from four Gaussian processes. For each one, specify

   - the kernel (either exponential, Brownian, squared exponential or Matérn 3/2),
   - if the process is centred (yes/no),
   - if the process is stationary (yes/no).



3. [1 pt] The figures bellow correspond to samples of a Matern 5/2 kernel with variance parameter $\sigma^2 \in \{0.1, 1, 10, 100\}$ and with length scale $\theta \in \{0.1, 1, 10\}$. For each figure, specify the corresponding set of parameters.



4. [1.5 pts] We are interested in computing the mean value of a function f using no more that 50 observations. What are the main steps you would go through for solving this problem.

Exercise 2 (7 pts)

Let us consider the 2-dimensional function: $f(x_1, x_2) = x_1 + x_2 + x_1x_2$.

The aim is to perform a global sensitivity analysis of $f(X_1, X_2)$ where X_1, X_2 are independent uniform random variables, with $X_1 \sim \mathcal{U}\left[-\frac{a}{2}, \frac{a}{2}\right]$ and $X_2 \sim \mathcal{U}[-\frac{1}{2}, \frac{1}{2}]$, and $a > 0$.

We recall that for a uniform random variable $Z \sim \mathcal{U}[s, t]$, we have $\mathrm{E}(Z) = \frac{s+t}{2}$ and $\mathrm{var}(Z) = \frac{(t-s)^2}{12}$.

1. [2 pts] By verifying that X_1, X_2 and X_1X_2 satisfy the centering and non-simplification conditions, show that the Sobol-Hoeffding decomposition of $f(X_1, X_2)$ is simply:
$$\mu_0 = 0, \quad \mu_1(X_1) = X_1, \quad \mu_2(X_2) = X_2, \quad \mu_{1,2}(X_1, X_2) = X_1X_2$$
2. [1.5 pts] Compute the partial variances $D_I = \mathrm{var}(\mu_I(X_I))$ for $I = \{1\}, \{2\}, \{1, 2\}$ and check that the global variance is $D = \mathrm{var}(f(X_1, X_2)) = \frac{1}{12}\left(1 + \frac{13}{12}a^2\right)$
3. [1.5 pts] Recall that Sobol indices are defined by $S_I = D_I/D$. Compute S_1 and S_2, and check that S_1 (resp. S_2) is an increasing (resp. decreasing) function of a. Interpretation?

We now assume that X_1, X_2 are independent random variables, with $X_1, X_2 \sim \mathcal{U}[0, 1]$.

4. [0.5 pt] Explain why it is now *wrong* that $\mu_1(X_1) = X_1$.
5. [1.5 pts] Compute the Sobol decomposition of $f(X_1, X_2)$.

Exercise 3: ANOVA kernels (8 pts)

ANOVA kernels are kernels over $\mathbb{R}^d \times \mathbb{R}^d$ of the form : $k(x, y) = \prod_{i=1}^{d}\left(1 + k_i(x_i, y_i)\right)$, where the k_i are symmetric positive semi-definite functions.

1. [1 pt] Using the results from the course, show that ANOVA kernels are valid covariance functions.

We now consider costly-to-evaluate function $f : [0, 1]^{10} \to \mathbb{R}$, a design of experiment X based on 100 points and the set of observations F. The knowledge we have about f is that it is a smooth function that is infinitely differentiable.

2. [1 pt] With such settings, which kernel would you choose and what kind of Gaussian process regression model would you consider (simple Kriging, ordinary Kriging, Universal Kriging).
3. [1 pt] Give the expressions of the mean predictor and of the 95% confidence intervals.
4. [1 pt] Show that the mean predictor can be interpreted as a sum of 2^d functions with increasing interaction order. Does this decomposition coincides with the Sobol decomposition of the mean predictor? Why ?
5. [1 pt] Each term of this decomposition can be interpreted as a Gaussian process conditional distribution. Detail which one and deduce some confidence intervals associated to each sub-model.
6. [2 pts] According to an expert, the mean value of f is 6 and the interactions of order higher than 2 can be neglected. What changes can you make in the model and in the kernel expression in order to account for these informations ?
7. [1 pt] We now consider a particular type for the univariate kernels k_i such that $\int_0^1 k_i(s, x)\, ds = 0$ for all $x \in [0, 1]$. Is there a link between the sub-models and the Sobol decomposition in this particular case?

Bonus: Detail how to obtain a kernel k_i such that $\int_0^1 k_i(s, x)\, ds = 0$ using the conditional distribution of a Gaussian process given it has zero integral.